axiom of determinacy

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When doing descriptive set theory, it is conventional to use either ω^ω or 2^ω as your space of "reals" (these spaces are homeomorphic to the irrationals and the Cantor set, respectively). Throughout this article, I will use the term "reals" to refer to ω^ω.

Let $X \subseteq \omega^\omega$ be given and consider the following game on X played between two players, I and II: I starts by saying a natural number; II hears this number and replies with another (or possibly the same one); I hears this and replies with another; etc. The sequence of numbers said (in the order they were said) is a point in ω^ω. I wins if this point is in X, otherwise II wins.

A map $\sigma : \omega^{<\omega} \to \omega$ is said to be a winning strategy for I if it has the following property: if, after the play has gone $n_0 n_1 \ldots n_M$, I plays $\sigma(n_0 \ldots n_M)$ for each move, then I wins. A winning strategy for II is defined analogously.

The *axiom of determinacy* (AD) states that every such game is determined, that is either I or II has a winning strategy.

Using choice, a non-determined game can be constructed directly: for $\alpha < \mathfrak{c}$, enumerate the uncountable closed subsets of the reals F_α. Now construct two sequences $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ and $\langle y_\alpha : \alpha < \mathfrak{c}\rangle$ by choosing x_α, y_α as distinct points from F_α which are not in $\{x_\gamma, y_\gamma : \gamma < \alpha\}$ (this is possible as each uncountable closed set has cardinality $\mathfrak{c}$). Then the game on the set of all x_αs is non-determined.

From ZF+AD, one may prove many nice facts about the reals, such as: any subset is Lebesgue measurable, any subset has a perfect subset and the continuum hypothesis. ZF+AD also proves the axiom of countable choice.

AD itself is not taken seriously by many set theorists as a genuine alternative to choice. However, there is a weakening of AD (the axiom of quasi-projective determinacy, or QPD, which states that all games in $\mathsf{L}[\mathbb{R}]$ are determined) which is consistent with ZFC (in fact, it's equiconsistent to a large cardinal axiom) which is a serious axiom candidate.