signature

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Entry type	Definition
Classification	msc 03C07
Synonym	language
Synonym	non-logical symbols
Defines	constant symbol
Defines	function symbol
Defines	relation symbol

A *signature* Σ is a set

$$\Sigma := \left(\bigcup_{n\in\omega} \mathcal{R}_n\right) \cup \left(\bigcup_{n\in\omega} \mathcal{F}_n\right) \cup \mathcal{C}$$

where for each natural number $n > 0$,

- $\mathcal{R}_n$ is a (usually countable) set of n-ary relation symbols.
- $\mathcal{F}_n$ is a (usually countable) set of n-ary function symbols.
- $\mathcal{C}$ is a (usually countable) set of constant symbols.

We require that all these sets be pairwise disjoint.

Given a signature Σ, a Σ-structure is then a structure $\mathcal{A}$, whose underlying set is some set A, with elements $\mathcal{A}_c \in A$ for each constant symbol $c \in \Sigma$, n-ary operations $\mathcal{A}_f$ on A for each n-ary function symbol $f \in \Sigma$, for each n, and m-ary relations $\mathcal{A}_R$ on A for each m-ary relation symbol $R \in \Sigma$.

On the other hand, every structure is associated with a signature. For every structure, it has an underlying set, together with a collection of "designated" objects that "define" the structure. These objects may be elements of the underlying set, operations on the set, or relations on the set. For each such "designated" object, pick a symbol for it. Make sure all symbols used are distinct from one another. Then the collection of all such symbols is a signature for the structure.

For most structures that we encounter, the set Σ is finite, but we allow it to be infinite, even uncountable, as this sometimes makes things easier, and just about everything still works when the signature is uncountable.

Examples:

- A signature of sets is the empty set.
- A signature of pointed sets is a singleton consisting of a constant symbol.
- A signature of groups is a set $\{e, ^{-1}, \cdot\}$, where
  1. e (group identity symbol) is a constant symbol,
  2. $^{-1}$ (group inverse symbol) is a unary function symbol, and
  3. $\cdot$ (group multiplication symbol) is a binary function symbol.

- A signature of fields is a set $\{0, 1, -, ^{-1}, +, \cdot\}$, where

    1. 0 (additive identity symbol) and 1 (multiplicative identity symbol) are constant symbols,
    2. $-$ (the additive inverse symbol) and $^{-1}$ (the multiplicative inverse symbol) are the unary function symbols, and
    3. $+$ (addition symbol) $\cdot$ (multiplication symbol) are binary function symbols.

- A signature of posets is a singleton $\{\leq\}$, where $\leq$ (partial ordering symbol) is a binary relation symbol.

- A signature of vector spaces over a fixed field k consists of the following

    1. 0 (additive identity symbol) is the constant symbol,
    2. $+$ (vector addition symbol) is the binary function symbol, and
    3. $\cdot_r$ (multiplication by scalar r symbol) is the unary function symbol, for *each* $r \in k$.

Remark. Given a signature Σ, the set L of logical symbols from first order logic, and a countably infinite set V of variables, we can form a first order language, consisting of all formulas built from these symbols (in $\Sigma \cup L \cup V$). The language so-created is uniquely determined by Σ. In the literature, it is a common practice to identify Σ both as a signature and the unique language it generates.

References

[1] W. Hodges, *A Shorter Model Theory*, Cambridge University Press, (1997).

[2] D. Marker, *Model Theory, An Introduction*, Springer, (2002).