one-to-one function from onto function

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Theorem. *Given an onto function from a set A to a set B, there exists a one-to-one function from B to A.*

Proof. Suppose $f : A \to B$ is onto, and define $\mathcal{F} = \big\{f^{-1}(\{b\}) : b \in B\big\}$; that is, $\mathcal{F}$ is the set containing the pre-image of each singleton subset of B. Since f is onto, no element of $\mathcal{F}$ is empty, and since f is a function, the elements of $\mathcal{F}$ are mutually disjoint, for if $a \in f^{-1}(\{b_1\})$ and $a \in f^{-1}(\{b_2\})$, we have $f(a) = b_1$ and $f(a) = b_2$, whence $b_1 = b_2$. Let $\mathscr{C} : \mathcal{F} \to \bigcup \mathcal{F}$ be a choice function, noting that $\bigcup \mathcal{F} = A$, and define $g : B \to A$ by $g(b) = \mathscr{C}\big(f^{-1}(\{b\})\big)$. To see that g is one-to-one, let $b_1, b_2 \in B$, and suppose that $g(b_1) = g(b_2)$. This gives $\mathscr{C}\big(f^{-1}(\{b_1\})\big) = \mathscr{C}\big(f^{-1}(\{b_2\})\big)$, but since the elements of $\mathcal{F}$ are disjoint, this implies that $f^{-1}(\{b_1\}) = f^{-1}(\{b_2\})$, and thus $b_1 = b_2$. So g is a one-to-one function from B to A. $\square$