deduction theorem holds for classical propositional logic

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In this entry, we prove that the deduction theorem holds for classical propositional logic. For the logic, we use the axiom system found in http://planetmath.org/AxiomSystemForPropositionalLogicthis entry. To prove the theorem, we use the theorem schema $A \to A$ (whose deduction can be found http://planetmath.org/AxiomSystemForPropositionalLogichere).

Proof. Suppose $A_1, \ldots, A_n$ is a deduction of B from $\Delta \cup \{A\}$. We want to find a deduction of $A \to B$ from Δ. There are two main cases to consider:

- If B is an axiom or in $\Delta \cup \{A\}$, then

  $$B, B \to (A \to B), A \to B$$

  is a deduction of $A \to B$ from Δ, where $A \to B$ is obtained by modus ponens applied to B and the axiom $B \to (A \to B)$. So $\Delta \vdash A \to B$.

- If B is obtained from A_i and A_j by modus ponens. Then A_j, say, is $A_i \to B$. We use induction on the length n of the deduction of B. Note that $n \geq 3$. If $n = 3$, then the first two formulas are C and $C \to B$.

  - If C is A, then $C \to B$ is either an axiom or in Δ. So $A \to B$, which is just $C \to B$, is a deduction of $A \to B$ from Δ.
  - If $C \to B$ is A, then C is either an axiom or in Δ, and

    $$\mathcal{E}_0, (A \to A) \to ((A \to C) \to (A \to B)), C \to (A \to C), C, A \to C, A \to B$$

    is a deduction of $A \to B$ from Δ, where $\mathcal{E}_0$ is a deduction of $A \to A$, followed by two axiom instances, followed by C, followed by results of two applications of modus ponens.
  - If both C and $C \to B$ are either axioms are in Δ, then

    $$C, C \to B, B, B \to (A \to B), A \to B$$

    is a deduction of $A \to B$ from Δ.

  Next, assume the deduction $\mathcal{E}$ of B has length $n > 3$. A subsequence of $\mathcal{E}$ is a deduction of $A_i \to B$ from $\Delta \cup \{A\}$. This deduction has length less than n, so by induction,

  $$\Delta \vdash A \to (A_i \to B),$$

and therefore by $(A \to (A_i \to B)) \to ((A \to A_i) \to (A \to B))$, an axiom instance, and modus ponens,

$$\Delta \vdash (A \to A_i) \to (A \to B).$$

Likewise, a subsequence of $\mathcal{E}$ is a deduction of A_i, so by induction, $\Delta \vdash A \to A_i$. Therefore, an application of modus ponens gives us $\Delta \vdash A \to B$.

In both cases, $\Delta \vdash A \to B$ and we are done. □

We record two corollaries:

Corollary 1. *(Proof by Contradiction). If $\Delta, A \vdash \perp$, then $\Delta \vdash \neg A$.*

Proof. From $\Delta, A \vdash \perp$, we have $\Delta \vdash A \to \perp$ by the deduction theorem. Since $\neg A \leftrightarrow \perp$, the result follows. □

Corollary 2. *(Proof by Contrapositive). If $\Delta, A \vdash \neg B$, then $\Delta, B \vdash \neg A$.*

Proof. If $\Delta, A \vdash \neg B$, then $\Delta, A, B \vdash \perp$ by the deduction thereom, and therefore $\Delta, B \vdash \neg A$ by the deduction theorem again. □

Remark The deduction theorem for the classical propositional logic can be used to prove the deduction theorem for the classical first and second order predicate logic.

References

[1] J. W. Robbin, *Mathematical Logic, A First Course*, Dover Publication (2006)