alternative definition of cardinality

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The concept of cardinality comes from the notion of equinumerosity of sets. To define the cardinality $|A|$ of a set A, one desirable property is that A is equinumerous to B precisely when $|A| = |B|$. The first attempt, due to Frege and Russel, is to define a relation $\sim$ on the class V of sets so that $A \sim B$ iff there is a bijection from A to B. This relation is an equivalence relation on V. Then we can define $|A|$ as the equivalence class containing the set A. However, $|A|$ is not a set, so we can't do much with $|A|$ in ZF.

The second attempt, due to Von Neumann, defines $|A|$ to be the smallest ordinal $\mathrm{card}(A)$ equinumerous to A. Now, $\mathrm{card}(A)$ exists if A is well-orderable. But in general, we do not know if A is well-orderable unless the well-ordering principle is applied, which is just another form of the axiom of choice. Thus, this definition depends on AC, and, in everyday mathematical usage (which assumes ZFC), $|A| := \mathrm{card}(A)$ suffices.

The third way, due to Scott, of looking at $|A|$, without AC, is to modify the first attempt somewhat, so that $|A|$ is a set. Recall that the rank of a set A is the least ordinal α such that $A \subseteq V_\alpha$ in the cumulative hierarchy. A set having a rank is said to be *grounded*. By the axiom of foundation, every set is grounded. For any set A, let $R(A) := \{\rho(B) \mid B \sim A\}$. Then $R(A)$, as a class of ordinals, has a least element $r(A)$. So $r(A) \leq \rho(A)$. Next, we define (borrowing the terminology used in the first reference below)

$$\mathrm{kard}(A) := \{B \mid B \sim A \text{ and } \rho(B) = r(A)\},$$

and set $|A| := \mathrm{kard}(A)$. Since every element in $\mathrm{kard}(A)$ is a subset of $V_{r(A)}$, $\mathrm{kard}(A) \subseteq V_{r(A)^+}$, so that $|A|$ is a set. This method is known as Scott's trick. It can also be used in defining other isomorphism types on sets. It is easy to see that $|A| = |B|$ iff $A \sim B$. However, with this definition, $\mathrm{kard}(n) \neq n$ in general, where n is a natural number.

Nevertheless, it is known that every finite set is well-orderable, and so we come to the fourth definition of the cardinality of a set: given a set A:

$$|A| := \begin{cases} \mathrm{card}(A) \text{ if } A \text{ is well-orderable,} \\ \mathrm{kard}(A) \text{ otherwise .} \end{cases}$$

The one big advantage of this definition is clear: it does not require AC, and with AC, it is identical to the second definition above. At the same time, it also resolves the conflict with our intuitive notion about cardinality: the cardinality of a finite set is the number of elements in the set. However, the one big disadvantage in this definition is that we do not have $A \sim |A|$ in general (of course, A is infinite). There is no way, without AC, to find a definition of $|A|$, such that $A \sim B$ iff $|A| = |B|$, and $A \sim |A|$ at the same time.

References

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[2] T. J. Jech, *Set Theory*, 3rd Ed., Springer, New York, (2002).

[3] A. Levy, *Basic Set Theory*, Dover Publications Inc., (2002).