Kripke semantics for modal propositional logic

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A *Kripe model* for a modal propositional logic PL_M is a triple $M := (W, R, V)$, where

1. W is a set, whose elements are called *possible worlds*,
2. R is a binary relation on W,
3. V is a function that takes each wff (well-formed formula) A in PL_M to a subset $V(A)$ of W, such that
   - $V(\perp) = \varnothing$,
   - $V(A \to B) = V(A)^c \cup V(B)$,
   - $V(\Box A) = V(A)^{\Box}$, where $S^{\Box} := \{s \mid {\uparrow}s \subseteq S\}$, and ${\uparrow}s := \{t \mid sRt\}$.

For derived connectives, we also have $V(A \wedge B) = V(A) \cap V(B)$, $V(A \vee B) = V(A) \cup V(B)$, $V(\neg A) = V(A)^c$, the complement of $V(A)$, and $V(\diamond A) = V(A)^{\diamond} := V(A)^{c\Box c}$.

One can also define a *satisfaction relation* $\models$ between W and the set L of wff's so that

$$\models_w A \qquad \text{iff} \qquad w \in V(A)$$

for any $w \in W$ and $A \in L$. It's easy to see that

- $\not\models_w \perp$ for any $w \in W$,
- $\models_w A \to B$ iff $\models_w A$ implies $\models_w B$,
- $\models_w A \wedge B$ iff $\models_w A$ and $\models_w B$,
- $\models_w A \vee B$ iff $\models_w A$ or $\models_w B$,
- $\models_w \neg A$ iff $\not\models_w A$,
- $\models_w \Box A$ iff for all u such that wRu, we have $\models_u A$,
- $\models_w \diamond A$ iff there is a u such that wRu and $\models_u A$.

When $\models_w A$, we say that A is true at world w.

The pair $\mathcal{F} := (W, R)$ in a Kripke model $M := (W, R, V)$ is also called a (Kripke) frame, and M is said to be a model based on the frame $\mathcal{F}$. The validity of a wff A at different levels (in a model, a frame, a collection of frames) is defined in the `http://planetmath.org/KripkeSemantics`parent entry.

For example, any tautology is valid in any model.

Now, to prove that any tautology is valid, by the completeness of PL_c, every tautology is a theorem, which is in turn the result of a deduction from axioms using modus ponens.

First, modus ponens preserves validity: for suppose $\models_w A$ and $\models_w A \to B$. Since $\models_w A$ implies $\models_w B$, and $\models_w A$ by assumption, we have $\models_w B$. Now, w is arbitrary, the result follows.

Next, we show that each axiom of PL_c is valid:

- $A \to (B \to A)$: If $\models_w A$ and $\models_w B$, then $\models_w A$, so $\models_w B \to A$.
- $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$: Suppose $\models_w A \to (B \to C)$, $\models_w A \to B$, and $\models_w A$. Then $\models_w B \to C$ and $\models_w B$, and therefore $\models_w C$.
- $(\neg A \to \neg B) \to (B \to A)$: we use a different approach to show this:

$$\begin{aligned} V((\neg A \to \neg B) \to (B \to A)) &= V(\neg A \to \neg B)^c \cup V(B \to A) \\ &= (V(\neg A) \cap V(\neg B)^c) \cup V(B)^c \cup V(A) \\ &= (V(A)^c \cap V(B)) \cup V(B)^c \cup V(A) \\ &= (V(A)^c \cup V(B)^c) \cup V(A) = W. \end{aligned}$$

In addition, the rule of necessitation preserves validity as well: suppose $\models_w A$ for all w, then certainly $\models_u A$ for all u such that wRu, and therefore $\models_w \Box A$.

There are also valid formulas that are not tautologies. Here's one:

$$\Box(A \to B) \to (\Box A \to \Box B)$$

Proof. Let w be any world in M. Suppose $\models_w \Box(A \to B)$. Then for all u such that wRu, $\models_u A \to B$, or $\models_u A$ implies $\models_u B$, or for all u such that wRu, $\models_u A$, implies that for all u such that wRu, $\models_u B$, or $\models_w \Box A$ implies $\models_w \Box B$, or $\models_w (\Box A \to \Box B)$. Therefore, $\models_w \Box(A \to B) \to (\Box A \to \Box B)$. $\Box$

From this, we see that Kripke semantics is appropriate only for normal modal logics.

Below are some examples of Kripke frames and their corresponding validating logics:

1. $A \to \Box A$ is valid in a frame (W, R) iff R weak identity: wRu implies $w = u$.

Proof. Let (W, R) be a frame validating $A \to \Box A$, and M a model based on (W, R), with $V(p) = \{w\}$. Then $\models_w p$. So $\models_w \Box p$, or $\models_u p$ for all u such that wRu. But then $u \in V(p)$, or $u = w$. Hence R is the relation: if wRu, then $w = u$.

Conversely, suppose (W, R) is weak identity, M based on (W, R), and w a world in M with $\models_w A$. If wRu, then $u = w$, which means $\models_u A$ for all u such that wRu. In other words, $\models_w \Box A$, and therefore, $\models_w A \to \Box A$. $\Box$

2. $\Box A$ is valid in a frame (W, R) iff $R = \varnothing$.

   Proof. First, suppose $\Box A$ is valid in (W, R), and M a model based on (W, R), with $V(p) = \varnothing$. Since $\models_w \Box p$, $\models_u p$ for any u such that wRu. Since no such u exists, and w is arbitrary, $R = \varnothing$.

   Conversely, given a model M based on $(W, \varnothing)$, and a world w in M, it is vacuously true that $\models_u A$ for any u such that wRu, since no such u exists. Therefore $\models_w \Box A$. $\Box$

A logic is said to be sound if every theorem is valid, and complete if every valid wff is a theorem. Furthermore, a logic is said to have the finite model property if any wff in the class of finite frames is a theorem.