properties of injective functions

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Theorem 1. *Suppose A, B, C are sets and $f\colon A \to B$, $g\colon B \to C$ are injective functions. Then the composition $g \circ f$ is an injection.*

Proof. Suppose that $(g \circ f)(x) = (g \circ f)(y)$ for some $x, y \in A$. By definition of composition, $g(f(x)) = g(f(y))$. Since g, is assumed injective, $f(x) = f(y)$. Since f is also assumed injective, $x = y$. Therefore, $(g \circ f)(x) = (g \circ f)(y)$ implies $x = y$, so $g \circ f$ is injective. $\square$

Theorem 2. *Suppose $f\colon A \to B$ is an injection, and $C \subseteq A$. Then the restriction $f|_C\colon C \to B$ is an injection.*

Proof. Suppose $(f|_C)(x) = (f|_C)(y)$ for some $x, y \in C$. By definition of restriction, $f(x) = f(y)$. Since f is assumed injective this, in turn, implies that $x = y$. Thus, $f|_C$ is also injective. $\square$

Theorem 3. *Suppose A, B, C are sets and that the functions $f\colon A \to B$ and $g\colon B \to C$ are such that $g \circ f$ is injective. Then f is injective.*

Proof. (*direct proof*) Let $x, y \in A$ be such that $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$. But as $g \circ f$ is injective, this implies that $x = y$, hence f is also injective. $\square$

Proof. (*proof by contradiction*) Suppose that f were not injective. Then there would exist $x, y \in A$ such that $f(x) = f(y)$ but $x \neq y$. Composing with g, we would then have $g(f(x)) = g(f(y))$. However, since $g \circ f$ is assumed injective, this would imply that $x = y$, which contradicts a previous statement. Hence f must be injective. $\square$

Theorem 4. *Suppose $f\colon A \to B$ is an injection. Then, for all $C \subseteq A$, it is the case that $f^{-1}(f(C)) = C$.*[1]

Proof. It follows from the definition of f^{-1} that $C \subseteq f^{-1}(f(C))$, whether or not f happens to be injective. Hence, all that need to be shown is that $f^{-1}(f(C)) \subseteq C$. Assume the contrary. Then there would exist $x \in f^{-1}(f(C))$ such that $x \notin C$. By defintion, $x \in f^{-1}(f(C))$ means $f(x) \in f(C)$, so there exists $y \in A$ such that $f(x) = f(y)$. Since f is injective, one would have $x = y$, which is impossible because y is supposed to belong to C but x is not supposed to belong to C. $\square$

[1] In this equation, the symbols "f" and "f^{-1}" as applied to sets denote the direct image and the inverse image, respectively

Theorem 5. *Suppose $f\colon A \to B$ is an injection. Then, for all $C, D \subseteq A$, it is the case that $f(C \cap D) = f(C) \cap f(D)$.*

Proof. Whether or not f is injective, one has $f(C \cap D) \subseteq f(C) \cap f(D)$; if x belongs to both C and D, then $f(x)$ will clearly belong to both $f(C)$ and $f(D)$. Hence, all that needs to be shown is that $f(C) \cap f(D) \subseteq f(C \cap D)$. Let x be an element of B which belongs to both $f(C)$ and $f(D)$. Then, there exists $y \in C$ such that $f(y) = x$ and $z \in D$ such that $f(z) = x$. Since $f(y) = f(z)$ and f is injective, $y = z$, so $y \in C \cap D$, hence $x \in f(C \cap D)$. $\square$