monadic algebra

Canonical name	MonadicAlgebra
Date of creation	2013-03-22 17:48:57
Last modified on	2013-03-22 17:48:57
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	9
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03G15
Classification	msc 06E25
Related topic	QuantifierAlgebra
Related topic	PolyadicAlgebra
Defines	existential quantifier operator
Defines	universal quantifier operator

Let B be a Boolean algebra. An *existential quantifier operator* on B is a function $\exists : B \to B$ such that

1. $\exists(0) = 0$,
2. $a \leq \exists(a)$, where $a \in B$, and
3. $\exists(a \wedge \exists(b)) = \exists(a) \wedge \exists(b)$, where $a, b \in B$.

A *monadic algebra* is a pair $(B, \exists)$, where B is a Boolean algebra and $\exists$ is an existential quantifier operator.

There is an obvious connection between an existential quantifier operator on a Boolean algebra and an existential quantifier in a first order logic:

1. A statement $\varphi(x)$ is false iff $\exists x\varphi(x)$ is false. For example, suppose x is a real number. Let $\varphi(x)$ be the statement $x = x + 1$. Then $\varphi(x)$ is false no matter what x is. Likewise, $\exists\varphi(x)$ is always false too.
2. $\varphi(x)$ implies $\exists x\varphi(x)$; in other words, if $\exists x\varphi(x)$ is false, then so is $\varphi(x)$. For example, let $\varphi(x)$ be the statement $1 < x$, where $x \in \mathbb{R}$. By itself, $\varphi(x)$ is neither true nor false. However $\exists x\varphi(x)$ is always true.
3. $\exists x(\varphi(x) \wedge \exists x\psi(x))$ iff $\exists x\varphi(x) \wedge \exists x\psi(x)$. For example, suppose again x is real. Let $\varphi(x)$ be the statement $x < 1$ and $\psi(x)$ the statement $x > 1$. Then both $\exists x\psi(x)$ and $\exists x\varphi(x)$ are true. It is easy to verify the equivalence of the two sentences in this example. Notice that, however, $\exists x(\varphi(x) \wedge \psi(x))$ is false.

Remarks

- One may replace condition 3. above with the following three conditions to get an equivalent definition of an existential quantifier operator:

  1. $\exists(\exists(a)) = \exists(a)$
  2. $\exists(a \vee b) = \exists(a) \vee \exists(b)$
  3. $\exists((\exists a)') = (\exists a)'$

  From this, it is easy to see that $\exists$ is a closure operator on B, and that $\exists a$ and $(\exists a)'$ are both closed under $\exists$.

- Like the Lindenbaum algebra of propositional logic, monadic algebra is an attempt at converting first order logic into an algebra so that a logical question may be turned into an algebraic one. However, the existential quantifier operator in a monadic algebra corresponds to existential quantifier applied to formulas with only one variable (hence the name monadic). Formulas with multiple variables, such as $x^2+y^2=1$, $x \leq y+z$, or $x_i = x_{i+1} + x_{i+2}$ where $i = 0, 1, 2, \ldots$ require further generalizations to what is known as a *polyadic algebra*. The notions of monadic and polyadic algebras were introduced by Paul Halmos.

Dual to the notion of an existential quantifier is that of a universal quantifier. Likewise, there is a dual of an existential quantifier operator on a Boolean algebra, a *universal quantifier operator*. Formally, a *universal quantifier operator* on a Boolean algebra B is a function $\forall : B \to B$ such that

1. $\forall(1) = 1$,
2. $\forall(a) \leq a$, where $a \in B$, and
3. $\forall(a \vee \forall(b)) = \forall(a) \vee \forall(b)$, where $a, b \in B$.

Every existential quantifier operator $\exists$ on a Boolean algebra B induces a universal quantifier operator $\forall$, given by

$$\forall(a) := (\exists(a'))'.$$

Conversely, every universal quantifier operator induces an existential quantifier by exchanging $\forall$ and $\exists$ in the definition above. This shows that the two operations are dual to one another.

References

[1] P. Halmos, S. Givant, *Logic as Algebra*, The Mathematical Association of America (1998).