signature of a permutation

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Defines	inversion
Defines	signature
Defines	parity
Defines	even permutation
Defines	odd permutation

Let X be a finite set, and let G be the group of permutations of X (see permutation group). There exists a unique homomorphism χ from G to the multiplicative group $\{-1, 1\}$ such that $\chi(t) = -1$ for any transposition (loc. sit.) $t \in G$. The value $\chi(g)$, for any $g \in G$, is called the *signature* or *sign* of the permutation g. If $\chi(g) = 1$, g is said to be of even *parity*; if $\chi(g) = -1$, g is said to be of odd parity.

Proposition: If X is totally ordered by a relation $<$, then for all $g \in G$,

$$\chi(g) = (-1)^{k(g)} \tag{1}$$

where $k(g)$ is the number of pairs $(x, y) \in X \times X$ such that $x < y$ and $g(x) > g(y)$. (Such a pair is sometimes called an *inversion* of the permutation g.)

Proof: This is clear if g is the identity map $X \to X$. If g is any other permutation, then for some *consecutive* $a, b \in X$ we have $a < b$ and $g(a) > g(b)$. Let $h \in G$ be the transposition of a and b. We have

$$\begin{aligned} k(g \circ h) &= k(g) - 1 \\ \chi(g \circ h) &= -\chi(g) \end{aligned}$$

and the proposition follows by induction on $k(g)$.