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modal logic S4

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Defines S4 Defines 4 The modal logic S4 is the smallest normal modal logic containing the following schemas:

- (T) $\Box A \rightarrow A$, and
- $(4) \square A \rightarrow \square \square A$.

In http://planetmath.org/ModalLogicTthis entry, we show that T is valid in a frame iff the frame is reflexive.

Proposition 1. 4 is valid in a frame \mathcal{F} iff \mathcal{F} is transitive.

Proof. First, suppose \mathcal{F} is a frame validating 4, with wRu and uRt. Let M be a model with $V(p) = \{v \mid wRv\}$, where p a propositional variable. So $\models_w \Box p$. By assumption, we have $\models_w \Box p \to \Box \Box p$. Then $\models_w \Box \Box p$. This means $\models_v \Box p$ for all v such that wRv. Since wRu, $\models_u \Box p$, which means $\models_s p$ for all s such that uRs. Since uRt, we have $\models_t p$, or $t \in V(p)$, or wRt. Hence R is transitive.

Conversely, let \mathcal{F} be a transitive frame, M a model based on \mathcal{F} , and w any world in M. Suppose $\models_w \Box A$. We want to show $\models_w \Box \Box A$, or for all u with wRu, we have $\models_u \Box A$, or for all u with wRu and all t with uRt, we have $\models_t A$. If wRu and uRt, wRt since R is transitive. Then $\models_t A$ by assumption. Therefore, $\models_w \Box A \to \Box \Box A$.

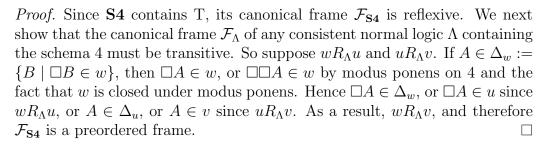
As a result,

Proposition 2. S4 is sound in the class of preordered frames.

Proof. Since any theorem in S4 is deducible from a finite sequence consisting of tautologies, which are valid in any frame, instances of T, which are valid in reflexive frames, instances of 4, which are valid in transitive frames by the proposition above, and applications of modus ponens and necessitation, both of which preserve validity in any frame, whence the result.

In addition, using the canonical model of S4, which is preordered, we have

Proposition 3. S4 is complete in the class of serial frames.



By a proper translation, one can map intuitionistic propositional logic PL_i into S4, so that a wff of PL_i is a theorem iff its translate is a theorem of S4.