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# Kripke semantics

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A *Kripke frame* (or simply a frame) $\mathcal{F}$ is a pair $(W, R)$ where $W$ is a non-empty set whose elements are called *worlds* or *possible worlds*, $R$ is a binary relation on $W$ called the *accessibility relation*. When $vRw$, we say that $w$ is *accessible* from $v$. A Kripke frame is said to have property $P$ if $R$ has the property $P$. For example, a symmetric frame is a frame whose accessibility relation is symmetric.

A *Kripke model* (or simply a model) $M$ for a propositional logical system (classical, intuitionistic, or modal) $\Lambda$ is a pair $(\mathcal{F}, V)$, where $\mathcal{F} := (W, R)$ is a Kripke frame, and $V$ is a function that takes each atomic formula of $\Lambda$ to a subset of $W$. If $w \in V(p)$, we say that $p$ is *true* at world $w$. We say that $M$ is a $\Lambda$-model *based on* the frame $\mathcal{F}$ if $M = (\mathcal{F}, V)$ is a model for the logic $\Lambda$.

**Remark**. Associated with each world $w$, we may also define a Boolean-valued valuation $V_w$ on the set of all wff's of $\Lambda$, so that $V_w(p) = 1$ iff $w \in V(p)$. In this sense, the Kripke semantics can be thought of as a generalization of the truth-value semantics for classical propositional logic. The truth-value semantics is just a Kripke model based on a frame with one world. Conversely, given a collection of valuations $\{V_w \mid w \in W\}$, we have model $(\mathcal{F}, V)$ where $w \in V(p)$ iff $V_w(p) = 1$.

Since the well-formed formulas (wff's) of $\Lambda$ are uniquely readable, $V$ may be inductively extended so it is defined on all wff's. The following are some examples:

- in classical propositional logic $\mathrm{PL}_c$, $V(A \to B) := V(A)^c \cup V(B)$, where $S^c := W - S$,
- in the modal propositional logic $K$, $V(\Box A) := V(A)^{\Box}$, where $S^{\Box} := \{u \mid \uparrow u \subseteq S\}$, and $\uparrow u := \{w \mid uRw\}$, and
- in intuitionistic propositional logic $\mathrm{PL}_i$, $V(A \to B) := (V(A) - V(B))^{\#}$, where $S^{\#} := (\downarrow S)^c$, and $\downarrow S := \{u \mid uRw, w \in S\}$.

Truth at a world can now be defined for wff's: a wff $A$ is *true* at world $w$ if $w \in V(A)$, and we write

$$M \models_w A \qquad \text{or} \qquad \models_w A$$

if no confusion arises. If $w \notin V(A)$, we write $M \not\models_w A$. The three examples above can be now interpreted as:

- $\models_w A \to B$ means $\models_w A$ implies $\models_w B$ in $\mathrm{PL}_c$,
- $\models_w \Box A$ means for all worlds $v$ with $wRv$, we have $\models_v A$ in $K$, and
- $\models_w A \to B$ means for all worlds $v$ with $wRv$, $\models_v A$ implies $\models_v B$ in $\mathrm{PL}_i$.

A wff $A$ is said to be *valid*

- in a model $M$ if $A$ in true at all possible worlds $w$ in $M$,
- in a frame if $A$ is valid in all models $M$ based on $\mathcal{F}$,
- in a collection $\mathcal{C}$ of frames if $A$ is valid in all frames in $\mathcal{C}$.

We denote

$$M \models A, \qquad \mathcal{F} \models A, \qquad \text{or} \qquad \mathcal{C} \models A$$

if $A$ is valid in $M, \mathcal{F}$, or $\mathcal{C}$ respectively.

A logic $\Lambda$, equipped with a deductive system, is *sound* in $\mathcal{C}$ if

$$\vdash A \qquad \text{implies} \qquad \mathcal{C} \models A.$$

Here, $\vdash A$ means that wff $A$ is a theorem deducible from the deductive system of $\Lambda$. Conversely, if

$$\mathcal{C} \models A \qquad \text{implies} \qquad \vdash A,$$

we say that $\Lambda$ is *complete* in $\mathcal{C}$.