deductions are Δ_1

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Using the example of Gödel numbering, we can show that $\mathrm{Proves}(a, x)$ (the statement that a is a proof of x, which will be formally defined below) is Δ_1.

First, $\mathrm{Term}(x)$ should be true if and only if x is the Gödel number of a term. Thanks to primitive recursion, we can define it by:

$$\begin{aligned}\mathrm{Term}(x) \leftrightarrow & \exists i < x[x = \langle 0, i\rangle]\vee \\ & x = \langle 5\rangle \vee \\ & \exists y < x[x = \langle 6, y\rangle \wedge \mathrm{Term}(y)]\vee \\ & \exists y, z < x[x = \langle 8, y, z\rangle \wedge \mathrm{Term}(y) \wedge \mathrm{Term}(z)]\vee \\ & \exists y, z < x[x = \langle 9, y, z\rangle \wedge \mathrm{Term}(y) \wedge \mathrm{Term}(z)]\end{aligned}$$

Then $\mathrm{AtForm}(x)$, which is true when x is the Gödel number of an atomic formula, is defined by:

$$\begin{aligned}\mathrm{AtForm}(x) \leftrightarrow & \exists y, z < x[x = \langle 1, y, z\rangle \wedge \mathrm{Term}(y) \wedge \mathrm{Term}(z)]\vee \\ & \exists y, z < x[x = \langle 7, y, z\rangle \wedge \mathrm{Term}(y) \wedge \mathrm{Term}(z)]\end{aligned}$$

Next, $\mathrm{Form}(x)$, which is true only if x is the Gödel number of a formula, is defined recursively by:

$$\begin{aligned}\mathrm{Form}(x) \leftrightarrow & \mathrm{AtForm}(x)\vee \\ & \exists i, y < x[x = \langle 2, i, y\rangle \wedge \mathrm{Form}(y)]\vee \\ & \exists y < x[x = \langle 3, y\rangle \wedge \mathrm{Form}(y)]\vee \\ & \exists y, z < x[x = \langle 4, y, z\rangle \wedge \mathrm{Form}(y) \wedge \mathrm{Form}(z)]\end{aligned}$$

The definition of $\mathrm{QFForm}(x)$, which is true when x is the Gödel number of a quantifier free formula, is defined the same way except without the second clause.

Next we want to show that the set of logical tautologies is Δ_1. This will be done by formalizing the concept of truth tables, which will require some development. First we show that $\mathrm{AtForms}(a)$, which is a sequence containing the (unique) atomic formulas of a is Δ_1. Define it by:

$$\begin{aligned}
\mathrm{AtForms}(a,t) \leftrightarrow &(\neg\,\mathrm{Form}(a) \wedge t = 0) \vee \\
&\mathrm{Form}(a) \wedge (\\
&\exists x, y < a[a = \langle 1, x, y\rangle t = a] \vee \\
&\exists x, y < a[a = \langle 7, x, y\rangle \wedge t = a] \vee \\
&\exists i, x < a[a = \langle 2, i, x\rangle \wedge t = \mathrm{AtForms}(x)] \vee \\
&\exists x < a[a = \langle 3, x\rangle \wedge t = \mathrm{AtForms}(x)] \vee \\
&\exists x, y < a[a = \langle 4, x, y\rangle \wedge t = \mathrm{AtForms}(x) *_u \mathrm{AtForms}(y)])
\end{aligned}$$

We say v is a *truth assignment* if it is a sequence of pairs with the first member of each pair being a atomic formula and the second being either 1 or 0:

$$TA(v) \leftrightarrow \forall i < \mathrm{len}(v) \exists x, y < (v)_i[(v)_i = \langle x, y\rangle \wedge \mathrm{AtForm}(x) \wedge (y = 1 \vee y = 0)]$$

Then v is a truth assignment for a if v is a truth assignment, a is quantifier free, and every atomic formula in a is the first member of one of the pairs in v. That is:

$$TAf(v,a) \leftrightarrow TA(v) \wedge \mathrm{QFForm}(a) \wedge \forall i < \mathrm{len}(\mathrm{AtForms}(a)) \exists j < \mathrm{len}(v)[((v)_j)_0 = (\mathrm{AtForms}(a))_i]$$

Then we can define when v makes a true by:

$$\begin{aligned}
\mathrm{True}(v,a) \leftrightarrow &TAf(v,a) \wedge \\
&\mathrm{AtForm}(a) \wedge \exists i < \mathrm{len}(v)[((v)_i)_0 = a \wedge ((v)_i)_1 = 1] \vee \\
&\exists y < x[x = \langle 3, y\rangle \wedge \mathrm{True}(v,y)] \vee \\
&\exists y, z < x[x = \langle 4, y, z\rangle \wedge \mathrm{True}(v,y) \rightarrow \mathrm{True}(v,z)]
\end{aligned}$$

Then a is a tautology if every truth assignment makes it true:

$$\mathrm{Taut}(a) \forall v < 2^{2^{\mathrm{AtForms}(a)}}[TAf(v,a) \rightarrow \mathrm{True}(v,a)]$$

We say that a number a is a *deduction* of ϕ if it encodes a proof of ϕ from a set of axioms Ax. This means that a is a sequence where for each $(a)_i$ either:

- $(a)_i$ is the Gödel number of an axiom

- $(a)_i$ is a logical tautology

  or

- there are some $j, k < i$ such that $(a)_j = \langle 4, (a)_k, (a)_i \rangle$ (that is, $(a)_i$ is a conclusion under modus ponens from $(a)_j$ and $(a)_k$).

and the last element of a is $\ulcorner \phi \urcorner$.

If Ax is Δ_1 (almost every system of axioms, including PA, is Δ_1) then Proves(a, x), which is true if a is a deduction whose last value is x, is also Δ_1. This is fairly simple to see from the above results (let $\mathrm{Ax}(x)$ be the relation specifying that x is the Gödel number of an axiom):

$$\mathrm{Proves}(a, x) \leftrightarrow \forall i < \mathrm{len}(a)\,[\mathrm{Ax}((a)_i) \vee \exists j, k < i[(a)_j = \langle 4, (a)_k, (a)_i \rangle] \vee \mathrm{Taut}((a)_i)]$$