Ψ is surjective if and only if Ψ^* is injective

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Suppose X is a set and V is a vector space over a field F. Let us denote by $M(X,V)$ the set of mappings from X to V. Now $M(X,V)$ is again a vector space if we equip it with pointwise multiplication and addition. In detail, if $f,g \in M(X,V)$ and $\mu,\lambda \in F$, we set

$$\mu f + \lambda g \colon x \quad \mapsto \quad \mu f(x) + \lambda g(x).$$

Next, let Y be another set, let $\Psi\colon X \to Y$ is a mapping, and let $\Psi^*\colon M(Y,V) \to M(X,V)$ be the pullback of Ψ as defined in `http://planetmath.org/Pullback2`this entry.

Proposition 1.

1. *Ψ^* is linear.*
2. *If V is not the zero vector space, then Ψ is surjective if and only if Ψ^* is injective.*

Proof. First, suppose $f,g \in M(Y,V)$, $\mu,\lambda \in F$, and $x \in X$. Then

$$\begin{aligned}
\Psi^*(\mu f + \lambda g)(x) &= (\mu f + \lambda g)(\Psi(x)) \\
&= \mu f \circ \Psi(x) + \lambda g \circ \Psi(x) \\
&= \left(\mu \Psi^*(f) + \lambda \Psi^*(g)\right)(x),
\end{aligned}$$

so $\Psi^*(\mu f + \lambda g) = \mu\Psi^*(f) + \lambda\Psi^*(g)$, and Ψ^* is linear. For the second claim, suppose Ψ is surjective, $f \in M(Y,V)$, and $\Psi^*(f) = 0$. If $y \in Y$, then for some $x \in X$, we have $\Psi(x) = y$, and $f(y) = f \circ \Psi(x) = \Psi^*(f)(x) = 0$, so $f = 0$. Hence, the kernel of Ψ^* is zero, and Ψ^* is an injection. On the other hand, suppose Ψ^* is a injection, and Ψ is not a surjection. Then for some $y' \in Y$, we have $y' \notin \Psi(X)$. Also, as V is not the zero vector space, we can find a non-zero vector $v \in V$, and define $f \in M(Y,V)$ as

$$f(y) = \begin{cases} v, & \text{if } y = y', \\ 0, & \text{if } y \neq y', y \in Y. \end{cases}$$

Now $f \circ \Psi(x) = 0$ for all $x \in X$, so $\Psi^* f = 0$, but $f \neq 0$. $\square$