properties of ranks of sets

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A set A is said to be *grounded*, if $A \subseteq V_\alpha$ in the cumulative hierarchy for some ordinal α. The smallest such α such that $A \subseteq V_\alpha$ is called the rank of A, and is denoted by $\rho(A)$.

In this entry, we list derive some basic properties of groundedness and ranks of sets. Proofs of these properties require an understanding of some of the basic properties of ordinals.

1. $\varnothing$ is grounded, whose rank is itself. This is obvious.

2. If A is grounded, so is every $x \in A$, and $\rho(x) < \rho(A)$.

   Proof. $A \subseteq V_{\rho(A)}$, so $x \in V_{\rho(A)}$, which means $x \subseteq V_\beta$ for some $\beta < \rho(A)$. This shows that x is grounded. Then $\rho(x) \leq \beta$, and hence $\rho(x) < \rho(A)$. □

3. If every $x \in A$ is grounded, so is A, and $\rho(A) = \sup\{\rho(x)^+ \mid x \in A\}$.

   Proof. Let $B = \{\rho(x)^+ \mid x \in A\}$. Then B is a set of ordinals, so that $\beta := \bigcup B = \sup B$ is an ordinal. Since each $x \in V_{\rho(x)^+}$, we have $x \in V_\beta$. So $A \subseteq V_\beta$, showing that A is grounded. If $\alpha < \beta$, then for some $x \in A$, $\alpha < \rho(x)^+$, which means $x \notin V_\alpha$, and therefore $A \nsubseteq V_\alpha$. This shows that $\rho(A) = \beta$. □

4. If A is grounded, so is $\{A\}$, and $\rho(\{A\}) = \rho(A)^+$. This is a direct consequence of the previous result.

5. If A, B are grounded, so is $A \cup B$, and $\rho(A \cup B) = \max(\rho(A), \rho(B))$.

   Proof. Since A, B are grounded, every element of $A \cup B$ is grounded by property 2, so that $A \cup B$ is also grounded by property 3. Then $\rho(A \cup B) = \sup\{\rho(x)^+ \mid x \in A \cup B\} = \max(\sup\{\rho(x)^+ \mid x \in A\}, \sup\{\rho(x)^+ \mid x \in B\}) = \max(\rho(A), \rho(B))$. □

6. If A is grounded, so is $B \subseteq A$, and $\rho(B) \leq \rho(A)$.

   Proof. Every element of B, as an element of the grounded set A, is grounded, and therefore B is grounded. So $\rho(B) = \sup\{\rho(x)^+ \mid x \in B\} \leq \sup\{\rho(x)^+ \mid x \in A\} = \rho(A)$. Since $\rho(B)$ and $\rho(A)$ are both ordinals, $\rho(B) \leq \rho(A)$. □

7. If A is grounded, so is $P(A)$, and $\rho(P(A)) = \rho(A)^+$.

   Proof. Every subset of A is grounded, since A is by property 6. So $P(A)$ is grounded. Furthermore, $P(A) = \sup\{\rho(x)^+ \mid x \in P(A)\}$. Since $\rho(B) \leq \rho(A)$ for any $B \in P(A)$, and $A \in P(A)$, we have $P(A) = \rho(A)^+$ as a result. $\square$

8. If A is grounded, so is $\bigcup A$, and $\rho(\bigcup A) = \sup\{\rho(x) \mid x \in A\}$.

   Proof. Since A is grounded, every $x \in A$ is grounded. Let $B = \{\rho(x) \mid x \in A\}$. Then $\beta := \bigcup B = \sup B$ is an ordinal. Since $\rho(x) \leq \beta$, $V_{\rho(x)} = V_\beta$ or $V_{\rho(x)} \in V_\beta$. In either case, $V_{\rho(x)} \subseteq V_\beta$, since V_α is a transitive set for any ordinal α. Since $x \subseteq V_{\rho(x)}$, $x \subseteq V_\beta$ for every $x \in A$. This means $\bigcup A \subseteq V_\beta$, showing that $\bigcup A$ is grounded. If $\alpha < \beta$, then $\alpha < \rho(x)$ for some $\rho(x) \leq \beta$, which means $x \nsubseteq V_\alpha$, or $\bigcup A \nsubseteq V_\alpha$ as a result. Therefore $\rho(\bigcup A) = \beta$. $\square$

9. Every ordinal is grounded, whose rank is itself.

   Proof. If $\alpha = 0$, then apply property 1. If α is a successor ordinal, apply properties 4 and 5, so that $\rho(\alpha) = \rho(\beta^+) = \rho(\beta \cup \{\beta\}) = \max(\rho(\beta), \rho(\{\beta\})) = \max(\rho(\beta), \rho(\beta)^+) = \rho(\beta)^+$. If α is a limit ordinal, then apply property 8 and transfinite induction, so that $\rho(\alpha) = \rho(\bigcup \alpha) = \sup\{\rho(\beta) \mid \beta < \alpha\} = \sup\{\beta \mid \beta < \alpha\} = \alpha$. $\square$

References

[1] H. Enderton, *Elements of Set Theory*, Academic Press, Orlando, FL (1977).

[2] A. Levy, *Basic Set Theory*, Dover Publications Inc., (2002).