Boolean ideal

Canonical name	BooleanIdeal
Date of creation	2013-03-22 17:01:59
Last modified on	2013-03-22 17:01:59
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03G05
Classification	msc 03G10
Related topic	BooleanRing
Defines	Boolean filter
Defines	prime Boolean ideal
Defines	maximal Boolean ideal

Let A be a Boolean algebra and B a subset of A. The following are equivalent:

1. If A is interpreted as a Boolean ring, B is a ring ideal.

2. If A is interpreted as a Boolean lattice, B is a lattice ideal.

Before proving this equivalence, we want to mention that a Boolean ring is equivalent to a Boolean lattice, and a more commonly used terminology is a Boolean algebra, which is also valid, as it is an algebra over the ring of integers. The standard way of characterizing the ring structure from the lattice structure is by defining $a + b := (a' \wedge b) \vee (a \wedge b')$ (called the symmetric difference) and $a \cdot b = a \wedge b$. From this, we can "solve" for $\vee$ in terms of $+$ and $\cdot$: $a \vee b = a + b + a \cdot b$.

Proof. First, suppose B is an ideal of the "ring" A. If $a, b \in B$, then $a \vee b = a + b + a \cdot b \in B$. Suppose now that $a \in B$ and $c \in A$ with $c \leq a$. Then $c = c \wedge a = c \cdot a \in B$ as well. So B is a lattice ideal of A.

Next, suppose B is an ideal of the "lattice" A. If $a, b \in B$, then both $a' \wedge b$ and $a \wedge b'$ are in B since the first one is less than or equal to b and the second less than or equal to a, so their join is in B as well, this means that $a - b = a + b = (a' \wedge b) \vee (a \wedge b') \in B$. Furthermore, if $a \in B$ and $c \in A$, then $c \cdot a = c \wedge a \leq a \in B$ as well. As a result, B is a ring ideal of A. $\qquad\square$

A subset of a Boolean algebra satisfying the two equivalent conditions above is called a *Boolean ideal*, or an *ideal* for short. A *prime Boolean ideal* is a prime lattice ideal, and a *maximal Boolean ideal* is a maximal lattice ideal. Again, these notions and their ring theoretic counterparts match exactly. In fact, one can say more about these ideals in the case of a Boolean algebra: prime ideals are precisely the maximal ideals. If A is a Boolean ring, and M is a maximal ideal of A, then A/M is isomorphic to $\{0, 1\}$.

Remark. The dual notion of a Boolean ideal is a *Boolean filter*, or a *filter* for short. A Boolean filter is just a lattice filter of the Boolean algebra when considered as a lattice. To see the connection between a Boolean ideal and a Boolean lattice, let us define, for any subset S of a Boolean algebra A, the set $S' := \{a' \mid a \in S\}$. It is easy to see that $S'' = S$. Now, if I is an ideal, then I' is a filter. Conversely, if F is a filter, F' is a ideal. In fact, given any Boolean algebra, there is a Galois connection

$$I \mapsto I', \qquad F \mapsto F'$$

between the set of Boolean ideals and the set of Boolean filters in A. In addition, I is prime iff I' is. As a result, a filter is prime iff it is an ultrafilter (maximal filter).