infinitesimal

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Let R be a real closed field, for example the reals thought of as a structure in L, the language of ordered rings. Let B be some set of parameters from R. Consider the following set of formulas in $L(B)$:

$$\{x < b : b \in B \wedge b > 0\}$$

Then this set of formulas is finitely satisfied, so by compactness is consistent. In fact this set of formulas extends to a unique type p over B, as it defines a Dedekind cut. Thus there is some model M containing B and some $a \in M$ so that the type of a over B is p.

Any such element will be called *B-infinitesimal*. In particular, suppose $B = \emptyset$. Then the definable closure of B is the intersection of the reals with the algebraic numbers. Then a $\emptyset$-infinitesimal (or simply *infinitesimal*) is any element of any real closed field that is positive but smaller than every real algebraic (positive) number.

As noted above such models exist, by compactness. One can construct them using ultraproducts; see the entry "`http://planetmath.org/Hyperreal`Hyperreal" for more details. This is due to Abraham Robinson, who used such fields to formulate nonstandard analysis.

Let K be any ordered ring. Then K contains $\mathbf{N}$. We say K is *archimedean* if and only if for every $a \in K$ there is some $n \in \mathbf{N}$ so that $a < n$. Otherwise K is *non-archimedean*.

Real closed fields with infinitesimal elements are non-archimedean: for any infinitesimal a we have $a < 1/n$ and thus $1/a > n$ for each $n \in \mathbf{N}$.

References

[1] Robinson, A., *Selected papers of Abraham Robinson. Vol. II. Nonstandard analysis and philosophy*, New Haven, Conn., 1979.