不定积分草稿2

第二类换元积分法

$$\int f(x)\,dx$$

[其中 $x=\varphi(t)$

$dx = d\varphi(t) = \varphi'(t)\,dt$]

$$= \int f(\varphi(t))\cdot\varphi'(t)\,dt$$

其中 $t=\varphi^{-1}(x)$

$$\int f(x)\,dx \overset{x=\varphi(t)}{=} \int f(\varphi(t))\varphi'(t)\,dt$$

$$= g(t) + C$$

$t=\varphi^{-1}(x)$ $\quad = g(\varphi^{-1}(x)) + C$

$\boxed{\sqrt{1-\sin^2 t}}$

$$\int \sqrt{a^2-x^2}\,dx$$

$\sin t = \frac{x}{a}$

$\Rightarrow t = \arcsin\frac{x}{a}$

令 $x = a\sin t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$

则 $dx = d(a\sin t) = a\cos t\,dt$

$\therefore$ 原式 $= \int a\cos t\cdot a\cos t\,dt$

∴原式 $= \int a\cos t \cdot a\cos t\,dt$

$= \int a^2\cos^2 t\,dt$

$= \frac{1}{2}a^2\int \frac{1+\cos 2t}{2}\,d2t$

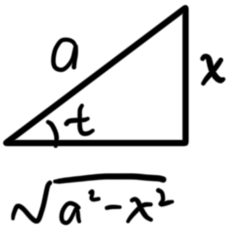
$= \frac{1}{4}a^2(2t+\sin 2t)+C$

$= \frac{1}{2}a^2 t+\frac{1}{4}a^2\sin 2t+C$

$= \frac{1}{2}a^2 t+\frac{1}{2}a^2\sin t\cos t+C$

$= \frac{1}{2}a^2\cdot\arcsin\frac{x}{a}+\frac{1}{2}\cancel{a^2}\frac{x}{\cancel{a}}\cdot\frac{\sqrt{a^2-x^2}}{\cancel{a}}+C$

$= \frac{1}{2}a^2\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$



$\boxed{1+\tan^2 t}$

$\int\frac{1}{\sqrt{x^2+a^2}}dx\ (a>0)$

令 $x = a\tan t,\ x\in(-\frac{\pi}{2},\frac{\pi}{2})$

$\Rightarrow dx = d(a\tan t)$

$= (a\cdot\sec^2 t)\,dt$

原式 $= \int\frac{1}{a\sqrt{(1+\tan^2 t)}}\cdot a\sec^2 t\,dt$

$= \int\frac{1}{\cancel{a}\cdot\cancel{\sec t}}\cdot\cancel{a}\sec^{\cancel{2}} t\,dt$

$= \int\sec t\,dt$

$= \ln|\sec t + \tan t| + C$

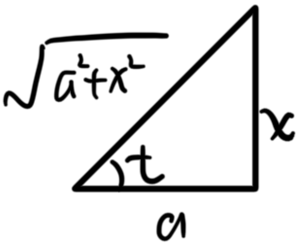
$\frac{1}{\cos t} + \tan t$

又∵ $x = a\tan t$

$\Rightarrow \tan t = \frac{x}{a}$

∴原式 $= \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C$

$= \ln(\sqrt{a^2+x^2} + x) + C$



$\cos t = \frac{a}{\sqrt{a^2+x^2}}$

$\boxed{\sec^2 x - 1}$

$\int \frac{1}{\sqrt{x^2-a^2}}\, dx\ (a>0)$

$x \in (0, \frac{\pi}{2})$

令 $x = a\cdot\sec t$ $\Rightarrow \sec t = \frac{1}{\cos t} = \frac{x}{a}$

$\Rightarrow \cos t = \frac{a}{x}$

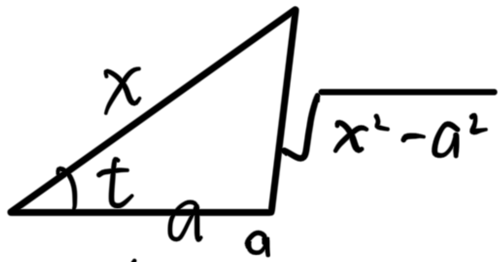
$\Rightarrow dx = d\, a\cdot\sec t$

$= \sec t\cdot\tan t\cdot a\, dt$

∴原式 $= \int \frac{1}{\cancel{a}\cancel{\sqrt{\sec^2 t - 1}}}\cdot \sec t\cdot \cancel{\tan t\cdot a}\, dt$

$= \int \sec t\, dt$

$$= \ln|\sec t + \tan t| + C$$



$\cos t = \frac{a}{x}$

$$\therefore 原式 = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C$$

① $\sqrt{a^2-x^2}$

令 $x = a\sin t$

$1 - \sin^2 t = \cos t$

② $\sqrt{a^2+x^2}$

令 $x = a\tan t$

$1 + \tan^2 t = \sec^2 t$

③ $\sqrt{x^2-a^2}$

令 $x = a\sec t$

$\sec^2 t - 1 = \tan^2 t$

$$\int \frac{\sqrt{a^2-x^2}}{x^4}\,dx$$

令 $x = \frac{1}{t} \Rightarrow dx = d\frac{1}{t} = -\frac{1}{t^2}dt$

$$\therefore 原式 = \int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}} \cdot \left(-\frac{1}{t^2}\right)dt$$

$$= -\int \sqrt{a^2-\frac{1}{t^2}} \cdot t^2\,dt$$

$$= -|t| \int \sqrt{a^2t^2-1}\,dt$$

$x>0$ 时, $t>0$

$$原式 = -\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\,d(a^2t^2-1)$$

$$= -\frac{1}{3a^2}(a^2t^2-1)^{\frac{3}{2}} + C$$

$$= -\frac{1}{3a^2}\left(\frac{a^2}{x^2}-1\right)^{\frac{3}{2}} + C$$

$$= -\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3} + C$$

$x<0$ 时，相同

$$\int \frac{\sqrt{a^2-x^2}}{x^4}dx$$

令 $x=a\sin t,\ x\in(-\frac{\pi}{2},\frac{\pi}{2})$

$$dx = d\,a\sin t = a\cos t\,dt$$

$$\therefore 原式 = \int \frac{a\cos t}{(a\sin t)^4}\cdot a\cos t\,dt$$

$$= \frac{1}{a^2}\int \frac{\cos^2 t}{\sin^4 t}dt$$

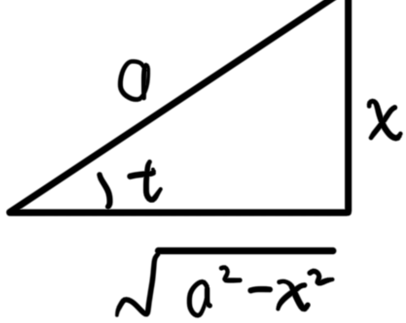
$$= \frac{1}{a^2}\int \frac{\cos^2 t}{\sin^2 t}\cdot\frac{1}{\sin^2 t}dt$$

$$= -\frac{1}{a^2}\int \cot^2 t\, d\cot t$$

$$= -\frac{1}{a^2}\cdot\frac{1}{3}\cot^3 t + C$$

$x=a\sin t$

$\Rightarrow \sin t = \frac{x}{a}$

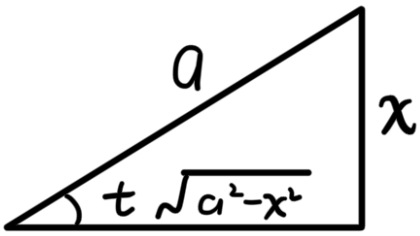


$$\therefore \cot t = \frac{1}{\tan t} = \frac{\sqrt{a^2-x^2}}{x}$$

$$\therefore 原式 = -\frac{1}{a^2} \cdot \frac{1}{3} \cdot \frac{(a^2-x^2)^{\frac{3}{2}}}{x^3} + C$$

$$= -\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3} + C$$

$$\sin t = \frac{x}{a}$$



$$\cot t = \frac{1}{\tan t} = \frac{\sqrt{a^2-x^2}}{x}$$

$$\int \tan x \, dx = -\ln|\cos x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan\frac{x}{a} + C$$

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a}+C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln(x+\sqrt{x^2+a^2})+C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\int \frac{dx}{\sqrt{4x^2+9}}$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{(2x)^2+3^2}}\,d(2x)$$

$$= \frac{1}{2}\ln(2x+\sqrt{4x^2+9})+C$$

$$\int \frac{dx}{\sqrt{1+x-x^2}}$$

$$= \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}} \, d\left(x-\frac{1}{2}\right)$$

$$= \arcsin \frac{x-\frac{1}{2}}{\frac{\sqrt{5}}{2}} + C$$

$$= \arcsin \frac{2x-1}{\sqrt{5}} + C$$

消根号!