axiom system for intuitionistic propositional logic

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There are several Hilbert-style axiom systems for intuitionistic propositional logic, or PL_i for short. One such a system is by Heyting, and is presented in `http://planetmath.org/IntuitionisticLogic`this entry. Here, we describe another, based on the one by Kleene. The language of the logic consists of a countably infinite set of propositional letters $p, q, r, \ldots$, and symbols for logical connectives $\to$, $\wedge$, $\vee$. Well-formed formulas (wff) are defined recursively as follows:

- propositional letters are wff
- if A, B are wff, so are $A \to B$, $A \wedge B$, $A \vee B$, and $\neg A$.

In addition, $A \leftrightarrow B$ (biconditional) is the abbreviation for $(A \to B) \wedge (B \to A)$, like PL_c (classical propositional logic),

We also use parentheses to avoid ambiguity. The axiom schemas for PL_i are

1. $A \to (B \to A)$.
2. $A \to (B \to A \wedge B)$.
3. $A \wedge B \to A$.
4. $A \wedge B \to B$.
5. $A \to A \vee B$.
6. $B \to A \vee B$.
7. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$.
8. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$.
9. $(A \to B) \to ((A \to \neg B) \to \neg A)$.
10. $\neg A \to (A \to B)$.

where A, B, and C are wff's. In addition, modus ponens is the only rule of inference for PL_i.

As usual, given a set Σ of wff's, a deduction of a wff A from Σ is a finite sequence of wff's $A_1, \ldots, A_n$ such that A_n is A, and A_i is either an axiom, a wff in Σ, or is obtained by an application of modus ponens on A_j to A_k where $j, k < i$. In other words, A_k is the wff $A_j \to A_i$. We write

$$\Sigma \vdash_i A$$

to mean that A is a deduction from Σ. When Σ is the empty set, we say that A is a theorem (of PL_i), and simply write $\vdash_i A$ to mean that A is a theorem.

As with PL_c, the deduction theorem holds for PL_i. Using the deduction theorem, one can derive the well-known theorem schemas listed below:

1. $A \wedge B \to B \wedge A$.
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
3. $A \wedge \neg A \to B$
4. $A \to \neg\neg A$
5. $\neg\neg\neg A \to \neg A$
6. $(A \to B) \to (\neg B \to \neg A)$
7. $\neg A \wedge \neg A$
8. $\neg\neg(A \vee \neg A)$

For example, the first schema can be proved as follows:

Proof. From the deduction,

1. $A \wedge B \to A$,
2. $A \wedge B \to B$,
3. $A \wedge B$,
4. A,
5. B,
6. $B \to (A \to B \wedge A)$,
7. $A \to B \wedge A$,
8. $B \wedge A$,

we have $A \wedge B \vdash_i B \wedge A$, and therefore $\vdash_i (A \wedge B) \to (B \wedge A)$ by the deduction theorem. □

Deductions of the other theorem schemas can be found `http://planetmath.org/SomeTheoremS` In fact, it is not hard to see that $\vdash_i X$ implies $\vdash_c X$ (that X is a theorem of PL_c). The converse is false. The following are theorems of PL_c, not PL_i:

1. $A \vee \neg A$.
2. $\neg\neg A \to A$
3. $(\neg A \to \neg B) \to (B \to A)$
4. $((A \to B) \to A) \to A$
5. $(\neg A \to B) \to ((\neg A \to \neg B) \to A)$

Remark. It is interesting to note, however, if any one of the above schemas were added to the list of axioms for PL_i, then the resulting system is an axiom system for PL_c. In notation,

$$\mathrm{PL}_i + X = \mathrm{PL}_c,$$

where X is one of the schemas above. When this equation holds for some X, it is necessary that $\vdash_c X$ and $\nvdash_i X$. However, this condition is not sufficient to imply the equation, even if $\mathrm{PL}_i + X$ is consistent (that is, the schema $C \wedge \neg C$ of wff's are not theorems). One such schema is $\neg\neg A \vee \neg A$. A logical system PL such that $\mathrm{PL}_i \leq \mathrm{PL} \leq \mathrm{PL}_c$ is called an *intermediate logic*. It can be shown that there are infinitely many such intermediate logics.

Remark. Yet another popular axiom system for PL_i uses the symbol $\perp$ (for falsity) instead of $\neg$. The wff's in this language consists of all propositional letters, the symbol $\perp$, and $A \wedge B$, $A \vee B$, and $A \to B$, whenever A and B are wff's. The axiom schemas consist of the first seven axiom schemas in the first system above, as well as

1. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ (the second theorem schema above)
2. $\perp \to A$.

$\neg A$ is the abbreviation for $A \to \perp$. The only rule of inference is modus ponens. Deductions and theorems are defined in the exact same way as above. Let us write $\vdash_{i1} A$ to mean wff A is a theorem in this axiom system. As mentioned, both axiom systems are equivalent, in that $\vdash_{i1} A$ implies $\vdash_i A$, and $\vdash_i A$ implies $\vdash_{i1} A^*$, where A^* is the wff obtained from A by replacing every occurrence of $\perp$ by the wff $(p \wedge \neg p)$, where p is a propositional letter not occurring in A.