proof of principle of transfinite induction

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To prove the transfinite induction theorem, we note that the class of ordinals is well-ordered by $\in$. So suppose for some Φ, there are ordinals α such that $\Phi(\alpha)$ is not true. Suppose further that Φ satisfies the hypothesis, i.e. $\forall\alpha(\forall\beta < \alpha(\Phi(\beta)) \Rightarrow \Phi(\alpha))$. We will reach a contradiction.

The class $C = \{\alpha : \neg\Phi(\alpha)\}$ is not empty. Note that it may be a proper class, but this is not important. Let $\gamma = \min(C)$ be the $\in$-minimal element of C. Then by assumption, for every $\lambda < \gamma$, $\Phi(\lambda)$ is true. Thus, by hypothesis, $\Phi(\gamma)$ is true, contradiction.