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complete Heyting algebra

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A Heyting algebra that is also a complete lattice is called a complete Heyting algebra. In the following, we give a lattice characterization of complete Heyting algebras without the relative pseudocomplementation operator $\to$.

Proposition 1. *Let H be a complete Heyting algebra, then*

$$x \to y = \bigvee \{z \mid z \wedge x \leq y\}$$

for any $x, y \in H$, and

$$x \wedge \bigvee A = \bigvee (x \wedge A),$$

where $x \wedge A := \{x \wedge y \mid y \in A\}$, for any $x \in H$, and any subset A of H.

Proof. We first prove the identity $x \wedge \bigvee A = \bigvee(x \wedge A)$. For any $y \in H$, we have $x \wedge \bigvee A \leq y$ iff $\bigvee A \leq x \to y$ iff $a \leq x \to y$ for all $a \in A$ iff $x \wedge a \leq y$ for all $a \in A$ iff $\bigvee(x \wedge A) \leq y$, hence $x \wedge \bigvee A = \bigvee(x \wedge A)$.

Next, we show $x \to y = \bigvee\{z \mid z \wedge x \leq y\}$. For any $a \in H$, we have $a \leq x \to y$ iff $a \wedge x \leq y$ iff $a \in \{z \mid z \wedge x \leq y\}$ iff $a \leq \bigvee\{z \mid z \wedge x \leq y\}$. $\square$

The converse of the above is also true.

Proposition 2. *Let H be a complete lattice such that*

$$x \wedge \bigvee A = \bigvee (x \wedge A),$$

where $x \wedge A := \{x \wedge y \mid y \in A\}$, for any $x \in H$, and any subset A of H. Then for any $x, y \in H$, defining

$$x \to y := \bigvee \{z \mid z \wedge x \leq y\}$$

turns H into a complete Heyting algebra.

Proof. We want to show that $a \leq x \to y$ iff $a \wedge x \leq y$ for any $a \in H$: $a \leq x \to y$ iff $a \leq \bigvee\{z \mid z \wedge x \leq y\}$. So $x \wedge a \leq x \wedge \bigvee\{z \mid z \wedge x \leq y\} = \bigvee\{x \wedge z \mid z \wedge x \leq y\} \leq \bigvee\{y\} = y$ $\square$

From this, one readily concludes that any finite distributive lattice is Heyting.

Remark. Since any complete lattice is bounded, a complete Brouwerian lattice is a complete Heyting algebra. A complete Heyting algebra is also called a *frame*.