ordinal arithmetic

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Ordinal arithmetic is the extension of normal arithmetic to the transfinite ordinal numbers. The successor operation Sx (sometimes written $x+1$, although this notation risks confusion with the general definition of addition) is part of the definition of the ordinals, and addition is naturally defined by recursion over this:

- $x + 0 = x$
- $x + Sy = S(x + y)$
- $x + \alpha = \sup_{\gamma<\alpha}(x + \gamma)$ for limit ordinal α

If x and y are finite then $x+y$ under this definition is just the usual sum, however when x and y become infinite, there are differences. In particular, ordinal addition is not commutative. For example,

$$\omega + 1 = \omega + S0 = S(\omega + 0) = S\omega$$

but

$$1 + \omega = \sup_{n<\omega} 1 + n = \omega$$

Multiplication in turn is defined by iterated addition:

- $x \cdot 0 = 0$
- $x \cdot Sy = x \cdot y + x$
- $x \cdot \alpha = \sup_{\gamma<\alpha}(x \cdot \gamma)$ for limit ordinal α

Once again this definition is equivalent to normal multiplication when x and y are finite, but is not commutative:

$$\omega \cdot 2 = \omega \cdot 1 + \omega = \omega + \omega$$

but

$$2 \cdot \omega = \sup_{n<\omega} 2 \cdot n = \omega$$

Both these functions are strongly increasing in the second argument and weakly increasing in the first argument. That is, if $\alpha < \beta$ then

- $\gamma + \alpha < \gamma + \beta$
- $\gamma \cdot \alpha < \gamma \cdot \beta$
- $\alpha + \gamma \leq \beta + \gamma$
- $\alpha \cdot \gamma \leq \beta \cdot \gamma$