well-ordering principle for natural numbers proven from the principle of finite induction

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Let S be a nonempty set of natural numbers. We show that there is an $a \in S$ such that for all $b \in S$, $a \leq b$. Suppose not, then

$$(*) \qquad \forall a \in S, \exists b \in S \quad b < a.$$

We will use the principle of finite induction (the strong form) to show that S is empty, a contradition.

Fix any natural number n, and suppose that for all natural numbers $m < n$, $m \in \mathbb{N} \setminus S$. If $n \in S$, then (*) implies that there is an element $b \in S$ such that $b < n$. This would be incompatible with the assumption that for all natural numbers $m < n$, $m \in \mathbb{N} \setminus S$. Hence, we conclude that n is not in S.

Therefore, by induction, no natural number is a member of S. The set is empty.