surjection and axiom of choice

Canonical name	SurjectionAndAxiomOfChoice
Date of creation	2013-03-22 18:44:37
Last modified on	2013-03-22 18:44:37
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	9
Author	CWoo (3771)
Entry type	Derivation
Classification	msc 03E25

In this entry, we show the statement that

(*) every surjection has a right inverse

is equivalent to the axiom of choice (AC).

Proposition 1. *AC implies (*).*

Proof. Let $f : A \to B$ be a surjection. Then the set $C := \{f^{-1}(y) \mid y \in B\}$ partitions A. By the axiom of choice, there is a function $g : C \to \bigcup C$ such that $g(f^{-1}(y)) \in f^{-1}(y)$ for every $y \in B$. Since $\bigcup C = A$, g is a function from C to A. Define $h : B \to A$ by $h(y) = g(f^{-1}(y))$. Then $h(y) \in f^{-1}(y)$, and therefore $(f \circ h)(y) = f(h(y)) = y$, implying that f has a right inverse. $\square$

Remark. The function h is easily seen to be an injection: if $h(y_1) = h(y_2)$, then $y_1 = f(h(y_1)) = f(h(y_2)) = y_2$.

Proposition 2. *(*) implies AC.*

Before proving this, let us remark that, in the collection C of non-empty sets of the axiom of choice, there is no assumption that the sets in C be pairwise disjoint. The statement

(**) given a set C of pairwise disjoint non-empty sets, there is a choice function $f : C \to \bigcup C$

seemingly weaker than AC, turns out to be equivalent to AC, and we will prove this fact first.

Proof. Obviously AC implies (**). Conversely, assume (**). Let C be a collection of non-empty sets. We assume $C \neq \varnothing$. For each $a \in C$, define a set $A_a := \{(x, a) \mid x \in a\}$. Since $a \neq \varnothing$, $A_a \neq \varnothing$. In addition, $A_a \cap A_b = \varnothing$ iff $a \neq b$ (true since elements of A_a and elements of A_b have distinct second coordinates). So the collection $D := \{A_a \mid a \in C\}$ is a set consisting of pairwise disjoint non-empty sets. By (**), there is a function $f : D \to \bigcup D$ such that $f(A_a) \in A_a$ for every $a \in C$. Now, define two functions $g : C \to D$ and $h : \bigcup D \to \bigcup C$ by $g(a) = A_a$ and $h(x, a) = x$ Then, for any $a \in C$, we have $(h \circ f \circ g)(a) = h(f(A_a))$. Since $f(A_a) \in A_a$, its first coordinate is an element of a. Therefore $h(f(A_a)) \in a$, and hence $h \circ f \circ g$ is the desired choice function. $\square$

Proof of Propositon 2. We show that (*) implies (**), and since (**) implies AC as shown above, the proof of Proposition 2 is then complete.

Let C be a collection of pairwise disjoint non-empty sets. Each element of $\bigcup C$ belongs to a unique set in C. Then the function $g : \bigcup C \to C$ taking each element of $\bigcup C$ to the set it belongs in C, is a well-defined function. It is clearly surjective. Hence, by assumption, there is a function $f : C \to \bigcup C$ such that $g \circ f = 1_C$ (a right inverse of g). For each $x \in C$, $g(f(x)) = x$, which is the same as saying that $f(x)$ is an element of x by the definition of g. □

Remark. In the category of sets, AC is equivalent to saying that every epimorophism is a split epimorphism. In general, a category is said to have the axiom of choice if every epimorphism is a split epimorphism.