complete ultrafilter and partitions

Canonical name	CompleteUltrafilterAndPartitions
Date of creation	2013-03-22 18:55:52
Last modified on	2013-03-22 18:55:52
Owner	yesitis (13730)
Last modified by	yesitis (13730)
Numerical id	4
Author	yesitis (13730)
Entry type	Definition
Classification	msc 03E02

If U is an ultrafilter on a set S, then

U *is κ-complete $\Leftrightarrow$ there is no partition of S into κ-many pieces for which each piece X_α of the partition is not in U.*

We prove the case of σ-completeness; the case of arbitrary infinite cardinality follows closely. For the $\Rightarrow$ direction, let P be a partition of S into ω many pieces, all of which do not belong to U, and write $S = \bigcup_{n=1}^{\omega} X_n$ to illustrate this partition. Now, $\varnothing = S^{\complement} = \bigcap_{n=1}^{\omega} X_n^{\complement}$. Since, by our assumption, each of the X_n do not belong to U, we have $X_n^{\complement} \in U$ for each $n < \omega$ as U is an ultrafilter. Thus, $\left(\bigcap_{n=1}^{\omega} X_n^{\complement}\right) \in U$ by σ-completeness. This, however, means $\varnothing \in U$, contradicting the definition of a filter.

Note that the converse states that every partition P of S into ω-many pieces has a (unique) piece $X_1 \in U$. To prove this, let Y_n be a collection of ω many members of U and let $Y = \bigcap_{n=1}^{\omega} Y_n$. Now consider the partition $\{P_\iota : \iota \leq \omega\}$ of $S \setminus Y$:

for each $s \in S \setminus Y$, put $s \in P_\iota$ if ι is the least index for which $s \notin Y_\iota$.

It is easy to verify that each $s \in S \setminus Y$ belongs to a *unique* P_ι, the collection of P_ι's is indeed a partition of $S \setminus Y$.

Along with Y, $\{P_\iota : \iota \leq \omega\}$ partitions S into $\aleph_0 = \omega$ many pieces. A (unique) piece of this partition belongs in U: $P_{\iota *} \in U$ or $Y \in U$. But, $P_\iota \cap Y_\iota = \varnothing \notin U$ by the definition of P_ι. This excludes the possibility for the former to belong in U (cf. alternative characterization of filter) and so $Y \in U$.

Thus, starting from an arbitrary collection $\{Y_n\}$ of ω-many members of U, we have identified a partition of S for which the unique piece which belongs to U is $\cap Y_n$. Therefore, U is σ-complete.