characterization of primitive recursive functions of one variable

Canonical name	CharacterizationOfPrimitiveRecursiveFunctionsOfOneVariable
Date of creation	2013-03-22 16:45:25
Last modified on	2013-03-22 16:45:25
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	9
Author	rspuzio (6075)
Entry type	Theorem
Classification	msc 03D20

It is possible to characterize primitive recursive functions of one variable in terms of operations involving only functions of a single variable. To describe how this goes, it is useful to first define some notation.

Definition 1. *Define the constant function $K\colon \mathbb{N} \to \mathbb{N}$ by $K(n) = 1$ for all n.*

Definition 2. *Define the identity function $I\colon \mathbb{N} \to \mathbb{N}$ by $I(n) = n$ for all n.*

Definition 3. *Define the excess over square function $E\colon \mathbb{N} \to \mathbb{N}$ by $E(n) = n - m^2$, where m is the largest integer such that $m^2 \leq n$.*

Definition 4. *Given a function $f\colon \mathbb{N} \to \mathbb{N}$, define the function $R(f)\colon \mathbb{N} \to \mathbb{N}$ by the following conditions:*

- $R(f)(0) = 0$
- $R(f)(n + 1) = f(R(f)(n))$ *for all integers* $n \geq 0$.

Theorem 1. *The class of primitive recursive functions of a single variable is the smallest class X of functions which contains the functions E and K defined above and is closed under the following three operations:*

1. *If $f \in X$ and $g \in X$, then $f \circ g \in X$.*
2. *If $f \in X$, then $f + I \in X$.* [1]
3. *If $f \in X$, then $R(f) \in X$.*

[1] Here $f + I$ has the usual meaning of pointwise addition — $(f + I)(x) = f(x) + I(x)$