truth-value semantics for classical propositional logic

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In classical propositional logic, an *interpretation* of a well-formed formula (wff) p is an assignment of truth (=1) or falsity (=0) to p. Any interpreted wff is called a proposition.

An *interpretation* of all wffs over the variable set V is then a Boolean function on $\overline{V}$. However, one needs to be careful, for we do not want both p and $\neg p$ be interpreted as true simultaneously (at least not in classical propositional logic)! The proper way to find an interpretation on the wffs is to start from the atoms.

Call any Boolean-valued function ν on V a *valuation* on V. We want to extend ν to a Boolean-valued function $\overline{\nu}$ on $\overline{V}$ of all wffs. The way this is done is similar to the construction of wffs; we build a sequence of functions, starting from ν on V_0, next ν_1 on V_1, and so on... Finally, we take the union of all these "approximations" to arrive at $\overline{\nu}$. So how do we go from ν to ν_1? We need to interpret $\neg p$ and $p \vee q$ from the valuations of atoms p and q. In other words, we must also interpret logical connectives too.

Before doing this, we define a truth function for each of the logical connectives:

- for $\neg$, define $f : \{0,1\} \to \{0,1\}$ given by $f(x) = 1 - x$.
- for $\vee$, define $g : \{0,1\}^2 \to \{0,1\}$ given by $g(x,y) = \max(x,y)$.

As we are trying to *interpret* $\neg$ (not) and $\vee$ (or), the choices for f and g are clear. The values $0,1$ are interpreted as the usual integers (so they can be subtracted and ordered, etc...). Hence f and g make sense.

Next, recall that V_i are sets of wffs built up from wffs in V_{i-1} (see construction of well-formed formulas for more detail). We define a function $\nu_i : V_i \to \{0,1\}$ for each i, as follows:

- $\nu_0 := \nu$
- suppose ν_i has been defined, we define $\nu_{i+1} : V_{i+1} \to \{0,1\}$ given by

$$\nu_{i+1}(p) := \begin{cases} \nu_i(p) & \text{if } p \in V_i, \\ f(\nu_i(q)) & \text{if } p = \neg q \text{ for some } q \in V_i, \\ g(\nu_i(q), \nu_i(r)) & \text{if } p = q \vee r \text{ for some } q, r \in V_i. \end{cases}$$

Finally, take $\overline{\nu}$ to be the union of all the approximations:

$$\overline{\nu} := \bigcup_{i=0}^{\infty} \nu_i.$$

Then, by unique readability of wffs, $\overline{\nu}$ is an interpretation on $\overline{V}$.

Remark. If $\perp$ is included in the language of the logic (as the symbol for falsity), we also require that $\nu_i(\perp) = \overline{\nu}(\perp) = 0$.

Remark $\overline{V}$ can be viewed as an inductive set over V with respect to the $\neg$ and $\vee$, viewed as operations on $\overline{V}$. Furthermore, $\overline{V}$ is freely generated by V, since each V_{i+1} can be partitioned into sets V_i, $\{(p \vee q) \mid p, q \in V_i\}$, and $\{(\neg p) \mid p \in V_i\}$, and each partition is non-empty. As a result, any valuation ν on V uniquely extends to a valuation $\overline{\nu}$ on $\overline{V}$.

Definitions. Let p, q be wffs in $\overline{V}$.

- p is *true* or *satisfiable* for some valuation ν if $\overline{\nu}(p) = 1$ (otherwise, it is *false* for ν).
- p is true for every valuation ν, then p is said to be *valid* (or *tautologous*). If p is false for every ν, it is *invalid.* If p is valid, we write $\models p$.
- p implies q for a valuation ν if $\overline{\nu}(p) = 1$ implies $\overline{\nu}(q) = 1$. p *semantically implies* if p implies q for every valuation ν, and is denoted by $p \models q$.
- p is *equivalent* to q for ν if $\overline{\nu}(p) = \overline{\nu}(q)$. They are *semantically equivalent* if they are equivalent for every ν, and written $p \equiv q$.

Semantical equivalence is an equivalence relation on $\overline{V}$.

The above can be easily generalized to sets of wffs. Let T be a set of propositions.

- T is true or satisfiable for ν if $\overline{\nu}(T) = \{1\}$ (otherwise, it is false for ν).
- T is valid if it is true for every ν; it is *invalid* if it is false for every ν. If T is valid, we write $\models T$.
- T implies p for ν if $\overline{\nu}(T) = \{1\}$ implies $\overline{\nu}(p) = 1$. T semantically implies p if T implies p for every ν, and is denoted by $T \models p$.
- T_1 implies T_2 for ν if, for every $p \in T_2$, T_1 implies p for ν. T_1 semantically implies T_2 if T_1 implies T_2 for every ν, and is denoted by $T_1 \models T_2$.
- T_1 is equivalent to T_2 for ν if for some valuation ν, T_1 implies T_2 for ν and T_2 implies T_1 for ν. T_1 and T_2 are semantically equivalent if $T_1 \models T_2$ and $T_2 \models T_1$, written $T_1 \equiv T_2$.

Clearly, $\models p$ iff $\varnothing \models p$, and $T \models p$ iff $T \models \{p\}$.