rational transducer

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Definition

A *rational transducer* is a generalization of a generalized sequential machine (gsm). Recall that a gsm M is a quadruple $(S, \Sigma, \Delta, \tau)$ where S is a finite set of states, Σ and Δ are the input and output alphabets respectively, and τ is the transition function taking an input symbol from one state to an output word in another state. A rational transducer has all of the components above, except that the transition function τ is more general: its domain consists of a pair of a state and an input word, rather than an input symbol.

Formally, a *rational transducer* M is a quadruple $(S, \Sigma, \Delta, \tau)$ where S, Σ, Δ are defined just as those in a gsm, except that the transition function τ has domain a finite subset of $S \times \Sigma^*$ such that $\tau(s, u)$ is finite for each $(s, u) \in \mathrm{dom}(\tau)$. One can think of τ as a finite subset of $S \times \Sigma^* \times S \times \Delta^*$, or equivalently a finite relation between $S \times \Sigma^*$ and $S \times \Delta^*$.

Like a gsm, a rational transducer can be turned into a language acceptor by fixing an initial state $s_0 \in S$ and a non-empty set F of finite states $F \subseteq S$. In this case, a rational transducer turns into a 6-tuple $(S, \Sigma, \Delta, \tau, s_0, F)$. An input configuration (s_0, u) is said to be *initial*, and an output configuration (t, v) is said to be *final* if $t \in F$. The language accepted by a rational transducer M is defined as the set

$$L(M) := \{u \in \Sigma^* \mid \tau(s_0, u) \text{ contains an final output configuration.}\}.$$

Rational Transductions

Additionally, like a gsm, a rational transducer can be made into a language translator. The initial state s_0 and the set F of final states are needed. Given a rational transducer M, for every input word u, let

$$\mathrm{RT}_M(u) := \{v \in \Delta^* \mid (t, v) \in \tau(s_0, u) \text{ is a final output configuration.}\}.$$

Thus, RT_M is a function from Σ^* to $P(\Delta^*)$, and is called the *rational transduction* (defined) for the rational transducer M. The rational transduction for M can be extended: for any language L over the input alphabet Σ,

$$\mathrm{RT}_M(L) := \bigcup\{\mathrm{RT}_M(u) \mid u \in L\}.$$

In this way, RT_M may be thought of as a language translator.

As with GSM mappings, one can define the inverse of a rational transduction, given a rational transduction RT_M:

$$\mathrm{RT}_M^{-1}(v) := \{u \mid v \in \mathrm{RT}_M(u)\} \quad \text{and} \quad \mathrm{RT}_M^{-1}(L) := \bigcup \{\mathrm{RT}_M^{-1}(v) \mid v \in L\}.$$

Here are some examples of rational transductions

- Every GSM mapping is clearly a rational transduction, since every gsm is a rational transducer. As a corollary, any homomorphism, as well as intersection with any regular language, is a rational transduction.

- The inverse of a rational transduction is a transduction. Given any rational transducer $M = (S, \Sigma, \Delta, \tau, s_0, F)$, define a rational transducer $M' = (S', \Delta, \Sigma, \tau', t_0, F')$ as follws: $S' = S \mathbin{\dot\cup} \{t_0\}$ (where $\dot\cup$ denotes disjoint union), $F' = \{s_0\}$, and $\tau' \subseteq S \times \Delta^* \times S \times \Sigma^*$ is given by

  $$\tau'(t, v) = \begin{cases} \{(s, v) \mid (s, v) \text{ is a final output configuration of } M\} & \text{if } t = t_0 \\ \{(s, u) \mid (t, v) \in \tau(s, u)\} & \text{otherwise.} \end{cases}$$

  As τ is finite, so is τ', so that M' is well-defined. In addition, $\mathrm{RT}_{M'} = \mathrm{RT}_M^{-1}$. As a corollary, the inverse homomorphism is a rational transduction.

- The composition of two rational transductions is a rational transduction. To see this, suppose $M_1 = (S_1, \Sigma_1, \Delta_1, \tau_1, s_1, F_1)$ and $M_2 = (S_2, \Sigma_2, \Delta_2, \tau_2, s_2, F_2)$ are two rational transducers such that $\Delta_1 \subseteq \Sigma_2$. Define $M = (S, \Sigma_1, \Delta_2, \tau, s_1, F_2)$ as follows: $S = S_1 \mathbin{\dot\cup} S_2$, and $\tau \subseteq S \times \Sigma_1^* \times S \times \Delta_2^*$ is given by

  $$\tau(s, u) = \begin{cases} \tau_1(s, u) & \text{if } (s, u) \in S_1 \times \Sigma_1^* \\ \tau_2(s, u) & \text{if } (s, u) \in S_2 \times \Sigma_2^* \\ \{(s_2, u)\} & \text{if } (s, u) \text{ is a final output configuration of } M_1 \\ \varnothing & \text{otherwise.} \end{cases}$$

  Again, since both τ_1 and τ_2 are finite, so is τ, and thus M well-defined. In addition, $\mathrm{RT}_M = \mathrm{RT}_{M_2} \circ \mathrm{RT}_{M_1}$.

A family $\mathscr{F}$ of languages is said to be closed under rational transduction if for every $L \in \mathscr{F}$, and any rational transducer M, we have $\mathrm{RT}_M(L) \in \mathscr{F}$. The three properties above show that if $\mathscr{F}$ is closed under rational transduction, it is a cone. The converse is also true, as it can be shown that every rational transduction can be expressed as a composition of inverse homomorphism, intersection with a regular language, and homomorphism. Thus, a family of languages being closed under rational transduction completely characterizes a cone.

References

[1] A. Salomaa *Computation and Automata, Encyclopedia of Mathematics and Its Applications, Vol. 25.* Cambridge (1985).