proof that forcing notions are equivalent to their composition

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This is a long and complicated proof, the more so because the meaning of Q shifts depending on what generic subset of P is being used. It is therefore broken into a number of steps. The core of the proof is to prove that, given any generic subset G of P and a generic subset H of $\hat{Q}[G]$ there is a corresponding generic subset $G*H$ of $P*Q$ such that $\mathfrak{M}[G][H] = \mathfrak{M}[G*H]$, and conversely, given any generic subset G of $P*Q$ we can find some generic G_P of P and a generic G_Q of $\hat{Q}[G_P]$ such that $\mathfrak{M}[G_P][G_Q] = \mathfrak{M}[G]$.

We do this by constructing functions using operations which can be performed within the forced universes so that, for example, since $\mathfrak{M}[G][H]$ has both G and H, $G*H$ can be calculated, proving that it contains $\mathfrak{M}[G*H]$. To ensure equality, we will also have to ensure that our operations are inverses; that is, given G, $G_P * G_H = G$ and given G and H, $(G*H)_P = P$ and $(G*H)_Q = H$.

The remainder of the proof merely defines the precise operations, proves that they give generic sets, and proves that they are inverses.

Before beginning, we prove a lemma which comes up several times:

Lemma: If G is generic in P and D is dense above some $p \in G$ then $G \cap D \neq \emptyset$

Let $D' = \{p' \in P \mid p' \in D \vee p' \text{ is incompatible with } p\}$. This is dense, since if $p_0 \in P$ then either p_0 is incompatible with p, in which case $p_0 \in D'$, or there is some p_1 such that $p_1 \leq p, p_0$, and therefore there is some $p_2 \leq p_1$ such that $p_2 \in D$, and therefore $p_2 \leq p_0$. So G intersects D'. But since a generic set is directed, no two elements are incompatible, so G must contain an element of D' which is not incompatible with p, so it must contain an element of D.

$G * H$ is a generic filter

First, given generic subsets G and H of P and $\hat{Q}[G]$, we can define:

$$G * H = \{\langle p, \hat{q}\rangle \mid p \in G \wedge \hat{q}[G] \in H\}$$

$G * H$ is closed

Let $\langle p_1, \hat{q}_1\rangle \in G*H$ and let $\langle p_1, \hat{q}_1\rangle \leq \langle p_2, \hat{q}_2\rangle$. Then we can conclude $p_1 \in G$, $p_1 \leq p_2$, $\hat{q}_1[G] \in H$, and $p_1 \Vdash \hat{q}_1 \leq \hat{q}_2$, so $p_2 \in G$ (since G is closed) and

$\hat{q}_2[G] \in H$ since $p_1 \in G$ and p_1 forces both $\hat{q}_1 \leq \hat{q}_2$ and that H is downward closed. So $\langle p_2, \hat{q}_2 \rangle \in G * H$.

$G * H$ is directed

Suppose $\langle p_1, \hat{q}_1 \rangle, \langle p_1, \hat{q}_1 \rangle \in G * H$. So $p_1, p_2 \in G$, and since G is directed, there is some $p_3 \leq p_1, p_2$. Since $\hat{q}_1[G], \hat{q}_2[G] \in H$ and H is directed, there is some $\hat{q}_3[G] \leq \hat{q}_1[G], \hat{q}_2[G]$. Therefore there is some $p_4 \leq p_3$, $p_4 \in G$, such that $p_4 \Vdash \hat{q}_3 \leq \hat{q}_1, \hat{q}_2$, so $\langle p_4, \hat{q}_3 \rangle \leq \langle p_1, \hat{q}_1 \rangle, \langle p_1, \hat{q}_1 \rangle$ and $\langle p_4, \hat{q}_3 \rangle \in G * H$.

$G * H$ is generic

Suppose D is a dense subset of $P * \hat{Q}$. We can project it into a dense subset of Q using G:

$$D_Q = \{\hat{q}[G] \mid \langle p, \hat{q} \rangle \in D\} \text{ for some } p \in G$$

Lemma: D_Q is dense in $\hat{Q}[G]$

Given any $\hat{q}_0 \in \hat{Q}$, take any $p_0 \in G$. Then we can define yet another dense subset, this one in G:

$$D_{\hat{q}_0} = \{p \mid p \leq p_0 \wedge p \Vdash \hat{q} \leq \hat{q}_0 \wedge \langle p, \hat{q} \rangle \in D\} \text{ for some } \hat{q} \in \hat{Q}$$

Lemma: $D_{\hat{q}_0}$ is dense above p_0 in P

Take any $p \in P$ such that $p \leq p_0$. Then, since D is dense in $P * \hat{Q}$, we have some $\langle p_1, \hat{q}_1 \rangle \leq \langle p, \hat{q}_0 \rangle$ such that $\langle p_1, \hat{q}_1 \rangle \in D$. Then by definition $p_1 \leq p$ and $p_1 \in D_{\hat{q}_0}$.

From this lemma, we can conclude that there is some $p_1 \leq p_0$ such that $p_1 \in G \cap D_{\hat{q}_0}$, and therefore some $\hat{q}_1$ such that $p_1 \Vdash \hat{q}_1 \leq \hat{q}_0$ where $\langle p_1, \hat{q}_1 \rangle \in D$. So D_Q is indeed dense in $\hat{Q}[G]$.

Since D_Q is dense in $\hat{Q}[G]$, there is some $\hat{q}$ such that $\hat{q}[G] \in D_Q \cap H$, and so some $p \in G$ such that $\langle p, \hat{q} \rangle \in D$. But since $p \in G$ and $\hat{q} \in H$, $\langle p, \hat{q} \rangle \in G * H$, so $G * H$ is indeed generic.

G_P is a generic filter

Given some generic subset G of $P * \hat{Q}$, let:

$$G_P = \{p \in P \mid p' \leq p \wedge \langle p', \hat{q} \rangle \in G\} \text{ for some } p' \in P \text{ and some } \hat{q} \in Q$$

G_P is closed

Take any $p_1 \in G_P$ and any p_2 such that $p_1 \leq p_2$. Then there is some $p' \leq p_1$ satisfying the definition of G_P, and also $p' \leq p_2$, so $p_2 \in G_P$.

G_P is directed

Consider $p_1, p_2 \in G_P$. Then there is some p'_1 and some $\hat{q}_1$ such that $\langle p'_1, \hat{q}_1 \rangle \in G$ and some p'_2 and some $\hat{q}_2$ such that $\langle p'_2, \hat{q}_2 \rangle \in G$. Since G is directed, there is some $\langle p_3, \hat{q}_3 \rangle \in G$ such that $\langle p_3, \hat{q}_3 \rangle \leq \langle p'_1, \hat{q}_1 \rangle, \langle p'_2, \hat{q}_2 \rangle$, and therefore $p_3 \in G_P$, $p_3 \leq p_1, p_2$.

G_P is generic

Let D be a dense subset of P. Then $D' = \{\langle p, \hat{q} \rangle \mid p \in D\}$. Clearly this is dense, since if $\langle p, \hat{q} \rangle \in P * \hat{Q}$ then there is some $p' \leq p$ such that $p' \in D$, so $\langle p', \hat{q} \rangle \in D'$ and $\langle p', \hat{q} \rangle \leq \langle p, \hat{q} \rangle$. So there is some $\langle p, \hat{q} \rangle \in D' \cap G$, and therefore $p \in D \cap G_P$. So G_P is generic.

G_Q is a generic filter

Given a generic subset $G \subseteq P * \hat{Q}$, define:

$$G_Q = \{\hat{q}[G_P] \mid \langle p, \hat{q} \rangle \in G\} \text{ for some } p \in P$$

(Notice that G_Q is dependant on G_P, and is a subset of $\hat{Q}[G_P]$, that is, the forcing notion inside $\mathfrak{M}[G_P]$, as opposed to the set of names Q which we've been primarily working with.)

G_Q is closed

Suppose $\hat{q}_1[G_P] \in G_Q$ and $\hat{q}_1[G_P] \leq \hat{q}_2[G_P]$. Then there is some $p_1 \in G_P$ such that $p_1 \Vdash \hat{q}_1 \leq \hat{q}_2$. Since $p_1 \in G_P$, there is some $p_2 \leq p_1$ such that for some $\hat{q}_3$, $\langle p_2, \hat{q}_3\rangle \in G$. By the definition of G_Q, there is some p_3 such that $\langle p_3, \hat{q}_1\rangle \in G$, and since G is directed, there is some $\langle p_4, \hat{q}_4\rangle \in G$ and $\langle p_4, \hat{q}_4\rangle \leq \langle p_3, \hat{q}_1\rangle, \langle p_2, \hat{q}_3\rangle$. Since G is closed and $\langle p_4, \hat{q}_4\rangle \leq \langle p_4, \hat{q}_2\rangle$, we have $\hat{q}_2[G_P] \in G_Q$.

G_Q is directed

Suppose $\hat{q}_1[G_P], \hat{q}_2[G_P] \in G_Q$. Then for some p_1, p_2, $\langle p_1, \hat{q}_1\rangle, \langle p_2, \hat{q}_2\rangle \in G$, and since G is directed, there is some $\langle p_3, \hat{q}_3\rangle \in G$ such that $\langle p_3, \hat{q}_3\rangle \leq \langle p_1, \hat{q}_1\rangle, \langle p_2, \hat{q}_2\rangle$. Then $\hat{q}_3[G_P] \in G_Q$ and since $p_3 \in G$ and $p_3 \Vdash \hat{q}_3 \leq \hat{q}_1, \hat{q}_2$, we have $\hat{q}_3[G_P] \leq \hat{q}_1[G_P], \hat{q}_2[G_P]$.

G_Q is generic

Let D be a dense subset of $Q[G_P]$ (in $\mathfrak{M}[G_P]$). Let $\hat{D}$ be a P-name for D, and let $p_1 \in G_P$ be a such that $p_1 \Vdash \hat{D}$ `is dense`. By the definition of G_P, there is some $p_2 \leq p_1$ such that $\langle p_2, \hat{q}_2\rangle \in G$ for some q_2. Then $D' = \{\langle p, \hat{q}\rangle \mid p \Vdash \hat{q} \in D \wedge p \leq p_2\}$.

Lemma: D' is dense (in G) above $\langle p_2, \hat{q}_2\rangle$

Take any $\langle p, \hat{q}\rangle \in P * Q$ such that $\langle p, \hat{q}\rangle \leq \langle p_2, \hat{q}_2\rangle$. Then $p \Vdash \hat{D}$ `is dense`, and therefore there is some $\hat{q}_3$ such that $p \Vdash \hat{q}_3 \in \hat{D}$ and $p \Vdash \hat{q}_3 \leq \hat{q}$. So $\langle p, \hat{q}_3\rangle \leq \langle p, \hat{q}\rangle$ and $\langle p, \hat{q}_3\rangle \in D'$.

Take any $\langle p_3, \hat{q}_3\rangle \in D' \cap G$. Then $p_3 \in G_P$, so $\hat{q}_3 \in D$, and by the definition of G_Q, $\hat{q}_3 \in G_Q$.

$G_P * G_Q = G$

If G is a generic subset of $P * Q$, observe that:

$$G_P * G_Q = \{\langle p, \hat{q}\rangle \mid p' \leq p \wedge \langle p', \hat{q}'\rangle \in G \wedge \langle p_0, \hat{q}\rangle \in G\} \text{ for some } p', \hat{q}', p_0$$

If $\langle p, \hat{q} \rangle \in G$ then obviously this holds, so $G \subseteq G_P * G_Q$. Conversely, if $\langle p, \hat{q} \rangle \in G_P * G_Q$ then there exist $p', \hat{q}'$ and p_0 such that $\langle p', \hat{q}' \rangle, \langle p_0, \hat{q} \rangle \in G$, and since G is directed, some $\langle p_1, \hat{q}_1 \rangle \in G$ such that $\langle p_1, \hat{q}_1 \rangle \leq \langle p', \hat{q}' \rangle, \langle p_0, \hat{q} \rangle$. But then $p_1 \leq p$ and $p_1 \Vdash \hat{q}_1 \leq \hat{q}$, and since G is closed, $\langle p, \hat{q} \rangle \in G$.

$(G * H)_P = G$

Assume that G is generic in P and H is generic in $Q[G]$.

Suppose $p \in (G * H)_P$. Then there is some $p' \in P$ and some $\hat{q} \in Q$ such that $p' \leq p$ and $\langle p', \hat{q} \rangle \in G * H$. By the definition of $G * H$, $p' \in G$, and then since G is closed $p \in G$.

Conversely, suppose $p \in G$. Then (since H is non-trivial), $\langle p, \hat{q} \rangle \in G * H$ for some $\hat{q}$, and therefore $p \in (G * H)_P$.

$(G * H)_Q = H$

Assume that G is generic in P and H is generic in $Q[G]$.

Given any $q \in H$, there is some $\hat{q} \in Q$ such that $\hat{q}[G] = q$, and so there is some p such that $\langle p, \hat{q} \rangle \in G * H$, and therefore $\hat{q}[G] \in H$.

On the other hand, if $q \in (G * H)_Q$ then there is some $\langle p, \hat{q} \rangle \in G * H$, and therefore some $\hat{q}[G] \in H$.