Graham's number

Canonical name	GrahamsNumber
Date of creation	2013-03-22 17:15:15
Last modified on	2013-03-22 17:15:15
Owner	PrimeFan (13766)
Last modified by	PrimeFan (13766)
Numerical id	4
Author	PrimeFan (13766)
Entry type	Definition
Classification	msc 00A05
Classification	msc 05A05
Classification	msc 68P30

Graham's number G is an upper bound in a problem in Ramsey theory, first mentioned in a paper by Ronald Graham and B. Rothschild. The *Guinness Book of World Records* calls it the largest number ever used in a mathematical proof. Graham's number is too difficult to write in scientific notation, so it is generally written using Knuth's arrow-up notation. In Graham's paper, the bound is given as

$$6 \le N(2) \le A(A(A(A(A(A(A(12,3),3),3),3),3),3),3),$$

where $N(2)$ is the least natural number such that $n \ge N(2)$ implies that given any arbitrary 2-coloring of the line segments between pairs of vertices of an n-dimensional box, there must exist a monochromatic rectangle in the box.

Here A is the function defined by (this is a direct quote):

$$A(1,n) = 2^n, A(m,2) = 4,\ m \ge 1,\ n \ge 2, A(m,n) = A(m-1, A(m, n-1)),\ m \ge 2,\ n \ge 3.$$

In the earlier paper Graham and Rothschild called this function F instead of A, and commented: "Clearly, there is some room for improvement here."

In Knuth's arrow-up notation, Graham's number is still cumbersome to write: we define the recurrence relation $g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3$ and $g_n = 3 \uparrow^{g_{n-1}} 3$. Graham's number is then $G = g_{64}$.

To help understand Graham's number from the more familiar viewpoint of standard exponentiation, Wikipedia offers the following chart:

$$g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow\uparrow (3 \uparrow\uparrow\uparrow 3) = 3 \uparrow\uparrow\uparrow \left(\underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}_{3^{3^3}} \quad \text{threes} \right) = \left. \begin{matrix} \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}} & \\ \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}} & \text{threes} \\ \vdots & \vdots \\ \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}} & \text{threes} \\ 3^{3^3} & \text{threes} \end{matrix} \right\} \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}_{3^{3^3}} \begin{matrix} \text{layers} \\ \text{threes} \end{matrix}$$

We don't know what the most significant base 10 digits of Graham's number are, but we do know that the least significant digit is 7 (and of course 0 in base 3).

Graham's number has its own entry in Wells's *Dictionary of Curious and Interesting Numbers*, it is the very last entry, right after Skewes' number, which it significantly dwarfs, and which was once also said to be the largest number ever used in a serious mathematical proof.

References

[1] M. P. Slone, PlanetMath message, March 19, 2007.

[2] R. L. Graham and B. L. Rothschild, "Ramsey's theorem for n-parameter sets", *Trans. Amer. Math. Soc.*, **159** (1971): 257 - 292

[3] R. L. Graham and B. L. Rothschild. Ramsey theory. *Studies in combinatorics*, ed. G.-C. Rota, Mathematical Association of America, 1978.