quasi-inverse of a function

Canonical name	QuasiinverseOfAFunction
Date of creation	2013-03-22 16:22:14
Last modified on	2013-03-22 16:22:14
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	11
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03E20
Synonym	quasi-inverse
Defines	quasi-inverse function

Let $f : X \to Y$ be a function from sets X to Y. A *quasi-inverse* g of f is a function g such that

1. $g : Z \to X$ where $\mathrm{ran}(f) \subseteq Z \subseteq Y$, and
2. $f \circ g \circ f = f$, where $\circ$ denotes functional composition operation.

Note that $\mathrm{ran}(f)$ is the range of f.

Examples.

1. If f is a real function given by $f(x) = x^2$. Then $g(x) = \sqrt{x}$ defined on $[0, \infty)$ and $h(x) = -\sqrt{x}$ also defined on $[0, \infty)$ are both quasi-inverses of f.
2. If $f(x) = 1$ defined on $[0, 1)$. Then $g(x) = \frac{1}{2}$ defined on $\mathbb{R}$ is a quasi-inverse of f. In fact, any $g(x) = a$ where $a \in [0, 1)$ will do. Also, note that $h(x) = x$ on $[0, 1)$ is also a quasi-inverse of f.
3. If $f(x) = [x]$, the step function on the reals. Then by the previous example, $g(x) = [x] + a$, any $a \in [0, 1)$, is a quasi-inverse of f.

Remarks.

- Every function has a quasi-inverse. This is just another form of the Axiom of Choice. In fact, if $f : X \to Y$, then for *every* subset Z of Y such that $\mathrm{ran}(f) \subseteq Z$, there is a quasi-inverse g of f whose domain is Z.
- However, a quasi-inverse of a function is in general not unique, as illustrated by the above examples. When it is unique, the function must be a bijection:

  If $\mathrm{ran}(f) \neq Y$, then there are at least two quasi-inverses, one with domain $\mathrm{ran}(f)$ and one with domain Y. So f is onto. To see that f is one-to-one, let g be the quasi-inverse of f. Now suppose $f(x_1) = f(x_2) = z$. Let $g(z) = x_3$ and assume $x_3 \neq x_1$. Define $h : Y \to X$ by $h(y) = g(y)$ if $y \neq z$, and $h(z) = x_1$. Then h is easily verified as a quasi-inverse of f that is different from g. This is a contradition. So $x_3 = x_1$. Similarly, $x_3 = x_2$ and therefore $x_1 = x_2$.

- Conversely, if f is a bijection, then the inverse of f is a quasi-inverse of f. In fact, f has only one quasi-inverse.

- The relation of being quasi-inverse is not symmetric. In other words, if g is a quasi-inverse of f, f need not be a quasi-inverse of g. In the second example above, h is a quasi-inverse of f, but not vice versa: $h(0) = 0$, but $hfh(0) = hf(0) = h(1) = 1 \neq h(0)$.

- Let g be a quasi-inverse of f, then the restriction of g to $\operatorname{ran}(f)$ is one-to-one. If g and f are quasi-inverses of one another, and g strictly includes $\operatorname{ran}(f)$, then g is *not* one-to-one.

- The set of real functions, with addition defined element-wise and multiplication defined as functional composition, is a ring. By remark 2, it is in fact a Von Neumann regular ring, as any quasi-inverse of a real function is also its pseudo-inverse as an element of the ring. Any space whose ring of continuous functions is Von Neumann regular is a P-space.

References

[1] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, Elsevier Science Publishing Company, (1983).