limit of sequence of sets

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Recall that lim sup and lim inf of a sequence of sets $\{A_i\}$ denote the and the of $\{A_i\}$, respectively. Please click http://planetmath.org/LimitSuperiorOfSetshere to see the definitions and http://planetmath.org/LimitSuperiorhere to see the specialized definitions when they are applied to the real numbers.

Theorem. Let $\{A_i\}$ be a sequence of sets with $i \in \mathbb{Z}^+ = \{1, 2, \ldots\}$. Then

1. for I ranging over all infinite subsets of $\mathbb{Z}^+$,
$$\limsup A_i = \bigcup_I \bigcap_{i \in I} A_i,$$
2. for I ranging over all subsets of $\mathbb{Z}^+$ with finite compliment,
$$\liminf A_i = \bigcup_I \bigcap_{i \in I} A_i,$$
3. $\liminf A_i \subseteq \limsup A_i$.

Proof.

1. We need to show, for I ranging over all infinite subsets of $\mathbb{Z}^+$,
$$\bigcup_I \bigcap_{i \in I} A_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_k. \tag{1}$$
Let x be an element of the LHS, the left hand side of Equation (1). Then $x \in \bigcap_{i \in I} A_i$ for some infinite subset $I \subseteq \mathbb{Z}^+$. Certainly, $x \in \bigcup_{i=1}^{\infty} A_i$. Now, suppose $x \in \bigcup_{i=k}^{\infty} A_i$. Since I is infinite, we can find an $l \in I$ such that $l > k$. Being a member of I, we have that $x \in A_l \subseteq \bigcup_{i=k+1}^{\infty} A_i$. By induction, we have $x \in \bigcup_{i=n}^{\infty} A_i$ for all $n \in \mathbb{Z}^+$. Thus x is an element of the RHS. This proves one side of the inclusion ($\subseteq$) in (1).

   To show the other inclusion, let x be an element of the RHS. So $x \in \bigcup_{i=n}^{\infty} A_i$ for all $n \in \mathbb{Z}^+$ In $\bigcup_{i=1}^{\infty} A_i$, pick the least element n_0 such that $x \in A_{n_0}$. Next, in $\bigcup_{i=n_0+1}^{\infty} A_i$, pick the least n_1 such that $x \in A_{n_1}$. Then the set $I = \{n_0, n_1, \ldots\}$ fulfills the requirement $x \in \bigcap_{i \in I} A_i$, showing the other inclusion ($\supseteq$).

2. Here we have to show, for I ranging over all subsets of $\mathbb{Z}^+$ with $\mathbb{Z}^+ - I$ finite,

$$\bigcup_I \bigcap_{i \in I} A_i = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_k. \tag{2}$$

Suppose first that x is an element of the LHS so that $x \in \bigcap_{i \in I} A_i$ for some I with $\mathbb{Z}^+ - I$ finite. Let n_0 be a upper bound of the finite set $\mathbb{Z}^+ - I$ such that for any $n \in \mathbb{Z}^+ - I$, $n < n_0$. This means that any $m \geq n_0$, we have $m \in I$. Therefore, $x \in \bigcap_{i=n_0}^{\infty} A_i$ and x is an element of the RHS.

Next, suppose x is an element of the RHS so that $x \in \bigcap_{k=n}^{\infty} A_k$ for some n. Then the set $I = \{n_0, n_0 + 1, \ldots\}$ is a subset of $\mathbb{Z}^+$ with finite complement that does the job for the LHS.

3. The set of all subsets (of $\mathbb{Z}^+$) with finite complement is a subset of the set of all infinite subsets. The third assertion is now clear from the previous two propositions. QED

Corollary. If $\{A_i\}$ is a decreasing sequence of sets, then

$$\liminf A_i = \limsup A_i = \lim A_i = \bigcap A_i.$$

Similarly, if $\{A_i\}$ is an increasing sequence of sets, then

$$\liminf A_i = \limsup A_i = \lim A_i = \bigcup A_i.$$

Proof. We shall only show the case when we have a descending chain of sets, since the other case is completely analogous. Let $A_1 \supseteq A_2 \supseteq \ldots$ be a descending chain of sets. Set $A = \bigcap_{i=1}^{\infty} A_i$. We shall show that

$$\limsup A_i = \liminf A_i = \lim A_i = A.$$

First, by the definition of of a sequence of sets:

$$\limsup A_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} A_n = A.$$

Now, by Assertion 3 of the above Theorem, $\liminf A_i \subseteq \limsup A_i = A$, so we only need to show that $A \subseteq \liminf A_i$. But this is immediate from the definition of A, being the intersection of all A_i with subscripts i taking on all values of $\mathbb{Z}^+$. Its complement is the empty set, clearly finite. Having shown both the existence and equality of the and of the A_i's, we conclude that the limit of A_i's exist as well and it is equal to A. QED