function

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Entry type	Definition
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Synonym	map
Related topic	Mapping
Related topic	InjectiveFunction
Related topic	Surjective
Related topic	Bijection
Related topic	Relation
Defines	domain
Defines	codomain
Defines	composition
Defines	image
Defines	range
Defines	composite function

A *function* is a triplet (f, A, B) where:

1. A is a set (called the *domain* of the function).
2. B is a set (called the *codomain* of the function).
3. f is a binary relation between A and B.
4. For every $a \in A$, there exists $b \in B$ such that $(a, b) \in f$.
5. If $a \in A$, $b_1, b_2 \in B$, and $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$.

The triplet (f, A, B) is usually written with the specialized notation $f\colon A \to B$. This notation visually conveys the fact that f maps elements of A into elements of B.

Other standard notations for functions are as follows:

- For $a \in A$, one denotes by $f(a)$ the unique element $b \in B$ such that $(a, b) \in f$.
- The *image* of (f, A, B), denoted $f(A)$, is the set

$$\{b \in B \mid f(a) = b \text{ for some } a \in A\}$$

  consisting of all elements of B which equal $f(a)$ for some element $a \in A$. Note that, by abuse of notation, the set $f(A)$ is almost always called the image of f, rather than the image of (f, A, B).
- In cases where the function f is clear from context, the notation $a \mapsto b$ is equivalent to the statement $f(a) = b$.
- Given two functions $f\colon A \to B$ and $g\colon B \to C$, there exists a unique function $g \circ f\colon A \to C$ satisfying the equation $g \circ f(a) = g(f(a))$. The function $g \circ f$ is called the *composition* of f and g, and a function constructed in this manner is called a *composite function*. Composition is associative, meaning that $h \circ (g \circ f) = (h \circ g) \circ f$ provided that either expression is defined.
- When a function $f\colon A \to A$ has its domain equal to its codomain, one often writes f^n for the n-fold composition

$$\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$$

where n is any natural number. Occasionally this can be confused with ordinary exponentiation (for example the function $x \mapsto (\sin x)(\sin x)$ is conventionally written as $\sin^2$); in such cases one usually writes $f^{[n]}$ to denote the n-fold composition.

There is no universal agreement as to the definition of the *range* of a function. Some authors define the range of a function to be equal to the codomain, and others define the range of a function to be equal to the image.

Remark. In set theory, a function is defined as a relation f, such that whenever $(a, b), (a, c) \in f$, then $b = c$. Notice that the sets A, B are not specified in advance, unlike the defintion given in the beginning of the article. The *domain* and *range* of the function f is the domain and range of f as a relation. Using this definition of a function, we may recapture the defintion at the top of the entry by saying that a function f *maps from a set A into a set B*, if the domain of f is A, and the range of f is a subset of B.