criterion of surjectivity

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Theorem. For surjectivity of a mapping $f\colon A \to B$, it's necessary and sufficient that

$$B \smallsetminus f(X) \subseteq f(A \smallsetminus X) \quad \forall\, X \subseteq A. \tag{1}$$

Proof. 1º. Suppose that $f\colon A \to B$ is surjective. Let X be an arbitrary subset of A and y any element of the set $B \smallsetminus f(X)$. By the surjectivity, there is an x in A such that $f(x) = y$, and since $y \notin f(X)$, the element x is not in X, i.e. $x \in A \smallsetminus X$ and thus $y = f(x) \in f(A \smallsetminus X)$. One can conclude that $B \smallsetminus f(X) \subseteq f(A \smallsetminus X)$ for all $X \subseteq A$.

2º. Conversely, suppose the condition (1). Let again X be an arbitrary subset of A and y any element of B. We have two possibilities:
a) $y \notin f(X)$; then $y \in B \smallsetminus f(X)$, and by (1), $y \in f(A \smallsetminus X)$. This means that there exists an element x of $A \smallsetminus X \subseteq A$ such that $f(x) = y$.
b) $y \in f(X)$; then there exists an $x \in X \subseteq A$ such that $f(x) = y$.
The both cases show the surjectivity of f.