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unique readability of parenthesized formulas

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Given a set V of propositional variables, and a set F of logical connectives, one may form the set \overline{V} of well-formed formulas, or wffs. The appearance of a wff depends on the formation rule of a well-formed formula. Auxiliary symbols may or may not be used in the construction of a wff. In the parent entry, we proved that every well-formed formula constructed without the aid of auxiliary symbols has a unique appearance. In this entry, we show that, with the aid of auxiliary symbols such as parentheses, specifically, unique readability is achieved as well. The specific formation rule we have in mind is the following: if p_1, \dots, p_n are existing wffs, and α is n -ary, then so is $(\alpha p_1 \cdots p_n)$ a wff.

Theorem 1. *Wffs constructed using the formation rule above have unique readability.*

We will only give a partial proof here, since the portion not proved can be easily produced by closely following the proof from the parent entry.

Sketch of Proof. A wff has one of the following three forms: v , $(\#)$, or $(\alpha p_1 \cdots p_n)$, where v is atomic (propositional variable), $\#$ and α are nullary and n -ary connectives respectively, with $n > 0$, and all p_i are existing wffs.

First, it is easy to see that every wff has the same number of left parentheses as right parentheses, and every proper non-trivial initial segment of a wff has more left than right parentheses.

Now, suppose p and q are wffs and $p = q$. We look at the following cases:

- If p is atomic, then so must be q , and vice versa.
- If $p = (\#)$, then q is either $(@)$ where $@$ is nullary, or $q = (\alpha p_1 \cdots p_n)$, where α is n -ary with $n > 0$. However, the second choice is not possible, as the length of the word $(\alpha p_1 \cdots p_n)$ is longer than $(\#)$. So we are left with $(@)$. Canceling the parentheses in the equation $(\#) = (@)$, we have $\# = @$.
- If $p = (\alpha p_1 \cdots p_n)$, then q must have the form $(\beta q_1 \cdots q_m)$. Equating the two expressions and canceling the parentheses, we get $\alpha = \beta$ and thus $m = n$. By an argument similar to the one given in the parent entry, we see that $p_n = q_n$, utilizing the function ϕ^* modified so that $\phi(() = \phi() = 0$. Continuing this, we see that $p_i = q_i$ for all $i = 1, \dots, n$.

In all three cases, we have unique readability, the proof is complete. \square

Commas are also often used as auxiliary symbols in forming wffs to improve comprehensibility without violating unique readability.

Theorem 2. *Wffs constructed using the following formula formation rule have unique readability: if p_1, \dots, p_n are existing wffs, and α is n -ary, then so is $\alpha(p_1, \dots, p_n)$ a wff.*

Sketch of Proof. Like the last proof, a wff also has one of the three forms: v , $(\#)$, or $\alpha(p_1, \dots, p_n)$, where v is atomic (propositional variable), $\#$ and α are nullary and n -ary connectives respectively, with $n > 0$, and all p_i are existing wffs.

The rest of the proof goes pretty much the same as the last one. The last case deserves a little more attention:

If $p = \alpha(p_1, \dots, p_n)$, and q has the form $\beta(q_1, \dots, q_m)$. We again have $\alpha = \beta$ and $m = n$, after equating p and q . Eliminating the parentheses, we get $p_1, \dots, p_n = q_1, \dots, q_n$. So p_n is a suffix of q_1, \dots, q_n , which means that it is either a suffix of q_n , or has the form r, q_k, \dots, q_n , where r is a suffix of q_{k-1} , and $k \leq n$. In the latter case, if r is the empty word, then $p_n = q_k, \dots, q_n$, which is impossible because no wffs begins with a comma. So r is a non-trivial suffix of q_{k-1} . Using ϕ^* (see the parent entry) modified so that $\phi(() = \phi(,) = \phi(,) = 0$, we again show that p_n can not be r, q_k, \dots, q_n . Therefore, p_n is a suffix of q_n . We will let the reader finish the rest. \square

Finally, another common practice is to infix a connective when it is binary. We again have unique readability:

Theorem 3. *Wffs constructed using the following formula formation rule have unique readability: if p_1, \dots, p_n are existing wffs, and α is n -ary, then so is $(\alpha p_1 \dots p_n)$ a wff if $n \neq 2$, and so is $(p_1 \alpha p_2)$ a wff if $n = 2$.*

Sketch of Proof. The proof is very much the same as before. A wff in this case has four forms: v , $(\#)$, $(\alpha p_1 \dots p_n)$, or $(q_1 \beta q_2)$. We only need to observe the following:

- if p has the form $(\alpha p_1 \dots p_n)$, and $p = q$, then q can not have the form $(q_1 \beta q_2)$, for otherwise q_1 begins with α , which is impossible, because a wff never begins with a connective symbol, as it either begins with a left parenthesis $($, or is an atom.

- so p has the form $(p_1\alpha p_2)$ and q has the form $(q_1\beta q_2)$, which means that p_1 is a prefix of $q_1\beta q_2$. Then p_1 is either a prefix of q_1 , or has the form $q_1\beta r$, where r is a prefix of q_2 . Let us look at the latter case. r can not be the empty word, for otherwise $p_1 = q_1\beta$, and no wffs end with a connective symbol. r can not be q_2 or q_2 , for otherwise $p_1\alpha p_2$ is longer (in length) than $q_1\beta q_2$. So r is a proper non-trivial prefix of q_2 , but this is also impossible, for then r has more left than right parentheses, which means $q_1\beta r$, or p_1 , a wff, has more left than right parentheses. This shows that p_1 is a prefix of q_1 .

The rest of the proof is now easy. □