elementary embedding

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Entry type	Definition
Classification	msc 03C99
Synonym	elementary monomorphism
Defines	elementary substructure
Defines	elementary extension
Defines	elementary chain

Let τ be a signature and $\mathcal{A}$ and $\mathcal{B}$ be two structures for τ such that $f : \mathcal{A} \to \mathcal{B}$ is an embedding. Then f is said to be *elementary* if for every first-order formula $\phi \in F(\tau)$, we have

$$\mathcal{A} \vDash \phi \quad \text{iff} \quad \mathcal{B} \vDash \phi.$$

In the expression above, $\mathcal{A} \vDash \phi$ means: if we write $\phi = \phi(x_1, \ldots, x_n)$ where the free variables of ϕ are all in $\{x_1, \ldots, x_n\}$, then $\phi(a_1, \ldots, a_n)$ holds in $\mathcal{A}$ for any $a_i \in \mathcal{A}$ (the underlying universe of $\mathcal{A}$).

If $\mathcal{A}$ is a substructure of $\mathcal{B}$ such that the inclusion homomorphism is an elementary embedding, then we say that $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$, or that $\mathcal{B}$ is an elementary extension of $\mathcal{A}$.

Remark. A chain $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_n \subseteq \cdots$ of τ-structures is called an *elementary chain* if $\mathcal{A}_i$ is an elementary substructure of $\mathcal{A}_{i+1}$ for each $i = 1, 2, \ldots$. It can be shown (Tarski and Vaught) that

$$\bigcup_{i<\omega} \mathcal{A}_i$$

is a τ-structure that is an elementary extension of $\mathcal{A}_i$ for every i.