permutation model

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A permutation model is a model of the axioms of set theory in which there is a non trivial automorphism of the set theoretic universe. Such models are used to show the consistency of the negation of the Axiom of Choice (AC).

A typical construction of a permutation model is done here. By ZF^- we denote the axioms of ZF minus the axiom of foundation. In particular we allow sets a such that $a = \{a\}$ which we will call atoms. Let A be an infinite set of atoms.

Define $V_\alpha(A)$ by induction on α as follows:

$$\begin{aligned} V_0(A) &= A \\ V_{\alpha+1}(A) &= \mathcal{P}(V_\alpha) \\ V_\alpha(A) &= \bigcup_{\gamma<\alpha} V_\gamma(A) \text{ for } \alpha \text{ limit} \end{aligned}$$

Finally define $V = \bigcup_{\alpha \in \mathrm{ON}} V_\alpha(A)$. Then we have

$$A = V_0(A) \subseteq V_1(A) \subseteq \cdots \subseteq V_\alpha(A) \cdots \subseteq V$$

For any $x \in V$ we can assign a rank,

$$\operatorname{rank}(x) = \text{ least } \alpha[x \in V_{\alpha+1}(A)]$$

Let G be the group of permutations of A. For $\pi \in G$ we extend π to a permutation of V by induction on $\in$ by defining

$$\pi(x) = \{\pi(y) : y \in x\}$$

and letting $\pi(\emptyset) = \emptyset$. Then G permutes V and fixes the well founded sets $WF \subseteq V$.

Lemma. *For all $x, y \in V$ and any $\pi \in G$.*

$$x \in y \iff \pi(x) \in \pi(y)$$

That is, π is an $\in$-automorphism of V. From this we can prove that $\pi(\{X, Y\}) = \{\pi(X), \pi(Y)\}$ and so

$$\begin{aligned} \pi((X, Y)) &= (\pi(X), \pi(Y)) \\ \pi((X, Y, Z)) &= (\pi(X), \pi(Y), \pi(Z)) \end{aligned}$$

Also by induction on α it is easy to show that

$$\text{rank}(x) = \text{rank}(\pi(x))$$

for all $x \in V$.

Let $a_1, \cdots, a_n \in A$ and define

$$[a_1, \cdots, a_n] = \{\pi \in G : \pi(a_i) = a_i, \text{ for } i = 1, \cdots, n\}$$

Call a set $X \in V$ symmetric if there exists $a_1, \cdots, a_n \in A$ such that $\pi(X) = X$ for all $\pi \in [a_1, \cdots, a_n]$. Define the class $HS \subseteq V$ of hereditarily symmetric sets

$$HS = \{x \in V : x \text{ is symmetric and } x \subseteq HS\}$$

Call a class N transitive if

$$\forall x \in N[x \subseteq N]$$

and call N almost universal if (for sets S)

$$\forall S \subseteq N[\exists Y \in N(S \subseteq Y)]$$

HS is transitive and almost universal.

To show that a class $N \models ZF^-$ is straightforward for most axioms of ZF^- except for the axiom of Comprehension. To show N is a model of Comprehension it suffices to show that N is closed under **Gödel Operations**:

$$\begin{aligned}
G_1(X, Y) &= \{X, Y\} \\
G_2(X, Y) &= X \setminus Y \\
G_3(X, Y) &= X \times Y \\
G_4(X) &= \text{dom}(X) \\
G_5(X) &= \in \cap X^2 \\
G_6(X) &= \{(a, b, c) : (b, c, a) \in X\} \\
G_7(X) &= \{(a, b, c) : (c, b, a) \in X\} \\
G_8(X) &= \{(a, b, c) : (a, c, b) \in X\}
\end{aligned}$$

Theorem. *(ZF) If N is transitive, almost universal and closed under Gödel Operations, then* $N \models ZF$.

HS is closed under Gödel operations and so $HS \models ZF^-$. The class HS is a permutation model. The set of atoms $A \in HS$ and furthermore:

Lemma. *Let $f : \omega \to A$ be a one to one function. Then $f \notin HS$ and so A cannot be well ordered in HS.*

Which proves the theorem:

Theorem. *$HS \models ZF^- + \neg AC$.*

which completes the proof that $\mathrm{Con}(ZF^-) \implies \mathrm{Con}(ZF^- + \neg AC)$. In particular we have that $ZF^- \nvdash AC$.