syntactic properties of a normal modal logic

Canonical name	SyntacticPropertiesOfANormalModalLogic
Date of creation	2013-03-22 19:34:18
Last modified on	2013-03-22 19:34:18
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	24
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03B45

Recall that a normal modal logic is a logic containing all tautologies, the schema K

$$\square(A \to B) \to (\square A \to \square B),$$

and closed under modus ponens and necessitation rules. Also, the modal operator diamond $\diamond$ is defined as

$$\diamond A := \neg\square\neg A.$$

Let Λ be any normal modal logic. We write $\vdash A$ to mean $\Lambda \vdash A$, or wff $A \in \Lambda$, or A is a theorem of Λ. In addition, for any set Δ, $\Delta \vdash A$ means there is a finite sequence of wff's such that each wff is either a theorem, a member of Δ, or obtained either by modus ponens or necessitation from earlier wff's in the sequence, and A is the last wff in the sequence. The sequence is called a deduction (of A) from Δ.

Below are some useful meta-theorems of Λ:

1. (RM) $\vdash A \to B$ implies $\vdash \square A \to \square B$

   Proof. By assumption and by necessitation, $\vdash \square(A \to B)$, by schema K and by modus ponens, we have the result. $\square$

2. As a result, $\vdash A \leftrightarrow B$ implies $\vdash \square A \leftrightarrow \square B$.

3. (substitution theorem). If $\vdash B_i \leftrightarrow C_i$ for $i = 1, \ldots, m$, then

   $$\vdash A[\overline{B}/\overline{p}] \leftrightarrow A[\overline{C}/\overline{p}],$$

   where $\overline{p} := (p_1, \ldots, p_m)$ is the tuple of all the propositional variables in A listed in order.

   Proof. For most of the proof, consult `http://planetmath.org/SubstitutionTheoremForPr` entry for more detail. What remains is the case when A has the form $\square D$. We do induction on the number n of $\square$'s in A. The case when $n = 0$ means that A is a wff of PL_c, and has already been proved. Now suppose A has $n+1$ $\square$'s. Then D has n $\square$'s, and so by induction, $\vdash D[B/p] \leftrightarrow D[C/p]$, and therefore $\vdash \square D[B/p] \leftrightarrow \square D[C/p]$ by 2. This means that $\vdash A[B/p] \leftrightarrow A[C/p]$. $\square$

4. $\vdash A \to B$ implies $\vdash \diamond A \to \diamond B$

Proof. By assumption, tautology $\vdash (A \to B) \to (\neg B \to \neg A)$, and modus ponens, we get $\vdash \neg B \to \neg A$. By 1, $\vdash \Box\neg B \to \Box\neg A$. By another instance of the above tautology and modus ponens, and the definition of $\diamond$, we get the result. $\Box$

5. $\vdash A \vee B$ implies $\vdash \diamond A \vee \Box B$

   Proof. Since $\vdash A \vee B \leftrightarrow (\neg A \to B)$, we have $\vdash \neg A \to B$, so $\vdash \Box\neg A \to \Box B$. By the tautology $C \leftrightarrow \neg\neg C$, we have $\vdash \neg\neg\Box\neg A \to \Box B$, or $\vdash \neg \diamond A \to \Box B$, and therefore $\vdash \diamond A \vee \Box B$. $\Box$

6. (RR) $\vdash A \wedge B \to C$ implies $\vdash \Box A \wedge \Box B \to \Box C$

   Proof. By assumption and 1, $\vdash \Box(A \wedge B) \to \Box C$. Since $\Box A \wedge \Box B \to \Box(A \wedge B)$ is a theorem (see `http://planetmath.org/SomeTheoremSchemasOfNormalModalLo` we get $\vdash \Box A \wedge \Box B \to \Box C$ by the law of syllogism. $\Box$

7. (RK) More generally, $\vdash A_1 \wedge \cdots \wedge A_n \to A$ implies $\vdash \Box A_1 \wedge \cdots \Box A_n \to \Box A$, where the case $n = 0$ is the necessitation rule.

   Proof. Cases $n = 1, 2$ are meta-theorems 1 and 6. If $\vdash A_1 \wedge \cdots \wedge A_n \wedge A_{n+1} \to A$, or $\vdash (A_1 \wedge \cdots \wedge A_n) \wedge A_{n+1} \to A$, then $\vdash \Box(A_1 \wedge \cdots \wedge A_n) \wedge \Box A_{n+1} \to \Box A$ by 6. But $\vdash \Box(A_1 \wedge \cdots \wedge A_n) \leftrightarrow \Box A_1 \wedge \cdots \wedge \Box A_n$, the result follows. $\Box$

8. Define a function s on $\{\neg, \lambda\}$, where λ is the empty word, such that $s(\neg) = \lambda$, the empty word, and $s(\lambda) = \neg$. Then for any wff A, and $\epsilon_1, \epsilon_2 \in \{\neg, \lambda\}$:
$$\vdash \epsilon_1 \Box^n \epsilon_2 A \leftrightarrow s(\epsilon_1) \diamond^n s(\epsilon_2) A.$$

   Technically speaking, this is really an infinite collection of theorem schemas.

   Proof. We will check the case when $\epsilon_1 = \neg$ and $\epsilon_2 = \lambda$ and leave the rest to the reader. We do induction on n. If $n = 0$, then we have the tautology $\neg A \leftrightarrow \neg A$. Suppose $\vdash \neg\Box^n A \leftrightarrow \diamond^n \neg A$. Then $\vdash \neg\Box^{n+1} A \leftrightarrow \diamond^n \neg\Box A$, by applying the induction case on wff $\Box A$. Since $A \leftrightarrow \neg\neg A$ is a tautology, $\vdash \neg\Box^{n+1} A \leftrightarrow \diamond^n \neg\Box\neg\neg A$ by the substitution theorem. By the definition of $\diamond$, we have $\vdash \neg\Box^{n+1} A \leftrightarrow \diamond^{n+1} \neg A$. $\Box$

9. Let $\Box\Delta := \{\Box A \mid A \in \Delta\}$. Then $\Delta \vdash A$ implies $\Box\Delta \vdash \Box A$.

   Proof. Induct on the length n of deduction of A from Δ. If $n = 0$, then either $\vdash A$, in which case $\vdash \Box A$ by necessitation, or $A \in \Delta$, in which case $\Box A \in \Box\Delta$. In either case, $\Box\Delta \vdash \Box A$. Next suppose the property holds for all deductions of length n, and there is a deduction $\mathcal{E}$ of A of length $n+1$. If A is obtained from Δ by necessitation, say A is $\Box B$, where B is in $\mathcal{E}$, then a subsequence of $\mathcal{E}$ is a deduction of B of length $\leq n$, from Δ. So by induction, $\Box\Delta \vdash \Box B$, or $\Box\Delta \vdash A$. By necessitation, $\Box\Delta \vdash \Box A$. Finally, if A is obtained by modus ponens, then there is a wff B such that $B, B \to A$ are both in $\mathcal{E}$. By induction, $\Box\Delta \vdash \Box B$ and $\Box\Delta \vdash \Box(B \to A)$, which, by K and modus ponens, $\Box\Delta \vdash \Box B \to \Box A$, and as a result, $\Box\Delta \vdash \Box A$ by modus ponens. $\Box$

Noticeably absent is the deduction theorem, for the necessitation rule says $A \vdash \Box A$, but this does not imply $\vdash A \to \Box A$. In fact, the wff $A \to \Box A$ is not a theorem in general, unless of course the logic includes the entire schema. All we can say is the following:

10. (deduction theorem) If $\Delta, A \vdash B$ and B is not of the form $\Box C$, then $\Delta \vdash A \to B$.

Remark. It can be shown that conversely, if a modal logic obeys metatheorem 7 above as an inference rule, then it is normal. For more detail, see `http://planetmath.org/EquivalentFormulationsOfNormality`here.