completeness theorem for propositional logic

Canonical name	CompletenessTheoremForPropositionalLogic1
Date of creation	2013-03-22 19:32:47
Last modified on	2013-03-22 19:32:47
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	21
Author	CWoo (3771)
Entry type	Theorem
Classification	msc 03B05

The completeness theorem of propositional logic is the statement that a wff is tautology iff it is a theorem. In symbol, we have

$$\models A \quad \text{iff} \quad \vdash A$$

for any wff A. The "if" part of the statement is the soundness theorem, and is proved `http://planetmath.org/TruthValueSemanticsForPropositionalLogicIsSound`here. We will prove the "only if" part, which is also known as the completeness portion of the theorem. We will give a constructive proof of this. Before proving this, we state and prove some preliminary facts:

1. $A, B \vdash A \to B$
2. $A, \neg B \vdash \neg(A \to B)$
3. $\neg A, B \vdash A \to B$
4. $\neg A, \neg B \vdash A \to B$
5. Let v be a valuation. For any wff A, let $v[A]$ be defined as follows:

   $$v[A] \text{ is } \begin{cases} A & \text{if } v(A) = 1, \\ \neg A & \text{if } v(A) = 0. \end{cases}$$

   Suppose $p_1, \ldots, p_n$ are the propositional variables in A. Then

   $$v[p_1], \ldots, v[p_n] \vdash v[A].$$

6. if $\Delta, A \vdash B$ and $\Delta, \neg A \vdash B$, then $\vdash B$.

Proof. Facts 1 and 3 come from the axiom schema $B \to (A \to B)$. From $\vdash B \to (A \to B)$, we have $C \vdash B \to (A \to B)$, so $C, B \vdash A \to B$. If C is A, we have fact 1, and if C is $\neg A$, we have fact 3.

Fact 2: this is proved `http://planetmath.org/SubstitutionTheoremForPropositionalLogi`

Fact 4: By ex falso quodlibet, $\vdash \perp \to B$, so $A \vdash \perp \to B$, and therefore $\vdash A \to (\perp \to B)$ by the deduction theorem. Now, $(A \to (\perp \to B)) \to ((A \to \perp) \to (A \to B))$ is an axiom instance, so $\vdash (A \to \perp) \to (A \to B)$, or $\vdash \neg A \to (A \to B)$, or $\neg A \vdash A \to B$, and

$$\neg A, \neg B \vdash A \to B$$

all the more so.

Fact 5: by induction on the number n of occurrences of $\to$ in A. If $n = 0$, then A is either $\perp$ or a propositional variable p. In the first case, $v[A]$ is $\neg\perp$, and from $\perp\vdash\perp$, we get $\vdash\perp\to\perp$, or $\vdash\neg\perp$. In the second case, $v[p]\vdash v[p]$. Now, suppose there are $n+1$ occurrences of $\to$ in A. Let $p_1,\dots,p_m$ be the propositional variables in A. By unique readability, A is $B\to C$ for some unique wff's B and C. Since each B and C has no more than n occurrences of $\to$, by induction, we have

$$v[p_{i(1)}],\dots,v[p_{i(s)}]\vdash v[B] \qquad \text{and} \qquad v[p_{j(1)}],\dots,v[p_{j(t)}]\vdash v[C],$$

where the propositional variables in B are $p_{i(1)},\dots,p_{i(s)}$, and in C are $p_{j(1)},\dots,p_{j(t)}$. So

$$v[p_1],\dots,v[p_m]\vdash v[B] \qquad \text{and} \qquad v[p_1],\dots,v[p_m]\vdash v[C].$$

Next, we want to show that $v[B],v[C]\vdash v[B\to C]$. We break this into four cases:

- if $v[B]$ is B and $v[C]$ is C: then $v[B\to C]$ is $B\to C$, and use Fact 1
- if $v[B]$ is B and $v[C]$ is $\neg C$: then $v[B\to C]$ is $\neg(B\to C)$, and use Fact 2
- if $v[B]$ is $\neg B$ and $v[C]$ is C: then $v[B\to C]$ is $B\to C$, and use Fact 3
- if $v[B]$ is $\neg B$ and $v[C]$ is $\neg C$: then $v[B\to C]$ is $B\to C$, and use Fact 4.

In all cases, we have by applying modus ponens,

$$v[p_1],\dots,v[p_m]\vdash v[B\to C].$$

Fact 6 is proved `http://planetmath.org/SubstitutionTheoremForPropositionalLogic`here

□

Theorem 1. *Propositional logic is complete with respect to truth-value semantics.*

Proof. Suppose A is a tautology. Let $p_1,\dots,p_n$ be the propositional variables in A. Then

$$v[p_1],\dots,v[p_n]\vdash v[A]$$

for any valuation v. Since $v[A]$ is A. We have

$$v[p_1], \dots, v[p_n] \vdash A.$$

If $n = 0$, then we are done. So suppose $n > 0$. Pick a valuation v such that $v(p_n) = 1$, and a valuation v' such that $v'(p_i) = v(p_i)$ and $v'(p_n) = 0$ Then

$$v[p_1], \dots, v[p_{n-1}], p_n \vdash A \qquad \text{and} \qquad v[p_1], \dots, v[p_{n-1}], \neg p_n \vdash A,$$

where the first deducibility relation comes from v and the second comes from v'. By Fact 6 above,

$$v[p_1], \dots, v[p_{n-1}] \vdash A.$$

So we have eliminated $v[p_n]$ from the left of $v[p_1], \dots, v[p_n] \vdash A$. Now, repeat this process until all of the $v[p_i]$ have been eliminated, and we have $\vdash A$. $\square$

The completeness theorem can be used to show that certain complicated wff's are theorems. For example, one of the distributive laws

$$\vdash (A \wedge B) \vee C \leftrightarrow (A \vee C) \wedge (B \vee C)$$

To see that this is indeed a theorem, by the completeness theorem, all we need to show is that it is true using the truth table:

$(A$	$\wedge$	$B)$	$\vee$	C	$\leftrightarrow$	$(A$	$\vee$	$C)$	$\wedge$	$(B$	$\vee$	$C)$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	T	T	F	T	T	T	F	T	T	T	F
T	F	F	T	T	T	T	T	T	T	F	T	T
T	F	F	F	F	T	T	T	F	F	F	F	F
F	F	T	T	T	T	F	T	T	T	T	T	T
F	F	T	F	F	T	F	F	F	F	T	T	F
F	F	F	T	T	T	F	T	T	T	F	T	T
F	F	F	F	F	T	F	F	F	F	F	F	F

Similarly, one can show $\vdash (A \vee B) \wedge C \leftrightarrow (A \wedge C) \vee (B \wedge C)$.