relation algebra

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It is a well-known fact that if A is a set, then $P(A)$ the power set of A, equipped with the intersection operation $\cap$, the union operation $\cup$, and the complement operation $'$ turns $P(A)$ into a Boolean algebra. Indeed, a Boolean algebra can be viewed as an abstraction of the family of subsets of a set with the usual set-theoretic operations. The algebraic abstraction of the set of binary relations on a set is what is known as a *relation algebra*.

Before defining formally what a relation algebra is, let us recall the following facts about binary relations on some given set A:

- binary relations are closed under $\cap, \cup$ and $'$, and in fact, satisfy all the axioms of a Boolean algebra. In short, the set $R(A)$ of binary relations on A is a Boolean algebra, where $A \times A$ is the top and $\varnothing \times \varnothing (= \varnothing)$ is the bottom of $R(A)$;

- if R and S are binary relations on A, then so are $R \circ S$, relation composition of R and S, and R^{-1}, the inverse of R;

- I_A, defined by $\{(a, a) \mid a \in A\}$, is a binary relation which is also the identity with respect to $\circ$, also known as the identity relation on A;

- some familiar identities:

  1. $R \circ (S \circ T) = (R \circ S) \circ T$
  2. $(R^{-1})^{-1} = R$
  3. $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$
  4. $(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$
  5. $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

In addition, there is also a rule that is true for all binary relations on A:

$$(R \circ S) \cap T = \varnothing \quad \text{iff} \quad (R^{-1} \circ T) \cap S = \varnothing \tag{1}$$

The verification of this rule is as follows: first note that sufficiency implies necessity, for if $(R^{-1} \circ T) \cap S = \varnothing$, then $(R \circ S) \cap T = ((R^{-1})^{-1} \circ S) \cap T = \varnothing$. To show sufficiency, suppose $(a, b) \in S$ is also an element of $R^{-1} \circ T$. Then there is $c \in A$ such that $(a, c) \in R^{-1}$ and $(c, b) \in T$. Therefore, $(c, a) \in R$ and $(c, b) = (c, a) \circ (a, b) \in R \circ S$ as well. This shows that $(R \circ S) \cap T \neq \varnothing$.

It can be shown that Rule (1) is equivalent to the following inclusion

$$R^{-1} \circ (R \circ S)' \subseteq S'. \tag{2}$$

Definition. A *relation algebra* is a Boolean algebra B with the usual Boolean operators $\vee, \wedge, '$, and additionally a binary operator $\;;$, a unary operator $^-$, and a constant i such that

1. $a\;;i = a$
2. $a\;;(b\;;c) = (a\;;b)\;;c$
3. $a^{--} = a$
4. $(a\;;b)^- = b^-\;;a^-$
5. $(a \vee b)\;;c = (a\;;c) \vee (b\;;c)$
6. $(a \vee b)^- = a^- \vee b^-$
7. $a^-\;;(a\;;b)' \leq b'$

where $\leq$ is the induced partial order in the underlying Boolean algebra.

Clearly, the set of all binary relations $R(A)$ on a set A is a relation algebra, as we have just demonstrated. Specifically, in $R(A)$, $\;;$ is the composition operator $\circ$, $^-$ is the inverse (or converse) operator $^{-1}$, and i is I_A.

A relation algebra is an algebraic system. As an algebraic system, we can define the usual algebraic notions, such as subalgebras of an algebra, homomorphisms between two algebras, etc... A relation algebra B that is a subalgebra of $R(A)$, the set of all binary relations on a set A, is called a *set relation algebra*.

References

[1] S. R. Givant, *The Structure of Relation Algebras Generated by Relativizations*, American Mathematical Society (1994).