principle of finite induction proven from the well-ordering principle for natural numbers

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We give a proof for the "strong" formulation.

Let S be a set of natural numbers such that n belongs to S whenever all numbers less than n belong to S (i.e., assume $\forall n (\forall m < n \; m \in S) \Rightarrow n \in S$, where the quantifiers range over all natural numbers). For indirect proof, suppose that S is not the set of natural numbers $\mathbb{N}$. That is, the complement $\mathbb{N} \setminus S$ is nonempty. The well-ordering principle for natural numbers says that $\mathbb{N} \setminus S$ has a smallest element; call it a. By assumption, the statement $(\forall m < a \; m \in S) \Rightarrow a \in S$ holds. Equivalently, the contrapositive statement $a \in \mathbb{N} \setminus S \Rightarrow \exists m < a \; m \in \mathbb{N} \setminus S$ holds. This gives a contradition since the element a is an element of $\mathbb{N} \setminus S$ and is, moreover, the *smallest* element of $\mathbb{N} \setminus S$.