cumulative hierarchy

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The *cumulative hierarchy* of sets is defined by transfinite recursion as follows: we define $V_0 = \varnothing$ and for each ordinal α we define $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ and for each limit ordinal δ we define $V_\delta = \bigcup_{\alpha\in\delta} V_\alpha$.

Every set is a subset of V_α for some ordinal α, and the least such α is called the *rank* of the set. It can be shown that the rank of an ordinal is itself, and in general the rank of a set X is the least ordinal greater than the rank of every element of X. For each ordinal α, the set V_α is the set of all sets of rank less than α, and $V_{\alpha+1} \setminus V_\alpha$ is the set of all sets of rank α.

Note that the previous paragraph makes use of the Axiom of Foundation: if this axiom fails, then there are sets that are not subsets of any V_α and therefore have no rank. The previous paragraph also assumes that we are using a set theory such as ZF, in which elements of sets are themselves sets.

Each V_α is a transitive set. Note that $V_0 = 0$, $V_1 = 1$ and $V_2 = 2$, but for $\alpha > 2$ the set V_α is never an ordinal, because it has the element $\{1\}$, which is not an ordinal.