p-morphism

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Let $\mathcal{F}_1 = (W_1, R_1)$ and $\mathcal{F}_2 = (W_2, R_2)$ be Kripke frames. A *p-morphism* from $\mathcal{F}_1$ to $\mathcal{F}_2$ is a function $f : W_1 \to W_2$ such that

- if uR_1w, then $f(u)R_2f(w)$,
- if sR_2t and $s = f(u)$ for some $u \in W_1$, then uR_1w and $t = f(w)$ for some $w \in W_1$,

We write $f : \mathcal{F}_1 \to \mathcal{F}_2$ to denote that f is a p-morphism from $\mathcal{F}_1$ to $\mathcal{F}_2$.

Let $M_1 = (\mathcal{F}_1, V_1)$ and $M_2 = (\mathcal{F}_2, V_2)$ be Kripke models of modal propositional logic PL_M. A *p-morphism* from M_1 to M_2 is a p-morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ such that

- $M_1 \models_w p$ iff $M_2 \models_{f(w)} p$ for any propositional variable p.

Proposition 1. *For any wff A, $M_1 \models_w A$ iff $M_2 \models_{f(w)} A$.*

Proof. Induct on the number n of logical connectives in A. When $n = 0$, A is either $\perp$ or a propositional variable. The case when A is $\perp$ is obvious, and the other case is definition. Next, suppose A is $B \to C$, then $M_1 \models_w A$ iff $M_1 \not\models_w B$ or $M_1 \models_w C$ iff $M_2 \not\models_{f(w)} B$ or $M_2 \models_{f(w)} C$ iff $M_2 \models_{f(w)} A$. Finally, suppose A is $\Box B$, and $M_1 \models_w A$. To show $M_2 \models_{f(w)} A$, let t be such that $f(w)R_2t$. Then there is a u such that $t = f(u)$ and wR_1u, so that $M_1 \models_u B$. By induction, $M_2 \models_{f(u)} B$, or $M_2 \models_t B$. Hence $M_2 \models_{f(w)} A$. Conversely, suppose $M_2 \models_{f(w)} A$. To show $M_1 \models_w A$, let u be such that wR_1u. So $f(w)R_2f(u)$, and therefore $M_2 \models_{f(u)} B$. By induction, $M_1 \models_u B$, whence $M_1 \models_w A$. $\Box$

Proposition 2. *If a p-morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ is one-to-one, then $\mathcal{F}_2 \models A$ implies $\mathcal{F}_1 \models A$ for any wff A.*

Proof. Suppose $\mathcal{F}_2 \models A$. Let $M = (W_1, R_1, V_1)$ be any model based on $\mathcal{F}_1$ and w any world in W_1. We want to show that $M \models_w A$.

Define a Kripke model $M' := (W_2, R_2, V_2)$ as follows: for any propositional variable p, let $V_2(p) := \{s \in W_2 \mid f^{-1}(s) \cap V_1(p) \neq \varnothing\}$. Then $M \models_w p$ iff $w \in V_1(p)$ iff $f^{-1}(f(w)) = \{w\} \subseteq V_1(p)$ (since f is one-to-one) iff $f^{-1}(f(w)) \cap V_1(p) \neq \varnothing$ iff $f(w) \in V_2(p)$ iff $M' \models_{f(w)} p$. This shows that f is a p-morphism from M to M'.

Now, let $w \in W_1$. Then $M' \models_{f(w)} A$ by assumption. By the last proposition, $M \models_w A$. $\Box$

Proposition 3. *If a p-morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ is onto, then $\mathcal{F}_1 \models A$ implies $\mathcal{F}_2 \models A$ for any wff A.*

Proof. Suppose $\mathcal{F}_1 \models A$. Let $M = (W_2, R_2, V_2)$ be any model based on $\mathcal{F}_2$ and s any world in W_2. We want to show that $M \models_s A$.

Define a Kripke model $M' := (W_1, R_1, V_1)$ as follows: for any propositional variable p, let $V_1(p) := \{w \in W_1 \mid f(w) \in V_2(p)\}$. Then $w \in V_1(p)$ iff $f(w) \in V_2(p)$, so f is a p-morphism from M' to M, and by assumption $M' \models A$ for any wff A.

Now, let $w \in W_1$ be a world such that $f(w) = s$. Since $M' \models A$, $M' \models_w A$ in particular, and therefore $M \models_{f(w)} A$ or $M \models_s A$ by the last proposition. $\square$

Corollary 1. *If $f : \mathcal{F}_1 \to \mathcal{F}_2$ is bijective, then $\mathcal{F}_1 \models A$ iff $\mathcal{F}_2 \models A$ for any wff A.*

A frame $\mathcal{F}'$ is said to be a p-morphic image of a frame $\mathcal{F}$ if there is an onto p-morphism $f : \mathcal{F} \to \mathcal{F}'$. Let $\mathcal{C}$ be the class of all frames validating a wff. Then by the third proposition above, $\mathcal{C}$ is closed under p-morphic images: if a frame is in $\mathcal{C}$, so is any of its p-morphic images. Using this property, we can show the following: if $\mathcal{C}$ is the class of all frames validating a wff A, then $\mathcal{C}$ can not be

- the class of all irreflexive frames
- the class of all asymetric frames
- the class of all anti-symmetric frames

Proof. Let $\mathcal{F}_1 = (\mathbb{N}, <)$ and $\mathcal{F}_2 = (\{0\}, R)$, where $0R0$. Notice that $\mathcal{F}_1$ is in both the class of irreflexive frames and the class of asymetric frames, but $\mathcal{F}_2$ is in neither. Let $f : \mathbb{N} \to \{0\}$ be the obvious surjection. Clearly, $m < n$ implies $f(m)Rf(n)$. Also, if $f(m)R0$, then $f(m)Rf(m+1)$. So f is a p-morphism. Suppose $\mathcal{C}$ is either the class of all irreflexive frames or the class of all asymetric frames. If A is validated by $\mathcal{C}$, A is validated by $\mathcal{F}_1$ in particular (since $\mathcal{F}_1$ is in $\mathcal{C}$), so that A is validated by $\mathcal{F}_2$ as well, which means $\mathcal{F}_2$ is $\mathcal{C}$ too, a contradiction. Therefore, no such an A exists.

Next, let $\mathcal{F}_3 = (\mathbb{N}, S)$, where $nS(n+1)$ for all $n \in \mathbb{N}$ and $\mathcal{F}_4 = (\{0,1\}, R)$, where $R = \{(0,1), (1,0)\}$. Let $\mathcal{C}$ be the class of all anti-symmetric frames. Then $\mathcal{F}_3$ is in $\mathcal{C}$ but $\mathcal{F}_4$ is not. Let $f : \mathcal{F}_3 \to \mathcal{F}_4$ be given by $f(n) = 0$ if n

is even and $f(n) = 1$ if n is odd. If aSb, then $f(a)$ and $f(b)$ differ by 1, so $f(a)Rf(b)$. On the other hand, if $f(a)Rx$, then x is either 0 or 1, depending on whether a odd or even. Pick $b = a + 1$, so aSb and $f(b) = x$. This shows that f is a p-morphism. By the same argument as in the last paragraph, no wff A is validated by precisely the members of $\mathcal{C}$. $\square$