satisfaction relation

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Alfred Tarski was the first mathematician to give a formal definition of what it means for a formula to be "true" in a structure. To do this, we need to provide a meaning to terms, and truth-values to the formulas. In doing this, free variables cause a problem : what value are they going to have ? One possible answer is to supply temporary values for the free variables, and define our notions in terms of these temporary values.

Let $\mathcal{A}$ be a structure with signature τ. Suppose $\mathcal{I}$ is an interpretation, and σ is a function that assigns elements of A to variables, we define the function $\mathrm{Val}_{\mathcal{I},\sigma}$ inductively on the construction of terms :

$$\begin{aligned}
\mathrm{Val}_{\mathcal{I},\sigma}(c) &= \mathcal{I}(c) \qquad c \text{ a constant symbol} \\
\mathrm{Val}_{\mathcal{I},\sigma}(x) &= \sigma(x) \qquad x \text{ a variable} \\
\mathrm{Val}_{\mathcal{I},\sigma}(f(t_1, ..., t_n)) &= \mathcal{I}(f)(\mathrm{Val}_{\mathcal{I},\sigma}(t_1), ..., \mathrm{Val}_{\mathcal{I},\sigma}(t_n)) \qquad f \text{ an } n\text{-ary function symbol}
\end{aligned}$$

Now we are set to define satisfaction. Again we have to take care of free variables by assigning temporary values to them via a function σ. We define the relation $\mathcal{A}, \sigma \models \varphi$ by induction on the construction of formulas :

$$\begin{aligned}
\mathcal{A}, \sigma &\models t_1 = t_2 \text{ if and only if } \mathrm{Val}_{\mathcal{I},\sigma}(t_1) = \mathrm{Val}_{\mathcal{I},\sigma}(t_2) \\
\mathcal{A}, \sigma &\models R(t_1, ..., t_n) \text{ if and only if } (\mathrm{Val}_{\mathcal{I},\sigma}(t_1), ..., \mathrm{Val}_{\mathcal{I},\sigma}(t_1)) \in \mathcal{I}(R) \\
\mathcal{A}, \sigma &\models \neg\varphi \text{ if and only if } \ \mathcal{A}, \sigma \not\models \varphi \\
\mathcal{A}, \sigma &\models \varphi \vee \psi \text{ if and only if either } \mathcal{A}, \sigma \models \psi \text{ or } \mathfrak{A}, \sigma \models \psi \\
\mathcal{A}, \sigma &\models \exists x.\varphi(x) \text{ if and only if for some } \ a \in A,\ \mathcal{A}, \sigma[x/a] \models \varphi
\end{aligned}$$

Here

$$\sigma[x/a](y) \begin{cases} a & \text{if} \quad x = y \\ \sigma(y) & \text{else.} \end{cases}$$

In case for some φ of L, we have $\mathcal{A}, \sigma \models \varphi$, we say that $\mathcal{A}$ **models**, or **is a model of**, or **satisfies** φ. If φ has the free variables $x_1, ..., x_n$, and $a_1, ..., a_n \in A$, we also write $\mathcal{A} \models \varphi(a_1, ..., a_n)$ or $\mathcal{A} \models \varphi(a_1/x_1, ..., a_n/x_n)$ instead of $\mathcal{A}, \sigma[x_1/a_1] \cdots [x_n/a_n] \models \varphi$. In case φ is a sentence (formula with no free variables), we write $\mathcal{A} \models \varphi$.