juxtaposition of automata

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Let $A = (S_1, \Sigma_1, \delta_1, I_1, F_1)$ and $B = (S_2, \Sigma_2, \delta_2, I_2, F_2)$ be two automata. We define the juxtaposition of A and B, written AB, as the sextuple $(S, \Sigma, \delta, I, F, \epsilon)$, as follows:

1. $S := S_1 \dot{\cup} S_1$, where $\dot{\cup}$ denotes disjoint union,
2. $\Sigma := (\Sigma_1 \cup \Sigma_2) \dot{\cup} \{\epsilon\}$,
3. $\delta : S \times \Sigma \to P(S)$ given by
   - $\delta(s, \epsilon) := I_2$ if $s \in F_1$, and $\delta(s, \epsilon) := \{s\}$ otherwise,
   - $\delta|(S_1 \times \Sigma_1) := \delta_1$,
   - $\delta|(S_2 \times \Sigma_2) := \delta_2$, and
   - $\delta(s, \alpha) := \varnothing$ otherwise (where $\alpha \neq \epsilon$).
4. $I := I_1$,
5. $F := F_2$.

Because S_1 and S_2 are considered as disjoint subsets of S, $I \cap F = \varnothing$. Also, from the definition above, we see that AB is an `http://planetmath.org/AutomatonWithEpsilonT` with ϵ-transitions.

The way AB works is as follows: a word $c = ab$, where $a \in \Sigma_1^*$ and $b \in \Sigma_2^*$, is fed into AB. AB first reads a as if it were read by A, via transition function δ_1. If a is accepted by A, then one of its accepting states will be used as the initial state for B when it reads b. The word c is accepted by AB when b is accepted by B.

Visually, the state diagram $G_{A_1A_2}$ of A_1A_2 combines the state diagram G_{A_1} of A_1 with the state diagram G_{A_2} of A_2 by adding an edge from each final node of A_1 to each of the start nodes of A_2 with label ϵ (the ϵ-transition).

Proposition 1. $L(AB) = L(A)L(B)$

Proof. Suppose $c = ab$ is a words such that $a \in \Sigma_1^*$ and $b \in \Sigma_2^*$. If $c \in L(AB)$, then $\delta(q, a\epsilon b) \cap F \neq \varnothing$ for some $q \in I = I_1$. Since $\delta(q, a\epsilon b) \cap F_2 = \delta(q, a\epsilon b) \cap F \neq \varnothing$ and $b \in \Sigma_2^*$, we have, by the definition of δ, that $\delta(q, a\epsilon b) = \delta(\delta(q, a\epsilon), b) = \delta_2(\delta(q, a\epsilon), b)$, which shows that $b \in L(B)$ and $\delta(q, a\epsilon) \cap I_2 \neq \varnothing$. But $\delta(q, a\epsilon) = \delta(\delta(q, a), \epsilon)$, by the definition of δ again, we also have $\delta(q, a) \cap F_1 \neq \varnothing$, which implies that $\delta(q, a) = \delta_1(q, a)$. As a result, $a \in L(A)$.

Conversely, if $a \in L(A)$ and $b \in L(B)$, then for any $q \in I = I_1$, $\delta(q, a) = \delta_1(q, a)$, which has non-empty intersection with F_1. This means that $\delta(q, a\epsilon) = \delta(\delta(q, a), \epsilon) = I_2$, and finally $\delta(q, a\epsilon b) = \delta(\delta(q, a\epsilon), b) = \delta(I_2, b)$, which has non-empty intersection with $F_2 = F$ by assumption. This shows that $a\epsilon b \in L((AB)_\epsilon)$, or $ab \in L(AB)$. $\square$