polyadic algebra with equality

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Defines	substitutive
Defines	reflexive
Defines	symmetric
Defines	transitive

Let $A = (B, V, \exists, S)$ be a polyadic algebra. An *equality predicate* on A is a function $E : V \times V \to B$ such that

1. $S(f) \circ E(x, y) = E(f(x), f(y))$ for any $f : V \to V$ and any $x, y \in V$
2. $E(x, x) = 1$ for every $x \in V$, and
3. $E(x, y) \wedge a \leq S(x/y)a$, where $a \in B$, and (x/y) denotes the function $V \to V$ that maps x to y, and constant everywhere else.

Heuristically, we can interpret the conditions above as follows:

1. if $x = y$ and if we replace x by, say x_1, and y by y_1, then $x_1 = y_1$.
2. $x = x$ for every variable x
3. if we have a propositional function a that is true, and $x = y$, then the proposition obtained from a by replacing all occurrences of x by y is also true.

The second condition is also known as the *reflexive property* of the equality predicate E, and the third is known as the *substitutive property* of E

A *polyadic algebra with equality* is a pair (A, E) where A is a polyadic algebra and E is an equality predicate on A. Paul Halmos introduced this concept and called this simply an *equality algebra*.

Below are some basic properties of the equality predicate E in an equality algebra (A, E):

- (symmetric property) $E(x, y) \leq E(y, x)$
- (transitive property) $E(x, y) \wedge E(y, z) \leq E(x, z)$
- $E(x, y) \wedge a = E(x, y) \wedge S(x, y)a$, where (x, y) in the S is the transposition on V that swaps x and y and leaves everything else fixed.
- if a variable $x \in V$ is not in the support of $a \in A$, then $a = \exists(x)(E(x, y) \wedge S(y/x)a)$.
- $\exists(x)(E(x, y) \wedge a) \wedge \exists(x)(E(x, y) \wedge a') = 0$ for all $a \in A$ and all $x, y \in V$ whenever $x \neq y$.
- $\exists(x)(E(x, y) \wedge E(x, z)) = E(y, z)$ for all $x, y, z \in V$ where $x \notin \{y, z\}$.

Remarks

- The degree and local finiteness of a polyadic algebra (A, E) are defined as the degree and the local finiteness and degree of its underlying polyadic algebra A.
- It can be shown that every locally finite polyadic algebra of infinite degree can be embedded (as a polyadic subalgebra) in a locally finite polyadic algebra with equality of infinite degree.
- Like cylindric algebras, polyadic algebras with equality is an attempt at "converting" a first order logic (with equality) into algebraic form, so that the logic can be studied using algebraic means.

References

[1] P. Halmos, *Algebraic Logic*, Chelsea Publishing Co. New York (1962).

[2] B. Plotkin, *Universal Algebra, Algebraic Logic, and Databases*, Kluwer Academic Publishers (1994).