properties of substitutability

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In this entry, we list some basic properties of substitutability in first order logic with respect to commutativity.

Proposition 1. *If x, y are distinct variables, then*

$$t[s/x][r/y] = t[r/y][s[r/y]/x],$$

provided that x does not occur in r.

Proof. Suppose x and y are distinct variables, and r, s, t are terms. We do induction on the complexity of t.

1. First, suppose t is a constant symbol. Then LHS and RHS are both t.

2. Next, suppose t is a variable.

   - If t is x, then LHS $= s[r/y]$, and since y is not x, RHS $= x[s[r/y]/x] = s[r/y]$.
   - If t is y, then LHS $= y[r/y] = r$, since x is not y, and RHS $= r[s[r/y]/x] = r$, since x does not occur in r.
   - If t is neither x nor y, then both sides are t.

   In all three cases, LHS = RHS.

3. Finally, suppose t is of the form $f(t_1, \ldots, t_n)$. Then LHS $= f(t_1[s/x], \ldots, t_n[s/x])[r/y] = f(t_1[s/x][r/y], \ldots, t_n[s/x][r/y])$, which, by induction, is

   $$f(t_1[r/y][s[r/y]/x], \ldots, t_n[r/y][s[r/y]/x])$$

   or $f(t_1[r/y], \ldots, t_n[r/y])[s[r/y]/x] = f(t_1, \ldots, t_n)[r/y][s[r/y]/x]$ =RHS.

□

Now, if s is y, then $t[y/x][r/y] = t[r/y][r/x]$, and we record the following corollary:

Corollary 1. *If x is not in r and y not in t, then $t[y/x][r/y] = t[r/x]$.*

The only thing we need to show is the case when x is y, but this is also clear, as $t[y/x][r/y] = t[x/x][r/x] = t[r/x]$.

With respect to formulas, we have a similar proposition:

Proposition 2. *If x, y are distinct variables, then*

$$A[t/x][s/y] = A[s/y][t[s/y]/x],$$

provided that x does not occur in s, and t and s are respectively free for x and y in A.

Proof. Suppose x and y are distinct variables, s, t terms, and A a wff. We do induction on the complexity of A.

1. First, suppose A is atomic.
   - A is of the form $t_1 = t_2$, then LHS is $t_1[t/x][s/y] = t_2[t/x][s/y]$ and we can apply the previous equation to both t_1 and t_2 to get RHS.
   - If A is of the form $R(t_1, \ldots, t_n)$, then LHS is $R(t_1[t/x][s/y], \ldots, t_n[t/x][s/y])$, and we again apply the previous equation to each t_i to get RHS.
2. Next, suppose A is of the form $B \to C$. Then LHS $= B[t/x][s/y] \to C[t/x][s/y]$, and, by induction, is $B[s/y][t[s/y]/x] \to C[s/y][t[s/y]/x] =$ RHS.
3. Finally, suppose A is of the form $\exists z B$.
   - x is z. Then $A[t/x][s/y] = A[s/y]$, and $A[s/y][t[s/y]/x] = A[s/y]$ since x is bound in $A[s/y]$.
   - x is not z. Then $A[t/x][s/y] = (\exists z B[t/x])[s/y]$.
     - y is z. Then $(\exists z B[t/x])[s/y] = \exists z B[t/x]$. On the other hand, $A[s/y][t[s/y]/x] = A[t[s/y]/x] = \exists z B[t[s/y]/x]$. By induction, t, s are free for x, y in B, and $B[t/x] = B[t[s/y]/x]$, the result follows.
     - y is not z. Then $(\exists z B[t/x])[s/y] = \exists z B[t/x][s/y]$. On the other hand, $A[s/y][t[s/y]/x] = (\exists z B[s/y])[t[s/y]/x] = \exists z B[s/y][t[s/y]/x]$ since x is not z. By induction again, t, s are free for x, y in B, and $B[t/x] = B[t[s/y]/x]$, the result follows once more.

□

Now, if t is y, then $A[y/x][s/y] = A[s/y][s/x]$, and we record the following corollary:

Corollary 2. *If y is not free in A, and is free for x in A, then $A[y/x][s/y] = A[s/x]$.*