equivalence of Zorn's lemma and the axiom of choice

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Let X be a set partially ordered by $<$ such that each chain has an upper bound. Define $p(x) = \{y \in X \mid x < y\} \in P(X)$. Let $p(X) = \{p(x) \mid x \in X\}$. If $p(x) = \emptyset$ then it follows that x is maximal.

Suppose no $p(x) = \emptyset$. Then by the `http://planetmath.org/AxiomOfChoice`axiom of choice there is a choice function f on $p(X)$, and since for each $p(x)$ we have $f(p(x)) \in p(x)$, it follows that $x < f(p(x))$. Define $f_\alpha(p(x))$ for all ordinals α by transfinite induction:

$f_0(p(x)) = x$

$f_{\alpha+1}(p(x)) = f(p(f_\alpha(p(x))))$

And for a limit ordinal α, let $f_\alpha(p(x))$ be an upper bound of $f_i(p(x))$ for $i < \alpha$.

This construction can go on forever, for any ordinal. Then we can easily construct an injective function from Ord to X by $g(\alpha) = f_\alpha(p(x))$ for an arbitrary $x \in X$. This must be injective, since $\alpha < \beta$ implies $g(\alpha) < g(\beta)$. But that requires that X be a proper class, in contradiction to the fact that it is a set. So there can be no such choice function, and there must be a maximal element of X.

For the reverse, assume Zorn's lemma and let C be any set of non-empty sets. Consider the set of functions $F = \{f \mid \forall a \in \mathrm{dom}(f)(a \in C \wedge f(a) \in a)\}$ partially ordered by inclusion. Then the union of any chain in F is also a member of F (since the union of a chain of functions is always a function). By Zorn's lemma, F has a maximal element f, and since any function with domain smaller than C can be easily expanded, $\mathrm{dom}(f) = C$, and so f is a choice function for C.