König's theorem

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König's Theorem is a theorem of cardinal arithmetic.

Theorem 1. *Let κ_i and λ_i be cardinals, for all i in some index set I. If $\kappa_i < \lambda_i$ for all $i \in I$, then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

The theorem can also be stated for arbitrary sets, as follows.

Theorem 2. *Let A_i and B_i be sets, for all i in some index set I. If $|A_i| < |B_i|$ for all $i \in I$, then*

$$\left|\bigcup_{i\in I} A_i\right| < \left|\prod_{i\in I} B_i\right|.$$

Proof. Let $\varphi\colon \bigcup_{i\in I} A_i \to \prod_{i\in I} B_i$ be a function. For each $i \in I$ we have $|\varphi(A_i)| \leq |A_i| < |B_i|$, so there is some $x_i \in B_i$ that is not equal to $(\varphi(a))(i)$ for any $a \in A_i$. Define $f\colon I \to \bigcup_{i\in I} B_i$ by $f(i) = x_i$ for all $i \in I$. For any $i \in I$ and any $a \in A_i$, we have $f(i) \neq (\varphi(a))(i)$, so $f \neq \varphi(a)$. Therefore f is not in the image of φ. This shows that there is no surjection from $\bigcup_{i\in I} A_i$ onto $\prod_{i\in I} B_i$. As $\prod_{i\in I} B_i$ is nonempty, this also means that there is no injection from $\prod_{i\in I} B_i$ into $\bigcup_{i\in I} A_i$. This completes the proof of Theorem 2. Theorem 1 follows as an immediate corollary. □

Note that the above proof is a diagonal argument, similar to the proof of Cantor's Theorem. In fact, Cantor's Theorem can be considered as a special case of König's Theorem, taking $\kappa_i = 1$ and $\lambda_i = 2$ for all i.

Also note that Theorem 2 is equivalent (in ZF) to the Axiom of Choice, as it implies that `http://planetmath.org/GeneralizedCartesianProduct`products of nonempty sets are nonempty. (Theorem 1, on the other hand, is not meaningful without the Axiom of Choice.)