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example of monadic algebra

Canonical name	ExampleOfMonadicAlgebra
Date of creation	2013-03-22 17:51:55
Last modified on	2013-03-22 17:51:55
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Example
Classification	msc 03G15
Defines	functional monadic algebra

The canonical example of a monadic algebra is what is known as a *functional monadic algebra*, which is explained in this entry.

Let A be a Boolean algebra and X be a non-empty set. Then A^X, the set of all functions from X into A, has a natural Boolean algebraic structure defined as follows:

$$(f \wedge g)(x) := f(x) \wedge g(x), \qquad (f')(x) := f(x)', \qquad 1(x) = 1$$

where $f, g : X \to A$ are functions, and $1 : X \to A$ is just the constant function mapping everything to $1 \in A$ (the abuse of notation here is harmless).

For each $f : X \to A$, let $f(X) \subseteq A$ be the range of f. Let B be the subset of A^X consisting of all functions f such that $\bigvee f(X)$ and $\bigwedge f(X)$ exist, where $\bigvee$ and $\bigwedge$ are the infinite join and infinite meet operations on A. In other words,

$$B := \{f \in A^X \mid \bigvee f(X) \in A \text{ and } \bigwedge f(X) \in A\}.$$

Proposition 1. *B defined above is a Boolean subalgebra of A^X.*

Proof. We need to show that, (1): $1 \in B$, (2): for any $f \in B$, $f' \in B$, and (3): for any $f, g \in B$, $f \wedge g \in B$.

1. $\bigvee 1(X) = \bigvee\{1\} = 1$ and $\bigwedge 1(X) = \bigwedge\{1\} = 1$ so $1 \in B$
2. Suppose $f \in B$. Then $\bigvee f'(X) = \bigvee\{f'(x) \mid x \in X\} = \bigvee\{f(x)' \mid x \in X\}$. By de Morgan's law on infinite joins, the last expression is $(\bigwedge\{f(x) \mid x \in X\})'$, which exists. Dually, $\bigwedge f'(X)$ exists by de Morgan's law on infinite meets. Therefore, $f' \in B$.
3. Suppose $f, g \in B$. Then

$$\begin{aligned} \bigwedge(f \wedge g)(X) &= \bigwedge\{f(x) \wedge g(x) \mid x \in X\} \\ &= \bigwedge\{f(x) \mid x \in X\} \wedge \bigwedge\{g(x) \mid x \in X\} \\ &= \bigwedge f(X) \wedge \bigwedge g(X), \end{aligned}$$

   which exists because both $\bigwedge f(X)$ and $\bigwedge g(X)$ do. In addition,

$$\bigvee(f \wedge g)(X) = \bigvee\{f(x) \wedge g(x) \mid x \in X\} = \bigvee f(X) \wedge \bigvee g(X).$$

   The last equality stems from the distributive law of infinite meets over finite joins. Since the last expression exists, $f \wedge g \in B$.

The three conditions are verified and the proof is complete. $\square$

Remark. Every constant function belongs to B.

For each $f \in B$, write $f^{\vee} := \bigvee f(X)$ and $f^{\wedge} := \bigwedge f(X)$. Define two functions $f^{\exists}, f^{\forall} \in A^X$ by

$$f^{\exists}(x) := f^{\vee} \qquad \text{and} \qquad f^{\forall}(x) := f^{\wedge}.$$

Since these are constant functions, they belong to B.

Now, we define operators $\exists, \forall$ on B by setting

$$\exists(f) := f^{\exists} \qquad \text{and} \qquad \forall(f) := f^{\forall}.$$

By the remark above, $\exists$ and $\forall$ are well-defined functions on B ($f^{\exists}, f^{\forall} \in B$).

Proposition 2. *$\exists$ is an existential quantifier operator on B and $\forall$ is its dual.*

Proof. The following three conditions need to be verified:

- $\exists(0) = 0$: $\exists(0)(x) = 0^{\exists}(x) = 0^{\vee} = \bigvee 0(X) = \bigvee 0 = 0$.
- $f \leq \exists(f)$: $f(x) \leq \bigvee f(X) = f^{\vee} = f^{\exists}(x) = \exists(f)(x)$.
- $\exists(f \wedge \exists(g)) = \exists(f) \wedge \exists(g)$:

$$\begin{aligned}
\exists(f \wedge \exists(g))(x) &= \bigvee \big(f \wedge \exists(g)\big)(X) = \bigvee \big(f(X) \wedge \exists(g)(X)\big) \\
&= \bigvee \big(f(X) \wedge \exists(g)(x)\big) = \bigvee \big(f(X) \wedge \bigvee g(X)\big) \\
&= \bigvee f(X) \wedge \bigvee g(X) = \exists(f)(x) \wedge \exists(g)(x) = \big(\exists(f) \wedge \exists(g)\big)(x).
\end{aligned}$$

Finally, to see that $\forall$ is the dual of $\exists$, we do the following computations:

$$\begin{aligned}
\forall(f)(x) &= \bigwedge\{f(x) \mid x \in X\} = \bigwedge\{f(x)'' \mid x \in X\} \\
&= \big(\bigvee\{f(x)' \mid x \in X\}\big)' = \big(\bigvee\{f'(x) \mid x \in X\}\big)' = \big(\exists(f')\big)'(x),
\end{aligned}$$

completing the proof. $\square$

Based on Propositions 1 and 2, $(B, \exists)$ is a monadic algebra, and is called the *functional monadic algebra* for the pair (A, X).