canonical ordering on pairs of ordinals

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The lexicographic ordering on **On**$\times$**On**, the class of all pairs of ordinals, is a well-order in the broad sense, in that every subclass of **On**$\times$**On** has a least element, as proposition 2 of the parent entry readily shows. However, with this type of ordering, we get initial segments which are not sets. For example, the initial segment of $(1,0)$ consists of all ordinal pairs of the form $(0,\alpha)$, where $\alpha \in$ **On**, and is easily seen to be a proper class. So the questions is: is there a way to order **On**$\times$**On** such that every initial segment of **On**$\times$**On** is a set? The answer is yes. The construction of such a well-ordering in the following discussion is what is known as the *canonical well-ordering* of **On**$\times$**On**.

To begin, let us consider a strictly linearly ordered set $(A,<)$. We construct a binary relation $\prec$ on $A\times A$ as follows:

$$(a_1,a_2)\prec(b_1,b_2) \quad \text{iff} \quad \begin{cases} \max\{a_1,a_2\}<\max\{b_1,b_2\}, \text{ or} \\ \max\{a_1,a_2\}=\max\{b_1,b_2\}, \text{ and } a_1<b_1, \text{ or} \\ \max\{a_1,a_2\}=\max\{b_1,b_2\}, \text{ and } a_1=b_1, \text{ and } a_2<b_2. \end{cases}$$

For example, consider the usual ordering on $\mathbb{Z}$. Given $(p,q)\in\mathbb{Z}\times\mathbb{Z}$. Suppose $p\le q$. Then the set of all $(m,n)\in\mathbb{Z}\times\mathbb{Z}$ such that $(m,n)\prec(p,q)$ is the union of the three pairwise disjoint sets $\{(m,n)\mid \max\{m,n\}<q\}\cup\{(m,q)\mid m<p\}\cup\{(p,n)\mid n<q\}$.

Proposition 1. . *$\prec$ is a strict linear ordering on $A\times A$.*

Proof. It is irreflexive because (a_1,a_2) is never comparable with itself. It is linear because, first of all, given $(a_1,a_2)\neq(b_1,b_2)$, exactly one of the three conditions is true, and hence either $(a_1,a_2)\prec(b_1,b_2)$, or $(b_1,b_2)\prec(a_1,a_2)$. It remains to show that $\prec$ is transitive, suppose $(a_1,a_2)\prec(b_1,b_2)$ and $(b_1,b_2)\prec(c_1,c_2)$.

The two cases

1. $\max\{a_1,a_2\}<\max\{b_1,b_2\}$ and $\max\{b_1,b_2\}\le\max\{c_1,c_2\}$,

2. $\max\{a_1,a_2\}\le\max\{b_1,b_2\}$ and $\max\{b_1,b_2\}<\max\{c_1,c_2\}$,

produce $\max\{a_1,a_2\}<\max\{c_1,c_2\}$. Now, assume $\max\{a_1,a_2\}=\max\{b_1,b_2\}=\max\{c_1,c_2\}$, which result in three more cases

1. $a_1<b_1$ and $b_1\le c_1$,

2. $a_1\le b_1$ and $b_1<c_1$,

3. $a_1 = b_1 = c_1$, and $a_2 < b_2$ and $b_2 < c_2$,

the first two produce $a_1 < c_1$, and the last $a_1 = c_1$ and $a_2 < c_2$. In all cases, we get $(a_1, a_2) \prec (c_1, c_2)$. $\square$

Proposition 2. *If $<$ is a well-order on A, then so is $\prec$ on $A \times A$.*

Proof. Let $R \subseteq A \times A$ be non-empty. Let

$$B := \{\max\{b_1, b_2\} \mid (b_1, b_2) \in R\}.$$

Then $\varnothing \neq B \subseteq A$, and therefore has a least element b, since $<$ is a well-order on A. Next, let

$$C := \{c_1 \mid \max\{c_1, c_2\} = b, \text{ where } (c_1, c_2) \in R\}.$$

Then $C \neq \varnothing$, and has a least element c. Finally, let

$$D := \{d_2 \mid \max\{c, d_2\} = b, \text{ where } (c, d_2) \in R\}.$$

Again, $D \neq \varnothing$, so has a least element d. So $(c, d) \in R$. We want to show that (c, d) is the least element of R.

Pick any $(x, y) \in R$ distinct from (c, d). Then $\max\{x, y\} \in B$ is at least $b = \max\{c, d\}$. If $b < \max\{x, y\}$, then $(c, d) \prec (x, y)$. Otherwise, $b = \max\{x, y\}$, so that $x \in C$ is at least c. If $c < x$, then again we have $(c, d) \prec (x, y)$. But if $c = x$, then $y \in D$, so that $d \leq y$. Since $(x, y) \neq (c, d)$, and $x = c$, $y \neq d$. Therefore $d < y$, and $(c, d) \prec (x, y)$ as a result. $\square$

The ordering relation above can be generalized to arbitrary classes. Since **On** is well-ordered by $\in$, the canonical ordering on **On**$\times$**On** is a well-ordering by proposition 2. Moreover,

Proposition 3. *Given the canonical ordering $\prec$ on **On**$\times$**On**, every initial segment is a set.*

Proof. Given ordinals $\alpha, \beta \in$**On**, suppose $\lambda = \max\{\alpha, \beta\}$. The initial segment of (α, β) is the union of the following collections

1. $\{(\gamma, \delta) \mid \max\{\gamma, \delta\} < \lambda\}$, which is a subcollection of $\lambda \times \lambda$,

2. $\{(\gamma, \delta) \mid \max\{\gamma, \delta\} = \lambda, \text{ and } \gamma < \alpha\}$, which again is a subcollection $\lambda \times \lambda$, and

3. $\{(\alpha, \delta) \mid \max\{\alpha, \delta\} = \lambda, \text{ and } \delta < \beta\}$, which is a subcollection of $\{\alpha\} \times \beta$.

Since $\lambda \times \lambda$ and $\{\alpha\} \times \beta$ are both sets, so is the initial segment of (α, β). $\square$

Remark. The canonical well-ordering on $\mathbf{On} \times \mathbf{On}$ can be used to prove a well-known property of alephs: $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$, for any ordinal α.