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Cantor's paradox

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Cantor's paradox demonstrates that there can be no largest cardinality. In particular, there must be an unlimited number of infinite cardinalities. For suppose that α were the largest cardinal. Then we would have $|\mathcal{P}(\alpha)| = |\alpha|$. (Here $\mathcal{P}(\alpha)$ denotes the power set of α.) Suppose $f : \alpha \to \mathcal{P}(\alpha)$ is a bijection proving their equicardinality. Then $X = \{\beta \in \alpha \mid \beta \notin f(\beta)\}$ is a subset of α, and so there is some $\gamma \in \alpha$ such that $f(\gamma) = X$. But $\gamma \in X \leftrightarrow \gamma \notin X$, which is a paradox.

The key part of the argument strongly resembles Russell's paradox, which is in some sense a generalization of this paradox.

Besides allowing an unbounded number of cardinalities as ZF set theory does, this paradox could be avoided by a few other tricks, for instance by not allowing the construction of a power set or by adopting paraconsistent logic.