saddle point approximation

Canonical name	SaddlePointApproximation
Date of creation	2013-03-22 13:38:07
Last modified on	2013-03-22 13:38:07
Owner	msihl (2134)
Last modified by	msihl (2134)
Numerical id	5
Author	msihl (2134)
Entry type	Topic
Classification	msc 00A05
Classification	msc 00A79
Synonym	stationary phase method

The saddle point approximation (SPA), a.k.a. stationary phase approximation, is a widely used method in quantum field theory (QFT) and related fields. Suppose we want to evaluate the following integral in the limit $\zeta \to \infty$:

$$\mathcal{I} = \lim_{\zeta \to \infty} \int_{-\infty}^{\infty} \mathrm{d}x \; \mathrm{e}^{-\zeta f(x)}. \tag{1}$$

The saddle point approximation can be applied if the function $f(x)$ satisfies certain conditions. Assume that $f(x)$ has a global minimum $f(x_0) = y_{min}$ at $x = x_0$, which is sufficiently separated from other local minima and whose value is sufficiently smaller than the value of those. Consider the Taylor expansion of $f(x)$ about the point x_0:

$$f(x) = f(x_0) + \partial_x f(x)\Big|_{x=x_0}(x - x_0) + \frac{1}{2}{\partial_x}^2 f(x)\Big|_{x=x_0}(x - x_0)^2 + O(x^3). \tag{2}$$

Since $f(x_0)$ is a (global) minimum, it is clear that $f'(x_0) = 0$. Therefore $f(x)$ may be approximated to quadratic order as

$$f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2. \tag{3}$$

The above assumptions on the minima of $f(x)$ ensure that the dominant contribution to (**??**) in the limit $\zeta \to \infty$ will come from the region of integration around x_0:

$$\begin{aligned}\mathcal{I} &\approx \lim_{\zeta \to \infty} \mathrm{e}^{-\zeta f(x_0)} \int_{-\infty}^{\infty} \mathrm{d}x \; \mathrm{e}^{-\frac{\zeta}{2} f''(x_0)(x-x_0)^2} \\ &\approx \lim_{\zeta \to \infty} \mathrm{e}^{-\zeta f(x_0)} \left(\frac{2\pi}{\zeta f''(x_0)} \right)^{1/2}.\end{aligned} \tag{4}$$

In the last step we have performed the Gaußian integral. The next nonvanishing higher order correction to (**??**) stems from the quartic term of the expansion (**??**). This correction may be incorporated into (**??**) to yield (after expanding part of the exponential):

$$\mathcal{I} \approx \lim_{\zeta \to \infty} \mathrm{e}^{-\zeta f(x_0)} \int_{-\infty}^{\infty} \mathrm{d}x \; \mathrm{e}^{-\frac{\zeta}{2} f''(x_0)(x-x_0)^2} \left(1 - \frac{\zeta}{4!} (\partial_x^4 f(x))|_{x=x_0} (x - x_0)^4 \right). \tag{5}$$

...to be continued with applications to physics...