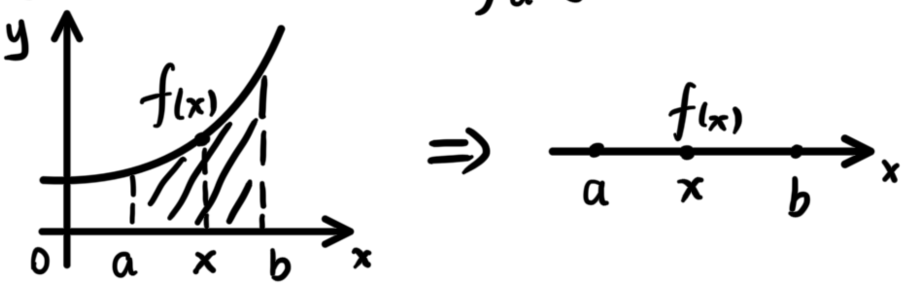
重积分(5)

三重积分

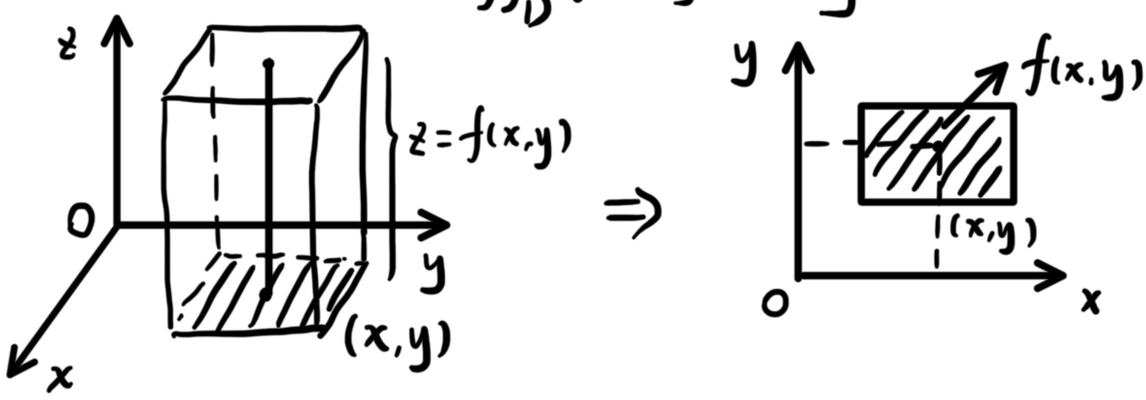
$$\iiint_{\Omega} f(x,y,z)\,dV = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta V_i$$

重新理解定积分

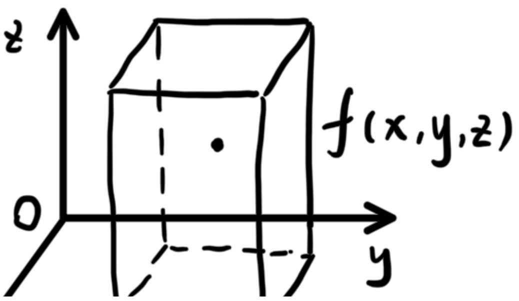
① 一重积分 $\int_a^b f(x)\,dx$

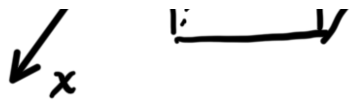


② 二重积分 $\iint_D f(x,y)\,dxdy$



② 三重积分 $\iiint_{\Omega} f(x,y,z)\,dxdydz$

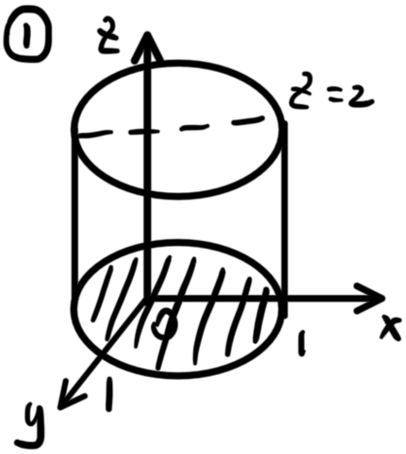




三重积分的计算

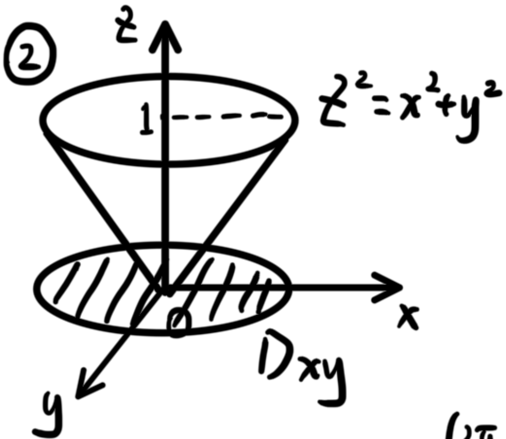
(一) 投影法 (先一后二)

①



$$\iiint_{\Omega} f(x,y,z)\,dV = \iint_{D_{xy}} dxdy \int_0^2 f(x,y,z)\,dz$$

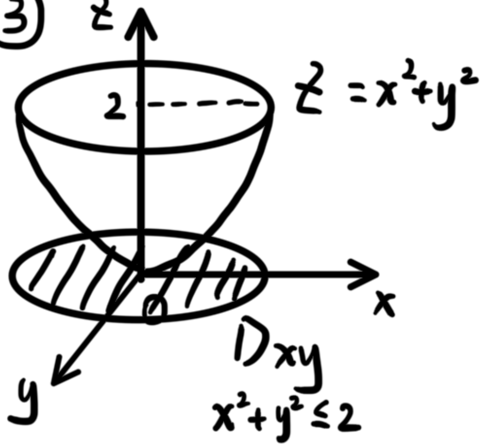
②



$$\iiint_{\Omega} f(x,y,z)\,dV = \iint_{D_{xy}} dxdy \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho\,d\rho \int_{\rho}^{1} f(\rho\cos\theta, \rho\sin\theta, z)\,dz$$

③



$$\iiint_{\Omega} 5\,dV$$

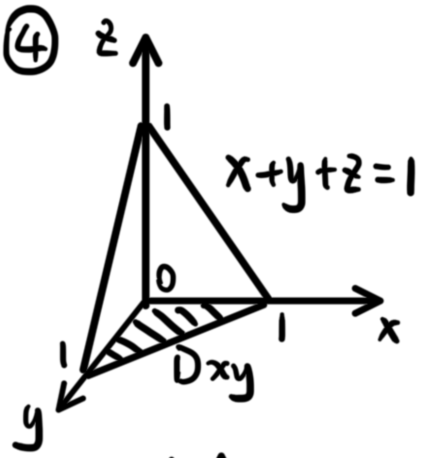
$$= \iint_{D_{xy}} dxdy \int_{x^2+y^2}^{2} 5\,dz$$

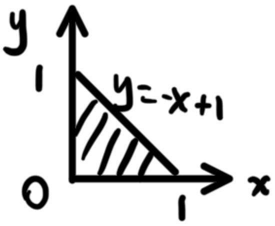
$$= \iint_{D_{xy}} 5(2-x^2-y^2)\,dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} 5(2-\rho^2)\rho\,d\rho$$

$$= \int_0^{2\pi} 5\, d\theta = 10\pi$$

④





$$\iiint_{\Omega} dx\,dy\,dz$$

$$= \iint_{D_{xy}} dx\,dy \int_0^{1-x-y} dz$$

$$= \iint_{D_{xy}} (1-x-y)\, dx\,dy$$

$$= \int_0^1 dx \int_0^{-x+1} (1-x-y)\, dy$$

$$= \int_0^1 \left[\frac{1}{2}(x-1)^2\right] d(x-1)$$

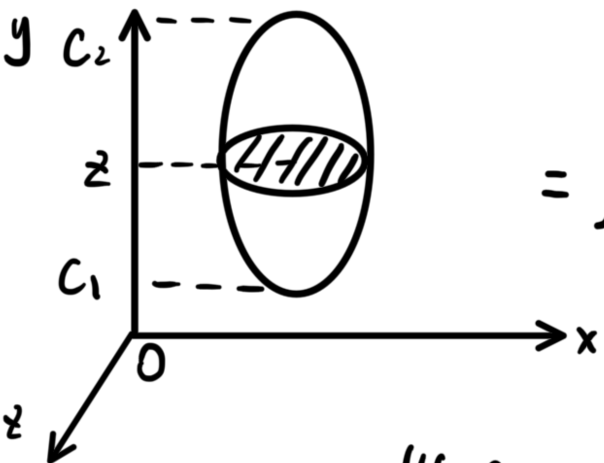
$$= \frac{1}{2}\left[\frac{1}{3}(x-1)^3\right]_0^1 = \frac{1}{6}$$

另法：$\iiint_{\Omega} dx\,dy\,dz$

$$= \iint_{D_{zx}} dx\,dz \int_0^{1-x-z} dy$$

$$= \int_0^1 dx \int_0^{1-x} dz \int_0^{1-x-z} dy$$

(二) 截面法（先二后一）



$$\iiint_{\Omega} dx\,dy\,dz$$

$$= \int_{C_1}^{C_2} dz \iint_{D_z} f(x,y,z)\, dx\,dy$$

① 上述圆柱：$\iiint_{\Omega} f(x,y,z)\, dV$

$$= \int_0^2 dz \iint_{D_z} f(x,y,z)\, dx\,dy$$

② 上述圆锥：设 $f(x,y,z)=x^2+y^2$

$$\iiint_{\Omega} f(x,y,z)\,dV$$

$$=\int_0^1 dz\iint_{D_z}(x^2+y^2)\,dxdy$$

$$=\int_0^1 dz\int_0^{2\pi}d\theta\int_0^{z}\rho^3\,d\rho$$

$$=\int_0^1 dz\int_0^{2\pi}\frac{1}{4}z^4\,d\theta$$

$$=\int_0^1\frac{1}{2}z^4\pi\,dz=\frac{\pi}{10}$$

③ 上述旋转抛物面：$\iiint_{\Omega}5\,dV$

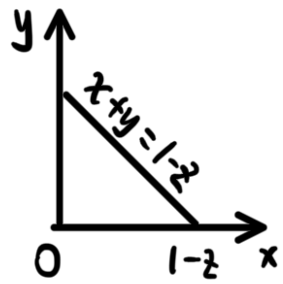
$$=\int_0^2 dz\iint_{D_z}5\,dxdy$$

$$=\int_0^2 dz\int_0^{2\pi}d\theta\int_0^{\sqrt{z}}5\rho\,d\rho$$

$$=\int_0^2 dz\int_0^{2\pi}\frac{5}{2}z\,d\theta$$

$$=5\pi\int_0^2 z\,dz=10\pi$$

④ 上述三棱锥：$\iiint_{\Omega}1\,dV$



$$=\int_0^1 dz\iint_{D_z}dxdy$$

$$=\int_0^1 dz\int_0^{1-z}dx\int_0^{1-x-z}dy$$

$$=\int_0^1 dz\int_0^{1-z}(1-x-z)\,dx$$

$$=\int_0^1\left[\frac{1}{2}(1-z)^2\right]dz=\frac{1}{6}$$