proof of Zermelo's well-ordering theorem

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Let X be any set and let f be a choice function on $\mathcal{P}(X) \setminus \{\emptyset\}$. Then define a function i by transfinite recursion on the class of ordinals as follows:

$$i(\beta) = f(X - \bigcup_{\gamma<\beta} \{i(\gamma)\}) \text{ unless } X - \bigcup_{\gamma<\beta} \{i(\gamma)\} = \emptyset \text{ or } i(\gamma) \text{ is undefined for some } \gamma < \beta$$

(the function is undefined if either of the unless clauses holds).

Thus $i(0)$ is just $f(X)$ (the least element of X), and $i(1) = f(X - \{i(0)\})$ (the least element of X other than $i(0)$).

Define by the axiom of replacement $\beta = i^{-1}[X] = \{\gamma \mid i(\gamma) = x \text{ for some } x \in X\}$. Since β is a set of ordinals, it cannot contain all the ordinals (by the Burali-Forti paradox).

Since the ordinals are well ordered, there is a least ordinal α not in β, and therefore $i(\alpha)$ is undefined. It cannot be that the second unless clause holds (since α is the least such ordinal) so it must be that $X - \bigcup_{\gamma<\alpha}\{i(\gamma)\} = \emptyset$, and therefore for every $x \in X$ there is some $\gamma < \alpha$ such that $i(\gamma) = x$. Since we already know that i is injective, it is a bijection between α and X, and therefore establishes a well-ordering of X by $x <_X y \leftrightarrow i^{-1}(x) < i^{-1}(y)$.

The reverse is simple. If C is a set of nonempty sets, select any well ordering of $\bigcup C$. Then a choice function is just $f(a) =$ `the least member of` a `under that well ordering`.