proof of partial order with chain condition does not collapse cardinals

Canonical name	ProofOfPartialOrderWithChainConditionDoesNotCollapseCardinals
Date of creation	2013-03-22 12:53:43
Last modified on	2013-03-22 12:53:43
Owner	Henry (455)
Last modified by	Henry (455)
Numerical id	4
Author	Henry (455)
Entry type	Proof
Classification	msc 03E35

Outline:

Given any function f purporting to violate the theorem by being surjective (or cofinal) on λ, we show that there are fewer than κ possible values of $f(\alpha)$, and therefore only $\max(\alpha, \kappa)$ possible elements in the entire range of f, so f is not surjective (or cofinal).

Details:

Suppose $\lambda > \kappa$ is a cardinal of $\mathfrak{M}$ that is not a cardinal in $\mathfrak{M}[G]$.

There is some function $f \in \mathfrak{M}[G]$ and some cardinal $\alpha < \lambda$ such that $f : \alpha \to \lambda$ is surjective. This has a name, $\hat{f}$. For each $\beta < \alpha$, consider

$$F_\beta = \{\gamma < \lambda \mid p \Vdash \hat{f}(\beta) = \gamma\} \text{ for some } p \in P$$

$|F_\beta| < \kappa$, since any two $p \in P$ which force different values for $\hat{f}(\beta)$ are incompatible and P has no sets of incompatible elements of size κ.

Notice that F_β is definable in $\mathfrak{M}$. Then the range of f must be contained in $F = \bigcup_{i<\alpha} F_i$. But $|F| \leq \alpha \cdot \kappa = \max(\alpha, \kappa) < \lambda$. So f cannot possibly be surjective, and therefore λ is not collapsed.

Now suppose that for some $\alpha \geq \lambda > \kappa$, $\mathrm{cf}(\alpha) = \lambda$ in $\mathfrak{M}$ and for some $\eta < \lambda$ there is a cofinal function $f : \eta \to \alpha$.

We can construct F_β as above, and again the range of f is contained in $F = \bigcup_{i<\eta} F_i$. But then $|\operatorname{range}(f)| \leq |F| \leq \eta \cdot \kappa < \lambda$. So there is some $\gamma < \alpha$ such that $f(\beta) < \gamma$ for any $\beta < \eta$, and therefore f is not cofinal in α.