properties of complement

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Let X be a set and A, B are subsets of X.

1. $(A^{\complement})^{\complement} = A$.

   Proof. $a \in (A^{\complement})^{\complement}$ iff $a \notin A^{\complement}$ iff $a \in A$. $\square$

2. $\emptyset^{\complement} = X$.

   Proof. $a \in \emptyset^{\complement}$ iff $a \notin \emptyset$ iff $a \in X$. $\square$

3. $X^{\complement} = \emptyset$.

   Proof. $a \in X^{\complement}$ iff $a \notin X$ iff $a \in \emptyset$. $\square$

4. $A \cup A^{\complement} = X$.

   Proof. $a \in A \cup A^{\complement}$ iff $a \in A$ or $a \in A^{\complement}$ iff $a \in A$ or $a \notin A$ iff $a \in X$. $\square$

5. $A \cap A^{\complement} = \emptyset$.

   Proof. $a \in A \cap A^{\complement}$ iff $a \in A$ and $a \in A^{\complement}$ iff $a \in A$ and $a \notin A$ iff $a \in \emptyset$. $\square$

6. $A \subseteq B$ iff $B^{\complement} \subseteq A^{\complement}$.

   Proof. Suppose $A \subseteq B$. If $a \in B^{\complement}$, then $a \notin B$, so $a \notin A$, or $a \in A^{\complement}$. This shows that $B^{\complement} \subseteq A^{\complement}$. On the other hand, if $B^{\complement} \subseteq A^{\complement}$, then by applying what's just been proved, $A = (A^{\complement})^{\complement} \subseteq (B^{\complement})^{\complement} = B$. $\square$

7. $A \cap B = \emptyset$ iff $A \subseteq B^{\complement}$.

   Proof. Suppose $A \cap B = \emptyset$. If $a \in A$, then $a \in B^{\complement}$, or $a \notin B$, which implies that $A \cap B = \emptyset$. Suppose next that $A \subseteq B^{\complement}$. If there is $a \in A \cap B$, then $a \in B$ and $a \in A$. But the second containment implies that $a \in B^{\complement}$, which contradicts the first containment. $\square$

8. $A \setminus B = A \cap B^{\complement}$, where the complement is taken in X.

   Proof. $a \in A \setminus B$ iff $a \in A$ and $a \notin B$ iff $a \in A$ and $a \in B^{\complement}$ iff $a \in A \cap B^{\complement}$. $\square$

9. (de Morgan's laws) $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$ and $(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$.

Proof. See `http://planetmath.org/DeMorgansLawsProof`here. $\square$