every proposition is equivalent to a proposition in DNF

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Theorem. *Given any proposition, there exists a proposition in disjunctive normal form which is equivalent to that proposition.*

Proof. Any two propositions are equivalent if and only if they determine the same truth function. Therefore, if one can exhibit a mapping which assigns to a given truth function f a proposition in disjunctive normal form such that the truth function of this proposition is f, the theorem follows immediately.

Let n denote the number of arguments f takes. Define

$$V(f) = \{X \in \{T, F\}^n | f(X) = T\}$$

For every $X \in \{T, F\}^n$, define $L_i(X) \colon \{T, F\}^n \to \{T, F\}$ as follows:

$$L_i(X)(Y) = \begin{cases} Y_i & X_i = T \\ \neg Y_i & X_i = F \end{cases}$$

Then, we claim that

$$f(Y) = \bigwedge_{X \in V(f)} \bigvee_{i=1}^{n} L_i(X)(Y)$$

On the one hand, suppose that $f(Y) = T$ for a certain $Y \in \{T, F\}^n$. By definition of $V(f)$, we have $Y \in V(f)$. By definition of L_i, we have

$$L_i(Y)(Y) = \begin{cases} Y_i & Y_i = T \\ \neg Y_i & Y_i = F \end{cases}$$

In either case, $L_i(Y)(Y) = T$. Since a conjunction equals T if and only if each term of the conjunction equals T, it follows that $\bigvee_{i=1}^{n} L_i(Y)(Y) = T$. Finally, since a disjunction equals T if and only if there exists a term which equals T, it follows the right hand side equals equals T when the left-hand side equals T.

On the one hand, suppose that $f(Y) = F$ for a certain $Y \in \{T, F\}^n$. Let X be any element of $V(f)$. Since $Y \notin V(f)$, there must exist an index i such that $X_i \neq Y_i$. For this choice of i, $Y_i = \neg X_i$ Then we have

$$L_i(X)(Y) = \begin{cases} \neg X_i & X_i = T \\ \neg\neg X_i & X_i = F \end{cases}$$

In either case, $L_i(X)(Y) = F$. Since a conjunction equals F if and only if there exists a term which evaluates to F, it follows that $\bigvee_{i=1}^{n} L_i(X)(Y) = F$

for all $X \in V(f)$. Since a disjunction equals F if and only if each term of the conjunction equals F, it follows that the right hand side equals equals F when the left-hand side equals F.

$\square$