term algebra

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Let Σ be a signature and V a set of variables. Consider the set of all terms of $T := T(\Sigma)$ over V. Define the following:

- For each constant symbol $c \in \Sigma$, c^T is the element c in T.
- For each n and each n-ary function symbol $f \in \Sigma$, f^T is an n-ary operation on T given by

$$f^T(t_1, \dots, t_n) = f(t_1, \dots, t_n),$$

  meaning that the evaluation of f^T at $(t_1, \dots, t_n)$ is the term $f(t_1, \dots, t_n) \in T$.
- For each relational symbol $R \in \Sigma$, $R^T = \varnothing$.

Then T, together with the set of constants and n-ary operations defined above is an Σ-`http://planetmath.org/Structure`structure. Since there are no relations defined on it, T is an algebraic system whose signature Σ' is the subset of Σ consisting of all but the relation symbols of Σ. The algebra T is aptly called the *term algebra* of the signature Σ (over V).

The prototypical example of a term algebra is the set of all well-formed formulas over a set V of propositional variables in classical propositional logic. The signature Σ is just the set of logical connectives. For each n-ary logical connective $\#$, there is an associated n-ary operation $[\#]$ on V, given by $[\#](p_1, \dots, p_n) = \#p_1 \cdots p_n$.

Remark. The term algebra T of a signature Σ over a set V of variables can be thought of as a *free structure* in the following sense: if A is any Σ-structure, then any function $\phi : V \to A$ can be extended to a unique structure homomorphism $\phi' : T \to A$. In this regard, V can be viewed as a free basis for the algebra T. As such, T is also called the *absolutely free Σ-structure with basis V*.