every congruence is the kernel of a homomorphism

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Let Σ be a fixed signature, and $\mathfrak{A}$ a structure for Σ. If $\sim$ is a congruence on $\mathfrak{A}$, then there is a homomorphism f such that $\sim$ is the kernel of f.

Proof. Define a homomorphism $f\colon \mathfrak{A} \to \mathfrak{A}/\sim : a \mapsto [\![a]\!]$. Observe that $a \sim b$ if and only if $f(a) = f(b)$, so $\sim$ is the kernel of f. To verify that f is a homomorphism, observe that

1. For each constant symbol c of Σ, $f(c^{\mathfrak{A}}) = [\![c^{\mathfrak{A}}]\!] = c^{\mathfrak{A}/\sim}$.

2. For each $n \in \mathbb{N}$ and each n-ary function symbol F of Σ,

$$\begin{aligned} f(F^{\mathfrak{A}}(a_1, \ldots a_n)) &= [\![F^{\mathfrak{A}}(a_1, \ldots a_n)]\!] \\ &= F^{\mathfrak{A}/\sim}([\![a_1]\!], \ldots [\![a_n]\!]) \\ &= F^{\mathfrak{A}/\sim}(f(a_1), \ldots f(a_n)). \end{aligned}$$ $\square$

3. For each $n \in \mathbb{N}$ and each n-ary relation symbol R of Σ, if $R^{\mathfrak{A}}(a_1, \ldots, a_n)$ then $R^{\mathfrak{A}/\sim}([\![a_1]\!], \ldots, [\![a_n]\!])$, so $R^{\mathfrak{A}/\sim}(f(a_1), \ldots, f(a_n))$.