equivalent formulation of substitutability

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Proposition 1. *Suppose a variable x occurs free in a wff A. A term t is free for x in A iff no variables in t are bound by a quantifier in $A[t/x]$.*

Proof. We do induction on the complexity of A.

- If A is atomic, then any t is free for x in A, and clearly $A[t/x]$ is just A, which has no bound variables.
- If A is of the form $B \to C$, then t is free for x in A iff t is free for x in both B and C iff no variables in t are bound in either B or C iff no variables in t are bound in A.
- Finally, suppose A is of the form $\exists y B$. Since x is free in A, x is not y, and t is free for x in A iff y is not in t and t is free for x in B iff, by induction, y is not in t and no variables of t are bound in $B[t/x]$ iff no variables of t are bound in $\exists y B[t/x]$, which is just $A[t/x]$ (since $x \neq y$).

□

If x does not occur free in A (either x occurs bound in A or not at all in A), then t is obviously free for x in A, but $A[t/x]$ is just A, and there is no guarantee that variables in t are bound in A or not.

In the special case where t is a variable y, we see that y is free for x in A iff y is not bound in $A[y/x]$, provided that x occurs free in A. In other words, y is free for x in A iff no free occurrences of x in A are in the scope of Qy, where Q is either $\exists$ or $\forall$. So if y is not bound in A, y is free for x in A, regardless of whether x is free or bound in A. Also, x is always free for x in A.