微分方程（4）

常系数齐次线性微分方程

$$y''+P(x)y'+Q(x)y=0$$

$$y''+py'+qy=0$$

例. $y''-2y'+y=0$

$\Rightarrow y=e^x$

$$y''-4y'+3y=0$$

$y=e^{rx}$, $y'=re^{rx}$, $y''=r^2e^{rx}$

$\Rightarrow r^2e^{rx}-4re^{rx}+3e^{rx}=0$

$\Rightarrow e^{rx}(r^2-4r+3)=0$

$\Rightarrow r^2-4r+3=0$

$\Rightarrow (r-1)(r-3)=0$

$\Rightarrow r=1$ 或 $r=3$

$$y''+py'+qy=0$$

令 $y=e^{rx}$

则 $(r^2+pr+q)e^{rx}=0$

$\Rightarrow r^2+pr+q=0$ (特征方程)

① $r_1 \neq r_2$，实根

$$y=C_1e^{r_1x}+C_2e^{r_2x}$$

② $r_1=r_2$，实根

$$y=(C_1+C_2x)e^{r_1x}$$

③ $\alpha \pm \beta i$，共轭复根

$$y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$$

例1. $y''-2y'-3y=0$

特征方程：$r^2-2r-3=0$

$\Rightarrow (r-3)(r+1)=0$

$\Rightarrow r_1=-1,\ r_2=3$

$r_1 \neq r_2$

$\therefore y=C_1e^{-x}+C_2e^{3x}$

例2. $s''+2s'+s=0$

特征方程：$r^2+2r+1=0$

$$\Rightarrow (r+1) = 0$$

$$\Rightarrow r = -1$$

$r_1 = r_2$

$\therefore s = (C_1 + C_2 t) e^{-t}$

满足 $s\big|_{t=0} = 4,\ s'\big|_{t=0} = -2$

又 $\because s' = -C_1 \cdot e^{-t} + C_2 \cdot e^{-t}$

$$- C_2 t \cdot e^{-t}$$

$$= (C_2 - C_1 - C_2 t) \cdot e^{-t}$$

$$\therefore \begin{cases} C_1 = 4 \\ C_2 - C_1 = -2 \end{cases} \Rightarrow \begin{cases} C_1 = 4 \\ C_2 = 2 \end{cases}$$

$\therefore s = (4 + 2t) e^{-t}$

例13. $y'' - 2y' + 5y = 0$

特征方程：$r^2 - 2r + 5 = 0$

$\Rightarrow r = \dfrac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$

$\therefore \alpha = 1,\ \beta = 2$

$\therefore y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$$= e^{x}(C_1 \cos 2x + C_2 \sin 2x)$$

另：已知解为 $y=C_1e^x+C_2e^{5x}$，求微分方程.

$r_1=1$，$r_2=5$

$\therefore$ 特征方程：$(r-1)(r-5)=0$

$\Rightarrow r^2-6r+5=0$

$\therefore$ 微分方程：$y''-6y'+5y=0$

$y=(C_1+C_2x)e^{3x}$

$r_1=r_2=3$

$\therefore \quad (r-3)^2=0$

$\Rightarrow r^2-6r+9=0$

$\Rightarrow y''-6y'+9y=0$

$y=e^x(C_1\cos 2x+C_2\sin 2x)$

$\alpha=1$，$\beta=2$

$\therefore \quad r_1=1+2i$，$r_2=1-2i$

(I) 令 $r^2+br+c=0$

韦达定理：$\begin{cases} b=-2 \\ c=5 \end{cases}$

$\Rightarrow \quad r^2-2r+5=0$

$\Rightarrow \quad y''-2y'+5y=0$

(II) 回写法：

$[r-(1+2i)][r-(1-2i)]=0$

$\Rightarrow r^2-r(1+2i)-r(1-2i)+(1+2i)(1-2i)=0$

$\Rightarrow r^2-2r+5=0$

$\Rightarrow y''-2y'+5y=0$

(III) $r=1\pm 2i$

$\Rightarrow r-1=\pm 2i$

$\Rightarrow (r-1)^2=(\pm 2i)^2$

$\Rightarrow r^2-2r+1=-4$

$\Rightarrow r^2-2r+5=0$

妙啊！

通解公式的证明过程：

① $r_1 \neq r_2$，实根

① ……，实根.

∵ $y_1=e^{r_1x}$，$y_2=e^{r_2x}$ 是两个解.

且 $\frac{y_2}{y_1}=\frac{e^{r_2x}}{e^{r_1x}}=e^{(r_2-r_1)x}\neq k$

∴通解为 $y=C_1e^{r_1x}+C_2e^{r_2x}$

② $r_1=r_2$，实根

一解为 $y_1=e^{r_1x}$

设 $y_2=e^{r_1x}u(x)$

$$y_2'=r_1e^{r_1x}\cdot u(x)+e^{r_2x}u'(x)$$

$$=e^{r_1x}(u'+r_1u)$$

$$y_2''=e^{r_1x}(u''+2r_1u'+r_1^2u)$$

∴代入微分方程得：

$$e^{r_1x}[(u''+2r_1u'+r_1^2u)+p(u'+r_1u)+qu]=0$$

$$\Rightarrow u''+\overset{=0}{\underbrace{(2r_1+p)}}u'+\overset{=0}{\underbrace{(r_1^2+pr_1+q)}}u=0$$

又∵ $(r-r_1)^2=0$

$$\left.\begin{array}{l}\Rightarrow r^2-2r_1r+r_1^2=0\\ \text{又又}\because\ r^2+pr+q=0\end{array}\right\}\Rightarrow p=-2r_1$$

$\Rightarrow 2r_1+p=0$

~~又~~~~又~~又 $\because r_1^2+pr_1+q=0$

$\therefore u''=0$

$\therefore u(x)=x$

$\therefore y_2=xe^{r_1x}$

$\therefore y=C_1y_1+C_2y_2$

$=C_1e^{r_1x}+C_2xe^{r_1x}$

$=(C_1+C_2x)e^{r_1x}$

③ $r_1=\alpha+\beta i$, $r_2=\alpha-\beta i$

$y_1=e^{(\alpha+\beta i)x}=e^{\alpha x}e^{\beta xi}$

$y_2=e^{(\alpha-\beta i)x}=e^{\alpha x}e^{-\beta xi}$

由欧拉公式得：

$y_1=e^{\alpha x}(\cos\beta x+i\sin\beta x)$

$y_2=e^{\alpha x}(\cos\beta x-i\sin\beta x)$

$\therefore \bar{y_1}=\frac{1}{2}(y_1+y_2)=e^{\alpha x}\cos\beta x$

$\bar{y_2}=\frac{1}{2}(y_1-y_2)=e^{\alpha x}\sin\beta x$

$\because \frac{\bar{y_1}}{\bar{y_2}}=\cot\beta x\neq k$

$\therefore y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$

常系数非齐次线性微分方程

$$y''+py'+qy=f(x)=e^{\lambda x}P_m(x)$$

$$y^*=R(x)e^{\lambda x}$$

$$y^{*\prime}=R'(x)e^{\lambda x}+\lambda R(x)e^{\lambda x}$$

$$=[\lambda R(x)+R'(x)]e^{\lambda x}$$

$$y^{*\prime\prime}=e^{\lambda x}[\lambda^2R(x)+2\lambda R'(x)+R''(x)]$$

$$\Rightarrow R''(x)+(2\lambda+p)R'(x)+(\lambda^2+p\lambda+q)R(x)=P_m(x)$$

① 当 λ 不是 $r^2+pr+q=0$ 的根时，

$$\Rightarrow \lambda^2+p\lambda+q\neq 0$$

$\therefore R(x)$ 为 m 次的多项式

例1. $y''-2y'-3y=3x+1$

$$=e^{0x}(3x+1)$$

$\lambda=0$，$y''-2y'-3y=0$

$$\Rightarrow r^2-2r-3=0$$

$$\Rightarrow (r-3)(r+1)=0$$

$\Rightarrow r_1 = 3,\ r_2 = -1$

$\lambda = 0$ 不是 $r^2 - 2r - 3 = 0$ 的解

$\therefore \quad R(x) = b_0 x + b_1$

$\Rightarrow y^* = e^{0x} \cdot R(x) = b_0 x + b_1$

$\Rightarrow y^{*\prime} = b_0$

$\Rightarrow y^{*\prime\prime} = 0$

$\therefore 0 - 2b_0 - 3(b_0 x + b_1) = 3x + 1$

$\Rightarrow -3b_0 x - 2b_0 - 3b_1 = 3x + 1$

$$\Rightarrow \begin{cases} -3b_0 = 3 \\ -2b_0 - 3b_1 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} b_0 = -1 \\ b_1 = \frac{1}{3} \end{cases}$$

$\therefore y^* = -x + \frac{1}{3}$

② 当 λ 是 $r^2 + pr + q$ 的单根时，

$\lambda^2 + p\lambda + q = 0$

$\because$ λ 是重根时 $2\lambda + p = 0$

$\therefore$ λ 是单根时 $2\lambda + p \neq 0$

$R'(x)$ 是 m 次

$R(x) = x R_m(x)$

例2. $y'' - 5y' + 6y = xe^{2x}$

$\lambda = 2$

特征方程为 $r^2 - 5r + 6 = 0$

$\Rightarrow (r-2)(r-3) = 0$

$\Rightarrow r_1 = 2, r_2 = 3$

齐次方程：$y'' - 5y' + 6y = 0$

$\Rightarrow Y = C_1 e^{2x} + C_2 e^{3x}$

$\because R(x) = x(b_0 x + b_1)$

$\therefore y^* = e^{2x} \cdot x(b_0 x + b_1)$

$= e^{2x}(b_0 x^2 + b_1 x)$

$\Rightarrow y^{*'} = 2e^{2x}(b_0 x^2 + b_1 x)$

$+ e^{2x}(2b_0 x + b_1)$

$= e^{2x}[2b_0 x^2 + 2(b_0 + b_1)x + b_1]$

$\Rightarrow y^{*''} = 2e^{2x} \cdots$

$\cdots\cdots$

$-2b_0 x + 2b_0 - b_1 = x$

$$\Rightarrow \begin{cases} -2b_0 = 1 \\ 2b_0 - b_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} b_0 = -\frac{1}{2} \\ b_1 = -1 \end{cases}$$

$\therefore y^* = x(-\frac{1}{2}x - 1)e^{2x}$

$\therefore y = C_1 e^{2x} + C_2 e^{3x} - \frac{1}{2}(x^2 + 2x)e^{2x}$

③ λ是$r^2 + pr + q = 0$的重根

$\lambda^2 + p\lambda + q = 0$ ， $2\lambda + p = 0$

$\therefore R''(x) = P_m(x)$