freely generated inductive set

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In the parent entry, we see that an *inductive set* is a set that is closed under the successor operator. If A is a non-empty inductive set, then $\mathbb{N}$ can be embedded in A.

More generally, fix a non-empty set U and a set F of finitary operations on U. A set $A \subseteq U$ is said to be *inductive* (with respect to F) if A is closed under each $f \in F$. This means, for example, if f is a binary operation on U and if $x, y \in A$, then $f(x, y) \in A$. A is said to be inductive over X if $X \subseteq A$. The intersection of inductive sets is clearly inductive. Given a set $X \subseteq U$, the intersection of all inductive sets over X is said to be the *inductive closure* of X. The inductive closure of X is written $\langle X \rangle$. We also say that X generates $\langle X \rangle$.

Another way of defining $\langle X \rangle$ is as follows: start with

$$X_0 := X.$$

Next, we "inductively" define each X_{i+1} from X_i, so that

$$X_{i+1} := X_i \cup \bigcup \{ f(X_i^n) \mid f \in F, f \text{ is } n\text{-ary} \}.$$

Finally, we set

$$\overline{X} := \bigcup_{i=0}^{\infty} X_i.$$

It is not hard to see that $\overline{X} = \langle X \rangle$.

Proof. By definition, $X \subseteq \overline{X}$. Suppose $f \in F$ is n-ary, and $a_1, \ldots, a_n \in \overline{X}$, then each $a_i \in X_{m(i)}$. Take the maximum m of the integers $m(i)$, then $a_i \in X_m$ for each i. Therefore $f(a_1, \ldots, a_n) \in X_{m+1} \subseteq \overline{X}$. This shows that $\overline{X}$ is inductive over X, so $\langle X \rangle \subseteq \overline{X}$, since $\langle X \rangle$ is minimal. On the other hand, suppose $a \in \overline{X}$. We prove by induction that $a \in \langle X \rangle$. If $a \in X$, this is clear. Suppose now that $X_i \subseteq \langle X \rangle$, and $a \in X_{i+1}$. If $a \in X_i$, then we are done. Suppose now $a \in X_{i+1} - X_i$. Then there is some n-ary operation $f \in F$, such that $a = f(a_1, \ldots, a_n)$, where each $a_j \in X_i$. So $a_j \in \langle X \rangle$ by hypothesis. Since $\langle X \rangle$ is inductive, $f(a_1, \ldots, a_n) \in \langle X \rangle$, and hence $a \in \langle X \rangle$ as well. This shows that $X_{i+1} \subseteq A$, and consequently $\overline{X} \subseteq \langle X \rangle$. □

The inductive set A is said to be freely generated by X (with respect to F), if the following conditions are satisfied:

1. $A = \langle X \rangle$,

2. for each n-ary $f \in F$, the restriction of f to A^n is one-to-one;

3. for each n-ary $f \in F$, $f(A^n) \cap X = \varnothing$;

4. if $f, g \in F$ are n, m-ary, then $f(A^n) \cap g(A^m) = \varnothing$.

For example, the set $\overline{V}$ of well-formed formulas (wffs) in the classical proposition logic is inductive over the set of V propositional variables with respect to the logical connectives (say, $\neg$ and $\vee$) provided. In fact, by unique readability of wffs, $\overline{V}$ is freely generated over V. We may readily interpret the above "freeness" conditions as follows:

1. $\overline{V}$ is generated by V,

2. for distinct wffs p, q, the wffs $\neg p$ and $\neg q$ are distinct; for distinct pairs (p, q) and (r, s) of wffs, $p \vee q$ and $r \vee s$ are distinct also

3. for no wffs p, q are $\neg p$ and $p \vee q$ propositional variables

4. for wffs p, q, the wffs $\neg p$ and $p \vee q$ are never the same

A characterization of free generation is the following:

Proposition 1. *The following are equivalent:*

1. *A is freely generated by X (with respect to F)*

2. *if $V \neq \varnothing$ is a set, and G is a set of finitary operations on V such that there is a function $\phi : F \to G$ taking every n-ary $f \in F$ to an n-ary $\phi(f) \in G$, then every function $h : X \to B$ has a unique extension $\overline{h} : A \to B$ such that*

$$\overline{h}(f(a_1, \ldots, a_n)) = \phi(f)(\overline{h}(a_1), \ldots, \overline{h}(a_n)),$$

where f is an n-ary operation in F, and $a_i \in A$.

References

[1] H. Enderton: *A Mathematical Introduction to Logic*, Academic Press, San Diego (1972).