some theorem schemas of intuitionistic propositional logic

Canonical name	SomeTheoremSchemasOfIntuitionisticPropositionalLogic
Date of creation	2013-03-22 19:31:21
Last modified on	2013-03-22 19:31:21
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	22
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03B20
Classification	msc 03F55
Related topic	AxiomSystemForIntuitionisticLogic

We present some theorem schemas of intuitionistic propositional logic and their deductions, based on the axiom system given in `http://planetmath.org/AxiomSystemForInt` entry.

1. $A \vee B \to B \vee A$

Proof. From the deduction

1. $A \to B \vee A$,
2. $B \to B \vee A$,
3. $(A \to B \vee A) \to ((B \to B \vee A) \to (A \vee B \to B \vee A))$,
4. $(B \to B \vee A) \to (A \vee B \to B \vee A)$,
5. $A \vee B \to B \vee A$,

so we get $A \vee B \vdash_i B \vee A$, and therefore $\vdash_i A \vee B \to B \vee A$ by the deduction theorem. $\square$

2. $(A \wedge B) \wedge C \to A \wedge (B \wedge C)$

Proof. From the deduction

1. $(A \wedge B) \wedge C \to A \to B$,
2. $(A \wedge B) \wedge C \to C$,
3. $(A \wedge B) \wedge C$,
4. $A \wedge B$,
5. C,
6. $A \wedge B \to A$,
7. $A \wedge B \to B$,
8. A,
9. B,
10. $B \to (C \to B \wedge C)$,
11. $C \to B \wedge C$,
12. $B \wedge C$,
13. $A \to (B \wedge C \to A \wedge (B \wedge C))$,
14. $B \wedge C \to A \wedge (B \wedge C)$,
15. $A \wedge (B \wedge C)$,

so $(A \wedge B) \wedge C \vdash_i A \wedge (B \wedge C)$, and therefore $\vdash_i (A \wedge B) \wedge C \to A \wedge (B \wedge C)$ by the deduction theorem. $\square$

3. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Proof. From the deduction

1. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$,
2. $A \to B$,
3. $(A \to (B \to C)) \to (A \to C)$,
4. $A \to (B \to C)$,
5. $A \to C$,

so $A \to (B \to C), A \to B \vdash_i A \to C$. By two applications of the deduction theorem, $\vdash_i (A \to (B \to C)) \to ((A \to B) \to (A \to C))$. □

4. $A \land \neg A \to B$

Proof. From the deduction

1. $A \land \neg A \to A$,
2. $A \land \neg A \to \neg A$,
3. $A \land \neg A$,
4. $\neg A$,
5. A,
6. $\neg A \to (A \to B)$,
7. $A \to B$,
8. B

so $A \land \neg A \vdash_i B$. By the deduction theorem, $\vdash_i A \land \neg A \to B$. □

5. $A \to \neg\neg A$

Proof. From the deduction

1. $A \to (\neg A \to A)$,
2. A,
3. $\neg A \to A$,
4. $(\neg A \to A) \to ((\neg A \to \neg A) \to \neg\neg A)$,
5. $(\neg A \to \neg A) \to \neg\neg A$,
6. $\neg A \to \neg A$,
7. $\neg\neg A$,

so $A \vdash_i \neg\neg A$. By the deduction theorem, $\vdash_i A \to \neg A \neg A$. □

In the proof above, we use the schema $B \to B$ in step 6 of the deduction, because $\vdash_i B \to B$, as a result of applying the deduction theorem to $B \vdash_i B$.

6. $\neg\neg\neg A \to \neg A$

Proof. From the deduction

1. $(A \to \neg\neg A) \to ((A \to \neg\neg\neg A) \to \neg A)$,
2. $A \to \neg\neg A$,
3. $(A \to \neg\neg\neg A) \to \neg A$,
4. $\neg\neg\neg A \to (A \to \neg\neg\neg A)$,
5. $\neg\neg\neg A$,
6. $A \to \neg\neg\neg A$,
7. $\neg A$,

so $A \to \neg\neg A, \neg\neg\neg A \vdash_i \neg A$. By the deduction theorem, $A \to \neg\neg A, \vdash_i \neg\neg\neg A \to \neg A$. Since $\vdash_i A \to \neg A \neg A$, $\vdash_i \neg\neg\neg A \to \neg A$ as a result. □

In the above proof, we use the fact that if $\vdash_i C$ and $C \vdash_i D$, then $\vdash_i D$. This is the result of the following fact: if $\vdash_i C$ and $\vdash_i C \to D$, then $\vdash_i D$.

7. $(A \to B) \to (\neg B \to \neg A)$

Proof. From the deduction

1. $(A \to B) \to ((A \to \neg B) \to \neg A)$,
2. $A \to B$,
3. $(A \to \neg B) \to \neg A$,
4. $\neg B \to (A \to \neg B)$,
5. $\neg B$,
6. $A \to \neg B$,
7. $\neg A$,

so $A \to B, \neg B \vdash_i \neg A$. Applying the deduction theorem twice gives us $\vdash_i (A \to B) \to (\neg B \to \neg A)$. □

8. $\neg(A \wedge \neg A)$

Proof. From the deduction

1. $(A \wedge \neg A \to B) \to ((A \wedge \neg A \to \neg B) \to \neg(A \wedge \neg A))$,
2. $A \wedge \neg A \to B$,
3. $(A \wedge \neg A \to \neg B) \to \neg(A \wedge \neg A)$,
4. $A \wedge \neg A \to \neg B$,
5. $\neg(A \wedge \neg A)$.

Since $\vdash_i A \wedge \neg A \to B$ and $\vdash_i A \wedge \neg A \to \neg B$ are instances of theorem schema 4 above, $\vdash_i \neg(A \wedge \neg A)$ as a result. □

This also shows that $\vdash_i B \to \neg(A \wedge \neg A)$, which is the result of applying modus ponens to $\neg(A \wedge \neg A)$ to $\neg(A \wedge \neg A) \to (B \to \neg(A \wedge \neg A))$.

9. $(B \to A \wedge \neg A) \to \neg B$

Proof. From the deduction

1. $B \to A \wedge \neg A$,
2. $(B \to A \wedge \neg A) \to ((B \to \neg(A \wedge \neg A)) \to \neg B)$,
3. $(B \to \neg(A \wedge \neg A)) \to \neg B$,
4. $B \to \neg(A \wedge \neg A)$,
5. $\neg B$,

so $B \to A \wedge \neg A \vdash_i \neg B$. Applying the deduction theorem gives us $\vdash_i (B \to A \wedge \neg A) \to \neg B$. □

10. $\neg\neg(A \vee \neg A)$

Proof. From the deduction

1. $A \to (A \vee \neg A)$,
2. $(A \to (A \vee \neg A)) \to (\neg(A \vee \neg A) \to \neg A)$,
3. $\neg(A \vee \neg A) \to \neg A$,
4. $\neg A \to (A \vee \neg A)$,
5. $(\neg A \to (A \vee \neg A)) \to (\neg(A \vee \neg A) \to \neg\neg A)$,
6. $\neg(A \vee \neg A) \to \neg\neg A$,
7. $(\neg(A \vee \neg A) \to \neg A) \to ((\neg(A \vee \neg A) \to \neg\neg A) \to \neg\neg(A \vee \neg A))$,
8. $(\neg(A \vee \neg A) \to \neg\neg A) \to \neg\neg(A \vee \neg A)$,
9. $\neg\neg(A \vee \neg A)$,

□

Remark. Again from `http://planetmath.org/AxiomSystemForIntuitionisticLogic`this entry, if we use the second axiom system instead, keeping in mind that $\neg A$ means $A \rightarrow \perp$, the following are theorem schemas:

1. $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$. The proof of this is essentially the same as the proof of the third theorem schema above.

2. $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$. This is just $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow \perp)) \rightarrow (A \rightarrow \perp))$, an instance of the theorem schema above.

3. $\neg A \rightarrow (A \rightarrow B)$.

Proof. From the deduction

1. $\neg A$ (which is $A \rightarrow \perp$),
2. A,
3. $\perp$,
4. $\perp \rightarrow B$,
5. B

so $\neg A, A \vdash_i B$. By two applications of the deduction theorem, we get $\vdash_i \neg A \rightarrow (A \rightarrow B)$. □