modal logic GL

Canonical name	ModalLogicGL
Date of creation	2013-03-22 19:35:33
Last modified on	2013-03-22 19:35:33
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	19
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03B45
Classification	msc 03F45

The modal logic **GL** (after Gödel and Löb) is the smallest normal modal logic containing the following schema:

- W: $\Box(\Box A \to A) \to \Box A$.

GL is also known as *provability logic*, because it is used to study the provability and consistency of first order Peano arithmetic.

Recall that 4 is the schema $\Box A \to \Box\Box A$.

Proposition 1. *In any normal modal logic, $\vdash W$ implies $\vdash 4$.*

The proof of this requires some http://planetmath.org/SomeTheoremSchemasOfNormalModalL and http://planetmath.org/SyntacticPropertiesOfANormalModalLogicmeta-theorems of a normal modal logic.

Proof. We start with the tautology $A \to ((\Box\Box A \wedge \Box A) \to (\Box A \wedge A))$, which an instance of the schema $X \to ((Y \wedge Z) \to (Z \wedge X))$. Since $\Box(\Box A \wedge A) \leftrightarrow \Box\Box A \wedge \Box A$ is a theorem in any normal modal logic, $A \to (\Box(\Box A \wedge A) \to (\Box A \wedge A))$ is a theorem by the substitution theorem. By the syntactic property RM, $\Box A \to \Box(\Box(\Box A \wedge A) \to (\Box A \wedge A))$ is a theorem. Since $\Box(\Box(\Box A \wedge A) \to (\Box A \wedge A)) \to \Box(\Box A \wedge A)$ is an instance of W, by law of syllogism, $\Box A \to \Box(\Box A \wedge A)$ is a theorem.

Next, from the tautology $\Box A \wedge A \to \Box A$, we have the theorem $\Box(\Box A \wedge A) \to \Box\Box A$ by RM. Combining this with the last theorem in the previous paragraph, we see that, by law of syllogism, $\Box A \to \Box\Box A$, or 4, is a theorem. $\Box$

Corollary 1. *4 is a theorem of **GL**.*

A binary relation is said to be *converse well-founded* iff its inverse is well-founded.

Proposition 2. *W is valid in a frame $\mathcal{F}$ iff $\mathcal{F}$ is transitive and converse well-founded.*

Proof. Suppose first that the schema W is valid in $\mathcal{F} = (U, R)$, then any theorem of **GL** is valid in $\mathcal{F}$, so in particular 4 is valid in $\mathcal{F}$, and hence $\mathcal{F}$ is transitive (see http://planetmath.org/ModalLogicS4here). We next show that R is converse well-founded. Suppose not. Then there is a non-empty subset $S \subseteq U$ such that S has no R-maximal element. We want to find a model (U, R, V) such that, for some propositional variable p and some world

u in U, $\not\models_u \Box(\Box p \to p) \to \Box p$, or equivalently, $\models_u \Box(\Box p \to p)$ and $\not\models_u \Box p$. Let V be the valuation such that $V(p) := \{w \in U \mid w \notin S\}$. Pick any $u \in S$. Suppose uRv. To show that $\models_u \Box(\Box p \to p)$, we want to show that $\models_v \Box p \to p$. There are two cases:

- If $v \in S$, then $\not\models_v p$. Furthermore, since S does not contain an R-maximal element, there is a $w \in S$ such that vRw. Since $w \in S$, $\not\models_w p$. Since vRw, $\not\models_v \Box p$. As a result, $\models_v \Box p \to p$.

- If $v \notin S$, then $\models_v p$, so that $\models_v \Box p \to p$.

Next, we want to show that $\not\models_u \Box p$. Since $u \in S$, and S does not have an R-maximal element, there is a $w \in S$ such that uRw. Since $w \in S$, $\not\models_w p$. But since uRw, $\not\models_u \Box p$.

Conversely, let $\mathcal{F}$ be a transitive and converse well-founded frame, M a model based on $\mathcal{F}$, and u a world in M. We want to show that $\models_u \Box(\Box p \to p) \to \Box p$. So suppose $\not\models_u \Box p$. Then the set $S := \{v \mid uRv \text{ and } \not\models_v p\}$ is not empty. Since R is converse well-founded, S has a R-maximal element, say w. So uRw and $\not\models_w p$. Now, if $\models_w \Box p \to p$, then $\not\models_w \Box p$, which means there is a v such that wRv and $\not\models_v p$. But since R is transitive and uRw, we get uRv, implying $v \in S$, contradicting the R-maximality of w. Therefore, $\not\models_w \Box p \to p$, or $\not\models_u \Box(\Box p \to p)$. As a result, $\models_u \Box(\Box p \to p) \to \Box p$. $\Box$

Proposition 2 immediately implies

Corollary 2. ***GL*** *is sound in the class of transitive and converse well-founded frames.*

Remark. However, unlike many other modal logics, **GL** *is not* complete in the class of transitive and converse well-founded frames. While its canonical model (hence the corresponding canonical frame) is transitive (because 4 is valid in it), it is not converse well-founded.

Instead, it can be shown that **GL** *is* complete in the restricted class of finite transitive and converse well-founded frames, or equivalently, finite transitive and irreflexive frames.