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some theorem schemas of propositional logic

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Based on the axiom system in http://planetmath.org/AxiomSystemForPropositionalLogic entry, we will exhibit some theorem schemas, as well as prove some meta-theorems of propositional logic. All of these are based on the important deduction theorem, which is proved http://planetmath.org/DeductionTheoremHoldsForClassica

First, some theorem schemas:

1. $A \to \neg\neg A$
2. $\neg\neg A \to A$
3. (law of the excluded middle) $A \vee \neg A$
4. (ex falso quodlibet) $\perp \to A$
5. $A \leftrightarrow A$
6. (law of double negation) $A \leftrightarrow \neg\neg A$
7. $A \wedge B \to A$ and $A \wedge B \to B$
8. (absorption law for $\wedge$) $A \leftrightarrow A \wedge A$
9. (commutative law for $\wedge$) $A \wedge B \leftrightarrow B \wedge A$
10. (associative law for $\wedge$) $(A \wedge B) \wedge C \leftrightarrow A \wedge (B \wedge C)$
11. (law of syllogism) $(A \to B) \to ((B \to C) \to (A \to C))$
12. (law of importation) $(A \to (B \to C)) \to (A \wedge B \to C)$
13. $(A \to B) \to ((B \to (C \to D)) \to (A \wedge C \to D))$
14. $(A \to B) \leftrightarrow (A \to (A \to B))$

Proof. Many of these can be easily proved using the deduction theorem:

1. we need to show $A \vdash \neg\neg A$, which means we need to show $A, \neg A \vdash \perp$. Since $\neg A$ is $A \to \perp$, by modus ponens, $A, \neg A \vdash \perp$.

2. we observe first that $\neg A \to \neg\neg\neg A$ is an instance of the above theorem schema, since $(\neg A \to \neg\neg\neg A) \to (\neg\neg A \to A)$ is an instance of one of the axiom schemas, we have $\vdash \neg\neg A \to A$ as a result.

3. since $A \vee \neg A$ is $\neg A \to \neg A$, to show $\vdash A \vee \neg A$, we need to show $\neg A \vdash \neg A$, but this is obvious.

4. we need to show $\perp \vdash A$. Since $\perp, \perp \to (A \to \perp), A \to \perp$ is a deduction of $A \to \perp$ from $\perp$, and the result follows.

5. this is because $\vdash A \to A$, so $\vdash (A \to A) \wedge (A \to A)$.

6. this is the result of the first two theorem schemas above.

For the next four schemas, we need the the following meta-theorems (see `http://planetmath.org/SomeMetatheoremsOfPropositionalLogic`here for proofs):

M1. $\Delta \vdash A$ and $\Delta \vdash B$ iff $\Delta \vdash A \wedge B$

M2. $\Delta \vdash A$ implies $\Delta \vdash B$ iff $\Delta \vdash A \to B$

7. If $\vdash A \wedge B$, then $\vdash A$ by M1, so $\vdash A \wedge B \to A$ by M2. Similarly, $\vdash A \wedge B \to B$.

8. $\vdash A \wedge A \to A$ comes from 7, and since $\vdash A$ implies $\vdash A \wedge A$ by M1, $\vdash A \to A \wedge A$ by M2. Therefore, $\vdash A \leftrightarrow A \wedge A$ by M1.

9. If $\vdash A \wedge B$, then $\vdash A$ and $\vdash B$ by M1, so $\vdash B \wedge A$ by M1 again, and therefore $\vdash A \wedge B \to B \wedge A$ by M2. Similarly, $\vdash B \wedge A \to A \wedge B$. Combining the two and apply M1, we have the result.

10. If $\vdash (A \wedge B) \wedge C$, then $\vdash A \wedge B$ and $\vdash C$, so $\vdash A$, $\vdash B$, and $\vdash C$ by M1. By M1 again, we have $\vdash A$ and $\vdash B \wedge C$, and another application of M1, $\vdash A \wedge (B \wedge C)$. Therefore, by M2, $\vdash (A \wedge B) \wedge C \to A \wedge (B \wedge C)$, Similarly, $\vdash A \wedge (B \wedge C) \to (A \wedge B) \wedge C$. Combining the two and applying M1, we have the result.

11. $A \to B, B \to C, A \vdash C$ by modus ponens 3 times.

12. $A \to (B \to C), A \wedge B, A \wedge B \to A, A, B \to C, A \wedge B \to B, B, C$ is a deduction of C from $A \to (B \to C)$ and $A \wedge B$.

13. $A \to B, B \to (C \to D), A \wedge C, A \wedge C \to A, A, B, C \to D, A \wedge C \to C, C, D$ is a deduction of D from $A \to B, B \to (C \to D)$, and $A \wedge C$.

14. $(A \to B) \to (A \to (A \to B))$ is just an axiom, while $\vdash (A \to (A \to B)) \to (A \to B)$ comes from two applications of the deduction theorem to $A \to (A \to B), A \vdash B$, which is the result of the deduction $A \to (A \to B), A, A \to B, B$ of B from $A \to (A \to B)$ and A.

$\square$