重积分(4)

二重积分的换元法

定理：设$f(x,y)$在xOy平面上的闭区域上连续，若变换T：$x=x(u,v)$，$y=y(u,v)$将平面上的闭区域D'变为xOy平面上的D，且满足

(1) $x(u,v)$，$y(u,v)$在D'上具有一阶连续偏导

(2) 在D'上雅可比式$J(u,v)=\dfrac{\partial(x,y)}{\partial(u,v)}\neq 0$

(3) 变换T：$D'\to D$是一对一的

则有$\iint\limits_D f(x,y)\,dxdy$

绝对值

$$=\iint\limits_{D'} f[x(u,v),y(u,v)]\,|J(u,v)|\,dudv$$

其中$J(u,v)=\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$

于是直角坐标→极坐标时，$\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}$

$$J=\begin{vmatrix}\cos\theta & -\rho\sin\theta\\ \sin\theta & \rho\cos\theta\end{vmatrix}=\rho\cos^2\theta+\rho\sin^2\theta$$

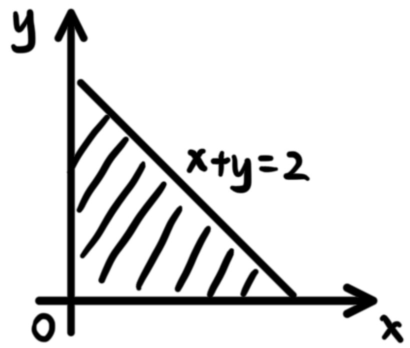
$$=\rho$$

$$\therefore\iint\limits_D f(x,y)\,d\sigma=\iint\limits_D f(\rho\cos\theta,\rho\sin\theta)\,\rho\,d\rho\,d\theta$$

(多一个ρ出来)

例7. 计算 $\iint\limits_{D} e^{\frac{y-x}{y+x}}dxdy$，其中D是由x轴、y轴和直线 $x+y=2$ 所围成的闭区域.

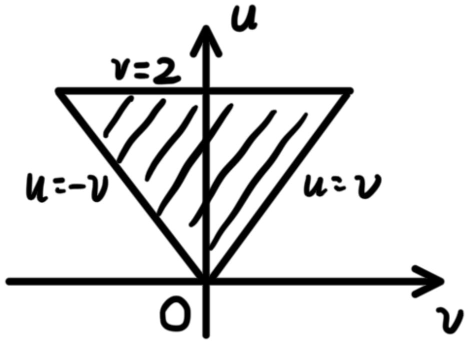
解：令 $u=y-x$，$v=y+x$



$\Rightarrow x=\frac{v-u}{2}$，$y=\frac{u+v}{2}$

$$J=\frac{\partial(x,y)}{\partial(u,v)}$$

$$=\begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}=-\frac{1}{4}-\frac{1}{4}=-\frac{1}{2}$$



$\therefore \iint\limits_{D} e^{\frac{y-x}{y+x}}dxdy$

$=\iint\limits_{D'} e^{\frac{u}{v}}\left|-\frac{1}{2}\right|dudv$

$=\frac{1}{2}\int_0^2 dv\int_{-v}^{v} e^{\frac{u}{v}}du$

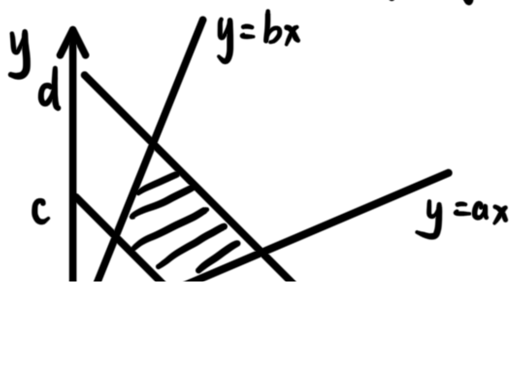
$=\frac{1}{2}\int_0^2 [v(e-e^{-1})]dv$

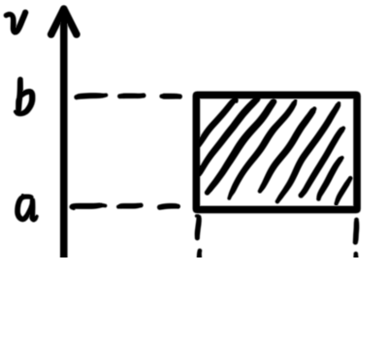
$=\frac{1}{2}\left[\frac{1}{2}v^2(e-e^{-1})\right]_0^2$

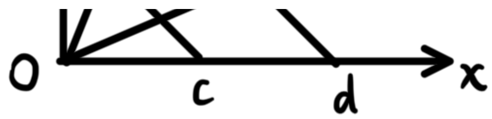
$=e-e^{-1}$

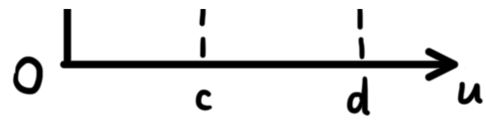
例8. 求由直线 $x+y=c$, $x+y=d$, $y=ax$, $y=bx$ $(0<c<d, 0<a<b)$ 所围成的闭区域D的面积.

解：









令 $u=x+y$，$v=\frac{y}{x}$，则 $x=\frac{u}{v+1}$，$y=\frac{uv}{v+1}$

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{v+1} & \frac{-u}{(v+1)^2}\\ \frac{v}{v+1} & \frac{u}{(v+1)^2}\end{vmatrix}$$

$$=\frac{u}{(v+1)^3}+\frac{uv}{(v+1)^3}=\frac{u}{(v+1)^2}$$

$$\therefore \iint_D dxdy=\iint_{D'}\frac{u}{(v+1)^2}dudv$$

$$=\int_a^b dv\int_c^d\frac{u}{(v+1)^2}du$$

$$=\int_a^b\left[\frac{u^2}{2(v+1)^2}\right]_c^d dv$$

$$=\int_a^b\left[\frac{d^2-c^2}{2(v+1)^2}\right]dv$$

$$=\int_a^b\left[\frac{d^2-c^2}{2}\cdot(v+1)^{-2}\right]d(v+1)$$

$$=\frac{d^2-c^2}{2}\left[-\frac{1}{v+1}\right]_a^b$$

$$=\frac{d^2-c^2}{2}\left(-\frac{1}{b+1}+\frac{1}{a+1}\right)$$

$$=\frac{(b-a)(d^2-c^2)}{2(a+1)(b+1)}$$

哪种情况需要换元？

① 被积函数不好积

② 积分区域不好表示

例9. 计算 $\iint\limits_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy$，其中D为椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 所围成的闭区域.

解：作广义极坐标换元 $\begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \end{cases}$

$\therefore D' = \{(\rho,\theta) \mid 0\le\rho\le 1,\ 0\le\theta\le 2\pi\}$

$$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & -a\rho\sin\theta \\ b\sin\theta & b\rho\cos\theta \end{vmatrix}$$

$$= ab\rho\cos^2\theta + ab\rho\sin^2\theta$$

$$= ab\rho$$

$$\therefore \iint\limits_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy$$

$$= \iint\limits_{D'} \sqrt{1-\rho^2}\cdot ab\rho\,d\rho\,d\theta$$

$$= -\frac{1}{2}ab\int_0^{2\pi} d\theta \int_0^1 (1-\rho^2)^{\frac{1}{2}}\,d(1-\rho^2)$$

$$= -\frac{1}{2}ab\int_0^{2\pi} \left[\frac{2}{3}(1-\rho^2)^{\frac{3}{2}}\right]_0^1 d\theta$$

$$= -\frac{1}{2}ab\int_0^{2\pi} -\frac{2}{3}\,d\theta$$

$$= \frac{2}{3}\pi ab$$