cofinality

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Defines	cofinal
Defines	regular cardinal
Defines	singular cardinal
Defines	regular
Defines	singular

Definitions

Let $(P, \leq)$ be a poset. A subset $A \subseteq P$ is said to be *cofinal* in P if for every $x \in P$ there is a $y \in A$ such that $x \leq y$. A function $f : X \to P$ is said to be *cofinal* if $f(X)$ is cofinal in P. The least cardinality of a cofinal set of P is called the *cofinality* of P. Equivalently, the cofinality of P is the least `http://planetmath.org/node/2787`ordinal α such that there is a cofinal function $f : \alpha \to P$. The cofinality of P is written $\mathrm{cf}(P)$, or $\mathrm{cof}(P)$.

Cofinality of totally ordered sets

If $(T, \leq)$ is a totally ordered set, then it must contain a well-ordered cofinal subset which is order-isomorphic to $\mathrm{cf}(T)$. Or, put another way, there is a cofinal function $f : \mathrm{cf}(T) \to T$ with the property that $f(x) < f(y)$ whenever $x < y$.

For any ordinal β we must have $\mathrm{cf}(\beta) \leq \beta$, because the identity map on β is cofinal. In particular, this is true for cardinals, so any cardinal κ either satisfies $\mathrm{cf}(\kappa) = \kappa$, in which case it is said to be *regular*, or it satisfies $\mathrm{cf}(\kappa) < \kappa$, in which case it is said to be *singular*.

The cofinality of any totally ordered set is necessarily a regular cardinal.

Cofinality of cardinals

0 and 1 are regular cardinals. All other finite cardinals have cofinality 1 and are therefore singular.

It is easy to see that $\mathrm{cf}(\aleph_0) = \aleph_0$, so $\aleph_0$ is regular.

$\aleph_1$ is regular, because the union of countably many countable sets is countable. More generally, all infinite successor cardinals are regular.

The smallest infinite singular cardinal is $\aleph_\omega$. In fact, the function $f : \omega \to \aleph_\omega$ given by $f(n) = \omega_n$ is cofinal, so $\mathrm{cf}(\aleph_\omega) = \aleph_0$. More generally, for any nonzero limit ordinal δ, the function $f : \delta \to \aleph_\delta$ given by $f(\alpha) = \omega_\alpha$ is cofinal, and this can be used to show that $\mathrm{cf}(\aleph_\delta) = \mathrm{cf}(\delta)$.

Let κ be an infinite cardinal. It can be shown that $\mathrm{cf}(\kappa)$ is the least cardinal μ such that κ is the sum of μ cardinals each of which is less than κ. This fact together with König's theorem tells us that $\kappa < \kappa^{\mathrm{cf}(\kappa)}$. Replacing κ by 2^κ in this inequality we can further deduce that $\kappa < \mathrm{cf}(2^\kappa)$. In particular,

$\mathrm{cf}(2^{\aleph_0}) > \aleph_0$, from which it follows that $2^{\aleph_0} \neq \aleph_\omega$ (this being the smallest uncountable aleph which is provably not the cardinality of the continuum).