elementary recursive function

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Informally, elementary recursive functions are functions that can be obtained from functions encountered in elementary schools: addition, multiplication, subtraction, and division, using basic operations such as substitutions and finite summation and product. Before stating the formal definition, the following notations are used:

> $\mathcal{F} = \bigcup\{F_k \mid k \in \mathbb{N}\}$, where for each $k \in \mathbb{N}$, $F_k = \{f \mid f\colon \mathbb{N}^k \to \mathbb{N}\}$.

Definition. The set of *elementary recursive functions*, or *elementary functions* for short, is the smallest subset $\mathcal{ER}$ of $\mathcal{F}$ where:

1. (addition) $\mathrm{add} \in \mathcal{ER} \cap F_2$, given by $\mathrm{add}(m,n) := m+n$;
2. (multiplication) $\mathrm{mult} \in \mathcal{ER} \cap F_2$, given by $\mathrm{mult}(m,n) := mn$;
3. (difference) $\mathrm{diff} \in \mathcal{ER} \cap F_2$, given by $\mathrm{diff}(m,n) := |m-n|$;
4. (quotient) $\mathrm{quo} \in \mathcal{ER} \cap F_2$, given by $\mathrm{quo}(m,n) := [m/n]$, which is the largest non-negative integer z such that $nz \le m$ (by convention, $\mathrm{quo}(0,0) := 1$);
5. (projection) $p^k_m \in \mathcal{ER} \cap F_k$, where $m \le k$, given by $p^k_m(n_1,\ldots,n_k) := n_m$;
6. $\mathcal{ER}$ is closed under composition: If $\{g_1,\ldots,g_m\} \subseteq \mathcal{ER} \cap F_k$ and $h \in \mathcal{ER} \cap F_m$, then $f \in \mathcal{ER} \cap F_k$, where
$$f(n_1,\ldots,n_k) = h(g_1(n_1,\ldots,n_k),\ldots,g_m(n_1,\ldots,n_k));$$
7. $\mathcal{ER}$ is closed under bounded sum: if $f \in \mathcal{ER} \cap F_k$, then $f_s \in \mathcal{ER} \cap F_k$, where
$$f_s(\boldsymbol{x},y) := \sum_{i=0}^{y} f(\boldsymbol{x},i);$$
8. $\mathcal{ER}$ is closed under bounded product: if $f \in \mathcal{ER} \cap F_k$, then $f_p \in \mathcal{ER} \cap F_k$, where
$$f_p(\boldsymbol{x},y) := \prod_{i=0}^{y} f(\boldsymbol{x},i).$$

Examples.

- The initial functions in the definition of primitive recursive functions are elementary:
  1. The zero function $z(x)$ is $\text{diff}(x, x)$.
  2. The constant function $\text{const}_1(x) := 1$ is $\text{quo}(x, x)$.
  3. The successor function $s(x)$ can be obtained by substituting (by composition) the constant function const_1 and the projection function p_1^1, into the addition function $\text{add}(p_1^1(x), \text{const}_1(x))$.
- Multiplication mult in 2 above may be removed from the definition, since

$$\text{mult}(x, y) = \text{diff}(f(x, y), p_1^2(x, y)), \quad \text{where } f(x, y) := \sum_{i=0}^{y} p_1^2(x, i)$$

- Many other basic functions, such as the restricted subtraction, exponential function, the i-th prime function, are all elementary. One may replace the difference function in 3 above by the restricted subtraction without changing $\mathcal{ER}$.

Remarks

- Consider the set $\mathcal{PR}$ of primitive recursive functions. The functions in the first five groups above are all in $\mathcal{PR}$. In addition, $\mathcal{PR}$ is closed under the operations in 6, 7, and 8 above, we see that $\mathcal{ER} \subseteq \mathcal{PR}$, since $\mathcal{ER}$, as defined, is the smallest such set.
- Furthermore, $\mathcal{ER} \neq \mathcal{PR}$. For example, the super-exponential function, given by $f(x, 0) = m$, and $f(x, n+1) = \exp(m, f(x, n))$, where $m > 1$, can be shown to be non-elementary.
- In addition, it can be shown that $\mathcal{ER}$ is the set of primitive recursive functions that can be obtained from the zero function, the successor function, and the projection functions via composition, and no more than three applications of primitive recursion.
- By taking the closure of $\mathcal{ER}$ with respect to unbounded minimization, one obtains $\mathcal{R}$, the set of all recursive functions (partial or total). In fact, by replacing bounded sum and bounded product with unbounded minimization, and the difference function with restricted subtraction, one obtains $\mathcal{R}$.