free and bound variables

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Defines	free variable
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Defines	bound occurrence
Defines	occurs free
Defines	occurs bound

In the entry `http://planetmath.org/TermsAndFormulas`first-order language, I have mentioned the use of variables without mentioning what variables really are. A variable is a symbol that is supposed to range over the universe of discourse. Unlike a constant, it has no fixed value.

There are two ways in which a variable can occur in a formula: **free** or **bound**. Informally, a variable is said to occur *free* in a formula φ if and only if it is not within the "scope" of a quantifier. For instance, x occurs free in φ if and only if it occurs in it as a symbol, and no subformula of φ is of the form $\exists x.\psi$. Here the x after the $\exists$ is to be taken literally : it is x and no other symbol.

Variables in Terms

To formally define free (resp. bound) variables in a formula, we start by defining variables occurring in terms, which can be easily done inductively: let t be a term (in a first-order language), then $\mathrm{Var}(t)$ is

- if t is a variable v, then $\mathrm{Var}(t)$ is $\{v\}$
- if t is $f(t_1, \ldots, t_n)$, where f is a function symbol of arity n, and each t_i is a term, then $\mathrm{Var}(t)$ is the union of all the $\mathrm{Var}(t_i)$.

Free Variables

Now, let φ be a formula. Then the set $\mathrm{FV}(\varphi)$ of *free variables* of φ is now defined inductively as follows:

- if φ is $t_1 = t_2$, then $\mathrm{FV}(\varphi)$ is $\mathrm{Var}(t_1) \cup \mathrm{Var}(t_2)$,
- if φ is $R(t_1, \ldots, t_n)$, then $\mathrm{FV}(\varphi)$ is $\mathrm{Var}(t_1) \cup \cdots \cup \mathrm{Var}(t_n)$
- if φ is $\neg\psi$, then $\mathrm{FV}(\varphi)$ is $\mathrm{FV}(\psi)$
- if φ is $\psi \vee \sigma$, then $\mathrm{FV}(\varphi)$ is $\mathrm{FV}(\psi) \cup \mathrm{FV}(\sigma)$, and
- if φ is $\exists x\psi$, then $\mathrm{FV}(\varphi)$ is $\mathrm{FV}(\psi) - \{x\}$.

If $\mathrm{FV}(\varphi) \neq \varnothing$, it is customary to write φ as $\varphi(x_1, \ldots, x_n)$, in order to stress the fact that there are some free variables left in φ, and that those free variables are among $x_1, \ldots, x_n$. When $x_1, \ldots, x_n$ appear free in φ, then they are considered as **place-holders**, and it is understood that we will have to supply "values" for them, when we want to determine the truth of φ. If $\mathrm{FV}(\varphi) = \varnothing$, then φ is called a **sentence**. Another name for a sentence is a *closed formula*. A formula that is not closed is said to be *open*.

Bound Variables

Bound variables in formulas are inductively defined as well: let φ be a formula. Then the set $\mathrm{BV}(\varphi)$ of *bound variables* of φ

- if φ is an atomic formula, then $\mathrm{BV}(\varphi)$ is $\varnothing$, the empty set,
- if φ is $\neg\psi$, then $\mathrm{BV}(\varphi)$ is $\mathrm{BV}(\psi)$
- if φ is $\psi \vee \sigma$, then $\mathrm{BV}(\varphi)$ is $\mathrm{BV}(\psi) \cup \mathrm{BV}(\sigma)$, and
- if φ is $\exists x\psi$, then $\mathrm{BV}(\varphi)$ is $\mathrm{BV}(\psi) \cup \{x\}$.

Thus, a variable x is bound in φ if and only if $\exists x\psi$ is a subformula of φ for some formula ψ.

The set of all variables occurring in a formula φ is denoted $\mathrm{Var}(\varphi)$, and is $\mathrm{FV}(\varphi) \cup \mathrm{BV}(\varphi)$.

Note that it is possible for a variable to be both free and bound. In other words, $\mathrm{FV}(\varphi)$ and $\mathrm{BV}(\varphi)$ are not necessarily disjoint. For example, consider the following formula φ of the lenguage $\{+, \cdot, 0, 1\}$ of ring theory :

$$x + 1 = 0 \wedge \exists x(x + y = 1)$$

Then $\mathrm{FV}(\varphi) = \{x, y\}$ and $\mathrm{BV}(\varphi) = \{x\}$: the variable x occurs both free and bound. However, the following lemma tells us that we can always avoid this situation :

Lemma 1. It is possible to rename the bound variables without affecting the truth of a formula. In other words, if $\varphi = \exists x(\psi)$, or $\forall x(\psi)$, and z is a variable not occurring in ψ, then $\vdash \varphi \Leftrightarrow \exists z(\psi[z/x])$, where $\psi[z/x]$ is the formula obtained from ψ by replacing every free occurrence of x by z.

As a result of the lemma above, we see that every formula is logically equivalent to a formula φ such that $\mathrm{FV}(\varphi) \cap \mathrm{BV}(\varphi) = \varnothing$.