Gödel's beta function

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Gödel's β function is the tool needed to express in a formal theory $\mathbf{Z}$ assertions about finite sequences of natural numbers. (For a description of $\mathbf{Z}$ see the PM entry `http://planetmath.org/BeyondFormalism`"beyond formalism: Gödel's incompleteness").

Let

$$1, \ldots, n.$$

be a sequence of natural numbers. Then there is the factorial $n!$ such that

$$n \cdot (n-1) \cdot \ldots \cdot 1 = n!.$$

This means that the factorial $n!$ is divided by each of the elements of

$$1, \ldots, n$$

so that we can build the derived sequence

$$1 \mid n!, 2 \mid n!, \ldots, n \mid n!.$$

Definition 1 **[Gödel's Γ sequence]**

If we denote $n!$ by l then the elements of Gödel's Γ sequence are the numbers of the form
$$(k+1) \cdot l + 1.$$

Its general representation is

$$\Gamma = 1 \cdot l + 1, 2 \cdot l + 1, \ldots, n \cdot l + 1, (n+1) \cdot l + 1.$$

Theorem 1:

The elements of Γ are pairwise relatively prime.

Proof: (*Reductio*)

1. We assume the existence of a prime number p such that p divides

$$(j+1) \cdot l + 1$$

and p divides also

$$(j+k+1) \cdot l + 1.$$

2. This gives by definition the congruence

$$[(j+k+1)\cdot l+1] \equiv [(j+1)\cdot l+1] \pmod p$$

3. Therefore p divides

$$[(j+k+1)\cdot l+1] - [(j+1)\cdot l+1].$$

4. Then p divides

$$j\cdot l + k\cdot l + l + 1 - j\cdot l - l - 1,$$

that is, p divides $k\cdot l$.

5. But

$$p \mid k\cdot l \to p \mid k \vee p \mid l.$$

Case I: **Assume** $p \mid l$

1. $p \mid l \to p \mid l\cdot(j+1)$.
2. But by hypothesis: $p \mid (j+1)\cdot l+1$.
3. Therefore: $(p \nmid l)$.

Case II: **Assume** $p \mid k$

1. $k \leq n \leq \max(1,\ldots,n)$.
2. But $(\forall k) k \mid l$.
3. $p \mid k \wedge k \mid l \to p \mid l$

Case I.3. and Case II.3. are the desired contradiction.
Definition 2: **[Gödel's β function]**

If $m, l,$ and k are natural numbers then the function
$$\beta(m, l, k)$$
computes the rest of the division of m by a term $(k+1) \cdot l + 1$
of the Γ-sequence.
Therefore: $\beta(m, l, k) = R[m, (k+1) \cdot l + 1]$.

Theorem 2: **[Sequences of natural numbers are representable by the β function.]**

$$< a_o, \ldots, a_n > \in \mathbb{N} \rightarrow (\exists m)(\exists l)[a_k = \beta(m, k, l)].$$

Proof:

1. Let

$$a_o, \ldots, a_n$$

   be a sequence of natural numbers.

2. Then there is a number l such that

$$\Gamma = 1 \cdot l + 1, 2 \cdot l + 1, \ldots, n \cdot l + 1, (n+1) \cdot l + 1.$$

3. If $l \geq \max(a_o, \ldots, a_n)$.

4. Then $a_k < (k+1) \cdot l + 1$.

5. But by the previous proposition the numbers

$$(k+1) \ldots l + 1$$

   are pairwise relatively prime.

6. This implies that the simultaneous congruences

$$x \equiv a_o[\pmod{1 \cdot l + 1}]$$

$$\vdots$$

$$x \equiv a_n[\pmod{(n+1)\cdot l+1}]$$

have a common solution m, by the chinese remainder theorem.

7. Therefore

$$m \equiv a_k[\pmod{(k+1)\cdot l+1}].$$

8. This means that

$$a_k = R[m, (k+1)\cdot l+1].$$

9. But that is

$$a_k = \beta(m, l, k).$$

It is easily seen that the β function is primitive recursive.
For that we only have to redenominated m, l and k as:

$$\begin{array}{rcl} m & = & x_1 \\ l & = & x_2 \\ k & = & x_3. \end{array}$$

Then $\beta(x_1, x_2, x_3) = R[x_1, (x_3+1)\cdot x_2+1]$.
But the functions "$+$", "$\cdot$" e "R" are primitive recursive.
Therefore β is primitive recursive.

References

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[3] Kleene, S., *Introduction to metamathematics*, North-Holland, Amsterdam, 1964.