Kripke semantics for intuitionistic propositional logic

Canonical name	KripkeSemanticsForIntuitionisticPropositionalLogic
Date of creation	2013-03-22 19:33:19
Last modified on	2013-03-22 19:33:19
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	35
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03B55
Classification	msc 03B20
Related topic	AxiomSystemForIntuitionisticLogic

A *Kripe model* for intuitionistic propositional logic PL_i is a triple $M := (W, \leq, V)$, where

1. W is a set, whose elements are called *possible worlds*,
2. $\leq$ is a preorder on W,
3. V is a function that takes each wff (well-formed formula) A in PL_i to a subset $V(A)$ of W, such that
   - if p is a propositional variable, $V(p)$ is upper closed,
   - $V(A \wedge B) = V(A) \cap V(B)$,
   - $V(A \vee B) = V(A) \cup V(B)$,
   - $V(\neg A) = V(A)^{\#}$,
   - $V(A \to B) = (V(A) - V(B))^{\#}$,

   where $S^{\#} := (\downarrow S)^c$, the complement of the lower closure of any $S \subseteq W$.

Remarks.

- If $\perp$ were used as a primitive symbol instead of $\neg$, then we require that $V(\perp) = \varnothing$. Then introducing $\neg$ by $\neg A := A \to \perp$, we get $V(\neg A) = V(A)^{\#}$.
- Some simple properties of $\#$: for any subset S of W, $S^{\#}$ is upper closed. This means that for any wff A, $V(A)$ is upper closed. Also, S and $S^{\#}$ are disjoint, which means that $V(A \wedge \neg A) = \varnothing$ for any A.

One can also define a *satisfaction relation* $\models$ between W and the set L of wff's so that

$$\models_w A \qquad \text{iff} \qquad w \in V(A)$$

for any $w \in W$ and $A \in L$. It's easy to see that

- for any propositional variable p, if $\models_w p$ and $w \leq u$, then $\models_u p$,
- $\models_w A \wedge B$ iff $\models_w A$ and $\models_w B$,
- $\models_w A \vee B$ iff $\models_w A$ or $\models_w B$,
- $\models_w \neg A$ iff for all u such that $w \leq u$, we have $\not\models_u A$

- $\models_w A \to B$ iff for all u such that $w \le u$, we have $\models_u A$ implies $\models_u B$.

When $\models_w A$, we say that A is true at world w.

Remark. Since $V(A)$ is upper closed, $\models_w A$ implies $\models_u A$ for any u such that $w \le u$. Now suppose $w \le u$ and $u \le w$, then $\models_w A$ iff $\models_u A$. This shows that, as far as validity of formulas is concerned, we can take $\le$ to be a partial order in the definition above.

Some examples of Kripke models:

1. Let M_1 be the model consisting of $W = \{w, u\}$, $\le = \{(w,w),(u,u),(w,u)\}$, with $V(p) = \{u\}$ and $V(q) = W$. Then $V(p)^{\#} = V(q)^{\#} = \varnothing$, and we have the following:

   - $V(p \vee \neg p) = \{u\}$.
   - $V(q \to p) = V(p)$, and $V(\neg p \to \neg q) = W$, so
   $$V((\neg p \to \neg q) \to (q \to p)) = \{w\}^{\#} = \{u\}.$$
   - $V(p \to q) = V(\neg q \to \neg p) = W$, so
   $$V((\neg q \to \neg p) \to (p \to q)) = \varnothing^{\#} = W.$$
   - $V((p \to q) \vee (q \to p)) = W$.
   - In fact, for any wff's A, B, either $V(A) \subseteq V(B)$ or $V(B) \subseteq V(A)$, since $\le$ is linearly ordered, so that
   $$V((A \to B) \vee (B \to A)) = V(A \to B) \cup V(B \to A) = W,$$
   assuming $V(A) \subseteq V(B)$.

2. Let M_2 be the model consisting of $W = \{w, u, v\}$, $\le = \{(w,w),(u,u),(v,v),(w,u),(w,v)\}$, with $V(p) = \{u\}$ and $V(q) = \{v\}$. Then

   - $V(\neg p) = V(p)^{\#} = \{v\}$,
   - $V(\neg\neg p) = V(\neg p)^{\#} = \{u\}$,
   - so $V(\neg p \vee \neg\neg p) = \{u, v\}$.
   - $V(p \to q) = V(p)^{\#} = \{v\}$,
   - $V(q \to p) = V(q)^{\#} = \{u\}$,

- so $V((p \to q) \vee (q \to p)) = \{u, v\}$.

3. Let M be an arbitrary model. Then

   - $V(A \wedge B \to A) = (V(A \wedge B) - V(A))^{\#} = W$,
   - $V(A \to A \vee B) = (V(A) - V(A \vee B))^{\#} = W$,
   - $V(A \to (B \to A)) = (V(A) - V(B \to A))^{\#} = (V(A) - (V(B) - V(A))^{\#})^{\#} = W$. The last equation comes from the fact that for any upper set S, $S \subseteq S^{c\#}$.
   - Suppose $V(A) = V(A \to B) = W$. Then $\varnothing = \downarrow(V(A) - V(B)) = \downarrow(V(B)^c)$. Since $V(B)$ is upper, $V(B)^c$ is lower, so $\varnothing = \downarrow(V(B)^c) = V(B)^c$, or $W = V(B)$. This shows that modus ponens preserves validity.

4. Let W be any set and $\leq = W^2$. Then for any wff A, either $V(A) = W$ or $V(A) = \varnothing$. Therefore, $V(\neg\neg A) = V(A)$, and $V(\neg\neg A \to A) = W$.

The pair $\mathcal{F} := (W, \leq)$ in a Kripke model $M := (W, \leq, V)$ is also called a (Kripke) frame, and M is said to be a model based on the frame $\mathcal{F}$. The validity of a wff A at various levels can be found in the parent entry. Furthermore, A is valid (with respect to Krikpe semantics) for PL_i if it is valid in the class of all frames.

Based on the examples above, we see that

1. $(\neg q \to \neg p) \to (p \to q)$ is valid in M_1, while $(\neg p \to \neg q) \to (q \to p)$ is not.

2. $(p \to q) \vee (q \to p)$ is valid in the class of linearly ordered frames, while it is not valid in M_2, and neither is $\neg p \vee \neg\neg p$.

3. It is not hard to see that $\neg A \vee \neg\neg A$ is valid in any weakly connected frame, that is, for any $w \in W$, the set $\{u \mid w \leq u\}$ is linear.

4. Any wff in any of the schemas $A \wedge B \to A$, $A \to A \vee B$, or $A \to (B \to A)$ is valid in PL_i. See remark below for more detail.

5. Any theorem in the classical propositional logic is valid in any universal frame, that is, a frame with a universal relation.

Remark. It can be shown that every theorem of PL_i is valid. This is the soundness theorem of PL_i. Conversely, every valid wff is a theorem. This is known as the completeness theorem of PL_i. Furthermore, a wff valid in the class of finite frames is a theorem. This is the finite model property of PL_i.