properties of Ackermann function

Canonical name	PropertiesOfAckermannFunction
Date of creation	2013-03-22 19:07:25
Last modified on	2013-03-22 19:07:25
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	9
Author	CWoo (3771)
Entry type	Result
Classification	msc 03D75

In this entry, we derive some basic properties of the Ackermann function $A(x,y)$, defined by double recursion, as follows:

$$A(0,y) = y+1, \quad A(x+1,0) = A(x,1), \quad A(x+1,y+1) = A(x, A(x+1,y)).$$

These properties will be useful in proving that A is not primitive recursive.

1. A is total ($\mathrm{dom}(A) = \mathbb{N}^2$).
2. $A(1,y) = y+2$.
3. $A(2,y) = 2y+3$.
4. $y < A(x,y)$.
5. $A(x,y) < A(x,y+1)$.
6. $A(x,y+1) \leq A(x+1,y)$.
7. $A(x,y) < A(x+1,y)$.
8. $A(r, A(s,y)) < A(r+s+2,y)$
9. For any r,s, $A(r,y) + A(s,y) < A(t,y)$ for some t not depending on y.

Most of the proofs are done by induction.

Proof.

1. Induct on x. First, $A(0,y) = y+1$ is well-defined, so $(0,y) \in \mathrm{dom}(A)$ for all y. Next, suppose that for a given x, $(x,y) \in \mathrm{dom}(A)$ for all y. We want to show that $(x+1,y) \in \mathrm{dom}(A)$ for all y. To do this, induct on y. First, $(x+1,0) \in \mathrm{dom}(A)$, since $A(x+1,0) = A(x,1)$ is well-defined. Next, assume that $(x+1,y) \in \mathrm{dom}(A)$. Then $A(x, A(x+1,y)) = A(x+1,y+1)$ is well-defined. so $(x+1,y+1) \in \mathrm{dom}(A)$ as well.

2. Induct on y. First, $A(1,0) = A(0,1) = 2$. Next, assume $A(1,y) = y+2$. Then $A(1,y+1) = A(0,y+2) = y+3 = (y+1)+2$.

3. Induct on y. First, $A(2,0) = A(1,1) = 1+2 = 3$. Next, assume $A(2,y) = 2y+3$. Then $A(2,y+1) = A(1,A(2,y)) = A(2,y)+2 = (2y+3)+2 = 2(y+1)+3$.

4. Induct on x. First, $y < y+1 = A(0,y)$. Next, assume $y < A(x,y)$, where $x > 0$. Then $y+1 \leq A(x,y) < A(x-1, A(x,y)) = A(x, y+1)$.

5. Induct on x. First, $A(0,y) = y+1 < y+2 = A(0,y+1)$. Next, assume that $A(x,y) < A(x,y+1)$. Then $A(x+1,y) < A(x, A(x+1,y)) = A(x+1,y+1)$.

6. Induct on y. First, $A(x,1) = A(x+1,0)$. Next, assume that $A(x,y+1) \leq A(x+1,y)$. Then $A(x,y+2) \leq A(x, A(x,y+1)) \leq A(x, A(x+1,y)) = A(x+1,y+1)$.

7. Induct on x. First, $A(0,y) = y+1 < y+2 = A(1,y)$. Next, assume that $A(x,y) < A(x+1,y)$. There are two cases: $y = 0$. Then $A(x+1,0) = A(x,1) < A(x+1,1)$. Otherwise, $y = z+1$, so that $A(x+1,y) = A(x+1,z+1) = A(x, A(x+1,z)) \leq A(x, A(x,z+1)) = A(x, A(x,y)) < A(x, A(x+1,y)) = A(x+1,y+1)$.

8. $A(r, A(s,y)) < A(r+s, A(s,y)) < A(r+s, A(r+s+1,y)) = A(r+s+1, y+1) \leq A(r+s+2, y)$.

9. Let $z = \max\{r,s\}$. Then $A(r,y) + A(s,y) \leq 2A(z,y) < 2A(z,y)+3 = A(2, A(z,y)) < A(4+z, y)$. The proof is completed by setting $t = 4+z$. □

With respect to the recursive property of A, we see that A is recursive, since, by Church's Thesis, A can be effectively computed (in fact, a program can be easily written to compute $A(x,y)$). We also have the following:

Proposition 1. *Define $A_n : \mathbb{N} \to \mathbb{N}$ by $A_n(m) = A(n,m)$. Then A_n is primitive recursive for each n.*

Proof. $A_0(m) = m+1 = s(m)$, so is primitive recursive. Now, assume A_n is primitive recursive, then $A_{n+1}(0) = A(n+1,0) = A(n,1) = A_n(1) = k$, and $A_{n+1}(m+1) = A(n+1,m+1) = A(n, A(n+1,m)) = A_n(A(n+1,m)) = A_n(A_{n+1}(m))$, so that A_{n+1} is defined by primitive recursion via the constant function const_k, and A_n, which is primitive recursive by the induction hypothesis. Therefore A_{n+1} is primitive recursive also. □

The most important fact about A concerning recursiveness is that A is not primitive recursive. Due to the length of its proof, it is demonstrated in http://planetmath.org/AckermannFunctionIsNotPrimitiveRecursivethis entry.