proof equivalence of formulation of foundation

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We show that each of the three formulations of the axiom of foundation given are equivalent.

$1 \Rightarrow 2$

Let X be a set and consider any function $f : \omega \to \mathrm{tc}(X)$. Consider $Y = \{f(n) \mid n < \omega\}$. By assumption, there is some $f(n) \in Y$ such that $f(n) \cap Y = \emptyset$, hence $f(n+1) \notin f(n)$.

$2 \Rightarrow 3$

Let ϕ be some formula such that $\phi(x)$ is true and for every X such that $\phi(X)$, there is some $y \in X$ such that $\phi(y)$. Then define $f(0) = x$ and $f(n+1)$ is some $y \in f(n)$ such that $\phi(y)$. This would construct a function violating the assumption, so there is no such ϕ.

$3 \Rightarrow 1$

Let X be a nonempty set and define $\phi(x) \equiv x \in X$. Then ϕ is true for some X, and by assumption, there is some y such that $\phi(y)$ but there is no $z \in y$ such that $\phi(z)$. Hence $y \in X$ but $y \cap X = \emptyset$.