transfinite recursion

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Transfinite recursion, roughly speaking, is a statement about the ability to define a function recursively using transfinite induction. In its most general and intuitive form, it says

Theorem 1. *Let G be a (class) function on V, the class of all sets. Then there exists a unique (class) function F on* **On***, the class of ordinals, such that*

$$F(\alpha) = G(F|\alpha)$$

where $F|\alpha$ is the function whose domain is $\mathrm{seg}(\alpha) := \{\beta \mid \beta < \alpha\}$ *and whose values coincide with F on every $\beta \in \mathrm{seg}(\alpha)$. In other words, $F|\alpha$ is the restriction of F to α.*

Notice that the theorem above is not provable in ZF set theory, as G and F are both classes, not sets. In order to prove this statement, one way of getting around this difficulty is to convert both G and F into formulas, and modify the statement, as follows:

Let $\varphi(x, y)$ be a formula such that

$$\forall x \exists y \forall z (\varphi(x, z) \leftrightarrow z = y).$$

Think of $G = \{(x, y) \mid \varphi(x, y)\}$. Then there is a unique formula $\psi(\alpha, z)$ (think of F as $\{(\alpha, z) \mid \psi(\alpha, z)\}$) such that the following two sentences are derivable using ZF axioms:

1. $\forall x \exists y \forall z \big(\mathbf{On}(x) \wedge (\psi(x, z) \leftrightarrow z = y)\big)$, where $\mathbf{On}(x)$ means "x is an ordinal",
2. $\forall x \forall y \Big(\mathbf{On}(x) \wedge \big(\psi(x, y) \leftrightarrow \exists f (A \wedge B \wedge C \wedge D)\big)\Big)$, where
   - A is the formula "f is a function",
   - B is the formula "$\mathrm{dom}(f) = x$",
   - C is the formula $\forall z \big(z \in x \wedge \varphi(f|z, f(z))\big)$, and
   - D is the formula $\varphi(f, y)$.

A stronger form of the transfinite recursion theorem says:

Theorem 2. *Let $\varphi(x, y)$ be any formula (in the language of set theory). Then the following is a theorem: assume that φ satisfies property that, for every x,* *there is a unique y such that $\varphi(x, y)$. If A be a well-ordered set (well-ordered by $\leq$), then there is a unique function f defined on A such that*

$$\varphi(f|\,\mathrm{seg}(s), f(s))$$

for every $s \in A$. Here, $\mathrm{seg}(s) := \{t \in A \mid t < s\}$*, the initial segment of s in A.*

The above theorem is actually a collection of theorems, or known as a theorem schema, where each theorem corresponds to a formula. The other difference between this and the previous theorem is that this theorem is provable in ZF, because the domain of the function f is now a set.