partition is equivalent to an equivalence relation

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Proposition 1. *There is a one-to-one correspondence between the set* $\mathrm{Part}(S)$ *of partitions of* S *and the set* $\mathrm{Equiv}(S)$ *of equivalence relations on* S*.*

Proof. Let S be a set.

Suppose $P = \{P_i \mid i \in I\}$ is a partition of S. Form $E = \bigcup\{P_i \times P_i \mid i \in I\}$. Given any $a \in S$, $a \in P_i$ for some $i \in I$. Then $(a, a) \in P_i \times P_i \in E$. If $(a, b) \in E$, then $(a, b) \in P_i \times P_i$ for some $i \in I$, so $a, b \in P_i$, whence $(b, a) \in P_i \times P_i \subseteq E$. Finally, suppose $(a, b), (b, c) \in E$. Then $(a, b) \in P_i \times P_i$ and $(b, c) \in P_j \times P_j$, or $a, b \in P_i$ and $b, c \in P_j$. Since $b \in P_i \cap P_j$, $P_i = P_j$ since P is a partition. So $a, c \in P_i = P_j$, or $(a, c) \in P_i \times P_i \subseteq E$.

Conversely, suppose E is an equivalence relation on S. For each $a \in S$, define

$$[a] = \{b \in S \mid (a, b) \in E\}.$$

Then each $a \in [a]$ since E is reflexive. If $b \in [a]$, then $(a, b) \in E$ or $(b, a) \in E$ as E is symmetric. So $a \in [b]$ as well. Next, pick any $c \in [b]$. Then $(b, c) \in E$. But $(a, b) \in E$. So $(a, c) \in E$ since E is transitive. Therefore $c \in [a]$, which implies $[b] \subseteq [a]$. Collect all the information we have so far, this implies $[a] = [b]$. Therefore $P = \{[a] \mid a \in S\}$ forms a partitition of S (P is being interpreted as a set, not a multiset). In fact, $E = \bigcup\{[a] \times [a] \mid a \in S\}$. $\square$

From the above proof, we also have

Corollary 1. *For every partition* $P = \{P_i \mid i \in I\}$ *of* S*,* $E = \bigcup\{P_i^2 \mid i \in I\}$ *is an equivalence relation. Conversely, every equivalence relation on* S *can be formed this way.*