algebraic definition of a lattice

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The http://planetmath.org/Latticeparent entry defines a lattice as a relational structure (a poset) satisfying the condition that every pair of elements has a supremum and an infimum. Alternatively and equivalently, a lattice L can be a defined directly as an algebraic structure with two binary operations called meet $\wedge$ and join $\vee$ satisfying the following conditions:

- (idempotency of $\vee$ and $\wedge$): for each $a \in L$, $a \vee a = a \wedge a = a$;
- (commutativity of $\vee$ and $\wedge$): for every $a, b \in L$, $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$;
- (associativity of $\vee$ and $\wedge$): for every $a, b, c \in L$, $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$; and
- (absorption): for every $a, b \in L$, $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

It is easy to see that this definition is equivalent to the one given in the parent, as follows: define a binary relation $\leq$ on L such that

$$a \leq b \quad \text{iff} \quad a \vee b = b.$$

Then $\leq$ is reflexive by the idempotency of $\vee$. Next, if $a \leq b$ and $b \leq a$, then $a = a \vee b = b$, so $\leq$ is anti-symmetric. Finally, if $a \leq b$ and $b \leq c$, then $a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c$, and therefore $a \leq c$. So $\leq$ is transitive. This shows that $\leq$ is a partial order on L. For any $a, b \in L$, $a \vee (a \vee b) = (a \vee a) \vee b = a \vee b$ so that $a \leq a \vee b$. Similarly, $b \leq a \vee b$. If $a \leq c$ and $b \leq c$, then $(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c$. This shows that $a \vee b$ is the supremum of a and b. Similarly, $a \wedge b$ is the infimum of a and b.

Conversely, if $(L, \leq)$ is defined as in the parent entry, then by defining

$$a \vee b = \sup\{a, b\} \quad \text{and} \quad a \wedge b = \inf\{a, b\},$$

the four conditions above are satisfied. For example, let us show one of the absorption laws: $a \vee (a \wedge b) = a$. Let $c = \inf\{a, b\} \leq a = a \wedge b$. Then $c \leq a$ so that $\sup\{a, c\} = a$, which precisely translates to $a = a \vee c = a \vee (a \wedge b)$. The remainder of the proof is left for the reader to try.