Stochastic approximation in mathematical finance

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Introduction: Framework

Let $\{S_t, t \in [0, T]\}$ be a geometric Brownian motion modelling the stock price.

The put option is defined as the function

$$f(X_\sigma) = e^{-rT}(K - X_\sigma)_+ \quad \text{with} \quad (x)_+ = max(x, 0).$$

Where X_σ is defined as,

- European: $X_\sigma := S_T$
- Asian: $X_\sigma := \overline{S}$, where $\overline{S} = \frac{1}{m}\sum_{i=1}^{m} S_{t_i}$ and $t_i = \frac{iT}{m}$.

The option price is defined as,

$$I(\sigma) = E[(f(X_\sigma)].$$

For a desired price I_{market}, we seek the root σ^*,

$$I(\sigma^*) - I_{market} = 0.$$

Introduction: Robin-Monro Algorithm

In the frame considered, I is an increasing function of σ. To find the root σ^*, we implemented the Robin-Monro algorithm:

$$\sigma_{n+1} = \sigma_n - \alpha_n J(\sigma_n) \quad \text{where} \quad \alpha_n := \frac{\alpha_0}{n^\rho}.$$

With an MC estimator of J,

$$\hat{J}_N(\sigma_n) := \hat{I}_N(\sigma_n) - I_{market} = \frac{1}{N}\sum_{i=1}^{N}(Z^{(i)} - I_{market}) = \frac{1}{N}\sum_{i=1}^{N}\tilde{Z}^{(i)}$$

And a estimator $\hat{I}(\sigma)$ as it it stochastic,

$$\hat{I}_N(\sigma) = \frac{1}{N}\sum_{i=1}^{N} Z^{(i)}.$$

RM Algorithm for pricing a European Put Option

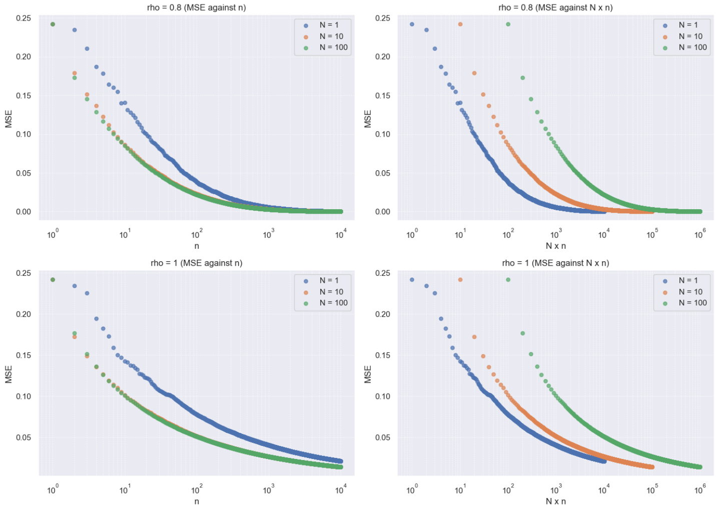
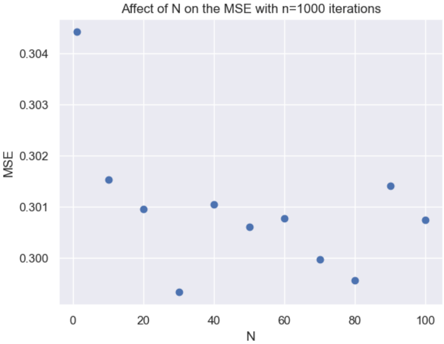


Figure: Plots of MSE $\mathbb{E}[(\sigma_n - \sigma_*)^2]$

Convergence rate

$$\log(\text{MSE}) = -\alpha \log(n) + \text{constant}$$

ρ	N	$\alpha =$ **Convergence Rate**
0.8	1	0.7487
0.8	10	0.7702
0.8	100	0.7844
1	1	0.2878
1	10	0.2899
1	100	0.2902



Asian Put Option

$$I(\sigma) = E[f(X_\sigma)] \quad \text{with } X_\sigma := \frac{1}{m}\sum_{i=1}^{m} S_{t_i}, \quad t_i = \frac{iT}{m}.$$

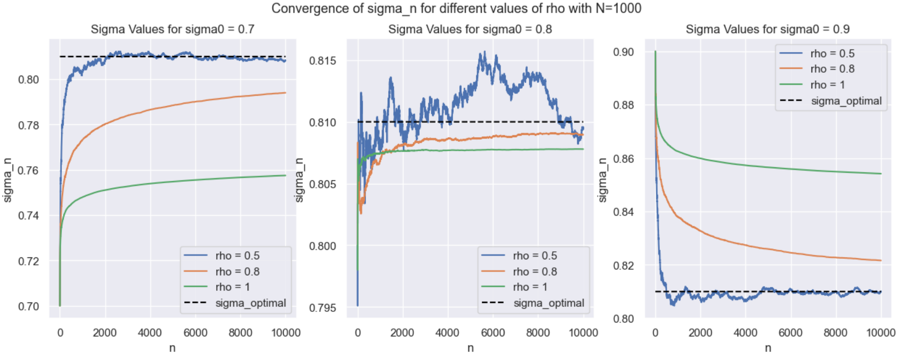
- No closed formula
- X_σ can only be expressed as the average sum of lognormal.
- Consider random vector $(S_{T/m}, S_{2T/m}, \ldots, S_T)$ (Markov chain) its joint pdf (product lognnormal pdf),

$$p(x_1, \ldots, x_m) = \prod_{i=1}^{m} \frac{1}{x_i\sigma\sqrt{T/m}}\varphi\left(\frac{\log[x_i/x_{i-1}] - (r - \sigma^2/2)(T/m)}{\sigma\sqrt{T/m}}\right)$$

- X_σ is heavy tailed and skewed (Super slow CLT & poor confidence intervals).

Algorithm: Speed or Precision?

$$\sigma_{n+1} = \sigma_n - \alpha_n \hat{J}_N(\sigma_n) \quad \text{where} \quad \alpha_n := \frac{\alpha_0}{n^\rho}, \; \alpha_0 := \frac{2}{K + S_0}.$$



Speed or Precision? "Burn-in" to reconcile.

Stopping Criterion

Stopping criteria:

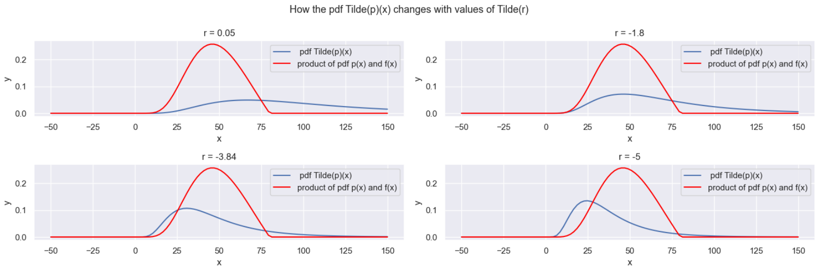
- Moving average of the last 1000 iterations of $\hat{J}(\sigma_n)$ is less than a tolerance t_1.
- Current value of $|\hat{J}(\sigma_n)| < t_2$.

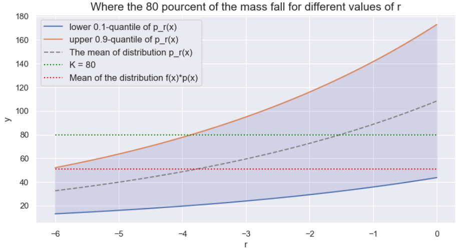
Statistic	Mean	Standard deviation
σ_n	0.81021	0.00107
iterations	15945.45	11704.99317

Figure: MC simulation of 20 samples, $\rho = 0.5$ constant, $t_1, t_2 = 10^{-2}$

Importance sampling

Figure: The optimal $\tilde{r}$ should be a function of $\sigma_n(I_{\text{market}}), S_0, K$





(a) Dynamic in 1 dimensional case, $\sigma = 1.2$, $K = 80$, $S_0 = 100$

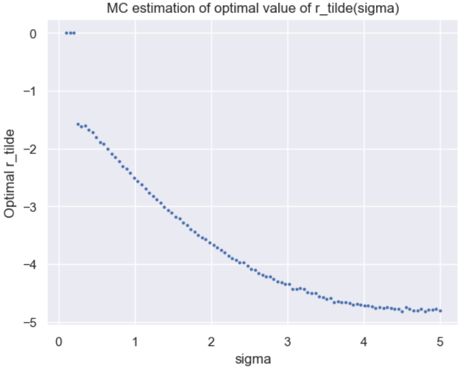
Sophisticated Importance Sampling

For a given σ, the optimal value of $\tilde{r}$ is the one that minimizes the variance of the IS estimator:

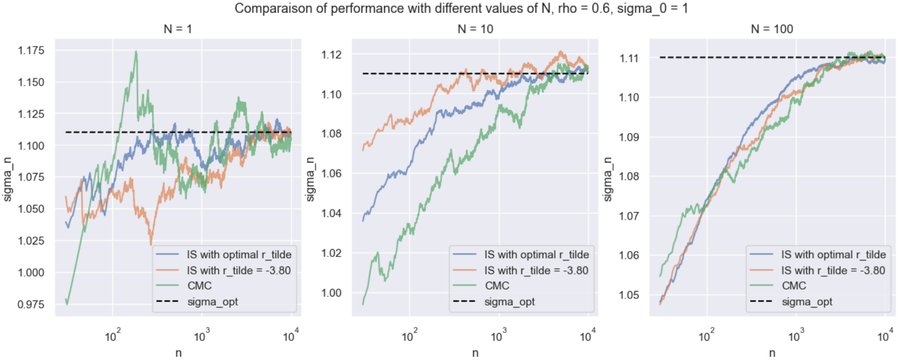
$$\tilde{r}(\sigma) = \arg\min_{\eta} \mathbb{E}_{p(\cdot;r,\sigma)} \left[f^2(S;r,\sigma) \frac{p(S;r,\sigma)}{p(S;\eta,\sigma)} \right]$$

$$= \arg\min_{\eta} \mathbb{E}_{p(\cdot;r,\sigma)} \left[f^2(S;r,\sigma) \left(\frac{S_T}{S_0} \right)^{\frac{r-\eta}{\sigma^2}} \exp\left(\frac{(r+\eta-\sigma^2)(\eta-r)T}{2\sigma^2} \right) \right]$$

Using a MC estimator

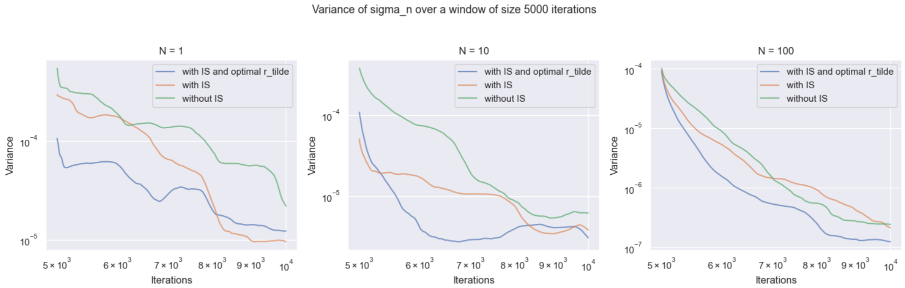


Comparison of different strategies



Comparison of different strategies

Algorithm	Average num. of iterations n	Mean value of σ_n	Standard deviation in σ_n
Standard RM	31632	1.11100	0.00352
RM with IS	12983	1.10952	0.00192
RM with IS and optimal $\tilde{r}$	8080	1.10958	0.00295



THANK YOU!