行列式(6)

行列式的计算

例1. 计算n阶行列式

$$D=\begin{vmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix}$$

D中每行(列)元素之和均相同，可利用"加倍"性质将其"集中"起来.

$$D=[a+(n-1)b]\cdot\begin{vmatrix} 1 & b & b & \cdots & b \\ 1 & a & & \cdots & b \\ \vdots & \vdots & a & \ddots & \vdots \\ 1 & b & \cdots & & a \end{vmatrix}$$

$$=[a+(n-1)b]\cdot\begin{vmatrix} 1 & & & & \\ & a-b & & & \\ & & a-b & & \\ & & & \ddots & \\ & & & & a-b \end{vmatrix}$$

$$=[a+(n-1)b](a-b)^{n-1}$$

例2. (爪形行列式)

计算:
$$\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

$$D = \begin{vmatrix} 1 & -2 & & & \\ \vdots & & -3 & & \\ & & & \ddots & \\ 1 & & & & -n \end{vmatrix}$$

$$= \begin{vmatrix} 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} & 1 & 1 & 1 & \cdots & 1 \\ & -2 & & & & \\ & & -3 & & & \\ & & & -4 & & \\ & & & & \ddots & \\ & & & & & -n \end{vmatrix}$$

$$= \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)(-1)^{n-1}\cdot n!$$

例3. 计算.

$$D = \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ 1 & \vdots & \ddots & \vdots \\ \vdots & & & \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} \quad \times(-1)$$

$$= \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ -a_1 & a_2 & & \\ -a_1 & & a_3 & \\ \vdots & & & \ddots \\ -a_1 & & & a_n \end{vmatrix}$$

$$\times\frac{a_1}{a_2} \quad \times\frac{a_1}{a_3} \quad \cdots$$

$$= \begin{vmatrix} 1+\frac{a_1}{a_2}+\cdots+\frac{a_1}{a_n} & 1 & 1 & \cdots & 1 \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{vmatrix}$$

$$= \left(1+\frac{a_1}{a_2}+\cdots+\frac{a_1}{a_n}\right) a_2 a_3 \cdots a_n$$

$= (1+\frac{1}{a_2}+\cdots+\frac{1}{a_n})\cdot a_2a_3\cdots a_n$

例4. 计算.

$$D_n=\begin{vmatrix} x & -1 & & & & \\ & x & -1 & & & \\ & & x & \ddots & & \\ & & & \ddots & -1 & \\ & & & & x & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & x+a_1 \end{vmatrix}$$

$$= (x^n+a_1x^{n-1}+\cdots+a_n)\times(-1)^{n+1}\times(-1)^{n-1}$$

$$= x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$$

例5. 范德蒙德行列式☆.

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1\le j<i\le n}(x_i-x_j)$$

(注: $\sum_{i=1}^{n} a_i = a_1+a_2+\cdots+a_n$

$\prod_{i=1}^{n} b_i = b_1b_2\cdots b_n$)

例6. 计算行列式

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \end{vmatrix}$$

$$D = \begin{vmatrix} a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

$$= \underline{(b-a)(c-a)(d-a)}\,\underline{(c-b)(d-b)}\,\underline{(d-c)}$$

$$D = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 1 & 9 \\ 4 & 1 & 16 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{vmatrix}$$

$$= -(3-2)(4-2)(4-3)$$

$$= -2$$

例7. 计算

$$D = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= (a+b+c)(b-a)(c-a)(c-b)$$

例19. 计算

$$D = \begin{vmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix}$$

法一：拉普拉斯定理：

$$D = \begin{vmatrix} a_2 & b_2 \\ b_3 & a_3 \end{vmatrix} \times (-1)^{2+3+2+3} \times \begin{vmatrix} a_1 & b_1 \\ b_4 & a_4 \end{vmatrix}$$

$$= (a_2a_3 - b_2b_3)(a_1a_4 - b_1b_4)$$

法二：

$$D = -\begin{vmatrix} a_1 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & a_2 \\ 0 & 0 & a_3 & b_3 \\ b_4 & a_4 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ b_4 & a_4 & 0 & 0 \\ 0 & 0 & a_3 & b_3 \\ 0 & 0 & b_2 & a_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 \\ b_4 & a_4 \end{vmatrix} \cdot \begin{vmatrix} a_3 & b_3 \\ b_2 & a_2 \end{vmatrix}$$

$$= (a_1a_4 - b_1b_4)(a_2a_3 - b_2b_3)$$

法三：

$$D = a_1 \times (-1)^{1+1} \begin{vmatrix} a_2 & b_2 & 0 \\ b_3 & a_3 & 0 \\ 0 & 0 & a_4 \end{vmatrix}$$

$$+ b_1 \times (-1)^{1+4} \begin{vmatrix} 0 & a_2 & b_2 \\ 0 & b_3 & a_3 \\ b_4 & 0 & 0 \end{vmatrix}$$

$$= a_1 \cdot \begin{vmatrix} a_2 & b_2 \\ b_3 & a_3 \end{vmatrix} \cdot a_4 - b_1 \cdot (-1)^{2\times 1} \cdot \begin{vmatrix} a_2 & b_2 \\ b_3 & a_3 \end{vmatrix} \cdot b_4$$

$$= (a_1a_4 - b_1b_4)(a_2a_3 - b_2b_3)$$