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cardinal arithmetic

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Defines	cardinal addition
Defines	cardinal multiplication
Defines	cardinal exponentiation
Defines	sum of cardinals
Defines	product of cardinals
Defines	addition
Defines	multiplication
Defines	exponentiation
Defines	sum
Defines	product

Definitions

Let κ and λ be cardinal numbers, and let A and B be disjoint sets such that $|A|=\kappa$ and $|B|=\lambda$. (Here $|X|$ denotes the cardinality of a set X, that is, the unique cardinal number equinumerous with X.) Then we define cardinal addition, cardinal multiplication and cardinal exponentiation as follows.

$$\begin{aligned}
\kappa+\lambda &= |A\cup B|. \\
\kappa\lambda &= |A\times B|. \\
\kappa^{\lambda} &= |A^{B}|.
\end{aligned}$$

(Here A^B denotes the set of all functions from B to A.) These three operations are well-defined, that is, they do not depend on the choice of A and B. Also note that for multiplication and exponentiation A and B do not actually need to be disjoint.

We also define addition and multiplication for arbitrary numbers of cardinals. Suppose I is an index set and κ_i is a cardinal for every $i\in I$. Then $\sum_{i\in I}\kappa_i$ is defined to be the cardinality of the union $\bigcup_{i\in I}A_i$, where the A_i are pairwise disjoint and $|A_i|=\kappa_i$ for each $i\in I$. Similarly, $\prod_{i\in I}\kappa_i$ is defined to be the cardinality of the `http://planetmath.org/GeneralizedCartesianProduct`Cartesian product $\prod_{i\in I}B_i$, where $|B_i|=\kappa_i$ for each $i\in I$.

Properties

In the following, κ, λ, μ and ν are arbitrary cardinals, unless otherwise specified.

Cardinal arithmetic obeys many of the same algebraic laws as real arithmetic. In particular, the following properties hold.

$$\begin{aligned}
\kappa+\lambda &= \lambda+\kappa. \\
(\kappa+\lambda)+\mu &= \kappa+(\lambda+\mu). \\
\kappa\lambda &= \lambda\kappa. \\
(\kappa\lambda)\mu &= \kappa(\lambda\mu). \\
\kappa(\lambda+\mu) &= \kappa\lambda+\kappa\mu. \\
\kappa^{\lambda}\kappa^{\mu} &= \kappa^{\lambda+\mu}. \\
(\kappa^{\lambda})^{\mu} &= \kappa^{\lambda\mu}. \\
\kappa^{\mu}\lambda^{\mu} &= (\kappa\lambda)^{\mu}.
\end{aligned}$$

Some special cases involving 0 and 1 are as follows:

$$
\begin{aligned}
\kappa + 0 &= \kappa. \\
0\kappa &= 0. \\
\kappa^0 &= 1. \\
0^\kappa &= 0, \text{ for } \kappa > 0. \\
1\kappa &= \kappa. \\
\kappa^1 &= \kappa. \\
1^\kappa &= 1.
\end{aligned}
$$

If at least one of κ and λ is infinite, then the following hold.

$$
\begin{aligned}
\kappa + \lambda &= \max(\kappa, \lambda). \\
\kappa\lambda &= \max(\kappa, \lambda), \text{ provided } \kappa \neq 0 \neq \lambda.
\end{aligned}
$$

Also notable is that if κ and λ are cardinals with λ infinite and $2 \leq \kappa \leq 2^\lambda$, then

$$\kappa^\lambda = 2^\lambda.$$

Inequalities are also important in cardinal arithmetic. The most famous is Cantor's theorem

$$\kappa < 2^\kappa.$$

If $\mu \leq \kappa$ and $\nu \leq \lambda$, then

$$
\begin{aligned}
\mu + \nu &\leq \kappa + \lambda. \\
\mu\nu &\leq \kappa\lambda. \\
\mu^\nu &\leq \kappa^\lambda, \text{ unless } \mu = \nu = \kappa = 0 < \lambda.
\end{aligned}
$$

Similar inequalities hold for infinite sums and products. Let I be an index set, and suppose that κ_i and λ_i are cardinals for every $i \in I$. If $\kappa_i \leq \lambda_i$ for every $i \in I$, then

$$
\begin{aligned}
\sum_{i \in I} \kappa_i &\leq \sum_{i \in I} \lambda_i. \\
\prod_{i \in I} \kappa_i &\leq \prod_{i \in I} \lambda_i.
\end{aligned}
$$

If, moreover, $\kappa_i < \lambda_i$ for all $i \in I$, then we have König's theorem.

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i.$$

If $\kappa_i = \kappa$ for every i in the index set I, then

$$\sum_{i \in I} \kappa_i = \kappa |I|.$$

$$\prod_{i \in I} \kappa_i = \kappa^{|I|}.$$

Thus it is possible to define exponentiation in terms of multiplication, and multiplication in terms of addition.