modal logic B

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The modal logic **B** (for Brouwerian) is the smallest normal modal logic containing the following schemas:

- (T) $\Box A \to A$, and
- (B) $A \to \Box \diamond A$.

In `http://planetmath.org/ModalLogicT`this entry, we show that T is valid in a frame iff the frame is reflexive.

Proposition 1. *B is valid in a frame $\mathcal{F}$ iff $\mathcal{F}$ is symmetric.*

Proof. First, suppose B is valid in a frame $\mathcal{F}$, and wRu. Let M be a model based on $\mathcal{F}$, with $V(p) = \{w\}$, p a propositional variable. Since $w \in V(p)$, $\models_w p$, and $\models_w p \to \Box \diamond p$ by assumption, $\models_v \diamond p$ for all v such that wRv. In particular, $\models_u \diamond p$, which means there is a t such that uRt and $\models_t p$. But this means that $t \in V(p)$, so $t = w$, whence uRw, and R is symmetric.

Conversely, let $\mathcal{F}$ be a symmetric frame, M a model based on $\mathcal{F}$, and w a world in M. Suppose $\models_w A$. If $\not\models_w \Box \diamond A$, then there is a u such that wRu, with $\not\models_u \diamond A$. This mean for no t with uRt, we have $\models_t A$. Since R is symmetric, uRw, so $\not\models_w A$, a contradiction. Therefore, $\models_w \Box \diamond A$, and $\models_w A \to \Box \diamond A$ as a result. □

As a result,

Proposition 2. ***B** is sound in the class of symmetric frames.*

Proof. Since any theorem in **B** is deducible from a finite sequence consisting of tautologies, which are valid in any frame, instances of B, which are valid in symmetric frames by the proposition above, and applications of modus ponens and necessitation, both of which preserve validity in any frame, whence the result. □

In addition, using the canonical model of **B**, we have

Proposition 3. ***B** is complete in the class of reflexive, symmetric frames.*

Proof. Since **B** contains T, its canonical frame $\mathcal{F}_{\mathbf{B}}$ is reflexive. We next show that any consistent normal logic Λ containing the schema B is symmetric. Suppose $wR_\Lambda u$. We want to show that $uR_\Lambda w$, or that $\Delta_u := \{B \mid \Box B \in u\} \subseteq w$. It is then enough to show that if $A \notin w$, then $A \notin \Delta_u$. If $A \notin w$, $\neg A \in w$ because w is maximal, or $\Box \diamond \neg A \in w$ by modus ponens on B, or

$\Box\neg\Box A \in w$ by the substitution theorem on $A \leftrightarrow \neg\neg A$, or $\neg\Box A \in \Delta_w$ by the definition of Δ_w, or $\neg\Box A \in u$ since $wR_\Lambda u$, or $\Box A \notin u$, since u is maximal, or $A \notin \Delta_u$ by the definition of Δ_u. So R_Λ is symmetric, and $R_{\mathbf{B}}$ is both reflexive and symmetric. $\Box$