symmetric difference on a finite number of sets

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Recall that the symmetric difference operation on sets is commutative and associative. Therefore, one can speak of the symmetric difference of a finite collection of sets. More precisely, let $A_1, A_2, \ldots, A_n$ be sets, not necessarily pairwise distinct. The set

$$A := \mathop{\triangle}_{i=1}^{n} A_i,$$

the symmetric difference of the sets A_i, is well-defined.

Let $A, A_1, \ldots, A_n$ be defined as above. There is a curious property on A:

Proposition 1. $x \in A$ *iff x belongs to an odd number of the sets A_i.*

Before proving this fact, let us make some quick observations. If there are two sets A_1, A_2, then $A = A_1 \triangle A_2 = (A_1 \cap A_2') \cup (A_1' \cap A_2)$ (here we are assuming that A_1 and A_2 are subsets of some universe U, so the the complement makes sense). So $x \in A$ iff $x \in A_1 \cap A_2'$ or $x \in A_1' \cap A_2$ iff $x \in A_1$ or $x \in A_2$, which conforms with the statement of the proposition above. If $n = 3$, then A has conjunctive normal form

$$A_1 \triangle A_2 \triangle A_3 = (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2' \cap A_3') \cup (A_1' \cap A_2 \cap A_3') \cup (A_1' \cap A_2' \cap A_3),$$

(for a proof of this, see `http://planetmath.org/ProofOfTheAssociativityOfTheSymmetricDiff` Then $x \in A$ iff x belongs any one of the four intersections in the CNF above. In each of the four cases, x belongs to an odd number of sets. For example, if $x \in A_1 \cap A_2' \cap A_3'$, then $x \in A_1$.

From the two examples above, it seems that the approach to proving the proposition is to express the symmetric difference in CNF, and this is indeed the case.

To facilitate with the proof, let us introduce some notations. Start with sets $A_1, \ldots, A_n$, which are assumed to be subsets of some set U. Let I and C be the identity and complementation operations taking $A \subseteq U$ to A and A' respectively. Let $\mathbf{n}$ be the set $\{1, \ldots, n\}$. Let F_n be the set of all functions from $\mathbf{n}$ into $\{I, C\}$. For every $f \in F_n$, we write f_i for $f(i)$. Finally, we partition F_n into two sets E_n and O_n, where E_n (O_n) consists of all functions f such that $|f^{-1}(I)|$ is even (odd), respectively.

The proposition can now be restated as a single equation:

$$\mathop{\triangle}_{i=1}^{n} A_i = \bigcup_{f \in O_n} \bigcap_{i=1}^{n} f_i(A_i)$$

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Proof. We prove this equation by induction on n, for $n \geq 2$. The case when $n = 2$ is already discussed above. Now,

$$\underset{i=1}{\overset{n+1}{\triangle}} A_i = \left(\underset{i=1}{\overset{n}{\triangle}} A_i \right) \triangle A_{n+1} = X \cup Y,$$

where

$$X = \left(\underset{i=1}{\overset{n}{\triangle}} A_i \right) \cap A'_{n+1} \qquad \text{and} \qquad Y = A_{n+1} \cap \left(\underset{i=1}{\overset{n}{\triangle}} A_i \right)'.$$

By the induction hypothesis,

$$\underset{i=1}{\overset{n}{\triangle}} A_i = \bigcup_{f \in O_n} \bigcap_{i=1}^{n} f_i(A_i),$$

so that

$$X = \left(\bigcup_{f \in O_n} \bigcap_{i=1}^{n} f_i(A_i) \right) \cap A'_{n+1} = \bigcup_{f \in O_n} \bigcap_{i=1}^{n+1} \hat{f}_i(A_i),$$

where $\hat{f} : (\mathbf{n+1}) \to \{I, C\}$ is given by $\hat{f}_i = f_i$ if $i \in \mathbf{n}$, and $\hat{f}_{n+1} = C$.

Now, for any $x \in U$, x is either in an even number of A_i's, or an odd number of A_i's. This means that

$$x \in \bigcup_{f \in E_n} \bigcap_{i=1}^{n} f_i(A_i) \qquad \text{or} \qquad x \in \bigcup_{f \in O_n} \bigcap_{i=1}^{n} f_i(A_i)$$

and never both. This shows that U can be partitioned into the two sets above. In other words,

$$\left(\bigcup_{f \in O_n} \bigcap_{i=1}^{n} f_i(A_i) \right)' = \bigcup_{f \in E_n} \bigcap_{i=1}^{n} f_i(A_i).$$

As a result,

$$Y = A_{n+1} \cap \left(\bigcup_{f \in E_n} \bigcap_{i=1}^{n} f_i(A_i) \right) = \bigcup_{f \in E_n} \bigcap_{i=1}^{n+1} \overline{f}_i(A_i),$$

where $\overline{f} : (\mathbf{n+1}) \to \{I, C\}$ is given by $\overline{f}_i = f_i$ if $i \in \mathbf{n}$, and $\hat{f}_{n+1} = I$.

Every function $g \in O_{n+1}$ can be obtained from a function in $f \in F_n$ so that $g_i = f_i$ for $i \in \mathbf{n}$. If $f \in O_n$, then $g_{n+1} = C$, and if $f \in E_n$, then $g_{n+1} = I$. This means that O_{n+1} can be partitioned into two sets O and E, where O (or E) contains all functions whose restriction to $\mathbf{n}$ are in O_n (or E_n).

Therefore,

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\begin{aligned}
\bigcup_{g \in O_{n+1}} \bigcap_{i=1}^{n+1} g_i(A_i) &= \left(\bigcup_{g \in O} \bigcap_{i=1}^{n+1} g_i(A_i) \right) \cup \left(\bigcup_{g \in E} \bigcap_{i=1}^{n+1} g_i(A_i) \right) \\
&= \left(\bigcup_{f \in O_n} \bigcap_{i=1}^{n+1} \hat{f}_i(A_i) \right) \cup \left(\bigcup_{f \in E_n} \bigcap_{i=1}^{n+1} \overline{f}_i(A_i) \right) \\
&= X \cup Y = \underset{i=1}{\overset{n+1}{\triangle}} A_i.
\end{aligned}
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$\square$