property

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Defines	unary relation
Defines	predicate

Let X be a set. A *property* p of X is a function

$$p\colon X \to \{\textit{true}, \textit{false}\}.$$

An element $x \in X$ is said to *have* or *does not have the property* p depending on whether $p(x) = \textit{true}$ or $p(x) = \textit{false}$. Any property gives rise in a natural way to the set

$$X(p) := \{x \in X \mid x \text{ has property } p\}$$

and the corresponding `http://planetmath.org/node/CharacteristicFunction`characteristic function $1_{X(p)}$. The identification of p with $X(p) \subseteq X$ enables us to think of a property of X as a 1-ary, or a *unary relation* on X. Therefore, one may treat all these notions equivalently.

Usually, a property p of X can be identified with a so-called *propositional function*, or a *predicate* $\varphi(v)$, where v is a variable or a tuple of variables whose values range over X. The values of a propositional function is a proposition, which can be interpreted as being either "true" or "false", so that $X(p) = \{x \mid \varphi(x) \text{ is } \textit{true}\}$.

Below are a few examples:

- Let $X = \mathbb{Z}$. Let $\varphi(v)$ be the propositional function "v is divisible by 3". If p is the property identified with $\varphi(v)$, then $X(p) = 3\mathbb{Z}$.

- Again, let $X = \mathbb{Z}$. Let $\varphi(v_1, v_2) :=$"v_1 is divisible by v_2" and p the corresponding property. Then

  $$X(p) = \{(m, n) \mid m = np, \text{ for some } p \in \mathbb{Z}\},$$

  which is a subset of $X \times X$. So p is a property of $X \times X$.

- The reflexive property of a binary relation on X can be identified with the propositional function $\varphi(V) :=$ "$\forall a \in X,\ (a, a) \in V$", and therefore

  $$X(p) = \{R \subseteq X \times X \mid \varphi(R) \text{ is } \textit{true}\},$$

  which is a subset of $2^{X \times X}$. Thus, p is a property of $2^{X \times X}$.

- In point set topology, we often encounter the finite intersection property on a family of subsets of a given set X. Let

  $$\varphi(\mathcal{V}) := \text{“}\forall n \in \mathbb{N}, \forall E_1 \in \mathcal{V}, \dots, \forall E_n \in \mathcal{V}, \exists x \in X (x \in E_1 \cap \cdots \cap E_n)\text{”}$$

and p the corresponding property, then

$$X(p) = \{\mathcal{F} \subseteq 2^X \mid \varphi(\mathcal{F}) \text{ is } \textit{true}\},$$

which is a subset of 2^{2^X}. Thus p is a property of 2^{2^X}.