proof that $C_\cup$ and $C_\cap$ are consequence operators

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Owner	rspuzio (6075)
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The proof that the operators $C_\cup$ and $C_\cap$ defined in the second example of section 3 of the `http://planetmath.org/ConsequenceOperator`parent entry are consequence operators is a relatively straightforward matter of checking that they satisfy the defining properties given there. For convenience, those definitions are reproduced here.

Definition 1. *Given a set L and two elements, X and Y, of this set, the function $C_\cap(X,Y)\colon \mathcal{P}(L) \to \mathcal{P}(L)$ is defined as follows:*

$$C_\cap(X,Y)(Z) = \begin{cases} X \cup Z & Y \cap Z \neq \emptyset \\ Z & Y \cap Z = \emptyset \end{cases}$$

Theorem 1. *For every choice of two elements, X and Y, of a given set L, the function $C_\cap(X,Y)$ is a consequence operator.*

Proof.

Property 1: Since Z is a subset of itself and of $X \cup Z$, it follows that $Z \subseteq C_\cap(X,Y)(Z)$ in either case.

Property 2: We consider two cases. If $Y \cap Z = \emptyset$, then $C_\cap(X,Y)(Z) = Z$, so

$$C_\cap(X,Y)(C_\cap(X,Y)(Z)) = C_\cap(X,Y)(Z).$$

If $Y \cap Z \neq \emptyset$, then

$$\begin{aligned} Y \cap C_\cap(X,Y)(Z) &= Y \cap (X \cup Z) \\ &= (Y \cap X) \cup (Y \cap Z). \end{aligned}$$

Again, since $Y \cap Z \neq \emptyset$, we also have $(Y \cap X) \cup (Y \cap Z) \neq \emptyset$, so

$$\begin{aligned} C_\cap(X,Y)(C_\cap(X,Y)(Z)) &= X \cup C_\cap(X,Y)(Z) \\ &= X \cup (X \cup Z) \\ &= X \cup Z \\ &= C_\cap(X,Y)(Z) \end{aligned}$$

So, in both cases, we find that

$$C_\cap(X,Y)(C_\cap(X,Y)(Z)) = C_\cap(X,Y)(Z).$$

Property 3: Suppose that Z and W are subsets of L and that Z is a subset of W. Then there are three possibilities:

1. $Y \cap Z = \emptyset$ and $Y \cap W = \emptyset$

In this case, we have $C_\cap(X,Y)(Z) = Z$ and $C_\cap(X,Y)(W) = W$, so $C_\cap(X,Y)(Z) \subseteq C_\cap(X,Y)(W)$.

2. $Y \cap Z = \emptyset$ but $Y \cap W \neq \emptyset$

In this case, $C_\cap(X,Y)(Z) = Z$ and $C_\cap(X,Y)(W) = X \cup W$. Since $Z \subseteq W$ implies $Z \subseteq X \cup W$, we have $C_\cap(X,Y)(Z) \subseteq C_\cap(X,Y)(W)$.

3. $Y \cap Z \neq \emptyset$ and $Y \cap W \neq \emptyset$

In this case, $C_\cap(X,Y)(Z) = X \cup Z$ and $C_\cap(X,Y)(W) = X \cup W$. Since $Z \subseteq W$ implies $X \cup Z \subseteq X \cup W$, we have $C_\cap(X,Y)(Z) \subseteq C_\cap(X,Y)(W)$.

□

Definition 2. *Given a set L and two elements, X and Y, of this set, the function $C_\cup(X,Y) \colon \mathcal{P}(L) \to \mathcal{P}(L)$ is defined as follows:*

$$C_\cup(X,Y)(Z) = \begin{cases} X \cup Z & Y \cup Z = Z \\ Z & Y \cup Z \neq Z \end{cases}$$

Theorem 2. *For every choice of two elements, X and Y, of a given set L, the function $C_\cup(X,Y)$ is a consequence operator.*

Proof.

Property 1: Since Z is a subset of itself and of $X \cup Z$, it follows that $Z \subseteq C_\cup(X,Y)(Z)$ in either case.

Property 2: We consider two cases. If $C_\cup(X,Y)(Z) = Z$, then

$$C_\cup(X,Y)(C_\cup(X,Y)(Z)) = C_\cup(X,Y)(Z).$$

If $C_\cup(X,Y)(Z) = X \cup Z$, then we note that, because $X \cup (X \cup Z) = X \cup Z$, we must have $C_\cup(X,Y)(X \cup Z) = X \cup Z$ whether or not $Y \cup (X \cup Z) = X \cup Z$, so

$$C_\cup(X,Y)(C_\cup(X,Y)(Z)) = C_\cup(X,Y)(Z).$$

Property 3: Suppose that Z and W are subsets of L and that Z is a subset of W. Then there are three possibilities:

1. $Y \cup Z = Z$ and $Y \cup W = W$

In this case, we have $C_\cup(X,Y)(Z) = X \cup Z$ and $C_\cup(X,Y)(W) = X \cup W$. Since $Z \subseteq W$ implies $X \cup Z \subseteq X \cup W$, we have $C_\cup(X,Y)(Z) \subseteq C_\cup(X,Y)(W)$.

2. $Y \cup Z \neq Z$ but $Y \cup W = W$

In this case, $C_\cup(X,Y)(Z) = Z$ and $C_\cup(X,Y)(W) = X \cup W$. Since $Z \subseteq W$ implies $Z \subseteq X \cup W$, we have $C_\cup(X,Y)(Z) \subseteq C_\cup(X,Y)(W)$.

3. $Y \cup Z \neq Z$ and $Y \cup W \neq W$

In this case, $C_{\cup}(X, Y)(Z) = Z$ and $C_{\cup}(X, Y)(W) = W$, so $C_{\cup}(X, Y)(Z) \subseteq C_{\cup}(X, Y)(W)$. $\square$