axiom of dependent choices

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The *axiom of dependent choices* (DC), or the *principle of dependent choices*, is the following statement:

> given a set A and a binary relation $R \neq \varnothing$ on A such that $\mathrm{ran}(R) \subseteq \mathrm{dom}(R)$, then there is a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that $a_n R a_{n+1}$.

Here, $\mathbb{N}$ is the set of all natural numbers.

The relation between DC, AC (axiom of choice), and CC (axiom of countable choice) are the following:

Proposition 1. *ZF+AC implies ZF+DC.*

We prove this by using one of the equivalents of AC: Zorn's lemma. For this proof, we define $\mathrm{seg}(n) := \{m \in \mathbb{N} \mid m \leq n\}$, the initial segment of $\mathbb{N}$ with the greatest element n. Before starting the proof, we need a fact about initial segments:

Lemma 1. *The union of initial segments of $\mathbb{N}$ is either $\mathbb{N}$ or an initial segment.*

Proof. Let S be a set of initial segments of $\mathbb{N}$. If $s := \bigcup S \neq \mathbb{N}$, then $B := \mathbb{N} - S \neq \varnothing$, so B has a least element r. As a result, none of $i \in \mathrm{seg}(r-1)$ is in B, and $\mathrm{seg}(r-1) \subseteq s$. If some $m \geq r$ is in s, then there is an initial segment $\mathrm{seg}(n) \in S$ with $m \in \mathrm{seg}(n)$, so that $r \in \mathrm{seg}(m) \subseteq \mathrm{seg}(n) \subseteq s$, contradicting $r \in B$. □

Remark. The fact that B in the proof above has a least element is a direct result of ZF, so the well-ordering principle (and hence AC) is not needed.

We are now ready for the proof of proposition 1.

Proof. Let R be a non-empty binary relation on a set A (of course non-empty). We want to find a function $f : \mathbb{N} \to \mathrm{dom}(R)$ such that $f(n)Rf(n+1)$.

Let P be the set of all partial functions $f : \mathbb{N} \Rightarrow \mathrm{dom}(R)$ such that $\mathrm{dom}(f)$ is either an initial segment of $\mathbb{N}$, or $\mathbb{N}$ itself, such that $f(n)Rf(n+1)$, whenever $n, n+1 \in \mathrm{dom}(f)$. Since $R \neq \varnothing$, some $(a, b) \in R$. Additionally, $b \in \mathrm{ran}(R) \subseteq \mathrm{dom}(R)$. Define function $g : \{1, 2\} \to \mathrm{dom}(R)$ by $g(1) = a$ and $g(2) = b$. Then $g(1)Rg(2)$, so that $g \in P$, or P is non-empty.

Partial order P by inclusion so it is a poset. Let C be a chain in P, then $h := \bigcup C$ is a partial function from $\mathbb{N}$ to A. Since $\mathrm{dom}(h)$ is the union of initial segments or $\mathbb{N}$, $\text{dom}(h)$ itself is either an initial segment or $\mathbb{N}$ by Lemma 1.

Now, suppose $m, m+1 \in \text{dom}(h)$, then $m+1 \in \text{dom}(s)$ for some $s \in C$, so $m \in \text{dom}(s)$ as well. Therefore $s(m)Rs(m+1)$. Since $h(i) = s(i)$ for any $i \in \text{dom}(s)$, we see that $h(m)Rh(m+1)$. This shows that $h \in P$, or that C has an upper bound in P.

By Zorn's lemma, P has a maximal element f. We claim that f is a total function. If not, then $\text{dom}(f) = \{1, \ldots, n\}$ for some n. Since $f(n) \in \text{dom}(R)$, there is some $d \in \text{ran}(R)$ such that $f(n)Rd$. Define a partial function $g : \mathbb{N} \Rightarrow \text{dom}(R)$ such that $\text{dom}(g) = \{1, \ldots, n+1\}$, and $g(i) = f(i)$ for all $i = 1, \ldots, n$, and $g(n+1) = b$. So $g \neq f$ extends f, contradicting the maximality of f. Hence, f is a total function, and we are done. $\square$

Proposition 2. *ZF+DC implies ZF+CC.*

Proof. Let C be a countable set of non-empty sets. We assume that C is countably infinite, for the finite case can be proved using ZF alone, and is left for the reader.

Since there is a bijection $\phi : C \to \mathbb{N}$, index each element in C by its image in $\mathbb{N}$, so that $C = \{A_i \mid i \in \mathbb{N}\}$. Let $A := \bigcup C$. We want to find a function $f : C \to A$ such that $f(A_i) \in A_i$ for every $i \in \mathbb{N}$.

Define a binary relation R on A as follows: aRb iff there is an $i \in \mathbb{N}$ such that $a \in A_i$ and $b \in A_{i+1}$. Since each $A_i \neq \varnothing$, $R \neq \varnothing$. Furthermore, if $b \in \text{ran}(R)$, then $b \in A_{i+1}$ for some $i \in \mathbb{N}$. Pick any $c \in A_{i+2}$ (since $A_{i+2} \neq \varnothing$), so that bRc, and therefore $b \in \text{dom}(R)$. This shows that $\text{ran}(R) \subseteq \text{dom}(R)$.

By DC, there is a function $g : \mathbb{N} \to \text{dom}(R)$ such that $g(i)Rg(i+1)$ for every $i \in \mathbb{N}$. Now, $g(1) \in A_j$ for some $j \in \mathbb{N}$. Define a function $h : \mathbb{N} \to A$ as follows, for each $i \in \text{seg}(j-1)$, pick $a_i \in A_i$ and set $h(i) := a_i$ (this can be done by induction), and for $i \geq j$, set $h(i) := g(j - i + 1)$ (arithmetic of finite cardinals is possible in ZF). Then $h(i) \in A_i$ for all $i \in \mathbb{N}$.

Finally, define $f : C \to A$ as follows: for each $A_i \in C$, set $f(A_i) := h(i)$. Then f has the desired property $f(A_i) \in A_i$, and the proof is complete. $\square$

However, the converses of both of these implications are false. Jensen proved the independence of DC from ZF+CC, and Mostowski and Jech proved the independence of AC from ZF+DC. In fact, it was shown that the weaker version of AC, which states that every set with cardinality at most $\aleph_1$ has a choice function, is independent from ZF+DC.

Remark. DC is related to Baire spaces in point-set topology. It can be shown that DC is equivalent to each of the following statements in ZF:

- Any complete pseudometric space is Baire under the topology induced by the pseudometric.
- Any product of compact Hausdorff spaces is Baire under the product topology.

References

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