有理函数的积分

有理分式 $\frac{P(x)}{Q(x)}$

$$\frac{x^3+x^2-x+1}{x^2+x+1}$$

$$\begin{array}{r|l} & \quad x \\ \hline x^2+x+1 & x^3+x^2-x+1 \\ & x^3+x^2+x \\ \hline & \qquad\qquad -2x+1 \end{array}$$

$$= x + \frac{-2x+1}{x^2+x+1}$$

$$\frac{x^3+x^2-5}{x^3-x^2+x-1}$$

$$= \frac{x^3-x^2+x-1+2x^2-x-4}{x^3-x^2+x-1}$$

$$= 1 + \frac{2x^2-x-4}{x^3-x^2+x-1}$$

$m<n$

① $\int \frac{1}{2x+1} dx$

$= \frac{1}{2} \int \frac{1}{2x+1} d(2x+1)$

$= \frac{1}{2} \ln|2x+1| + C$

② $\int \frac{1}{x^2-3x+2} dx$

$= \int \frac{1}{(x-1)(x-2)} dx$

$= \int \left(\frac{1}{x-2} - \frac{1}{x-1}\right) dx$

$= \ln\left|\frac{x-2}{x-1}\right| + C$

③ $\int \frac{1}{x^2-2x+4} dx$

$= \int \frac{1}{(x-1)^2+3} d(x-1)$

$= \frac{1}{\sqrt{3}} \int \frac{1}{\left(\frac{x-1}{\sqrt{3}}\right)^2+1} d\left(\frac{x-1}{\sqrt{3}}\right)$

$= \frac{1}{\sqrt{3}} \arctan\frac{x-1}{\sqrt{3}} + C$

④ $\int \frac{1}{x^2-2x-3} dx$

$= \int \frac{1}{(x-1)^2-4} dx$

$= \int \frac{1}{(x+1)(x-3)} dx$

$= \frac{1}{4}\int \left(\frac{1}{x-3} - \frac{1}{x+1} \right) dx$

$= \frac{1}{4} \ln \left| \frac{x-3}{x+1} \right| + C$

另：令 $x-1=t$

原式 $= \int \frac{1}{t^2-4} dt$

$= \frac{1}{4}\int \left(\frac{1}{t-2} - \frac{1}{t+2} \right) dt$

$= \frac{1}{4} \ln \left| \frac{t-2}{t+2} \right| + C$

$= \frac{1}{4} \ln \left| \frac{x-3}{x+1} \right| + C$

⑤ $\int \frac{x}{x^2+2x+2} dx$

$= \frac{1}{2} \int \frac{2x+2-2}{x^2+2x+2} dx$

$= \frac{1}{2} \int \frac{1}{x^2+2x+2} d(x^2+2x+2)$

$$-\int \frac{1}{x^2+2x+2}\,dx$$

$$= \frac{1}{2}\ln\left|x^2+2x+2\right| + C - \int \frac{1}{(x+1)^2+1}\,d(x+1)$$

$$= \frac{1}{2}\ln\left|x^2+2x+2\right| - \arctan(x+1) + C$$

$$= \frac{1}{2}\ln(x^2+2x+2) - \arctan(x+1) + C$$

⑥ $$\int \frac{1}{(1+2x)(1+x^2)}\,dx$$

$$= \int \left(\frac{A}{1+2x} + \frac{Bx+C}{1+x^2}\right)dx$$

通分

$$A(1+x^2)+(1+2x)(Bx+C)$$

$$= Ax^2+A+Bx+C+2Bx^2+2Cx$$

$$= (A+2B)x^2+(B+2C)x+A+C$$

$$\begin{cases} A+2B=0 \\ B+2C=0 \\ A+C=1 \end{cases}$$

$$\Rightarrow \quad \begin{aligned} &2A+2C=2 \\ &B+2C=0 \end{aligned} \downarrow -$$

$$2A-B=2$$

$$\Rightarrow A-\frac{1}{2}B=1$$

$$A+2B=0 \downarrow -$$

$$-\frac{5}{2}B=1 \Rightarrow B=-\frac{2}{5}$$

$$A=\frac{4}{5} \quad C=\frac{1}{5}$$

$$\therefore \begin{cases} A=\frac{4}{5} \\ B=-\frac{2}{5} \\ C=\frac{1}{5} \end{cases}$$

$$\therefore \text{原式} = \int \left(\frac{\frac{4}{5}}{1+2x} + \frac{-\frac{2}{5}x+\frac{1}{5}}{1+x^2} \right) dx$$

$$= \frac{2}{5}\int \frac{1}{1+2x} d(1+2x)$$

$$-\frac{1}{5}\int \frac{2x-1}{1+x^2} dx$$

$$= \frac{2}{5}\ln|1+2x| + C - \frac{1}{5}\int \frac{2x}{1+x^2} dx$$

$$+\frac{1}{5}\int \frac{1}{1+x^2} dx$$

$$= \frac{2}{5}\ln|1+2x| + C - \frac{1}{5}\int \frac{1}{1+x^2} d(1+x^2)$$

$$+\frac{1}{5}\int \frac{1}{1+x^2} dx$$

$$= \frac{2}{5}\ln|1+2x| - \frac{1}{5}\ln(x^2+1)$$

$$=\frac{2}{5}\ln|1+2x|-\frac{1}{5}\ln(x^2+1)$$

$$+\frac{1}{5}\arctan x+C$$

$$=\frac{1}{5}\ln\frac{(2x+1)^2}{x^2+1}+\frac{1}{5}\arctan x+C$$

待定系数法

$$\frac{1}{(1+2x)(1+x^2)}$$

$$=\frac{A}{1+2x}+\frac{Bx+C}{1+x^2}$$

$$\therefore A(1+x^2)+(Bx+C)(1+2x)=1$$

$$\Rightarrow A+Ax^2+Bx+2Bx^2+C+2Cx=1$$

$$\Rightarrow\begin{cases}A+C=1\\A+2B=0\\B+2C=0\end{cases}$$

⑦ $\displaystyle\int\frac{x^2+1}{(x+2)(x+1)^2}dx$

比较系数法

待定系数法

$$\frac{A}{x+2}+\frac{Bx+C}{(x+1)^2}$$

$$\Rightarrow A(x+1)^2+(Bx+C)(x+2)=x^2+1$$

令 $x=0$ 令 $x=-1$

$$\Rightarrow \left.\begin{array}{l} A+2C=1 \\ -B+C=2 \\ A=5 \end{array}\right\}$$ 令 $x=-2$

$$\Rightarrow \begin{cases} A=5 \\ B=-4 \\ C=-2 \end{cases}$$

$$\frac{5}{x+2}+\frac{-4x-2}{(x+1)^2}$$

$$\frac{5}{x+2}-\frac{4}{x+1}+\frac{2}{(x+1)^2}$$

$$\frac{-4(x+1)+2}{(x+1)^2}=\frac{-4}{x+1}+\frac{2}{(x+1)^2}$$

$$\therefore \text{原式}=5\int\frac{dx}{x+2}-4\int\frac{dx}{x+1}+2\int(x+1)^{-2}d(x+1)$$

$= 5\ln|x+2| - 4\ln|x+1|$

$+ 2\times(-1)(x+1)^{-1} + C$

$= 5\ln|x+2| - 4\ln|x+1| - \frac{2}{x+1} + C$

$\int \frac{\sqrt{x-1}}{x} dx$

令 $t = \sqrt{x-1}$

$\Rightarrow t^2 = x-1$

$\Rightarrow x = t^2+1$ ⟨ $dx = 2t\,dt$

则原式 $= \int \frac{t}{t^2+1} \cdot 2t\,dt$

$= \int \frac{2t^2}{t^2+1} dt$

$= 2\int \frac{t^2+1-1}{t^2+1} dt$

$= 2\int \left(1 - \frac{1}{t^2+1}\right) dt$

$= 2(t - \arctan t) + C$

$= 2t - 2\arctan t + C$

$= 2\sqrt{x-1} - 2\arctan\sqrt{x-1} + C$

$$\int \frac{1}{1+\sqrt[3]{x+2}}\,dx$$

令 $t=(x+2)^{\frac{1}{3}}$

$\Rightarrow t^3 = x+2$

$\Rightarrow x = t^3-2$

$\therefore dx = d(t^3-2)$

$= 3t^2\,dt$

$\therefore$ 原式 $=\int \frac{3t^2}{1+t}\,dt$

$$= 3\int \frac{t^2-1+1}{t+1}\,dt$$

$$=3\int\left(t-1+\frac{1}{t+1}\right)dt$$

$$=3\left(\frac{1}{2}t^2-t+\ln|t+1|\right)+C$$

$$=\frac{3}{2}\sqrt[3]{(x+2)^2}-3\sqrt[3]{x+2}$$

$$+3\ln\left|1+\sqrt[3]{x+2}\right|+C$$

$$\int \frac{1}{(1+\sqrt[3]{x})\sqrt{x}}dx = \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{5}{6}}}dx$$

令 $t = x^{\frac{1}{6}}$

$\Rightarrow x = t^6$

$\therefore dx = dt^6 = 6t^5 dt$

$$\therefore 原式 = \int \frac{6t^5}{t^3+t^5}dt$$

$$= 6\int \frac{1+t^2-1}{1+t^2}dt$$

$$= 6\int \left(1-\frac{1}{1+t^2}\right)dt$$

$$= 6(t - \arctan t) + C$$

$$= 6(x^{\frac{1}{6}} - \arctan x^{\frac{1}{6}}) + C$$