properties of bijections

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Owner	CWoo (3771)
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Let A, B, C, D be sets. We write $A \sim B$ when there is a bijection from A to B. Below are some properties of bijections.

1. $A \sim A$. The identity function is the bijection from A to A.

2. If $A \sim B$, then $B \sim A$. If $f : A \to B$ is a bijection, then its inverse function $f^{-1} : B \to A$ is also a bijection.

3. If $A \sim B$, $B \sim C$, then $A \sim C$. If $f : A \to B$ and $g : B \to C$ are bijections, so is the composition $g \circ f : A \to C$.

4. If $A \sim B$, $C \sim D$, and $A \cap C = B \cap D = \varnothing$, then $A \cup B \sim C \cup D$.

   Proof. If $f : A \to B$ and $g : C \to D$ are bijections, so is $h : A \cup C \to B \cup D$, defined by
   $$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in C. \end{cases}$$
   Since $A \cap C = \varnothing$, h is a well-defined function. h is onto since both f and g are. Since f, g are one-to-one, and $B \cap D = \varnothing$, h is also one-to-one. $\square$

5. If $A \sim B$, $C \sim D$, then $A \times C \sim B \times D$. If $f : A \to B$ and $g : C \to D$ are bijections, so is $h : A \times C \to B \times D$, given by $h(x, y) = (f(x), g(y))$.

6. $A \times B \sim B \times A$. The function $f : A \times B \to B \times A$ given by $f(x, y) = (y, x)$ is a bijection.

7. If $A \sim B$ and $C \sim D$, then $A^C \sim B^D$.

   Proof. Suppose $\phi : A \to B$ and $\sigma : C \to D$ are bijections. Define $F : A^C \to B^D$ as follows: for any function $f : A \to C$, let $F(f) = \sigma \circ f \circ \phi^{-1} : B \to D$. F is a well-defined function. It is one-to-one because σ and ϕ are bijections (hence are cancellable). For any $g : B \to D$, it is easy to see that $F(\sigma^{-1} \circ g \circ \phi) = g$, so that F is onto. Therefore F is a bijection from A^C to B^D. $\square$

8. Continuing from property 8, using the bijection F, we have $\mathrm{Mono}(A, B) \sim \mathrm{Mono}(C, D)$, $\mathrm{Epi}(A, B) \sim \mathrm{Epi}(C, D)$, and $\mathrm{Iso}(A, B) \sim \mathrm{Iso}(C, D)$, where $\mathrm{Mono}(A, B)$, $\mathrm{Epi}(A, B)$, and $\mathrm{Iso}(A, B)$ are the sets of injections, surjections, and bijections from A to B.

9. $P(A) \sim 2^A$, where $P(A)$ is the powerset of A, and 2^A is the set of all functions from A to $2 = \{0, 1\}$.

   Proof. For every $B \subseteq A$, define $\varphi_B : A \to 2$ by

$$\varphi_B(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

   Then $\varphi : P(A) \to 2^A$, defined by $\varphi(B) = \varphi_B$ is a well-defined function. It is one-to-one: if $\varphi_B = \varphi_C$ for $B, C \subseteq A$, then $x \in B$ iff $x \in C$, so $B = C$. It is onto: suppose $f : A \to 2$, then by setting $B = \{x \in A \mid f(x) = 1\}$, we see that $\varphi_B = f$. As a result, φ is a bijection. $\square$

Remark. As a result of property 9, we sometimes denote 2^A the powerset of A.