properties of ordinal arithmetic

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Let **On** be the class of ordinals, and $\alpha, \beta, \gamma, \delta \in \textbf{On}$. Then the following properties are satisfied:

1. (additive identity): $\alpha+0 = 0+\alpha = \alpha$ (`http://planetmath.org/ExampleOfTransfiniteIndu`
2. (associativity of addition): $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
3. (multiplicative identity): $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$
4. (multiplicative zero): $\alpha \cdot 0 = 0 \cdot \alpha = 0$
5. (associativity of multiplication): $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
6. (left distributivity): $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
7. (existence and uniqueness of subtraction): if $\alpha \leq \beta$, then there is a unique γ such that $\alpha + \gamma = \beta$
8. (existence and uniqueness of division): for any α, β with $\beta \neq 0$, there exists a unique pair of ordinals γ, δ such that $\alpha = \beta \cdot \delta + \gamma$ and $\gamma < \beta$.

Conspicuously absent from the above list of properties are the commutativity laws, as well as right distributivity of multiplication over addition. Below are some counterexamples:

- $\omega + 1 \neq 1 + \omega = \omega$, for the former has a top element and the latter does not.
- $\omega \cdot 2 \neq 2 \cdot \omega$, for the former is $\omega + \omega$, which consists an element α such that $\beta < \alpha$ for all $\beta < \omega$, and the latter is $2 \cdot \sup\{n \mid n < \omega\} = \sup\{2 \cdot n \mid n < \omega\} = \sup\{n \mid n < \omega\}$, which is just ω, and which does not consist such an element α
- $(1+1) \cdot \omega \neq 1 \cdot \omega + 1 \cdot \omega$, for the former is $2 \cdot \omega$ and the latter is $\omega \cdot 2$, and the rest of the follows from the previous counterexample.

All of the properties above can be proved using transfinite induction. For a proof of the first property, please see `http://planetmath.org/ExampleOfTransfiniteInductio` link.

For properties of the arithmetic regarding exponentiation of ordinals, please refer to `http://planetmath.org/OrdinalExponentiation`this link.