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natural numbers are well-ordered

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In many proofs, one needs the following property of positive and nonnegative integers:

Theorem. Any non-empty set of natural numbers contains a least number.

Proof. Let A be an arbitrary non-empty subset of $\mathbb{N}$. Denote

$$C \;=\; \{x \in \mathbb{N} \vdots\; x \leq a \;\forall a \in A\}.$$

Then of course, $0 \in C$. There exists surely an element c of C such that $c{+}1 \notin C$, since otherwise the induction property would imply that $C = \mathbb{N}$. Because $c{+}1 \notin C$, there is a number a_0 of the set A such that $a_0 < c{+}1$. On the other , we must have $c \leq a_0$. Consequently, $c = a_0$ and therefore

$$a_0 \;=\; c \;\leq\; a \;\; \forall a \in A.$$

Hence, A has the least number a_0. Q.E.D.