axiom schema of separation

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Let $\phi(u,p)$ be a formula. For any X and p, there exists a set $Y = \{u \in X : \phi(u,p)\}$.

The Axiom Schema of Separation is an axiom schema of Zermelo-Fraenkel set theory. Note that it represents infinitely many individual axioms, one for each formula ϕ. In symbols, it reads:

$$\forall X \forall p \exists Y \forall u (u \in Y \leftrightarrow u \in X \wedge \phi(u,p)).$$

By Extensionality, the set Y is unique.

The Axiom Schema of Separation implies that ϕ may depend on more than one parameter p.

We may show by induction that if $\phi(u, p_1, \ldots, p_n)$ is a formula, then

$$\forall X \forall p_1 \cdots \forall p_n \exists Y \forall u (u \in Y \leftrightarrow u \in X \wedge \phi(u, p_1, \ldots, p_n))$$

holds, using the Axiom Schema of Separation and the Axiom of Pairing.

Another consequence of the Axiom Schema of Separation is that a subclass of any set is a set. To see this, let $\mathbf{C}$ be the class $\mathbf{C} = \{u : \phi(u, p_1, \ldots, p_n)\}$. Then

$$\forall X \exists Y (\mathbf{C} \cap X = Y)$$

holds, which means that the intersection of $\mathbf{C}$ with any set is a set. Therefore, in particular, the intersection of two sets $X \cap Y = \{x \in X : x \in Y\}$ is a set. Furthermore the difference of two sets $X - Y = \{x \in X : x \notin Y\}$ is a set and, provided there exists at least one set, which is guaranteed by the Axiom of Infinity, the empty set is a set. For if X is a set, then $\emptyset = \{x \in X : x \neq x\}$ is a set.

Moreover, if $\mathbf{C}$ is a nonempty class, then $\bigcap \mathbf{C}$ is a set, by Separation. $\bigcap \mathbf{C}$ is a subset of every $X \in \mathbf{C}$.

Lastly, we may use Separation to show that the class of all sets, V, is not a set, i.e., V is a proper class. For example, suppose V is a set. Then by Separation

$$V' = \{x \in V : x \notin x\}$$

is a set and we have reached a Russell paradox.