Schröder Bernstein Theorem: Proof

Canonical name	SchroderBernsteinTheoremProof
Date of creation	2013-03-22 16:40:36
Last modified on	2013-03-22 16:40:36
Owner	sauravbhaumik (15615)
Last modified by	sauravbhaumik (15615)
Numerical id	10
Author	sauravbhaumik (15615)
Entry type	Proof
Classification	msc 03E20

Let A and B be two nonempty sets; and let there be, in addition, two one-one functions $f : A \rightarrowtail B$ and $g : B \rightarrowtail A$. We propose to show that A and B are equinumerous i.e., they are in one to one correspondence.

Consider the notation:

$$\begin{array}{rl} g^{-1}(x), & \text{if} \quad x \in g(B) \\ f^{-1}(g^{-1}(x)), & \text{if} \quad g^{-1}(x) \in f(A) \\ g^{-1}(f^{-1}(g^{-1}(x))), & \text{if} \quad f^{-1}(g^{-1}(x)) \in g(B) \\ \ldots\ldots & \end{array}$$

Define, for each $x \in A$, the order of it, denoted by $\circ(x)$, to be the number of such preimage(s) which exist. In a similer way, we'd be able to define the order of an element $y \in B$, i.e., by considering the sequence $f^{-1}(y), g^{-1}(f^{-1}(y)), \ldots.$

Now define, for each $x \in A$,

$$\begin{aligned} \phi(x) : &= f(x), \quad \circ(x) = \infty \\ &= f(x), \quad \circ(x) = 2n, \text{ for some } n \in \omega \\ &= b, \quad \circ(x) = 2n + 1, i.e., \exists b \in B : g(b) = x \end{aligned}$$

Notice that if the order is infinite, $\phi(x) = f(x)$ is also infinite. Because, otherwise x would have to have a finite order. On the other hand, if $y \in B$ and $\circ(y)$ is infinite, then $f^{-1}(y)$ exists and has an infinite order; call the latter one x. This means, ϕ maps the infinite order elements of A bijectively onto the infinite order elements of B.

Next, if $\circ(x) = 2n$, then the order of $f(x)$ is sheer $2n + 1$. Similer to the above para, if for $y \in B$, the order is $2n + 1$, as the order is non-zero, $f^{-1}(y)$ exists and it must have order $2n$. To formally show it you need a tedious inductive reasoning!

Last, if $\circ(x) = 2n + 1$, then the order is ≥ 1, and so, $g^{-1}(x)$ exists and the order of $g^{-1}(x)$ is sheer $2n$(looking upon $g^{-1}(x)$ as an element of B). Conversely, similer to above, if there is an element y of order $2n$ in B, take $x = g(y)$ and the order of x is indeed $2n + 1$. All that you need to convince a sceptic is a long, tedious, involved induction!!

What we learn from what precedes is that the one-one function ϕ maps the infinite order elements onto infinite order elements, odd order onto odd order, and even order onto even order; a simple set theory reveals that ϕ is a one-one map from A onto B. This completes the proof.