planetmath.org

Math for the people, by the people.

proof that a relation is union of functions if and only if AC

Canonical name	ProofThatARelationIsUnionOfFunctionsIfAndOnlyIfAC
Date of creation	2013-03-22 16:34:23
Last modified on	2013-03-22 16:34:23
Owner	ratboy (4018)
Last modified by	ratboy (4018)
Numerical id	4
Author	ratboy (4018)
Entry type	Proof
Classification	msc 03E25

Theorem. *A relation R is the union of a set of functions each of which has the same domain as R if and only if for each set A of nonempty sets, there is a choice function on A.*

Proof. Suppose that R is a relation with $\mathrm{dom}(R) = A$ and $\mathrm{rng}(R) \subseteq B$. Let $g\colon A \to \mathcal{P}(B)$ be given by $a \mapsto R[\{a\}]$. There is be a choice function c on $g[A]$. Let $f = c \circ g$, and for each pair $\langle a, b\rangle \in R$, let f_{ab} send a to b and agree with f elsewhere. Let $F = \{f_{ab} \mid \langle a, b\rangle \in R\}$. Clearly $\bigcup F \subseteq A \times B$, so suppose $\langle u, v\rangle \in \bigcup F$; then there is a pair $\langle a, b\rangle \in R$ such that $\langle u, v\rangle \in f_{ab} \in F$. Either $\langle u, v\rangle = \langle a, b\rangle$, or $v = f(u) = c \circ g(u) = c(R[\{u\}]) \in R[\{u\}]$. In each case, $\langle u, v\rangle \in R$. Thus, $\bigcup F \subseteq R$. For each pair $\langle a, b\rangle \in R$, $\langle a, b\rangle \in f_{ab} \in F$, so $R \subseteq \bigcup F$. Therefore, $R = \bigcup F$.
Suppose that A is set of nonempty sets. Let $R = \bigcup\{\{a\} \times a \mid a \in A\}$. A set x is an element of $\mathrm{dom}(R)$ if and only if $x \in \{a\}$ for some $a \in A$. Thus, $\mathrm{dom}(R) = A$. There is a set F of functions, each of which has domain A, such that $R = \bigcup F$. Let $f \in F$; then $\mathrm{dom}(f) = A$, and for each pair $\langle a, f(a)\rangle \in f$, $\langle a, f(a)\rangle \in \{a\} \times a$; i.e., $f(a) \in a$. Each such f is, thus, a choice function on A. $\square$