proof of delta system lemma

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Since there are only $\aleph_0$ possible cardinalities for any element of S, there must be some n such that there are an uncountable number of elements of S with cardinality n. Let $S^* = \{a \in S \mid |a| = n\}$ for this n. By induction, the lemma holds:

If $n = 1$ then there each element of S^* is distinct, and has no intersection with the others, so $X = \emptyset$ and $S' = S^*$.

Suppose $n > 1$. If there is some x which is in an uncountable number of elements of S^* then take $S^{**} = \{a \setminus \{x\} \mid x \in a \in S^*\}$. Obviously this is uncountable and every element has $n - 1$ elements, so by the induction hypothesis there is some $S' \subseteq S^{**}$ of uncountable cardinality such that the intersection of any two elements is X. Obviously $\{a \cup \{x\} \mid a \in S'\}$ satisfies the lemma, since the intersection of any two elements is $X \cup \{x\}$.

On the other hand, if there is no such x then we can construct a sequence $\langle a_i \rangle_{i<\omega_1}$ such that each $a_i \in S^*$ and for any $i \neq j$, $a_i \cap a_j = \emptyset$ by induction. Take any element for a_0, and given $\langle a_i \rangle_{i<\alpha}$, since α is countable, $A = \bigcup_{i<\alpha} a_i$ is countable. Obviously each element of A is in only a countable number of elements of S^*, so there are an uncountable number of elements of S^* which are candidates for a_α. Then this sequence satisfies the lemma, since the intersection of any two elements is $\emptyset$.