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equivalence class of equinumerous sets is not a set

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Recall that two sets are equinumerous iff there is a bijection between them.

Proposition 1. *Let A be a non-empty set, and $E(A)$ the class of all sets equinumerous to A. Then $E(A)$ is a proper class.*

Proof. $E(A) \neq \varnothing$ since A is in $E(A)$. Since $A \neq \varnothing$, pick an element $a \in A$, and let $B = A - \{a\}$. Then $C := \{y \mid y \text{ is a set, and } y \notin B\}$ is a proper class, for otherwise $C \cup B$ would be the "set" of all sets, which is impossible. For each y in C, the set $F(y) := B \cup \{y\}$ is in one-to-one correspondence with A, with the bijection $f : F(y) \to A$ given by $f(x) = x$ if $x \in B$, and $f(y) = a$. Therefore $E(A)$ contains $F(y)$ for every y in the proper class C. Furthermore, since $F(y_1) \neq F(y_2)$ whenever $y_1 \neq y_2$, we have that $E(A)$ is a proper class as a result. $\square$

Remark. In the proof above, one can think of F as a class function from C to $E(A)$, taking every $y \in C$ into $F(y)$. This function is one-to-one, so C embeds in $E(A)$, and hence $E(A)$ is a proper class.