曲线积分与曲面积分(6)

对坐标的曲面积分

$$\iint_{\Sigma} R(x,y,z)\,dx\,dy = \lim_{\lambda\to 0}\sum_{i=1}^{n} R(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{xy}$$

$$(\Delta S)_{xy} = \begin{cases} (\Delta\sigma)_{xy}, & \cos\gamma > 0 \\ -(\Delta\sigma)_{xy}, & \cos\gamma < 0 \\ 0, & \cos\gamma \equiv 0 \end{cases}$$

例1. 计算曲面积分 $\iint_{\Sigma} x^2dydz + y^2dzdx + z^2dxdy$,
其中 Σ 是长方体 Ω 的整个表面的外侧,
$\Omega = \{(x,y,z) \mid 0\le x\le a,\ 0\le y\le b,\ 0\le z\le c\}$.

解: $\iint_{\Sigma} x^2dydz$

$= \iint_{\Sigma_1} x^2dydz + \iint_{\Sigma_2} x^2dydz$

$= \iint_{\Sigma_2} a^2dydz$

$= \iint_{D_{yz}} a^2dydz$

$= a^2bc$

同理: $\iint_{\Sigma} y^2dzdx = b^2ac$, $\iint_{\Sigma} z^2dxdy = c^2ab$

$\therefore$ 原式 $= (a+b+c)abc$

例2. 计算曲面积分 $\iint_{\Sigma} xyz\,dxdy$，其中 Σ 是球面 $x^2+y^2+z^2=1$ 外侧在 $x\geq 0, y\geq 0$ 的部分.

解：$\iint_{\Sigma} xyz\,dxdy$

$$= \iint_{\Sigma_1} xyz\,dxdy + \iint_{\Sigma_2} xyz\,dxdy$$

$$= \iint_{D_{xy}} xy\sqrt{1-x^2-y^2}\,dxdy - \iint_{D_{xy}} xy(-\sqrt{1-x^2-y^2})\,dxdy$$

$$= 2\iint_{D_{xy}} xy\sqrt{1-x^2-y^2}\,dxdy$$

$$= 2\iint_{D_{xy}} \rho^2 \sin\theta\cos\theta\sqrt{1-\rho^2}\,d\rho d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta \int_0^1 \rho^3\sqrt{1-\rho^2}\,d\rho$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\,d(2\theta) \int_0^1 \frac{1}{2}\rho^2\sqrt{1-\rho^2}\,d\rho^2$$

$$= \frac{1}{4}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}} \int_1^0 (1-t^2)t\,d(1-t^2)$$

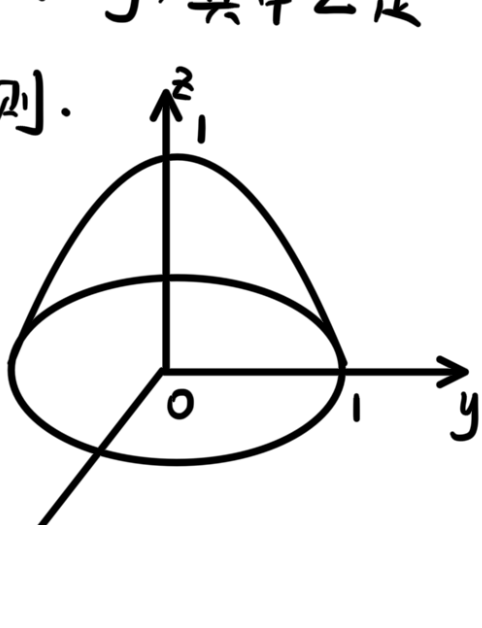
$$= \frac{1}{2}\times\frac{4}{15} = \frac{2}{15}$$

例3. 计算曲面积分

$I = \iint_{\Sigma} xz\,dydz + 2zy\,dzdx + 3xy\,dxdy$，其中 Σ 是曲面 $z=1-x^2-y^2\ (0\leq z\leq 1)$ 的下侧.

解：补充 $\Sigma_1: z=0\ (x^2+y^2\leq 1)$

取上侧



$$\oiint_{\Sigma+\Sigma_1} P\,dydz + Q\,dzdx + R\,dxdy$$

$$= \iiint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

$$= \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz \qquad x$$

其中 $P = xz$, $Q = 2zy$, $R = 3xy$

$$\frac{\partial P}{\partial x} = z, \quad \frac{\partial Q}{\partial y} = 2z, \quad \frac{\partial R}{\partial z} = 0$$

$$\therefore \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

$$= \iiint_\Omega 3z \, dxdydz$$

$$= \int_0^1 3z \, dz \iint_{D_z} dxdy$$

$$= \int_0^1 3z^2 \pi (1-z) \, dz$$

$$= \frac{1}{4}\pi$$

$$\therefore \text{原式} = \iiint_\Omega 3z \, dxdydz - \iint_\Sigma 3xy \, dxdy$$

$$= \frac{1}{4}\pi - 0 = \frac{1}{4}\pi$$