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partial algebraic system

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Defines	partial operation
Defines	partial groupoid

Let λ be a cardinal. A partial function $f : A^\lambda \to A$ is called a *partial operation* on A. λ is called the arity of f. When λ is finite, f is said to be *finitary*. Otherwise, it is *infinitary*. A nullary partial operation is an element of A and is called a constant.

Definition. A *partial algebraic system* (or *partial algebra* for short) is defined as a pair (A, O), where A is a set, usually non-empty, and called the underlying set of the algebra, and O is a set of finitary partial operations on A. The partial algebra (A, O) is sometimes denoted by $\boldsymbol{A}$.

Partial algebraic systems sit between algebraic systems and relational systems; they are generalizations of algebraic systems, but special cases of relational systems.

The *type* of a partial algebra is defined exactly the same way as that of an algebra. When we speak of a partial algebra $\boldsymbol{A}$ of type τ, we typically mean that $\boldsymbol{A}$ is *proper*, meaning that the partial operation $f_{\boldsymbol{A}}$ is non-empty for every function symbol $f \in \tau$, and if f is a constant symbol, $f_{\boldsymbol{A}} \in A$.

Below is a short list of partial algebras.

1. Every algebraic system is automatically a partial algebraic system.

2. A division ring $(D, \{+,\ \cdot,\ -,\ ^{-1},\ 0,\ 1\})$ is a prototypical example of a partial algebra that is not an algebra. It has type $\langle 2, 2, 1, 1, 0, 0\rangle$. It is not an algebra because the unary operation $^{-1}$ (multiplicative inverse) is only partial, not defined for 0.

3. Let A be the set of all non-negative integers. Let "$-$" be the ordinary subtraction. Then $(A, \{-\})$ is a partial algebra.

4. A *partial groupoid* is a partial algebra of type $\langle 2\rangle$. In other words, it is a set with a partial binary operation (called the product) on it. For example, a small category may be viewed as a partial algebra. The product ab is only defined when the source of a matches with the target of b. Special types of small categories are `http://planetmath.org/GroupoidCategoryTheoretic` (category theoretic), and Brandt groupoids, all of which are partial.

5. A small category can also be thought of as a partial algebra of type $\langle 2, 1, 1\rangle$, where the two (total) unary operators are the source and target operations.

Remark. Like algebraic systems, one can define subalgebras, direct products, homomorphisms, as well as congruences in partial algebras.

References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).