proof of alternative characterization of filter

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First, suppose that $\mathbf{F}$ is a filter. We shall show that, for any two elements A and B of $\mathbf{F}$, it is the case that $A \cap B \in \mathbf{F}$ if and only if $A \in \mathbf{F}$ and $B \in \mathbf{F}$.

By the definition of filter, if $A \in \mathbf{F}$ and $B \in \mathbf{F}$ then $A \cap B \in \mathbf{F}$. Since $A \supseteq A \cap B$ and $\mathbf{F}$ is a filter, $A \cap B \in \mathbf{F}$ implies $A \in \mathbf{F}$. Likewise, $A \cap B \in \mathbf{F}$ implies $B \in \mathbf{F}$.

Next, we shall show that any proper subset $\mathbf{F}$ of the power set of X such that $A \cap B \in \mathbf{F}$ if and only if $A \in \mathbf{F}$ and $B \in \mathbf{F}$ is a filter.

If the empty set were to belong to $\mathbf{F}$ then for any $A \subset X$, we would have $A \cap \emptyset = \emptyset \in \mathbf{F}$. This would imply that every subset of X belongs to $\mathbf{F}$, contrary to our hypothesis that $\mathbf{F}$ is a proper subset of the power set of X.

If $A \subseteq B \subseteq X$ and $A \in \mathbf{F}$, then $A \cap B = A \in \mathbf{F}$. By our hypothesis, $B \in \mathbf{F}$.

The third defining property of a filter — If $A \in \mathbf{F}$ and $B \in \mathbf{F}$ then $A \cap B \in \mathbf{F}$ — is part of our hypothesis.