well-foundedness and axiom of foundation

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Recall that a relation R on a class C is well-founded if

1. For any $x \in C$, the collection $\{y \in C \mid yRx\}$ is a set, and
2. for any non-empty $B \subseteq C$, there is an element $z \in B$ such that if yRz, then $y \notin B$.

z is called an R-minimal element of B. It is clear that the membership relation $\in$ in the class of all sets satisfies the first condition above.

Theorem 1. *Given ZF, $\in$ is a well-founded relation iff the Axiom of Foundation (AF) is true.*

We will prove this using one of the equivalent versions of AF: for every non-empty set A, there is an $x \in A$ such that $x \cap A = \varnothing$.

Proof. Suppose $\in$ is well-founded and A a non-empty set. We want to find $x \in A$ such that $x \cap A = \varnothing$. Since $\in$ is well-founded, there is a $\in$-minimal set x such that $x \in A$. Since no set y such that $y \in x$ and $y \in A$ (otherwise x would not be $\in$-minimal), we have that $x \cap A = \varnothing$.

Conversely, suppose that AF is true. Let A be any non-empty set. We want to find a $\in$-minimal element in A. Let $x \in A$ such that $x \cap A = \varnothing$. Then x is $\in$-minimal in A, for otherwise there is $y \in A$ such that $y \in x$, which implies $y \in x \cap A = \varnothing$, a contradiction. $\square$