# well-founded recursion

| | |
|---|---|
| Canonical name | WellfoundedRecursion |
| Date of creation | 2013-03-22 17:54:52 |
| Last modified on | 2013-03-22 17:54:52 |
| Owner | CWoo (3771) |
| Last modified by | CWoo (3771) |
| Numerical id | 8 |
| Author | CWoo (3771) |
| Entry type | Theorem |
| Classification | msc 03E10 |
| Classification | msc 03E45 |
| Related topic | TransfiniteRecursion |

**Theorem 1.** *Let $G$ be a binary (class) function on $V$, the class of all sets. Let $A$ be a well-founded set (with $R$ the well-founded relation). Then there is a unique function $F$ such that*

$$F(x) = G(x, F|\operatorname{seg}(x)),$$

*where* $\operatorname{seg}(x) := \{y \in A \mid yRx\}$*, the initial segment of $x$.*

**Remark**. Since every well-ordered set is well-founded, the well-founded recursion theorem is a generalization of the transfinite recursion theorem. Notice that the $G$ here is a function in two arguments, and that it is necessary to specify the element $x$ in the first argument (in contrast with the $G$ in the transfinite recursion theorem), since it is possible that $\operatorname{seg}(a) = \operatorname{seg}(b)$ for $a \neq b$ in a well-founded set.

By converting $G$ into a formula ($\varphi(x, y, z)$ such that for all $x, y$, there is a unique $z$ such that $\varphi(x, y, z)$), then the above theorem can be proved in ZF (with the aid of the well-founded induction). The proof is similar to the proof of the transfinite recursion theorem.