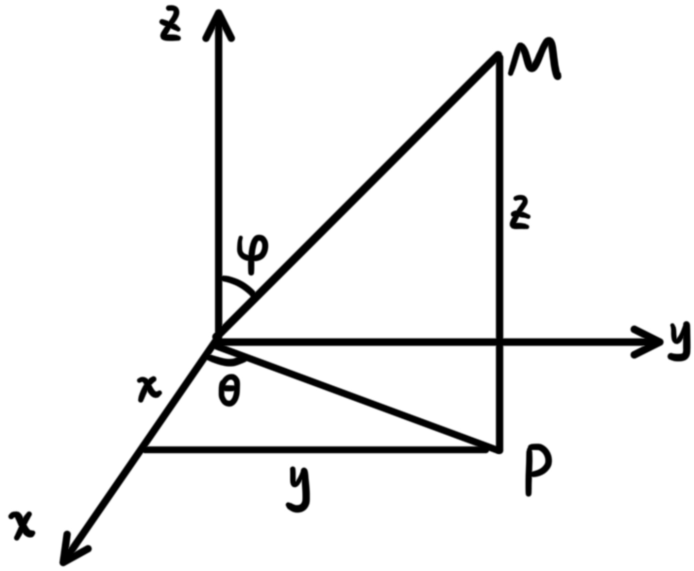
重积分 (6)

利用球面坐标计算三重积分



$$\begin{cases} 0 \le r < +\infty \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\begin{cases} x = OP\cos\theta = r\sin\varphi\cos\theta \\ y = OP\sin\theta = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

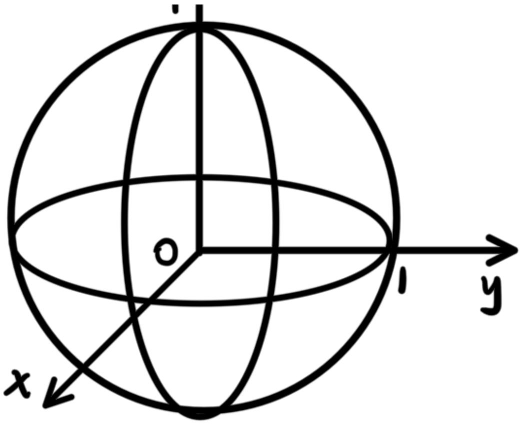
二级结论： $x^2+y^2=r^2\sin^2\varphi$

$x^2+y^2+z^2=r^2$

换元求积分： $\iiint\limits_{\Omega} f(x,y,z)\,dxdydz$

$= \iiint\limits_{\Omega} F(r,\varphi,\theta)\,\underline{\underline{r^2\sin\varphi}}\,drd\varphi d\theta$

$\iiint dxdydz$

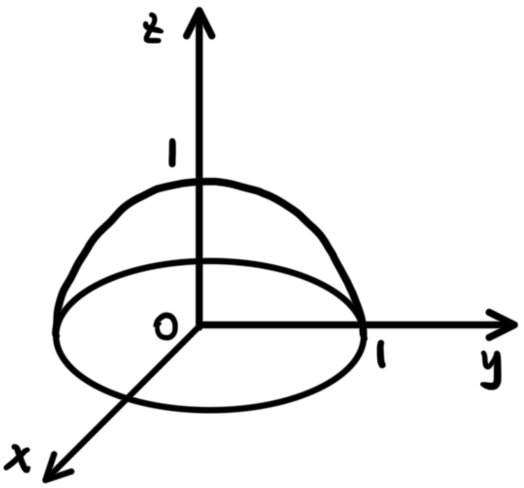


$$= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 r^2 \sin\varphi \, dr$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \frac{1}{3} \sin\varphi \, d\varphi$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} \cos\varphi \right]_0^{\pi} d\theta$$

$$= -\frac{1}{3} \times (-1-1) \times 2\pi = \frac{4}{3}\pi$$



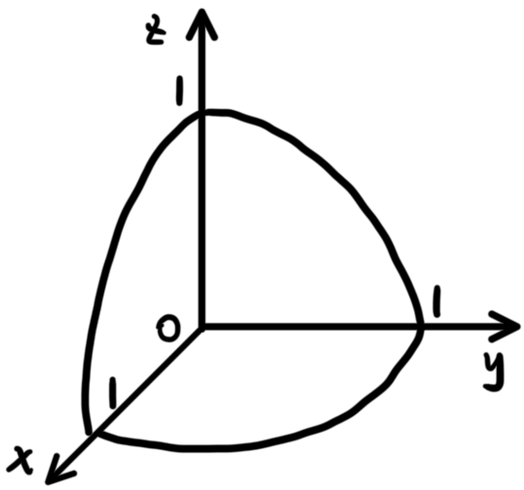
$$\iiint_{\Omega} (x^2+y^2) \, dxdydz$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r^2 \sin^2\varphi \, r^2 \sin\varphi \, dr$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \sin^3\varphi \, d\varphi \cdot \int_0^1 r^4 dr$$

$$= 2\pi \cdot \frac{2}{3} \cdot 1 \cdot \left[\frac{1}{5} r^5 \right]_0^1$$

$$= \frac{4}{15}\pi$$



$$\iiint_{\Omega} z \, dxdydz$$

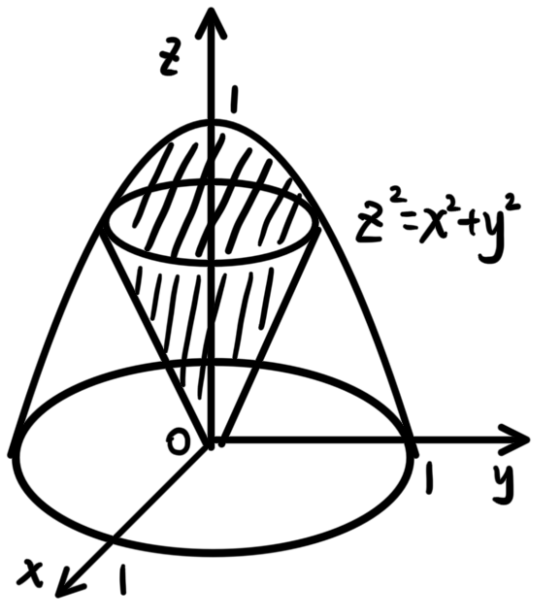
$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r\cos\varphi \cdot r^2 \sin\varphi \, dr$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi \, d\sin\varphi \int_0^1 r^3 dr$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{8} \sin^2\varphi \right]_0^{\frac{\pi}{2}} d\theta$$

π

$$= \frac{\pi}{6}$$



$$\iiint_{\Omega} dxdydz$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^1 r^2 \sin\varphi dr$$

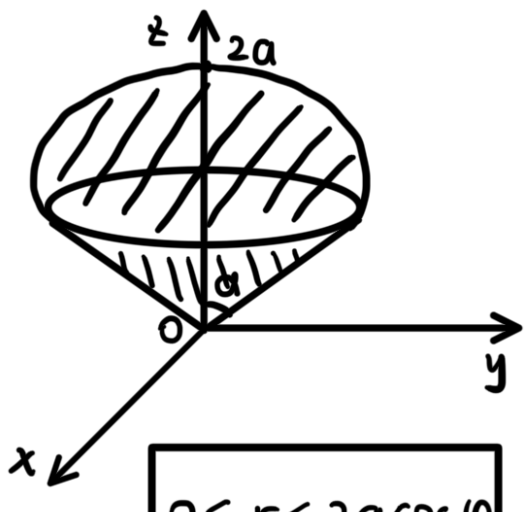
$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \frac{1}{3}\sin\varphi \, d\varphi$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\cos\varphi\right]_0^{\frac{\pi}{4}} d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left(\frac{\sqrt{2}}{2}-1\right) d\theta$$

$$= \frac{(2-\sqrt{2})\pi}{3}$$

例4. 求半径为a的球面与半顶角为α的内接锥面所围成的立体的体积.



$$\boxed{\begin{array}{l} 0 \le r \le 2a\cos\varphi \\ 0 \le \varphi \le \alpha \\ 0 \le \theta \le 2\pi \end{array}}$$

$$V = \iiint_{\Omega} r^2 \sin\varphi dr d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\alpha} d\varphi \int_0^{2a\cos\varphi} r^2 \sin\varphi dr$$

$$= 2\pi \cdot \int_0^{\alpha} \sin\varphi \left[\frac{1}{3}r^3\right]_0^{2a\cos\varphi} d\varphi$$

$$= 2\pi \cdot \int_0^{\alpha} \left(\frac{8}{3}a^3 \sin\varphi \cos^3\varphi\right) d\varphi$$

$$= -\frac{16}{3}\pi a^3 \cdot \int_0^{\alpha} \cos^3\varphi \, d\cos\varphi$$

$$= -\frac{16}{3}\pi a^3 \cdot \left[\frac{1}{4}\cos^4\varphi\right]_0^{\alpha}$$

$$= \frac{4}{3}\pi a^3 (1-\cos^4\alpha)$$