定积分的应用（体积与弧长）

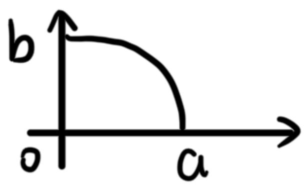
体积

旋转体：

例6. $P(h,r)$ $y=\frac{r}{h}x$

$$V=\int_0^h \pi y^2 dx$$
$$=\frac{\pi r^2}{h^2}\int_0^h x^2 dx$$
$$=\frac{\pi r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$
$$=\frac{1}{3}\pi r^2 h$$

例7.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

在第一象限：$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$$\Rightarrow y^2=\frac{b^2}{a^2}(a^2-x^2)$$
$$\Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}$$

$$\therefore V=2\int_0^a \pi\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2 dx$$
$$=2\pi\int_0^a \left(b^2-\frac{b^2}{a^2}x^2\right)dx$$
$$=\left[2\pi b^2 x\right]_0^a-\left[\frac{2\pi b^2}{a^2}\times\frac{1}{3}x^3\right]_0^a$$
$$=2\pi ab^2-\frac{2}{3}\pi ab^2$$

$= \frac{4}{3}\pi a b^2$

特殊地，当 $a=b$ 时，

此时为球的体积 $V=\frac{4}{3}\pi a^2$

绕 y 轴旋转：

$$V = \pi \int_c^d [\varphi(y)]^2 dy$$

$$y=f(x) \Rightarrow x=\varphi(y)$$

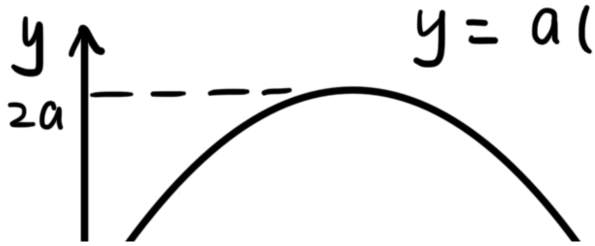
"截面为圆环型"求体积：

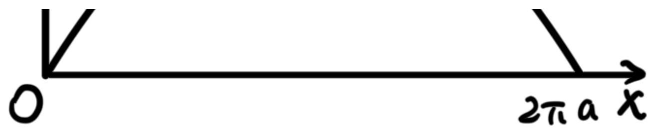
$$V = \int_a^b \pi f_1^2(x)\,dx - \int_a^b \pi f_2^2(x)\,dx$$
$$= \int_a^b \pi [f_1^2(x) - f_2^2(x)]\,dx$$

$$V = \int_c^d \pi [\varphi_2^2(y) - \varphi_1^2(y)]\,dy$$

例18. 摆线 $x=a(t-\sin t)$

$y=a(1-\cos t)$





绕x轴旋转：

$$V=\int_0^{2\pi a}\pi y^2 dx$$

又∵ $y=a(1-\cos t)$

$$dx=d\,a(t-\sin t)$$

$$=a(1-\cos t)dt$$

$x: 0\to 2\pi a$

$t: 0\to 2\pi$

$$\therefore V=\int_0^{2\pi}\pi a^2(1-\cos t)^2\cdot a(1-\cos t)dt$$

$$=\pi a^3\int_0^{2\pi}(1-\cos t)^3 dt$$

$$=\pi a^3\int_0^{2\pi}(1-3\cos t+3\cos^2 t-\cos^3 t)dt$$

$$=\pi a^3\left[t-3\sin t\right]_0^{2\pi}$$

$$+\frac{3}{4}\pi a^3\int_0^{2\pi}(1+\cos 2t)d(2t)$$

$$-\pi a^3\int_0^{2\pi}(1-\sin^2 t)d\sin t$$

$$=2\pi^2a^3+\frac{3}{4}\pi a^3\left[2t+\sin 2t\right]_0^{2\pi}$$

$$-\pi a^3\left[\sin t-\frac{1}{3}\sin^3 t\right]_0^{2\pi}$$

$$=2\pi^2a^3+3\pi^2a^3-0$$

$$=5\pi^2a^3$$

绕y轴旋转：

$$V=\int_0^{2a}\pi x_2^2(y)\,dy-\int_0^{2a}\pi x_1^2(y)\,dy$$

$$=\int_{2\pi}^{\pi}\pi a^2(t-\sin t)^2 a\sin t\,dt$$

$$-\int_0^{\pi}\pi a^2(t-\sin t)^2 a\sin t\,dt$$

$$=-\pi a^3\int_0^{2\pi}(t-\sin t)^2\sin t\,dt$$

$$=6\pi^3a^3$$

平行截面面积为已知的立体：

$$x^2+y^2=R^2$$

$$V=\int_{-R}^{R}\frac{1}{2}y\cdot y\tan\alpha\,dx$$

$$=\int_{-R}^{R}\frac{1}{2}(R^2-x^2)\tan\alpha\,dx$$

$$=\int_0^{R}(R^2-x^2)\tan\alpha\,dx$$

$$=\left[(R^2x-\frac{1}{3}x^3)\tan\alpha\right]_0^R$$

$$=\frac{2}{3}R^3\tan\alpha$$

例10.

正劈锥体：

$$V=\int_{-R}^{R}\frac{1}{\cancel{2}}h\cancel{2}y\,dx$$

$$=2\int_0^R hy\,dx$$

$$= 2\int_0^R h\sqrt{R^2-x^2}\,dx$$

令 $x = R\sin t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$

$\therefore dx = R\cos t\,dt$

$x: 0\to R,\ t: 0\to\frac{\pi}{2}$

$$\therefore V = 2hR^2\int_0^{\frac{\pi}{2}}\cos^2 t\,dt$$

$$= \frac{1}{2}hR^2\int_0^{\frac{\pi}{2}}(1+\cos 2t)\,d2t$$

$$= \left[\frac{1}{2}hR^2\cdot 2t\right]_0^{\frac{\pi}{2}} + \left[\frac{1}{2}hR^2\cdot\sin 2t\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}hR^2\pi + 0$$

$$= \frac{\pi R^2 h}{2}$$

或用"点火公式"

$$V = 2hR^2\int_0^{\frac{\pi}{2}}\cos^2 t\,dt$$

$$= 2hR^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi R^2 h}{2}$$

平面曲线的弧长

$$\begin{cases} x = \varphi(t) & dx = \varphi'(t)\,dt \\ y = \psi(t) & dy = \psi'(t)\,dt \end{cases}$$

$ds = \sqrt{(dx)^2+(dy)^2}$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{\varphi'^2(t)(dt)^2 + \psi'^2(t)(dt)^2}$$

$$= \sqrt{\varphi'^2(t) + \psi'^2(t)} \cdot dt$$

$$\therefore S = \int_\alpha^\beta \sqrt{\varphi'^2(t) + \psi'^2(t)} \cdot dt$$

$$\begin{cases} x = x \\ y = f(x) \end{cases}$$

$$S = \int_a^b \sqrt{1 + y'^2}\, dx$$

↳ 参考弧微分公式：

$$ds = \sqrt{1 + y'^2}\, dx$$

$$\rho = \rho(\theta),\ \alpha \le \theta \le \beta$$

$$\begin{cases} x = \rho(\theta)\cos\theta \\ y = \rho(\theta)\sin\theta \end{cases}$$

$$\Rightarrow \quad x' = \rho'(\theta)\cos\theta - \rho(\theta)\sin\theta$$

$$y' = \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta$$

$$\therefore x'^2 + y'^2 = \rho'^2(\theta) + \rho^2(\theta)$$

$$\therefore S = \int_\alpha^\beta \sqrt{\rho^2(\theta) + \rho'^2(\theta)}\, d\theta$$

例11. $y=\frac{2}{3}x^{\frac{3}{2}},\ a\le x\le b$

$y'=x^{\frac{1}{2}}$

$$S=\int_a^b\sqrt{1+y'^2}\,dx$$
$$=\int_a^b(1+x)^{\frac{1}{2}}\,d(1+x)$$
$$=\frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_a^b$$
$$=\frac{2}{3}\left[(1+b)^{\frac{3}{2}}-(1+a)^{\frac{3}{2}}\right]$$

例12. $\begin{cases}x=a(\theta-\sin\theta)\\y=a(1-\cos\theta)\end{cases}$

$0\le\theta\le2\pi$

$$S=\int_0^{2\pi}\sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$
$$=a\int_0^{2\pi}\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta$$
$$=a\int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta$$
$$=\sqrt{2}a\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta$$
$$=\sqrt{2}a\int_0^{2\pi}\sqrt{2\sin^2\frac{\theta}{2}}\,d\theta$$
$$=2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta$$
$$=4a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\frac{\theta}{2}$$
$$=-4a\left[\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$= -4a(\cos\pi - \cos 0)$

$= -4a(-1-1)$

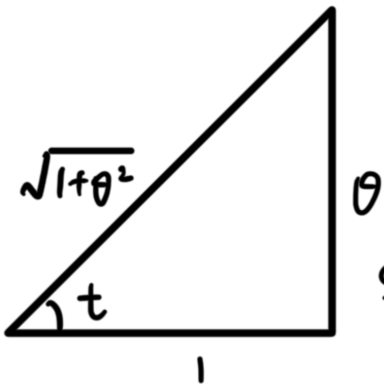
$= 8a$

例13. $\rho = a\theta,\ 0 \le \theta \le 2\pi$

$S = \int_0^{2\pi} \sqrt{a^2\theta^2 + a^2}\, d\theta$

$= a \int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta$

令 $\theta = \tan t$



$\sec t = \frac{1}{\cos t} = \sqrt{1+\theta^2}$

$\therefore d\theta = d\tan t$

$\therefore \int \sqrt{1+\theta^2}\, d\theta$

$= \int \sec t\, d\tan t = \int \sec^3 t\, dt$

$= \sec t \cdot \tan t - \int \tan t\, d\sec t$

$= \sec t \cdot \tan t - \int \sec t \cdot \tan^2 t\, dt$

$= \sec t \cdot \tan t - \int \sec t(\sec^2 t - 1)\, dt$

$= \sec t \cdot \tan t - \int \sec^3 t\, dt - \int \sec t\, dt$

$\therefore y = \sec t \cdot \tan t - y - \ln|\sec t + \tan t| + C$

$\Rightarrow y = \frac{1}{2}(\sec t \cdot \tan t - \ln|\sec t + \tan t|) + C$

$= \frac{1}{2}(\sqrt{1+\theta^2} \cdot \theta - \ln|\sqrt{1+\theta^2} + \theta|) + C$

$\therefore S = \frac{a}{2}\left[\sqrt{1+\theta^2} \cdot \theta - \ln|\sqrt{1+\theta^2} + \theta|\right]_0^{2\pi}$

$$= \frac{a}{2}\left[\sqrt{1+4\pi^2}\cdot 2\pi - \ln\left(\sqrt{1+4\pi^2}+2\pi\right)\right]$$