forcings are equivalent if one is dense in the other

Canonical name	ForcingsAreEquivalentIfOneIsDenseInTheOther
Date of creation	2013-03-22 12:54:43
Last modified on	2013-03-22 12:54:43
Owner	Henry (455)
Last modified by	Henry (455)
Numerical id	6
Author	Henry (455)
Entry type	Result
Classification	msc 03E35
Classification	msc 03E40

Suppose P and Q are forcing notions and that $f : P \to Q$ is a function such that:

- $p_1 \leq_P p_2$ implies $f(p_1) \leq_Q f(p_2)$
- If $p_1, p_2 \in P$ are incomparable then $f(p_1), f(p_2)$ are incomparable
- $f[P]$ is `http://planetmath.org/DenseInAPoset`dense in Q

then P and Q are equivalent.

Proof

We seek to provide two operations (computable in the appropriate universes) which convert between generic subsets of P and Q, and to prove that they are inverses.

$F(G) = H$ where H is generic

Given a generic $G \subseteq P$, consider $H = \{q \mid f(p) \leq q\}$ for some $p \in G$.

If $q_1 \in H$ and $q_1 \leq q_2$ then $q_2 \in H$ by the definition of H. If $q_1, q_2 \in H$ then let $p_1, p_2 \in P$ be such that $f(p_1) \leq q_1$ and $f(p_2) \leq q_2$. Then there is some $p_3 \leq p_1, p_2$ such that $p_3 \in G$, and since f is order preseving $f(p_3) \leq f(p_1) \leq q_1$ and $f(p_3) \leq f(p_2) \leq q_2$.

Suppose D is a dense subset of Q. Since $f[P]$ is dense in Q, for any $d \in D$ there is some $p \in P$ such that $f(p) \leq d$. For each $d \in D$, assign (using the axiom of choice) some $d_p \in P$ such that $f(d_p) \leq d$, and call the set of these D_P. This is dense in P, since for any $p \in P$ there is some $d \in D$ such that $d \leq f(p)$, and so some $d_p \in D_P$ such that $f(d_p) \leq d$. If $d_p \leq p$ then D_P is dense, so suppose $d_p \not\leq p$. If $d_p \leq p$ then this provides a member of D_P less than p; alternatively, since $f(d_p)$ and $f(p)$ are compatible, d_p and p are compatible, so $p \leq d_p$, and therefore $f(p) = f(d_p) = d$, so $p \in D_P$. Since D_P is dense in P, there is some element $p \in D_P \cap G$. Since $p \in D_P$, there is some $d \in D$ such that $f(p) \leq d$. But since $p \in G$, $d \in H$, so H intersects D.

G can be recovered from $F(G)$

Given H constructed as above, we can recover G as the set of $p \in P$ such that $f(p) \in H$. Obviously every element from G is included in the new set, so consider some p such that $f(p) \in H$. By definition, there is some $p_1 \in G$ such that $f(p_1) \leq f(p)$. Take some dense $D \in Q$ such that there is no $d \in D$ such that $f(p) \leq d$ (this can be done easily be taking any dense subset and removing all such elements; the resulting set is still dense since there is some d_1 such that $d_1 \leq f(p) \leq d$). This set intersects $f[G]$ in some q, so there is some $p_2 \in G$ such that $f(p_2) \leq q$, and since G is directed, some $p_3 \in G$ such that $p_3 \leq p_2, p_1$. So $f(p_3) \leq f(p_1) \leq f(p)$. If $p_3 \not\leq p$ then we would have $p \leq p_3$ and then $f(p) \leq f(p_3) \leq q$, contradicting the definition of D, so $p_3 \leq p$ and $p \in G$ since G is directed.

$F^{-1}(H) = G$ where G is generic

Given any generic H in Q, we define a corresponding G as above: $G = \{p \in P \mid f(p) \in H\}$. If $p_1 \in G$ and $p_1 \leq p_2$ then $f(p_1) \in H$ and $f(p_1) \leq f(p_2)$, so $p_2 \in G$ since H is directed. If $p_1, p_2 \in G$ then $f(p_1), f(p_2) \in H$ and there is some $q \in H$ such that $q \leq f(p_1), f(p_2)$.

Consider D, the set of elements of Q which are $f(p)$ for some $p \in P$ and either $f(p) \leq q$ or there is no element greater than both $f(p)$ and q. This is dense, since given any $q_1 \in Q$, if $q_1 \leq q$ then (since $f[P]$ is dense) there is some p such that $f(p) \leq q_1 \leq q$. If $q \leq q_1$ then there is some p such that $f(p) \leq q \leq q_1$. If neither of these and q there is some $r \leq q_1, q$ then any p such that $f(p) \leq r$ suffices, and if there is no such r then any p such that $f(p) \leq q$ suffices.

There is some $f(p) \in D \cap H$, and so $p \in G$. Since H is directed, there is some $r \leq f(p), q$, so $f(p) \leq q \leq f(p_1), f(p_2)$. If it is not the case that $f(p) \leq f(p_1)$ then $f(p) = f(p_1) = f(p_2)$. In either case, we confirm that H is directed.

Finally, let D be a dense subset of P. $f[D]$ is dense in Q, since given any $q \in Q$, there is some $p \in P$ such that $p \leq q$, and some $d \in D$ such that $d \leq p \leq q$. So there is some $f(p) \in f[D] \cap H$, and so $p \in D \cap G$.

H can be recovered from $F^{-1}(H)$

Finally, given G constructed by this method, $H = \{q \mid f(p) \leq q\}$ for some $p \in G$. To see this, if there is some $f(p)$ for $p \in G$ such that $f(p) \leq q$ then $f(p) \in H$ so $q \in H$. On the other hand, if $q \in H$ then the set of $f(p)$ such that either $f(p) \leq q$ or there is no $r \in Q$ such that $r \leq q, f(p)$ is dense (as shown above), and so intersects H. But since H is directed, it must be that there is some $f(p) \in H$ such that $f(p) \leq q$, and therefore $p \in G$.