# example of a universal structure

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Let $L$ be the first order language with the binary relation $\leq$. Consider the following sentences:

- $\forall x, y((x \leq y \vee y \leq x) \wedge ((x \leq y \wedge y \leq x) \leftrightarrow x = y))$
- $\forall x, y, z(x \leq y \wedge y \leq z \rightarrow x \leq z)$

Any $L$-structure satisfying these is called a linear order. We define the relation $<$ so that $x < y$ iff $x \leq y \wedge x \neq y$. Now consider these sentences:

1. $\forall x, y((x < y \rightarrow \exists z(x < z < y))$
2. $\forall x \exists y, z(y < x < z)$

A linear order that satisfies 1. is called `http://planetmath.org/DenseTotalOrder`dense. We say that a linear order that satisfies 2. is *without endpoints*. Let $T$ be the theory of dense linear orders without endpoints. This is a complete theory.

We can see that $(\mathbb{Q}, \leq)$ is a model of $T$. It is actually a rather special model.

**Theorem 1** *Let $(S, \leq)$ be any finite linear order. Then $S$ embeds in $(\mathbb{Q}, \leq)$.*

Proof: By induction on $|S|$, it is trivial for $|S| = 1$.

Suppose that the statement holds for all linear orders with cardinality less than or equal to $n$. Let $|S| = n + 1$, then pick some $a \in S$, let $S'$ be the structure induced by $S$ on $S \setminus a$. Then there is some embedding $e$ of $S'$ into $\mathbb{Q}$.

- Now suppose $a$ is less than every member of $S'$, then as $\mathbb{Q}$ is without endpoints, there is some element $b$ less than every element in the image of $e$. Thus we can extend $e$ to map $a$ to $b$ which is an embedding of $S$ into $\mathbb{Q}$.
- We work similarly if $a$ is greater than every element in $S'$.
- If neither of the above hold then we can pick some maximum $c_1 \in S'$ so that $c_1 < a$. Similarly we can pick some minimum $c_2 \in S'$ so that $c_2 < a$. Now there is some $b \in \mathbb{Q}$ with $e(c_1) < b < e(c_2)$. Then extending $e$ by mapping $a$ to $b$ is the required embedding. $\square$

It is easy to extend the above result to countable structures. One views a countable structure as a the union of an increasing chain of finite substructures. The necessary embedding is the union of the embeddings of the substructures. Thus $(\mathbb{Q}, \leq)$ is universal countable linear order.

**Theorem 2** $(\mathbb{Q}, \leq)$ *is homogeneous.*

Proof: The following type of proof is known as a *back and forth* argument. Let $S_1$ and $S_2$ be two finite substructures of $(\mathbb{Q}, \leq)$. Let $e : S_1 \to S_2$ be an isomorphism. It is easier to think of two disjoint copies $B$ and $C$ of $\mathbb{Q}$ with $S_1$ a substructure of $B$ and $S_2$ a substructure of $C$.

Let $b_1, b_2, \ldots$ be an enumeration of $B \setminus S_1$. Let $c_1, c_2, \ldots, c_n$ be an enumeration of $C \setminus S_2$. We iterate the following two step process:

**The $i$th forth step** If $b_i$ is already in the domain of $e$ then do nothing. If $b_i$ is not in the domain of $e$. Then as in proposition **??**, either $b_i$ is less than every element in the domain of $e$ or greater than or it has an immediate successor and predecessor in the range of $e$. Either way there is an element $c$ in $C \setminus \mathrm{range}(e)$ relative to the range of $e$. Thus we can extend the isomorphism to include $b_i$.

**The $i$th back step** If $c_i$ is already in the range of $e$ then do nothing. If $c_i$ is not in the domain of $e$. Then exactly as above we can find some $b \in B \setminus \mathrm{dom}(e)$ and extend $e$ so that $e(b) = c_i$.

After $\omega$ stages, we have an isomorphism whose range includes every $b_i$ and whose domain includes every $c_i$. Thus we have an isomorphism from $B$ to $C$ extending $e$. $\Box$

A similar back and forth argument shows that any countable dense linear order without endpoints is isomorphic to $(\mathbb{Q}, \leq)$ so $T$ is $\aleph_0$-categorical.