# example of transfinite induction


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Suppose we are interested in showing the property $\Phi(\alpha)$ holds for all ordinals $\alpha$ using transfinite induction. The proof basically involves three steps:

1. (first ordinal) show that $\Phi(0)$ holds;
2. (successor ordinal) if $\Phi(\beta)$ holds, then $\Phi(S\beta)$ holds;
3. (limit ordinal) if $\Phi(\gamma)$ holds for all $\gamma < \beta$ and $\beta = \sup\{\gamma \mid \gamma < \beta\}$, then $\Phi(\beta)$ holds.

Below is an example of a proof using transfinite induction.

**Proposition 1.** *$0 + \alpha = \alpha$ for any ordinal $\alpha$.*

*Proof.* Let $\Phi(\alpha)$ be the property: $0 + \alpha = \alpha$. We follow the three steps outlined above.

1. Since $0 + 0 = 0$ by definition, $\Phi(0)$ holds.
2. Suppose $0 + \beta = \beta$. By definition $0 + S\beta = S(0 + \beta)$, which is equal to $S\beta$ by the induction hypothesis, so $\Phi(S\beta)$ holds.
3. Suppose $\beta = \sup\{\gamma \mid \gamma < \beta\}$ and $0 + \gamma = \gamma$ for all $\gamma < \beta$. Then

   $$0 + \beta = 0 + \sup\{\gamma \mid \gamma < \beta\} = \sup\{0 + \gamma \mid \gamma < \beta\}.$$

   The second equality follows from definition. Furthermore, the last expression above is equal to $\sup\{\gamma \mid \gamma < \beta\} = \beta$ by the induction hypothesis. So $\Phi(\beta)$ holds.

Therefore $\Phi(\alpha)$ holds for every ordinal $\alpha$, which is the statement of the theorem, completing the proof. $\square$