functional completeness

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Recall that in classical propositional logic, well-formed formulas (wffs) can be built up (recursively) from propositional variables via logical connectives. There are several choices for the logical connectives used:

- $F_1 = \{\neg, \vee\}$,
- $F_2 = \{\neg, \wedge\}$,
- $F_3 = \{\neg, \rightarrow\}$,
- $F_4 = \{\neg, \vee, \wedge\}$,
- $F_5 = \{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$.

For a given set V of (propositional) variables, and a set F of logical connectives, denote $\overline{V}(F)$ the set of all wffs built from V with respect to F. From the choices above, we see that $\overline{V}(F_i) \subset \overline{V}(F_5)$ for all $i < 5$, and $\overline{V}(F_j) \subset \overline{V}(F_4)$ for all $j < 3$.

However, we know that, intuitively, some of the connectives are "redundant" in that they can be "defined" using existing connectives. For example, the connective $\leftrightarrow$ can be defined in terms of $\rightarrow$ and $\vee$:

$$p \leftrightarrow q := (p \rightarrow q) \vee (q \rightarrow p),$$

and $\rightarrow$ can in turn be defined in terms of $\vee$ and $\neg$:

$$p \rightarrow q := (\neg p) \vee q,$$

etc... This means that, although $\overline{V}(F_5)$ is a much larger set than, say, $\overline{V}(F_1)$, every extra wff in $\overline{V}(F_5)$ is in some way *equivalent* to an wff in $\overline{V}(F_1)$. This equivalence is the familiar semantic equivalence. If fact, we can show that $\neg$ and $\vee$ are all we need: "any" logical connective can be "defined" in terms of them, not just the ones mentioned above. This is the notion of *truth functional completeness*, or *functional completeness* for short. To make this precise, we have the following:

Definition A set F of logical connectives is said to be *truth functionally complete*, or *functionally complete* if, given logical connective ϕ, every wff in $\overline{V}(F \cup \{\phi\})$ is semantically equivalent to a wff in $\overline{V}(F)$, considered as a subset of $\overline{V}(F \cup \{\phi\})$.

It is clear that if F is functionally complete, so is any of its superset. Also, given a set F of logical connectives, if there is a functionally complete set G

of logical connectives such that every wff in $\overline{V}(G)$ is semantically equivalent to a wff in $\overline{V}(F)$, then F is functionally complete.

For example, it can be shown that F_1 above is functionally complete, and as an easy corollary, so is each of the rest of F_i above.

References

[1] D. van Dalen: *Logic and Structure*, Springer, 4th Ed., Berlin (2008).

[2] H. Enderton: *A Mathematical Introduction to Logic*, Academic Press, San Diego (1972).