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Canonical name	ConstructionOfBanachLimitUsingLimitAlongAnUltrafilter
Date of creation	2013-03-22 15:32:29
Last modified on	2013-03-22 15:32:29
Owner	kompik (10588)
Last modified by	kompik (10588)
Numerical id	8
Author	kompik (10588)
Entry type	Application
Classification	msc 03E99
Classification	msc 40A05
Related topic	BanachLimit

Construction of Banach limit using limit along an ultrafilter

The existence of Banach limit is proved in mathematical analysis usually by Hahn-Banach theorem. (This proof can be found e.g. in [?], [?] or [?].) Here we will show another approach using limit along a filter. In fact we define it as an $\mathcal{F}$-limit of (y_n), where (y_n) is the Cesàro mean of the sequence (x_n) and $\mathcal{F}$ is an arbitrary ultrafilter on $\mathbb{N}$.

Theorem 1. *Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$. Let (x_n) be a* `http://planetmath.org/Boundedbound` *real sequence. Then the functional $\varphi : \ell_\infty \to \mathbb{R}$*

$$\varphi(x_n) = \mathcal{F}\text{-}\lim \frac{x_1 + \ldots + x_n}{n}$$

is a Banach limit.

Proof. We first observe that φ is defined. Let us denote $y_n := \frac{x_1+\ldots+x_n}{n}$. Since (x_n) is bounded, the sequence (y_n) is bounded as well. Every bounded sequence has a limit along any ultrafilter. This means, that $\varphi(x_n) = \mathcal{F}\text{-}\lim y_n$ exists.

To prove that φ is a Banach limit, we should verify its continuity, positivity, linearity, shift-invariance and to verify that it extends limits.

We first show the shift-invariance. By Sx we denote the sequence x_{n+1} and we want to show $\varphi(Sx) = \varphi(x)$. We observe that $\frac{x_1+\ldots+x_n}{n} - \frac{(Sx)_1+\ldots+(Sx)_n}{n} = \frac{x_1+\ldots+x_n}{n} - \frac{x_2+\ldots+x_{n+1}}{n} = \frac{x_1-x_{n+1}}{n}$. As the sequence (x_n) is bounded, the last expression converges to 0. Thus $\varphi(x) - \varphi(Sx) = \mathcal{F}\text{-}\lim \frac{x_1-x_{n+1}}{n} = 0$ and $\varphi(x) = \varphi(Sx)$.

The rest of the proof is relatively easy, we only need to use the basic properties of a limit along a filter and of Cesàro mean.

Continuity: $\|x\| \le 1 \Rightarrow |x_n| \le 1 \Rightarrow |y_n| \le 1 \Rightarrow |\varphi(x)| \le 1$.

Positivity and linearity follow from positivity and linearity of $\mathcal{F}$-limit.

Extends limit: If (x_n) is a convergent sequence, then its Cesàro mean (y_n) is convergent to the same limit. □

References

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