intersection structure

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Intersection structures

An *intersection structure* is a set C such that

1. C is a subset of the powerset $P(A)$ of some set A, and

2. intersection of a non-empty family $\mathcal{F}$ of elements of C is again in C.

If order C by set inclusion, then C becomes a poset.

There are numerous examples of intersection structures. In algebra, the set of all subgroups of a group, the set of all ideals of a ring, and the set of all subspaces of a vector space. In topology, the set of all closed sets of a topological space is an intersection structure. Finally, in functional analysis, the set of all convex subsets of a topological vector space is also an intersection structure.

The set of all partial orderings on a set is also an intersection structure. A final example can be found in domain theory: let C be the set of all partial functions from a non-empty set X to a non-empty set Y. Since each partial function is a subset of $X \times Y$, C is a subset of $P(X \times Y)$. Let $\mathcal{F} := \{f_i \mid i \in I\}$ be an arbitrary collection of partial functions in C and $f = \bigcap \mathcal{F}$. f is clearly a relation between X and Y. Suppose x is in the domain of f. Let $E = \{y \in Y \mid xfy\}$. Then xf_iy for each f_i where x is in the domain of f_i. Since f_i is a partial function, $y = f_i(x)$, so that y is uniquely determined. This means that E is a singleton, hence f is a partial function, so that $\bigcap \mathcal{F} \in C$, meaning that C is an intersection structure.

The main difference between the last two examples and the previous examples is that in the last two examples, C is rarely a complete lattice. For example, let $\leq$ be a partial ordering on a set P. Then its dual $\leq^\partial$ is also a partial ordering on P. But the join of $\leq$ and $\leq^\partial$ does not exist. Here is another example: let $X = \{1\}$ and $Y = \{2, 3\}$. Then $C = \{\varnothing, (1, 2), (1, 3)\}$. $(1, 2)$ and $(1, 3)$ are the maximal elements of C, but the join of these two elements does not exist.

Topped intersection strucutres

If, in condition 2 above, we remove the requirement that $\mathcal{F}$ be non-empty, then we have an intersection structure called a *topped intersection structure*.

The reason for calling them topped is because the top element of such an intersection structure always exists; it is the intersection of the empty family.

In addition, a topped intersection structure is always a complete lattice. For a proof of this fact, see this http://planetmath.org/CriteriaForAPosetToBeACompleteLatticelin

As a result, for example, to show that the subgroups of a group form a complete lattice, it is enough to observe that arbitrary intersection of subgroups is again a subgroup.

Remarks.

- A topped intersection structure is also called a *closure system*. The reason for calling this is that every topped intersection structure $C \subseteq P(X)$ induces a closure operator cl on $P(X)$, making X a closure space. $\mathrm{cl}: P(X) \to P(X)$ given by

  $$\mathrm{cl}(A) = \bigcap \{B \in C \mid A \subseteq B\}$$

  is well-defined.

- Conversely, it is not hard to see that every closure space (X, cl) gives rise to a closure system $C := \{\mathrm{cl}(A) \mid A \in P(X)\}$.

- An intersection structure C is said to be *algebraic* if for every directed set $B \subseteq C$, we have that $\bigcup B \in C$. All of the examples above, except the set of closed sets in a topological space, are algebraic intersection structures. A topped intersection structure that is algebraic is called an *algebraic closure system* if,

- Every algebraic closure system is an algebraic lattice.

References

[1] B. A. Davey, H. A. Priestley, *Introduction to Lattices and Order*, 2nd Edition, Cambridge (2003)

[2] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).