Peano arithmetic

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Defines	Peano's axioms
Defines	successor
Defines	axiom of induction

Peano's axioms are a definition of the set of natural numbers, denoted $\mathbb{N}$. From these axioms *Peano arithmetic* on natural numbers can be derived.

1. $0 \in \mathbb{N}$ (0 is a natural number)
2. For each $x \in \mathbb{N}$, there exists exactly one $x' \in \mathbb{N}$, called the *successor* of x
3. $x' \neq 0$ (0 is not the successor of any natural number)
4. $x = y$ if and only if $x' = y'$.
5. (*axiom of induction*) If $M \subseteq \mathbb{N}$ and $0 \in M$ and $x \in M$ implies $x' \in M$, then $M = \mathbb{N}$.

The *successor* of x is sometimes denoted Sx instead of x'. We then have $1 = S0$, $2 = S1 = SS0$, and so on.

Peano arithmetic consists of statements derived via these axioms. For instance, from these axioms we can define addition and multiplication on natural numbers. Addition is defined as

$$\begin{aligned} x + 1 &= x' \quad \text{for all } x \in \mathbb{N} \\ x + y' &= (x + y)' \quad \text{for all } x, y \in \mathbb{N} \end{aligned}$$

Addition defined in this manner can then be proven to be both associative and commutative.

Multiplication is

$$\begin{aligned} x \cdot 1 &= x \quad \text{for all } x \in \mathbb{N} \\ x \cdot y' &= x \cdot y + x \quad \text{for all } x, y \in \mathbb{N} \end{aligned}$$

This definition of multiplication can also be proven to be both associative and commutative, and it can also be shown to be distributive over addition.