models constructed from constants

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The definition of a structure and of the satisfaction relation is nice, but it raises the following question : how do we get models in the first place? The most basic construction for models of first-order theory is the construction that uses constants. Throughout this entry, L is a fixed first-order language.

Let C be a set of constant symbols of L, and T be a theory in L. Then we say C is a *set of witnesses* for T if and only if for every formula φ with at most one free variable x, we have $T \vdash \exists x(\varphi) \Rightarrow \varphi(c)$ for some $c \in C$.

Lemma. Let T is any consistent set of sentences of L, and C is a set of new symbols such that $|C| = |L|$. Let $L' = L \cup C$. Then there is a consistent set $T' \subseteq L'$ extending T and which has C as set of witnesses.

Lemma. If T is a consistent theory in L, and C is a set of witnesses for T in L, then T has a model whose elements are the constants in C.

Proof: Let Σ be the signature for L. If T is a consistent set of sentences of L, then there is a maximal consistent $T' \supseteq T$. Note that T' and T have the same sets of witnesses. As every model of T' is also a model of T, we may assume T is maximal consistent.

We let the universe of $\mathfrak{M}$ be the set of equivalence classes $C/\sim$, where $a \sim b$ if and only if "$a = b$" $\in T$. As T is maximal consistent, this is an equivalence relation. We interpret the non-logical symbols as follows :

1. $[a] =^{\mathfrak{M}} [b]$ if and only if $a \sim b$;

2. Constant symbols are interpreted in the obvious way, i.e. if $c \in \Sigma$ is a constant symbol, then $c^{\mathfrak{M}} = [c]$;

3. If $R \in \Sigma$ is an n-ary relation symbol, then $([a_1], ..., [a_n]) \in R^{\mathfrak{M}}$ if and only if $R(a_1, ..., a_n) \in T$;

4. If $F \in \Sigma$ is an n-any function symbol, then $F^{\mathfrak{M}}([a_0], ..., [a_n]) = [b]$ if and only if "$F(a_1, ..., a_n) = b$" $\in T$.

From the fact that T is maximal consistent, and $\sim$ is an equivalence relation, we get that the operations are well-defined (it is not so simple, i'll write it out later). The proof that $\mathfrak{M} \models T$ is a straightforward induction on the complexity of the formulas of T. $\diamondsuit$

Corollary. (The extended completeness theorem) A set T of formulas of L is consistent if and only if it has a model (regardless of whether or not L has witnesses for T).

Proof: First add a set C of new constants to L, and expand T to T' in such a way that C is a set of witnesses for T'. Then expand T' to a maximal consistent set T''. This set has a model $\mathfrak{M}$ consisting of the constants in C, and $\mathfrak{M}$ is also a model of T. $\diamondsuit$

Corollary. (Compactness theorem) A set T of sentences of L has a model if and only if every finite subset of T has a model.

Proof: Replace "has a model" by "is consistent", and apply the syntactic compactness theorem. $\diamondsuit$

Corollary. (Gödel's completeness theorem) Let T be a consistent set of formulas of L. Then A sentence φ is a theorem of T if and only if it is true in every model of T.

Proof: If φ is not a theorem of T, then $\neg\varphi$ is consistent with T, so $T \cup \{\neg\varphi\}$ has a model $\mathfrak{M}$, in which φ cannot be true. $\diamondsuit$

Corollary. (Downward Löwenheim-Skolem theorem) If $T \subseteq L$ has a model, then it has a model of power at most $|L|$.

Proof: If T has a model, then it is consistent. The model constructed from constants has power at most $|L|$ (because we must add at most $|L|$ many new constants). $\diamondsuit$

Most of the treatment found in this entry can be read in more details in Chang and Keisler's book *Model Theory*.