definition by cases

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Definition A (total) function $f : \mathbb{N}^k \to \mathbb{N}$ is said to be defined by cases if there are functions $f_1, \dots, f_m : \mathbb{N}^k \to \mathbb{N}$, and predicates $\Phi_1(\boldsymbol{x}), \dots, \Phi_m(\boldsymbol{x})$, which are pairwise exclusive

$$S(\Phi_i) \cap S(\Phi_j) = \varnothing$$

for $i \neq j$, such that

$$f(\boldsymbol{x}) := \begin{cases} f_1(\boldsymbol{x}) & \text{if } \Phi_1(\boldsymbol{x}), \\ \cdots & \\ f_m(\boldsymbol{x}) & \text{if } \Phi_m(\boldsymbol{x}). \end{cases}$$

Since f is a total function (domain is all of $\mathbb{N}^k$), we see that $S(\Phi_1) \cup \cdots \cup S(\Phi_m) = \mathbb{N}^k$.

Proposition 1. *As above, if the functions $f_1, \dots, f_m : \mathbb{N}^k \to \mathbb{N}$, as well as the predicates $\Phi_1(\boldsymbol{x}), \dots, \Phi_m(\boldsymbol{x})$, are primitive recursive, then so is the function $f : \mathbb{N}^k \to \mathbb{N}$ defined by cases from the f_i and Φ_j.*

To see this, we first need the following:

Lemma 1. *If functions $f_1, \dots, f_m : \mathbb{N}^k \to \mathbb{N}$ are primitive recursive, so is $f_1 + \cdots + f_m$.*

Proof. By induction on m. The case when $m = 1$ is clear. Suppose the statement is true for $m = n$. Then $f_1 + \cdots + f_n + f_{n+1} = \text{add}(f_1 + \cdots + f_n, f_{n+1})$, which is primitive recursive, since add is, and that primitive recursiveness is preserved under functional composition. $\square$

Proof of Proposition 1. f is just

$$f(\boldsymbol{x}) := \begin{cases} f_1(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in S(\Phi_1), \\ \cdots & \\ f_m(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in S(\Phi_m). \end{cases}$$

which can be re-written as

$$f = \varphi_{S(\Phi_1)} f_1 + \cdots + \varphi_{S(\Phi_m)} f_m,$$

where φ_S denotes the characteristic function of set S. By assumption, both f_i and $\varphi_{S(\Phi_i)}$ are primitive recursive, so is their product, and hence the sum of these products. As a result, f is primitive recursive too. $\square$