truth-value semantics for intuitionistic propositional logic

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A *truth-value semantic system* for intuitionistic propositional logic consists of the set $V_n := \{0, 1, \ldots, n\}$, where $n \geq 1$, and a function v from the set of wff's (well-formed formulas) to V_n satisfying the following properties:

1. $v(A \wedge B) = \min\{v(A), v(B)\}$
2. $v(A \vee B) = \max\{v(A), v(B)\}$
3. $v(A \to B) = n$ if $v(A) \leq v(B)$, and $v(B)$ otherwise
4. $v(\neg A) = n$ if $v(A) = 0$, and 0 otherwise.

This function v is called an *interpretation* for the propositional logic. A wff A is said to be *true* for (V_n, v) if $v(A) = n$, and a *tautology* for V_n if A is true for (V_n, v) for all interpretations v. When A is a tautology for V_n, we write $\models_n A$. It is not hard see that any truth-value semantic system is sound, in the sense that $\vdash_i A$ (A is a theorem) implies $\models_n A$, for any n. A proof of this fact can be found `http://planetmath.org/truthvaluesemanticsforintuitionisticpropositionallogic`

(V_n, v) is a generalization of the truth-value semantics for classical propositional logic. Indeed, when $n = 1$, we have the truth-value system for classical propositional logic.

However, unlike the truth-value semantic system for classical propositional logic, no truth-value semantic systems for intuitionistic propositional logic are complete: there are tautologies that are not theorems for each n. For example, for each n, the wff

$$\bigvee_{j=1}^{n+2} \bigvee_{i=j}^{n+1} (p_j \leftrightarrow p_{i+1})$$

is a tautology for V_n that is not a theorem, where each p_i is a propositional letter. The formula $\bigvee_{k=1}^{m} A_i$ is the abbreviation for $(\cdots(A_1 \vee A_2) \vee \cdots) \vee A_m$, where each A_i is a formula. The following is a proof of this fact:

Proof. Let A be the $\bigvee_{j=1}^{n+2} \bigvee_{i=j}^{n+1} (p_j \leftrightarrow p_{i+1})$. Note that $p_1, \ldots, p_{n+2}$ are all the proposition letters in A. However, there are only $n+1$ elements in V_n, so for every interpretation v, there are some p_i and p_j such that $v(p_i) = v(p_j)$ by the pigeonhole principle. Then $v(p_i \leftrightarrow p_j) = n$, and hence $v(A) = n$, implying that A is a tautology for V_n. However, A is not a tautology for V_{n+1}: let v be the interpretation that maps p_i to $i-1$. Then $v(p_i \leftrightarrow p_j) = \min\{i, j\} - 1$, so that $v(A) = n \neq n+1$. Therefore, A is not a theorem. □

Nevertheless, the truth-value semantic systems are useful in showing that certain theorems of the classical propositional logic are not theorems of the intuitionistic propositional logic. For example, the wff $p \vee \neg p$ (law of the excluded middle) is not a theorem, because it is not a tautology for V_2, for if $v(p) = 1$, then $v(p \vee \neg p) = 1 \neq 2$. Similarly, neither $\neg\neg p \to p$ (law of double negation) nor $((p \to q) \to p) \to p$ (Peirce's law) are theorems of the intuitionistic propositional logic.

Remark. The linearly ordered set $V_n := \{0, 1, \dots, n\}$ turns into a Heyting algebra if we define the relative pseudocomplementation operation $\to$ by $x \to y := n$ if $x \leq y$ and $x \to y := y$ otherwise. Then the pseudocomplement x^* of x is just $x \to 0$. This points to the connection of the intuitionistic propositional logic and Heyting algebra.