normal modal logic

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The study of modal logic is based on the concept of a *logic*, which is a set Λ of wff's satisfying the following:

- contains all tautologies, and
- is closed under modus ponens.

The last condition means: if A and $A \to B$ are in Λ, so is B in Λ.

A *normal modal logic* is a modal logic Λ that includes the *law of distribution* K (after Kripke):

$$\Box(A \to B) \to (\Box A \to \Box B)$$

as an axiom schema, and obeying the *rule of necessitation RN*:

$$\text{from } \vdash A \text{, we may infer } \vdash \Box A\text{: if } A \in \Lambda \text{, then } \Box A \in \Lambda.$$

Normal modal logics are the most widely studied modal logics. The smallest normal modal logic is called **K**. Other normal modal logics are built from **K** by attaching wff's as axiom schemas. Below is a list of schemas used to form some of the most common normal modal logics:

- 4: $\Box A \to \Box\Box A$
- 5: $\Diamond A \to \Box\Diamond A$
- D: $\Box A \to \Diamond A$
- T: $\Box A \to A$
- B: $A \to \Box\Diamond A$
- C: $\Box(A \wedge \Box B) \to \Box(A \wedge B)$
- M: $\Box(A \wedge B) \to \Box A \wedge \Box B$
- G: $\Diamond\Box A \to \Box\Diamond A$
- L: $\Box(A \wedge \Box A \to B) \vee \Box(B \wedge \Box B \to A)$
- W: $\Box(\Box A \to A) \to \Box A$

For example, the normal modal logic **D** is the smallest normal modal logic containing D as its axiom schema.

Notation. The smallest normal modal logic containing schemas $\Sigma_1, \ldots, \Sigma_n$ is typically denoted

$$\mathbf{K\Sigma_1 \cdots \Sigma_n}.$$

It is easy to see that $\mathbf{K\Sigma_1 \cdots \Sigma_n}$ can be built from the "bottom up": call a finite sequence of wff's a deduction if each wff is either a tautology, an instance of Σ_i for some i, or as a result of an application of modus ponens or necessitation on earlier wff's in the sequence. A wff is deducible from if it is the last member of some deduction. Let Λ_k be the set of all wff's deducible from deductions of lengths at most k. Then

$$\mathbf{K\Sigma_1 \cdots \Sigma_n} = \bigcup_{i=1}^{\infty} \Lambda_i$$

Below are some of the most common normal modal logics:

name	**D**	**T**	**B**	**S4**	**S5**	**GL**	**K4.3**	**S4.3**
notation	**KD**	**KT**	**KTB**	**KT4**	**KT5**	**KW**	**K4L**	**KT4L**

Remarks

- **D** is commonly used in the study of deontic logic (logic of obligation). Extensions of **D** such as **KD4** and **KD45** are used in the study of doxastic logic (logic of belief).
- **GL** is known as provability logic, where $\Box A$ means A is provable in Peano arithmetic.
- **S4** and **S5** are two of the Lewis' 5 modal logical systems. They are commonly used in the study of epistemic logic (logic of knowledge). The modal logics **S1**, **S2**, and **S3** are non-normal.

Semantics

The dominant semantics for normal modal logic is the Kripke semantics, or relational semantics. More on this can be found `http://planetmath.org/node/12541`here. A logic is *sound* in a class of frames if every theorem is valid in every frame in the class, and *complete* if any formula valid in every frame in the class is a theorem. When a logic Λ is both sound and complete in a class $\mathcal{C}$ of frames, we say that $\mathcal{C}$ *describes* Λ.

The following table lists the logics $\mathbf{K\Sigma}$ and the corresponding sound and complete classes of (Kripke) frames:

Σ in $\mathbf{K\Sigma}$	frame $\mathbf{K\Sigma}$ is sound in	frame $\mathbf{K\Sigma}$ is complete in
4	transitive	transitive
5	Euclidean	Euclidean
D	serial	serial
T	reflexive	reflexive
B	symmetric	symmetric
G	weakly directed	weakly directed
L	weakly connected	weakly connected
W	transitive and converse well-founded	finite transitive and irreflexive