定积分的应用

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

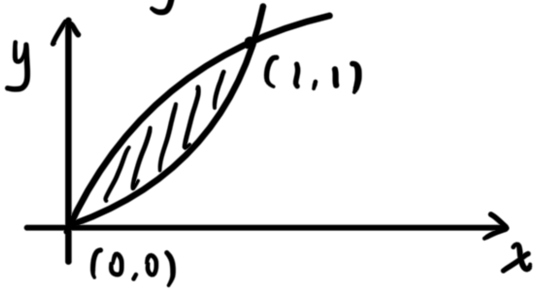
1、定积分的元素法

$$S_{扇} = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}\theta r^2$$

2、定积分在几何学上的应用

$$A = \int_a^b f(x)\,dx$$

$y^2 = x$，$y = x^2$



$$\begin{cases} y^2 = x \\ y = x^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} 或 \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x)-g(x)]\,dx$$

$$\left(\int_a^b [上-下]\,dx \right)$$

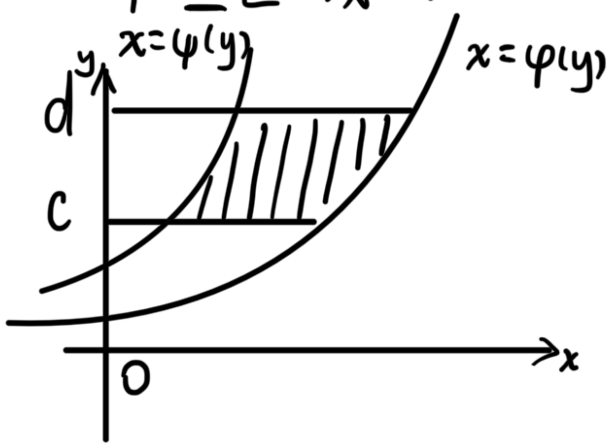
"X型区域"

$$\therefore S = \int_0^1 (x^{\frac{1}{2}} - x^2)\,dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \right]_0^1$$

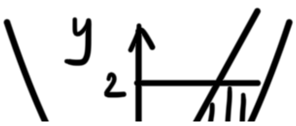
$$= \frac{1}{3}$$

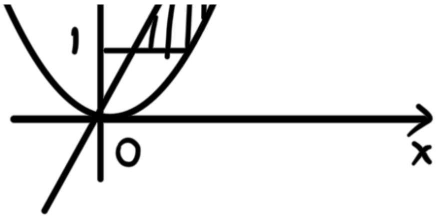
"Y型区域":



$$\int_c^d [\varphi(y) - \psi(y)]\,dy$$

$$= \int_c^d [右-左]\,dy$$





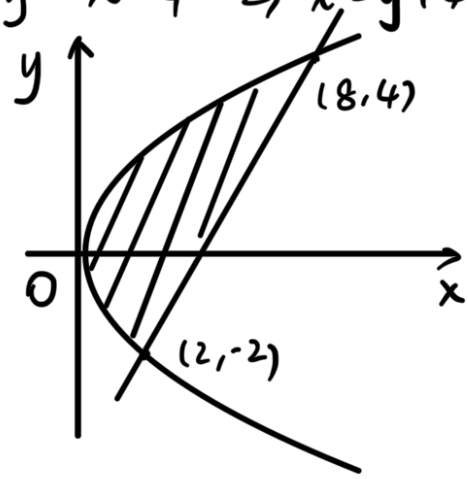
$y=3x \Rightarrow x=\frac{1}{3}y$

$y=x^2 \Rightarrow x=\sqrt{y}$

$\therefore \int_1^2 (\sqrt{y}-\frac{1}{3}y)\,dy$

例2：$y^2=2x \Rightarrow x=\frac{1}{2}y^2$

$y=x-4 \Rightarrow x=y+4$



Y型区域做法：

$\begin{cases} y^2=2x \\ y=x-4 \end{cases} \Rightarrow 2x=(x-4)^2$

$\Rightarrow 2x=x^2-8x+16$

$\Rightarrow x^2-10x+16=0$

$\Rightarrow (x-2)(x-8)=0$

$\Rightarrow x=2$ 或 $x=8$

$\Rightarrow \begin{cases} x=2 \\ y=-2 \end{cases}$ 或 $\begin{cases} x=8 \\ y=4 \end{cases}$

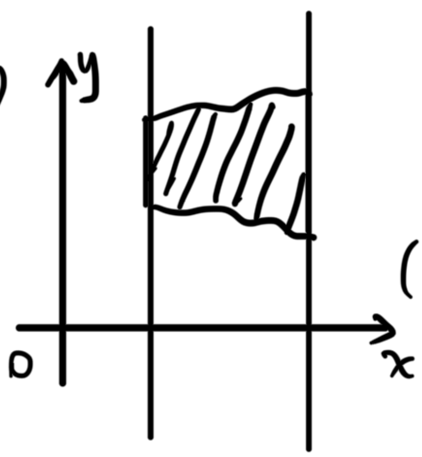
$$\int_{-2}^{4} \left(y+4-\frac{1}{2}y^2 \right) dy$$
$$= \left[\frac{1}{2}y^2+4y-\frac{1}{6}y^3 \right]_{-2}^{4}$$
$$= 8+16-\frac{32}{3}-\left(2-8+\frac{4}{3}\right)$$
$$= 18$$

X型区域做法：

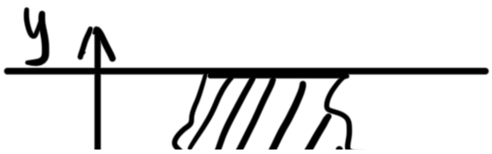
$$S = \int_0^2 \left[\sqrt{2x}-(-\sqrt{2x})\right]dx + \int_2^8 \left[\sqrt{2x}-(x-4)\right]dx$$
$$= 18$$

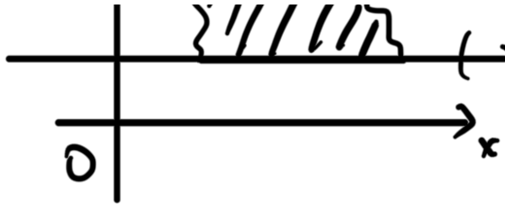
1) 画图

2)

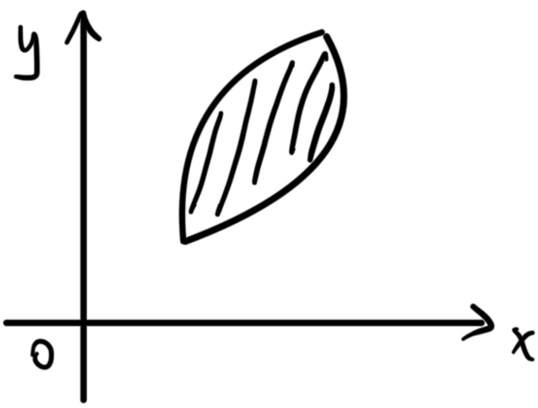


(X型)





(Y型)



☆.具体问题具体分析

$$y=\frac{1}{2}x^2$$

$$y=\frac{1}{1+x^2}$$

$$x\in[-\sqrt{3},\sqrt{3}]$$

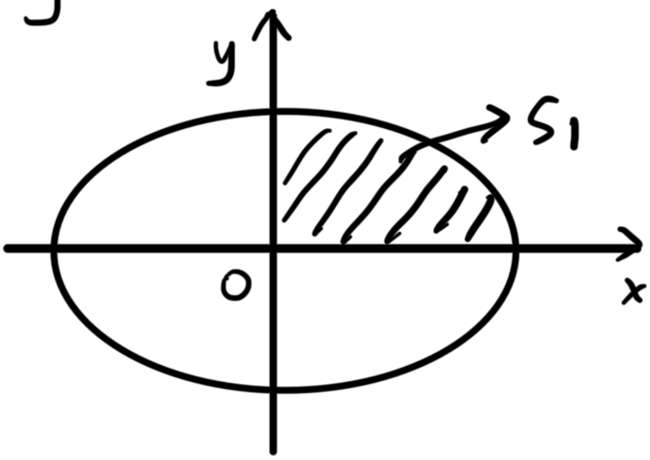
$$S=2\left[\int_{1}^{\sqrt{3}}\left(\frac{1}{2}x^2-\frac{1}{1+x^2}\right)dx\right.$$

$$\left.+\int_{0}^{1}\left(\frac{1}{1+x^2}-\frac{1}{2}x^2\right)dx\right]$$

例3：

椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\begin{cases} x=a\cos t \\ y=b\sin t \end{cases} \quad \left(0\le t\le \frac{\pi}{2}\right)$$



$$S_1=\int_0^a (y-0)\,dx$$

$$=\int_{\frac{\pi}{2}}^{0} b\sin t\, d(a\cos t)$$

$$=-ab\int_{\frac{\pi}{2}}^{0}\sin^2 t\,dt$$

$x: 0\to a$

$t: \frac{\pi}{2}\to 0$

$$=\frac{1}{2}ab\int_{0}^{\frac{\pi}{2}}\frac{1-\cos 2t}{2}\,d(2t)$$

$$= \frac{1}{4}ab\left[2t - \sin 2t\right]_0^{\pi}$$

$$= \frac{1}{4}ab(\pi - 0)$$

$$= \frac{1}{4}ab\pi$$

$\therefore S_{椭圆} = \pi ab$

不用参数方程法：

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

$$\therefore S_1 = \int_0^a \left(\frac{b}{a}\sqrt{a^2 - x^2} - 0\right)dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

令 $x = a\sin t$，$t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore dx = da\sin t = a\cos t\,dt$

$x: 0 \to a$，$t: 0 \to \frac{\pi}{2}$

$$\therefore S_1 = \frac{b}{a}\int_0^{\frac{\pi}{2}} a^2\cos^2 t\,dt$$

$$= \frac{ab}{4}\int_0^{\frac{\pi}{2}} (1 + \cos 2t)\,d(2t)$$

$$= \frac{ab}{4}\left[2t + \sin 2t\right]_0^{\frac{\pi}{2}}$$

$$= \frac{ab}{4}(\pi - 0)$$

$= \frac{ab}{4}\pi$

$\therefore S_{椭圆} = \pi ab$