weakly compact cardinals and the tree property

Canonical name	WeaklyCompactCardinalsAndTheTreeProperty
Date of creation	2013-03-22 12:52:51
Last modified on	2013-03-22 12:52:51
Owner	Henry (455)
Last modified by	Henry (455)
Numerical id	7
Author	Henry (455)
Entry type	Result
Classification	msc 03E10
Related topic	TreeProperty
Related topic	Aronszajn

A cardinal is weakly compact if and only if it is inaccessible and has the tree property.

Weak compactness implies tree property

Let κ be a weakly compact cardinal and let $(T, <_T)$ be a κ tree with all levels smaller than κ. We define a theory in $L_{\kappa,\kappa}$ with for each $x \in T$, a constant c_x, and a single unary relation B. Then our theory Δ consists of the sentences:

- $\neg [B(c_x) \wedge B(c_y)]$ for every incompatible $x, y \in T$
- $\bigvee_{x \in T(\alpha)} B(c_x)$ for each $\alpha < \kappa$

It should be clear that B represents membership in a cofinal branch, since the first class of sentences asserts that no incompatible elements are both in B while the second class states that the branch intersects every level.

Clearly $|\Delta| = \kappa$, since there are κ elements in T, and hence fewer than $\kappa \cdot \kappa = \kappa$ sentences in the first group, and of course there are κ levels and therefore κ sentences in the second group.

Now consider any $\Sigma \subseteq \Delta$ with $|\Sigma| < \kappa$. Fewer than κ sentences of the second group are included, so the set of x for which the corresponding c_x must all appear in $T(\alpha)$ for some $\alpha < \kappa$. But since T has branches of arbitrary height, $T(\alpha) \models \Sigma$.

Since κ is weakly compact, it follows that Δ also has a model, and that model obviously has a set of c_x such that $B(c_x)$ whose corresponding elements of T intersect every level and are compatible, therefore forming a cofinal branch of T, proving that T is not Aronszajn.