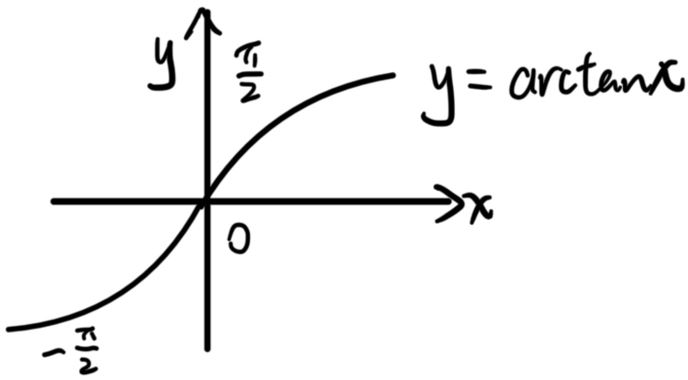
反常积分

1. 无穷限的反常积分

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$$

$$= \left[\arctan x\right]_{-\infty}^{+\infty}$$

$$= \pi$$



$$\int_0^{+\infty} te^{-pt}\,dt \quad (p>0)$$

$$= \frac{1}{-p}\int_0^{+\infty} t\cdot(-pe^{-pt})\,dt$$

$$= \frac{1}{-p}\int_0^{+\infty} t\,d(e^{-pt})$$

$$= -\frac{1}{p}\left[te^{-pt}\right]_0^{+\infty} + \frac{1}{p}\int_0^{+\infty} e^{-pt}\,dt$$

$$= -\frac{1}{p}\lim_{t\to+\infty} te^{-pt} - \frac{1}{p^2}\left[e^{-pt}\right]_0^{+\infty}$$

$$= -\frac{1}{p}\lim_{t\to+\infty} \frac{t}{e^{pt}} + \frac{1}{p^2}$$

$$= -\frac{1}{p}\lim_{t\to+\infty} \frac{1}{p\cdot e^{pt}} + \frac{1}{p^2}$$

$$= 0 + \frac{1}{\ }$$

$$= 0 \cdot \frac{1}{p^2}$$

$$= \frac{1}{p^2}$$

$$-\frac{1}{p^2}\left(\int_0^{+\infty} e^{-pt}\, d(-pt)\right)$$

$$= -\frac{1}{p^2}\left[e^{-pt}\right]_0^{+\infty}$$

$$= -\frac{1}{p^2}(0-1)$$

$$= \frac{1}{p^2}$$

$$\int_a^{+\infty} \frac{dx}{x^p} \quad (a>0)$$

① 当 $p=1$ 时，

$$\int_a^{+\infty} \frac{1}{x}\, dx = \left[\ln x\right]_0^{+\infty} = +\infty$$

② 当 $p \neq 1$ 时，

$$\int_a^{+\infty} x^{-p}\, dx = \frac{1}{1-p}\left[x^{1-p}\right]_a^{+\infty}$$

$$= \begin{cases} +\infty, & p<1 \\ \dfrac{a^{1-p}}{p-1}, & p>1 \end{cases}$$

2. 无界函数的反常积分

b为瑕点、$\int_a^b f(x)\,dx = \lim\limits_{t \to b^-} \int_a^t f(x)\,dx$

a为瑕点、$\int_a^b f(x)\,dx = \lim\limits_{t \to a^+} \int_t^b f(x)\,dx$

c为瑕点、$\int_a^b f(x)\,dx$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

牛顿-莱布尼兹公式

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

$$= \left[F(x) \right]_a^{b^-}$$

$$= F(b^-) - F(a)$$

例 4. $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \quad (a > 0)$

令 $x = a\sin t$

$dx = a\cos t\, dt$

$x: 0 \to a,\ t: 0 \to \frac{\pi}{2}$

$\therefore$ 原式 $= \int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\cos t} dt$

$= \int_0^{\frac{\pi}{2}} 1\, dt$

$= [t]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$

或 原式 $= \left[\arcsin \frac{x}{a}\right]_0^a$

$= \arcsin 1$

$= \frac{\pi}{2}$

例5. $\int_{-1}^{1} \frac{1}{x^2} dx$

$= 2\int_{-1}^{0} \frac{1}{x^2} dx$

$= 2\left[-\frac{1}{x}\right]_{-1}^{0}$

$= 2\left[\lim_{x\to 0^-}\left(-\frac{1}{x}\right) - 1\right]$

$= +\infty$

$\therefore \int_{-1}^{1} \frac{1}{x^2} dx$ 发散

例7. $\int_0^{+\infty} \frac{1}{\sqrt{x(x+1)^3}} dx$

令 $\frac{1}{x} = t \Rightarrow x = \frac{1}{t}$

$$dx = d\frac{1}{t} = -\frac{1}{t^2}dt$$

$x: 0 \to +\infty$, $t: \infty \to 0$

$$\therefore 原式 = \int_{+\infty}^{0} \frac{1}{\sqrt{\frac{1}{t}(\frac{1}{t}+1)^3}} \cdot \left(-\frac{1}{t^2}\right)dt$$

$$= -\int_{+\infty}^{0} \frac{1}{\sqrt{t^3(\frac{1}{t}+1)^3}}dt$$

$$= -\int_{+\infty}^{0} (1+t)^{-\frac{3}{2}} d(1+t)$$

$$= 2\left[(1+t)^{-\frac{1}{2}}\right]_{+\infty}^{0}$$

$$= 2(1-0)$$

$$= 2$$

Γ函数

$$\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1} dx$$

$(s>0)$

递推公式 $\Gamma(s+1) = s\Gamma(s)$

证: $\Gamma(s+1)$

证：$\Gamma(s+1)$

$$= \int_0^{+\infty} e^{-x} x^s dx$$

$$= -\int_0^{+\infty} x^s de^{-x}$$

$$= -[x^s e^{-x}]_0^{+\infty} + \int_0^{+\infty} e^{-x} dx^s$$

$$= 0 + s\int_0^{+\infty} e^{-x} x^{s-1} dx$$

$$= s\Gamma(s)$$

$$\Gamma(1) = \int_0^{+\infty} e^{-x} dx$$

$$= -[e^{-x}]_0^{+\infty}$$

$$= -(0-1)$$

$$= 1$$

…

$$\Gamma(n+1) = n!$$