well-founded induction

Canonical name	WellfoundedInduction
Date of creation	2013-03-22 12:42:17
Last modified on	2013-03-22 12:42:17
Owner	ratboy (4018)
Last modified by	ratboy (4018)
Numerical id	22
Author	ratboy (4018)
Entry type	Theorem
Classification	msc 03B10
Related topic	PrincipleOfFiniteInduction

Definition. Let S be a non-empty set, and R be a binary relation on S. Then R is said to be a **well-founded** relation if and only if every nonempty subset $X \subseteq S$ has an `http://planetmath.org/RMinimalElement`R-minimal element. When R is well-founded, we also call the underlying set S well-founded.

Note that R is by no means required to be a total order, or even a partial order. When R is a partial order, then R-minimality is the same as minimality (of the partial order). A classical example of a well-founded set that is not totally ordered is the set $\mathbb{N}$ of natural numbers ordered by division, i.e. aRb if and only if a divides b, and $a \neq 1$. The R-minimal elements of $\mathbb{N}$ are the prime numbers.

Let Φ be a property defined on a well-founded set S. The principle of well-founded induction states that if the following is true :

1. Φ is true for all the R-minimal elements of S
2. for every a, if for every x such that xRa, we have $\Phi(x)$, then we have $\Phi(a)$

then Φ is true for every $a \in S$.

As an example of application of this principle, we mention the proof of the fundamental theorem of arithmetic : every natural number has a unique factorization into prime numbers. The proof goes by well-founded induction in the set $\mathbb{N}$ ordered by division.