constant functions and continuity

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It is easy to see that every constant function between topological spaces is continuous. A converse result is as follows.

Theorem. *Suppose X is path connected and D is a countable discrete topological space. If $f\colon X \to D$ is continuous, then f is a constant function.*

Proof. By `http://planetmath.org/FiniteAndCountableDiscreteSpaces`this result we can assume that D is either $\{1, \dots, n\}$, $n \geq 2$ or $\mathbb{Z}$, and these are equipped with the subspace topology of $\mathbb{R}$. Suppose $f(X)$ has at least two distinct elements, say $\alpha, \beta \in \mathbb{Z}$ so that

$$f(x) = \alpha, \quad f(y) = \beta$$

for some $x, y \in X$. Since X is path connected there is a continuous path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then $f \circ \gamma\colon [0,1] \to D$ is continuous. Since D has the subspace topology of $\mathbb{R}$, `http://planetmath.org/ContinuityIsPreserve` result implies that also $f \circ \gamma\colon [0,1] \to \mathbb{R}$ is continuous. Since $f \circ \gamma$ achieves two different values, it achieves uncountably many values, by the intermediate value theorem. This is a contradiction since $f \circ \gamma([0,1])$ is countable. $\square$