techniques in mathematical proofs

Canonical name	TechniquesInMathematicalProofs
Date of creation	2013-03-22 14:46:15
Last modified on	2013-03-22 14:46:15
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	18
Author	CWoo (3771)
Entry type	Feature
Classification	msc 00A35
Classification	msc 03F07
Related topic	IrrationalToAnIrrationalPowerCanBeRational
Related topic	ExistentialTheorem
Related topic	IABIsInvertibleIfAndOnlyIfIBAIsInvertible
Defines	existential proof
Defines	existence proof
Defines	constructive proof

The following example (from ring theory) illustrates the one aspect of proofs in mathematics: proving the existence of certain mathematical .

Statement: Let R be a ring such that $1-ab$ is right invertible, with $a,b \in R$. Then $1-ba$ is right invertible.

This statement will be proven here using two methods. The first method is called an *existential proof* (also known as an *existence proof*), in which one only seeks to prove that the mathematical in question exists, *not* to show how to obtain it. The second method is called a *constructive proof*, in which one actually shows how to obtain the mathematical in question.

Existential proof: Since $1-ab \in R$ is right invertible, $(1-ab)R = R$. Now,

$$(1-ba)R \supseteq (1-ba)bR = b(1-ab)R = bR.$$

So

$$(ba)R = b(aR) \subseteq bR \subseteq (1-ba)R,$$

and consequently,

$$R = (1-ba)R + (ba)R \subseteq (1-ba)R,$$

showing that $1 \in (1-ba)R$. $\blacksquare$

Notice, we merely demonstrated the existence of a right inverse of $1-ba$ without actually finding such an . The next proof in fact finds a right inverse of $1-ba$.

Constructive proof: Since $1-ab \in R$ is right invertible, let $c \in R$ be a right inverse so that $1 = (1-ab)c$. We seek to construct a right inverse of $1-ba$ in terms of a, b, and c. Rewriting the equation, we have $abc = c-1$. Then,

$$(1-ba)bc = bc - babc = bc - b(c-1) = b.$$

We have just expressed b in terms of $1-ba$. Next, multiply a on the right to each term on both sides of the equation, to get

$$ba = (1-ba)bca.$$

Then, negate both terms and add 1, to get

$$1-ba = 1-(1-ba)bca.$$

Finally, rearranging the terms and we have

$$1 = (1-ba) + (1-ba)bca = (1-ba)(1+bca),$$

showing that a right inverse of $1-ba$ exists by explicitly constructing one. $\blacksquare$

Many other techniques are used in proving mathematical statements. Proof by mathematical induction, proof by contradiction, proof by contrapositive, and proof by exhaustion are just some of the major techniques (a is in the entry "irrational to an irrational power can be rational").

As this entry is still in its very rough form, PM users are welcome and encouraged to refine and provide additional techniques with interesting and illustrative examples!