重积分(7)

重积分的应用

一、曲面的面积

$$dA=\frac{d\sigma}{\cos\gamma}$$

其中 $\cos\gamma=\dfrac{1}{\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}}$

$\therefore\ dA=\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,d\sigma$

即 $A=\iint\limits_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$

例1. 求半径为a的球的表面积.

解：取上半球面 $z=\sqrt{a^2-x^2-y^2}$

$D_1=\left\{(x,y)\mid x^2+y^2\leq a^2\right\}$

$\frac{\partial z}{\partial x}=\frac{1}{2}\cdot\frac{-2x}{\sqrt{a^2-x^2-y^2}}=\frac{-x}{\sqrt{a^2-x^2-y^2}}$ ， $\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{a^2-x^2-y^2}}$

$$\therefore\ \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{\frac{a^2-x^2-y^2+x^2+y^2}{a^2-x^2-y^2}}$$

$$=\frac{a}{\sqrt{a^2-x^2-y^2}}$$

$\therefore\ A_1=\iint \frac{a}{\sqrt{\quad}}\,dxdy$

$$= \int_0^{2\pi} d\theta \int_0^{b} \frac{a}{\sqrt{a^2-\rho^2}} \rho d\rho \quad (令 b \to a)$$

$$= -\frac{a}{2} \cdot 2\pi \cdot \int_0^{b} (a^2-\rho^2)^{-\frac{1}{2}} d(a^2-\rho^2)$$

$$= -a\pi \cdot \left[2(a^2-\rho^2)^{\frac{1}{2}} \right]_0^{b}$$

$$= -a\pi \cdot (2\sqrt{a^2-b^2} - 2a)$$

$$= 2\pi a (a - \sqrt{a^2-b^2})$$

又 $\because \lim\limits_{b \to a} A_1 = \lim\limits_{b \to a} 2\pi a (a - \sqrt{a^2-b^2}) = 2\pi a^2$

$\therefore A = 2A_1 = 4\pi a^2$

答：半径为 a 的球的表面积为 $4\pi a^2$.

二、质心

$$\bar{x} = \frac{M_y}{M} = \frac{\iint\limits_D x \mu(x,y) d\sigma}{\iint\limits_D \mu(x,y) d\sigma}, \quad \bar{y} = \frac{M_x}{M} = \frac{\iint\limits_D y \mu(x,y) d\sigma}{\iint\limits_D \mu(x,y) d\sigma}$$

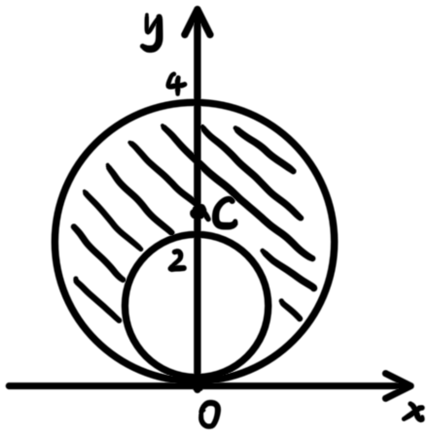
其中在点 (x,y) 处的面密度为 $\mu(x,y)$

若均匀（ $\mu(x,y) = \mu$ ）时，

$$\bar{x} = \frac{1}{A} \iint\limits_D x d\sigma, \quad \bar{y} = \frac{1}{A} \iint\limits_D y d\sigma$$

例3. 求位于两圆 $\rho=2\sin\theta$ 和 $\rho=4\sin\theta$ 之间的均匀薄片的质心.

解: $\bar{x}=0$



$$\bar{y}=\frac{1}{A}\iint_D y\,d\sigma$$

$$=\frac{1}{3\pi}\int_0^{\pi}d\theta\int_{2\sin\theta}^{4\sin\theta}\rho\sin\theta\,\rho\,d\rho$$

$$=\frac{1}{3\pi}\int_0^{\pi}\left[\frac{1}{3}\rho^3\sin\theta\right]_{2\sin\theta}^{4\sin\theta}d\theta$$

$$=\frac{1}{3\pi}\int_0^{\pi}\left(\frac{56}{3}\sin^4\theta\right]d\theta$$

$$=\frac{56}{9\pi}\times 2\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta$$

$$=\frac{112}{9\pi}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$=\frac{7}{3}$$

∴ 所求质心为 $C(0,\frac{7}{3})$

三、转动惯量

$$I_x=\iint_D y^2\mu(x,y)\,d\sigma$$

$$I_y=\iint_D x^2\mu(x,y)\,d\sigma$$

例5. 求半径为 a 的均匀半圆薄片(面密度为常量 μ)对于其直径边的转动惯量.

解: $I_x=\iint_D y^2\mu\,d\sigma$

$$= \mu \int_0^{\pi} d\theta \int_0^{a} \rho^2 \sin^2\theta \cdot \rho \, d\rho$$

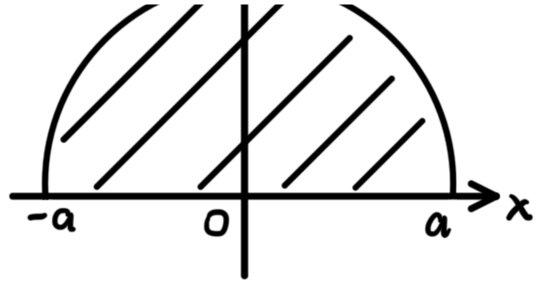
$$= \mu \int_0^{\pi} \left[\frac{1}{4}\rho^4 \sin^2\theta \right]_0^{a} d\theta$$

$$= \mu \int_0^{\pi} \left(\frac{1}{4} a^4 \sin^2\theta \right) d\theta$$

$$= \frac{1}{4}\mu a^4 \cdot 2\int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta$$

$$= \frac{1}{4}\mu a^4 \cdot \frac{\pi}{2}$$

$$= \frac{1}{4} M a^2 \quad \left(M = \frac{1}{2}\pi a^2 \mu \text{ 为半圆薄片的质量} \right)$$



$D = \{(x,y) \mid x^2+y^2 \le a^2, y \ge 0\}$

四、引力

$$dF = (dF_x, dF_y, dF_z)$$

$$= \left(G\frac{\rho(x,y,z)(x-x_0)}{r^3}dV, G\frac{\rho(x,y,z)(y-y_0)}{r^3}dV, G\frac{\rho(x,y,z)(z-z_0)}{r^3}dV \right)$$

$$\therefore F = (F_x, F_y, F_z)$$

$$= \left(\iiint_\Omega \frac{G\rho(x,y,z)(x-x_0)}{r^3}dV, \iiint_\Omega \frac{G\rho(x,y,z)(y-y_0)}{r^3}dV, \iiint_\Omega \frac{G\rho(x,y,z)(z-z_0)}{r^3}dV \right)$$

例7. 设半径为R的质量均匀的球占有空间闭区域 $\Omega = \{(x,y,z) \mid x^2+y^2+z^2 \le R^2\}$. 求它对位于 $M_0(0,0,a)$ $(a>R)$ 处的单位质量的质点的引力.

解: $F_x = F_y = 0$

$$F_z = \iiint_\Omega G\rho_0 \frac{z-a}{[x^2+y^2+(z-a)^2]^{\frac{3}{2}}} dV$$

$$= G\rho_0 \int_{-R}^{R} dz \iint_{x^2+y^2\le R^2-z^2} \frac{z-a}{[x^2+y^2+(z-a)^2]^{\frac{3}{2}}}\, dxdy$$

$$= G\rho_0 \int_{-R}^{R} (z-a)\, dz \int_0^{2\pi} d\theta \int_0^{\sqrt{R^2-z^2}} \frac{1}{[\rho^2+(z-a)^2]^{\frac{3}{2}}}\, \rho\, d\rho$$

$$= 2\pi G\rho_0 \int_{-R}^{R} (z-a) \left(\frac{1}{a-z} - \frac{1}{\sqrt{R^2-2az+a^2}} \right) dz$$

$$= 2\pi G\rho_0 \left[-2R + \frac{1}{a} \int_{-R}^{R} (z-a)\, d\sqrt{R^2-2az+a^2} \right]$$

$$= 2\pi G\rho_0 \left(-2R+2R-\frac{2R^3}{3a^2} \right)$$

$$= -G\cdot \frac{4\pi R^3}{3}\rho_0 \cdot \frac{1}{a^2}$$

$$= -G\cdot \frac{M}{a^2}$$ （其中 $M=\frac{4\pi R^3}{3}\rho_0$ 为球的质量）