proof of complete partial orders do not add small subsets

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Take any $x \in \mathfrak{M}[G]$, $x \subseteq \kappa$. Let $\hat{x}$ be a name for x. There is some $p \in G$ such that

$$p \Vdash \hat{x} \text{ is a subset of } \kappa \text{ bounded by } \lambda < \kappa$$

Outline:

For any $q \leq p$, we construct by induction a series of elements q_α stronger than p. Each q_α will determine whether or not $\alpha \in \hat{x}$. Since we know the subset is bounded below κ, we can use the fact that P is κ complete to find a single element stronger than q which fixes the exact value of $\hat{x}$. Since the series is definable in $\mathfrak{M}$, so is $\hat{x}$, so we can conclude that above any element $q \leq p$ is an element which forces $\hat{x} \in \mathfrak{M}$. Then p also forces $\hat{x} \in \mathfrak{M}$, completing the proof.

Details:

Since forcing can be described within $\mathfrak{M}$, $S = \{q \in P \mid q \Vdash \hat{x} \in V\}$ is a set in $\mathfrak{M}$. Then, given any $q \leq p$, we can define $q_0 = q$ and for any q_α ($\alpha < \lambda$), $q_{\alpha+1}$ is an element of P stronger than q_α such that either $q_{\alpha+1} \Vdash \alpha + 1 \in \hat{x}$ or $q_{\alpha+1} \Vdash \alpha + 1 \notin \hat{x}$. For limit α, let q'_α be any upper bound of q_β for $\alpha < \beta$ (this exists since P is κ-complete and $\alpha < \kappa$), and let q_α be stronger than q'_α and satisfy either $q_{\alpha+1} \Vdash \alpha \in \hat{x}$ or $q_{\alpha+1} \Vdash \alpha \notin \hat{x}$. Finally let q^* be the upper bound of q_α for $\alpha < \lambda$. $q^* \in P$ since P is κ-complete.

Note that these elements all exist since for any $p \in P$ and any (first-order) sentence ϕ there is some $q \leq p$ such that q forces either ϕ or $\neg\phi$.

q^* not only forces that $\hat{x}$ is a bounded subset of κ, but for every ordinal it forces whether or not that ordinal is contained in $\hat{x}$. But the set $\{\alpha < \lambda \mid q^* \Vdash \alpha \in \hat{x}\}$ is defineable in $\mathfrak{M}$, and is of course equal to $\hat{x}[G^*]$ in any generic G^* containing q^*. So $q^* \Vdash \hat{x} \in \mathfrak{M}$.

Since this holds for any element stronger than p, it follows that $p \Vdash \hat{x} \in \mathfrak{M}$, and therefore $\hat{x}[G] \in \mathfrak{M}$.