bisimulation

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Given two labelled state transition systems (LTS) $M = (S_1, \Sigma, \to_1)$, $N = (S_2, \Sigma, \to_2)$, a binary relation $\approx \subseteq S_1 \times S_2$ is called a *simulation* if whenever $p \approx q$ and $p \xrightarrow{\alpha}_1 p'$, then there is a q' such that $p' \approx q'$ and $q \xrightarrow{\alpha}_2 q'$. An LSTS $M = (S_1, \Sigma, \to_1)$ is *similar* to an LTS $N = (S_2, \Sigma, \to_2)$ if there is a simulation $\approx S_1 \times S_2$.

For example, any LTS is similar to itself, as the identity relation on the set of states is a simulation. In addition,

1. if M is similar to N and N is similar to P, then M is similar to P:

   Proof. Let $M = (S_1, \Sigma, \to_1)$, $N = (S_2, \Sigma, \to_2)$, and $P = (S_3, \Sigma, \to_3)$ be LSTS, and suppose $p \approx_1 \circ \approx_2 q$ with $p \xrightarrow{\alpha}_1 p'$, where $\approx_1$ and $\approx_2$ are simulations. Then there is an r such that $p \approx_1 r$ and $r \approx_2 q$. Since $\approx_1$ is a simulation, there is an r' such that $r \xrightarrow{\alpha}_2 r'$. But then since $\approx_2$ is a simulation, there is a q' such that $q \xrightarrow{\alpha}_3 q'$. As a result, $\approx_1 \circ \approx_2$ is a simulation. □

2. a union of simulations is a simulation.

   Proof. Let $\approx$ be the union of simulations $\approx_i$, where $i \in I$, and suppose $p \approx q$, with $p \xrightarrow{\alpha} p'$. Then $p \approx_i q$ for some i. Since $\approx_i$ is a simulation, there is a state q' such that $p' \approx_i q'$ and $q \xrightarrow{\alpha} q'$. So $p' \approx q'$ and therefore $\approx$ is a simulation. □

A binary relation $\approx$ between S_1 and S_2 is a *bisimulation* if both $\approx$ and its converse $\approx^{-1}$ are simulations. A bisimulation is also called a *strong bisimulation*, in contrast with *weak bisimulation*. When there is a bisimulation between the state sets of two LTS, we say that the two systems are *bisimilar*, or *strongly bisimilar*. By abuse of notation, we write $M \approx N$ to denote that M is bisimilar to N.

An equivalent formulation of bisimulation is given by extending the binary relation on the sets to a binary relation on the corresponding power sets. Here's how: let $\approx \subseteq S_1 \times S_2$. For any $A \subseteq S_1$ and $B \subseteq S_2$, define

$$C(A) := \{b \in S_2 \mid a \approx b \text{ for some } a \in A\} \text{ and } C(B) := \{a \in S_1 \mid a \approx b \text{ for some } b \in B\}.$$

Then the binary relation $\approx$ can be extended to a binary relation from $P(S_1)$ to $P(S_2)$, still denoted by $\approx$, as

$$A \approx B \quad \text{iff} \quad A \subseteq C(B) \text{ and } B \subseteq C(A),$$

for any $A \subseteq S_1$ and $B \subseteq S_2$. In other words, $A \approx B$ iff for any $a \in A$, there is a $b \in B$ such that $a \approx b$ and for any $b \in B$, there is an $a \in A$ such that $a \approx b$. Now, for any $p \in S_1$ and $\alpha \in \Sigma$, let

$$\delta_1(p, a) = \{q \in S_1 \mid p \xrightarrow{\alpha}_1 q\}.$$

We can similar define function $\delta_2 : S_2 \times \Sigma \to P(S_2)$. So a binary relation $\approx$ between S_1 and S_2 is a bisimilation iff for any $(p, q) \in S_1 \times S_2$ such that $p \approx q$, we have $\delta_1(p, a) \approx \delta_2(q, a)$ for any $a \in \Sigma$.

Let $M = (S_1, \Sigma, \rightarrow_1)$, $N = (S_2, \Sigma, \rightarrow_2)$, and $P = (S_3, \Sigma, \rightarrow_3)$ be LTS. The following are some basic properties of bisimulation:

1. The identity relation $=$ is a bisimilation on any LTS, since $=$ is a simulation and $=^{-1}$ is just $=$.

2. If M is bisimilar to N via $\approx$, then N is bisimilar to M via $\approx^{-1}$, since both $\approx^{-1}$ and $(\approx^{-1})^{-1} =\approx$ are simulations.

3. If $M \approx_1$ and $N \approx_2 P$, then $M \approx_1 \circ \approx_2 P$, since $\approx_1 \circ \approx_2$ and $(\approx_1 \circ \approx_2)^{-1} =\approx_2^{1-} \circ \approx_1^{-1}$ are both simulations according to the argument above.

4. A union of bisimilations is a bisimilation.

   Proof. Let $\approx$ be the union of bisimulations $\approx_i$, where $i \in I$. Then $\approx$ is a simulation by the argument above. Now, suppose $p \approx^{-1} q$ and $p \xrightarrow{\alpha} p'$, then $q \approx p$. Then $q \approx_i p$ for some $i \in I$. So $p \approx_i^{-1} q$. Since $\approx_i$ is a bisimulation, so is $\approx_i^{-1}$, and therefore for some state q', $p' \approx_i^{-1} q'$ and $q \xrightarrow{\alpha} q'$. This means that $p' \approx^{-1} q'$, implying that $\approx^{-1}$ is a simulation. Hence $\approx$ is a bisimulation. $\square$

5. The union of all bisimilations on an LTS is a bisimulation that is also an equivalence relation.

   Proof. For an LTS M, let $\approx_M$ be the union of all bisimulations on M. Then $\approx_M$ is a bisimulation by the previous result. Since $=$ is a bisimulation on M, $\approx_M$ is reflexive. If $p \approx_M q$, $p \approx q$ for some bisimulation $\approx$ on M. Then $\approx^{-1}$ is a bisimulation and therefore $q \approx^{-1} p$ implies that $q \approx_M p$, so that $\approx_M$ is symmetric. Finally, if $p \approx_M q$ and $q \approx_M r$, then $p \approx_1 q$ and $q \approx_2 r$ for some bisimulations $\approx_1$ and $\approx_2$. So

$\approx_1 \circ \approx_2$ is a bisimulation. Since $p \approx_1 \circ \approx_2 r$, $p \approx_M r$ and therefore $\approx_M$ is transitive. $\square$

For an LTS $M = (S, \Sigma, \to)$, let $\approx_M$ be the maximal bisimulation on M as defined above. Since $\approx_M$ is an equivalence relation, we can form an equivalence class $[p]$ for each state $p \in S$. Let $[S]$ be the set of all such equivalence classes: $[S] := \{[p] \mid p \in S\}$. Define $[\to]$ on $S \times \Sigma \times S$ by

$$[p] \overset{\alpha}{[\to]} [q] \qquad \text{iff} \qquad p \overset{\alpha}{\to} q.$$

This is a well-defined ternary relation, for if $p \approx_M p'$ and $q \approx_M q'$, we have $p' \overset{\alpha}{\to} q'$. Now, $[M] := ([S], \Sigma, [\to])$ is an LSTS, and M is bisimilar to it, with bisimulation given by the relation $\{(p, [p]) \mid p \in S\}$.