de Morgan's laws for sets (proof)

Canonical name	DeMorgansLawsForSetsproof
Date of creation	2013-03-22 13:32:16
Last modified on	2013-03-22 13:32:16
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	7
Author	mathcam (2727)
Entry type	Proof
Classification	msc 03E30

Let X be a set with subsets $A_i \subset X$ for $i \in I$, where I is an arbitrary index-set. In other words, I can be finite, countable, or uncountable. We first show that

$$\big(\cup_{i\in I} \, A_i \big)' \;\; = \;\; \cap_{i\in I} A_i',$$

where A' denotes the complement of A.

Let us define $S = \big(\cup_{i\in I} \, A_i \big)'$ and $T = \cap_{i\in I} A_i'$. To establish the equality $S = T$, we shall use a standard argument for proving equalities in set theory. Namely, we show that $S \subset T$ and $T \subset S$. For the first claim, suppose x is an element in S. Then $x \notin \cup_{i\in I} A_i$, so $x \notin A_i$ for any $i \in I$. Hence $x \in A_i'$ for all $i \in I$, and $x \in \cap_{i\in I} A_i' = T$. Conversely, suppose x is an element in $T = \cap_{i\in I} A_i'$. Then $x \in A_i'$ for all $i \in I$. Hence $x \notin A_i$ for any $i \in I$, so $x \notin \cup_{i\in I} A_i$, and $x \in S$.

The second claim,

$$\big(\cap_{i\in I} \, A_i \big)' \;\; = \;\; \cup_{i\in I} A_i',$$

follows by applying the first claim to the sets A_i'.