ordinal exponentiation

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Let α, β be ordinals. We define α^β as follows:

$$\alpha^\beta := \begin{cases} 1 & \text{if } \beta = 0, \\ \alpha^\gamma \cdot \alpha & \text{if } \beta \text{ is a successor ordinal and } \beta = S\gamma, \\ \sup\{\alpha^\gamma \mid \gamma < \beta\} & \text{if } \beta \text{ is a limit ordinal and } \beta = \sup\{\gamma \mid \gamma < \beta\}. \end{cases}$$

Some properties of exponentiation:

1. $0^\alpha = 0$ if $\alpha > 0$
2. $1^\alpha = 1$
3. $\alpha^1 = \alpha$
4. $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$
5. $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$
6. For any ordinals α, β with $\alpha > 0$ and $\beta > 1$, there exists a unique triple $(\gamma, \delta, \epsilon)$ of ordinals such that

$$\alpha = \beta^\gamma \cdot \delta + \epsilon$$

where $0 < \delta < \beta$ and $\epsilon < \beta^\delta$.

All of these properties can be proved using transfinite induction.