# the Cartesian product of a finite number of countable sets is countable

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| Canonical name | TheCartesianProductOfAFiniteNumberOfCountableSetsIsCountable |
| Date of creation | 2013-03-22 15:19:45 |
| Last modified on | 2013-03-22 15:19:45 |
| Owner | BenB (9643) |
| Last modified by | BenB (9643) |
| Numerical id | 13 |
| Author | BenB (9643) |
| Entry type | Theorem |
| Classification | msc 03E10 |
| Synonym | The product of a finite number of countable sets is countable |
| Related topic | CardinalityOfACountableUnion |
| Related topic | AlgebraicNumbersAreCountable |
| Related topic | CardinalityOfTheRationals |

**Theorem 1** *The Cartesian product of a finite number of countable sets is countable.*

*Proof:* Let $A_1, A_2, \ldots, A_n$ be countable sets and let $S = A_1 \times A_2 \times \cdots \times A_n$. Since each $A_i$ is countable, there exists an injective function $f_i \colon A_i \to \mathbb{N}$. The function $h \colon S \to \mathbb{N}$ defined by

$$h(a_1, a_2, \ldots a_n) = \prod_{i=1}^{n} p_i^{f_i(a_i)}$$

where $p_i$ is the $i$th prime is, by the fundamental theorem of arithmetic, a bijection between $S$ and a subset of $\mathbb{N}$ and therefore $S$ is also countable.

Note that this result does *not* (in general) extend to the Cartesian product of a countably infinite collection of countable sets. If such a collection contains more than a finite number of sets with at least two elements, then Cantor's diagonal argument can be used to show that the product is not countable.

For example, given $B = \{0, 1\}$, the set $F = B \times B \times \cdots$ consists of all countably infinite sequences of zeros and ones. Each element of $F$ can be viewed as a binary fraction and can therefore be mapped to a unique real number in $[0, 1)$ and $[0, 1)$ is, of course, not countable.