filter

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Entry type	Definition
Classification	msc 03E99
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Related topic	Ultrafilter
Related topic	KappaComplete
Related topic	KappaComplete2
Related topic	Net
Related topic	LimitAlongAFilter
Related topic	UpperSet
Related topic	OrderIdeal
Defines	principal filter
Defines	nonprincipal filter
Defines	non-principal filter
Defines	free filter
Defines	fixed filter
Defines	neighbourhood filter
Defines	principal element
Defines	convergent filter

Let X be a set. A filter on X is a set $\mathbb{F}$ of subsets of X such that

- $X \in \mathbb{F}$
- The intersection of any two elements of $\mathbb{F}$ is an element of $\mathbb{F}$.
- $\emptyset \notin \mathbb{F}$ (some authors do not include this axiom in the definition of filter)
- If $F \in \mathbb{F}$ and $F \subset G \subset X$ then $G \in \mathbb{F}$.

The first two axioms can be replaced by one:

- Any finite intersection of elements of $\mathbb{F}$ is an element of $\mathbb{F}$.

with the usual understanding that the intersection of an empty family of subsets of X is the whole set X.

A filter $\mathbb{F}$ is said to be *fixed* or *principal* if there is $F \in \mathbb{F}$ such that no proper subset of F belongs to $\mathbb{F}$. In this case, $\mathbb{F}$ consists of all subsets of X containing F, and F is called a *principal element* of $\mathbb{F}$. If $\mathbb{F}$ is not principal, it is said to be *non-principal* or *free*.

If x is any point (or any subset) of any topological space X, the set $\mathcal{N}_x$ of neighbourhoods of x in X is a filter, called the *neighbourhood filter* of x. If $\mathbb{F}$ is any filter on the space X, $\mathbb{F}$ is said to *converge* to x, and we write $\mathbb{F} \to x$, if $\mathcal{N}_x \subset \mathbb{F}$. If every neighbourhood of x meets every set of $\mathbb{F}$, then x is called an *accumulation point* or *cluster point* of $\mathbb{F}$.

Remarks: The notion of filter (due to H. Cartan) has a simplifying effect on various proofs in analysis and topology. Tychonoff's theorem would be one example. Also, the two kinds of limit that one sees in elementary real analysis – the limit of a sequence at infinity, and the limit of a function at a point – are both special cases of the limit of a filter: the Fréchet filter and the neighbourhood filter respectively. The notion of a Cauchy sequence can be extended with no difficulty to any uniform space (but not just a topological space), getting what is called a Cauchy filter; any convergent filter on a uniform space is a Cauchy filter, and if the converse holds then we say that the uniform space is *complete*.