factor embeddable

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Defines	factor embeddable class
Defines	factor embedding
Defines	strong factor embedding

Let K be a class of models (structures) of a given signature. Consider a (non-empty) family of structures $\{A_i : i \in I\}$ in K. If $j \in I$ and $f : A_j \to \prod_{i\in I} A_i$ is an embedding, we say that f is a *factor embedding.* [?, ?] If additionally f satisfies the condition that $\pi_j \circ f$ is the identity on A_i, where $\pi_j : \prod_{i\in I} A_i \to A_j$ is the jth projection, then f is said to be a *strong factor embedding.* [?] K is said to be a *factor embeddable* class iff for every (non-empty) family of structures $\{A_i : i \in I\}$ in K and every $j \in I$ there is a factor embedding $f : A_j \to \prod_{i\in I} A_i$. [?, ?]

The definition above does not require the product $\prod_{i\in I} A_i$ to be a member of K, however many interesting examples of factor embeddable classes are in fact closed under products. Factor embeddable classes that are closed under finite products (or equivalently under binary products) have the joint embedding property. Factor embeddable classes closed under arbitrary products have the strong joint embedding property.

0.0.1 Characterization

Factor embeddable classes have an easy to prove but somewhat unintuitive characterization which does not mention the concepts of product or embedding:

The following are equivalent for a class K of models [?]:

1. K is factor embeddable.
2. For every pair of models $A, B \in K$ there exists a homomorphism $f : A \to B$.

To see the above, suppose K is factor embeddable and consider models $A, B \in K$. Then there exists a factor embedding from A into the product $A \times B$. Composing this embedding with the projection onto B gives a homomorphism $f : A \to B$. Conversely suppose such a homomorphism $f : A \to B$ exists for all $A, B \in K$ and consider a family $\{A_i : i \in I\}$ in K and $j \in I$. We can define a strong factor embedding $f : A_j \to \prod_{i\in I} A_i$ by choosing homomorphisms $f_i : A_j \to A_i$ for all $i \in I$ with f_j the identity map on A_j, and then for all $a \in A$ setting $f(a)_i = f_i(a)$ for each $i \in I$. [?]

The above proof shows that the factor embeddings guaranteed to exist for a factor embeddable class can always be chosen to be strong factor emebeddings. [?]

A corollory of the above is that if there exists a model which is a retract of every member of a class K then, K is factor embeddable - in particular if the members of K have one element submodels, then K is factor embeddable. [**?**] (A retract of a model is a submodel which is also a quotient model such that the quotient map composed with the submodel embedding is the identity map.)

0.0.2 Examples

The following are examples of factor embeddable classes:

- The variety of all groups (the trivial group is a one element subalgebra of every group)
- The variety of all lattices (every lattice has one element sublattices)
- The class of all non-trivial Boolean algebras (the two element Boolean algebra is a retract of all non-trivial Boolean algebras)

The class of all Boolean algebras is an example of a class which is not factor embeddable - there is no way to embed the trivial Boolean algebra into a product of itself with any non-trivial Boolean algebras. (The trivial Boolean algebra satisfies the identity $0 = 1$ which is not satisfied by any Boolean algebra having more than one element.)

References

[1] Peter Bruyns, Henry Rose: *Varieties with cofinal sets: examples and amalgamation*, Proc. Amer. Math. Soc. 111 (1991), 833-840

[2] Colin Naturman, Henry Rose: *Ultra-universal models*, Quaestiones Mathematicae, 15(2), 1992, 189-195