product of non-empty set of non-empty sets is non-empty

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In this entry, we show that the statement:

> (*) the non-empty generalized cartesian product of non-empty sets is non-empty

is equivalent to the axiom of choice (AC).

Proposition 1. *AC implies (*).*

Proof. Suppose $C = \{A_j \mid j \in J\}$ is a set of non-empty sets, with $J \neq \varnothing$. We want to show that

$$B := \prod_{j \in J} A_j$$

is non-empty. Let $A = \bigcup C$. Then, by AC, there is a function $f : C \to A$ such that $f(X) \in X$ for every $X \in C$. Define $g : J \to A$ by $g(j) := f(A_j)$. Then $g \in B$ as a result, B is non-empty. □

Remark. The statement that if $J \neq \varnothing$, then $B \neq \varnothing$ implies $A_j \neq \varnothing$ does not require AC: if B is non-empty, then there is a function $g : J \to A$, and, as $J \neq \varnothing$, $g \neq \varnothing$, which means $A \neq \varnothing$, or that $A_j \neq \varnothing$ for some $j \in J$.

Proposition 2. *(*) implies AC.*

Proof. Suppose C is a set of non-empty sets. If C itself is empty, then the choice function is the empty set. So suppose that C is non-empty. We want to find a (choice) function $f : C \to \bigcup C$, such that $f(x) \in x$ for every $x \in C$. Index elements of C by C itself: $A_x := x$ for each $x \in C$. So $A_x \neq \varnothing$ by assumption. Hence, by (*), the (non-empty) cartesian product B of the A_x is non-empty. But an element of B is just a function f whose domain is C and whose codomain is the union of the A_x, or $\bigcup C$, such that $f(A_x) \in A_x$, which is precisely $f(x) \in x$. □