maximally consistent

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A set Δ of well-formed formulas (wff) is maximally consistent if Δ is consistent and any consistent superset of it is itself: $\Delta \subseteq \Gamma$ with Γ consistent implies $\Gamma = \Delta$.

Below are some basic properties of a maximally consistent set Δ:

1. Δ is deductively closed (Δ is a theory): $\Delta \vdash A$ iff $A \in \Delta$.

2. Δ is *complete*: $\Delta \vdash A$ or $\Delta \vdash \neg A$ for any wff A.

3. for any wff A, either $A \in \Delta$ or $\neg A \in \Delta$.

4. If $A \notin \Delta$, then $\Delta \cup \{A\}$ is not consistent.

5. Δ is a logic: Δ contains all theorems and is closed under modus ponens.

6. $\perp \notin \Delta$.

7. $A \to B \in \Delta$ iff $A \in \Delta$ implies $B \in \Delta$.

8. $A \wedge B \in \Delta$ iff $A \in \Delta$ and $B \in \Delta$.

9. $A \vee B \in \Delta$ iff $A \in \Delta$ or $B \in \Delta$.

Proof.

1. If $A \in \Delta$, then clearly $\Delta \vdash A$. Conversely, suppose $\Delta \vdash A$. Let $\mathcal{E}$ be a deduction of A from Δ, and $\Gamma := \Delta \cup \{A\}$. Suppose $\Gamma \vdash B$. Let $\mathcal{E}_1$ be a deduction of B from Γ, then $\mathcal{E}, \mathcal{E}_1$ is a deduction of B from Δ, so $\Delta \vdash B$. Since $\Delta \nvdash \perp$, $\Gamma \nvdash \perp$, so Γ is consistent. Since Δ is maximal, $\Gamma = \Delta$, or $A \in \Delta$.

2. Suppose $\Delta \nvdash A$, then $A \notin \Delta$ by 1. Then $\Delta \cup \{A\}$ is not consistent (since Δ is maximal), which means $\Delta, A \vdash \perp$, or $\Delta \vdash A \to \perp$, or $\Delta \vdash \neg A$.

3. If $A \notin \Delta$, then $\Delta \nvdash A$ by 1, so $\Delta \vdash \neg A$ by 2, and therefore $\neg A \in \Delta$ by 1 again.

4. If $A \notin \Delta$, then $\neg A \in \Delta$ by 3., so that $\neg A, A, \perp$ is a deduction of $\perp$ from $\Delta \cup \{A\}$, showing that $\Delta \cup \{A\}$ is not consistent.

5. If A is a theorem, then $\Delta \vdash A$, so that $A \in \Delta$ by 1. If $A \in \Delta$ and $A \to B \in \Delta$, then $A, A \to B, B$ is a deduction of B from Δ, so $B \in \Delta$ by 1.

6. This is true for any consistent set.

7. Suppose $A \to B \in \Delta$. If $A \in \Delta$, then $B \in \Delta$ since Δ is closed under modus ponens. Conversely, suppose $A \in \Delta$ implies $B \in \Delta$. This means that $\Delta, A \vdash B$. Then $\Delta \vdash A \to B$ by the deduction theorem, and therefore $A \to B \in \Delta$ by 1.

8. Suppose $A \wedge B \in \Delta$, then by modus ponens on theorems $A \wedge B \to A$ and $A \wedge B \to B$, we get $A, B \in \Delta$, since Δ is a logic by 5. Conversely, suppose $A, B \in \Delta$, then by modus ponens twice on theorem $A \to (B \to A \wedge B)$, we get $A \wedge B \in \Delta$ by 5.

9. Suppose $A \vee B \in \Delta$. Then $\neg(\neg A \wedge \neg B) \in \Delta$ by the definition of $\vee$, so $\neg A \wedge \neg B \notin \Delta$ by 3., which means $\neg A \notin \Delta$ or $\neg B \notin \Delta$ by the contrapositive of 8, or $A \in \Delta$ or $B \in \Delta$ by 3. Conversely, suppose $A \in \Delta$ or $B \in \Delta$. Then by modus ponens on theorems $A \to A \vee B$ or $B \to A \vee B$ respectively, we get $A \vee B \in \Delta$ by 5.

$\square$

The converses of 2 and 3 above are true too, and they provide alternative definitions of maximal consistency.

1. any complete consistent theory is maximally consistent.

2. any consistent set satisfying the condition in 3 above is maximally consistent.

Proof. Suppose Δ is complete consistent. Let Γ be a consistent superset of Δ. Γ is also complete. If $A \in \Gamma - \Delta$, then $\Gamma \vdash A$, so $\Gamma \nvdash \neg A$ since Γ is consistent. But then $\Delta \nvdash \neg A$ since Γ is a superset of Δ, which means $\Delta \vdash A$ since Δ is complete. But then $A \in \Delta$ since Δ is deductively closed, which is a contradiction. Hence Δ is maximal.

Next, suppose Δ is consistent satisfying the condition: either $A \in \Delta$ or $\neg A \in \Delta$ for any wff A. Suppose Γ is a consistent superset of Δ. If $A \in \Gamma - \Delta$, then $\neg A \in \Delta$ by assumption, which means $\neg A \in \Gamma$ since Γ is a superset of Δ. But then both A and $\neg A$ are deducible from Γ, contradicting the assumption that Γ is consistent. Therefore, Γ is not a proper superset of Δ, or $\Gamma = \Delta$. $\square$

Remarks.

- In the converse of 2, we require that Δ be a theory, for there are complete consistent sets that are not deductively closed. One such an example is the set V of all propositional variables: it can be shown that for every wff A, exactly one of $V \vdash A$ or $V \vdash \neg A$ holds.

- So far, none of the above actually tell us that a maximally consistent set exists. However, by Zorn's lemma, it is not hard to see that such a set does exist. For more detail, see `http://planetmath.org/LindenbaumsLemma`here.

- There is also a semantic characterization of a maximally consistent set: a set is maximally consistent iff there is a unique valuation v such that $v(A) = 1$ for every wff A in the set (see `http://planetmath.org/CompactnessTheoremForCl`