多元函数微分法及其应用（2）

偏导数

① 对 x 的偏导数：

令 $y=y_0$，若 $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$ 存在，则称此极限为函数 $z=f(x,y)$ 在点 (x_0,y_0) 处对 x 的偏导数，记作 $\left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}$ 或 $\left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}$ 或 $\left.z_x\right|_{\substack{x=x_0\\y=y_0}}$ 或 $f_x(x_0,y_0)$

② 对 y 的偏导数：

令 $x=x_0$，若 $\lim\limits_{\Delta y\to 0}\dfrac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$ 存在，则称此极限为函数 $z=f(x,y)$ 在点 (x_0,y_0) 处对 y 的偏导数.

函数 $z=f(x,y)$ 对自变量 x 的偏导函数可记作：

$\dfrac{\partial z}{\partial x}$ 或 $\dfrac{\partial f}{\partial x}$ 或 z_x 或 z'_x 或 $f_x(x,y)$ 或 $f'_x(x,y)$

（y 同理）

求 z_x 时，可将 y 视为常数，对 x 求导.（y 同理）

例1. 求 $z=x^2+3xy+y^2$ 在 $(1,2)$ 处的偏导数.

解. $\dfrac{\partial z}{\partial x}=2x+3y$，$\dfrac{\partial z}{\partial y}=3x+2y$

解: $\frac{\partial z}{\partial x} = 2x+3y$, $\frac{\partial z}{\partial y} = 3x+2y$

$\therefore \left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=2}} = 8$, $\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=2}} = 7$

例2. $z = x^2 \sin 2y$

$\frac{\partial z}{\partial x} = 2x\sin 2y$, $\frac{\partial y}{\partial x} = 2x^2\cos 2y$

例3. 设 $z = x^y (x>0, x\neq 1)$, 求证: $\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z$.

解: 证明: $\frac{\partial z}{\partial x} = yx^{y-1}$

$\frac{\partial z}{\partial y} = x^y \ln x$

$\therefore \frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = \frac{x}{\cancel{y}}\cdot \cancel{y}\, x^{y-1} + \frac{1}{\ln x} x^y \ln x$

$= x^y + x^y = 2x^y = 2z$

证明完毕

偏导数的几何意义

① 对 x 求偏导时, $y=y_0$ 不变, $z=f(x,y_0)$

② 对 y 求偏导时, $x=x_0$ 不变, $z=f(x_0,y)$

曲线在该点的斜率

(参考 P_{66} 图 9-5)

一元函数：可导必连续

多元函数：偏导数都存在未必连续

$$z=f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

其中 $\left.\dfrac{\partial z}{\partial x}\right|_{(0,0)}=\lim\limits_{\Delta x\to 0}\dfrac{f(0+\Delta x,0)-f(0,0)}{\Delta x}$

$=\lim\limits_{\Delta x\to 0}\dfrac{0}{\Delta x}=\lim\limits_{\Delta x\to 0}0=0$

$\left.\dfrac{\partial z}{\partial y}\right|_{(0,0)}=\lim\limits_{\Delta y\to 0}\dfrac{f(0,0+\Delta y)-f(0,0)}{\Delta y}$

$=\lim\limits_{\Delta y\to 0}\dfrac{0}{\Delta y}=\lim\limits_{\Delta y\to 0}0=0$

又 ∵ $\lim\limits_{\substack{y=kx\\ x\to 0}}=\dfrac{kx^2}{x^2+k^2x^2}=\dfrac{k}{1+k^2}$ 不为常数

∴ z 的极限不存在

∴ 对于该二元函数来说，偏导都存在未必连续

高阶偏导数

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f''_{xx}(x,y)=z''_{xx}$$

$$\left.\begin{matrix}\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)\\[2ex] \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)\end{matrix}\right\}\text{混合偏导数}$$

例6. 设 $z=x^3y^2-3xy^3-xy+1$.

$$\frac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 - y \quad , \frac{\partial z}{\partial y} = 2yx^3 - 9y^2x - x$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = 6xy^2 \ , \ \frac{\partial^2 z}{\partial y \partial x} = 6yx^2 - 9y^2 - 1 ,$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6yx^2 - 9y^2 - 1 \ , \ \frac{\partial^2 z}{\partial y^2} = 2x^3 - 18yx \ , \ \frac{\partial^3 z}{\partial x^3} = 6y^2$$

定理：如果函数 $z = f(x, y)$ 的二阶偏导数 $\frac{\partial^2 z}{\partial y \partial x}$ 及 $\frac{\partial^2 z}{\partial x \partial y}$ 在区域 D 内连续，那么在该区域内这两个二阶混合偏导数必相等。

例7. 验证 $z = \ln\sqrt{x^2+y^2}$ 满足 $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

解：$z = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{2} \times 2x \times \frac{1}{x^2+y^2} = \frac{x}{x^2+y^2} \ , \ \frac{\partial z}{\partial y} = \frac{y}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{x^2+y^2-2x \cdot x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \ , \ \frac{\partial^2 z}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

练习：① $z = x^3y - y^3x$, $z_x = 3x^2y - y^3$, $z_y = x^3 - 3xy^2$

② $z = (1+xy)^y$, $z_x = y^2(1+xy)^{y-1}$;

$\because z = e^{\ln(1+xy)^y} = e^{y\ln(1+xy)}$

$\therefore z'_y = (1+xy)^y \left[\ln(1+xy) + \frac{xy}{1+xy} \right]$

③ $u = x^{\frac{y}{z}}$, $u_x = \frac{y}{z} x^{\frac{y}{z}-1}$, $u_y = \frac{x^{\frac{y}{z}} \ln x}{z}$,

$$u_z = -\frac{y\left(x^{\frac{y}{z}} \ln x\right)}{z^2}$$