truth-value semantics for intuitionistic propositional logic is sound

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Proposition 1. *The truth-value semantics for intuitionistic propositional logic is sound.*

Proof. We show that, for each positive integer n, every theorem of intuitionistic propositional logic is a tautology for V_n. This amounts to showing that, for every interpretation v on V_n,

- each axiom is true, and
- modus ponens preserves truth.

Let us take care of the second one first. Suppose $v(A) = v(A \to B) = n$. If $v(A) \leq v(B)$, then $v(B) = n$. Otherwise, $v(B) < v(A)$. But this means that $n = v(A \to B) = v(B)$, forcing $v(B) = n$. Therefore, $v(B) = n$.

Now, we verify that each of the axiom schemas below are true:

1. $(A \wedge B) \to A$ and $(A \wedge B) \to B$.

   Since $v(A \wedge B) = \min\{v(A), v(B)\} \leq v(A)$, we get $v((A \wedge B) \to A) = n$. The other one is proved similarly.

2. $A \to (A \vee B)$ and $B \to (A \vee B)$.

   Since $v(A) \leq \max\{v(A), v(B)\} = v(A \vee B)$, we get $v(A \to (A \vee B)) = n$. The other one is proved similarly.

3. $A \to (B \to A)$.

   If $v(B) \leq v(A)$, $v(B \to A) = n$, so that $v(A \to (B \to A)) = n$ as well. If $v(A) < v(B)$, then $v(B \to A) = v(A)$, so that $v(A \to (B \to A)) = n$.

4. $\neg A \to (A \to B)$.

   If $v(A) \leq v(B)$, $v(A \to B) = n$, so that $v(\neg A \to (A \to B)) = n$ as well. If $v(B) < v(A)$, then $v(A \to B) = v(B)$. Also, $v(B) < v(A)$ implies that $v(A) > 0$, so that $v(\neg A) = 0$, and $v(\neg A \to (A \to B)) = n$ as a result.

5. $A \to (B \to (A \wedge B))$.

   If $v(B) = v(A \wedge B)$, then $v(B) \leq v(A)$ and $v(B \to (A \wedge B)) = n$, so that $v(A \to (B \to (A \wedge B)) = n$ also. If on the other hand $v(A \wedge B) < v(B)$, then

   $$v(A) = v(A \wedge B) \quad \text{and} \quad v(B \to (A \wedge B)) = v(A \wedge B),$$

so that

$$v(A \to (B \to (A \wedge B))) = v(A \to (A \wedge B)) = n.$$

6. $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$.
   If $v(B) = v(A \vee B)$, then $v(A) \leq v(B)$, and

   $$v((B \to C) \to ((A \vee B) \to C)) = v((B \to C) \to (B \to C)) = n,$$

   so that

   $$v((A \to C) \to ((B \to C) \to ((A \vee B) \to C))) = n$$

   as well. Otherwise, $v(B) < v(A) = v(A \vee B)$. This means that

   $$v((B \to C) \to ((A \vee B) \to C))) = v((B \to C) \to (A \to C)),$$

   and therefore

   $$v((A \to C) \to ((B \to C) \to ((A \vee B) \to C))) = v((A \to C) \to ((B \to C) \to (A \to C))) = n$$

   by 3 above.

7. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$.
   It is clear that $v(C) \leq v(B \to C)$. If $v(C) = v(B \to C)$, then

   $$v((A \to (B \to C)) \to (A \to C)) = v((A \to C) \to (A \to C)) = n,$$

   so that

   $$v((A \to B) \to ((A \to (B \to C)) \to (A \to C))) = n$$

   too. If $v(C) < v(B \to C)$, then $v(B \to C) = n$, which implies $v(B) \leq v(C)$. This in turn implies that $v(A \to B) \leq v(A \to C)$, so that

   $$v((A \to C) \to ((A \to (B \to C)) \to (A \to C))) \leq v((A \to B) \to ((A \to (B \to C)) \to (A \to C$$

   But by 3 above,

   $$v((A \to C) \to ((A \to (B \to C)) \to (A \to C))) = n,$$

   hence

   $$v((A \to B) \to ((A \to (B \to C)) \to (A \to C))) = n$$

   as a result.

8. $(A \to B) \to ((A \to \neg B) \to \neg A)$.

   Pick any C such that $v(C) = 0$, such as $D \wedge \neg D$. Then $v(\neg B) = v(B \to C)$, so that

   $$v((A \to B) \to ((A \to \neg B) \to \neg A)) = v((A \to B) \to ((A \to (B \to C)) \to (A \to C))) = n$$

   by 7.

□

Note that the proofs of the axioms employ some elementary facts, for any wff's A, B, C:

- If $v(B) = n$ or $v(A) = 0$, then $v(A \to B) = n$.
- if $v(B) = 0$, then $v(A \to B) = v(\neg A)$.
- if $v(A) = n$, then $v(A \to B) = v(B)$.
- $v(B) \leq v(A \to B)$.
- if $v(B) \leq v(C)$, then
  - $v(A \vee B) \leq v(A \vee C)$,
  - $v(A \wedge B) \leq v(A \wedge C)$,
  - $v(A \to B) \leq v(A \to C)$,
  - $v(C \to A) \leq v(B \to A)$.

From the facts above, one readily deduces:

- if $v(B) \leq v(C)$, then $v(\neg C) \leq v(\neg B)$,
- if $v(B) = v(C)$, then
  - $v(A \vee B) = v(A \vee C)$,
  - $v(A \wedge B) = v(A \wedge C)$,
  - $v(A \to B) = v(A \to C)$,
  - $v(C \to A) = v(B \to A)$,
  - $v(\neg B) = v(\neg C)$.