example of uncountable family of subsets of a countable set with finite intersections

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We wish to give an answer to the following:

Problem. Assume, that X is a countable set. Is there a family $\{X_i\}_{i \in I}$ of subsets of X such that I is an uncountable set, but for any $i \neq j \in I$ the intersection $X_i \cap X_j$ is finite?

Example. Let $x \in [1, 2)$ be a real number. Express x using digits

$$x = 1.x_1x_2x_3x_4 \cdots = 1 + \sum_{i=1}^{\infty} x_i \cdot 10^{-i}$$

where each $x_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. With x we associate the following natural numbers

$$\beta_n(x) = 1x_1x_2x_3 \cdots x_{n-1}x_n = 10^{n+1} + \sum_{i=1}^{n} x_i \cdot 10^{n-i+1}.$$

Now define $A : [1, 2) \to \mathrm{P}(\mathbb{N})$ (here $\mathrm{P}(X)$ stands for ,,the power set of X") by

$$A(x) = \{\beta_1(x), \beta_2(x), \beta_3(x), \ldots\}.$$

A is injective. Indeed, note that for any $x, y \in [1, 2)$ if $\beta_i(x) = \beta_j(y)$, then $i = j$ (this is because equal β numbers have equal ,,length" and this is because each β has 1 at the begining, zeros are not the problem). Therefore, if $A(x) = A(y)$ for some x, y, then it follows, that $\beta_i(x) = \beta_i(y)$ for each i, but this implies that corresponding digits of x and y are equal. Thus $x = y$.

This shows, that $\{A(x)\}_{x \in [1,2)}$ is an uncountable family of subsets of $\mathbb{N}$. Now in order to prove that $A(x) \cap A(y)$ is finite whenever $x \neq y$ it is enough to show that we can uniquely reconstruct x from any infinite sequence of numbers from $A(x)$. This can be proved by using similar techniques as before and we leave it as a simple exercise.