example of antisymmetric

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Owner	Algeboy (12884)
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The axioms of a partial ordering demonstrate that every partial ordering is antisymmetric. That is: the relation $\leq$ on a set S forces

$$a \leq b \text{ and } b \leq a \text{ implies } a = b$$

for every $a, b \in S$.

For a concrete example consider the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ (as defined by the `http://planetmath.org/PeanoArithmetic`Peano postulates). Take the relation set to be:

$$R = \{(a, a+n) : a, n \in \mathbb{N}\} \subset \mathbb{N} \times \mathbb{N}.$$

Then we denote $a \leq b$ if $(a, b) \in R$. That is, $5 \leq 7$ because $(5, 7) = (5, 5+2)$ and both $5, 2 \in \mathbb{N}$.

We can prove this relation is antisymmetric as follows: Suppose $a \leq b$ and $b \leq a$ for some $a, b \in \mathbb{N}$. Then there exist $n, m \in \mathbb{N}$ such that $a + n = b$ and $b + m = a$. Therefore

$$b = a + n = b + m + n$$

so by the cancellation property of the natural numbers, $0 = m + n$. But by the first piano postulate, 0 has no predecessor, meaning $0 \neq m + n$ unless $m = n = 0$.