proof of pigeonhole principle

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Proof. It will first be proven that, if a bijection exists between two finite sets, then the two sets have the same number of elements. Let S and T be finite sets and $f\colon S \to T$ be a bijection. The claim will be proven by induction on $|S|$.

If $|S| = 0$, then $S = \emptyset$, and $f\colon \emptyset \to T$ can only be surjective if $T = \emptyset$.

Assume the statement holds for any set S with $|S| = n$. Let $|S| = n + 1$. Let $x_1, \ldots, x_{n+1} \in S$ with $S = \{x_1, \ldots, x_{n+1}\}$. Let $R = S \setminus \{x_{n+1}\}$. Then $|R| = n$.

Define $g\colon R \to T \setminus \{f(x_{n+1})\}$ by $g(x) = f(x)$. Since $R \subset S$, $f(x) \in T$ for all $x \in R$. Thus, to show that g is well-defined, it only needs to be verified that $f(x) \neq f(x_{n+1})$ for all $x \in R$. This follows immediately from the facts that $x_{n+1} \notin R$ and f is injective. Therefore, g is well-defined.

Now it need to be proven that g is a bijection. The fact that g is injective follows immediately from the fact that f is injective. To verify that g is surjective, let $y \in T \setminus \{f(x_{n+1})\}$. Since f is surjective, there exists $x \in S$ with $f(x) = y$. Since $f(x) = y \neq f(x_{n+1})$ and f is injective, $x \neq x_{n+1}$. Thus, $x \in R$. Hence, $g(x) = f(x) = y$. It follows that g is a bijection.

By the induction hypothesis, $|R| = |T \setminus \{f(x_{n+1})\}|$. Thus, $n = |R| = |T \setminus \{f(x_{n+1})\}| = |T| - 1$. Therefore, $|T| = n + 1 = |S|$. $\square$