planetmath.org

Math for the people, by the people.

Martin's axiom and the continuum hypothesis

Canonical name	MartinsAxiomAndTheContinuumHypothesis
Date of creation	2013-03-22 12:55:05
Last modified on	2013-03-22 12:55:05
Owner	Henry (455)
Last modified by	Henry (455)
Numerical id	4
Author	Henry (455)
Entry type	Result
Classification	msc 03E50

$MA_{\aleph_0}$ always holds

Given a countable collection of dense subsets of a partial order, we can selected a set $\langle p_n \rangle_{n<\omega}$ such that p_n is in the n-th dense subset, and $p_{n+1} \leq p_n$ for each n. Therefore CH implies MA.

If MA_κ then $2^{\aleph_0} > \kappa$, and in fact $2^\kappa = 2^{\aleph_0}$

$\kappa \geq \aleph_0$, so $2^\kappa \geq 2^{\aleph_0}$, hence it will suffice to find an surjective function from $\mathrm{P}(\aleph_0)$ to $\mathrm{P}(\kappa)$.

Let $A = \langle A_\alpha \rangle_{\alpha<\kappa}$, a sequence of infinite subsets of ω such that for any $\alpha \neq \beta$, $A_\alpha \cap A_\beta$ is finite.

Given any subset $S \subseteq \kappa$ we will construct a function $f : \omega \to \{0,1\}$ such that a unique S can be recovered from each f. f will have the property that if $i \in S$ then $f(a) = 0$ for finitely many elements $a \in A_i$, and if $i \notin S$ then $f(a) = 0$ for infinitely many elements of A_i.

Let P be the partial order (under inclusion) such that each element $p \in P$ satisfies:

- p is a partial function from ω to $\{0,1\}$
- There exist $i_1, \ldots, i_n \in S$ such that for each $j < n$, $A_{i_j} \subseteq \mathrm{dom}(p)$
- There is a finite subset of ω, w_p, such that $w_p = \mathrm{dom}(p) - \bigcup_{j<n} A_{i_j}$
- For each $j < n$, $p(a) = 0$ for finitely many elements of A_{i_j}

This satisfies ccc. To see this, consider any uncountable sequence $S = \langle p_\alpha \rangle_{\alpha<\omega_1}$ of elements of P. There are only countably many finite subsets of ω, so there is some $w \subseteq \omega$ such that $w = w_p$ for uncountably many $p \in S$ and $p \upharpoonright w$ is the same for each such element. Since each of these function's domain covers only a finite number of the A_α, and is 1 on all but a finite number of elements in each, there are only a countable number of different combinations available, and therefore two of them are compatible.

Consider the following groups of dense subsets:

- $D_n = \{p \in P \mid n \in \mathrm{dom}(p)\}$ for $n < \omega$. This is obviously dense since any p not already in D_n can be extended to one which is by adding $\langle n, 1 \rangle$

- $D_\alpha = \{p \in P \mid \mathrm{dom}(p) \supseteq A_\alpha\}$ for $\alpha \in S$. This is dense since if $p \notin D_\alpha$ then $p \cup \{\langle a, 1\rangle \mid a \in A_\alpha \setminus \mathrm{dom}(p)\}$ is.

- For each $\alpha \notin S$, $n < \omega$, $D_{n,\alpha} = \{p \in P \mid m \geq n \wedge p(m) = 0\}$ for some $m < \omega$. This is dense since if $p \notin D_{n,\alpha}$ then $\mathrm{dom}(p) \cap A_\alpha = A_\alpha \cap \left(w_p \cup \bigcup_j A_{i_j}\right)$. But w_p is finite, and the intersection of A_α with any other A_i is finite, so this intersection is finite, and hence bounded by some m. A_α is infinite, so there is some $m \leq x \in A_\alpha$. So $p \cup \{\langle x, 0\rangle\} \in D_{n,\alpha}$.

By MA_κ, given any set of κ dense subsets of P, there is a generic G which intersects all of them. There are a total of $\aleph_0 + |S| + (\kappa - |S|) \cdot \aleph_0 = \kappa$ dense subsets in these three groups, and hence some generic G intersecting all of them. Since G is directed, $g = \bigcup G$ is a partial function from ω to $\{0, 1\}$. Since for each $n < \omega$, $G \cap D_n$ is non-empty, $n \in \mathrm{dom}(g)$, so g is a total function. Since $G \cap D_\alpha$ for $\alpha \in S$ is non-empty, there is some element of G whose domain contains all of A_α and is 0 on a finite number of them, hence $g(a) = 0$ for a finite number of $a \in A_\alpha$. Finally, since $G \cap D_{n,\alpha}$ for each $n < \omega$, $\alpha \notin S$, the set of $n \in A_\alpha$ such that $g(n) = 0$ is unbounded, and hence infinite. So g is as promised, and $2^\kappa = 2^{\aleph_0}$.