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axiom system for propositional logic

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The language of (classical) propositional logic PL_c consists of a set of propositional letters or variables, the symbol $\perp$ (for falsity), together with two symbols for logical connectives $\neg$ and $\to$. The well-formed formulas (wff's) of PL_c are inductively defined as follows:

- each propositional letter is a wff
- $\perp$ is a wff
- if A and B are wff's, then $A \to B$ is a wff

We also use parentheses (and) to remove ambiguities. The other familiar logical connectives may be defined in terms of $\to$: $\neg A$ is $A \to \perp$, $A \vee B$ is the abbreviation for $\neg A \to B$, $A \wedge B$ is the abbreviation for $\neg(A \to \neg B)$, and $A \leftrightarrow B$ is the abbreviation for $(A \to B) \wedge (B \to A)$.

The axiom system for PL_c consists of sets of wffs called *axiom schemas* together with a rule of inference. The axiom schemas are:

1. $A \to (B \to A)$,
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,
3. $(\neg A \to \neg B) \to (B \to A)$,

and the rule of inference is modus ponens (MP): from $A \to B$ and A, we may infer B.

A *deduction* is a finite sequence of wff's $A_1, \ldots, A_n$ such that each A_i is either an instance of one of the axiom schemas above, or as a result of applying rule MP to earlier wff's in the sequence. In other words, there are $j, k < i$ such that A_k is the wff $A_j \to A_i$. The last wff A_n in the deduction is called a *theorem* of PL_c. When A is a theorem of PL_c, we write

$$\vdash_c A \qquad \text{or simply} \qquad \vdash A.$$

For example, $\vdash A \to A$, whose deduction is

1. $(A \to ((B \to A) \to A)) \to ((A \to (B \to A)) \to (A \to A))$ by Axiom II,
2. $A \to ((B \to A) \to A)$ by Axiom I,
3. $(A \to (B \to A)) \to (A \to A)$ by modus ponens on 2 to 1,

4. $A \to (B \to A)$ by Axiom I,
5. $A \to A$ by modus ponens on 4 to 3.

More generally, given a set Σ of wff's, we write

$$\Sigma \vdash A$$

if there is a finite sequence of wff's such that each wff is either an axiom, a member of Σ, or as a result of applying MP to earlier wff's in the sequence. An important (meta-)theorem called the deduction theorem, states: if $\Sigma, A \vdash B$, then $\Sigma \vdash A \to B$. The deduction theorem holds for PL_c (proof `http://planetmath.org/deductiontheoremholdsforclassicalpropositionallogic`here)

Remark. The axiom system above was first introduced by Polish logician Jan Łukasiewicz. Two axiom systems are said to be *deductively equivalent* if every theorem in one system is also a theorem in the other system. There are many axiom systems for PL_c that are deductively equivalent to Łukasiewicz's system. One such system consists of the first two axiom schemas above, but the third axiom schema is $\neg\neg A \to A$, with MP its sole inference rule.