uniqueness of cardinality

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Theorem. *The cardinality of a set is unique.*

Proof. We will verify the result for finite sets only. Suppose A is a finite set with cardinality $\mid A \mid= n$ and $\mid A \mid= m$. Then there exist bijections $f : \mathbb{N}_n \to A$ and $g : \mathbb{N}_m \to A$. Since g is a bijection, it is invertible, and $g^{-1} : A \to \mathbb{N}_m$ is a bijection. Then the composition $g^{-1} \circ f$ is a bijection from $\mathbb{N}_n$ to $\mathbb{N}_m$. We will show by induction on n that $m = n$. In the case $n = 1$ we have $\mathbb{N}_n = \mathbb{N}_1 = \{1\}$, and it must be that $\mathbb{N}_m = \{1\}$ as well, whence $m = 1 = n$. Now let $n \geq 1 \in \mathbb{N}$, and suppose that, for all $m \in \mathbb{N}$, the existence of a bijection $f : \mathbb{N}_n \to \mathbb{N}_m$ implies $n = m$. Let $m \in \mathbb{N}$ and suppose $h : \mathbb{N}_{n+1} \to \mathbb{N}_m$ is a bijection. Let $k = h(n+1)$, and notice that $1 \leq k \leq m$. Since h is onto there exists some $j \in \mathbb{N}_{n+1}$ such that $f(j) = m$. There are two cases to consider. If $j = n+1$, then we may define $\phi : \mathbb{N}_n \to \mathbb{N}_{m-1}$ by $\phi(i) = h(i)$ for all $i \in \mathbb{N}_n$, which is clearly a bijection, so by the inductive hypothesis $n = m-1$, and thus $n+1 = m$. Now suppose $j \neq n+1$, and define $\phi : \mathbb{N}_n \to \mathbb{N}_{m-1}$ by

$$\phi(i) = \begin{cases} h(i) & \text{if } i \neq j \\ k & \text{if } i = j \end{cases}.$$

We first show that ϕ is one-to-one. Let $i_1, i_2 \in \mathbb{N}_n$, where $i_1 \neq i_2$. First consider the case where neither i_1 nor i_2 is equal to j. Then, since h is one-to-one, we have

$$\phi(i_1) = h(i_1) \neq h(i_2) = \phi(i_2).$$

In the case $i_1 = j$, again because h is one-to-one, we have

$$\phi(i_1) = k = h(n+1) \neq h(i_2) = \phi(i_2).$$

Similarly it can be shown that, if $i_2 = j$, $\phi(i_1) \neq \phi(i_2)$, so ϕ is one-to-one. We now show that ϕ is onto. Let $l \in \mathbb{N}_{m-1}$. If $l = k$, then we may take $i = j$ to have $\phi(i) = \phi(j) = k = l$. If $l \neq k$, then, because h is onto, there exists some $i \in \mathbb{N}_n$ such that $h(i) = l$, so $g(i) = h(i) = l$. Thus g is onto, and our inductive hypothesis again gives $n = m-1$, hence $n+1 = m$. So by the principle of induction, the result holds for all n. Returning now to our set A and our bijection $g^{-1} \circ f : \mathbb{N}_n \to \mathbb{N}_m$, we may conclude that $m = n$, and consequently that the cardinality of A is unique, as desired. □

Corollary. *There does not exist a bijection from a finite set to a proper subset of itself.*

Proof. Without a loss of generality we may assume the proper subset is nonempty, for if the set is empty then the corollary holds vacuously. So let A be as above with cardinality $n \in \mathbb{N}$, B a proper subset of A with cardinality $m < n \in \mathbb{N}$, and suppose $f : A \to B$ is a bijection. By hypothesis there exist bijections $g : \mathbb{N}_n \to A$ and $h : \mathbb{N}_m \to B$; but then $h^{-1} \circ (f \circ g)$ is a bijection from $\mathbb{N}_n$ to $\mathbb{N}_m$, whence by the preceding result $n = m$, contrary to assumption. □

The corollary is also known more popularly as the pigeonhole principle.