重积分(3)

二重积分的计算法(极坐标)

$$\Delta\sigma_i = \frac{1}{2}(\rho_i+\Delta\rho_i)^2\cdot\Delta\theta_i - \frac{1}{2}\rho_i^2\cdot\Delta\theta_i$$

$$= \frac{1}{2}(2\rho_i+\Delta\rho_i)\Delta\rho_i\cdot\Delta\theta_i$$

$$= \frac{\rho_i+(\rho_i+\Delta\rho_i)}{2}\cdot\Delta\rho_i\cdot\Delta\theta_i$$

$$= \overline{\rho_i}\cdot\Delta\rho_i\cdot\Delta\theta_i$$

$$\iint_D f(x,y)\,d\sigma = \iint_D f(\rho\cos\theta,\rho\sin\theta)\,\rho\, d\rho\, d\theta$$

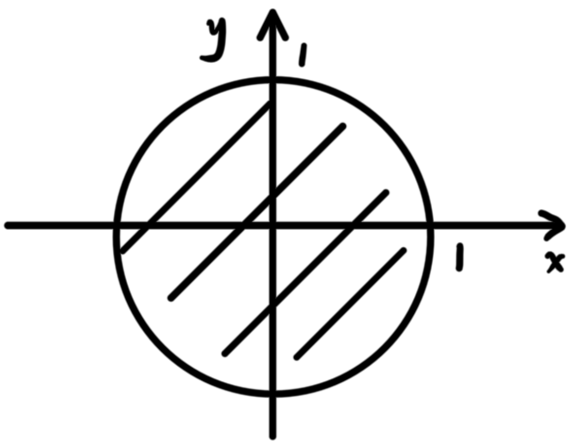
(多一个ρ出来)

例如，$\iint_D (x^2+y^2)\,d\sigma$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho^2\cdot\rho\, d\rho$$

$$= \int_0^{2\pi} \left[\frac{1}{4}\rho^4\right]_0^1 d\theta$$

$$= \frac{\pi}{2}$$



$\iint_D x\,dx\,dy$

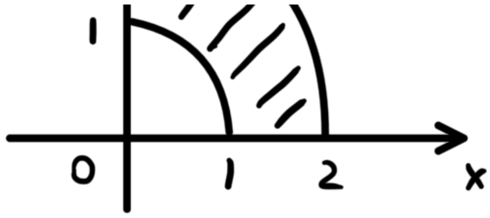
$$= \int_0^{\frac{\pi}{2}} d\theta \int_1^2 \rho\cos\theta\cdot\rho\, d\rho$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{3}\rho^3\cos\theta\right]_1^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{7}{3}\cos\theta \right] d\theta$$

$$= \left[\frac{7}{3}\sin\theta \right]_0^{\frac{\pi}{2}}$$

$$= \frac{7}{3}$$



$$\iint_D (x^2+y^2)\,dxdy$$

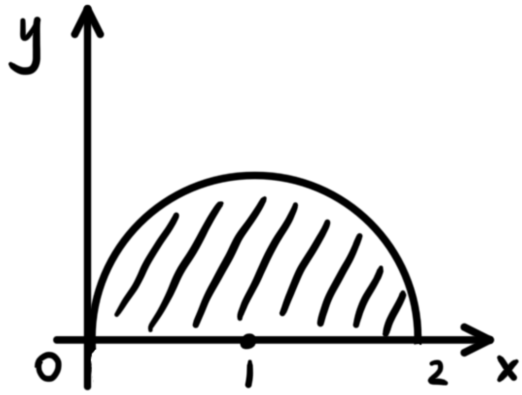
$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \rho^2 \cdot \rho\, d\rho$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{4}\rho^4 \right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[4\cos^4\theta \right] d\theta$$

$$= 4 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{3}{4}\pi$$



$(x-1)^2+y^2=1$

$\Rightarrow x^2+y^2=2x$

$\Rightarrow \rho = 2\cos\theta$

例5. 计算 $\iint_D e^{-x^2-y^2}dxdy$，其中 D 是由圆心在原点，半径为 a 的圆周所围成的闭区域.

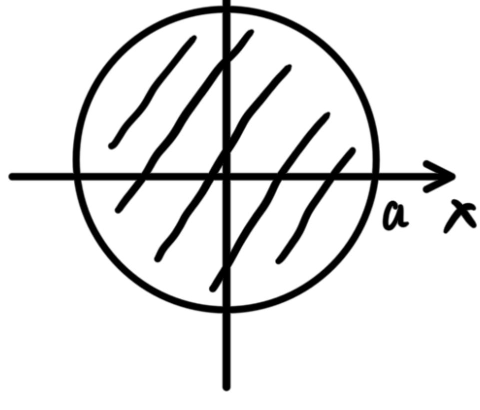
解：$\iint_D e^{-x^2-y^2}dxdy$

$$= \int_0^{2\pi} d\theta \int_0^a e^{-\rho^2}\rho\, d\rho$$

$$= -\frac{1}{2}\int_0^{2\pi} d\theta \int_0^a e^{-\rho^2} d(-\rho^2)$$

$$= -\frac{1}{2}\int_0^{2\pi} \left[e^{-\rho^2} \right]_0^a d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} \left[e^{-a^2}-1 \right] d\theta$$



$$= \pi(1-e^{-\infty})$$

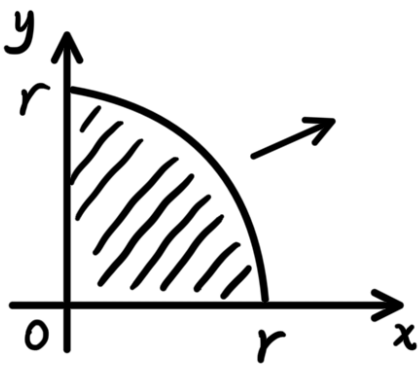
$$\int_0^{+\infty} e^{-x^2}dx \cdot \int_0^{+\infty} e^{-y^2}dy$$

$$= \iint_D e^{-(x^2+y^2)}dxdy$$

$$= \frac{1}{4}\pi(1-e^{-r^2}) \quad (r \to +\infty)$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

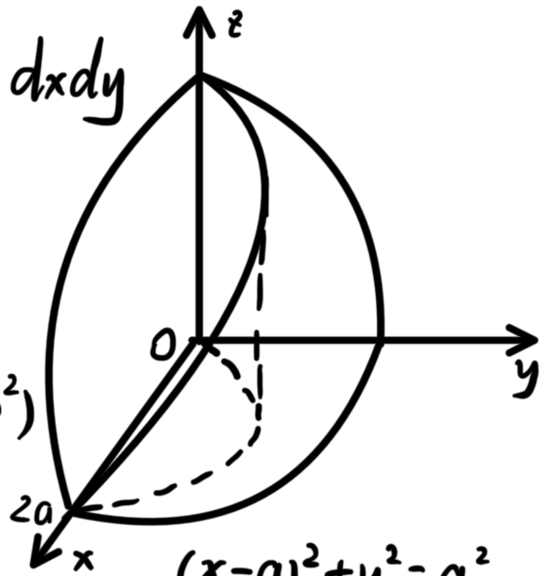


例6. 求球体 $x^2+y^2+z^2 \leq 4a^2$ 被圆柱面 $x^2+y^2=2ax\ (a>0)$ 所截得的(含在圆柱面内的部分)立体的体积.

解: $V = 4\iint_D \sqrt{4a^2-x^2-y^2}\,dxdy$

$$= 4\int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} \sqrt{4a^2-\rho^2}\,\rho\, d\rho$$

$$= -2\int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} \sqrt{4a^2-\rho^2}\, d(4a^2-\rho^2)$$



$$(x-a)^2+y^2=a^2$$

$$\Rightarrow x^2+y^2=2ax$$

$$\Rightarrow \rho = 2a\cos\theta$$

$$= -2\int_0^{\frac{\pi}{2}} \left[\frac{2}{3}(4a^2-\rho^2)^{\frac{3}{2}}\right]_0^{2a\cos\theta} d\theta$$

$$= -\frac{4}{3}\int_0^{\frac{\pi}{2}} [8a^3(\sin^3\theta-1)]\, d\theta$$

$$= \frac{32}{3}a^3\left[\int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta\right]$$

$$= \frac{32}{3}a^3\left(\frac{\pi}{2} - \frac{2}{3}\right)$$