recursive set

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Entry type	Definition
Classification	msc 03B25
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Synonym	decidable set
Synonym	computable set
Synonym	decidable predicate
Synonym	computable predicate
Defines	recursively enumerable set
Defines	recursive predicate
Defines	recursively enumerable predicate
Defines	recursive language

A subset S of the natural numbers $\mathbb{N}$ is said to be *recursive* if its characteristic function

$$\chi_S(x) \quad := \quad \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \in \mathbb{N} - S \end{cases}$$

is recursive (computable). In other words, there is an algorithm (via Turing machine for example) that determines whether an element is in S or not in S.

More generally, a subset $S \subseteq \mathbb{N}^n$ is *recursive* if its characteristic function f_S is recursive.

A recursive set is also known as a *decidable set* or a *computable set*.

Examples of recursive sets are finite subset of $\mathbb{N}$, the set $\mathbb{N}$ itself, the set of even integers, the set of Fibonacci numbers, the set of pairs (a, b) where a divides b, and the set of prime numbers. In the last example, one may use the Sieve of Eratosthenes as an algorithm to determine the primality of an integer.

A set $S \subseteq \mathbb{N}$ is *recursively enumerable* if the partial function

$$f(x) \quad := \quad \begin{cases} 1 & \text{if } x \in S \\ \text{undefined} & \text{if } x \in \mathbb{N} - S \end{cases}$$

is computable. In other words, there is an algorithm that halts (and returns 1) only when an element in S is used as an input.

Remarks

- A special case of a recursive set is that of a *primitive recursive set*. A set is *primitive recursive* if its characteristic function is `http://planetmath.org/PrimitiveRecu` recursive. All of the examples cited above are primitive recursive.

- On the other hand, one can broaden the scope of recursiveness to sets which are not necessarily subsets of $\mathbb{N}^n$. Below are two examples:

  - Since $\mathbb{Z}$ can be effectively embedded in $\mathbb{N}$, so the notion of recursive sets be extended to subsets of $\mathbb{Z}$.

  - Since every finite set Σ can be encoded by the natural numbers, we can define a *recursive language* over Σ to be a subset $L \subseteq \Sigma^*$ such that L, when encoded by the natural numbers, is a recursive set. Equivalently, L is recursive iff there is a Turing machine that decides L (accepts L and rejects $\Sigma^* - L$).

- Similarly, recursive enumerability can be defined on languages: a language L over Σ is *recursively enumerable* if its encoding by the natural numbers is a recursively enumerable set. This is equivalent to saying that L is accepted by a Turing machine.

- Using the above definition, one can define a *recursive predicate* or a *recursively enumerable predicate* $\varphi(x)$ according to whether $\{x \mid \varphi(x)\}$ is a recursive or recursively enumerable set respectively.