substitution theorem for propositional logic

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In this entry, we will prove the substitution theorem for propositional logic based on the axiom system found `http://planetmath.org/AxiomSystemForPropositionalLogic`. Besides the deduction theorem, below are some additional results we will need to prove the theorem:

1. If $\Delta \vdash A \to B$ and $\Gamma \vdash B \to C$, then $\Delta, \Gamma \vdash A \to C$.
2. $\Delta \vdash A$ and $\Delta \vdash B$ iff $\Delta \vdash A \land B$.
3. $\vdash A \leftrightarrow A$.
4. $\vdash A \leftrightarrow \neg\neg A$ (law of double negation).
5. $\perp \to A$ (ex falso quodlibet)
6. $\Delta \vdash A$ implies $\Delta \vdash B$ iff $\Delta \vdash A \to B$.

The proofs of these results can be found `http://planetmath.org/SomeTheoremSchemasOfProposi`

Theorem 1. *(Substitution Theorem) Suppose $p_1, \ldots, p_m$ are all the propositional variables, not necessarily distinct, that occur in order in A, and if $B_1, \ldots, B_m, C_1, \ldots, C_m$ are wff's such that $\vdash B_i \leftrightarrow C_i$, then*

$$\vdash A[B_1/p_1, \ldots, B_m/p_m] \leftrightarrow A[C_1/p_1, \ldots, C_m/p_m]$$

where $A[X_1/p_1, \ldots, X_m/p_m]$ is the wff obtained from A by replacing p_i by the wff X_i via simultaneous substitution.

Proof. We do induction on the number n of $\to$ in wff A.

If $n = 0$, A is either a propositional variable, say p, or $\perp$, which respectively means that $A[B/p] \leftrightarrow A[C/p]$ is either $B \leftrightarrow C$ or $\perp\leftrightarrow\perp$. The former is the assumption and the latter is a theorem.

Suppose now A has $n+1$ occurrences of $\to$. We may write A as $X \to Y$ uniquely by unique readability. Also, both X and Y have at most n occurrences of $\to$.

Let A_1 be $A[B_1/p_1, \ldots, B_m/p_m]$ and A_2 be $A[C_1/p_1, \ldots, C_m/p_m]$. Then A_1 is $X_1 \to Y_1$ and $X_2 \to Y_2$, where X_1 is $X[B_1/p_1, \ldots, B_k/p_k]$, Y_1 is $Y[B_{k+1}/p_{k+1}, \ldots, B_m/p_m]$, X_2 is $X[C_1/p_1, \ldots, C_k/p_k]$, and Y_2 is $Y[C_{k+1}/p_{k+1}, \ldots, C_m/p_m]$.

Then

by induction $\vdash X_1 \leftrightarrow X_2$ (1)

by 2 above $\vdash X_1 \to X_2$ and $\vdash X_2 \to X_1$ (2)

by induction $\vdash Y_1 \leftrightarrow Y_2$ (3)

by 2 above $\vdash Y_1 \to Y_2$ and $\vdash Y_2 \to Y_1$ (4)

since A_1 is $X_1 \to Y_1$ $A_1 \vdash X_1 \to Y_1$ (5)

by applying 1 to $\vdash X_2 \to X_1$ and (5) $A_1 \vdash X_2 \to Y_1$ (6)

by applying 1 to (6) and $\vdash Y_1 \to Y_2$ $A_1 \vdash X_2 \to Y_2$ (7)

by the deduction theorem $\vdash A_1 \to A_2$ (8)

by a similar reasoning as above $\vdash A_2 \to A_1$ (9)

by applying 2 to (8) and (9) $\vdash A_1 \leftrightarrow A_2$ (10)

□

As a corollary, we have

Corollary 1. *If $\vdash B \leftrightarrow C$, then $\vdash A[B/s(p)] \leftrightarrow A[C/s(p)]$, where p is a propositional variable that occurs in A, $s(p)$ is a set of positions of occurrences of p in A, and the wff $A[X/s(p)]$ is obtained by replacing all p that occur in the positions in $s(p)$ in A by wff X.*

Proof. For any propositional variable q not being replaced, use the corresponding theorem $\vdash q \leftrightarrow q$, and then apply the substitution theorem. □

Remark. What about $\vdash B[A/p] \leftrightarrow C[A/p]$, given $\vdash B \leftrightarrow C$? Here, $B[A/p]$ and $C[A/p]$ are wff's obtained by uniform substitution of p (all occurrences of p) in B and C respectively. Since $B[A/p] \leftrightarrow C[A/p]$ is just

$(B \leftrightarrow C)[A/p]$, an instance of the schema $B \leftrightarrow C$ by assumption, the result follows directly if we assume $B \leftrightarrow C$ is a theorem schema.

Using the substitution theorem, we can easily derive more theorem schemas, such as

7. $(A \to B) \leftrightarrow (\neg B \to \neg A)$ (Law of Contraposition)

8. $A \to (\neg B \to \neg(A \to B))$

9. $((A \to B) \to A) \to A$ (Peirce's Law)

Proof. 7. Since $(\neg B \to \neg A) \to (A \to B)$ is already a theorem schema, we only need to show $\vdash (A \to B) \to (\neg B \to \neg A)$. By law of double negation (4 above) and the substitution theorem, it is enough to show that $\vdash (\neg\neg A \to \neg\neg B) \to (\neg B \to \neg A)$. But this is just an instance of an axiom schema. Combining the two schemas, we get $\vdash (A \to B) \leftrightarrow (\neg B \to \neg A)$.

8. First, observe that $A, A \to B \vdash B$ by modus ponens. Since $\vdash B \leftrightarrow \neg\neg B$, we have $A, A \to B \vdash \neg\neg B$ by the substitution theorem. So $A, A \to B, \neg B \vdash \perp$ by the deduction theorem, and $A, \neg B \vdash (A \to B) \to \perp$ by the deduction theorem again. Apply the deduction two more times, we get $\vdash A \to (\neg B \to \neg(A \to B))$.

9. To show $\vdash ((A \to B) \to A) \to A$, it is enough to show $\vdash \neg A \to \neg((A \to B) \to A)$ by 7 and modus ponens, or $\neg A \vdash \neg((A \to B) \to A)$ by the deduction theorem. Now, since $\vdash X \wedge Y \leftrightarrow \neg(X \to \neg Y)$ (as they are the same thing, and because $C \leftrightarrow C$ is a theorem schema), by the law of double negation and the substitution theorem, $\vdash X \wedge \neg Y \leftrightarrow \neg(X \to Y)$, and we have $\vdash (A \to B) \wedge \neg A \leftrightarrow \neg((A \to B) \to A)$. So to show $\neg A \vdash \neg((A \to B) \to A)$, it is enough to show $\neg A \vdash (A \to B) \wedge \neg A$, which is enough to show that $\neg A \vdash A \to B$ and $\neg A \vdash \neg A$, according to a meta-theorem found `http://planetmath.org/SomeTheoremSchemasOfPropositionalLogic`here. To show $\neg A \vdash A \to B$, it is enough to show $\{\neg A, A\} \vdash B$, and $A, \neg A, \perp, \perp \to B, B$ is such a deduction. The second statement $\neg A \vdash \neg A$ is clear. $\square$

As an application, we prove the following useful meta-theorems of propositional logic:

Proposition 1. *There is a wff A such that $\Delta \vdash A$ and $\Delta \vdash \neg A$ iff $\Delta \vdash \perp$*

Proof. Assume the former. Let $\mathcal{E}_1$ be a deduction of A from Δ and $\mathcal{E}_2$ a deduction of $\neg A$ from Δ, then

$$\mathcal{E}_1, \mathcal{E}_2, \perp$$

is a deduction of $\perp$ from Δ. Conversely, assume the later. Pick any wff A (if necessary, pick $\perp$). Then $\perp \to A$ by ex falso quodlibet. By modus ponens, we have $\Delta \vdash A$. Similarly, $\Delta \vdash \neg A$. □

Proposition 2. *If $\Delta, A \vdash B$ and $\Delta, \neg A \vdash B$, then $\Delta \vdash B$*

Proof. By assumption, we have $\Delta \vdash A \to B$ and $\Delta \vdash \neg A \to B$. Using modus ponens and the theorem schema $(A \to B) \to (\neg B \to \neg A)$, we have $\Delta \vdash \neg B \to \neg A$, or

$$\Delta, \neg B \vdash \neg A.$$

Similarly, $\Delta \vdash \neg B \to \neg\neg A$. By the law of double negation and the substitution theorem, we have $\Delta \vdash \neg B \to A$, or

$$\Delta, \neg B \vdash A.$$

By the previous proposition, $\Delta, \neg B \vdash \perp$, or $\Delta \vdash \neg\neg B$. Applying the substitution theorem and the law of double negation, we have

$$\Delta \vdash B.$$

□