Kleene star of an automaton

Canonical name	KleeneStarOfAnAutomaton
Date of creation	2013-03-22 18:04:06
Last modified on	2013-03-22 18:04:06
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	16
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03D05
Classification	msc 68Q45

Let $A = (S, \Sigma, \delta, I, F)$ be an automaton. Define the Kleene star A^* of A as an http://planetmath.org/AutomatonWithEpsilonTransitionsautomaton with ϵ-transitions $(S_1, \Sigma, \delta_1, I_1, F_1, \epsilon)$ where

1. $S_1 = S \dot{\cup} \{q\}$ (we either assume that q is not a symbol in Σ, or we take the disjoint union)
2. δ_1 is a function from $S_1 \times (\Sigma \dot{\cup} \{\epsilon\})$ to $P(S_1)$ given by:
   - $\delta_1(s, \epsilon) := I$ if $s = q$, $\delta_1(s, \epsilon) := \{q\}$ for any $s \in F$,
   - $\delta_1(s, \alpha) := \delta(s, \alpha)$, for any $(s, \alpha) \in S \times \Sigma$, and
   - $\delta_1(s, \alpha) := \varnothing$ otherwise.
3. $I_1 = F_1 = \{q\}$

Basically, we throw into A^* all transitions in A. In addition, we add to A^* transitions taking s to any initial states of A, as well as transitions taking any final states of A to s. Visually, the state diagram G_{A^*} of A^* is obtained by adding a node q to the state diagram G_A of A, and making q both the start and the final node of G_{A^*}. Furthermore, add edges from q to the start nodes of G_A, and edges from the final nodes of G_A to q, and let ϵ be the label for all of the newly added edges.

Proposition 1. $L(A)^* = L(A^*)$.

Proof. Clearly $\lambda \in L(A)*$. In addition, since $I_1 = F_1$, $\lambda \in L(A^*_\epsilon) = L(A^*)$. This proves the case when the word is empty. Now, we move to the case when the word has non-zero length.

- $L(A)^* \subseteq L(A^*)$.

  Suppose $a = a_1 a_2 \cdots a_n$ is a word such that $a_i \in L(A)$, then we claim that $b = b_1 b_2 \cdots b_n$, where $b_i = \epsilon a_i \epsilon$, is accepted by A^*_ϵ. This can be proved by induction on n:

  1. First, $n = 1$. Then

     $$\delta_1(q, b_1) = \delta_1(\delta_1(q, \epsilon), a_1\epsilon) = \delta_1(I, a_1\epsilon) = \delta_1(\delta_1(I, a_1), \epsilon).$$

     Since a_1 is accepted by A, $\delta_1(I, a_1) = \delta(I, a_1)$ contains an accepting state $s \in F$, so that

     $$q \in \delta_1(s, \epsilon) \subseteq \delta_1(\delta_1(I, a_1), \epsilon).$$

     Hence b_1 is accepted by A^*_ϵ.

2. Next, suppose that given $b = b_1 \cdots b_n$, the subword $b_1 \cdots b_{n-1}$ (induction step) is accepted by A^*_ϵ. This means that $q \in \delta_1(q, b_1 \cdots b_{n-1})$, so that

$$\delta_1(q, b_n) \subseteq \delta_1(\delta_1(q, b_1 \cdots b_{n-1}), b_n) = \delta_1(q, b_1 \cdots b_n).$$

But $\delta_1(q, b_n)$ contains q as was shown in step 1 above. As a result, $b_1 \cdots b_n$ is accepted by A^*_ϵ.

This shows that $L(A)^* \subseteq L(A^*)$.

- $L(A^*) \subseteq L(A)^*$.

Suppose now that a is a word over Σ accepted by A^*. This means that, for some i_j, $j = 0, 1, \ldots, n$, the word

$$b := \epsilon^{i_0} a_1 \epsilon^{i_1} \cdots \epsilon^{i_{n-1}} a_n \epsilon^{i_n}$$

is accepted by A^*_ϵ, where each a_j is a word over Σ with $a = a_1 \cdots a_n$. We want to show that each a_j is accepted by A.

The main thing is to notice is that if $q \in \delta_1(J, \epsilon^i)$ and $J \subseteq S$, where i is a positive integer, then J must contain a state in F. Otherwise, $J \subseteq S - F$, so that $\delta_1(J, \epsilon) = \varnothing$, and we must have $\delta_1(J, \epsilon^i) = \delta_1(\delta_1(J, \epsilon), \epsilon^{i-1}) = \delta_1(\varnothing, \epsilon^{i-1}) = \varnothing$.

Set $J = \delta_1(q, \epsilon^{i_0} a_1 \epsilon^{i_1} \cdots \epsilon^{i_{n-1}} a_n)$ and $K = \delta_1(q, \epsilon^{i_0} a_1 \epsilon^{i_1} \cdots \epsilon^{i_{n-1}})$. Then $J = \delta_1(K, a_n) = \delta(K, a_n) \subseteq S$. Furthermore, by assumption $q \in \delta_1(q, b) = \delta_1(J, \epsilon^{i_n})$. Therefore, J must contain a state in F. Thus, a_n is accepted by A. The fact that the remaining a_i's are accepted by A is proved inductively.

This shows that $L(A^*) \subseteq L(A)^*$.

This completes the proof. $\square$