a property of truth-value semantics for intuitionistic propositional logic

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In this entry, we show the following: if $\neg A$ is a tautology of V_n, then $\neg A$ is a theorem. First, we need the following lemma, which is the intuitionistic version of one for classical propositional logic, found `http://planetmath.org/CompletenessTheoremF`
Given an interpretation v, define

$$v[A] \text{ is } \begin{cases} \neg A & \text{if } v(A)=0, \\ \neg\neg A & \text{otherwise.} \end{cases}$$

It is easy to see that for any A, $v(v[A]) = n$ for any v, so that $v[A]$ is always true. In addition, we have the following table:

$v(A)$	$v[A]$	$v(B)$	$v[B]$	$v(A \wedge B)$	$v[A \wedge B]$	$v(A \vee B)$	$v[A \vee B]$	$v(A \to B)$	$v[A$
0	$\neg A$	0	$\neg B$	0	$\neg(A \wedge B)$	0	$\neg(A \vee B)$	0	$\neg\neg(A$
0	$\neg A$	$\neq 0$	$\neg\neg B$	0	$\neg(A \wedge B)$	$\neq 0$	$\neg\neg(A \vee B)$	n	$\neg\neg(A$
$\neq 0$	$\neg\neg A$	0	$\neg B$	0	$\neg(A \wedge B)$	$\neq 0$	$\neg\neg(A \vee B)$	0	$\neg(A$
$\neq 0$	$\neg\neg A$	$\neq 0$	$\neg\neg B$	$\neq 0$	$\neg\neg(A \wedge B)$	$\neq 0$	$\neg\neg(A \vee B)$	$\neq 0$	$\neg\neg(A$

The proofs of the following lemmas use instances of the theorem schemas below (proofs `http://planetmath.org/SomeTheoremSchemasOfIntuitionisticPropositionalLog`

1	2	3
$(C \to D) \to (\neg D \to \neg C)$	$\neg\neg\neg C \to \neg C$	$C \to \neg\neg C$

Lemma 1. $v[A], v[B] \vdash v[A \wedge B]$.

Proof. Since $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, by modus ponens and instances of the theorem schema 1 above, we have $\vdash \neg A \to \neg(A \wedge B)$ and $\vdash \neg B \to \neg(A \wedge B)$. This proves the first three cases.

For the last case, we start with the axiom $A \to (B \to A \wedge B)$, or $A \vdash B \to A \wedge B$ by the deduction theorem. Apply modus ponens twice to instances of schema 1, we get $A \vdash \neg\neg B \to \neg\neg(A \wedge B)$, or $\neg\neg B \vdash A \to \neg\neg(A \wedge B)$ by the deduction theorem twice. Again, applying modus ponens twice to instances of 1, we have $\neg\neg B \vdash \neg\neg A \to \neg\neg\neg\neg(A \wedge B)$, or $\neg\neg B, \neg\neg A \vdash \neg\neg\neg\neg(A \wedge B)$ by the deduction theorem. One application of modus ponens to an instance of schema 2, we have $\neg\neg B, \neg\neg A \vdash \neg\neg(A \wedge B)$, as desired. □

Lemma 2. $v[A], v[B] \vdash v[A \vee B]$.

Proof. Since $\vdash A \to A \vee B$ and $\vdash B \to A \vee B$, by modus ponens twice to instances of the schema 1, we have $\vdash \neg\neg A \to \neg\neg(A \vee B)$ and $\vdash \neg\neg B \to \neg\neg(A \vee B)$. This settles the last three cases.

For the first case, we use the axiom $(A \to\perp) \to ((B \to\perp) \to ((A\vee B) \to\perp))$, which is just $\neg A \to (\neg B \to \neg(A \vee B))$, or $\neg A, \neg B \vdash \neg(A \vee B)$ by the deduction theorem twice. □

Lemma 3. $v[A], v[B] \vdash v[A \to B]$.

Proof. For the first two, all we need is $\neg A \vdash \neg\neg(A \to B)$. To see this, we have deduction

$$A \to\perp, A, \perp, \perp\to B, B,$$

so $\neg A, A \vdash B$, or $\neg A \vdash A \to B$ by the deduction theorem. Since $(A \to B) \to \neg\neg(A \to B)$ is an instance of schema 3, by modus ponens, $\neg A \vdash \neg\neg(A \to B)$ as desired.

For the third, by the deduction theorem, it is enough to show $\neg\neg A, \neg B, A \to B \vdash\perp$. Now,

$$\neg A \to\perp, \neg B, A \to B, (A \to B) \to (\neg B \to \neg A), \neg B \to \neg A, \neg A, \perp$$

is a deduction of $\perp$ from $\neg\neg A, \neg B$, and $A \to B$, where $(A \to B) \to (\neg B \to \neg A)$ is a theorem.

For the last, all we need to show is $\neg\neg B \vdash \neg\neg(A \to B)$. We start with $B \to (A \to B)$, which is an axiom. Applying modus ponens twice to instances of 1, we have $\vdash \neg\neg B \to \neg\neg(A \to B)$, or $\neg\neg B \vdash \neg\neg(A \to B)$. □

Lemma 4. *Suppose $p_1, \ldots, p_n$ are all the propositional variables in a wff A. Then*

$$v[p_1], \ldots, v[p_m] \vdash v[A].$$

Proof. We use induction on the number n of primitive logical connectives ($\wedge, \vee$, and $\to$) in A. If $n = 0$, then A is either $\perp$ or a propositional variable p. If A is $\perp$, then $\perp\vdash\perp$, or $\vdash \neg \perp$, or $\vdash v[\perp]$. If A is p, then clearly $v[p] \vdash v[p]$. Now, if A has $n+1$ connectives, and is either $B \wedge C$, $B \vee C$, or $B \to C$, then B and C has no more than n connectives. By induction,

$$v[p_{i(1)}], \ldots, v[p_{i(s)}] \vdash v[B] \qquad \text{and} \qquad v[p_{j(1)}], \ldots, v[p_{j(t)}] \vdash v[C]$$

or

$$v[p_1], \ldots, v[p_m] \vdash v[B] \qquad \text{and} \qquad v[p_1], \ldots, v[p_m] \vdash v[C]$$

By the first three lemmas above, $v[B], v[C] \vdash v[A]$, so by modus ponens twice,

$$v[p_1], \ldots, v[p_m] \vdash v[A].$$

□

We are now ready for the main result:

Theorem 1. *If A is a tautology of V_n, then $\vdash \neg\neg A$.*

Proof. Let v be any interpretation, then $v[p_1], \ldots, v[p_m] \vdash v[A]$ by the last lemma, where $p_1, \ldots, p_m$ are all the propositional variables in A. Since A is a tautology,

$$v[p_1], \ldots, v[p_m] \vdash \neg\neg A.$$

If $m = 0$, then we are done. Otherwise, let v_1 and v_2 be two interpretations such that $v_1[p_i] = v_2[p_i]$ for $i = 1, \ldots, m-1$, and $v_1[p_m] = \neg p_m$ and $v_2[p_m] = \neg\neg p_m$, so that

$$v[p_1], \ldots, v[p_{m-1}], \neg p_m \vdash \neg\neg A \qquad \text{and} \qquad v[p_1], \ldots, v[p_{m-1}], \neg\neg p_m \vdash \neg\neg A.$$

By applying the deduction theorem twice to each of the above deductive relations, we get

$$v[p_1], \ldots, v[p_{m-1}], \neg A \vdash \neg\neg p_m \qquad \text{and} \qquad v[p_1], \ldots, v[p_{m-1}], \neg A \vdash \neg\neg\neg p_m.$$

Apply schema 2 to the second deductive relation above, we get

$$v[p_1], \ldots, v[p_{m-1}], \neg A \vdash \neg p_m.$$

By the deduction theorem once more, we have

$$v[p_1], \ldots, v[p_{m-1}] \vdash \neg A \to \neg\neg p_m \qquad \text{and} \qquad v[p_1], \ldots, v[p_{m-1}] \vdash \neg A \to \neg p_m.$$

With the axiom instance $(\neg A \to \neg p_m) \to ((\neg A \to \neg\neg p_m) \to \neg\neg A)$, apply modus ponens to each of the last two deductive relations, we get

$$v[p_1], \ldots, v[p_{m-1}] \vdash \neg\neg A,$$

so that $v[p_m]$ is removed from the original deductive relation. Continue this process until all of the $v[p_i]$ are removed on the left, and we get

$$\vdash \neg\neg A.$$

□

We record to immediate corollaries:

Corollary 1. *If* $\neg A$ *is a tautology of* V_n*, then* $\vdash \neg A$.

Proof. By the theorem, $\vdash \neg\neg\neg A$. But $\vdash \neg\neg\neg A \to \neg A$, $\vdash \neg A$ by modus ponens. $\square$

In the next corollary, we use $\vdash_c A$ and $\vdash_i$ to distinguish that A is a theorem of classical and intuitionistic propositional logic respectively.

Corollary 2. *(Glivenko's Theorem)* $\vdash_c A$ *iff* $\vdash_i \neg\neg A$.

Proof. If $\vdash_c A$, then by the soundness theorem of classical propositional logic, A is a tautology of truth-value semantics, which is just V_2, and therefore by the theorem above, $\vdash_i \neg\neg A$.

Conversely, if $\vdash_i \neg\neg A$, then certainly $\vdash_c \neg\neg A$, as `http://planetmath.org/IntuitionisticPro` is a subsystem of PL_c. Since $\neg\neg A \to A$ is a theorem of PL_c, we get $\vdash_c A$ by modus ponens. $\square$

In particular, $\vdash_c \perp$ iff $\vdash_i \perp$, since $\vdash_i \neg\neg \perp \leftrightarrow \perp$. In other words, PL_c is consistent iff PL_i is.