negative translation

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Synonym	double negative translation
Defines	Kolmogorov negative translation
Defines	Godel negative translation
Defines	Kuroda negative translation
Defines	Krivine negative translation

It is well-known that classical propositional logic PL_c can be considered as a subsystem of intuitionistic propositional logic PL_i by translating any wff A in PL_c into $\neg\neg A$ in PL_i. According to Glivenko's theorem, A is a theorem of PL_c iff $\neg\neg A$ is a theorem of PL_i. This translation, however, fails to preserve theoremhood in the corresponding predicate logics. For example, if A is of the form $\exists x B$, then $\vdash_c A$ no longer implies $\vdash_i \neg\neg A$. A number of translations have been devised to overcome this defect. They are collective known as *negative translations* or *double negative translations* of classical logic into intuitionistic logic. Below is a list of the most commonly mentioned negative translations:

- *Kolmogorov negative translation*:

  1. $p^\circ := \neg\neg p$ with p atomic and $\bot^\circ{:=}\bot$
  2. $(A \wedge B)^\circ := \neg\neg(A^\circ \wedge B^\circ)$
  3. $(A \vee B)^\circ := \neg\neg(A^\circ \vee B^\circ)$
  4. $(A \to B)^\circ := \neg\neg(A^\circ \to B^\circ)$
  5. $(\forall x A)^\circ := \neg\neg\forall x A^\circ$
  6. $(\exists x A)^\circ := \neg\neg\exists x A^\circ$

- *Godël negative translation*:

  1. $p^- := \neg\neg p$ with p atomic and $\bot^-{:=}\bot$
  2. $(A \wedge B)^- := A^- \wedge B^-$
  3. $(A \vee B)^- := \neg\neg(A^- \vee B^-)$
  4. $(A \to B)^- := A^- \to B^-$
  5. $(\forall x A)^- := \forall x A^-$
  6. $(\exists x A)^- := \neg\neg\exists x A^-$

- *Kuroda negative translation*:

  1. $p_u := p$ with p atomic and $\bot_u{:=}\bot$
  2. $(A \wedge B)_u := A_u \wedge B_u$
  3. $(A \vee B)_u := A_u \vee B_u$
  4. $(A \to B)_u := A_u \to B_u$
  5. $(\forall x A)_u := \forall x \neg\neg A_u$
  6. $(\exists x A)_u := \exists x A_u$

  And then $A^u := \neg\neg A_u$.

- *Krivine negative translation*:

1. $p_r := \neg p$ with p atomic and $\perp_r := \neg \perp$
2. $(A \wedge B)_r := A_r \vee B_r$
3. $(A \vee B)_r := A_r \wedge B_r$
4. $(A \to B)_r := \neg A_r \wedge B_r$
5. $(\forall x A)_r := \exists x A_r$
6. $(\exists x A)_r := \neg \exists x \neg A_r$

And then $A^r := \neg A_r$.

Remark. It can be shown that for any wff A:

$$\vdash_i A^* \leftrightarrow A^\# \qquad \text{and} \qquad \vdash_c A \quad \text{iff} \quad \vdash_i A^*$$

where $*, \# \in \{\circ, -, u, r\}$.