number of ultrafilters

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Theorem. *Let X be a set. The* **http://planetmath.org/CardinalNumber***number of* **http://planetmath.org/Ultrafilter***ultrafilters on X is $|X|$ if X is* **http://planetmath.org/Finite***finite, and $2^{2^{|X|}}$ if X is* **http://planetmath.org/Infinite***infin*

Proof. If X is finite then each ultrafilter on X is principal, and so there are exactly $|X|$ ultrafilters. In the rest of the proof we will assume that X is infinite.

Let F be the set of all finite subsets of X, and let Φ be the set of all finite subsets of F.

For each $A \subseteq X$ define $B_A = \{(f, \phi) \in F \times \Phi \mid A \cap f \in \phi\}$, and $B_A^{\complement} = (F \times \Phi) \setminus B_A$. For each $\mathcal{S} \subseteq \mathcal{P}(X)$ define $\mathcal{B}_{\mathcal{S}} = \{B_A \mid A \in \mathcal{S}\} \cup \{B_A^{\complement} \mid A \notin \mathcal{S}\}$.

Let $\mathcal{S} \subseteq \mathcal{P}(X)$, and suppose $A_1, \ldots, A_m \in \mathcal{S}$ and $D_1, \ldots, D_n \in \mathcal{P}(X) \setminus \mathcal{S}$, so that we have $B_{A_1}, \ldots, B_{A_m}, B_{D_1}^{\complement}, \ldots, B_{D_n}^{\complement} \in \mathcal{B}_{\mathcal{S}}$. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ let $a_{i,j}$ be such that either $a_{i,j} \in A_i \setminus D_j$ or $a_{i,j} \in D_j \setminus A_i$. This is always possible, since $A_i \neq D_j$. Let $f = \{a_{i,j} \mid 1 \leq i \leq m,\, 1 \leq j \leq n\}$, and put $\phi = \{A_i \cap f \mid 1 \leq i \leq m\}$. Then $(f, \phi) \in B_{A_i}$, for $i = 1, \ldots, m$. If for some $j \in \{1, \ldots, n\}$ we have $D_j \cap f \in \phi$, then $D_j \cap f = A_i \cap f$ for some $i \in \{1, \ldots, m\}$, which is impossible, as $a_{i,j}$ is in one of these sets but not the other. So $D_j \cap f \notin \phi$, and therefore $(f, \phi) \in B_{D_j}^{\complement}$. So $(f, \phi) \in B_{A_1} \cap \cdots \cap B_{A_m} \cap B_{D_1}^{\complement} \cap \cdots \cap B_{D_n}^{\complement}$. This shows that any finite subset of $\mathcal{B}_{\mathcal{S}}$ has nonempty intersection, and therefore $\mathcal{B}_{\mathcal{S}}$ can be extended to an ultrafilter $\mathcal{U}_{\mathcal{S}}$.

Suppose $\mathcal{R}, \mathcal{S} \subseteq \mathcal{P}(X)$ are distinct. Then, relabelling if necessary, $\mathcal{R} \setminus \mathcal{S}$ is nonempty. Let $A \in \mathcal{R} \setminus \mathcal{S}$. Then $B_A \in \mathcal{U}_{\mathcal{R}}$, but $B_A \notin \mathcal{U}_{\mathcal{S}}$ since $B_A^{\complement} \in \mathcal{U}_{\mathcal{S}}$. So $\mathcal{U}_{\mathcal{R}}$ and $\mathcal{U}_{\mathcal{S}}$ are distinct for distinct $\mathcal{R}$ and $\mathcal{S}$. Therefore $\{\mathcal{U}_{\mathcal{S}} \mid \mathcal{S} \subseteq \mathcal{P}(X)\}$ is a set of $2^{2^{|X|}}$ ultrafilters on $F \times \Phi$. But $|F \times \Phi| = |X|$, so $F \times \Phi$ has the same number of ultrafilters as X. So there are at least $2^{2^{|X|}}$ ultrafilters on X, and there cannot be more than $2^{2^{|X|}}$ as each ultrafilter is an element of $\mathcal{P}(\mathcal{P}(X))$. □

Corollary. *The number of topologies on an infinite set X is $2^{2^{|X|}}$.*

Proof. Let X be an infinite set. By the theorem, there are $2^{2^{|X|}}$ ultrafilters on X. If $\mathcal{U}$ is an ultrafilter on X, then $\mathcal{U} \cup \{\varnothing\}$ is a topology on X. So there are at least $2^{2^{|X|}}$ topologies on X, and there cannot be more than $2^{2^{|X|}}$ as each topology is an element of $\mathcal{P}(\mathcal{P}(X))$. □