club filter

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If κ is a regular uncountable cardinal then club(κ), the filter of all sets containing a club subset of κ, is a κ-complete filter closed under diagonal intersection called the *club filter*.

To see that this is a filter, note that $\kappa \in \text{club}(\kappa)$ since it is obviously both closed and unbounded. If $x \in \text{club}(\kappa)$ then any subset of κ containing x is also in club(κ), since x, and therefore anything containing it, contains a club set.

It is a κ complete filter because the intersection of fewer than κ club sets is a club set. To see this, suppose $\langle C_i \rangle_{i<\alpha}$ is a sequence of club sets where $\alpha < \kappa$. Obviously $C = \bigcap C_i$ is closed, since any sequence which appears in C appears in every C_i, and therefore its limit is also in every C_i. To show that it is unbounded, take some $\beta < \kappa$. Let $\langle \beta_{1,i} \rangle$ be an increasing sequence with $\beta_{1,1} > \beta$ and $\beta_{1,i} \in C_i$ for every $i < \alpha$. Such a sequence can be constructed, since every C_i is unbounded. Since $\alpha < \kappa$ and κ is regular, the limit of this sequence is less than κ. We call it β_2, and define a new sequence $\langle \beta_{2,i} \rangle$ similar to the previous sequence. We can repeat this process, getting a sequence of sequences $\langle \beta_{j,i} \rangle$ where each element of a sequence is greater than every member of the previous sequences. Then for each $i < \alpha$, $\langle \beta_{j,i} \rangle$ is an increasing sequence contained in C_i, and all these sequences have the same limit (the limit of $\langle \beta_{j,i} \rangle$). This limit is then contained in every C_i, and therefore C, and is greater than β.

To see that club(κ) is closed under diagonal intersection, let $\langle C_i \rangle$, $i < \kappa$ be a sequence, and let $C = \Delta_{i<\kappa} C_i$. Since the diagonal intersection contains the intersection, obviously C is unbounded. Then suppose $S \subseteq C$ and $\sup(S \cap \alpha) = \alpha$. Then $S \subseteq C_\beta$ for every $\beta \geq \alpha$, and since each C_β is closed, $\alpha \in C_\beta$, so $\alpha \in C$.