planetmath.org

Math for the people, by the people.

union of countable sets

Canonical name	UnionOfCountableSets
Date of creation	2013-03-22 19:02:56
Last modified on	2013-03-22 19:02:56
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	11
Author	CWoo (3771)
Entry type	Result
Classification	msc 03E10

In this entry, we prove a useful property of countability which will gives us many more examples of countable sets.

Proposition 1. *$A \cup B$ is countable iff A and B are countable.*

Proof. Clearly, if $A \cup B$ is countable, then A and B are each countable, as they are subsets of a countable set.

Conversely, suppose $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ are two surjections. Let $C = A \cup B$. Define $h : \mathbb{N} \to C$ as follows: $h(2n+1) = f(n)$ for $n = 0, 1, \ldots$, and $h(2n) = g(n)$, for $n = 1, 2, \ldots$. Then h is a well-defined function, for each $i \in \mathbb{N}$ is either even or odd, so $h(i)$ is defined in either case. Finally, h is onto, for if $c \in C$, then $c \in A$ or $c \in B$. If $c \in A$, then $h(2p+1) = c$ for some p, and if $c \in B$, then $h(2q) = c$ for some q. Hence C is countable. $\square$

The idea behind the above proof is to realize that we can list elements of C in the following manner:

$$\begin{array}{cccccc} f(1) & f(2) & f(3) & f(4) & f(5) & \cdots \\ g(1) & g(2) & g(3) & g(4) & g(5) & \cdots \end{array}$$

Therefore, h is defined so its first value is $f(1)$, its second value is $g(1)$, third is $f(2)$, fourth $g(2)$, etc... In the end, all of the elements of C are exhausted by this way of counting.

As a corollary, we have

Corollary 1. *$A_1 \cup A_2 \cup \cdots \cup A_n$ is countable iff each A_i is.*

Proof. This is true by induction. $\square$

The property can easily be extended to the countably infinite case, the proof of which is just a variant of the above methodology:

Proposition 2. *$\bigcup\{A_i \mid i \in \mathbb{N}\}$ is countable iff each A_i is.*

Proof. Again, one direction is obvious, so we concentrate on the other direction.

Let $A = A_1 \cup A_2 \cup \cdots$. Suppose we have surjections $f_i : \mathbb{N} \to A_i$ for $i = 1, 2, \ldots$. Then listing elements of A in the following manner (table below on the left)

$$
\begin{array}{c|cccc}
i\backslash j & 1 & 2 & 3 & \cdots \\
\hline
1 & f_1(1) & f_1(2) & f_1(3) & \cdots \\
2 & f_2(1) & f_2(2) & f_2(3) & \cdots \\
3 & f_3(1) & f_3(2) & f_3(3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\qquad
\begin{array}{c|cccc}
i\backslash j & 1 & 2 & 3 & \cdots \\
\hline
1 & f(1) & f(2) & f(4) & \cdots \\
2 & f(3) & f(5) & f(8) & \cdots \\
3 & f(6) & f(9) & f(13) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

provides a surjection $f : \mathbb{N} \to A$ (table above on the right). The first few values of f are

$$f(1) = f_1(1), \quad f(2) = f_1(2), \quad f(3) = f_2(1), \quad f(4) = f_1(3), \quad f(5) = f_2(2), \quad \dots$$

$\square$

Notice the similarity between the function f above, and the pairing function used in the proof that $\mathbb{N}^2$ is countable `http://planetmath.org/ProductOfCountableSets`here

Remark. However, the property fails when there are uncountably many sets to deal with. For example, the union of $\{r\}$ for each $r \in \mathbb{R}$ is uncountable.