properties of symmetric difference

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Owner	CWoo (3771)
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Author	CWoo (3771)
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Recall that the symmetric difference of two sets A, B is the set $A \cup B - (A \cap B)$. In this entry, we list and prove some of the basic properties of $\triangle$.

1. (commutativity of $\triangle$) $A \triangle B = B \triangle A$, because $\cup$ and $\cap$ are commutative.

2. If $A \subseteq B$, then $A \triangle B = B - A$, because $A \cup B = B$ and $A \cap B = A$.

3. $A \triangle \varnothing = A$, because $\varnothing \subseteq A$, and $A - \varnothing = A$.

4. $A \triangle A = \varnothing$, because $A \subseteq A$ and $A - A = \varnothing$.

5. $A \triangle B = (A - B) \cup (B - A)$ (hence the name symmetric difference).

   Proof. $A \triangle B = (A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)' = (A \cup B) \cap (A' \cup B') = ((A \cup B) \cap A') \cup ((A \cup B) \cap B') = (B \cap A') \cup (A \cap B') = (B - A) \cup (A - B)$. □

6. $A' \triangle B' = A \triangle B$, because $A' \triangle B' = (A' - B') \cup (B' - A') = (A' \cap B) \cup (B' \cap A) = (B - A) \cap (A - B) = A \triangle B$.

7. (distributivity of $\cap$ over $\triangle$) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

   Proof. $A \cap (B \triangle C) = A \cap ((B \cup C) - (B \cap C))$, which is $(A \cap (B \cup C)) - (A \cap (B \cap C))$, one of the properties of set difference (see proof `http://planetmath.org/PropertiesOfSetDifference`here). This in turns is equal to $((A \cap B) \cup (A \cap C)) - ((A \cap B) \cap (A \cap C)) = (A \cap B) \triangle (A \cap C)$. □

8. (associativity of $\triangle$) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

   Proof. Let U be a set containing A, B, C as subsets (take $U = A \cup B \cup C$ if necessary). For a given B, let $f : P(U) \times P(U) \to P(U)$ be a function defined by $f(A, C) = (A \triangle B) \triangle C$. Associativity of $\triangle$ is then then same as showing that $f(A, C) = f(C, A)$, since $A \triangle (B \triangle C) = (B \triangle C) \triangle A = (C \triangle B) \triangle A$.

By expanding $f(A,C)$, we have

$$\begin{aligned}
(A\triangle B)\triangle C &= ((A\triangle B)-C)\cup(C-(A\triangle B)) \\
&= (((A-B)\cup(B-A))\cap C')\cup(C-((A\cup B)-(A\cap B))) \\
&= (((A\cap B')\cup(B\cap A'))\cap C')\cup((C\cap A\cap B)\cup(C-(A\cup B)) \\
&= ((A\cap B'\cap C')\cup(B\cap A'\cap C'))\cup((C\cap A\cap B)\cup(C\cap A'\cap B')) \\
&= (B\cap A'\cap C')\cup(B\cap A\cap C)\cup(B'\cap A\cap C')\cup(B'\cap A'\cap C).
\end{aligned}$$

It is now easy to see that the last expression does not change if one exchanges A and C. Hence, $f(A,C)=f(C,A)$ and this shows that $\triangle$ is associative. $\square$

Remark. All of the properties of $\triangle$ on sets can be generalized to http://planetmath.org/DerivedBooleanOperations$\triangle$ on Boolean algebras.