first isomorphism theorem

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Owner	almann (2526)
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Author	almann (2526)
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Let Σ be a fixed signature, and $\mathfrak{A}$ and $\mathfrak{B}$ structures for Σ. If $f\colon \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then there is a unique bimorphism $\phi\colon \mathfrak{A}/\ker(f) \to \operatorname{im}(f)$ such that for all $a \in \mathfrak{A}$, $\phi([\![a]\!]) = f(a)$. Furthermore, if f has the additional property that for each $n \in \mathbb{N}$ and each n-ary relation symbol R of Σ,

$$R^{\mathfrak{B}}(f(a_1), \ldots, f(a_n)) \Rightarrow \exists a_i'[f(a_i) = f(a_i') \wedge R^{\mathfrak{A}}(a_1', \ldots, a_n')],$$

then ϕ is an isomorphism.

Proof. Since the homomorphic image of a Σ-structure is also a Σ-structure, we may assume that $\operatorname{im}(f) = \mathfrak{B}$.

Let $\sim\, = \ker(f)$. Define a bimorphism $\phi\colon \mathfrak{A}/{\sim} \to \mathfrak{B} : [\![a]\!] \mapsto f(a)$. To verify that ϕ is well defined, let $a \sim a'$. Then $\phi([\![a]\!]) = f(a) = f(a') = \phi([\![a']\!])$. To show that ϕ is injective, suppose $\phi([\![a]\!]) = \phi([\![a']\!])$. Then $f(a) = f(a')$, so $a \sim a'$. Hence $[\![a]\!] = [\![a']\!]$. To show that ϕ is a homomorphism, observe that for any constant symbol c of Σ we have $\phi([\![c^{\mathfrak{A}}]\!]) = f(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$. For each $n \in \mathbb{N}$ and each n-ary function symbol F of Σ,

$$\begin{aligned}
\phi(F^{\mathfrak{A}/\sim}([\![a_1]\!], \ldots, [\![a_n]\!])) &= \phi([\![F^{\mathfrak{A}}(a_1, \ldots, a_n)]\!]) \\
&= f(F^{\mathfrak{A}}(a_1, \ldots, a_n)) \\
&= F^{\mathfrak{B}}(f(a_1), \ldots, f(a_n)) \\
&= F^{\mathfrak{B}}(\phi([\![a_1]\!], \ldots, \phi([\![a_n]\!])).
\end{aligned}$$

For each $n \in \mathbb{N}$ and each n-ary relation symbol R of Σ,

$$\begin{aligned}
R^{\mathfrak{A}/\sim}([\![a_1]\!], \ldots, [\![a_n]\!]) &\Rightarrow R^{\mathfrak{A}}(a_1, \ldots, a_n) \\
&\Rightarrow R^{\mathfrak{B}}(f(a_1), \ldots, f(a_n)) \\
&\Rightarrow R^{\mathfrak{B}}(\phi([\![a_1]\!], \ldots, \phi([\![a_n]\!])).
\end{aligned}$$

Thus ϕ is a bimorphism.

Now suppose f has the additional property mentioned in the statement of the theorem. Then

$$\begin{aligned}
R^{\mathfrak{B}}(\phi([\![a_1]\!]), \ldots, \phi([\![a_n]\!])) &\Rightarrow R^{\mathfrak{B}}(f(a_1), \ldots, f(a_n)) \\
&\Rightarrow \exists a_i'[a_i \sim a_i' \wedge R^{\mathfrak{A}}(a_1', \ldots, a_n')] \\
&\Rightarrow R^{\mathfrak{A}/\sim}([\![a_1]\!], \ldots, [\![a_n]\!]).
\end{aligned}$$

Thus ϕ is an isomorphism. $\square$