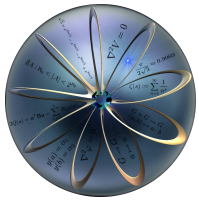

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Knuth's up arrow notation

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Knuth's up arrow noation is a way of writing numbers which would be unwieldy in standard decimal notation. It expands on the exponential notation $m \uparrow n = m^n$. Define $m \uparrow\uparrow 0 = 1$ and $m \uparrow\uparrow n = m \uparrow (m \uparrow\uparrow [n-1])$.

Obviously $m \uparrow\uparrow 1 = m^1 = m$, so $3 \uparrow\uparrow 2 = 3^{3\uparrow\uparrow 1} = 3^3 = 27$, but $2 \uparrow\uparrow 3 = 2^{2\uparrow\uparrow 2} = 2^{2^{2\uparrow\uparrow 1}} = 2^{(2^2)} = 16$.

In general, $m \uparrow\uparrow n = m^{m^{\cdots^m}}$, a tower of height n.

Clearly, this process can be extended: $m \uparrow\uparrow\uparrow 0 = 1$ and $m \uparrow\uparrow\uparrow n = m \uparrow\uparrow (m \uparrow\uparrow\uparrow [n-1])$.

An alternate notation is to write $m^{(i)}n$ for $m \underbrace{\uparrow \cdots \uparrow}_{i-2 \text{ times}} n$. ($i-2$ times because then $m^{(2)}n = m \cdot n$ and $m^{(1)}n = m + n$.) Then in general we can define $m^{(i)}n = m^{(i-1)}(m^{(i)}(n-1))$.

To get a sense of how quickly these numbers grow, $3 \uparrow\uparrow\uparrow 2 = 3 \uparrow\uparrow 3$ is more than seven and a half trillion, and the numbers continue to grow much more than exponentially.