Kripke submodel

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Let $\mathcal{F} = (W, R)$ be a Kripke frame. A *subframe* of $\mathcal{F}$ is a pair $\mathcal{F}' = (W', R')$ such that W' is a subset of W and $R' = R \cap (W' \times W')$. A *submodel* of a Kripke model $M = (W, R, V)$ of a modal propositional logic PL_M is a triple (W', R', V') where (W', R') is a subframe of (W, R), and $V'(p) = V(p) \cap W'$ for each propositional variable p in PL_M.

The most common submodel of a Kripke model $M = (W, R, V)$ is constructed by taking $W_w := \{u \mid wR^*u\}$, where R^* is the reflexive transitive closure of R, for any $w \in W$. The submodel $M_w := (W_w, R_w, V_w)$ is called the submodel of M *generated* by the world w, and $\mathcal{F}_w$ the subframe of $\mathcal{F}$ generated by w. A submodel of M is called a *generated submodel* if it is generated by some world w in M.

Proposition 1. *For any wff A and any $u \in W_w$, $M \models_u A$ iff $M_w \models_u A$.*

Proof. We do induction on the number n of connectives in A.

If $n = 0$, then A is either $\perp$ or a propositional variable. Clearly, $M \not\models_u \perp$ iff $M_w \not\models_u \perp$, or $M \models_u \perp$ iff $M_w \models_u \perp$. If A is some propositional variable p, then $M \models_u p$ iff $u \in V(p)$ iff $u \in V(p)$ iff $u \in V(p)$ and $u \in W_w$ (by assumption) iff $u \in V(p) \cap M_w$ iff $u \in V_w(p)$ iff $M_w \models_u p$.

Next, suppose A is $B \to C$. Then $M \models_u A$ iff $u \in V(B)^c \cup V(C)$ iff $u \in (W - V(B)) \cup V(C)$ and $u \in W_w$ (by assumption) iff $u \in W_w - V(B)$ or $u \in V_w(C)$ iff $u \in W_w - V_w(B)$ or $u \in V_w(C)$ iff $M_w \models_u A$.

Finally, suppose A is $\Box B$. First let $M \models_u A$. To show $M_w \models_u A$, pick any $v \in W_w$ such that uR_wv. Then uRv, so that $M \models_v B$, or $v \in V(B)$. Since $v \in W_w$, $v \in V_w(B)$, or $M_w \models_v B$, and thus $M_w \models_u A$. Conversely, let $M_w \models_u A$. To show $M \models_u A$, pick any $v \in W$ such that uRv. Since wR^*u, wR^*v so that $v \in W_w$. Furthermore $(u, v) \in R \cap (W_w \times W_w) = R_w$. So $M_w \models_u B$, or $v \in V_w(B) \subseteq V(B)$, which means $M \models_u A$. $\square$

Corollary 1. *$M \models A$ iff $M_w \models A$ for all $w \in W$.*

Proof. If $M \models A$, then $M \models_u A$ for all $u \in W$, so that $M_w \models_u A$ for all $u \in W_w$ and all $w \in W$, or $M_w \models A$ for all $w \in W$. Conversely, if $M_w \models A$ for all $w \in W$, then in particular $M_w \models_w A$ (since R^* is reflexive) for all $w \in W$, or $M \models_w A$ for all $w \in W$, or $M \models A$. $\square$

Corollary 2. *$\mathcal{F} \models A$ iff $\mathcal{F}_w \models A$ for all $w \in W$.*

Proof. If $\mathcal{F} \models A$, then $M \models A$ for all M based on $\mathcal{F}$, or $M_w \models A$, where M_w is based on $\mathcal{F}_w$, for all $w \in W$ by the last corollary. Since any model based on $\mathcal{F}_w$ is of the form M_w, $\mathcal{F}_w \models A$. Conversely, suppose $\mathcal{F}_w \models A$ for all $w \in W$. Let M be any model based on $\mathcal{F}$. Then M_w is based on $\mathcal{F}_w$, and therefore $M_w \models A$. Since w is arbitrary, $M \models A$ by the last corollary, so $\mathcal{F} \models A$. $\square$