FS iterated forcing preserves chain condition

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Let κ be a regular cardinal and let $\langle \hat{Q}_\beta \rangle_{\beta<\alpha}$ be a finite support iterated forcing where for every $\beta < \alpha$, $\Vdash_{P_\beta} \hat{Q}_\beta$ `has the` κ `chain condition`.

By induction:

P_0 is the empty set.

If P_α satisfies the κ chain condition then so does $P_{\alpha+1}$, since $P_{\alpha+1}$ is equivalent to $P_\alpha * Q_\alpha$ and composition preserves the κ chain condition for regular κ.

Suppose α is a limit ordinal and P_β satisfies the κ chain condition for all $\beta < \alpha$. Let $S = \langle p_i \rangle_{i<\kappa}$ be a subset of P_α of size κ. The domains of the elements of p_i form κ finite subsets of α, so if $\mathrm{cf}(\alpha) > \kappa$ then these are bounded, and by the inductive hypothesis, two of them are compatible.

Otherwise, if $\mathrm{cf}(\alpha) < \kappa$, let $\langle \alpha_j \rangle_{j<\mathrm{cf}(\alpha)}$ be an increasing sequence of ordinals cofinal in α. Then for any $i < \kappa$ there is some $n(i) < \mathrm{cf}(\alpha)$ such that $\mathrm{dom}(p_i) \subseteq \alpha_{n(i)}$. Since κ is regular and this is a partition of κ into fewer than κ pieces, one piece must have size κ, that is, there is some j such that $j = n(i)$ for κ values of i, and so $\{p_i \mid n(i) = j\}$ is a set of conditions of size κ contained in P_{α_j}, and therefore contains compatible members by the induction hypothesis.

Finally, if $\mathrm{cf}(\alpha) = \kappa$, let $C = \langle \alpha_j \rangle_{j<\kappa}$ be a strictly increasing, continuous sequence cofinal in α. Then for every $i < \kappa$ there is some $n(i) < \kappa$ such that $\mathrm{dom}(p_i) \subseteq \alpha_{n(i)}$. When $n(i)$ is a limit ordinal, since C is continuous, there is also (since $\mathrm{dom}(p_i)$ is finite) some $f(i) < i$ such that $\mathrm{dom}(p_i) \cap [\alpha_{f(i)}, \alpha_i) = \emptyset$. Consider the set E of elements i such that i is a limit ordinal and for any $j < i$, $n(j) < i$. This is a club, so by Fodor's lemma there is some j such that $\{i \mid f(i) = j\}$ is stationary.

For each p_i such that $f(i) = j$, consider $p'_i = p_i \upharpoonright j$. There are κ of these, all members of P_j, so two of them must be compatible, and hence those two are also compatible in P.