曲线积分与曲面积分(2)

对坐标的曲线积分

$$\int_L P(x,y)dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} P(\xi_i,\eta_i)\Delta x_i$$

$$\int_L Q(x,y)dy = \lim_{\lambda\to 0}\sum_{i=1}^{n} Q(\xi_i,\eta_i)\Delta y_i$$

以上两个积分也称为"第二类曲线积分"

三维：

$$\int_\Gamma P(x,y,z)dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} P(\xi_i,\eta_i,\zeta_i)\Delta x_i$$

$$\int_\Gamma Q(x,y,z)dy = \lim_{\lambda\to 0}\sum_{i=1}^{n} Q(\xi_i,\eta_i,\zeta_i)\Delta y_i$$

$$\int_\Gamma R(x,y,z)dz = \lim_{\lambda\to 0}\sum_{i=1}^{n} R(\xi_i,\eta_i,\zeta_i)\Delta z_i$$

$$\int_L P(x,y)dx + \int_L Q(x,y)dy$$

$$= \int_L P(x,y)dx + Q(x,y)dy$$

$$= \int_L \vec{F}(x,y)\cdot d\vec{r}$$

其中 $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ 为向量值函数

$d\vec{r} = dx\vec{i} + dy\vec{j}$

性质：

① $\int_L [\alpha\vec{F_1}(x,y) + \beta\vec{F_2}(x,y)]\cdot d\vec{r}$

$= \alpha\int_L \vec{F_1}(x,y)\cdot d\vec{r} + \beta\int_L \vec{F_2}(x,y)\cdot d\vec{r}$

② $\int_L \vec{F}(x,y)\cdot d\vec{r} = \int_{L_1} \vec{F}(x,y)\cdot d\vec{r} + \int_{L_2} \vec{F}(x,y)\cdot d\vec{r}$

③ 设 L^- 是 L 的反向曲线弧，则

$$\int_{L^-} \vec{F}(x,y)\cdot d\vec{r} = -\int_L \vec{F}(x,y)\cdot d\vec{r}$$

对坐标的曲线积分的计算法

参数方程 $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$

$t: \alpha \to \beta$（α 不一定比 β 小，α 为起点，β 为终点）

则 $\int_L P(x,y)dx + Q(x,y)dy$

$$= \int_\alpha^\beta \{P[\varphi(t),\psi(t)]\varphi'(t) + Q[\varphi(t),\psi(t)]\psi'(t)\}dt$$

对比对弧长的曲线积分：

$$\int_L f(x,y)ds = \int_\alpha^\beta f[\varphi(t),\psi(t)]\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt \ (\alpha<\beta)$$

特别地，

① $y=\psi(x)$ 时，（$x=x$）

$\int_L P(x,y)dx + Q(x,y)dy$

$$= \int_a^b \{P[x,\psi(x)] + Q[x,\psi(x)]\psi'(x)\}dx$$

② $x=\varphi(y)$ 时，（$y=y$）

$\int_L P(x,y)dx + Q(x,y)dy$

$$\int_L P(x,y)dx+Q(x,y)dy$$

$$=\int_a^b\{P[\varphi(y),y]\varphi'(y)+Q[\varphi(y),y]\}dy$$

③ $x=\varphi(t),\ y=\psi(t),\ z=\omega(t)$ 时,

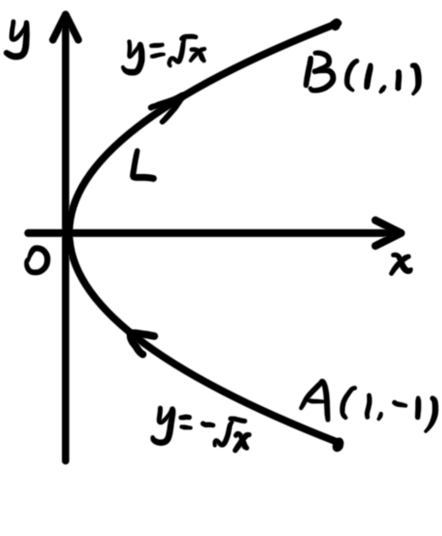
$$\int_\Gamma P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$$

$$=\int_\alpha^\beta\{P[\varphi(t),\psi(t),\omega(t)]\varphi'(t)$$

$$+Q[\varphi(t),\psi(t),\omega(t)]\psi'(t)$$

$$+R[\varphi(t),\psi(t),\omega(t)]\omega'(t)\}dt$$

例1. 计算 $\int_L xy\,dx$,其中 L 为抛物线 $y^2=x$ 上从点 $(1,-1)$ 到点 $B(1,1)$ 的一段弧.



解:法一: $A\to O,\ O\to B$

$$\int_L xy\,dx=\int_{AO}xy\,dx+\int_{OB}xy\,dx$$

$$=\int_1^0 x(-\sqrt{x})dx+\int_0^1 x\sqrt{x}\,dx$$

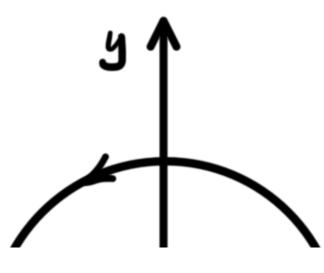
$$=2\int_0^1 x^{\frac{3}{2}}dx$$

$$=\frac{4}{5}$$

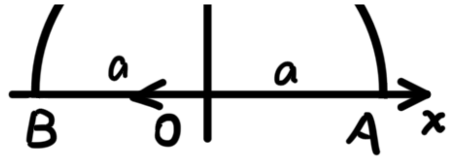
法二: $\int_L xy\,dx=\int_{-1}^1 y^3 d(y^2)=2\int_{-1}^1 y^4 dy=\frac{4}{5}$

例2. $\int_L y^2 dx$

(1) 积分区域上半圆周



令 $\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}$ $(0 \le \theta \le \pi)$



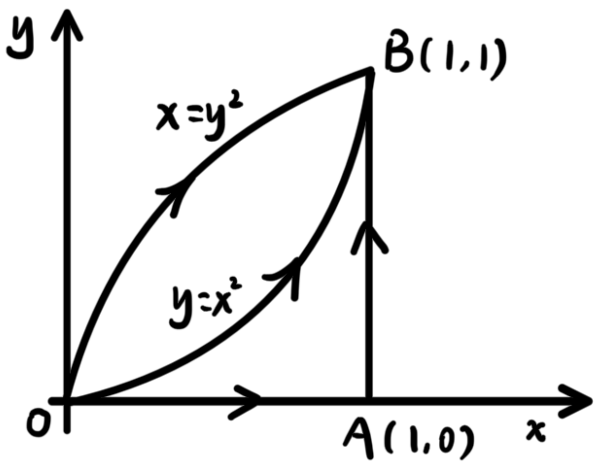
$$\therefore \int_L y^2 dx = \int_0^{\pi} (a\sin\theta)^2 d(a\cos\theta)$$
$$= -\int_0^{\pi} a^3 \sin^3\theta \, d\theta$$
$$= -2a^3 \int_0^{\frac{\pi}{2}} \sin^3\theta \, d\theta$$
$$= -\frac{4}{3}a^3$$

(2) 积分区域线段AB (A→B)

$$\int_L y^2 dx = \int_a^{-a} 0 \, dx = 0$$

例3. $\int_L 2xy\,dx + x^2 dy$ (保守力做功)

(1) $y = x^2$



$$\int_L 2xy\,dx + x^2 dy$$
$$= \int_0^1 [2x \cdot x^2 + x^2 \cdot (x^2)'] dx$$
$$= \int_0^1 4x^3 dx = 1$$

(2) $x = y^2$

$$\int_L 2xy\,dx + x^2 dy$$
$$= \int_0^1 [2y^2 \cdot y \cdot (y^2)' + y^4] dy = \int_0^1 5y^4 dy = 1$$

(3) 有向折线OAB

$$\int_L 2xy\,dx + x^2 dy$$

$= \int_0^1 (2x \cdot 0 + x^2 \cdot 0)dx + \int_0^1 (2x \cdot 0 + 1)dy$

$= 0 + 1 = 1$

例4. 计算 $\int_\Gamma x^3dx + 3zy^2dy - x^2ydz$，其中 Γ 是从点 $A(3,2,1)$ 到点 $B(0,0,0)$ 的直线段 AB.

解：l_{AB}：$\frac{x}{3} = \frac{y}{2} = \frac{z}{1}$

参数化：$\begin{cases} x = 3t \\ y = 2t \\ z = t \end{cases}$　$t: 1 \to 0$

$\therefore \int_\Gamma x^3dx + 3zy^2dy - x^2ydz$

$= \int_1^0 [(3t)^3 \cdot 3 + 3t \cdot (2t)^2 \cdot 2 - (3t)^2 \cdot 2t \cdot 1]dt$

$= \int_1^0 87t^3dt = -\frac{87}{4}$