cardinality of the continuum

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The *cardinality of the continuum*, often denoted by $\mathfrak{c}$, is the cardinality of the set $\mathbb{R}$ of real numbers. A set of cardinality $\mathfrak{c}$ is said to have *continuum many* elements.

Cantor's diagonal argument shows that $\mathfrak{c}$ is uncountable. Furthermore, it can be shown that $\mathbb{R}$ is equinumerous with the power set of $\mathbb{N}$, so $\mathfrak{c} = 2^{\aleph_0}$. It can also be shown that $\mathfrak{c}$ has uncountable cofinality.

It can also be shown that

$$\mathfrak{c} = \mathfrak{c}^{\aleph_0} = \aleph_0\mathfrak{c} = \mathfrak{c}\mathfrak{c} = \mathfrak{c} + \kappa = \mathfrak{c}^n$$

for all finite cardinals $n \geq 1$ and all cardinals $\kappa \leq \mathfrak{c}$. See the article on cardinal arithmetic for some of the basic facts underlying these equalities.

There are many properties of $\mathfrak{c}$ that independent of ZFC, that is, they can neither be proved nor disproved in ZFC, assuming that ZF is consistent. For example, for every nonzero natural number n, the equality $\mathfrak{c} = \aleph_n$ is independent of ZFC. (The case $n = 1$ is the well-known `http://planetmath.org/ContinuumHypothesis`Co Hypothesis.) The same is true for most other alephs, although in some cases equality can be ruled out on the grounds of cofinality, e.g., $\mathfrak{c} \neq \aleph_\omega$. In particular, $\mathfrak{c}$ could be either $\aleph_1$ or $\aleph_{\omega_1}$, so it could be either a successor cardinal or a limit cardinal, and either a regular cardinal or a singular cardinal.