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axiom

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In a nutshell, the logico-deductive method is a system of inference where conclusions (new knowledge) follow from premises (old knowledge) through the application of sound arguments (syllogisms, rules of inference). Tautologies excluded, nothing can be deduced if nothing is assumed. Axioms and postulates are the basic assumptions underlying a given body of deductive knowledge. They are accepted without demonstration. All other assertions (theorems, if we are talking about mathematics) must be proven with the aid of the basic assumptions.

The logico-deductive method was developed by the ancient Greeks, and has become the core principle of modern mathematics. However, the interpretation of mathematical knowledge has changed from ancient times to the modern, and consequently the terms *axiom* and *postulate* hold a slightly different meaning for the present day mathematician, then they did for Aristotle and Euclid.

The ancient Greeks considered geometry as just one of several sciences, and held the theorems of geometry on par with scientific facts. As such, they developed and used the logico-deductive method as a means of avoiding error, and for structuring and communicating knowledge. Aristotle's `http://classics.mit.edu/Aristotle/posterior.1.i.html`Posterior Analytics is a definitive exposition of the classical view.

"Axiom", in classical terminology, referred to a self-evident assumption common to many branches of science. A good example would be the assertion that

> *When an equal amount is taken from equals, an equal amount results.*

At the foundation of the various sciences lay certain basic hypotheses that had to be accepted without proof. Such a hypothesis was termed a *postulate*. The postulates of each science were different. Their validity had to be established by means of real-world experience. Indeed, Aristotle warns that the content of a science cannot be successfully communicated, if the learner is in doubt about the truth of the postulates.

The classical approach is well illustrated by Euclid's elements, where we see a list of axioms (very basic, self-evident assertions) and postulates (common-sensical geometric facts drawn from our experience).

A1 Things which are equal to the same thing are also equal to one another.

A2 If equals be added to equals, the wholes are equal.

A3 If equals be subtracted from equals, the remainders are equal.

A4 Things which coincide with one another are equal to one another.

A5 The whole is greater than the part.

P1 It is possible to draw a straight line from any point to any other point.

P2 It is possible to produce a finite straight line continuously in a straight line.

P3 It is possible to describe a circle with any centre and distance.

P4 It is true that all right angles are equal to one another.

P5 It is true that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The classical view point is explored in more detail `http://www.mathgym.com.au/history/pythago`

A great lesson learned by mathematics in the last 150 years is that it is useful to strip the meaning away from the mathematical assertions (axioms, postulates, propositions, theorems) and definitions. This abstraction, one might even say formalization, makes mathematical knowledge more general, capable of multiple different meanings, and therefore useful in multiple contexts.

In structuralist mathematics we go even further, and develop theories and axioms (like field theory, group theory, topology, vector spaces) without *any* particular application in mind. The distinction between an "axiom" and a "postulate" disappears. The postulates of Euclid are profitably motivated by saying that they lead to a great wealth of geometric facts. The truth of these complicated facts rests on the acceptance of the basic hypotheses. However by throwing out postulate 5, we get theories that have meaning in wider contexts, hyperbolic geometry for example. We must simply be prepared to use labels like "line" and "parallel" with greater flexibility. The development of hyperbolic geometry taught mathematicians that postulates should be regarded as purely formal statements, and not as facts based on experience.

When mathematicians employ the axioms of a field, the intentions are even more abstract. The propositions of field theory do not concern any one particular application; the mathematician now works in complete abstraction. There are many examples of fields; field theory gives correct knowledge in all contexts.

It is not correct to say that the axioms of field theory are "propositions that are regarded as true without proof." Rather, the Field Axioms are a set of constraints. If any given system of addition and multiplication tolerates these constraints, then one is in a position to instantly know a great deal of extra information about this system. There is a lot of bang for the formalist buck.

Modern mathematics formalizes its foundations to such an extent that mathematical theories can be regarded as mathematical objects, and logic itself can be regarded as a branch of mathematics. Frege, Russell, Poincaré, Hilbert, and Gödel are some of the key figures in this development.

In the modern understanding, a set of axioms is any collection of formally stated assertions from which other formally stated assertions follow by the application of certain well-defined rules. In this view, logic becomes just another formal system. A set of axioms should be consistent; it should be impossible to derive a contradiction from the axiom. A set of axioms should also be non-redundant; an assertion that can be deduced from other axioms need not be regarded as an axiom.

It was the early hope of modern logicians that various branches of mathematics, perhaps all of mathematics, could be derived from a consistent collection of basic axioms. An early success of the formalist program was Hilbert's formalization of Euclidean geometry, and the related demonstration of the consistency of those axioms.

In a wider context, there was an attempt to base all of mathematics on Cantor's set theory. Here the emergence of Russell's paradox, and similar antinomies of naive set theory raised the possibility that any such system could turn out to be inconsistent.

The formalist project suffered a decisive setback, when in 1931 Gödel showed that it is possible, for any sufficiently large set of axioms (Peano's axioms, for example) to construct a statement whose truth is independent of that set of axioms. As a corollary, Gödel proved that the consistency of a theory like Peano arithmetic is an unprovable assertion within the scope of that theory.

It is reasonable to believe in the consistency of Peano arithmetic because it is satisfied by the system of natural numbers, an infinite but intuitively accessible formal system. However, at this date we have no way of demonstrating the consistency of modern set theory (Zermelo-Frankel axioms). The axiom of choice, a key hypothesis of this theory, remains a very controversial assumption. Furthermore, using techniques of forcing (Cohen) one can show that the continuum hypothesis (Cantor) is independent of the Zermelo-Frankel axioms. Thus, even this very general set of axioms cannot be regarded as the definitive foundation for mathematics.