向量代数与空间解析几何 (4)

空间直线及其方程

一般式: $\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$

$\vec{s}=\vec{n_1}\times\vec{n_2}$

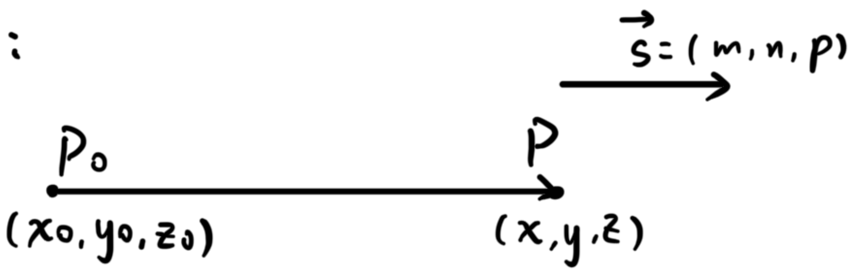
点向式: $P_0(x,y,z)$, $\vec{s}=(m,n,p)$

$$\Rightarrow \frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}$$

过点 $(1,2,3)$, $\vec{s}=(-5,6,0)$

$\therefore$ 点向式方程为 $\frac{x-1}{-5}=\frac{y-2}{6}=\frac{z-3}{0}$

证明:



$\overrightarrow{P_0P}=(x-x_0,y-y_0,z-z_0)$, $\overrightarrow{P_0P}\parallel\vec{s}$

$$\Rightarrow \frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}$$

参数方程:

设 $\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}=t$

$$\Rightarrow \begin{cases} x=x_0+mt \\ y=y_0+nt \end{cases}$$

$$\begin{cases} y = y_0 + nt \\ z = z_0 + pt \end{cases}$$

例1. $\begin{cases} x+y+z+1=0 \\ 2x-y+3z+4=0 \end{cases}$

令 $x=1 \Rightarrow \begin{cases} y+z=-2 \\ y-3z=6 \end{cases} \Rightarrow \begin{cases} y=0 \\ z=-2 \end{cases}$

过点 $(1,0,-2)$，又 $\because \vec{n_1}=(1,1,1)$，$\vec{n_2}=(2,-1,3)$

$$\therefore \vec{s} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = (4,-1,-3)$$

$\therefore$ 直线的对称式方程：$\dfrac{x-1}{4} = \dfrac{y}{-1} = \dfrac{z+2}{-3}$

令 $\dfrac{x-1}{4} = \dfrac{y}{-1} = \dfrac{z+2}{-3} = t$

$$\Rightarrow \begin{cases} x = 1+4t \\ y = -t \\ z = -2-3t \end{cases}$$

两直线的夹角 $(0 \le \theta \le \frac{\pi}{2})$

$$\cos\theta = \frac{|\vec{s_1} \cdot \vec{s_2}|}{|\vec{s_1}| \cdot |\vec{s_2}|}$$

两直线的位置关系

① 垂直 $l_1 \perp l_2 \Rightarrow \vec{s_1} \cdot \vec{s_2} = 0$

② 平行 $l_1 \parallel l_2 \Rightarrow \vec{s_1} \parallel \vec{s_2}$ 且 ... 不成立

③ 重合 $l_1 \parallel l_2$ 且一直线上取一点代入另一直线成立

④ 异面但不垂直

例 2. $l_1: \dfrac{x-1}{1} = \dfrac{y}{-4} = \dfrac{z+3}{1}$

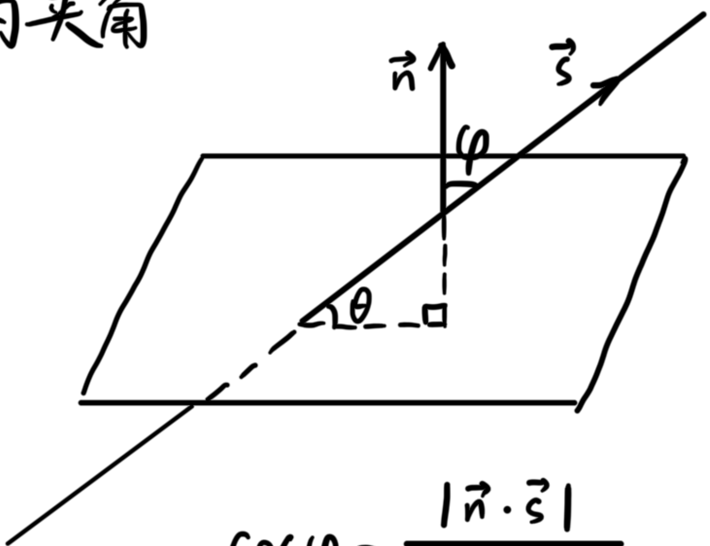
$l_2: \dfrac{x}{2} = \dfrac{y+2}{-2} = \dfrac{z}{-1}$

$\vec{s_1} = (1, -4, 1)$, $\vec{s_2} = (2, -2, -1)$

$$\cos\theta = \frac{|\vec{s_1} \cdot \vec{s_2}|}{|\vec{s_1}| \cdot |\vec{s_2}|} = \frac{9}{3\sqrt{2} \times 3} = \frac{\sqrt{2}}{2}$$

$\Rightarrow \theta = \frac{\pi}{4}$

直线与平面的夹角



$$\cos\varphi = \frac{|\vec{n} \cdot \vec{s}|}{|\vec{n}| \cdot |\vec{s}|}$$

$\sin\theta = \cos\varphi$

梦回高中!

有关夹角的公式:

① 两向量 $\vec{a}$, $\vec{b}$, $0 \le \theta \le \pi$, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$

② 两平面, $0 \le \theta \le \frac{\pi}{2}$, $\cos\theta = \dfrac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}|\cdot|\vec{n_2}|}$

③ 两直线, $0 \le \theta \le \frac{\pi}{2}$, $\cos\theta = \dfrac{|\vec{s_1}\cdot\vec{s_2}|}{|\vec{s_1}|\cdot|\vec{s_2}|}$

④ 线与面, $\sin\theta = \dfrac{|\vec{s}\cdot\vec{n}|}{|\vec{s}|\cdot|\vec{n}|}$

特殊地, 若线 ⊥ 面, 则 $\vec{s} \parallel \vec{n}$

若线 $\parallel$ 面, 则 $\vec{s} \perp \vec{n}$

例3. 求过 $(1,-2,4)$, 与面 $2x-3y+z-4=0$ 垂直的直线方程.

$\vec{n} = (2,-3,1)$

$\therefore$ 方程为 $\dfrac{x-1}{2} = \dfrac{y+2}{-3} = \dfrac{z-4}{1}$

杂例

例4. 求与 $\overset{(1,0,-4)}{x-4z=3}$, $\overset{(2,-1,-5)}{2x-y-5z=1}$ 交线平行, 过点 $(-3,2,5)$ 的直线方程.

$$\vec{s}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -4\\ 2 & -1 & -5\end{vmatrix}=(-4,-3,-1)$$

$\therefore$ 方程为 $\dfrac{x+3}{4}=\dfrac{y-2}{3}=\dfrac{z-5}{1}$

例5. 求线 $\dfrac{x-2}{1}=\dfrac{y-3}{1}=\dfrac{z-4}{2}$ 与面 $2x+y+z-6=0$ 交点.

直线的参数方程为 $\begin{cases} x=2+t \\ y=3+t \\ z=4+2t \end{cases}$

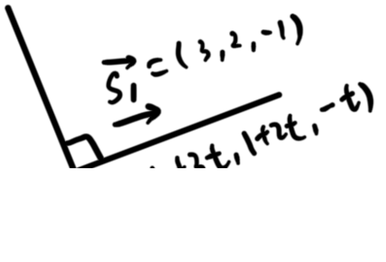
代入面得：$2(2+t)+(3+t)+(4+2t)-6=0$

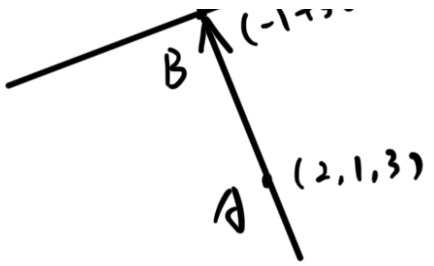
$\Rightarrow t=-1$ 代入参数方程.

$\therefore$ 交点为 $(1,2,2)$

例6. 求过 $(2,1,3)$ 且与线 $\dfrac{x+1}{3}=\dfrac{y-1}{2}=\dfrac{z}{-1}$ 垂直相交的直线方程.

所给直线参数方程为 $\begin{cases} x=-1+3t \\ y=1+2t \\ z=-t \end{cases}$





设所给直线任意一点B

∴ $\vec{AB} = (-3+3t, 2t, -t-3)$

由题： $\vec{AB} \perp \vec{s_1} \Rightarrow \vec{AB} \cdot \vec{s_1} = 0$

$\Rightarrow 3(-3+3t) + 2 \times 2t + t + 3 = 0$

$\Rightarrow 14t = 6$

$\Rightarrow t = \frac{3}{7}$

∴ $B(\frac{2}{7}, \frac{13}{7}, -\frac{3}{7})$

∴ $\vec{AB} = (-\frac{12}{7}, \frac{6}{7}, -\frac{24}{7}) = -\frac{6}{7}(2, -1, 4)$

∴ $l_{AB}: \frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-3}{4}$