# equivalence of Hausdorff's maximum principle, Zorn's lemma and the well-ordering theorem


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**Hausdorff's maximum principle implies Zorn's lemma.** Consider a partially ordered set $X$, where every chain has an upper bound. According to the maximum principle there exists a maximal totally ordered subset $Y \subseteq X$. This then has an upper bound, $x$. If $x$ is not the largest element in $Y$ then $\{x\} \cup Y$ would be a totally ordered set in which $Y$ would be properly contained, contradicting the definition. Thus $x$ is a maximal element in $X$.

**Zorn's lemma implies the well-ordering theorem.** Let $X$ be any non-empty set, and let $\mathcal{A}$ be the collection of pairs $(A, \leq)$, where $A \subseteq X$ and $\leq$ is a well-ordering on $A$. Define a relation $\preceq$, on $\mathcal{A}$ so that for all $x, y \in \mathcal{A} : x \preceq y$ iff $x$ equals an initial of $y$. It is easy to see that this defines a partial order relation on $\mathcal{A}$ (it inherits reflexibility, anti symmetry and transitivity from one set being an initial and thus a subset of the other).

For each chain $C \subseteq \mathcal{A}$, define $C' = (R, \leq')$ where R is the union of all the sets $A$ for all $(A, \leq) \in C$, and $\leq'$ is the union of all the relations $\leq$ for all $(A, \leq) \in C$. It follows that $C'$ is an upper bound for $C$ in $\mathcal{A}$.

According to Zorn's lemma, $\mathcal{A}$ now has a maximal element, $(M, \leq_M)$. We postulate that $M$ contains all members of $X$, for if this were not true we could for any $a \in X - M$ construct $(M_*, \leq_*)$ where $M_* = M \cup \{a\}$ and $\leq_*$ is extended so $S_a(M_*) = M$. Clearly $\leq_*$ then defines a well-order on $M_*$, and $(M_*, \leq_*)$ would be larger than $(M, \leq_M)$ contrary to the definition.

Since $M$ contains all the members of $X$ and $\leq_M$ is a well-ordering of $M$, it is also a well-ordering on $X$ as required.

**The well-ordering theorem implies Hausdorff's maximum principle.** Let $(X, \preceq)$ be a partially ordered set, and let $\leq$ be a well-ordering on $X$. We define the function $\phi$ by transfinite recursion over $(X, \leq)$ so that

$$\phi(a) = \begin{cases} \{a\} & \text{if } \{a\} \cup \bigcup_{b<a} \phi(b) \text{ is totally ordered under } \preceq . \\ \emptyset & \text{otherwise.} \end{cases} .$$

It follows that $\bigcup_{x \in X} \phi(x)$ is a maximal totally ordered subset of $X$ as required.