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computing the Ackermann function

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Recall that the Ackermann function (the modern version) $A(x, y)$ is given by the following double recursive relation:

$$A(0, y) = y+1, \quad A(x+1, 0) = A(x, 1), \quad A(x+1, y+1) = A(x, A(x+1, y)).$$

From the equations above, we see that computing the Ackermann function involves first deciding whether the pair (x, y) is such that

$$x = 0, \quad \text{or} \quad x > 0 \text{ and } y = 0, \quad \text{or} \quad x > 0 \text{ and } y > 0,$$

which dictates which one of the three equations above to use. Let us illustrate this by a simple example: $x = 1$ and $y = 1$:

$$A(1, 1) = A(0, A(1, 0)) = A(0, A(0, 1)) = A(0, 2) = 3.$$

If $x = 2$, then quite a few more steps are involved:

$$\begin{aligned} A(2, 1) &= A(1, A(2, 0)) = A(1, A(1, 1)) = A(1, A(0, A(1, 0))) = A(1, A(0, A(0, 1))) \\ &= A(1, A(0, 2)) = A(1, 3) = A(0, A(1, 2)) = A(0, A(0, A(1, 1))) \\ &= A(0, A(0, A(0, A(1, 0)))) = A(0, A(0, A(0, A(0, 1)))) \\ &= A(0, A(0, A(0, 2))) = A(0, A(0, 3)) = A(0, 4) = 5. \end{aligned}$$

When $x > 2$, computations of $A(x, y)$ becomes unwieldy, mainly due to the number of steps involved, and partially due to the number of A 's and the parentheses that need to be written down.

Nevertheless, based on the computations of $A(1, 1)$ and $A(2, 1)$ above, we see an algorithm emerging for computing $A(x, y)$ in general. First, notice that in each of the expression $A(\dots, \dots)$, the right parentheses all occur at the right end of the expression. Therefore, there is no ambiguity involved if the A 's and the parentheses were removed. Formalizing this notion, we have

Definition. Suppose z is in the range of A . We say that a sequence x_1, \dots, x_n is an *Ackermann sequence* for z if

$$A(x_1, A(x_2, \dots, A(x_{n-1}, x_n) \dots)) = z.$$

In particular, the sequence z of length 1 is an Ackermann sequence for z .

Therefore, computing $A(x, y) = z$ is just a process of transforming the Ackermann sequence x, y to the Ackermann sequence z , for z . Transforming one sequence \mathbf{x} to another sequence \mathbf{x}' can be formalized as follows:

Definition. Suppose \mathbf{x} is a finite non-empty sequence of non-negative integers. A sequence \mathbf{x}' is said to be *immediately derivable* from \mathbf{x} , written $\mathbf{x} \longrightarrow \mathbf{x}'$, if exactly one of the following conditions holds:

1. \mathbf{x} consists of one number, and $\mathbf{x}' = \mathbf{x}$;
2. $\mathbf{x} = \mathbf{y}, 0, z$, and $\mathbf{x}' = \mathbf{y}, z + 1$;
3. $\mathbf{x} = \mathbf{y}, y, 0$, with $y > 0$, and $\mathbf{x}' = \mathbf{y}, y - 1, 1$; or
4. $\mathbf{x} = \mathbf{y}, y, z$, with $y > 0, z > 0$, and $\mathbf{x}' = \mathbf{y}, y - 1, y, z - 1$,

where \mathbf{y} may be the empty sequence.

It is clear that conditions 2-4 correspond to the three equations defining the Ackermann function.

We also write $\mathbf{x} \Longrightarrow \mathbf{x}'$ to mean a finite chain of sequences

$$\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m = \mathbf{x}'$$

such that either $m = 1$, or $m > 1$, and $\mathbf{x}_i \rightarrow \mathbf{x}_{i+1}$ for $i = 1, \dots, m - 1$.

From the definition above, we can also describe the entire computational process rigorously:

1. start with a sequence x, y , call the sequence *derived at step* $k = 0$;
2. If \mathbf{x} is derived at step k , and $\mathbf{x} \rightarrow \mathbf{x}'$, then \mathbf{x}' is derived at step $k + 1$.

For example, the computation of $A(1, 1)$ can be written simply as

$$1, 1 \rightarrow 0, 1, 0 \rightarrow 0, 0, 1 \rightarrow 0, 2 \rightarrow 3 \rightarrow 3 \rightarrow \dots$$

A number of easy consequences of \rightarrow can now be listed:

- if $\mathbf{x} \rightarrow \mathbf{x}'$, then $\mathbf{x} \neq \mathbf{x}'$ unless \mathbf{x} consists of only one number.
- if $\mathbf{x} \rightarrow \mathbf{x}'$, then \mathbf{x} is an Ackermann sequence for z iff \mathbf{x}' is.
- if $\mathbf{x} \rightarrow z$, where $z \in \mathbb{N}$, then \mathbf{x} is an Ackermann sequence for z .
- if $x > 0$, then there is t such that $x, y \Longrightarrow x - 1, t$.
- any pair x, y is an Ackermann sequence for some z .