classes of ordinals and enumerating functions

Canonical name	ClassesOfOrdinalsAndEnumeratingFunctions
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Entry type	Definition
Classification	msc 03F15
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Defines	order type
Defines	enumerating function
Defines	closed
Defines	kappa-closed
Defines	continuous
Defines	kappa-continuous
Defines	continuous function
Defines	kappa-continuous function
Defines	closed class
Defines	kappa-closed class
Defines	normal function
Defines	kappa-normal function
Defines	normal
Defines	kappa-normal
Defines	unbounded
Defines	unbounded clas

A *class of ordinals* is just a subclass of the `http://planetmath.org/Class`class **On** of all ordinals. For every class of ordinals M there is an *enumerating function* f_M defined by transfinite recursion:

$$f_M(\alpha) = \min\{x \in M \mid f(\beta) < x \text{ for all } \beta < \alpha\},$$

and we define the *order type* of M by $\operatorname{otype}(M) = \operatorname{dom}(f)$. The possible values for this value are either **On** or some ordinal α. The above function simply lists the elements of M in order. Note that it is not necessarily defined for all ordinals, although it is defined for a segment of the ordinals. If $\alpha < \beta$ then $f_M(\alpha) < f_M(\beta)$, so f_M is an order isomorphism between $\operatorname{otype}(M)$ and M.

For an ordinal κ, we say M is *κ-closed* if for any $N \subseteq M$ such that $|N| < \kappa$, also $\sup N \in M$.

We say M is *κ-unbounded* if for any $\alpha < \kappa$ there is some $\beta \in M$ such that $\alpha < \beta$.

We say a function $f\colon M \to \mathbf{On}$ is *κ-continuous* if M is κ-closed and

$$f(\sup N) = \sup\{f(\alpha) \mid \alpha \in N\}$$

A function is *κ-normal* if it is order preserving ($\alpha < \beta$ implies $f(\alpha) < f(\beta)$) and continuous. In particular, the enumerating function of a κ-closed class is always κ-normal.

All these definitions can be easily extended to all ordinals: a class is *closed* (resp. *unbounded*) if it is κ-closed (unbounded) for all κ. A function is *continuous* (resp. *normal*) if it is κ-continuous (normal) for all κ.