微分方程（1）

微分方程的基本概念

例1. 过 $(1,2)$ $k=2x$

$y=x^2+C$

$\Rightarrow y=x^2+1$

含导数

$F(x,y,y',\cdots,y^{(n)})=0$

通解：

含常数的个数 = 阶

$y''=3 \Rightarrow y'=3x+C_1$

$\Rightarrow y=\frac{3}{2}x^2+C_1x+C_2$

可分离变量的微分方程

$g(y)dy=f(x)dx$

$$\Rightarrow \int g(y)\,dy = \int f(x)\,dx$$

$$\frac{dy}{dx} = 2x$$

$$\Rightarrow dy = 2x\,dx$$

$$\Rightarrow \int dy = \int 2x\,dx$$

$$\Rightarrow y = x^2 + C$$

例1. $\frac{dy}{dx} = 2xy$

$$\Rightarrow \int \frac{1}{y}\,dy = \int 2x\,dx$$

$$\Rightarrow \ln|y| = x^2 + C$$

$$\Rightarrow |y| = e^{x^2+C}$$

$$\Rightarrow y = \pm\, e^{x^2} e^{C}$$

$$= Ce^{x^2}$$

例2. $\frac{dm}{dt} = -\lambda m$

$$\Rightarrow \int \frac{1}{m}\,dm = \int -\lambda\,dt$$

$$\Rightarrow \ln m = -\lambda t + C$$

...

例5. $m\frac{dv}{dt} = mg - kv$

$\Rightarrow \int \frac{1}{mg-kv}\,dv = \int \frac{1}{m}\,dt$

$\Rightarrow -\frac{1}{k}\ln(mg-kv) = \frac{t}{m} + C$

……

齐次方程

$$\frac{dy}{dx} = \frac{1+\left(\frac{y}{x}\right)^2}{1-2\left(\frac{y}{x}\right)^2} \quad *$$

① $u = \frac{y}{x}$

② $y = ux$

③ $\frac{dy}{dx} = u + x\frac{du}{dx}$

③代入*得：

$u + x\frac{du}{dx} = \frac{1+u^2}{1-2u^2}$

$\Rightarrow x\frac{du}{dx} = \frac{1+u^2}{1-2u^2} - u$

…… 可分离变量

例1. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{xy - x^2}$$

$$= \frac{\left(\frac{y}{x}\right)^2}{\frac{y}{x} - 1}$$

令 $u = \frac{y}{x}$,

则 $\cancel{u} + x\frac{du}{dx} = \frac{u^2}{u-1} - u$

$$\Rightarrow x\frac{du}{dx} = \frac{u}{u-1}$$

$$\Rightarrow \frac{u-1}{u} du = \frac{1}{x} dx$$

$$\Rightarrow \int \left(1 - \frac{1}{u}\right) du = \int \frac{1}{x} dx$$

$$\Rightarrow u - \ln|u| = \ln|x| + C$$

$$\Rightarrow \ln|xu| = u + C$$

$$\Rightarrow \ln|y| = \frac{y}{x} + C$$

齐次方程 /remake

齐次: $\frac{dy}{dx} = \frac{1+\left(\frac{y}{x}\right)^2}{1-2\left(\frac{y}{x}\right)^2}$ *

($\frac{y}{x}$整体出现)

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$$

① $u=\frac{y}{x}$

② $y=xu$

③ $\frac{dy}{dx}=1\times u+x\cdot u'$

$=\frac{y}{x}+x\frac{du}{dx}$

代入*式得:

$$x\frac{du}{dx}=\frac{1+u^2}{1-2u^2}-u$$

$\Rightarrow \frac{x}{dx}\cdot du=\frac{3u^2-u+1}{1-2u^2}$

$\Rightarrow \frac{1-2u^2}{3u^2-u+1}du=\frac{1}{x}dx$

可解

例1. $y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$

$\Rightarrow y^2=(xy-x^2)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{y^2}{xy-x^2}$

$=\frac{(\frac{y}{x})^2}{\frac{y}{x}-1}$

令 $u=\frac{y}{x}$，则 $y=u\cdot x$,

则 $\frac{dy}{dx}=u'\cdot x+u=u+x\frac{du}{dx}$

$\Rightarrow x\frac{du}{dx}=\frac{u^2}{\ }-u$

$\Rightarrow \quad \cdots dx \cdots u-1$

$\Rightarrow \quad \frac{x}{dx}du = \frac{u}{u-1}$

$\Rightarrow \int (1-\frac{1}{u})du = \int \frac{1}{x}dx$

$\Rightarrow \quad u - \ln|u| = \ln|x| + C$

$\Rightarrow \quad u = \ln|xu| + C$

$\Rightarrow \frac{y}{x} = \ln|y| + C$

$\Rightarrow \quad \ln|y| = \frac{y}{x} + C$

$\Rightarrow y = Ce^{\frac{y}{x}}$

可化为齐次的方程

$$\frac{dy}{dx} = \frac{ax+by+C}{a_1x+b_1y+C_1}$$

令 $x = X+h, \ y = Y+k$

$$\therefore \frac{dY}{dX} = \frac{aX+bY+ah+bk+C}{a_1X+b_1Y+a_1h+b_1k+C_1}$$

若 $\begin{cases} ah+bk+C=0 \\ a_1h+b_1k+C_1=0 \end{cases}$

① $\frac{a_1}{a} \neq \frac{b_1}{b} \Rightarrow \begin{cases} h = \cdots \\ k = \cdots \end{cases}$

② $\frac{a_1}{a} = \frac{b_1}{b}$

② $\frac{a_1}{a} = \frac{b_1}{b} = \lambda$

$\Rightarrow a_1 = \lambda a,\ b_1 = \lambda b$

$\therefore \frac{dy}{dx} = \frac{ax+by+c}{\lambda(ax+by)+c_1}$

令 $V = ax+by$

$\therefore \frac{dv}{dx} = a + b\frac{dy}{dx}$

$\Rightarrow \frac{dv}{dx} = b\frac{V+c}{\lambda V+c_1} + a$

即为可分离变量方程

可解.