every ϵ-automaton is equivalent to an automaton

Canonical name	EveryepsilonautomatonIsEquivalentToAnAutomaton
Date of creation	2013-03-22 19:02:03
Last modified on	2013-03-22 19:02:03
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	8
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03D05
Classification	msc 68Q45

In this entry, we show that an automaton with `http://planetmath.org/EpsilonTransition`ϵ-transitions is no more power than one without. Having ϵ-transitions is purely a matter of convenience.

Proposition 1. *Every* ***`http://planetmath.org/EpsilonAutomaton`***ϵ*-automaton is equivalent to an automaton.*

For the proof, we use the following setup (see the parent entry for more detail):

- $E = (S, \Sigma, \delta, I, F, \epsilon)$ is an ϵ-automaton, and E_ϵ is the automaton associated with E,
- $h : (\Sigma \cup \{\epsilon\})^* \to \Sigma^*$ is the homomorphism that erases ϵ (it takes ϵ to the empty word, also denoted by ϵ). From the parent entry, $L(E) := h(L(E_\epsilon))$.

Proof. Define a function $\delta_1 : S \times \Sigma \to P(S)$, as follows: for each pair $(s, a) \in S \times \Sigma$, let

$$\delta_1(s, a) = \bigcup \{\, \delta(s, u) \mid h(u) = a \,\}.$$

In other words, $\delta_1(s, a)$ is the set of all states reachable from s by words of the form $\epsilon^m a \epsilon^n$. As usual, we extend δ_1 so its domain is $S \times \Sigma^*$. By abuse of notation, we use δ_1 again for this extension. First, we set $\delta_1(s, \epsilon) := \{s\}$.

Then we inductively define $\delta_1(s, ua) = \delta_1(\delta_1(s, u), a)$. Using induction,

$$\begin{aligned}
\delta_1(s, ua) &= \delta_1(\delta_1(s, u), a) \\
&= \delta_1(\bigcup_{h(v)=u} \delta(s, v), a) \\
&= \bigcup_{h(v)=u} \delta_1(\delta(s, v), a) \\
&= \bigcup_{h(v)=u} \bigcup_{t\in\delta(s,v)} \delta_1(t, a) \\
&= \bigcup_{h(v)=u} \bigcup_{t\in\delta(s,v)} \bigcup_{h(w)=a} \delta(t, w) \\
&= \bigcup_{h(v)=u} \bigcup_{h(w)=a} \bigcup_{t\in\delta(s,v)} \delta(t, w) \\
&= \bigcup_{h(v)=u} \bigcup_{h(w)=a} \delta(\delta(s, v), w) \\
&= \bigcup_{h(v)=u} \bigcup_{h(w)=a} \delta(s, vw) \\
&= \bigcup\{\delta(s, vw) \mid h(v) = u \text{ and } h(w) = a\} \\
&= \bigcup\{\delta(s, x) \mid h(x) = ua\}
\end{aligned}$$

So for any non-empty word u, we have the following equation:

$$\delta_1(s, u) = \bigcup\{\, \delta(s, v) \mid h(v) = u \,\}. \tag{1}$$

In other words, if $u = a_1a_2 \cdots a_n$, then $\delta_1(s, u)$ is the set of all states reachable from s by words of the form

$$\epsilon^{i_0}a_1\epsilon^{i_1}a_2\epsilon^{i_2} \cdots \epsilon^{i_{n-1}}a_n\epsilon^{i_n}. \tag{2}$$

Now, define A to be the automaton $(S, \Sigma, \delta_1, I, F)$. Then, from equation (1) above, a word

$$u = a_1a_2 \cdots a_n$$

is accepted by A iff some word v of the form (2) is accepted by E_ϵ iff $u = h(v)$ is accepted by E, proving the proposition. $\square$

Remark. Another approach is to use the concept of http://planetmath.org/EpsilonClosure closure. The proof is very similar to the one given above, and the resulting equivalent automaton is a DFA.