Ackermann function is not primitive recursive

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In this entry, we show that the Ackermann function $A(x,y)$, given by

$$A(0,y)=y{+}1, \qquad A(x{+}1,0)=A(x,1), \qquad A(x{+}1,y{+}1)=A(x,A(x{+}1,y))$$

is not primitive recursive. We will utilize the properties of A listed in `http://planetmath.org/PropertiesOfAckermannFunction`this entry.

The key to showing that A is not primitive recursive, is to find a properties shared by all primitive recursive functions, but not by A. One such property is in showing that A in some way "grows" faster than any primitive recursive function. This is formalized by the notion of "majorization", which is explained `http://planetmath.org/SuperexponentiationIsNotElementary`here.

Proposition 1. *Let $\mathcal{A}$ be the set of all functions majorized by A. Then $\mathcal{PR} \subseteq \mathcal{A}$.*

Proof. We break this up into three parts: show all initial functions are in $\mathcal{A}$, show $\mathcal{A}$ is closed under functional composition, and show $\mathcal{A}$ is closed under primitive recursion. The proof is completed by realizing that $\mathcal{PR}$ is the smallest set satisfying the three conditions.

In the proofs below, $\boldsymbol{x}$ denotes some tuple of non-negative integers $(x_1,\ldots,x_n)$ for some n, and $x=\max\{x_1,\ldots,x_n\}$. Likewise for $\boldsymbol{y}$ and y.

1. We show that the zero function, the successor function, and the projection functions are in $\mathcal{A}$.
   - $z(n)=0<n+1=A(0,n)$, so $z\in\mathcal{A}$.
   - $s(n)=n+1<n+2=A(1,n)$, so $s\in\mathcal{A}$.
   - $p^k_m(x_1,\ldots,x_k)=x_m\le x<x+1=A(0,x)$, so $p^k_m\in\mathcal{A}$.
2. Next, suppose $g_1,\ldots,g_m$ are k-ary, and h is m-ary, and that each g_i, and h are in $\mathcal{A}$. This means that $g_i(\boldsymbol{x})<A(r_i,x)$, and $h(\boldsymbol{y})<A(s,y)$. Let

   $$f=h(g_1,\ldots,g_m), \quad \text{and} \quad g_j(\boldsymbol{x})=\max\{g_i(\boldsymbol{x})\mid i=1,\ldots,m\}.$$

   Then $f(\boldsymbol{x})<A(s,g_j(\boldsymbol{x}))<A(s,A(r_j,x))<A(s+r_j+2,x)$, showing that $f\in\mathcal{A}$.
3. Finally, suppose g is k-ary and h is $(k+2)$-ary, and that $g,h\in\mathcal{A}$. This means that $g(\boldsymbol{x})<A(r,x)$ and $h(\boldsymbol{y})<A(s,y)$. We want to show that f, defined by primitive recursion via functions g and h, is in $\mathcal{A}$.

We first prove the following claim:

$$f(\boldsymbol{x}, n) < A(q, n+x), \quad \text{for some } q \text{ not depending on } x \text{ and } n.$$

Pick $q = 1 + \max\{r, s\}$, and induct on n. First, $f(\boldsymbol{x}, 0) = g(\boldsymbol{x}) < A(r, x) < A(q, x)$. Next, suppose $f(\boldsymbol{x}, n) < A(q, n+x)$. Then $f(\boldsymbol{x}, n+1) = h(\boldsymbol{x}, n, f(\boldsymbol{x}, n)) < A(s, z)$, where $z = \max\{x, n, f(\boldsymbol{x}, n)\}$. By the induction hypothesis, together with the fact that $\max\{x, n\} \leq n+x < A(q, n+x)$, we see that $z < A(q, n+x)$. Thus, $f(\boldsymbol{x}, n+1) < A(s, z) < A(s, A(q, n+x)) \leq A(q-1, A(q, n+x)) = A(q, n+1+x)$, proving the claim.

To finish the proof, let $z = \max\{x, y\}$. Then, by the claim, $f(\boldsymbol{x}, y) < A(q, x+y) \leq A(q, 2z) < A(q, 2z+3) = A(q, A(2, z)) = A(q+4, z)$, showing that $f \in \mathcal{A}$.

Since $\mathcal{PR}$ is by definition the smallest set containing the initial functions, and closed under composition and primitive recursion, $\mathcal{PR} \subseteq \mathcal{A}$. $\square$

As a corollary, we have

Corollary 1. *The Ackermann function A is not primitive recursive.*

Proof. Otherwise, $A \in \mathcal{A}$, which is impossible. $\square$