surreal number

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The surreal numbers are a generalization of the reals. Each surreal number consists of two parts (called the left and right), each of which is a set of surreal numbers. For any surreal number N, these parts can be called N_L and N_R. (This could be viewed as an ordered pair of sets, however the surreal numbers were intended to be a basis for mathematics, not something to be embedded in set theory.) A surreal number is written $N = \langle N_L \mid N_R \rangle$.

Not every number of this form is a surreal number. The surreal numbers satisfy two additional properties. First, if $x \in N_R$ and $y \in N_L$ then $x \not\leq y$. Secondly, they must be well founded. These properties are both satisfied by the following construction of the surreal numbers and the $\leq$ relation by mutual induction:

$\langle | \rangle$, which has both left and right parts empty, is 0.

Given two (possibly empty) sets of surreal numbers R and L such that for any $x \in R$ and $y \in L$, $x \not\leq y$, $\langle L \mid R \rangle$.

Define $N \leq M$ if there is no $x \in N_L$ such that $M \leq x$ and no $y \in M_R$ such that $y \leq N$.

This process can be continued transfinitely, to define infinite and infinitesimal numbers. For instance if $\mathbb{Z}$ is the set of integers then $\omega = \langle \mathbb{Z} \mid \rangle$. Note that this does not make equality the same as identity: $\langle 1 \mid 1 \rangle = \langle | \rangle$, for instance.

It can be shown that N is "sandwiched" between the elements of N_L and N_R: it is larger than any element of N_L and smaller than any element of N_R.

Addition of surreal numbers is defined by

$$N+M = \langle \{N+x \mid x \in M_L\} \cup \{M+x \mid y \in N_L\} \mid \{N+x \mid x \in M_R\} \cup \{M+x \mid y \in N_R\} \rangle$$

It follows that $-N = \langle -N_R \mid -N_L \rangle$.

The definition of multiplication can be written more easily by defining $M \cdot N_L = \{M \cdot x \mid x \in N_L\}$ and similarly for N_R.

Then

$$\begin{aligned} N \cdot M = &\langle M \cdot N_L + N \cdot M_L - N_L \cdot M_L, M \cdot N_R + N \cdot M_R - N_R \cdot M_R \mid \\ &M \cdot N_L + N \cdot M_R - N_L \cdot M_R, M \cdot N_R + N \cdot M_L - N_R \cdot M_L \rangle \end{aligned}$$

The surreal numbers satisfy the axioms for a field under addition and multiplication (whether they really are a field is complicated by the fact that they are too large to be a set).

The integers of surreal mathematics are called the *omnific integers*. In general positive integers n can always be written $\langle n-1 \mid \rangle$ and so $-n = \langle \mid 1-n \rangle = \langle \mid (-n)+1 \rangle$. So for instance $1 = \langle 0 \mid \rangle$.

In general, $\langle a \mid b \rangle$ is the simplest number between a and b. This can be easily used to define the dyadic fractions: for any integer a, $a+\frac{1}{2} = \langle a \mid a+1 \rangle$. Then $\frac{1}{2} = \langle 0 \mid 1 \rangle$, $\frac{1}{4} = \langle 0 \mid \frac{1}{2} \rangle$, and so on. This can then be used to locate non-dyadic fractions by pinning them between a left part which gets infinitely close from below and a right part which gets infinitely close from above.

Ordinal arithmetic can be defined starting with ω as defined above and adding numbers such as $\langle \omega \mid \rangle = \omega + 1$ and so on. Similarly, a starting infinitesimal can be found as $\langle 0 \mid 1, \frac{1}{2}, \frac{1}{4} \ldots \rangle = \frac{1}{\omega}$, and again more can be developed from there.