properties of set difference

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Let A, B, C, D, X be sets.

1. $A \setminus B \subseteq A$. This is obvious by definition.

2. If $A, B \subseteq X$, then

   $$A \setminus B = A \cap B^{\complement}, \qquad (A \setminus B)^{\complement} = A^{\complement} \cup B, \qquad \text{and} \qquad A^{\complement} \setminus B^{\complement} = B \setminus A$$

   where $^{\complement}$ denotes complement in X.

   Proof. For the first equation, see `http://planetmath.org/PropertiesOfComplement`here. The second equation comes from the first: $(A \setminus B)^{\complement} = (A \cap B^{\complement})^{\complement} = (A^{\complement}) \cup (B^{\complement})^{\complement} = A^{\complement} \cup B$. The last equation also follows from the first: $A^{\complement} \setminus B^{\complement} = A^{\complement} \cap (B^{\complement})^{\complement} = B \cap A^{\complement} = B \setminus A$. $\square$

3. $A \subseteq B$ iff $A \setminus B = \emptyset$.

   Proof. Since $A \subseteq B$, $B^{\complement} \subseteq A^{\complement}$. Then $A \setminus B = A \cap B^{\complement} \subseteq A \cap A^{\complement} = \emptyset$. On the other hand, suppose $A \setminus B = \emptyset$. Then $A \cap B^{\complement} = \emptyset$ by property 1, which means $A \subseteq (B^{\complement})^{\complement} = B$. $\square$

4. $A \cap B = \emptyset$ iff $A \setminus B = A$.

   Proof. Suppose first that $A \cap B = \emptyset$. If $a \in A$, then $a \notin B$, so $a \in A \setminus B$, and hence $A \subseteq A \setminus B$. The equality is shown by applying property 1. Next suppose $A \setminus B = A$. If $a \in A$, then $a \in A \setminus B$, so $a \notin B$, which means $A \subseteq B^{\complement}$, or $A \cap B = \emptyset$. $\square$

5. $A \setminus \emptyset = A$ and $A \setminus A = \emptyset = \emptyset \setminus A$.

   Proof. The first equation follows from property 4 and the last two equations from property 3. $\square$

6. (de Morgan's laws on set difference):

   $$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) \qquad \text{and} \qquad A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

Proof. These laws follow from property 2 and the de Morgan's laws on set complement. For example, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) = A \cap (B \cap C)^{\complement} = A \cap (B^{\complement} \cup C^{\complement}) = (A \cap B^{\complement}) \cup (A \cap C^{\complement}) = (A \setminus B) \cup (A \setminus C)$. The other equation is proved similarly. $\square$

7. $A \setminus (A \cap B) = A \setminus B = (A \cup B) \setminus B$.

   Proof. The first equation follows from property 6: $A \setminus (A \cap B) = (A \setminus A) \cup (A \setminus B) = A \setminus B$ by property 5. Next, $(A \cup B) \setminus B = (A \cup B) \cap B^{\complement} = (A \cap B^{\complement}) \cup (B \cap B^{\complement}) = A \cap B^{\complement} = A \setminus B$, proving the second equation. $\square$

8. $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.

   Proof. Using property 2, we get $(A \cap B) \setminus C = (A \cap B) \cap C^{\complement} = (A \cap C^{\complement}) \cap (B \cap C^{\complement}) = (A \setminus C) \cap (B \setminus C)$. $\square$

9. $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

   Proof. $(A \cap B) \setminus (A \cap C) = (A \cap B) \cap (A \cap C)^{\complement} = (A \cap B) \cap (A^{\complement} \cup C^{\complement}) = ((A \cap B) \cap A^{\complement}) \cup ((A \cap B) \cap C^{\complement}) = (A \cap B) \cap C^{\complement} = A \cap (B \cap C^{\complement}) = A \cap (B \setminus C)$. $\square$

10. $(A \setminus B) \cap (C \setminus D) = (C \setminus B) \cap (A \setminus D)$

    Proof. Expanding the LHS, we get $A \cap B^{\complement} \cap C \cap D^{\complement}$. Expanding the RHS, we get the same thing. $\square$

11. $(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D)$.

    Proof. Starting from the RHS: $(A \cap C) \setminus (B \cup D) = ((A \cap C) \setminus B) \cap ((A \cap C) \setminus D) = (A \setminus B) \cap (C \setminus B) \cap (A \setminus D) \cap (C \setminus D) = (A \setminus B) \cap (C \setminus D)$, where the last equality comes from property 10. $\square$

Remarks.

1. Many of the proofs above use the properties of the set complement. Please see this `http://planetmath.org/PropertiesOfComplement`link for more detail.

2. All of the properties of $\setminus$ on sets can be generalized to `http://planetmath.org/DerivedBoole` subtraction on Boolean algebras.