重积分(2)

二重积分的计算法(直角坐标)

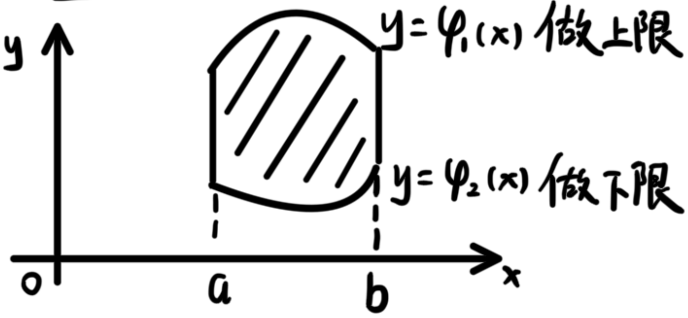
$$A(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$

$$V = \int_a^b A(x)\,dx = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right] dx$$

$$\Rightarrow \iint_D f(x,y)\,d\sigma = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right] dx$$

也记作 $\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$ (二次积分)

X型积分:

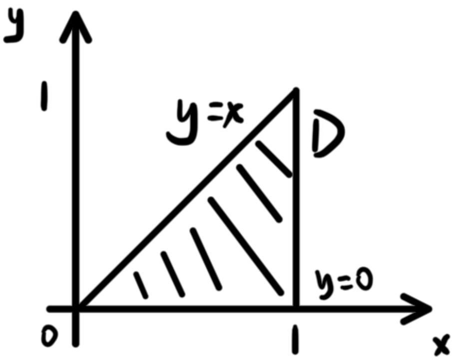


$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$

比如 $\iint_D xy\,dxdy = \int_0^1 dx \int_0^x xy\,dy$

$$= \int_0^1 \left[\frac{1}{2}xy^2\right]_0^x dx$$

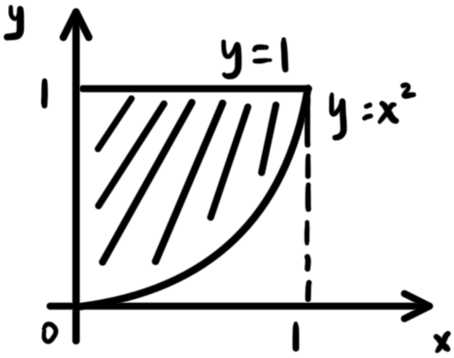
$$= \int_0^1 \frac{1}{2}x^3\,dx = \left[\frac{1}{8}x^4\right]_0^1 = \frac{1}{8}$$



$0 \le x \le 1$, $0 \le y \le x$

再如 $\iint (x+y)\,dxdy = \int_0^1 dx \int^1 (x+y)\,dy$

再如 D ... $\int_0 dx \int_{x^2} (x+y)\,dy$



$$= \int_0^1 \left[xy + \frac{1}{2}y^2 \right]_{x^2}^{1} dx$$

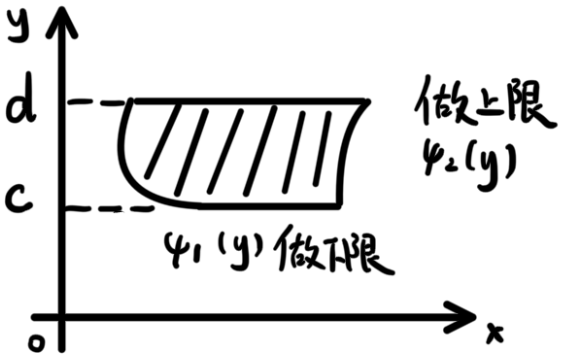
$$= \int_0^1 \left(x + \frac{1}{2} - x^3 - \frac{1}{2}x^4 \right) dx$$

$$= \left[\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4}x^4 - \frac{1}{10}x^5 \right]_0^1$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} - \frac{1}{10} = \frac{13}{20}$$

$0 \le x \le 1,\ x^2 \le y \le 1$

Y型积分

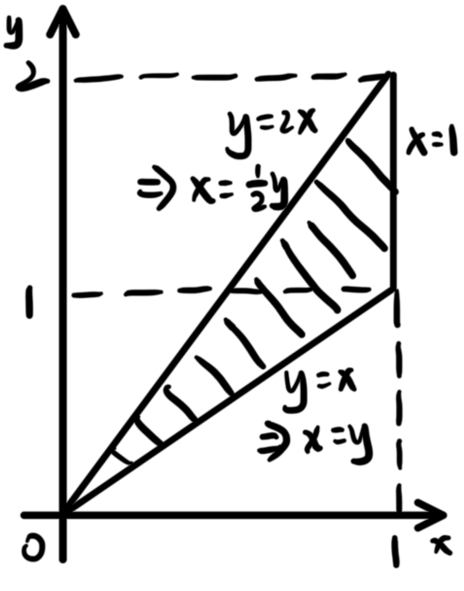


$$\int_c^d dy \int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,dx$$

题目是 $y=f(x)$，改写成

$x=g(y)$

$c \le y \le d,\ \varphi_1(y) \le x \le \varphi_2(y)$



X型：$0 \le x \le 1,\ x \le y \le 2x$

Y型：
$$\begin{cases} 0 \le y \le 1, & \frac{1}{2}y \le x \le y \\ 1 \le y \le 2, & \frac{1}{2}y \le x \le 1 \end{cases}$$

X型：$\iint_D (x+y)\,dx\,dy = \int_0^1 dx \int_x^{2x} (x+y)\,dy$

$$= \int_0^1 \left[xy+\frac{1}{2}y^2\right]_x^{2x} dx$$

$$= \int_0^1 \left(\frac{5}{2}x^2\right)dx$$

$$= \left[\frac{5}{6}x^3\right]_0^1$$

$$= \frac{5}{6}$$

Y型： $\iint_D (x+y)\,dxdy$

$$= \int_0^1 dy \int_{\frac{1}{2}y}^{y} (x+y)\,dx + \int_1^2 dy \int_{\frac{1}{2}y}^{1} (x+y)\,dx$$

$$= \int_0^1 \left[\frac{1}{2}x^2+xy\right]_{\frac{1}{2}y}^{y} dy + \int_1^2 \left[\frac{1}{2}x^2+xy\right]_{\frac{1}{2}y}^{1} dy$$

$$= \int_0^1 \left(\frac{7}{8}y^2\right)dy + \int_1^2 \left(\frac{1}{2}+y-\frac{5}{8}y^2\right)dy$$

$$= \left[\frac{7}{24}y^3\right]_0^1 + \left[\frac{1}{2}y+\frac{1}{2}y^2-\frac{5}{24}y^3\right]_1^2$$

$$= \frac{7}{24}+1+2-\frac{5}{3}-\frac{1}{2}-\frac{1}{2}+\frac{5}{24}$$

$$= \frac{5}{6}$$

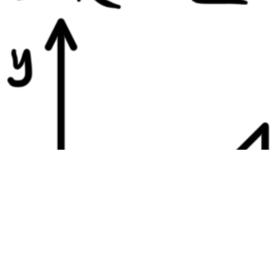
如何决定按X型、Y型积分？

① 尽量写一个积分

② 看函数边缘

③ 某一种积分积不出来时换一种

例1. 计算 $\iint_D xy\,d\sigma$，其中D是由直线 $y=1$，$x=2$ 及 $y=x$ 所围成的闭区域.



解：$\iint\limits_D xy\,d\sigma$

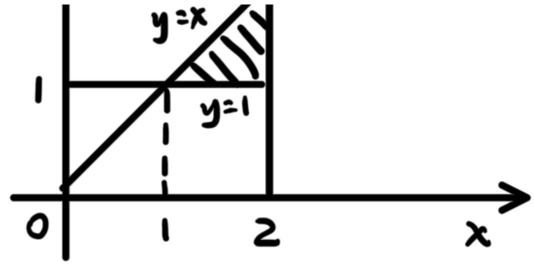
$= \int_1^2 dx \int_1^x xy\,dy$

$= \int_1^2 \left[\frac{1}{2}xy^2\right]_1^x dx$

$= \int_1^2 \left[\frac{1}{2}x^3 - \frac{1}{2}x\right] dx$

$= \left[\frac{1}{8}x^4 - \frac{1}{4}x^2\right]_1^2$

$= \frac{9}{8}$



$\boxed{1 \leq x \leq 2,\ 1 \leq y \leq x}$

例2. 计算 $\iint\limits_D y\sqrt{1+x^2-y^2}\,d\sigma$，其中 D 是由直线 $y=x$，$x=-1$ 和 $y=1$ 所围成的闭区域.

解：$\iint\limits_D y\sqrt{1+x^2-y^2}\,d\sigma$

$= \int_{-1}^1 dx \int_x^1 y\sqrt{1+x^2-y^2}\,dy$

$= -\frac{1}{2}\int_{-1}^1 dx \int_x^1 \sqrt{1+x^2-y^2}\,d(1+x^2-y^2)$

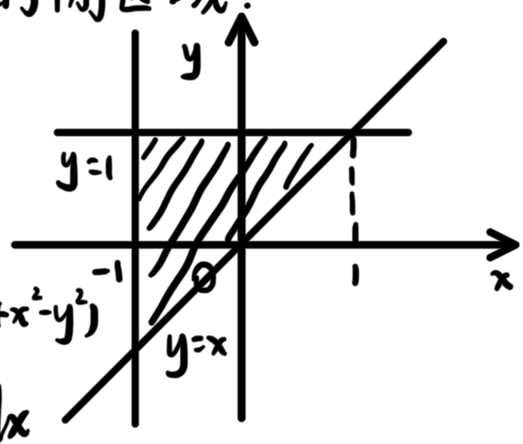
$= -\frac{1}{2}\int_{-1}^1 \left[\frac{2}{3}(1+x^2-y^2)^{\frac{3}{2}}\right]_x^1 dx$

$= -\frac{1}{3}\int_{-1}^1 \left[|x|^3 - 1\right] dx$

$= -\frac{2}{3}\int_0^1 \left[x^3 - 1\right] dx$

$= -\frac{2}{3}\left[\frac{x^4}{4} - x\right]_0^1$

$= \frac{1}{2}$

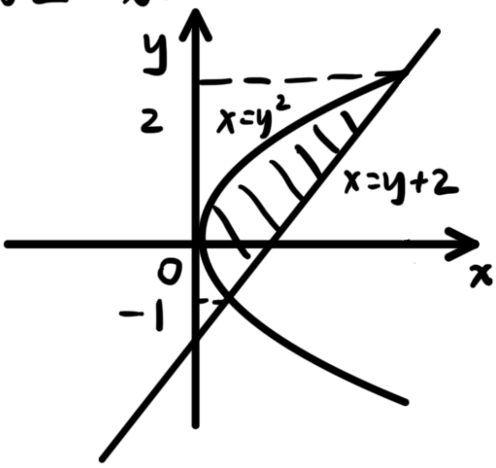


$\boxed{-1 \leq x \leq 1,\ x \leq y \leq 1}$

例3. 计算 $\iint\limits_D xy\,d\sigma$，其中 D 是由抛物线 $y^2=x$

及直线 $y=x-2$ 所围成的闭区域.

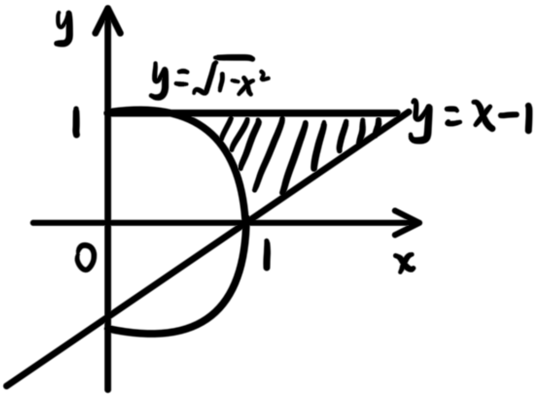
解：$\iint\limits_D xy\,d\sigma$



$\boxed{-1\le y\le 2,\ y^2\le x\le y+2}$

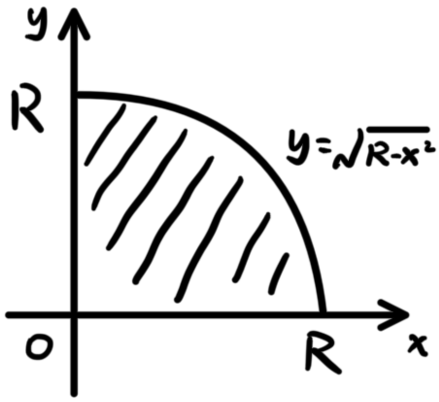
$$= \int_{-1}^{2} dy \int_{y^2}^{y+2} xy\,dx$$
$$= \int_{-1}^{2} \left[\frac{1}{2}x^2 y \right]_{y^2}^{y+2} dy$$
$$= \int_{-1}^{2} \left[\frac{1}{2}y^3 + 2y^2 + 2y - \frac{1}{2}y^5 \right] dy$$
$$= \left[\frac{1}{8}y^4 + \frac{2}{3}y^3 + y^2 - \frac{1}{12}y^6 \right]_{-1}^{2}$$
$$= 6 - \frac{3}{8} = \frac{45}{8}$$

交换积分次序：



$$\int_0^1 dy \int_{\sqrt{1-y^2}}^{y+1} f(x,y)\,dx$$
$$= \int_0^1 dx \int_{\sqrt{1-x^2}}^{1} f(x,y)\,dy$$
$$+ \int_1^2 dx \int_{x-1}^{1} f(x,y)\,dy$$

例4.



解：$x^2+y^2=R^2,\ x^2+z^2=R^2$

$$\Rightarrow z=\sqrt{R^2-x^2}$$

$\boxed{0\le x\le R,\ 0\le y\le \sqrt{R-x^2}}$

$$V_1 = \iint\limits_D \sqrt{R^2-x^2}\,d\sigma$$
$$= \int_0^R dx \int_0^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2}\,dy$$
$$= \int_0^R \left[R^2-x^2 \right] dx$$

$$= \int_0^R [R^2 - x^2] dx$$

$$= \left[R^2 x - \frac{1}{3} x^3 \right]_0^R$$

$$= \frac{2}{3} R^3$$

$$\therefore \; V = 8V_1 = \frac{16}{3} R^3$$