# arithmetical hierarchy is a proper hierarchy


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By definition, we have $\Delta_n = \Pi_n \cap \Sigma_n$. In addition, $\Sigma_n \cup \Pi_n \subseteq \Delta_{n+1}$.

This is proved by vacuous quantification. If $R$ is equivalent to $\phi(\vec{n})$ then $R$ is equivalent to $\forall x \phi(\vec{n})$ and $\exists x \phi(\vec{n})$, where $x$ is some variable that does not occur free in $\phi$.

More significant is the proof that all containments are proper. First, let $n \geq 1$ and $U$ be universal for 2-ary $\Sigma_n$ relations. Then $D(x) \leftrightarrow U(x,x)$ is obviously $\Sigma_n$. But suppose $D \in \Delta_n$. Then $D \in Pi_n$, so $\neg D \in \Sigma_n$. Since $U$ is universal, ther is some $e$ such that $\neg D(x) \leftrightarrow U(e,x)$, and therefore $\neg D(e) \leftrightarrow U(e,e) \leftrightarrow \neg U(e,e)$. This is clearly a contradiction, so $D \in \Sigma_n \setminus \Delta_n$ and $\neg D \in \Pi_n \setminus \Delta_n$.

In addition the recursive join of $D$ and $\neg D$, defined by

$$D \oplus \neg D(x) \leftrightarrow (\exists y < x[x = 2 \cdot y] \wedge D(x)) \vee (\neg \exists y < x[x = 2 \cdot y] \wedge \neg D(x))$$

Clearly both $D$ and $\neg D$ can be recovered from $D \oplus \neg D$, so it is contained in neither $\Sigma_n$ nor $\Pi_n$. However the definition above has only unbounded quantifiers except for those in $D$ and $\neg D$, so $D \oplus \neg D(x) \in \Delta_{n+1} \setminus \Sigma_n \cup \Pi_n$