consequence operator determined by a class of subsets

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Theorem 1. *Let L be a set and let K be a subset of $\mathcal{P}(L)$. The the mapping $C\colon \mathcal{P}(L) \to \mathcal{P}(L)$ defined as $C(X) = \cap\{Y \in K \mid X \subseteq Y\}$ is a consequence operator.*

Proof. We need to check that C satisfies the defining properties.

Property 1: Since every element of the set $\{Y \in K \mid X \subseteq Y\}$ contains X, we have $X \subseteq C(X)$.

Property 2: For every element Y of K such that $X \subseteq Y$, it also is the case that $C(X) \subseteq Y$ because an intersection of a family of sets is a subset of any member of the family. In other words (or rather, symbols),

$$\{Y \in K \mid X \subseteq Y\} \subseteq \{Y \in K \mid C(X) \subseteq Y\},$$

hence $C(C(X)) \subseteq C(X)$. By the first property proven above, $C(X) \subseteq C(C(X))$ so $C(C(X)) = C(X)$. Thus, $C \circ C = C$.

Property 3: Let X and Y be two subsets of L such that $X \subseteq Y$. Then if, for some other subset Z of L, we have $Y \subset Z$, it follows that $X \subset Z$. Hence,

$$\{Z \in K \mid Y \subseteq Z\} \subseteq \{Z \in K \mid X \subseteq Z\},$$

so $C(X) \subseteq C(Y)$.

□