Lindenbaum-Tarski algebra

Canonical name	LindenbaumTarskiAlgebra
Date of creation	2013-03-22 12:42:40
Last modified on	2013-03-22 12:42:40
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	31
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03G05
Classification	msc 03B05
Classification	msc 03B10
Synonym	Lindenbaum algebra

Lindenbaum-Tarski algebra of a propositional langauge

Let L be a classical propositional language. We define the equivalence relation $\sim$ over formulas of L by $\varphi \sim \psi$ if and only if $\vdash \varphi \Leftrightarrow \psi$. Let $B = L/\sim$ be the set of equivalence classes. We define the operations join $\vee$, meet $\wedge$, and complementation, denoted $[\varphi]'$ on B by :

$$\begin{aligned} [\varphi] \vee [\psi] &:= [\varphi \vee \psi] \\ [\varphi] \wedge [\psi] &:= [\varphi \wedge \psi] \\ [\varphi]' &:= [\neg\varphi] \end{aligned}$$

We let $0 = [\varphi \wedge \neg\varphi]$ and $1 = [\varphi \vee \neg\varphi]$. Then the structure $(B, \vee, \wedge, ', 0, 1)$ is a Boolean algebra, called the *Lindenbaum-Tarski algebra* of the propositional language L.

Intuitively, this algebra is an algebra of logical statements in which logically equivalent formulations of the same statement are not distinguished. One can develop intuition for this algrebra by considering a simple case. Suppose our language consists of a number of statement symbols P_i and the connectives $\vee, \wedge, \neg$ and that $\vdash$ denotes tautologies. Then our algebra consists of statements formed from these connectives with tautologously equivalent satements reckoned as the same element of the algebra. For instance, "$\neg(P_1 \wedge P_2)$" is considered the same as "$\neg P_1 \vee \neg P_2$". Furthermore, since any statement of propositional calculus may be recast in disjunctive normal form, we may view this particular Lindenbaum-Tarski algebra as a Boolean analogue of polynomials in the P_i's and their negations.

It can be shown that the Lindenbaum-Tarski algebra of the propositional language L is a free Boolean algebra freely generated by the set of all elements $[p]$, where each p is a propositional variable of L

Lindenbaum-Tarski algebra of a first order langauge

Now, let L be a first order language. As before, we define the equivalence relation $\sim$ over formulas of L by $\varphi \sim \psi$ if and only if $\vdash \varphi \Leftrightarrow \psi$. Let $B = L/\sim$ be the set of equivalence classes. The operations $\vee$ and $\wedge$ and complementation on B are defined exactly the same way as previously. The resulting algebra is the Lindenbaum-Tarski algebra of the first order language

L. It may be shown that

$$\bigvee_{t\in T}[\varphi(t)] := [\exists x\varphi(x)]$$
$$\bigwedge_{t\in T}[\varphi(t)] := [\forall x\varphi(x)]$$

where T is the set of all terms in the language L. Basically, these results say that the statement $\exists x\varphi(x)$ is equivalent to taking the supremum of all statements $\varphi(x)$ where x ranges over the entire set V of variables. In other words, if one of these statements is true (with truth value 1, as opposed to 0), then $\exists x\varphi(x)$ is true. The statement $\forall x\varphi(x)$ can be similarly analyzed.

Remark. It may possible to define the Lindenbaum-Tarski algebra on logical languages other than the classical ones mentioned above, as long as there is a notion of formal proof that can allow the definition of the equivalence relation. For example, one may form the Lindenbaum-Tarski algebra of an intuitionistic propositional language (or predicate language) or a normal modal propositional language. The resulting algebra is a Heyting algebra (or a complete Heyting algebra) for intuitionistic propositional language (or predicate language), or a Boolean algebras with an operator for normal modal propositional languages.