substitutability

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In any logical system, the way to obtain (well-formed) formulas from existing ones is by attaching logical connectives to existing ones. For example, in classical propositional logic, if φ and ψ are formulas, the following

$$\varphi \vee \psi, \quad \neg\varphi, \quad \psi \wedge \varphi$$

are formed by attaching logical connectives $\vee, \neg$, and $\wedge$ appropriately to φ and ψ.

Another convenient device is substitution:

> replace all occurrences of some symbol x in a formula φ by an expression ψ.

We denote the resulting expression by

$$\varphi[\psi/x].$$

For example, in classical propositional logic, if φ is $p \wedge (\neg r \vee p)$, then $\varphi[(p \vee q)/p]$ is

$$(p \vee q) \wedge (\neg r \vee (p \vee q))$$

On the other hand, the expressions

- $\varphi[\neg/p]$, which is $\neg \wedge (\neg r \vee \neg)$, and
- $\varphi[q/\wedge]$, which is $p\ q(\neg r \vee q)$,

are not formulas. Thus, one must be careful when performing substitutions on formulas lest the resulting expressions are ill-formed. In other words, conditions must be placed on x and ψ in $\varphi[\psi/x]$ in order that $\varphi[\psi/x]$ is a (well-formed) formula. These conditions are called the *substitutability conditions*. In this entry, we will concentrate on substitutability conditions on predicate logic. Details on substitions in propositional logic can be found in `http://planetmath.org/SubstitutionsInLogic`here.

Substitution in First-Order Logic

Substitution works pretty much the same way for first-order logic as in propositional logic. However, the substitutability conditions are more subtle. Take a look at the following example:

$$\exists x\ (x = 1)$$

If we replace x by 0, we end up with

$$\exists 0\ (0 = 1),$$

which is non-sensical (not a wff). This is because x occurs in the formula as a bound variable. (one reason why we distinguish the variables occurring in first order formulas into two types: free and bound).

We now formalize the notion of substitution in first-order logic. There are two parts: substitution for terms, and substitution for formulas.

Definition. For any term t, any symbol x, and any expression s, define $t[s/x]$ inductively, as follows:

1. if t is an individual variable or a constant symbol, then $t[s/x]$ is s if t is x, and $t[s/x]$ is t otherwise;

2. if t is $f(t_1, \ldots, t_n)$, where f is an n-ary function symbol, and each t_i is a term, then $t[s/x]$ is $f(t_1[s/x], \ldots, t_n[s/x])$.

For example, if t is $x + y$, then $t[(x - y)/x]$ is $(x - y) + y$.

It is easy to see that s is a term and x an individual variable, then $t[s/x]$ is a term. In addition, by induction, one can easily show that if the formula $s_1 = s_2$ is true, so is the formula $t[s_1/x] = t[s_2/x]$.

Next, we define substitution for formulas. In light of the last example at the beginning of this section, we need to be a little careful.

Definition. Let φ be a formula, x a symbol, and s an expression. The expression $\varphi[s/x]$ is again define inductively:

1. if φ is $t_1 = t_2$, then $\varphi[s/x]$ is $t_1[s/x] = t_2[s/x]$;

2. if φ is $R(t_1, \ldots, t_n)$, then $\varphi[s/x]$ is $R(t_1[s/x], \ldots, t_n[s/x])$;

3. if φ is $\neg\psi$, then $\varphi[s/x]$ is $\neg(\psi[s/x])$;

4. if φ is $\psi \vee \sigma$, then $\varphi[s/x]$ is $\psi[s/x] \vee \sigma[s/x]$;

5. if φ is $\exists y\psi$, then $\varphi[s/x]$ is $\exists y(\psi[s/x])$ if $x \neq y$, and $\varphi[s/x]$ is φ otherwise.

Again, substitutions involving logical connectives $\to$, $\wedge$, and the universal quantifier $\forall$ can be derived from the rules given above.

For example, if φ is $\exists x(x = y \vee y = z)$, then $\varphi[t/y]$ is $\exists x(x = t \vee t = z)$, whereas $\varphi[t/x]$ is just φ.

Given that φ is a formula, it is easy to see that if x is an individual variable, and s is a term, then $\varphi[s/x]$ is a formula.

In addition, it is easy to see that sentences are not affected by substitutions: if φ is a sentence, then $\varphi[s/x]$ is just φ. In other words, sentences can not be changed into formulas with free variables.

Conversely, can a formula with free variables be changed into a sentence by substitution? Certainly. For example, if φ is

$$\exists x(y < x),$$

then $\varphi[x/y]$ is

$$\exists x(x < x).$$

Although syntactically correct, this is undesirable in many situations, particularly when we are interested in the interpretations of these formulas. In the example above, we have changed $\exists x(y < x)$, which many very well be true in many interpretations, into $\exists x(x < x)$, something with a fixed meaning (and always false if $<$ is interpreted as the usual less than relation).

The problem with the situation described in the last paragraph arises because a free variable in t becomes bound in $\varphi[t/x]$. To eliminate this undesirable situation, we define the notion of "free for":

Definition. Let x be an individual variable, t a term, and φ a formula. We define the relation t *is free for, or substitutable for* x *in* φ, inductively, as follows:

1. φ is an atomic formula;
2. φ is $\neg\psi$, and t is free for x in ψ;
3. φ is $\psi \vee \sigma$, and t is free for x in ψ and in σ;
4. φ is $\exists y\psi$, and either
   - $x \notin \mathrm{FV}(\varphi)$ (x does not occur free in φ), or
   - y does not occur in t, and t is free for x in ψ.

In words, t is free for x in φ iff whenever z is a variable in t, no literal subformula of φ of the form $\exists z\psi$ contains an occurrence of x which is free in φ.

For example, $f(x, y)$ is free for x in the following formulas:

$$P(x,y), \qquad P(x)\vee\neg Q(z), \qquad \neg\exists x\,\neg R(x,y), \qquad \text{and} \qquad \neg S(y)\vee\exists y T(y,z),$$

but not in the following formulas:

$$\exists y P(x,y), \qquad Q(x)\vee\exists z\exists y R(x,y), \qquad \text{and} \qquad S(y)\to\forall y(T(y,y)\wedge\neg Q(x)).$$

Given any formula φ, we again write $\varphi(x)$ to mean that variable x occurs in φ. A *substitution instance* of $\varphi(x)$ is just $\varphi[t/x]$, or $\varphi(t)$ for short. Furthermore, if t is free for x in φ, then $\varphi(t)$ is called a *free substitution instance* of $\varphi(x)$.

It is easy, by induction, to show that if terms t_1 and t_2 are free for x in φ, and that the formula $t_1 = t_2$ is true, the substitution instances $\varphi(t_1)$ and $\varphi(t_2)$ are logically equivalent, as intended.