many-sorted structure

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Defines	many-sorted interpretation
Defines	many-sorted algebra

Let L be a many-sorted language and S the set of sorts. A *many-sorted structure* M for L, or simply an L-structure consists of the following:

1. for each sort $s \in S$, a non-empty set A_s,
2. for each function symbol f of sort type $(s_1, \ldots, s_n)$:
   - if $n > 1$, a function $f_M : A_{s_1} \times \cdots \times A_{s_{n-1}} \to A_{s_n}$
   - if $n = 1$ (constant symbol), an element $f_M \in A_{s_1}$
3. for each relation symbol r of sort type $(s_1, \ldots, s_n)$, a relation (or subset)

$$r_M \subseteq A_{s_1} \times \cdots \times A_{s_n}.$$

A *many-sorted algebra* is a many-sorted structure without any relations.

Remark. A many-sorted structure is a special case of a more general concept called a *many-sorted interpretation*, which consists all of items 1-3 above, as well as the following:

4. an element $x_M \in A_s$ for each variable x of sort s.

Examples.

1. A left module over a ring can be thought of as a two-sorted algebra (say, with sorts $\{s_1, s_2\}$), for there are
   - there are two non-empty sets M (corresponding to sort s_1) and R (corresponding to sort s_2), where
   - M has the structure of an abelian group (equipped with three operations: $0, -, +$, corresponding to function symbols of sort types $(s_1), (s_1, s_1)$, and (s_1, s_1, s_1))
   - R has the structure of a ring (equipped with at least four operations: $0, -, +, \times$, corresponding to function symbols of sort types $(s_2), (s_2, s_2)$ and (s_2, s_2, s_2) for $+$ and $\times$, and possibly a fifth operation 1 of sort type (s_2))
   - a function $\cdot : R \times M \to M$, which corresponds to a function symbol of sort type (s_2, s_1, s_1). Clearly, $\cdot$ is the scalar multiplication on the module M.

For a right module over a ring, one merely replaces the sort type of the last function symbol by the sort type (s_1, s_2, s_1).

2. A deterministic semiautomaton $A = (S, \Sigma, \delta)$ is a two-sorted algebra, where

   - S and Σ are non-empty sets, corresponding to sorts, say, s_1 and s_2,
   - $\delta : S \times \Sigma \to S$ is a function corresponding to a function symbol of sort type (s_1, s_2, s_1).

3. A deterministic automaton $B = (S, \Sigma, \delta, \sigma, F)$ is a two-sorted structure, where

   - (S, Σ, δ) is a semiautomaton discussed earlier,
   - σ is a constant corresponding to a nullary function symbol of sort type (s_1),
   - F is a unary relation corresponding to a relation symbol of sort type (s_1).

   Because F is a relation, B is not an algebra.

4. A complete sequential machine $M = (S, \Sigma, \Delta, \delta, \lambda)$ is a three-sorted algebra, where

   - (S, Σ, δ) is a semiautomaton discussed earlier,
   - Δ is a non-empty sets, corresponding to sort, say, s_3,
   - $\lambda : S \times \Sigma \to \Delta$ is a function corresponding to a function symbol of sort type (s_1, s_2, s_3).

References

[1] J. D. Monk, *Mathematical Logic*, Springer, New York (1976).