iterated forcing and composition

Canonical name	IteratedForcingAndComposition
Date of creation	2013-03-22 12:54:51
Last modified on	2013-03-22 12:54:51
Owner	Henry (455)
Last modified by	Henry (455)
Numerical id	6
Author	Henry (455)
Entry type	Result
Classification	msc 03E35
Classification	msc 03E40

There is a function satisfying forcings are equivalent if one is dense in the other $f : P_\alpha * Q_\alpha \to P_{\alpha+1}$.

Proof

Let $f(\langle g, \hat{q}\rangle) = g \cup \{\langle \alpha, \hat{q}\rangle\}$. This is obviously a member of $P_{\alpha+1}$, since it is a partial function from $\alpha + 1$ (and if the domain of g is less than α then so is the domain of $f(\langle g, \hat{q}\rangle))$, if $i < \alpha$ then obviously $f(\langle g, \hat{q}\rangle)$ applied to i satisfies the definition of iterated forcing (since g does), and if $i = \alpha$ then the definition is satisfied since $\hat{q}$ is a name in P_i for a member of Q_i.

f is order preserving, since if $\langle g_1, \hat{q}_1\rangle \leq \langle g_2, \hat{q}_2\rangle$, all the appropriate characteristics of a function carry over to the image, and $g_1 \upharpoonright \alpha \Vdash_{P_i} \hat{q}_1 \leq \hat{q}_2$ (by the definition of $\leq$ in $*$).

If $\langle g_1, \hat{q}_1\rangle$ and $\langle g_2, \hat{q}_2\rangle$ are incomparable then either g_1 and g_2 are incomparable, in which case whatever prevents them from being compared applies to their images as well, or $\hat{q}_1$ and $\hat{q}_2$ aren't compared appropriately, in which case again this prevents the images from being compared.

Finally, let g be any element of $P_{\alpha+1}$. Then $g \upharpoonright \alpha \in P_\alpha$. If $\alpha \notin \mathrm{dom}(g)$ then this is just g, and $f(\langle g, \hat{q}\rangle) \leq g$ for any $\hat{q}$. If $\alpha \in \mathrm{dom}(g)$ then $f(\langle g \upharpoonright \alpha, g(\alpha)\rangle) = g$. Hence $f[P_\alpha * Q_\alpha]$ is dense in $P_{\alpha+1}$, and so these are equivalent.