ring of sets

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Ring of Sets

Let S be a set and 2^S be the power set of S. A subset $\mathcal{R}$ of 2^S is said to be a *ring of sets* of S if it is a lattice under the intersection and union operations. In other words, $\mathcal{R}$ is a ring of sets if

- for any $A, B \in \mathcal{R}$, then $A \cap B \in \mathcal{R}$,
- for any $A, B \in \mathcal{R}$, then $A \cup B \in \mathcal{R}$.

A ring of sets is a distributive lattice. The word "ring" in the name has nothing to do with the ordinary ring found in algebra. Rather, it is an abelian semigroup with respect to each of the binary set operations. If $S \in \mathcal{R}$, then $(\mathcal{R}, \cap, S)$ becomes an abelian monoid. Similarly, if $\varnothing \in \mathcal{R}$, then $(\mathcal{R}, \cup, \varnothing)$ is an abelian monoid. If both $S, \varnothing \in \mathcal{R}$, then $(\mathcal{R}, \cup, \cap)$ is a commutative semiring, since $\varnothing \cap A = A \cap \varnothing = \varnothing$, and $\cap$ distributes over $\cup$. Dualizing, we see that $(\mathcal{R}, \cap, \cup)$ is also a commutative semiring. It is perhaps with this connection that the name "ring of sets" is so chosen.

Since S is not required to be in $\mathcal{R}$, a ring of sets can in theory be the empty set. Even if $\mathcal{R}$ may be non-empty, it may be a singleton. Both cases are not very interesting to study. To avoid such examples, some authors, particularly measure theorists, define a ring of sets to be a non-empty set with the first condition above replaced by

- for any $A, B \in \mathcal{R}$, then $A - B \in \mathcal{R}$.

This is indeed a stronger condition, as $A \cap B = A - (A - B) \in \mathcal{R}$. However, we shall stick with the more general definition here.

Field of Sets

An even stronger condition is to insist that not only is $\mathcal{R}$ non-empty, but that $S \in \mathcal{R}$. Such a ring of sets is called a field, or algebra of sets. Formally, given a set S, a *field of sets* $\mathcal{F}$ of S satisfies the following criteria

- $\mathcal{F}$ is a ring of sets of S,
- $S \in \mathcal{F}$, and
- if $A \in \mathcal{F}$, then the complement $\overline{A} \in \mathcal{F}$.

The three conditions above are equivalent to the following three conditions:

- $\varnothing \in \mathcal{F}$,
- if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$, and
- if $A \in \mathcal{F}$, then $\overline{A} \in \mathcal{F}$.

A field of sets is also known as an *algebra of sets*.

It is easy to see that $\mathcal{F}$ is a distributive complemented lattice, and hence a Boolean lattice. From the discussion earlier, we also see that $\mathcal{F}$ (of S) is a commutative semiring, with S acting as the multiplicative identity and $\varnothing$ both the additive identity and the multiplicative absorbing element.

Remark. Two remarkable theorems relating to of certain lattices as rings or fields of sets are the following:

1. a lattice is distributive iff it is `http://planetmath.org/LatticeIsomorphism`lattice isomorphic to a ring of sets (G. Birkhoff and M. Stone);
2. a lattice is `http://planetmath.org/BooleanLattice`Boolean iff it is lattice to a field of sets (M. Stone).

References

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[2] P. R. Halmos: *Measure Theory*, Springer-Verlag (1974).

[3] G. Grätzer: *General Lattice Theory*, Birkhäuser, (1998).