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consequence operator

Canonical name	ConsequenceOperator
Date of creation	2013-03-22 16:28:48
Last modified on	2013-03-22 16:28:48
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	40
Author	rspuzio (6075)
Entry type	Definition
Classification	msc 03G25
Classification	msc 03G10
Classification	msc 03B22
Synonym	closure operator
Defines	finitary consequence operator
Defines	finite consequence operator
Defines	algebraic consequence operator
Defines	axiomatic consequence operator
Defines	axiomless consequence operator

1 Definition

Let L be a set. A *consequence operator* on L is a mapping[1] $C\colon \mathcal{P}(L) \to \mathcal{P}(L)$ which satisfies the following three properties:

1. For all $X \subseteq L$, it happens that $X \subseteq C(X)$.
2. $C \circ C = C$
3. For all $X, Y \subseteq L$, if $X \subseteq Y$, then $C(X) \subseteq C(Y)$

If, in addition, the following condition is satisfied, then a consequence operator C is known as *finitary* . (Synonyms are "finite consequence operator" and "algebraic consequence operator".)

- For all $X \in L$, it happens that $C(X) = \bigcup_{Y \in \mathcal{F}(X)} C(Y)$.

It is worth noting that, if the above condition is satisfied, then the third condition of last paragraph becomes superfluous — as shown in `http://planetmath.org/node/8678`an attachment, it automatically follows from conitions 1 and 2 of last paragraph and the condition stated above.

A consequence operator C such that $C(\emptyset) = \emptyset$ is called *axiomless*. A consequence operator C such that $C(\emptyset) \neq \emptyset$ is called *axiomatic*.

2 Motivation

Alfred Tarksi introduced consequence operators as a way of discussing the notion of conclusions following from premises in a general fashion. Suppose that the set L consists of statements in some language. Then, given a set of statements X, let $C(X)$ be the set of all statements which can be inferred form statements of X.

The defining properties of "consequence operator" given above then express some fundamental facts about the process of inferring conclusions from premises: Any statement can be concluded from itself. If a statement s follows from a set of premises X and Y is a superset of X, then s also follows from Y. If one augments a set of premises by conclusions derived from those premises, then one can only draw conclusions from the larger set which could

[1] Here, "$\mathcal{P}$" denotes the power set and $\mathcal{F}$ denotes the finite power set.

have been drawn from the original set of premises. Note that these conditions hold for a large class of logics, not just classical logic of Aristotle, Boole, and Frege. However, they do not hold for all logics — in particular, there are the so-called nonmonotonic logics in which it is not always the case that, if $X \subset Y$, then $C(X) \subseteq C(Y)$.

In terms of this usage in logic, it is easy to understand the origin of the terms "axiomatic" and "axiomless". An axiom in a logical theory is a statement which is assumed true without having to prove it from any other statement. Hence, an axiom is a consequence of the empty set, so we call consequence operators which allow one to deduce conclusions from an empty set of premises axiomatic.

The distinction of finitary consequence operators has to do with whether one is permitted to draw a conclusion from an infinite set of premises which could not be drawn from any finite subset thereof. As for why one might want to do this, consider the following example. Suppose X consists of the following statements:

- $1 = 1^2$
- $2 = 1^2 + 1^2$
- $3 = 1^2 + 1^2 + 1^2$
- $4 = 2^2$
- $5 = 2^2 + 1^2$
- $6 = 2^2 + 1^2 + 1^2$
- $7 = 2^2 + 1^2 + 1^2 + 1^2$
- $8 = 2^2 + 2^2$
- $9 = 3^2$
- $10 = 3^2 + 1^2$
- $11 = 3^2 + 1^2 + 1^2$
- $12 = 2^2 + 2^2 + 2^2$
- $13 = 3^2 + 2^2$

- $14 = 3^2 + 2^2 + 1^2$
- $\cdots$

From X one would like to be able to draw the conclusion "Any positive integer can be expressed as the sum of at most four squares.". This conclusion, however, cannot be inferred from any proper subset of X, in particular, from any finite subset of X. To make this conclusion would require a consequence operator which is not finitary.

3 Examples

1. To begin, there are two trivial consequence operators defined on any set. One is the identity operator $I\colon \mathcal{P}(L) \to \mathcal{P}(L)$ defined as $I(X) = X$. The other is the constant operator $U\colon \mathcal{P}(L) \to \mathcal{P}(L)$ defined as $U(X) = L$. It is perfectly straightforward to check that these two operators satisfy the defining properties of consequence operator and, furthermore, that they are both finitary consequence operators and that I is axiomless whilst U is axiomatic. Trivial though they may be, these operators play an important role as exrtremal elements in the lattice of all consequence operators over a given set.

2. Next, we consider some less trivial consequence operators which can be defined over an arbitrary set. Let X and Y be any two subsets of L. Then we may define operators $C_\cap(X,Y)\colon \mathcal{P}(L) \to \mathcal{P}(L)$ and $C_\cup(X,Y)\colon \mathcal{P}(L) \to \mathcal{P}(L)$ as follows:

$$
\begin{aligned}
C_\cap(X,Y)(Z) &= \begin{cases} X \cup Z & Y \cap Z \neq \emptyset \\ Z & Y \cap Z = \emptyset \end{cases} \\
C_\cup(X,Y)(Z) &= \begin{cases} X \cup Z & Y \cup Z = Z \\ Z & Y \cup Z \neq Z \end{cases}
\end{aligned}
$$

   It is shown that these are indeed consequence operators in an attachment to this entry.

3. A much larger class of consequence operators may be defined as follows. Let K be a subset of $\mathcal{P}(L)$ which includes L. Then, as shown in an

http://planetmath.org/node/8671attachment, the map $C\colon \mathcal{P}(L) \to \mathcal{P}(L)$, defined as

$$C(X) = \cap\{Y \in K \mid X \subseteq Y\},$$

is a consequence operator. As we shall see, all consequence operators can be obtained by this construction. In particular, the examples discussed above can be obtained as follows: To obtain I, set $K = \mathbf{P}(L)$; to obtain U, set $K = \{L\}$; to obtain $C_\cap(X,Y)$, set

$$K = \{Z \subseteq L \mid Y \cap Z = \emptyset\} \cup \{X \cup Z \mid Z \subseteq L \wedge Y \cap Z \neq \emptyset\};$$

to obtain $C_\cup(X,Y)$, set

$$K = \{Z \subseteq L \mid Y \cup Z \neq \emptyset\} \cup \{X \cup Z \mid Z \subseteq L \wedge Y \cup Z = \emptyset\}.$$

4. Turning to more specific examples, we have the example which inspired the definintion in the first place. Let L be a set of logical expressions constructed from some set of sentence letters and predicate letters and the usual connectives and quantifiers. Given a subset $X \subseteq L$, let $C(X)$ be the set of all expressions ψ for which there exists a finite set of expressions $\phi_1, \dots, \phi_n$ such that $\ulcorner \phi_1 \wedge \cdots \wedge \phi_n \Rightarrow \psi \urcorner$ is a tautology. Note that this is a finitary consequence operator — it does not enable one to make the sort of deductions from infinite sets of premises described above.

5. This notion of consequence operator also applies to areas of mathematics other than logic. For instance, suppose that L is a vector space. Then the operator which assigns to a subset of L the vector subspace which it spans is a consequence operator. This particular consequence operator is finitary because if a vector v belongs to the span of a set X, then v can be expressed as a linear combination of a finite number of elements of X.

6. The closure operator in topology is a consequence operator. It is worth pointing out that not every consequence operator can be expressed as the closure operator for some topology because the closure operator satisfies some extra conditions beyond those which define consequence operators. Typically, the closure operator is not finitary because infinite subsets of topological spaces may have limit points.

4 Alternative Definition and Generalization

A consequence operator can be characterized by its fixed points. Given a consequence operator $C\colon \mathcal{P}(L) \to \mathcal{P}(L)$, set $K = \{X \subseteq L \mid C(X) = X\}$. By the second defining property of consequence operator, we have $K = \{C(X) \mid X \in L\}$. One can show that

$$C(X) = \cap\{Y \in K \mid X \subseteq Y\}.$$

Conversely, suppose that K is a subset of L with the following minimum property:

- For every $X \in L$, there exists a $Y \in K$ such that $X \subseteq Y$ and if, for any $Z \in K$, if $X \subseteq Z$, then $Y \subseteq Z$.

Then the operator C defined as

$$C(X) = \cap\{Y \in K \mid X \subseteq Y\}$$

is a consequence operator with K as its set of fixed points.

One may also define consequence operators in the more general context of a partially ordered set which may not be the power set of any set. Suppose that $\langle S, \leq \rangle$ is a partially ordered set. Then we may define a consequence operator on this ordered set to be a map $C\colon S \to S$ which satisfies the following three properties:

1. For all $X \in S$, it happens that $X \leq C(X)$.
2. $C \circ C = C$
3. For all $X, Y \in S$, if $X \leq Y$, then $C(X) \leq C(Y)$

Such more general consequence operators arise frequently when we restrict attention to distinguished subsets of a set. As an example, we may consider the following situation. Let S be the set of linear subspaces of a Banach space, ordered by inclusion. Then the operator $C\colon S \to S$ which assigns to each subspace its Cauchy completion is a consequence operator.

As an example which does not arise this way, let $S = \mathbb{R}$ with the usual order. Then the ceiling function $\lceil \cdot \rceil \colon \mathbb{R} \to \mathbb{R}$ is a consequence operator.

For another example, let S be the set of all fields with a countable number of elements. This set may be ordered as follows: $E \leq F$ if and only if there exists a non-trivial morphism of E into F. Then the operator which sends each field to its algebraic closure is a consequence operator.