定积分的换元法
和分部积分法

换元法：

$$\int_0^4 \frac{x+2}{\sqrt{2x+1}}dx$$

令 $t=\sqrt{2x+1}$

$\Rightarrow t^2 = 2x+1$

$\Rightarrow x = \frac{t^2-1}{2}$

$\therefore dx = d\,\frac{1}{2}t^2 = t\,dt$

☆. $x: 0\to 4$, $t: 1\to 3$

$$\therefore 原式 = \int_1^3 \frac{\frac{t^2-1}{2}+2}{\cancel{t}}\cancel{t}\,dt$$

$$= \int_1^3 \left(\frac{1}{2}t^2+\frac{3}{2}\right)dt$$

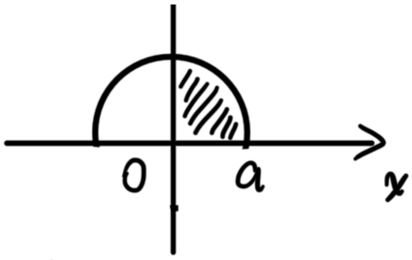
$$= \left[\frac{1}{6}t^3+\frac{3}{2}t\right]_1^3$$

$$= \frac{1}{6}\times 27+\frac{3}{2}\times 3-\frac{1}{6}-\frac{3}{2}$$

$$= \frac{22}{3}$$

$$\int_0^a \sqrt{a^2-x^2}\,dx \quad (a>0)$$

$y\uparrow$



$= \frac{1}{4}\pi a^2$

换元法：令 $x = a\sin t$

$\therefore dx = d\,a\sin t = a\cos t\,dt$

$x: 0 \to a$, $t: 0 \to \frac{\pi}{2}$

$\therefore$ 原式 $= \int_0^{\frac{\pi}{2}} a^2\cos^2 t\,dt$

$= \frac{1}{4}a^2\int_0^{\frac{\pi}{2}}(1+\cos 2t)\,d(2t)$

$= \frac{1}{4}a^2\left[2t+\sin 2t\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{4}a^2(\pi+0-0)$

$= \frac{1}{4}\pi a^2$

令 $x = a\cos t$

$x: 0 \to a$, $t: \frac{\pi}{2} \to 0$

$\int_0^{\frac{\pi}{2}} \cos^5 x \sin x\,dx$

令

令 $t = \cos x$

$x: 0 \to \frac{\pi}{2}$ ， $t: 1 \to 0$

$$\therefore \text{原式} = -\int_0^{\frac{\pi}{2}} \cos^5 x \, d\cos x$$
$$= \int_0^1 t^5 dt$$
$$= \left[\frac{1}{6} t^6\right]_0^1$$
$$= \frac{1}{6}$$

另： $$\text{原式} = -\int_0^{\frac{\pi}{2}} \cos^5 x \, d\cos x$$
$$= \left[\frac{1}{6}\cos^6 x\right]_{\frac{\pi}{2}}^0$$
$$= \frac{1}{6}$$

$$\int_0^{\frac{\pi}{3}} e^{\sin x} \cos x \, dx$$
$$= \int_0^{\frac{\pi}{3}} e^{\sin x} d\sin x$$
$$= \left[e^{\sin x}\right]_0^{\frac{\pi}{3}}$$
$$= e^{\sin\frac{\pi}{3}} - e^{\sin 0}$$
$$= e^{\frac{\sqrt{3}}{2}} - 1$$

$$\int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x}\, dx$$
$$= \int_0^{\pi} \sqrt{\sin^3 x \cdot \cos^2 x}\, dx$$
$$= \int_0^{\pi} |\cos x| \cdot (\sin x)^{\frac{3}{2}} dx$$
$$= \int_0^{\frac{\pi}{2}} \cos x \cdot (\sin x)^{\frac{3}{2}} dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) \cdot (\sin x)^{\frac{3}{2}} dx$$
$$= \int_0^{\frac{\pi}{2}} (\sin x)^{\frac{3}{2}} d(\sin x) - \int_{\frac{\pi}{2}}^{\pi} (\sin x)^{\frac{3}{2}} d(\sin x)$$
$$= \left[\frac{2}{5}(\sin x)^{\frac{5}{2}}\right]_0^{\frac{\pi}{2}} - \left[\frac{2}{5}(\sin x)^{\frac{5}{2}}\right]_{\frac{\pi}{2}}^{\pi}$$
$$= \frac{2}{5} - \left(-\frac{2}{5}\right) = \frac{4}{5}$$

$f(x)$ 在 $[-a, a]$ 为偶：
$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$$
$f(x)$ 在 $[-a, a]$ 为奇：
$$\int_{-a}^{a} f(x)\, dx = 0$$

$$\int_{-a}^{0} f(x)\, dx$$
令 $x = -t \Rightarrow dx = -dt$

$x: -a \to 0$, $t: a \to 0$

$\therefore \int_{-a}^{0} f(x)\, dx$

$$= -\int_a^0 f(-t)\,dt$$
$$= \int_0^a f(-x)\,dx$$

$$\int_{-1}^{1} x^2 \frac{\sin x}{\cos^2 x}\,dx = 0$$

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

令 $x = \pi - t$, $dx = -dt$

左 $= -\int_{\pi}^{0} (\pi - t) f(\sin(\pi - t))\,dt$

$$= \int_0^{\pi} (\pi - x) f(\sin x)\,dx$$
$$= \pi\int_0^{\pi} f(\sin x)\,dx - \int_0^{\pi} x f(\sin x)\,dx$$

$\Rightarrow$ 左 $= \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$
$$= \int_0^{\pi} x \cdot \frac{\sin x}{1+\cos^2 x}\,dx$$
$$= -\frac{\pi}{2}\int^{\pi} \frac{1}{\quad}\,d\cos x$$

$$= -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}\, d\cos x$$

$$= -\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\left(\arctan(-1) - \arctan 1\right)$$

$$= -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{4}$$

$f(x)$ 连续，周期为T

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

$$\varphi(a) = \int_a^{a+T} f(x)\,dx$$

$$\Rightarrow \varphi'(a) = f(a+T) - f(a) = 0$$

$$\varphi(a) = \varphi(0)$$

$$\Rightarrow \int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

$$\int_a^{a+T} f(x)\,dx$$

$$= ①\int_a^0 f(x)\,dx + ②\int_0^T f(x)\,dx + ③\int_T^{a+T} f(x)\,dx$$

③ 令 $x = u + T \quad dx = du$

$x: T \to T+a, \; u: 0 \to a$

$$③ = \int_0^a f(u)\,du = -\int_a^0 f(x)\,dx = -①$$

$$\int_a^{a+T} f \ldots \quad \int^T f \ldots$$

$$\therefore \int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

$$\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx$$

$$\int_0^{n\pi} \sqrt{1+\sin 2x}\,dx$$

$$= n \int_0^{\pi} \sqrt{(\sin x+\cos x)^2}\,dx$$

$$= n\int_0^{\pi} |\sin x+\cos x|\,dx$$

$$= \sqrt{2}\,n \int_0^{\pi} \left|\sin\left(x+\frac{\pi}{4}\right)\right| dx$$

令 $t = x+\frac{\pi}{4}$

$\Rightarrow \quad dx = dt$

$x:\ 0 \to \pi \qquad t:\ \frac{\pi}{4} \to \frac{5}{4}\pi$

$$\therefore 原式 = \sqrt{2}\,n \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} |\sin t|\,dt$$

$$= \sqrt{2}\,n \int_0^{\pi} \sin t\,dt$$

$$= -\sqrt{2}\,n \left[\cos t\right]_0^{\pi}$$

$$= -\sqrt{2}\,n\,(-1-1)$$

$$= 2\sqrt{2}\,n$$

$$f(x)=\begin{cases}\dfrac{1}{1+\cos x}, & -\pi<x<0\\ xe^{-x^2}, & x\geq 0\end{cases}$$

求 $\int_1^4 f(x-2)\,dx$

令 $t=x-2$

$\Rightarrow x=t+2 \quad \therefore dx=dt$

$x: 1\to 4,\ t: -1\to 2$

$\therefore$ 原式 $=\int_{-1}^{2} f(t)\,dt$

$=\int_{-1}^{0} f(t)\,dt+\int_0^2 f(t)\,dt$

$=\int_{-1}^{0}\frac{1}{1+\cos t}\,dt+\int_0^2 te^{-t^2}\,dt$

$=\int_{-1}^{0}\frac{1}{\cos^2\frac{t}{2}}\,d\frac{t}{2}-\frac{1}{2}\int_0^2 e^{-t^2}\,d(-t^2)$

$=\left[\tan\frac{t}{2}\right]_{-1}^{0}-\frac{1}{2}\left[e^{-t^2}\right]_0^2$

$=\tan(\frac{1}{2})-\frac{1}{2}e^{-4}+\frac{1}{2}$

分部积分法

不定积分：

$$\int u\,dv=uv-\int v\,du$$

定积分：

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

$$\int_0^{\frac{1}{2}} \arcsin x\,dx$$

$$= [x\arcsin x]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x\,d\arcsin x$$

$$= \frac{1}{2}\arcsin\frac{1}{2} + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,d(1-x^2)$$

$$= \frac{\pi}{12} + \frac{1}{2}\left[2(1-x^2)^{\frac{1}{2}}\right]_0^{\frac{1}{2}}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$\int_0^1 e^{\sqrt{x}}\,dx$$

令 $t = \sqrt{x}$，则 $x = t^2$

$x: 0 \to 1$， $t: 0 \to 1$

$\therefore dx = dt^2 = 2t\,dt$

$\therefore$ 原式 $= \int_0^1 e^t \cdot 2t\,dt$

$$= \int_0^1 2t\,de^t$$

$$= [2t \cdot e^t]_0^1 - 2\int_0^1 e^t\,dt$$

$$= 2e - 2[e^t]_0^1$$

$$= 2e - (2e - 2)$$

$= 2$

先换元，再分部积分

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \underbrace{\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}}_{4321,\frac{\pi}{2}}, & n \text{ 为正偶} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \underbrace{\frac{4}{5} \cdot \frac{2}{3} \cdot 1}_{54321}, & n > 1, \text{ 正奇} \end{cases}$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x \, dx = \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \, dx = \frac{4}{5} \times \frac{2}{3} \times 1$$

$$\int_0^{\frac{\pi}{2}} \sin^7 x \, dx = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1$$

补充：

微积分基本公式

积分上限函数

$$\Phi(x) = \int_a^x f(t)\,dt$$

则 $\Phi'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$\Phi(x) = \int_a^x f(t)\,dt$ 是 $f(x)$ 的一个原函数

$$\left(\int_a^x f(t)\,dt\right)' = f(x)$$

$$\left(\int_a^{\varphi(x)} f(t)\,dt\right)' = f(\varphi(x))\cdot\varphi'(x)$$

$$\left(\int_{\psi(x)}^{\varphi(x)} f(t)\,dt\right)' = f(\varphi(x))\varphi'(x) - f(\psi(x))\psi'(x)$$

牛顿-莱布尼茨公式

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$$

$\int f(x)\,dx = \sin x + C \Rightarrow f(x) = \cos x$

求 $\displaystyle\int \frac{f(\arcsin x)}{\sqrt{1-x^2}}\,dx$

$= \int f(\arcsin x)\,d\arcsin x$

$= \int \cos(\arcsin x)\,d\arcsin x$

$= \int \cos t\,dt$

$= \sin t + C$

$= \sin(\arcsin x) + C$

$= x + C.$

$\arcsin x = \sin^{-1} x$

$\sin\left(\frac{1}{\sin x}\right)$

求证：$\sin(\underbrace{\arcsin x}_{=y})$ $\quad \sin y = x$

令 $y = \arcsin x$

则 $\sin y = x$

$\therefore$ 原式 $= \sin y = x$

$\sin y$

$x = \arcsin 1 = \frac{\pi}{4}$

$\sin x = 1$

$\Rightarrow x = \frac{\pi}{4}$