



proof of Hausdorff paradox

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We start by outlining the general ideas, followed by the strict proof.

General approach

The unit sphere S^2 in the Euclidean space \mathbb{R}^3 is finite in the sense that it is <http://planetmath.org/node/4826> bounded and has a finite volume: $\frac{4}{3}\pi$.

On the other hand, S^2 is an infinite point set. As everyone who has ever visited Hilbert's Hotel knows, it is possible to split an infinite set, such as the natural numbers, in pieces and all pieces are, in a sense, equal to the original set, for example if

$$\mathbb{N}_3 := \{n \in \mathbb{N} \mid n \text{ is divisible by } 3\},$$

then naïvely \mathbb{N}_3 has a third of the size of \mathbb{N} , and half of the size of $\mathbb{N} \setminus \mathbb{N}_3$. There exists, however, a bijection from \mathbb{N} to $\mathbb{N} \setminus \mathbb{N}_3$, so \mathbb{N}_3 has, again naïvely, both half and a third of the size of \mathbb{N} .

To do the same thing with S^2 , bijections won't suffice, however: we need isometries to establish congruence. We can *almost* reduce this problem to the case of bijections in the following way. The group R of rotations (rotations are isometries) in \mathbb{R}^3 acts on S^2 (say, from the right). Given an countably infinite subgroup G of R which acts freely (or almost so) on S^2 , take a set of <http://planetmath.org/node/1517> orbit representatives M of the action of G on S^2 (this requires the <http://planetmath.org/node/310> axiom of choice). Then a disjoint decomposition G_1, \dots, G_n of G into countable sets acting on M yields a disjoint decomposition of S^2 . Doing similar juggling with G_1, \dots, G_n as we did with \mathbb{N} , \mathbb{N}_3 and $\mathbb{N} \setminus \mathbb{N}_3$ above should yield the result, *if* all the G_1, \dots, G_n are related by fixed isometries, for example if $G_1 = G_2\varphi$ for some isometry φ , and similar relations for the other pieces. This last restriction will make the proof possible only in three or more dimensions, and indeed, Banach and Tarski later showed in [?] that analogous theorems on the line or in the plane do not hold.

The proof

Let G be the subgroup of rotations of \mathbb{R}^3 generated by a half rotation φ and a one third rotation ψ around different axes. We fix an orthonormal basis for the rest of the proof, so we may identify φ and ψ with their rotation matrices

acting from the right on the row vectors of \mathbb{R}^3 , such that without restriction

$$\psi = \begin{pmatrix} \lambda & \mu & 0 \\ -\mu & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi = \begin{pmatrix} -\cos \vartheta & 0 & \sin \vartheta \\ 0 & -1 & 0 \\ \sin \vartheta & 0 & \cos \vartheta \end{pmatrix},$$

where $\lambda = \cos \frac{2\pi}{3} = -\frac{1}{2}$, $\mu = \sin \frac{2\pi}{3} = \frac{1}{2}\sqrt{3}$ and ϑ is arbitrary for now.

Since $\varphi^2 = 1$ and $\psi^3 = 1$, there exists for each element g from G other than the unity, ϕ , ψ or ψ^2 a positive integer n and numbers $m_k \in \{1, 2\}$, $1 \leq k \leq n$, such that g can be written in one of the following ways:

$$\begin{aligned} (\alpha) & \left(\prod_{k=1}^n \varphi \psi^{m_k} \right), \\ (\beta) & \psi^{m_1} \left(\prod_{k=2}^n \varphi \psi^{m_k} \right) \varphi, \\ (\gamma) & \left(\prod_{k=1}^n \varphi \psi^{m_k} \right) \varphi, \\ (\delta) & \psi^{m_1} \left(\prod_{k=2}^n \varphi \psi^{m_k} \right) \quad (n \geq 2). \end{aligned}$$

To have complete control over the structure of G , we fix ϑ in such a way, that g can be written *uniquely* in one of the ways (α) – (δ) , with fixed n and m_k . In other words: we fix ϑ such that the unity 1 cannot be written in one of the ways (α) – (δ) .

To see how to do that, let us see to where the vector $(0, 0, 1)$ is transported by a transformation of type (α) . It is easily verified that

$$\varphi\psi = \begin{pmatrix} -\lambda \cos \vartheta & -\mu \cos \vartheta & \sin \vartheta \\ \mu & -\lambda & 0 \\ \lambda \sin \vartheta & \mu \sin \vartheta & \cos \vartheta \end{pmatrix} \text{ and } \varphi\psi^2 = \begin{pmatrix} \lambda \cos \vartheta & -\mu \cos \vartheta & \sin \vartheta \\ \mu & \lambda & 0 \\ -\lambda \sin \vartheta & \mu \sin \vartheta & \cos \vartheta \end{pmatrix},$$

so that $(0, 0, 1)\varphi\psi = (\lambda \sin \vartheta, \mu \sin \vartheta, \cos \vartheta)$ and $(0, 0, 1)\varphi\psi^2 = (-\lambda \sin \vartheta, \mu \sin \vartheta, \cos \vartheta)$.

More generally, let n be a positive integer, p_1, p_2 polynomials of degree $n-1$ and p_3 a polynomial of degree n , then

$$\begin{aligned} & (p_1(\cos \vartheta) \sin \vartheta, p_2(\cos \vartheta) \sin \vartheta, p_3(\cos \vartheta))\varphi\psi = \\ & ((\lambda p_3(\cos \vartheta) + \mu p_2(\cos \vartheta) - \lambda \cos \vartheta p_1(\cos \vartheta)) \sin \vartheta, (\mu p_3(\cos \vartheta) - \lambda p_2(\cos \vartheta) - \mu \cos \vartheta p_1(\cos \vartheta)) \sin \vartheta, (1 - \end{aligned}$$

$$(p_1(\cos \vartheta) \sin \vartheta, p_2(\cos \vartheta) \sin \vartheta, p_3(\cos \vartheta)) \varphi \psi^2 = \\ ((-\lambda p_3(\cos \vartheta) + \mu p_2(\cos \vartheta) + \lambda \cos \vartheta p_1(\cos \vartheta)) \sin \vartheta, (\mu p_3(\cos \vartheta) + \lambda p_2(\cos \vartheta) - \mu \cos \vartheta p_1(\cos \vartheta)) \sin \vartheta, (1$$

By induction, it follows that $(0, 0, 1)$ is transported by a transformation of type (α) to a vector whose third component is a nonconstant polynomial p in $\cos \vartheta$. If we restrict ϑ such that $0 \leq \vartheta \leq \pi$, there are only finitely many values for ϑ such that $p(\cos \vartheta) = 1$. Given all possible combinations of n and m_k , $1 \leq k \leq n$, there are in total only countably many problematic values for ϑ , so we can easily fix hereby an unproblematic one. Now that the case (α) has been dealt with, so have been the others automatically. For assume one could write a 1 of type (γ) , one could convert it to type (δ) by $1 = \varphi 1 \varphi$, and from type δ to type (α) or type (β) by $1 = \psi^{3-m_1} 1 \psi^{m_1}$, and lastly from type (β) to type (α) by $1 = \varphi 1 \varphi$, completing the contradiction.

Now that we know the structure of G , how does it act on S^2 ? Certainly not freely, since every rotation other than the unity has its axis as fixed point set, which in case of the sphere makes two fixed points per rotation. The product of two rotations is again a rotation. Furthermore G is finitely generated and so is countable. So there are only countably many points of S^2 where the action of G fails to be free. Denote the set of these points by D , so that G acts freely on $E := S^2 \setminus D$. The action creates a partition on E into orbits. This allows us to choose a set M , such that M meets any orbit in precisely one element. We have then the disjoint union

$$E = \bigcup_{g \in G} Mg,$$

where Mg is the image of M under the action of the group element g . We now define three sets A, B, C to be the smallest sets satisfying:

- $M1 = M \subseteq A$;
- if Mg is a subset of A, B or C , then $Mg\varphi$ is a subset of B, A or A , respectively;
- if Mg is a subset of A, B or C , then $Mg\psi$ is a subset of B, C or A , respectively;
- if Mg is a subset of A, B or C , then $Mg\psi^2$ is a subset of C, A or B , respectively.

The sets A , B and C are well-defined because we ensured the uniqueness of the representations (α) – (δ) above.

The sets A , B , C and $B \cup C$ are all congruent by virtue of

$$A\psi = B, \quad A\psi^2 = C \quad \text{and} \quad A\varphi = B \cup C.$$

Since $S^2 = A \cup B \cup C \cup D$, a disjoint union, and D is countable, the theorem is proven.

References

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- [H] F. HAUSDORFF, Bemerkung über den Inhalt von Punktmengen, *Math. Ann.* 75, 428–433, (1915).