truth-value semantics for propositional logic is sound

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The soundness theorem of propositional logic says the following: every theorem is a tautology. In symbol: $\vdash A$ implies $\models A$ for any wff A.

Theorem 1. *Propositional logic is sound with respect to truth-value semantics.*

Proof. Basically, we need to show that every axiom is a tautology, and that the inference rule modus ponens preserves truth. Since theorems are deduced from axioms and by applications of modus ponens, they are tautologies as a result.

Using truth tables, one easily verifies that every axiom is true (under any valuation).

First, let us verify that $(A \to B) \to (\neg B \to \neg A)$ is a tautology. The corresponding truth table is

A	B	$\neg A$	$\neg B$	$A \to B$	$\neg B \to \neg A$	$(A \to B) \to (\neg B \to \neg A)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Checking the truth values in the last column confirms that $(A \to B) \to (\neg B \to \neg A)$ is a tautology.

Next, let us check that $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ is a tautology. This time, we use a "reduced" truth table.

$(A$	$\to$	$(B$	$\to$	$C))$	$\to$	$((A$	$\to$	$B)$	$\to$	$(A$	$\to$	$C))$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	F	T	F	F	T	T	T	T	F	T	F	F
T	T	F	T	T	T	T	F	F	T	T	T	T
T	T	F	T	F	T	T	F	F	T	T	F	F
F	T	T	T	T	T	F	T	T	T	F	T	T
F	T	T	F	F	T	F	T	T	T	F	T	F
F	T	F	T	T	T	F	T	F	T	F	T	T
F	T	F	T	F	T	F	T	F	T	F	T	F

Notice that the truth values under the third $\to$ are all T, hence $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ is a tautology.

Finally, we check that $A \to (B \to A)$ is a tautology. This can be done without truth tables. Let v be a valuation. We may assume $v(A) = 1$, since

$v(A \to (B \to A)) = 1$ otherwise. If $v(A) = 1$, then $v(B \to A) = 1$ no matter what $v(B)$ is. Therefore, $v(A \to (B \to A)) = 1$, and $A \to (B \to A)$ is a tautology.

Next, we show that modus ponens preserves truths. In other words, $V(A) = V(A \to B) = 1$ imply $V(B) = 1$. But if not, then either $V(A) = 0$, or $V(A \to B) = 0$. $\square$

The soundness theorem can be used to prove that certain wff's of propositional logic are not theorems. For example, we show that the schema $A \to (A \wedge B)$ is not a theorem schema (an instance of it is not a theorem). Pick two distinct propositional variables p and q, and use the truth table:

A	$\to$	$(A$	$\wedge$	$B)$
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	T	F	F	F

Since the second column contains an F, $p \to (p \wedge q)$ is not true, and therefore $\nvdash A \to (A \wedge B)$ by the soundness theorem. As another example, we show that the *disjunction property*

$$\text{if} \vdash A \vee B, \text{ then} \vdash A \text{ or} \vdash B$$

is not true in classical propositional logic (it is true, however, in intuitionistic logic). To see this, let A be $p \to q$ and B be $q \to p$, where p, q are propositional variables. Then $A \vee B$ is an instance of the theorem schema $(C \to D) \vee (D \to C)$. However, neither $\vdash A$ nor $\vdash B$, as illustrated in the following truth table:

p	q	$p \to q$	$q \to p$	$(p \to q) \vee (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Notice that both the third and the fourth columns contain an F, and therefore by the soundness theorem, A and B are not theorems.