微分方程（2）

一阶线性微分方程

$$\frac{dy}{dx} + p(x)y = Q(x)$$

① $Q(x) \equiv 0$ 齐次

$$\Rightarrow \frac{dy}{dx} = -p(x)y$$

$$\Rightarrow \int \frac{dy}{y} = \int -p(x) \cdot dx$$

$$\Rightarrow \ln|y| = -\int p(x)dx + C$$

$$\Rightarrow |y| = e^{-\int p(x)dx} \cdot e^{C_1}$$

$$\Rightarrow y = Ce^{-\int p(x)dx}$$

② 非齐次：

令 $y = ue^{-\int p(x)dx}$

$$\frac{dy}{dx} + p(x)y = Q(x)$$

$$\Rightarrow u' \cdot e^{-\int p(x)dx} + \cancel{u \cdot e^{-\int p(x)dx} \cdot (-p(x))}$$

$$\cancel{+ p(x) \cdot ue^{-\int p(x)dx}} = Q(x)$$

$$\Rightarrow u' \cdot e^{-\int p(x)dx} = Q(x)$$

$$\Rightarrow u' = Q(x)\cdot e^{\int P(x)dx}$$

$$\Rightarrow u = \int Q(x)\cdot e^{\int P(x)dx}dx + C$$

$$\Rightarrow y = e^{-\int P(x)dx}\left(\int Q(x)\cdot e^{\int P(x)dx}dx + C\right)$$

例1. $\frac{dy}{dx} - \frac{2y}{x+1} = (x+1)^{\frac{5}{2}}$

公式：$\frac{dy}{dx} + P(x)y = Q(x)$

其中 $P(x) = -\frac{2}{x+1}$

$Q(x) = (x+1)^{\frac{5}{2}}$

$$\therefore y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + C\right)$$

$$= e^{2\int \frac{1}{x+1}d(x+1)}\left(\int (x+1)^{\frac{5}{2}} e^{-2\int \frac{1}{x+1}d(x+1)}dx + C\right)$$

$$= e^{2\ln|x+1|}\left(\int (x+1)^{\frac{5}{2}} e^{-2\ln|x+1|}dx + C\right)$$

$$= (x+1)^2\left(\int (x+1)^{\frac{5}{2}}\cdot(x+1)^{-2}dx + C\right)$$

$$= (x+1)^2\left(\int (x+1)^{\frac{1}{2}}d(x+1) + C\right)$$

$$= (x+1)^2\left[\frac{2}{3}(x+1)^{\frac{3}{2}} + C\right]$$

例3. $\frac{dy}{dx} = \frac{1}{x+y}$

令 $x+y=u \Rightarrow y=u-x$

$$\therefore \frac{dy}{dx}=\frac{du}{dx}-1$$

$$\Rightarrow \frac{du}{dx}-1=\frac{1}{u}$$

$$\Rightarrow \frac{du}{dx}=\frac{1+u}{u}$$

$$\Rightarrow \frac{u}{1+u}du=dx$$

$$\Rightarrow \int \frac{1+u-1}{1+u}du=x+C$$

$$\Rightarrow \int\left(1-\frac{1}{1+u}\right)du=x+C$$

$$\Rightarrow u-\ln|1+u|=x+C$$

$$\Rightarrow \cancel{x}+y-\ln|1+x+y|=\cancel{x}+C$$

$$\Rightarrow y=\ln|1+x+y|+C$$

$$\Rightarrow e^{y}=(1+x+y)\cdot C$$

$$\Rightarrow x+y+1=Ce^{y}$$

$$\Rightarrow x=Ce^{y}-y-1$$

伯努利方程

$$\frac{dy}{dx}+P(x)y=Q(x)y^{n}$$

$(n\neq 0,1)$

$$\Rightarrow y^{-n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x)$$

令 $y^{1-n}=z$

$$\therefore \frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{1-n}\cdot\frac{dz}{dx}+P(x)z=Q(x)$$

$$\Rightarrow \frac{dz}{dx}+(1-n)P(x)z=(1-n)Q(x)$$

便得线性方程.

例4. $\frac{dy}{dx}+\frac{y}{x}=a(\ln x)y^2$

$$\Rightarrow y^{-2}\frac{dy}{dx}+y^{-1}\frac{1}{x}=a\ln x$$

$$\Rightarrow -\frac{dy^{-1}}{dx}+\frac{1}{x}y^{-1}=a\ln x$$

$$\Rightarrow \frac{dy^{-1}}{dx}-\frac{1}{x}y^{-1}=-a\ln x$$

$$\therefore y^{-1}=e^{\int\frac{1}{x}dx}\left(\int -a\ln x e^{\int-\frac{1}{x}dx}dx+C\right)$$

$$=x\left(\int -a\ln x\cdot e^{-\ln x}dx+C\right)$$

$$=x\left(-a\int\frac{1}{x}\ln x\,dx+c\right)$$

其中 $\int\frac{1}{x}\ln x\,dx$

$$=\int\ln x\,d\ln x$$

$$=(\ln x)^2-\int\ln x\,d\ln x$$

$$\Rightarrow \int\frac{1}{x}\ln x\,dx=\frac{1}{2}(\ln x)^2$$

$$\therefore y^{-1}=x\left[-\frac{a}{2}(\ln x)^2+c\right]$$

$$\Rightarrow yx\left[C-\frac{a}{2}(\ln x)^2\right]=1$$

注意：$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$\neq -x$

可降阶的高阶微分方程

一、$y^{(n)} = f(x)$

$$y^{(n-1)} = \int f(x)\,dx + C$$

例1. $y''' = e^{2x} - \cos x$

$\Rightarrow y'' = \frac{1}{2}e^{2x} - \sin x + C_1$

$\Rightarrow y' = \frac{1}{4}e^{2x} + \cos x + C_1 x + C_2$

$\Rightarrow y = \frac{1}{8}e^{2x} + \sin x + \frac{1}{2}C_1 x^2 + C_2 x + C_3$

二、$y'' = f(x, y') \rightarrow$ 无 y

令 $y' = p$

则 $y'' = p'$

$\therefore p' = f(x, p)$

例3. $(1+x^2)y'' = 2xy'$

$(1+x^2)y''=2xy'$

令 $y'=p \Rightarrow y''=p'$

$\Rightarrow (1+x^2)p'=2xp$

$\Rightarrow \frac{1+x^2}{x}\cdot\frac{dp}{dx}=2p$

$\Rightarrow \int\frac{1}{2p}dp=\int\frac{x}{1+x^2}dx$

$\Rightarrow \frac{1}{2}\ln|p|=\frac{1}{2}\int\frac{1}{1+x^2}d(1+x^2)$

$\Rightarrow \ln p=\ln(1+x^2)+C$

$\Rightarrow p=C_1(1+x^2)$

$\Rightarrow y'=C_1(1+x^2)$

满足初始条件 $y|_{x=0}=1$ ①

$y'|_{x=0}=3$ ②

② 代入 $y'=C_1(1+x^2)$ 得：$C_1=3$

$\therefore y'=3x^2+3$

$\Rightarrow y=\int 3x^2+3\,dx$

$=x^3+3x+C_2$

代入 ① 得：$C_2=1$

$\therefore y=x^3+3x+1$

三、$y''=f(y,y')\rightarrow$ 无 x

令 $y'=p$

$\Rightarrow y''=\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}$

$=p\cdot\frac{dp}{dy}$

$\Rightarrow p\cdot\frac{dp}{dy}=f(y,p)$

例15. $yy''-y'^2=0$

令 $y'=p$，则 $y''=\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}$

$=p\cdot\frac{dp}{dy}$

$\therefore y\cdot p\cdot\frac{dp}{dy}-p^2=0$

$\Rightarrow yp\frac{dp}{dy}=p^2$

在 $y\neq0$, $p\neq0$ 时，$y\frac{dp}{dy}=p$

$\Rightarrow \frac{1}{y}dy=\frac{1}{p}dp$

$\Rightarrow \ln|y|=\ln|p|+C$

$\Rightarrow p=C_1y$

$\Rightarrow y'=C_1y$

$\Rightarrow \frac{dy}{dx}=C_1y$

$\Rightarrow \int\frac{1}{y}dy=C_1\int1dx$

$\Rightarrow \ln|y|=C_1x+C_2'$

$\Rightarrow y=C_2e^{C_1x}$