product of countable sets

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Proposition 1. $\mathbb{N}^2$ *is countable.*

This is actually proved in `http://planetmath.org/AlternativeDefinitionsOfCountable`this entry, by finding either a surjection $\mathbb{N} \to \mathbb{N}^2$, or an injection $\mathbb{N}^2 \to \mathbb{N}$. In the following proof, we are going to get a bijection.

Proof. There are many ways to prove this. One way is to place the integer pairs in a two-dimensional array indicated by the table on the left below:

$i \backslash j$	1	2	3	$\cdots$
1	$(0,0)$	$(0,1)$	$(0,2)$	$\cdots$
2	$(1,0)$	$(1,1)$	$(1,2)$	$\cdots$
3	$(2,0)$	$(2,1)$	$(2,2)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$i \backslash j$	1	2	3	$\cdots$
1	1	2	4	$\cdots$
2	3	5	8	$\cdots$
3	6	9	13	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Let $C(i,j)$ be the content of cell (i,j), located in the i-th row and j-th column. For example, $C(1,1) = (0,0)$, and $C(3,2) = (2,1)$.

Now, let us construct a list of the pairs, which essentially amounts to constructing a bijection $h : \mathbb{N}^2 \to \mathbb{N}$ (the table on the right above). We start at cell $(1,1)$. If cell (i,j) has been counted, the next cell to be counted is $(i+1, j-1)$ if $j > 1$, or $(1, i+1)$ if $j = 1$. Thus, the first several pairs on the list are

$$(0,0),\ (0,1),\ (1,0),\ (0,2),\ (1,1),\ (2,0),\ (0,3),\ \ldots$$

We leave it to the reader to find the bijection h (hint: see the entry on pairing function). Therefore, $\mathbb{N}^2$ is countable. $\square$

Proposition 2. *If A and B are countable, so is $A \times B$.*

Proof. Suppose $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$ are injections. Then $h := (f,g) : A \times B \to \mathbb{N}^2$ is an injection. Since $\mathbb{N}^2$ is countable, so is $A \times B$. $\square$

Proposition 3. *Let n be a positive integer, and $A_1, \ldots, A_n$ sets. Then $A_1 \times \cdots \times A_n$ is countable iff each A_i is.*

Proof. Again, if one of A_i is empty, so is the product, and vice versa. The countability follows immediately. So we assume that none of A_i is empty. Set $A := A_1 \times \cdots A_n$.

Suppose first that $A_1, \ldots, A_n$ are countable. We do induction on n. The case where $n = 1$ is clear. Suppose now that $n = k$ is true. Then $A_1 \times \cdots \times$

$A_k \times A_{k+1}$ is just the product of two countable sets $A_1 \times \cdots \times A_k$ and A_{k+1}, which we know is countable by the proposition above.

Conversely, suppose A is countable. Let $g : A \to \mathbb{N}$ be an injection. Since $A_i \neq \varnothing$, fix $a_i \in A_i$ for each $i = 1, \ldots, n$. Now, for any A_i, define a map $e_i : A_i \to A$ so that the i-th component of $e_i(a)$ is a, and the j-th component is the fixed element $a_j \in A_j$, if $j \neq i$. Clearly, $e_i : A_i \to A$ is an injection, so its composition with g is also an injection from A to $\mathbb{N}$, showing that A_i is countable. $\square$

Corollary 1. *For any positive integer n, A is countable iff A^n is.*

Remark. However, infinite products of sets are in general uncountable, even if each of the sets is finite. In particular, $\{0, 1\}^{\mathbb{N}}$ is uncountable. The proof uses Cantor's diagonalization argument.