idempotency of infinite cardinals

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In this entry, we show that every infinite cardinal is idempotent with respect to cardinal addition and cardinal multiplication.

Theorem 1. *$\kappa \cdot \kappa = \kappa$ for any infinite cardinal κ.*

Proof. For any non-zero cardinal λ, we have $\lambda = 1 \cdot \lambda \leq \lambda \cdot \lambda$. So given an infinite cardinal κ, either $\kappa = \kappa \cdot \kappa$ or $\kappa < \kappa \cdot \kappa$. Let $\mathscr{C}$ be the class of infinite cardinals that fail to be idempotent (with respect to $\cdot$). Suppose $\mathscr{C} \neq \varnothing$. We shall derive a contradiction. Since $\mathscr{C}$ consists entirely of ordinals, it is therefore well-ordered, and has a least member κ.

Let $K = \kappa \times \kappa$. As K is a collection of ordered pairs of ordinals, it has the canonical well-ordering inherited from the canonical ordering on $\mathbf{On} \times \mathbf{On}$. Let α be the ordinal isomorphic to K. Since $\kappa < \kappa \cdot \kappa = |K|$, there is an initial segment L of K that is order isomorphic to κ.

Since L is an initial segment of K, $L = \{(\beta_1, \beta_2) \mid (\beta_1, \beta_2) \prec (\alpha_1, \alpha_2)\}$ for some $(\alpha_1, \alpha_2) \in K$. The well-order $\preceq$ denotes the canonical ordering on K. Let $\lambda = \max(\alpha_1, \alpha_2)$. Since $L \subset K = \kappa \times \kappa$, $\alpha_1 < \kappa$ and $\alpha_2 < \kappa$, and therefore $\lambda < \kappa$.

For any $(\beta_1, \beta_2) \in L$, we have $(\beta_1, \beta_2) \prec (\alpha_1, \alpha_2)$, which implies that $\max(\beta_1, \beta_2) \leq \lambda$. Therefore $L \subseteq \lambda^+ \times \lambda^+$, or $|L| \leq |\lambda^+ \times \lambda^+| \leq |\lambda^+| \cdot |\lambda^+|$. There are two cases to discuss:

1. If λ is finite, so is $\lambda^+ \times \lambda^+$, contradicting that L is (order) isomorphic to κ, an infinite set.

2. If λ is infinite, so is $|\lambda^+|$. Since $\lambda < \kappa$, and κ is a limit ordinal, $|\lambda^+| < k$ as well, which means $|\lambda^+| \notin \mathscr{C}$, or $|\lambda^+| \cdot |\lambda^+| = |\lambda^+|$. Therefore $|L| \leq |\lambda^+| \cdot |\lambda^+| = |\lambda^+| \leq \lambda^+ < \kappa$, again contradicting that L is (order) isomorphic to κ.

Therefore, the assumption $\mathscr{C} \neq \varnothing$ is false, and the proof is complete. $\square$

Corollary 1. *If $0 < \lambda \leq \kappa$ and κ is infinite, then $\lambda \cdot \kappa = \kappa$.*

Proof. $\kappa = 1 \cdot \kappa \leq \lambda \cdot \kappa \leq \kappa \cdot \kappa = \kappa$. By Schroder-Bernstein's Theorem, $\lambda \cdot \kappa = \kappa$. $\square$

Corollary 2. *If $\lambda \leq \kappa$ and κ is infinite, then $\lambda + \kappa = \kappa$.*

Proof. $\kappa = 0 + \kappa \leq \lambda + \kappa \leq \kappa + \kappa = 2 \cdot \kappa \leq \kappa \cdot \kappa = \kappa$ by the corollary above (since $2 \leq \kappa$). Another application of Schroder-Bernstein gives $\kappa = \lambda + \kappa$. $\square$

Since $\kappa \leq \kappa$, we get the following:

Corollary 3. $\kappa + \kappa = \kappa$ *for any infinite cardinal.*

Remark. No cardinal greater than 1 is idempotent with respect to cardinal exponentiation. This is a direct consequence of Cantor's theorem: $\kappa < 2^\kappa \leq \kappa^\kappa$.