composition preserves chain condition

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Let κ be a regular cardinal. Let P be a forcing notion satisfying the κ chain condition. Let $\hat{Q}$ be a P-name such that $\Vdash_P \hat{Q}$ `is a forcing notion satisfying the` κ `chain condition`. Then $P * Q$ satisfies the κ chain condition.

Proof:

Outline

We prove that there is some p such that any generic subset of P including p also includes κ of the p_i. Then, since $Q[G]$ satisfies the κ chain condition, two of the corresponding $\hat{q}_i$ must be compatible. Then, since G is directed, there is some p stronger than any of these which forces this to be true, and therefore makes two elements of S compatible.

Let $S = \langle p_i, \hat{q}_i \rangle_{i<\kappa} \subseteq P * Q$.

Claim: There is some $p \in P$ such that $p \Vdash |\{i \mid p_i \in \hat{G}\}| = \kappa$

(Note: $\hat{G} = \{\langle p, p \rangle \mid p \in P\}$, hence $\hat{G}[G] = G$)

If no p forces this then every p forces that it is not true, and therefore $\Vdash_P |\{i \mid p_i \in G\}| \leq \kappa$. Since κ is regular, this means that for any generic $G \subseteq P$, $\{i \mid p_i \in G\}$ is bounded. For each G, let $f(G)$ be the least α such that $\beta < \alpha$ implies that there is some $\gamma > \beta$ such that $p_\gamma \in G$. Define $B = \{\alpha \mid \alpha = f(G)\}$ for some G.

Claim: $|B| < \kappa$

If $\alpha \in B$ then there is some $p_\alpha \in P$ such that $p \Vdash f(\hat{G}) = \alpha$, and if $\alpha, \beta \in B$ then p_α must be incompatible with p_β. Since P satisfies the κ chain condition, it follows that $|B| < \kappa$.

Since κ is regular, $\alpha = \mathrm{sub}(B) < \kappa$. But obviously $p_{\alpha+1} \Vdash p_{\alpha+1} \in \hat{G}$. This is a contradiction, so we conclude that there must be some p such that $p \Vdash |\{i \mid p_i \in \hat{G}\}| = \kappa$.

If $G \subseteq P$ is any generic subset containing p then $A = \{\hat{q}_i[G] \mid p_i \in G\}$ must have cardinality κ. Since $Q[G]$ satisfies the κ chain condition, there exist $i, j < \kappa$ such that $p_i, p_j \in G$ and there is some $\hat{q}[G] \in Q[G]$ such that

$\hat{q}[G] \leq \hat{q}_i[G], \hat{q}_j[G]$. Then since G is directed, there is some $p' \in G$ such that $p' \leq p_i, p_j, p$ and $p' \Vdash \hat{q}[G] \leq \hat{q}_1[G], \hat{q}_2[G]$. So $\langle p', \hat{q} \rangle \leq \langle p_i, \hat{q}_i \rangle, \langle p_j, \hat{q}_j \rangle$.