example of Aronszajn tree

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Construction 1: If κ is a singular cardinal then there is a construction of a κhttp://planetmath.org/KappaAronszjanTree-Aronszajn tree. Let $\langle k_\beta \rangle_{\beta<\iota}$ with $\iota < \kappa$ be a sequence cofinal in κ. Then consider the tree where $T = \{(\alpha, k_\beta) \mid \alpha < k_\beta \wedge \beta < \iota\}$ with $(\alpha_1, k_{\beta_1}) <_T (\alpha_2, k_{\beta_2})$ iff $\alpha_1 < \alpha_2$ and $k_{\beta_1} = k_{\beta_2}$.

Note that this is similar to (indeed, a subtree of) the construction given for a tree with no cofinal branches. It consists of ι disjoint branches, with the β-th branch of height k_β. Since $\iota < \kappa$, every level has fewer than κ elements, and since the sequence is cofinal in κ, T must have height and cardinality κ.

Construction 2: We can construct an Aronszajn tree out of the compact subsets of $\mathbb{Q}^+$. $<_T$ will be defined by $x <_T y$ iff y is an end-extension of x. That is, $x \subseteq y$ and if $r \in y \setminus x$ and $s \in x$ then $s < r$.

Let $T_0 = \{[0]\}$. Given a level T_α, let $T_{\alpha+1} = \{x \cup \{q\} \mid x \in T_\alpha \wedge q > \max x\}$. That is, for every element x in T_α and every rational number q larger than any element of x, $x \cup \{q\}$ is an element of $T_{\alpha+1}$. If $\alpha < \omega_1$ is a limit ordinal then each element of T_α is the union of some branch in $T(\alpha)$.

We can show by induction that $|T_\alpha| < \omega_1$ for each $\alpha < \omega_1$. For the case, T_0 has only one element. If $|T_\alpha| < \omega_1$ then $|T_{\alpha+1}| = |T_\alpha| \cdot |\mathbb{Q}| = |T_\alpha| \cdot \omega = \omega < \omega_1$. If $\alpha < \omega_1$ is a limit ordinal then $T(\alpha)$ is a countable union of countable sets, and therefore itself countable. Therefore there are a countable number of branches, so T_α is also countable. So T has countable levels.

Suppose T has an uncountable branch, $B = \langle b_0, b_1, \ldots \rangle$. Then for any $i < j < \omega_1$, $b_i \subset b_j$. Then for each i, there is some $x_i \in b_{i+1} \setminus b_i$ such that x_i is greater than any element of b_i. Then $\langle x_0, x_1, \ldots \rangle$ is an uncountable increasing sequence of rational numbers. Since the rational numbers are countable, there is no such sequence, so T has no uncountable branch, and is therefore Aronszajn.