operations on multisets

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In this entry, we view multisets as functions whose ranges are the class K of cardinal numbers. We define operations on multisets that mirror the operations on sets.

Definition. Let $f : A \to K$ and $g : B \to K$ be multisets.

- The union of f and g, denoted by $f \cup g$, is the multiset whose domain is $A \cup B$, such that

  $$(f \cup g)(x) := \max(f(x), g(x)),$$

  keeping in mind that $f(x) := 0$ if x is not in the domain of f.

- The intersection of f and g, denoted by $f \cap g$, is the multiset, whose domain is $A \cap B$, such that

  $$(f \cap g)(x) := \min(f(x), g(x)).$$

- The sum (or disjoint union) of f and g, denoted by $f+g$, is the multiset whose domain is $A \cup B$ (not the disjoint union of A and B), such that

  $$(f + g)(x) := f(x) + g(x),$$

  again keeping in mind that $f(x) := 0$ if x is not in the domain of f.

Clearly, all of the operations described so far are commutative. Furthermore, if $+$ is cancellable on both sides: $f+g = f+h$ implies $g = h$, and $g+f = h+f$ implies $g = h$.

Subtraction on multisets can also be defined. Suppose $f : A \to K$ and $g : B \to K$ are multisets. Let C be the set $\{x \in A \cap B \mid f(x) > g(x)\}$. Then

- the complement of g in f, denoted by $f - g$, is the multiset whose domain is $D := (A - B) \cup C$, such that

  $$(f - g)(x) := f(x) - g(x)$$

  for all $x \in D$.

For example, writing finite multisets (those with finite domains and finite multiplicities for all elements) in their usual notations, if $f = \{a, a, b, b, b, c, d, d\}$ and $g = \{b, b, c, c, c, d, d, e\}$, then

- $f \cup g = \{a, a, b, b, b, c, c, c, d, d, e\}$
- $f \cap g = \{b, b, c, d, d\}$
- $f + g = \{a, a, b, b, b, b, b, c, c, c, c, d, d, d, d, e\}$
- $f - g = \{a, a, b\}$

We may characterize the union and intersection operations in terms of multisubsets.

Definition. A multiset $f : A \to K$ is a *multisubset* of a multiset $g : B \to K$ if

1. A is a subset of B, and
2. $f(a) \leq g(a)$ for all $a \in A$.

We write $f \subseteq g$ to mean that f is a multisubset of g.

Proposition 1. *Given multisets f and g.*

- $f \cup g$ is the smallest multiset such that f and g are multisubsets of it. In other words, if $f \subseteq h$ and $g \subseteq h$, then $f \cup g \subseteq h$.
- $f \cap g$ is the largest multiset that is a multisubset of f and g. In other words, if $h \subseteq f$ and $h \subseteq g$, then $h \subseteq f \cap g$.

Remark. One may also define the powerset of a multiset f: the multiset such that each of its elements is a multisubset of f. However, the resulting multiset is just a set (the multiplicity of each element is 1).