modal logic S4

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The modal logic **S4** is the smallest normal modal logic containing the following schemas:

- (T) $\Box A \to A$, and
- (4) $\Box A \to \Box\Box A$.

In `http://planetmath.org/ModalLogicT`this entry, we show that T is valid in a frame iff the frame is reflexive.

Proposition 1. *4 is valid in a frame $\mathcal{F}$ iff $\mathcal{F}$ is transitive.*

Proof. First, suppose $\mathcal{F}$ is a frame validating 4, with wRu and uRt. Let M be a model with $V(p) = \{v \mid wRv\}$, where p a propositional variable. So $\models_w \Box p$. By assumption, we have $\models_w \Box p \to \Box\Box p$. Then $\models_w \Box\Box p$. This means $\models_v \Box p$ for all v such that wRv. Since wRu, $\models_u \Box p$, which means $\models_s p$ for all s such that uRs. Since uRt, we have $\models_t p$, or $t \in V(p)$, or wRt. Hence R is transitive.

Conversely, let $\mathcal{F}$ be a transitive frame, M a model based on $\mathcal{F}$, and w any world in M. Suppose $\models_w \Box A$. We want to show $\models_w \Box\Box A$, or for all u with wRu, we have $\models_u \Box A$, or for all u with wRu and all t with uRt, we have $\models_t A$. If wRu and uRt, wRt since R is transitive. Then $\models_t A$ by assumption. Therefore, $\models_w \Box A \to \Box\Box A$. $\Box$

As a result,

Proposition 2. ***S4** is sound in the class of preordered frames.*

Proof. Since any theorem in **S4** is deducible from a finite sequence consisting of tautologies, which are valid in any frame, instances of T, which are valid in reflexive frames, instances of 4, which are valid in transitive frames by the proposition above, and applications of modus ponens and necessitation, both of which preserve validity in any frame, whence the result. $\Box$

In addition, using the canonical model of **S4**, which is preordered, we have

Proposition 3. ***S4** is complete in the class of serial frames.*

Proof. Since **S4** contains T, its canonical frame $\mathcal{F}_{\mathbf{S4}}$ is reflexive. We next show that the canonical frame $\mathcal{F}_\Lambda$ of any consistent normal logic Λ containing the schema 4 must be transitive. So suppose $wR_\Lambda u$ and $uR_\Lambda v$. If $A \in \Delta_w := \{B \mid \Box B \in w\}$, then $\Box A \in w$, or $\Box\Box A \in w$ by modus ponens on 4 and the fact that w is closed under modus ponens. Hence $\Box A \in \Delta_w$, or $\Box A \in u$ since $wR_\Lambda u$, or $A \in \Delta_u$, or $A \in v$ since $uR_\Lambda v$. As a result, $wR_\Lambda v$, and therefore $\mathcal{F}_{\mathbf{S4}}$ is a preordered frame. □

By a proper translation, one can map intuitionistic propositional logic PL_i into **S4**, so that a wff of PL_i is a theorem iff its translate is a theorem of **S4**.