first-order theory

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Defines	theory
Defines	complete theory
Defines	axiomatizable theory
Defines	deductively closed
Defines	finitely axiomatizable theory

In what follows, references to sentences and sets of sentences are all relative to some fixed first-order language L.

Definition. A **theory** T is a *deductively closed* set of sentences in L; that is, a set T such that for each sentence φ, $T \vdash \varphi$ only if $\varphi \in T$.

Remark. Some authors do not require that a theory be deductively closed. Therefore, a theory is simply a set of sentences. This is not a cause for alarm, since every theory T under this definition can be "extended" to a deductively closed theory $T^{\vdash} := \{\varphi \in L \mid T \vdash \varphi\}$. Furthermore, $T^{\vdash}$ is unique (it is the smallest deductively closed theory including T), and any structure M is a model of T iff it is a model of $T^{\vdash}$.

Definition. A theory T is *consistent* if and only if for some sentence φ, $T \not\vdash \varphi$. Otherwise, T is *inconsistent*. A sentence φ is *consistent with* T if and only if the theory $T \cup \{\varphi\}$ is consistent.

Definition. A theory T is *complete* if and only if T is consistent and for each sentence φ, either $\varphi \in T$ or $\neg\varphi \in T$.

Lemma. A consistent theory T is complete if and only if T is maximally consistent. That is, T is complete if and only if for each sentence φ, $\varphi \notin T$ only if $T \cup \{\varphi\}$ is inconsistent. See `http://planetmath.org/MaximallyConsistent`this entry for a proof.

Theorem. (Tarski) Every consistent theory T is included in a complete theory.

Proof : Use Zorn's lemma on the set of consistent theories that include T.

Remark. A theory T is *axiomatizable* if and only if T includes a `http://planetmath.org/Deci` subset Δ such that $\Delta \vdash T$ (every sentence of T is a logical consequence of Δ), and *finitely axiomatizable* if Δ can be made finite. Every complete axiomatizable theory T is decidable; that is, there is an algorithm that given a sentence φ as input yields 0 if $\varphi \in T$, and 1 otherwise.