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set theory

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Set theory is special among mathematical theories, in two ways: It plays a central role in putting mathematics on a reliable axiomatic foundation, and it provides the basic language and apparatus in which most of mathematics is expressed.

1 Axiomatic set theory

I will informally list the undefined notions, the axioms, and two of the "schemes" of set theory, along the lines of Bourbaki's account. The axioms are closer to the von Neumann-Bernays-Gödel model than to the equivalent ZFC model. (But some of the axioms are identical to some in ZFC; see the entry `http://planetmath.org/ZermeloFraenkelAxioms`ZermeloFraenkelAxioms.) The intention here is just to give an idea of the level and scope of these fundamental things.

There are three undefined notions:

1. the relation of equality of two sets

2. the relation of membership of one set in another ($x \in y$)

3. the notion of an ordered pair, which is a set comprised from two other sets, in a specific order.

Most of the eight schemes belong more properly to logic than to set theory, but they, or something on the same level, are needed in the work of formalizing any theory that uses the notion of equality, or uses quantifiers such as $\exists$. Because of their formal nature, let me just (informally) state two of the schemes:

S6. If A and B are sets, and $A = B$, then anything true of A is true of B, and conversely.

S7. If two properties $F(x)$ and $G(x)$ of a set x are equivalent, then the "generic" set having the property F, is the same as the generic set having the property G.

(The notion of a generic set having a given property, is formalized with the help of the Hilbert τ symbol; this is one way, but not the only way, to incorporate what is called the Axiom of Choice.)

Finally come the five axioms in this axiomatization of set theory. (Some are identical to axioms in ZFC, q.v.)

A1. Two sets A and B are equal iff they have the same elements, i.e. iff the relation $x \in A$ implies $x \in B$ and vice versa.

A2. For any two sets A and B, there is a set C such that the $x \in C$ is equivalent to $x = A$ or $x = B$.

A3. Two ordered pairs (A, B) and (C, D) are equal iff $A = C$ and $B = D$.

A4. For any set A, there exists a set B such that $x \in B$ is equivalent to $x \subset A$; in other words, there is a set of all subsets of A, for any given set A.

A5. There exists an infinite set.

The word "infinite" is defined in terms of Axioms A1-A4. But to formulate the definition, one must first build up some definitions and results about functions and ordered sets, which we haven't done here.

2 Product sets, relations, functions, etc.

Moving away from foundations and toward applications, all the more complex structures and relations of set theory are built up out of the three undefined notions. (See the entry "Set".) For instance, the relation $A \subset B$ between two sets, means simply "if $x \in A$ then $x \in B$".

Using the notion of ordered pair, we soon get the very important structure called the product $A \times B$ of two sets A and B. Next, we can get such things as equivalence relations and order relations on a set A, for they are subsets of $A \times A$. And we get the critical notion of a function $A \to B$, as a subset of $A \times B$. Using functions, we get such things as the product $\prod_{i \in I} A_i$ of a family of sets. ("Family" is a variation of the notion of function.)

To be strictly formal, we should distinguish between a function and the graph of that function, and between a relation and its graph, but the distinction is rarely necessary in practice.

3 Some structures defined in terms of sets

The natural numbers provide the first example. Peano, Zermelo and Fraenkel, and others have given axiom-lists for the set $\mathbb{N}$, with its addition, multiplication, and order relation; but nowadays the custom is to define even the natural numbers in terms of sets. In more detail, a natural number is the order-type of a finite well-ordered set. The relation $m \leq n$ between $m, n \in \mathbb{N}$ is defined with the aid of a certain theorem which says, roughly, that for any two well-ordered sets, one is a segment of the other. The sum or product of two natural numbers is defined as the cardinal of the sum or product, respectively, of two sets. (For an extension of this idea, see surreal numbers.)

(The term "cardinal" takes some work to define. The "type" of an ordered set, or any other kind of structure, is the "generic" structure of that kind, which is defined using τ.)

Groups provide another simple example of a structure defined in terms of sets and ordered pairs. A group is a pair (G, f) in which G is just a set, and f is a mapping $G \times G \to G$ satisfying certain axioms; the axioms (associativity etc.) can all be spelled out in terms of sets and ordered pairs, although in practice one uses algebraic notation to do it. When we speak of (e.g.) "the" group S_3 of permutations of a 3-element set, we mean the "type" of such a group.

Topological spaces provide another example of how mathematical structures can be defined in terms of, ultimately, the sets and ordered pairs in set theory. A topological space is a pair (S, U), where the set S is arbitrary, but U has these properties:

– any element of U is a subset of S

– the union of any family (or set) of elements of U is also an element of U

– the intersection of any *finite* family of elements of U is an element of U.

Many special kinds of topological spaces are defined by enlarging this list of restrictions on U.

Finally, many kinds of structure are based on more than one set. E.g. a left module is a commutative group M together with a ring R, plus a mapping $R \times M \to M$ which satisfies a specific set of restrictions.

4 Categories, homological algebra

Although set theory provides some of the language and apparatus used in mathematics generally, that language and apparatus have expanded over time, and now include what are called "categories" and "functors". A category is not a set, and a functor is not a mapping, despite similarities in both cases. A category comprises all the structured sets of the same kind, e.g. the groups, and contains also a definition of the notion of a morphism from one such structured set to another of the same kind. A functor is similar to a morphism but compares one category to another, not one structured set to another. The classic examples are certain functors from the category of topological spaces to the category of groups.

"Homological algebra" is concerned with sequences of morphisms within a category, plus functors from one category to another. One of its aims is to get structure theories for specific categories; the homology of groups and the cohomology of Lie algebras are examples. For more details on the categories and functors of homological algebra, I recommend a search for "Eilenberg-Steenrod axioms".