motivation for von Neumann ordinals

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The idea of the von Neumann ordinal can be traced back to the following well-known fact: for any natural number n, there are exactly n natural numbers which are less than n. For instance, the set

$$\{0, 1, 2, 3, 4\}$$

has 5 elements, the set

$$\{0, 1, 2, 3, 4, 5, 6\}$$

has 7 elements, etc.

To obtain von Neumann ordinals, we turn this idea around. Instead of taking it for granted that numbers exist (and have certain properties), we want to start with the more primitive notion of set and define numbers (and derive their properties). The way to define a number is as a set of objects which have that number of elements. For instance, consider counting on fingers — in that case, a set of fingers stands for a number. We will apply the same idea here in a more sophisticated form — the counters we will use are not going to be fingers or beads on an abacus, but abstract elements of an abstract set.

To do this, we turn the observation made earlier around and *define* a natural number to be the set of all natural numbers less than it. At first sight, this definition appears circular, but upon closer examination, we see that it is legitimate. The reason is that to define a particular number, we only need to make use of the numbers smaller than it as counters, so cn use our definition repeatedly to express numbers as sets.

To begin, we notice that, since there are no natural numbers smaller than zero, we represent zero by the empty set. Next, since the only number smaller than 1, is zero, which corresponds to the empty set, we see that 1 corresponds to the set whose only element is the empty set, i.e. $1 = \{0\} = \{\emptyset\}$. Then we can go on to express all other numbers in terms of the empty set in a manner which may be explained with a typical example:

$$\begin{aligned}
4 &= \{0, 1, 2, 3\} \\
&= \{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}\} \\
&= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{0, 1, \{0, 1\}\} \\
&= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}.
\end{aligned}$$

As we already see in this example, this representation of integers in terms of the empty set is extremely clumsy. While it is of little use in practical application (even tally marks or Roman numerals are more concise) it is of use theoretically because it is easy to define the basic operations on numbers in terms of set-theoretical operations.

As an example of such a definition, we note that the ordering relation — given two numbers m and n, we have $m < n$ exactly when $m \subset n$ as sets. As it turns out, our numbers are totally ordered, in fact well-ordered under this relation, so our numbers are ordinal numbers, hence the name "von Neumann ordinals".

Another important example is the successor function. Thinking for a minute about how we define a number as the set of numbers smaller than itself, we see that the next number is gotten by adding the set denoting the previous element to itself as an element, in symbols, $n + 1 = n \cup \{n\}$. For example, we have

$$4 + 1 = 4 \cup \{4\} = \{0, 1, 2, 3\} \cup \{4\} = \{0, 1, 2, 3, 4\} = 5.$$

Using this definition, one may do things like derive the Peano axioms from the axioms of set theory. From a foundational point of view, that derivation is important because it shows that it is not necessary to separately postulate natural numbers, but that they arise naturally from set theory.

Finally, this definition applies equally well to transfinite numbers. For instance, consider the first transfinite ordinal ω. By definition, this is the ordinal number of the ordered set of natural numbers. In our scheme, we simply define ω to be the set of all natural numbers. Furthermore, given *any* well-ordered set, one can show by transfinite induction that it is isomorphic to some von Neumann ordinal, so all ordinal numbers are represented.