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alternative definitions of countable

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The following are alternative ways of characterizing a countable set.

Proposition 1. *Let A be a set and $\mathbb{N}$ the set of natural numbers. The following are equivalent:*

1. *there is a surjection from $\mathbb{N}$ to A.*
2. *there is an injection from A to $\mathbb{N}$.*
3. *either A is finite or there is a bijection between A and $\mathbb{N}$.*

Proof. First notice that if A were the empty set, then any map to or from A is empty, so (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) vacuously. Now, suppose that $A \neq \varnothing$.

(1) $\Rightarrow$ (2). Suppose $f : \mathbb{N} \to A$ is a surjection. For each $a \in A$, let $f^{-1}(a)$ be the set $\{n \in \mathbb{N} \mid f(n) = a\}$. Since $f^{-1}(a)$ is a subset of $\mathbb{N}$, which is well-ordered, $f^{-1}(a)$ itself is well-ordered, and thus has a least element (keep in mind $A \neq \varnothing$, the existence of $a \in A$ is guaranteed, so that $f^{-1}(a) \neq \varnothing$ as well). Let $g(a)$ be this least element. Then $a \mapsto g(a)$ is a well-defined mapping from A to $\mathbb{N}$. It is one-to-one, for if $g(a) = g(b) = n$, then $a = f(n) = b$.

(2) $\Rightarrow$ (1). Suppose $g : A \to \mathbb{N}$ is one-to-one. So $g^{-1}(n)$ is at most a singleton for every $n \in \mathbb{N}$. If it is a singleton, identify $g^{-1}(n)$ with that element. Otherwise, identify $g^{-1}(n)$ with a designated element $a_0 \in A$ (remember A is non-empty). Define a function $f : \mathbb{N} \to A$ by $f(n) := g^{-1}(n)$. By the discussion above, $g^{-1}(n)$ is a well-defined element of A, and therefore f is well-defined. f is onto because for every $a \in A$, $f(g(a)) = a$.

(3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Let $g : A \to \mathbb{N}$ be an injection. Then $g(A)$ is either finite or infinite. If $g(A)$ is finite, so is A, since they are equinumerous. Suppose $g(A)$ is infinite. Since $g(A) \subseteq \mathbb{N}$, it is well-ordered. The (induced) well-ordering on $g(A)$ implies that $g(A) = \{n_1, n_2, \ldots\}$, where $n_1 < n_2 < \cdots$.

Now, define $h : \mathbb{N} \to A$ as follows, for each $i \in \mathbb{N}$, $h(i)$ is the element in A such that $g(h(i)) = n_i$. So h is well-defined. Next, h is injective. For if $h(i) = h(j)$, then $n_i = g(h(i)) = g(h(j)) = n_j$, implying $i = j$. Finally, h is a surjection, for if we pick any $a \in A$, then $g(a) \in g(A)$, meaning that $g(a) = n_i$ for some i, so $h(i) = g(a)$. $\square$

Therefore, countability can be defined in terms of either of the above three statements.

Note that the axiom of choice is not needed in the proof of (1) $\Rightarrow$ (2), since the selection of an element in $f^{-1}(a)$ is definite, not arbitrary.

For example, we show that $\mathbb{N}^2$ is countable. By the proposition above, we either need to find a surjection $f : \mathbb{N} \to \mathbb{N}^2$, or an injection $g : \mathbb{N}^2 \to \mathbb{N}$. Actually, in this case, we can find both:

1. the function $f : \mathbb{N} \to \mathbb{N}^2$ given by $f(a) = (m, n)$ where $a = 2^m(2n+1)$ is surjective. First, the function is well-defined, for every positive integer has a unique representation as the product of a power of 2 and an odd number. It is surjective because for every (m, n), we see that $f(2^m(2n + 1)) = (m, n)$.

2. the function $g : \mathbb{N}^2 \to \mathbb{N}$ given by $f(m, n) = 2^m 3^n$ is clearly injective.

Note that the injectivity of g, as well as f being well-defined, rely on the unique factorization of integers by prime numbers. In this `http://planetmath.org/ProductOfCou`
we actually find a bijection between $\mathbb{N}$ and $\mathbb{N}^2$.

As a corollary, we record the following:

Corollary 1. *Let A, B be sets, $f : A \to B$ a function.*

- *If f is an injection, and B is countable, so is A.*
- *If f is a surjection, and A countable, so is B.*

The proof is left to the reader.