Thue system

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A semi-Thue system $\mathfrak{S} = (\Sigma, R)$ is said to be a *Thue system* if R is a symmetric relation on Σ^*. In other words, if $x \to y$ is a defining relation in R, then so is $y \to x$.

Like a semi-Thue system, we can define the concepts of immediately derivable and derivable pairs. Let R' and R'' be the respective collections of these pairs. Since R is symmetric, so is R' and consequently R''. Similarly, the notations are: for elements $(a, b) \in R'$, we write $a \Rightarrow b$, and for elements $(c, d) \in R''$, we write $a \overset{*}{\Rightarrow} b$.

If we regard Σ^* as a free monoid with concatenation $\cdot$ as multiplication and the empty word λ as the multiplicative identity, then $\overset{*}{\Rightarrow}$ is a congruence relation on Σ^*: it is an equivalence relation and respects concatenation, meaning that if $a \overset{*}{\Rightarrow} b$ and $c \overset{*}{\Rightarrow} d$, then $ac \overset{*}{\Rightarrow} bd$. Therefore, we can take the quotent $\Sigma^*/\overset{*}{\Rightarrow}$ and the resulting set of equivalence classes is again a monoid with $[\lambda]$ as the multiplicative identity. It is a monoid generated by $[a]$ whenever $a \in \Sigma$ with relations $[u] = [v]$ whenever $u \to v$ is a defining relation in R. Thus, two elements are in the same equivalence class if one is derivable from another. Let us denote this monoid by $[\Sigma]_{\mathfrak{S}}$.

Now let $\mathfrak{S} = (\Sigma, R)$ be a Thue system. Then $\mathfrak{S}$ is called a *group system* if there exists an involution $^{-1}$ on Σ given by $a \mapsto a^{-1}$, and that for every $a \in \Sigma$, $aa^{-1} \to \lambda$ is a defining relation in R. Since $^{-1}$ is an involution, if b is the symbol in Σ such that $b = a^{-1}$, then $b^{-1} = a$. So $a^{-1}a = ba = bb^{-1} \to \lambda$ also. In fact, it is not hard to see that for a group system $\mathfrak{S}$, $[\Sigma]_{\mathfrak{S}}$ is the group with generators $[a]$ whenever $a \in \Sigma$ and with relators $[u][v]^{-1}$ whenever $u \to v$ is a defining relation in R. Every non-trivial element in $[\Sigma]_{\mathfrak{S}}$ has an expression $[a_1]^{p_1} \cdots [a_n]^{p_n}$, where each a_i is a letter in Σ such that it is distinct from its neighbors ($a_i \neq a_{i+1}$), and p_i are non-zero integers. This expression is unique in the sense that it is "reduced". See reduced words for more detail.

Remark. Like the word problem for semi-Thue systems, the word problem for Thue systems and group systems can be similarly posed. It can be shown that the word problem for Thue systems and group systems are both unsolvable. As a result, the corresponding word problems for semigroups and for groups are also unsolvable.

References

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