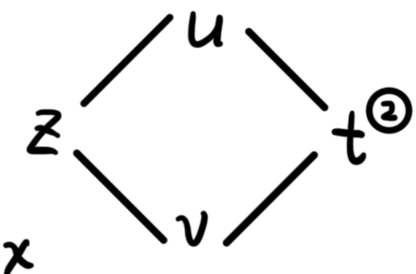
多元函数微分法及其应用(4)

多元复合函数的求导法则

1. 一元与多元复合

$z=f(u,v)$, $u=\varphi(t)$, $v=\psi(t)$

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$$



例如, $z=u^2+v$, $u=x^3$, $v=\sin x$

法①: $\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}$

$$=2u\cdot 3x^2+\cos x$$

$$=6x^5+\cos x$$

法②: $z=x^6+\sin x$

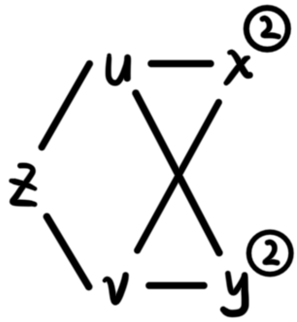
$$\frac{dz}{dx}=6x^5+\cos x$$

2. 多元与多元复合

$z=f(u,v)$, $u=\varphi(x,y)$, $v=\psi(x,y)$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$



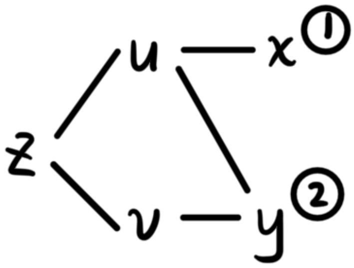
例1. 设 $z=e^u\sin v$, 而 $u=xy$, $v=x+y$.

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$=e^u\sin v\cdot y+e^u\cos v\cdot 1$$
$$=e^u(y\sin v+\cos v)$$
$$=e^{xy}\left[y\sin(x+y)+\cos(x+y)\right]$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$
$$=e^u\sin v\cdot x+e^u\cos v\cdot 1$$
$$=e^u(x\sin v+\cos v)$$
$$=e^{xy}\left[x\sin(x+y)+\cos(x+y)\right]$$

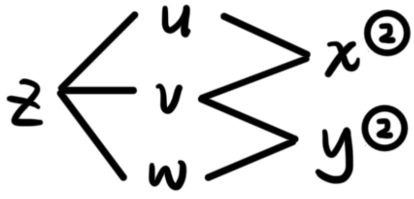
3. 其他情形

$z=f(u,v)$, $u=\varphi(x,y)$, $v=\varphi(y)$



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}$$
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dy}$$

$z=f(u,v,w)$, $u=\varphi(x)$, $v=\psi(x,y)$, $w=g(y)$

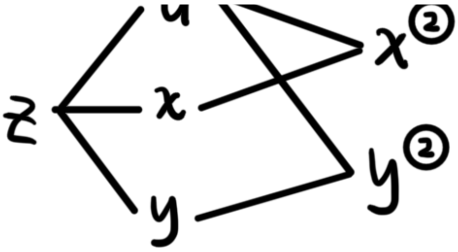


$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\cdot\frac{dw}{dy}$$

$z=f(u,v,w),\ u=\psi(x,y)$

$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x}$

$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y}$

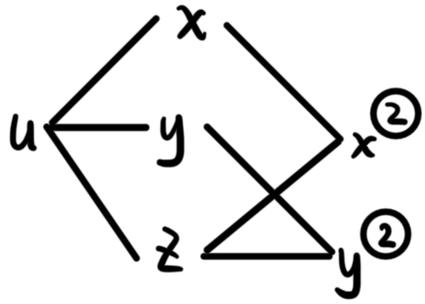


例12. 设 $u=f(x,y,z)=e^{x^2+y^2+z^2}$，而 $z=x^2\sin y$.

$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$

$=2x\cdot e^{x^2+y^2+z^2}+2z\cdot e^{x^2+y^2+z^2}\cdot 2x\sin y$

$=2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$



$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$

$=2y\cdot e^{x^2+y^2+z^2}+2z\cdot e^{x^2+y^2+z^2}\cdot x^2\cos y$

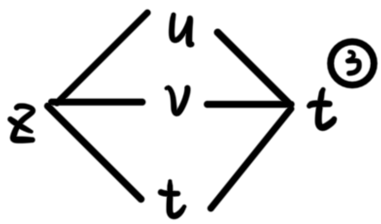
$=2(y+x^4\sin y\cos y)e^{x^2+y^2+x^4\sin^2 y}$

例13. 设 $z=f(u,v,t)=uv+\sin t$，而 $u=e^t, v=\cos t$.

法①：$\frac{dz}{dt}=\frac{\partial f}{\partial u}\cdot\frac{du}{dt}+\frac{\partial f}{\partial v}\cdot\frac{dv}{dt}+\frac{\partial f}{\partial t}$

$=v\cdot e^t+u\cdot(-\sin t)+\cos t$

$=e^t(\cos t-\sin t)+\cos t$



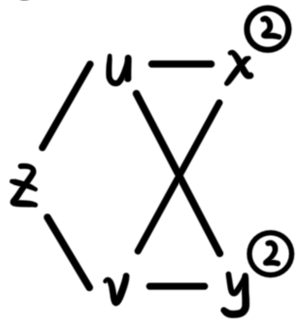
法②：$z=e^t\cos t+\sin t$

$\frac{dz}{dt}=e^t(\cos t-\sin t)+\cos t$

例3.5. $z=f(u,v)$, $u=x^2y$, $v=2x+3y$.

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=2xyf_u+2f_v$$



$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}(2xyf_u+2f_v)$$

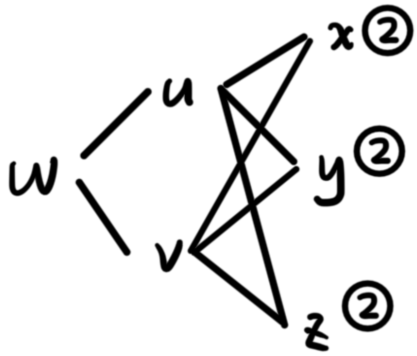
$$=2yf_u+2xy\frac{\partial f_u}{\partial x}+2\frac{\partial f_v}{\partial x}$$

$$=2yf_u+2xy(f_{uu}\cdot 2xy+f_{uv}\cdot 2)$$

$$+2(f_{vu}\cdot 2xy+f_{vv}\cdot 2)$$

例4. 设 $w=f(x+y+z,xyz)$，f 具有二阶连续偏导数，求 $\frac{\partial w}{\partial x}$ 及 $\frac{\partial^2 w}{\partial x\partial z}$.

解：设 $u=x+y+z$, $v=xyz$.



$$\frac{\partial w}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=f_u+yzf_v=f_1'+yzf_2'$$

$$\frac{\partial^2 w}{\partial x\partial z}=\frac{\partial}{\partial z}(f_u+yzf_v)$$

$$=\frac{\partial f_u}{\partial z}+yf_v+yz\frac{\partial f_v}{\partial z}$$

$$=\frac{\partial f_u}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial f_u}{\partial v}\cdot\frac{\partial v}{\partial z}+yf_v+yz\left(\frac{\partial f_v}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial f_v}{\partial v}\cdot\frac{\partial v}{\partial z}\right)$$

$$=f_{uu}+xyf_{uv}+yf_v+yz(f_{vu}+xyf_{vv})$$

$$=f_{uu}+y(x+z)f_{uv}+xy^2zf_{vv}+yf_v$$

$$=f_{11}''+y(x+z)f_{12}''+xy^2zf_{22}''+yf_2'$$

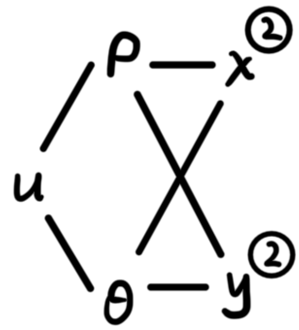
极坐标形式

$u=f(x,y)$，其中 $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$，

则 $\rho=\sqrt{x^2+y^2}$，$\theta=\arctan\frac{y}{x}$.

$\therefore u=f(x,y)=f(\rho\cos\theta,\rho\sin\theta)=F(\rho,\theta)$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \rho}\cdot\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial \theta}+\frac{\partial \theta}{\partial x}$$



$$=\frac{\partial u}{\partial \rho}\cdot\frac{2x}{2\sqrt{x^2+y^2}}-\frac{\partial u}{\partial \theta}\cdot\frac{\frac{y}{x^2}}{1+(\frac{y}{x})^2}$$

$$=\frac{\partial u}{\partial \rho}\cdot\frac{x}{\rho}-\frac{\partial u}{\partial \theta}\cdot\frac{y}{\rho^2}$$

$$=\frac{\partial u}{\partial \rho}\cos\theta-\frac{\partial u}{\partial \theta}\cdot\frac{\sin\theta}{\rho}$$

同理：$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \rho}\sin\theta+\frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{\rho}$

全微分形式不变性

① $z=f(u,v)$

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$$

② $u=\varphi(x,y)$，$v=\psi(x,y)$，$z=f(\varphi(x,y),\psi(x,y))$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

$$=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy$$

$$=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right)$$

$$=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$$

$\partial u \, du + \partial v \, dv$

无论 u 和 v 是自变量还是中间变量，函数 $z=f(u,v)$ 的全微分形式是一样的.

例6. $z=e^u \sin v$，$u=xy$，$v=x+y$.

$$dz = d(e^u \sin v) = e^u d\sin v + \sin v\, de^u$$

$$= e^u \cos v\, dv + e^u \sin v\, du$$

又 $\because du = d(xy) = xdy + ydx$

$$dv = d(x+y) = dx + dy$$

$$\therefore dz = e^u \cos v (dx+dy) + e^u \sin v (xdy + ydx)$$

$$= e^u(\cos v + y\sin v)dx + e^u(\cos v + x\sin v)dy$$

$$= e^{xy}[\cos(x+y) + y\sin(x+y)]dx + e^{xy}[\cos(x+y) + x\sin(x+y)]dy$$