arithmetical hierarchy

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Entry type	Definition
Classification	msc 03B10
Synonym	arithmetic hierarchy
Synonym	arithmetic
Synonym	arithmetical
Synonym	arithmetic formula
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Related topic	AnalyticHierarchy
Defines	sigma n
Defines	sigma-n
Defines	pi n
Defines	pi-n
Defines	delta n
Defines	delta-n
Defines	recursive
Defines	recursively enumerable
Defines	delta-0
Defines	delta 0
Defines	delta-1
Defines	delta 1
Defines	arithmetical

The *arithmetical hierarchy* is a hierarchy of either (depending on the context) formulas or relations. The relations of a particular level of the hierarchy are exactly the relations defined by the formulas of that level, so the two uses are essentially the same.

The first level consists of formulas with only bounded quantifiers, the corresponding relations are also called the Primitive Recursive relations (this definition is equivalent to the definition from computer science). This level is called any of Δ^0_0, Σ^0_0 and Π^0_0, depending on context.

A formula ϕ is Σ^0_n if there is some Δ^0_0 formula ψ such that ϕ can be written:

$$\phi(\vec{k}) = \exists x_1 \forall x_2 \cdots Q x_n \psi(\vec{k}, \vec{x})$$

where Q is either $\forall$ or $\exists$, whichever maintains the pattern of alternating quantifiers

The Σ^0_1 relations are the same as the Recursively Enumerable relations.

Similarly, ϕ is a Π^0_n relation if there is some Δ^0_0 formula ψ such that:

$$\phi(\vec{k}) = \forall x_1 \exists x_2 \cdots Q x_n \psi(\vec{k}, \vec{x})$$

where Q is either $\forall$ or $\exists$, whichever maintains the pattern of alternating quantifiers

A formula is Δ^0_n if it is both Σ^0_n and Π^0_n. Since each Σ^0_n formula is just the negation of a Π^0_n formula and vice-versa, the Σ^0_n relations are the complements of the Π^0_n relations.

The relations in $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$ are the Recursive relations.

Higher levels on the hierarchy correspond to broader and broader classes of relations. A formula or relation which is Σ^0_n (or, equivalently, Π^0_n) for some integer n is called *arithmetical*.

The superscript 0 is often omitted when it is not necessary to distinguish from the analytic hierarchy.

Functions can be described as being in one of the levels of the hierarchy if the graph of the function is in that level.