cardinality of disjoint union of finite sets

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To begin we will need a lemma.

Lemma. *Suppose A, B, C, and D are sets, with $A \cap B = C \cap D = \emptyset$, and suppose there exist bijective maps $f_1 : A \to C$ and $f_2 : B \to D$. Then there exists a bijective map from $A \cup B$ to $C \cup D$.*

Proof. Define the map $g : A \cup B \to C \cup D$ by

$$g(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in B \end{cases}. \tag{1}$$

To see that g is injective, let $x_1, x_2 \in A \cup B$, where $x_1 \neq x_2$. If $x_1, x_2 \in A$, then by the injectivity of f_1 we have

$$g(x_1) = f_1(x_1) \neq f_1(x_2) = g(x_2). \tag{2}$$

Similarly if $x_1, x_2 \in B$, $g(x_1) \neq g(x_2)$ by the injectivity of f_2. If $x_1 \in A$ and $x_2 \in B$, then $g(x_1) = f_1(x_1) \in C$, while $g(x_2) = f_2(x_2) \in D$, whence $g(x_1) \neq g(x_2)$ because C and D are disjoint. If $x_1 \in B$ and $x_2 \in A$, then $g(x_1) \neq g(x_2)$ by the same reasoning. Thus g is injective. To see that g is surjective, let $y \in C \cup D$. If $y \in C$, then by the surjectivity of f_1 there exists some $x \in A$ such that $f_1(x) = y$, hence $g(x) = y$. Similarly if $y \in D$, by the surjectivity of f_2 there exists some $x \in B$ such that $f_2(x) = y$, hence $g(x) = y$. Thus g is surjective. □

Theorem. *If A and B are finite sets with $A \cap B = \emptyset$, then $\mid A \cup B \mid = \mid A \mid + \mid B \mid$.*

Proof. Let A and B be finite, disjoint sets. If either A or B is empty, the result holds trivially, so suppose A and B are nonempty with $\mid A \mid = n \in \mathbb{N}$ and $\mid B \mid = m \in \mathbb{N}$. Then there exist bijections $f : \mathbb{N}_n \to A$ and $g : \mathbb{N}_m \to B$. Define $h : \mathbb{N}_n \to \mathbb{N}_{n+m} \setminus \mathbb{N}_m$ by $h(i) = m + i$ for each $i \in \mathbb{N}_n$. To see that h is injective, let $i_1, i_2 \in \mathbb{N}$, and suppose $h(i_1) = h(i_2)$. Then $m + i_1 = m + i_2$, whence $i_1 = i_2$. Thus h is injective. To see that h is surjective, let $k \in \mathbb{N}_{n+m} \setminus \mathbb{N}_m$. By construction, $m + 1 \leq k \leq m + n$, and consequently $1 \leq k - m \leq n$, so $k - m \in \mathbb{N}_n$; therefore we may take $i = k - m$ to have $h(i) = k$, so h is surjective. Then, again by construction, the composition $f \circ h^{-1}$ is a bijection from $\mathbb{N}_{n+m} \setminus \mathbb{N}_m$ to A, and as such, by the preceding lemma, the map $\phi : \mathbb{N}_{n+m} \setminus \mathbb{N}_m \cup \mathbb{N}_m \to A \cup B$ defined by

$$\phi(i) = \begin{cases} (f \circ h^{-1})(i) & \text{if } i \in \mathbb{N}_{n+m} \setminus \mathbb{N}_m \\ g(i) & \text{if } i \in \mathbb{N}_m \end{cases}, \tag{3}$$

is a bijection. Of course, the domain of ϕ is simply $\mathbb{N}_{n+m}$, so $\mid A \cup B \mid = n+m$, as asserted. $\square$

Corollary. *Let* $\{A_k\}_{k=1}^{n}$ *be a family of mutually disjoint, finite sets. Then* $\mid \bigcup_{k=1}^{n} A_k \mid = \sum_{k=1}^{n} \mid A_k \mid$.

Proof. We proceed by induction on n. In the case $n = 2$, the the preceding result applies, so let $n \geq 2 \in \mathbb{N}$, and suppose $\mid \bigcup_{k=1}^{n} A_k \mid = \sum_{k=1}^{n} \mid A_k \mid$. Then by our inductive hypothesis and the preceding result, we have

$$\left| \bigcup_{k=1}^{n+1} A_k \right| = \left| \bigcup_{k=1}^{n} A_k \cup A_{n+1} \right| = \left| \bigcup_{k=1}^{n} A_k \right| + \mid A_{n+1} \mid = \sum_{k=1}^{n} \mid A_k \mid + \mid A_{n+1} \mid = \sum_{k=1}^{n+1} \mid A_k \mid . \tag{4}$$

Thus the result holds for $n+1$, and by the principle of induction, for all n. $\square$