primitive recursive vector-valued function

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Recall that a primitive recursive function is an operation on $\mathbb{N}$, the set of all natural numbers (0 included here). In other words, it is a mapping into the set $\mathbb{N}$, mimicking a computation whose outcome has only one value. However, it is possible to extend this notion, so that a computation yields an outcome with several values simultaneously (imagine a URM where the contents of the output are found in first n tape cells). This is equivalent to a mapping $\mathbb{N}^m \to \mathbb{N}^n$ (provided with m inputs).

For this entry, $\mathcal{S} := \{f : \mathbb{N}^m \to \mathbb{N} \mid \text{ any } m \geq 1\}$, $\mathcal{V} := \{f : \mathbb{N}^m \to \mathbb{N}^n \mid \text{ any } m, n \geq 1\}$, and for specific $m, n \geq 1$, we set $\mathcal{V}(m, n) := \{f : \mathbb{N}^m \to \mathbb{N}^n\}$.

Definition. A function $f \in \mathcal{V}$ is *primitive recursive* iff each of its components is. In other words, if $f = (f_1, \ldots, f_n)$, where each $f_i : \mathbb{N}^m \to \mathbb{N}$, then f is primitive recursive if each f_i is (in the traditional sense).

It is evident that primitive recursiveness in this generalized version is closed under composition: if $f \in \mathcal{V}(m, n)$ and $g \in \mathcal{V}(n, p)$ are primitive recursive, then so is $g \circ f$.

In addition, it is also closed under iterated composition:

Proposition 1. *If $f \in \mathcal{V}(n, n)$ is primitive recursive, then so is the function $g \in \mathcal{V}(n+1, n)$ such that*

$$g(\boldsymbol{x}, 0) = \boldsymbol{x} \qquad \textit{and} \qquad g(\boldsymbol{x}, n) = f^n(\boldsymbol{x}).$$

The technique of mutual recursion is used to prove this fact.

Proof. Suppose $f = (f_1, \ldots, f_n)$ with each $f_i \in \mathcal{V}(n, 1)$, and $g = (g_1, \ldots, g_n)$ with each $g_i \in \mathcal{V}(n+1, 1)$. Then

$$f^n(\boldsymbol{x}) = (g_1(\boldsymbol{x}, n), \ldots, g_n(\boldsymbol{x}, n)).$$

From this we compute

$$\begin{aligned} & (g_1(\boldsymbol{x}, n+1), \ldots, g_n(\boldsymbol{x}, n+1)) \\ = \quad & f^{n+1}(\boldsymbol{x}) = f(f^n(\boldsymbol{x})) \\ = \quad & (f_1(g_1(\boldsymbol{x}, n), \ldots, g_n(\boldsymbol{x}, n)), \ldots, f_n(g_1(\boldsymbol{x}, n), \ldots, g_n(\boldsymbol{x}, n))). \end{aligned}$$

So $g_i(\boldsymbol{x}, n+1) = f_i(g_1(\boldsymbol{x}, n), \ldots, g_n(\boldsymbol{x}, n))$. This shows that the functions g_i are defined by mutual recursion via the functions f_i and the identity function p_1^1. Since each of the f_i is primitive recursive, so is each of the g_i, and hence so is g. □

Example. We give an alternative proof that the function $F(n)$ denoting the n-th Fibonacci number is primitive recursive.

Let

$$f(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = (x, x + y).$$

Clearly, f is primitive recursive. Now define $g(x, y, n)$ such that $g(x, y, 0) = (x, y)$ and $g(x, y, n) = f^n(x, y)$. By what we have just shown, g is primitive recursive too. Since

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} F(n-1) & F(n) \\ F(n) & F(n+1) \end{bmatrix}$$

for any $n > 1$, we see that

$$g(x, y, n) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} F(n-1) & F(n) \\ F(n) & F(n+1) \end{bmatrix} = (xF(n-1)+yF(n), xF(n)+yF(n+1)).$$

Therefore, the function $h(x, y, n) = xF(n)+yF(n+1)$ is primitive recursive, and hence $F(n) = h(1, 0, n)$ is primitive recursive also.

The example above is but a more general phenomenon: the operation of matrix multiplication is primitive recursive. To see what this means, we define:

Definition. Let $M(\boldsymbol{x})$ be an $m \times n$ matrix whose cells $M_{ij}(\boldsymbol{x}) \in \mathcal{V}(k, 1)$. We say that $M(\boldsymbol{x})$ is *primitive recursive* if each of its cells is primitive recursive.

If $M(\boldsymbol{x})$ and $N(\boldsymbol{x})$ are $m \times n$ and $n \times p$ primitive recursive matrices, then clearly so is their product $M(\boldsymbol{x})N(\boldsymbol{x})$, for each of its cell is just a finite sum of products of primitive recursive functions. More over, we have the follow:

Proposition 2. *If $M(\boldsymbol{x})$ is an $m \times m$ primitive recursive matrix, so is its n-fold power $M(\boldsymbol{x})^n$, where each cell is a function of $\boldsymbol{x}$ and n.*

Proof. We employ the same trick given in the example. Define, for any $\boldsymbol{y} \in \mathbb{N}^m$, the function $f(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{y}M(\boldsymbol{x})$, where the right hand side is product of the matrices $\boldsymbol{y}$ (considered as a $1 \times m$ matrix) and $M(\boldsymbol{x})$. So f is primitive recursive because M is. Next, define $g(\boldsymbol{x}, \boldsymbol{y}, n)$ such that $g(\boldsymbol{x}, \boldsymbol{y}, 0) = (\boldsymbol{x}, \boldsymbol{y})$, and $g(\boldsymbol{x}, \boldsymbol{y}, n) = f^n(\boldsymbol{x}, \boldsymbol{y})$. Then g is primitive recursive by proposition 1 above. But then $g(\boldsymbol{x}, \boldsymbol{y}, n)$ is just $\boldsymbol{y}M(\boldsymbol{x})^n$. Applying $\boldsymbol{y} = (1, 0, \ldots, 0)$ to g, we get $g(\boldsymbol{x}, 1, \ldots, 0, n)$, which is just the first row of $M(\boldsymbol{x})^n$, and is clearly primitive recursive. By the same token, the other rows are primitive recursive too, completing the proof. $\square$