多元函数微分法及其应用(5)

隐函数的求导公式

一、一个方程的情形

$F(x,y)=0$

直接法(y是x的函数)：两边同时对x求导.

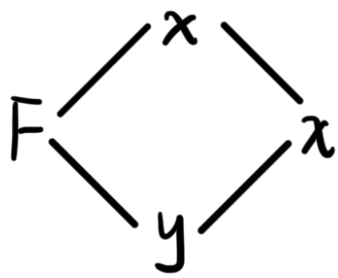
例如，$e^y+x^2y=6$

$\Rightarrow e^y\cdot y'+2xy+x^2y'=0$

$\Rightarrow y'=\dfrac{-2xy}{e^y+x^2}$

定理1：$F(x,y)=0$，$\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$.

$F(x,y)=0$，对x求导



$\Rightarrow \dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial y}\cdot\dfrac{dy}{dx}=0$

$\Rightarrow \dfrac{\partial F}{\partial y}\cdot\dfrac{dy}{dx}=-\dfrac{\partial F}{\partial x}$

$\Rightarrow \dfrac{dy}{dx}=-\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\dfrac{F_x}{F_y}$

例如，$e^y+x^2y=6$，$F(x,y)=e^y+x^2y-6$，

则 $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}=-\dfrac{2xy}{e^y+x^2}$

例1. $x^2+y^2-1=0$.

法①：两边同时求导得：$2x+2y\cdot y'=0$

$$\Rightarrow y'=-\frac{x}{y}$$

$$\Rightarrow y''=-\frac{y-xy'}{y^2}=-\frac{y+x\cdot\frac{x}{y}}{1-x^2}=-\frac{1}{y^3}$$

法②：$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{x}{y}$

$\frac{d^2y}{dx^2}=-\frac{y-xy'}{y^2}=-\frac{1}{y^3}$（该步骤同法①）

定理2：$F(x,y,z)=0$，$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$，$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$.

例如，$x^2+yz+z^2=0$.

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x}{y+2z}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{z}{y+2z}$$

另法（直接求导法）：$z=z(x,y)$

两边对x求偏导得：

$2x+y\cdot z_x+2z\cdot z_x=0$

$$\Rightarrow z_x=-\frac{2x}{y+2z}$$

两边对y求偏导得：

$z+y\cdot z_y+2z\cdot z_y=0$

$$\Rightarrow z_y=-\frac{z}{y+2z}$$

例2. $x^2+y^2+z^2-4z=0$.

设$F(x,y,z)=x^2+y^2+z^2-4z$.

∂z $\quad F_x$ $\quad 2x$ $\quad x$

$$\frac{}{\partial x} = -\frac{}{F_z} = -\frac{}{2z-4} = \frac{}{2-z}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2-z-x\left(-\frac{\partial z}{\partial x}\right)}{(2-z)^2} = \frac{2-z+\frac{x^2}{2-z}}{(2-z)^2} = \frac{(2-z)^2+x^2}{(2-z)^3}$$

二、方程组的情形

$$\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$$

$u=\varphi(x,y)$, $v=\psi(x,y)$

对x求偏导:

$$\begin{cases} u + xu_x - yv_x = 0 \\ yu_x + v + xv_x = 0 \end{cases}$$

$$\Rightarrow \begin{cases} xu_x - yv_x = -u \\ yu_x + xv_x = -v \end{cases}$$

$$\therefore u_x = \frac{\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}, \quad v_x = \frac{\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}$$

同样地,对y求偏导:

$$\begin{cases} xu_y - v - yv_y = 0 \\ u + yu_y + xv_y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} xu_y - yv_y = v \\ yu_y + xv_y = -u \end{cases}$$

$$\therefore u_y = \frac{\begin{vmatrix} v & -y \\ -u & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}, \quad v_y = \frac{\begin{vmatrix} x & v \\ y & -u \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}$$