an injection between two finite sets of the same cardinality is bijective

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Lemma. *Let A, B be two finite sets of the same cardinality. If $f\colon A \to B$ is an injective function then f is bijective.*

Proof. In order to prove the lemma, it suffices to show that if f is an injection then the cardinality of $f(A)$ and A are equal. We prove this by induction on $n = \mathrm{card}(A)$. The case $n = 1$ is trivial. Assume that the lemma is true for sets of cardinality n and let A be a set of cardinality $n+1$. Let $a \in A$ so that $A_1 = A - \{a\}$ has cardinality n. Thus, $f(A_1)$ has cardinality n by the induction hypothesis. Moreover, $f(a) \notin f(A_1)$ because $a \notin A_1$ and f is injective. Therefore:

$$f(A) = f(\{a\} \cup A_1) = \{f(a)\} \cup f(A_1)$$

and the set $\{f(a)\} \cup f(A_1)$ has cardinality $1 + n$, as desired. $\square$