some meta-theorems of propositional logic

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Based on the axiom system in `http://planetmath.org/AxiomSystemForPropositionalLogic` entry, we will prove some meta-theorems of propositional logic. In the discussion below, Δ and Γ are sets of well-formed formulas (wff's), and $A, B, C, \ldots$ are wff's.

1. (Deduction Theorem) $\Delta, A \vdash B$ iff $\Delta \vdash A \to B$.
2. (Proof by Contradiction) $\Delta, A \vdash \perp$ iff $\Delta \vdash \neg A$.
3. (Proof by Contrapositive) $\Delta, A \vdash \neg B$ iff $\Delta, B \vdash \neg A$.
4. (Law of Syllogism) If $\Delta \vdash A \to B$ and $\Gamma \vdash B \to C$, then $\Delta, \Gamma \vdash A \to C$.
5. $\Delta \vdash A$ and $\Delta \vdash B$ iff $\Delta \vdash A \wedge B$.
6. $\Delta \vdash A \leftrightarrow B$ iff $\Delta, A \vdash B$ and $\Delta, B \vdash A$.
7. If $\Delta \vdash A \leftrightarrow B$, then $\Delta \vdash B \leftrightarrow A$.
8. If $\Delta \vdash A \leftrightarrow B$ and $\Delta \vdash B \leftrightarrow C$, then $\Delta \vdash A \leftrightarrow C$.
9. $\Delta \vdash A \wedge B \to C$ iff $\Delta \vdash A \to (B \to C)$.
10. $\Delta \vdash A$ implies $\Delta \vdash B$ iff $\Delta \vdash A \to B$. This is a useful restatement of the deduction theorem.
11. (Substitution Theorem) If $\vdash B_i \leftrightarrow C_i$, then $\vdash A[\overline{B}/\overline{p}] \leftrightarrow A[\overline{C}/\overline{p}]$.
12. $\Delta \vdash \perp$ iff there is a wff A such that $\Delta \vdash A$ and $\Delta \vdash \neg A$.
13. If $\Delta, A \vdash B$ and $\Delta, \neg A \vdash B$, then $\Delta \vdash B$.

Remark. The theorem schema $A \to \neg\neg A$ is used in the proofs below.

Proof. The first three are proved `http://planetmath.org/DeductionTheoremHoldsForClassical` and the last three are proved `http://planetmath.org/SubstitutionTheoremForPropositionalL` We will prove the rest here, some of which relies on the deduction theorem.

4. From $\Delta \vdash A \to B$, by the deduction theorem, we have $\Delta, A \vdash B$. Let $\mathcal{E}_1$ be a deduction of B from $\Delta \cup \{A\}$, and $\mathcal{E}_2$ be a deduction of $B \to C$ from Γ, then
$$\mathcal{E}_1, \mathcal{E}_2, C$$
is a deduction of C from $\Delta \cup \{A\} \cup \Gamma$, so $\Delta, A, \Gamma \vdash C$, and by the deduction theorem again, we get $\Delta, \Gamma \vdash A \to C$.

5. $(\Rightarrow)$. Since $A \wedge B$ is $\neg(A \to \neg B)$, by the deduction theorem, it is enough to show $\Delta, A \to \neg B \vdash \perp$. Suppose $\mathcal{E}_1$ is a deduction of A from Δ and $\mathcal{E}_2$ is a deduction of B from Δ, then

$$\mathcal{E}_1, \mathcal{E}_2, A \to \neg B, \neg B, \perp$$

is a deduction of $\perp$ from $\Delta \cup \{A \to \neg B\}$.

$(\Leftarrow)$. We first show that $\Delta \vdash B$. Now, $\neg B \to (A \to \neg B)$ is an axiom and $\vdash (A \to \neg B) \to \neg\neg(A \to \neg B)$ is a theorem, $\vdash \neg B \to \neg\neg(A \to \neg B)$, so that by modus ponens, $\vdash \neg(A \to \neg B) \to B$, using axiom schema $(\neg C \to \neg D) \to (D \to C)$. Since by assumption $\Delta \vdash \neg(A \to \neg B)$, by modus ponens again, we get $\Delta \vdash B$.

We next show that $\Delta \vdash A$. From the deduction $A, A \to \perp, \perp$, we have $A, \neg A \vdash \perp$, so certainly $\Delta, \neg A, A, B \vdash \perp$. By three applications of the deduction theorem, we get $\Delta \vdash \neg A \to (A \to \neg B)$. By theorem $(A \to \neg B) \to \neg\neg(A \to \neg B)$, $\Delta \vdash \neg A \to \neg\neg(A \to \neg B)$. By axiom schema $(\neg C \to \neg D) \to (D \to C)$ and modus ponens, we get $\Delta \vdash \neg(A \to \neg B) \to A$. Since $\Delta \vdash \neg A \to \neg B$ by assumption, $\Delta \to A$ as a result.

6. $\Delta \vdash A \leftrightarrow B$ iff $\Delta \vdash A \to B$ and $\Delta \vdash B \to A$ iff $\Delta, A \vdash B$ and $\Delta, B \vdash A$.

7. Apply 6 to $\Delta \vdash A \to B$ and $\Delta \vdash B \to A$.

8. Apply 5 and 6.

9. Since $\Delta, A \vdash B \to A \wedge B$ by the theorem schema $\vdash A \to (B \to A \wedge B)$, together with $\Delta \vdash A \wedge B \to C$, we have $\Delta, A \vdash B \to C$ by law of syllogism, or equivalently $\Delta \vdash A \to (B \to C)$, by the deduction theorem. Conversely, $\Delta, A \vdash B \to C$ and theorem schema $A \wedge B \to B$ result in $\Delta, A \vdash A \wedge B \to C$ by law of syllogism. So $\Delta \vdash A \to (A \wedge B \to C)$ by the deduction theorem. But $A \wedge B \to A$ is a theorem schema, $\Delta \vdash A \wedge B \to (A \wedge B \to C)$, and therefore $\Delta \vdash A \wedge B \to C$ by the theorem schema $(X \to (X \to Y)) \leftrightarrow (X \to Y)$.

10. Assume the former. Then a deduction of B from Δ may or may not contain A. In either case, $\Delta, A \vdash B$, so $\Delta \vdash A \to B$ by the deduction theorem. Next, assume the later. Let $\mathcal{E}_1$ be a deduction of $A \to B$ from

Δ. Then if $\mathcal{E}_2$ is a deduction of A from Δ, then $\mathcal{E}_1, \mathcal{E}_2, B$ is a deduction of B from Δ, and therefore $\Delta \vdash B$.

To see the last meta-theorem implies the deduction theorem, assume $\Delta, A \vdash B$. Suppose $\Delta \vdash A$. Let $\mathcal{E}_1$ be a deduction of A from Δ, and $\mathcal{E}_2$ a deduction of B from $\Delta \cup \{A\}$. Then $\mathcal{E}_1, \mathcal{E}_2$ is a deduction of B from Δ. So $\Delta \vdash B$. As a result $\Delta A \to B$ by the statement of the meta-theorem. $\square$

Remark. Meta-theorems 7 and 8, together with the theorem schema $A \leftrightarrow A$, show that $\leftrightarrow$ defines an equivalence relation on the set of all wff's of propositional logic. Formally, for any two wff's A, B, if we define $A \sim B$ iff $\vdash A \leftrightarrow B$, then $\sim$ is an equivalence relation.