chain

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Defines	join of chains
Defines	chain homomorphism

Introduction

Let A be a poset ordered by $\leq$. A subset B of A is called a *chain* in A if any two elements of B are comparable. In other words, if $a, b \in B$, then either $a \leq b$ or $b \leq a$, that is, $\leq$ is a total order on B, or that B is a linearly ordered subset of A. When $a \leq b$, we also write $b \geq a$. When $a \leq b$ and $a \neq b$, we write $a < b$. When $a \geq b$ and $a \neq b$, then we write $a > b$. A poset is a *chain* if it is a chain as a subset of itself. The *cardinality* of a chain is the cardinality of the underlying set.

Below are some common examples of chains:

1. $\mathbf{n} := \{1, 2, \ldots, n\}$ is a chain under the usual order ($a \leq b$ iff $b - a$ is non-negative). This is an example of a *finite chain*: a chain whose cardinality is finite. Any finite set A can be made into a chain, since there is a bijection from $\mathbf{n}$ onto A, and the total order on A is induced by the order on $\mathbf{n}$. A chain that is not finite is called an *infinite chain*.

2. $\mathbb{N}$, the set of natural numbers, is a chain under the usual order. Here we have an example of a well-ordered set: a chain such that every non-empty subset has a minimal element (as in a poset). Other well-ordered sets are the set $\mathbb{Q}^+$ of positive rationals and $\mathbb{R}^+$ of positive reals. The well-ordering principle states that every set can be well-ordered. It can be shown that the axiom of choice is equivalent to the well-ordering principle.

3. $\mathbb{Z}$, the set of integers, is a chain under the usual order.

4. $\mathbb{Q}$, the set of rationals, is a chain under the usual order. This is an example of a *dense chain*: a chain such that for every pair of distinct elements $a < b$, there is an element c such that $a < c < b$.

5. $\mathbb{R}$, the set of reals, is a chain under the usual order. This is an example of a *Dedekind complete chain*: a chain such that every non-empty bounded subset has a supremum and an infimum.

Constructing chains

One easy way to produce a new chain from an existing one is to form the *dual* of the existing chain: if A is a chain, form A^∂ so that $a \leq b$ in A^∂ iff $b \leq a$ in A.

Another way to produce new chains from existing ones is to form a *join* of chains. Given two chains A, B, we can form a new chain $A \coprod B$. The basic idea is to form the disjoint union of A and B, and order this newly constructed set so that the order among elements of A is preserved, and similarly for B. Furthermore, any element of A is always less than any element of B (See `http://planetmath.org/AlternativeTreatmentOfConcatenation`here for detail).

With these two methods, one can construct many more examples of chains:

1. Take $\mathbb{R}$, and form $A = \mathbb{R} \coprod \{a\}$. Then A is a chain with a top element. If we take $B = \{b\} \coprod A$, we get a chain with both a top and a bottom element. In fact, B is an example of a complete chain: a chain such that every subset has a supremum and an infimum. Observe that any finite chain is complete.

2. We can form $\mathbb{N}^{\partial}$ which is a set with 1 as the top element. We can also form $\mathbb{N}^{\partial} \coprod \mathbb{N}$, which has neither top nor bottom, or $\mathbb{N} \coprod \mathbb{N}^{\partial}$, which has both a top and a bottom element, but is not complete, as $\mathbb{N}$, considered as a subset, has no top. Likewise, $\mathbb{N}^{\partial}$ is bottomless.

The idea of joining two chains can be generalized. Let $\{A_i \mid i \in I\}$ be a family of chains indexed by I, itself a chain. We form $\coprod_{i \in I} A_i$ as follows: take the disjoint union of A_i, which we also write as $\coprod_{i \in I} A_i$. Then $(a, i) \leq (b, j)$ iff either $i = j$ and $a \leq b$, or $i < j$.

For example, let $I = \mathbb{R}$ and $A_i = \mathbb{R}$, with $i \in I$. Then $\coprod_{i \in I} A_i$ is a chain, whose total order is the lexicographic order on $\mathbb{R}^2$. If we well-order $I = \mathbb{R}$, then $\coprod_{i \in I} A_i$ is another chain called the long line.

Chain homomorphisms

Let A, B be chains. A function f from A to B is said to be a *chain homomorphism* if it is a poset homomorphism (it preserves order). $f(A)$ is the homomorphic image of A in B. Two chains are homomorphic if there is a chain homomorphism from one to another. A chain homomorphism is an embedding if it is one-to-one. If A embeds in B, we write $A \subseteq B$. A strict embedding is an embedding that is not onto. If A strictly embeds in B, we write $A \subset B$. An onto embedding is also called an isomorphism. If A is isomorphic to B, we write $A \cong B$.

Some properties:

- Two finite chains are isomorphic iff they have the same cardinality.
- Top and bottom elements are preserved by chain isomorphisms. In other words, if $f : A \to B$ is a chain isomorphism and if $a \in A$ is the top (bottom) element, then $f(a)$ is the top (bottom) element in B.
- In addition, the properties of being well-ordered, dense, Dedekind complete, and complete are all preserved under a chain isomorphism.
- $A \subseteq A \coprod B$. More generally $A_i \subseteq \coprod_{i \in I} A_i$.
- $(A \coprod B) \coprod C \cong A \coprod (B \coprod C)$.
- If k is the bottom element of I, and A_k has a top element x, then there is a chain homomorphism $f : \coprod_{i \in I} A_i \to A_k$ given by $f(a, i) = a$ if $i = k$ and $f(a, i) = x$ if $i > k$.
- Dually, if k is the top of I and A_k has a bottom x, then there is a chain homomorphism $f : \coprod_{i \in I} A_i \to A_k$ given by $f(a, i) = a$ if $i = k$ and $f(a, i) = x$ if $i < k$.
- If A_k has both a bottom x and a top y, then we may define $f : \coprod_{i \in I} A_i \to A_k$ by $f(a, i) = a$ if $i = k$, $f(a, i) = x$ if $i < k$ and $f(a, i) = y$ if $k < i$.

Some examples:

- $\mathbf{n} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
- $\mathbf{n} \coprod \mathbf{m} \cong \mathbf{p}$ iff $p = m + n$ for any non-negative integers m, n and p.
- $\mathbf{n} \coprod \mathbb{N} \cong \mathbb{N}$ for any non-negative integer n.
- $\mathbb{N} \coprod \mathbb{N}^\partial \ncong \mathbb{N}^\partial \coprod \mathbb{N}$.
- Let I be the chain over $\mathbb{R}$ under the usual order, and J the chain over $\mathbb{R}$ under a well-ordering. Then $\coprod_{i \in I} \mathbb{R} \ncong \coprod_{j \in J} \mathbb{R}$.