completing the square

Canonical name	CompletingTheSquare
Date of creation	2013-03-22 13:36:27
Last modified on	2013-03-22 13:36:27
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	14
Author	mathcam (2727)
Entry type	Algorithm
Classification	msc 00A20
Related topic	SquareOfSum

Let us consider the expression x^2+xy, where x and y are real (or complex) numbers. Using the formula

$$(x+y)^2 = x^2 + 2xy + y^2$$

we can write

$$\begin{aligned} x^2 + xy &= x^2 + xy + 0 \\ &= x^2 + xy + \frac{y^2}{4} - \frac{y^2}{4} \\ &= \left(x + \frac{y}{2}\right)^2 - \frac{y^2}{4}. \end{aligned}$$

This manipulation is called *completing the square* [**?**] in x^2+xy, or completing the square x^2.

Replacing y by $-y$, we also have

$$x^2 - xy = \left(x - \frac{y}{2}\right)^2 - \frac{y^2}{4}.$$

Here are some applications of this method:

- `http://planetmath.org/DerivationOfQuadraticFormula`Derivation of the solution formula to the quadratic equation.
- Putting the general equation of a circle, ellipse, or hyperbola into standard form, e.g. the circle

  $$x^2 + y^2 + 2x + 4y = 5 \Rightarrow (x+1)^2 + (y+2)^2 = 10,$$

  from which it is frequently easier to read off important information (the center, radius, etc.)
- Completing the square can also be used to find the extremal value of a quadratic polynomial [**?**] without calculus. Let us illustrate this for the polynomial $p(x) = 4x^2 + 8x + 9$. Completing the square yields

  $$\begin{aligned} p(x) &= (2x+2)^2 - 4 + 9 \\ &= (2x+2)^2 + 5 \\ &\geq 5, \end{aligned}$$

  since $(2x+2)^2 \geq 0$. Here, equality holds if and only if $x = -1$. Thus $p(x) \geq 5$ for all $x \in \mathbb{R}$, and $p(x) = 5$ if and only if $x = -1$. It follows that $p(x)$ has a global minimum at $x = -1$, where $p(-1) = 5$.

- Completing the square can also be used as an integration technique to integrate, for example the function $\dfrac{1}{4x^2 + 8x + 9}$ [**?**].

References

[1] R. Adams, *Calculus, a complete course*, Addison-Wesley Publishers Ltd, 3rd ed.

[2] *Matematiklexikon* (in Swedish), J. Thompson, T. Martinsson, Wahlström & Widstrand, 1991.

(Anyone has an English reference?)