creating an infinite model

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From the syntactic compactness theorem for first order logic, we get this nice (and useful) result:

Let $\mathbf{T}$ be a theory of first-order logic. If $\mathbf{T}$ has finite models of unboundedly large sizes, then $\mathbf{T}$ also has an infinite model.

Proof. Define the propositions

$$\Phi_n \equiv \underline{\exists x_1 \cdots \exists x_n . (x_1 \neq x_2) \wedge \cdots \wedge (x_1 \neq x_n) \wedge (x_2 \neq x_3) \wedge \cdots \wedge (x_{n-1} \neq x_n)}$$

(Φ_n says "*there exist (at least)* n different *elements in the world*"). Note that

$$\cdots \vdash \Phi_n \vdash \cdots \vdash \Phi_2 \vdash \Phi_1 .$$

Define a new theory

$$\mathbf{T}_\infty = \mathbf{T} \cup \{\Phi_1, \Phi_2, \ldots\} .$$

For any *finite* subset $\mathbf{T}' \subset \mathbf{T}_\infty$, we claim that $\mathbf{T}'$ is consistent: Indeed, $\mathbf{T}'$ contains axioms of $\mathbf{T}$, along with finitely many of $\{\Phi_n\}_{n \geq 1}$. Let Φ_m correspond to the largest index appearing in $\mathbf{T}'$. If $\mathcal{M}_m \models \mathbf{T}$ is a model of $\mathbf{T}$ with at least m elements (and by hypothesis, such as model exists), then $\mathcal{M}_m \models \mathbf{T} \cup \{\Phi_m\} \vdash \mathbf{T}'$.

So every finite subset of $\mathbf{T}_\infty$ is consistent; by the compactness theorem for first-order logic, $\mathbf{T}_\infty$ is consistent, and by Gödel's completeness theorem for first-order logic it has a model $\mathcal{M}$. Then $\mathcal{M} \models \mathbf{T}_\infty \vdash \mathbf{T}$, so $\mathcal{M}$ is a model of $\mathbf{T}$ with infinitely many elements ($\mathcal{M} \models \Phi_n$ for any n, so $\mathcal{M}$ has at least $\geq n$ elements for all n). $\square$