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proof of downward Lowenheim-Skolem theorem

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We present here a proof of the Downward Lowenheim Skolem Theorem. The idea is to construct a submodel that meets the requirements of the DLS Theorem: we take K and close it under a procedure of choosing appropriate witnesses for the existential formulas satisfied by $\mathcal{A}$. Choosing the appropriate witnesses is done with the help of the so-called *Skolem functions* and thus rests upon the *choice function*.

Proof: First of all we introduce a usefull tool for the proof. Lemma: (*Tarski's Lemma*) If $\mathcal{A}$ and $\mathcal{B}$ are L-structures with the domain of $\mathcal{A}$ being a subset of the domain of $\mathcal{B}$ then $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ if for every L-formula $\phi(x, y)$ with $a \in A$ and $b \in B$, we have $\mathcal{B} \models \phi(a, b)$ ¡-¿ For some $a' \in A$ we have $\mathcal{A} \models \phi(a, a')$

Proof. Let's supposes the biconditionnal holds, then we need to show that $\mathcal{A}$ is a substructure of $\mathcal{B}$. This means that we need to show that every formula that is true in $\mathcal{A}$ is true in $\mathcal{B}$. The proof is straighforward with an induction on the complexity of formulas (connectives, negation, quantifiers).

Now, fix a point $p \in dom(\mathcal{A})$. For each L-formula $\phi(x, y)$, define the *Skolem function* of ϕ, $g_\phi : A^n \to A$ (with $A = dom(\mathcal{A})$) by:

$p \in A^n \to$ some $p' \in A$ such that $\mathcal{A} \models \phi(p, p')$ if such a p' exists and p otherwise. The set of *Skolem functions* has a cardinality equal to that of L.

The purpose here is to construct a model $\mathcal{B}$ whose domain B is closed under the skolem functions. This means that the domain of $\mathcal{A}$ contains all the witnesses we've appropriately choosen. If we take an existential formula $\exists y \phi(x, y)$ and $b \in B^k$ and if we apply the Skolem function to b we will have a witness for $\exists x \phi(b, x)$. In other words, this means that $\mathcal{A} \models \exists x \phi(b, x) \to \mathcal{A} \models \phi(b, g_\phi(b))$. By construction $g_\phi(b)$ is in B and thus $\mathcal{B}$ meets the requirements of Tarski's Lemma. We can find an elementary substructure of $\mathcal{A}$.

Let's take K of above and set $K_0 := K$ and K_{i+1} is the set of the $g_\phi(p), p \in K_i \wedge g_\phi$ (with g_ϕ a Skolem function). Let $B := \bigcup K_i$. Then B is closed under Skolem functions. And we have $|K| \leq \omega.|L| + |K| = |L| + |K|$. This comes from the fact that $|L| + |K_i| = |SF| + |K_i| = \sum_{k \in \mathbb{N}} |(SF)_k| + |K_i| = \sum_{k \in \mathbb{N}} |(SF)_k| * |K_i|$ but we have $|K_{i+1}| \leq \sum_{k \in \mathbb{N}} |(SF)_k| * |K_i|$. We need now to provide interpretations of relations, predicates, functions and constants so it can fit $\mathcal{A}$.

We have because B is closed under L-terms and for an L-function symbol f, the Skolem function of the L-formula $f(x) = y$ takes the value $f^{\mathcal{A}}(p)$ at p:

for any n-ary relation symbol P: $P^{\mathcal{B}} = P^{\mathcal{A}} \bigcap B^n$

for an m-ary function symbol f and $p \in B^m, p' \in B$ we have $f^{\mathcal{A}}(p) = p'$

¡-¿ $f^{\mathcal{B}}(p) = p'$.

for a constant symbol c, we have $c^{\mathcal{A}} = c^{\mathcal{B}}$ We have constructed a substructure $\mathcal{B}$ of $\mathcal{A}$.