superexponentiation is not elementary

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In this entry, we will show that the superexponetial function $f:\mathbb{N}^2\to\mathbb{N}$, given by

$$f(m,0)=m, \qquad f(m,n+1)=m^{f(m,n)}$$

is not elementary recursive (we set $f(0,n):=0$ for all n). We will use the properties of f (listed `http://planetmath.org/PropertiesOfSuperexponentiation`here) to complete this task.

The idea behind the proof is to find a property satisfied by all elementary recursive functions but not by f. The particular property we have in mind is the "growth rate" of a function. We want to demonstrate that f, in some way, grows faster than any elementary function g. This idea is similar to showing that 2^x is larger than, say, x^{100} for large enough x. Formally,

Definition. A function $h:\mathbb{N}^2\to\mathbb{N}$ is said to *majorize* $g:\mathbb{N}^k\to\mathbb{N}$ if there is a $b\in\mathbb{N}$, such that for any $a_1,\ldots,a_k\in\mathbb{N}$:

$$g(a_1,\ldots,a_k)<h(a,b), \qquad \text{where } a=\max\{a_1,\ldots,a_k\}>1.$$

It is easy to see that no binary function majorizes itself:

Proposition 1. *$h:\mathbb{N}^2\to\mathbb{N}$ does not majorize h.*

Proof. Otherwise, there is a b such that for any x,y, $h(x,y)<h(a,b)$ where $a=\max\{x,y\}>1$. Let $c=\max\{a,b\}>1$. Then $h(c,b)<h(\max\{b,c\},b)=h(c,b)$, a contradiction. □

Let $\mathcal{ER}$ be the set of all elementary recursive functions.

Proposition 2. *Let $\mathcal{A}$ be the set of all functions majorized by f. Then $\mathcal{ER}\subseteq\mathcal{A}$.*

Proof. We simply show that $\mathcal{A}$ contains the addition, multiplication, difference, quotient, and the projection functions, and that $\mathcal{A}$ is closed under composition, bounded sum, and bounded product. And since $\mathcal{ER}$ is the smallest such set, the proof completes.

- For addition, multiplication, and difference: suppose $t=\max\{x,y\}>1$. Then $x+y\le 2t=2f(t,0)\le f(t,1)<f(t,2)$, and $xy\le t^2=f(t,0)^2\le f(t,1)<f(t,2)$. Moreover, $|x-y|\le t=f(t,0)<f(t,1)$, and $\operatorname{quo}(x,y)\le t=f(t,0)<f(t,1)$.

- For projection functions p^k_m, suppose $t=\max\{x_1,\ldots,x_k\}>1$. Then $p^k_m(\boldsymbol{x})=x_m\le t=f(t,0)<f(t,1)$.

- Suppose $g_1, \ldots, g_m \in A$ are n-ary, and $h \in A$ is m-ary. Let $u = h(g_1, \ldots, g_m)$ and suppose $x = \{x_1, \ldots, x_n\} > 1$. Given $u(\boldsymbol{x}) = h(g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x}))$, let $z = \max\{g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x})\}$. We have two cases:

  1. $z \leq 1$. Let $y = \max\{h(y_1, \ldots, y_m) \mid y_i \in \{0,1\}\}$. Then $u(\boldsymbol{x}) \leq y < f(x, y)$.
  2. $z > 1$. Then, for some i, $z = g_i(\boldsymbol{x}) < f(x, s)$ for some s, since $g_i \in A$. Then $u(\boldsymbol{x}) = h(g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x})) \leq f(z, t)$ for some t since $h \in \mathcal{A}$. Now, $f(z, t) = f(g_i(\boldsymbol{x}), t) < f(f(x, s), t) \leq f(x, s + 2t)$. As a result, $u(\boldsymbol{x}) < f(x, s + 2t)$.

  In either case, let $r = \max\{y, s + 2t\}$. We see that $u(\boldsymbol{x}) < f(x, r)$.

- For the next two parts, suppose $g \in A$ is $(n+1)$-ary. For any $\boldsymbol{x} = (x_1, \ldots, x_n)$, let $x = \max\{x_1, \ldots, x_n\}$, and for any y, assume $z = \max\{x, y\} > 1$. Since $g \in A$, let $t \in \mathbb{N}$ be such that $g(\boldsymbol{x}, y) \leq f(z, t)$, where z is as described above.

  Let $g_s(\boldsymbol{x}, y) := \sum_{i=0}^{y} g(\boldsymbol{x}, i)$. We break this down into cases:

  1. $x > 1$. Then $g(\boldsymbol{x}, i) < f(z_i, t)$ where $z_i = \max\{x, i\} > 1$ for each i. Let $f(z_j, t)$ be the maximum among the $f(z_i, t)$. Then $g_s(\boldsymbol{x}, y) \leq (y+1) f(z_j, t) \leq (y+1) f(z, t)$, as $j \leq y$. Since $y+1 \leq z+1 < 2z = 2f(z, 0) \leq f(z, 1)$, we see that $g_s(\boldsymbol{x}, y) < f(z, 1) f(z, t) \leq f(z, t_1)$, where $t_1 = 1 + \max\{1, t\}$.
  2. $x = 1$. Then $y > 1$. So $g_s(\boldsymbol{x}, y) = h(\boldsymbol{x}) + \sum_{i=2}^{y} g(\boldsymbol{x}, i)$, where $h(\boldsymbol{x}) = g(\boldsymbol{x}, 0) + g(\boldsymbol{x}, 1)$. Let $v = \max\{h(v_1, \ldots, v_n) \mid v_i \in \{0,1\}\}$. Then $g_s(\boldsymbol{x}, y) \leq v + \sum_{i=2}^{y} g(\boldsymbol{x}, i)$. As before, $g(\boldsymbol{x}, i) \leq f(z_i, t)$ for each $i \leq 2$, so pick the largest $f(z_j, t)$ among the $f(z_i, t)$. Then $\sum_{i=2}^{y} g(\boldsymbol{x}, i) \leq (y-1) f(z_j, t) \leq (y-1) f(z, t) < z f(z, t) = f(z, 0) f(z, t) \leq f(z, t+1)$. Therefore, $g_s(\boldsymbol{x}, y) < v + f(z, t+1) < f(z, v) + f(z, t+1) \leq f(z, t_2)$, where $t_2 = 1 + \max\{v, t+1\}$.

  In each case, pick $t_3 = \max\{t_1, t_2\}$, so that $g_s(\boldsymbol{x}, y) < f(z, t_3)$.

- Let $g_p(\boldsymbol{x}, y) := \prod_{i=0}^{y} g(\boldsymbol{x}, i)$. We again break down the proof into cases:

  1. $x > 1$. Then each $g(\boldsymbol{x}, i) < f(z_i, t)$ where $z_i = \max\{x, i\} > 1$. Let $f(z_j, t)$ be the maximum among the $f(z_i, t)$. Then $g_s(\boldsymbol{x}, y) \leq f(z_j, t)^{(y+1)} \leq f(z, t)^{(y+1)}$. Since $y + 1 \leq z + 1 < 2z = 2f(z, 0) \leq$

$f(z,1)$, we see that $g_s(\boldsymbol{x},y) < f(z,t)^{f(z,1)} \leq f(z,t_1)$, where $t_1 = 2+\max\{1,t\}$.

2. $x=1$. Then $y>1$. So $g_p(\boldsymbol{x},y) = h(\boldsymbol{x})\prod_{i=2}^{y} g(\boldsymbol{x},i)$, where $h(\boldsymbol{x}) = g(\boldsymbol{x},0)g(\boldsymbol{x},1)$. Let $v = \max\{h(v_1,\ldots,v_n) \mid v_i \in \{0,1\}\}$. Then $g_p(\boldsymbol{x},y) \leq v\prod_{i=2}^{y} g(\boldsymbol{x},i)$. As before, each $g(\boldsymbol{x},i) \leq f(z_i,t)$, so pick the largest $f(z_j,t)$ among the $f(x_i,t)$. Then $\prod_{i=2}^{y} g(\boldsymbol{x},i) \leq f(z_j,t)^{(y-1)} \leq f(z,t)^{(y-1)} < f(z,t)^z = f(z,t)^{f(z,0)} \leq f(z,t+2)$. Therefore, $g_p(\boldsymbol{x},y) < vf(z,t+2) < f(z,v)f(z,t+2) \leq f(z,t_2)$, where $t_2 = 1+\max\{v,t+2\}$.

In each case, pick $t_3 = \max\{t_1,t_2\}$, so that $g_p(\boldsymbol{x},y) < f(z,t_3)$.

As a result, $\mathcal{ER} \subseteq \mathcal{A}$. In other words, every elementary function is majorized by f. $\square$

In conclusion, we have

Corollary 1. *f is not elementary.*

Proof. If it were, it would majorize itself, which is impossible. $\square$

Remark. Although f is not elementary recursive, it is easy to see that, for any n, the function $f_n : \mathbb{N} \to \mathbb{N}$ defined by $f_n(m) := f(m,n)$ is elementary. This can be done by induction on n:

> $f_0(m) = f(m,0) = m = p_1^1(m)$ is elementary, and if $f_n(m)$ is elementary, so is $f_{n+1}(m) = f(m,n+1) = \exp(m, f(m,n)) = \exp(p_1^1(m), f_n(m))$, since exp is elementary, and elementary recursiveness preserves composition.

Using this fact, one may in fact show that $\mathcal{ER} = \mathcal{A} \cap \mathcal{PR}$, where $\mathcal{PR}$ is the set of all primitive recursive functions.