properties of Heyting algebras

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Proposition 1. *Let H be a Brouwerian lattice. The following properties hold:*

1. $a \to a = 1$
2. $a \wedge (a \to b) = a \wedge b$
3. $b \wedge (a \to b) = b$
4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$

Proof. The first three equations are proved in this `http://planetmath.org/BrouwerianLatticeen` We prove the last equation here. For any $x \in H$, $x \leq a \to (b \wedge c)$ iff $x \wedge a \leq b \wedge c$ iff $x \wedge a \leq b$ and $x \wedge a \leq c$ iff $x \leq a \to b$ and $x \leq a \to c$ iff $x \leq (a \to b) \wedge (a \to c)$. Hence the equation holds. □

Proposition 2. *Conversely, a lattice with a binary operation $\to$ satisfying the four conditions above is a Brouwerian lattice.*

Proof. Let H be a lattice with a binary operation $\to$ on it satisfying the identities above. We want to show that $x \leq a \to b$ iff $x \wedge a \leq b$ for any $x \in H$. First, suppose $x \leq a \to b$. Then $x \wedge a \leq a \wedge (a \to b) = a \wedge b \leq b$. Conversely, suppose $x \wedge a \leq b$. Then $a \to (x \wedge a) \leq a \to b$ by the property 6 in `http://planetmath.org/BrouwerianLattice`this entry. As a result, $x = x \wedge (a \to x) \leq (a \to a) \wedge (a \to x) = a \to (a \wedge x) \leq a \to b$. □

Corollary 1. *The class of Brouwerian lattices is equational. The class of Heyting algebras is equational.*

Proof. The first fact is the result of the two propositions above. The second comes from the fact that 0 is not used in the proofs of the propositions. □

Proposition 3. *Let H be a Heyting algebra. Then $a \vee a^* = 1$ iff $a^{**} = a$ for all $a \in H$.*

Proof. Suppose $a \vee a^* = 1$. Since $a \leq a^{**}$ in any Heyting algebra, we only need to show that $a^{**} \leq a$. Since H is distributive, we have $a^{**} = a^{**} \wedge (a \vee a^*) = (a^{**} \wedge a) \vee (a^{**} \wedge a^*) = a^{**} \wedge a$. The last equation comes from the fact that $a^{**} \wedge a^* = 0$. As a result, $a^{**} \leq a$. Conversely, suppose $a^{**} = a$. Now, $(a \vee a^*)^* \leq a^* \wedge a^{**} = 0$, and therefore $a \vee a^* = (a \vee a^*)^{**} = 0^* = 1$. □

Note, the last inequality in the proof above comes from the inequality $(a \vee b)^* \leq a^* \wedge b^*$, which is a direct consequence of the fact that pseudocomplementation is order-reversing: $x \leq y$ implies that $y^* \leq x^*$.

Corollary 2. *A Heyting algebra where psuedocomplentation $*$ satisfies the equivalent conditions above is a Boolean algebra. Conversely, a Boolean algebra with $a \to b := a^* \vee b$ is a Heyting algebra.*

Proof. Since $a \wedge a^* = 0$ and $a \vee a^* = 1$, the pseudocomplementation operation $*$ is the complementation operation. And because any Heyting algebra is distributive, it is Boolean as a result. Conversely, assume B is Boolean. Then $c \leq a \to b = a^* \vee b$, so that $c \wedge a \leq a \wedge (a^* \vee b) = a \wedge b \leq b$. On the other hand, if $c \wedge a \leq b$, then $c \leq c \vee a^* = (c \wedge a) \vee a^* \leq a^* \vee b = a \to b$. $\square$

Proposition 4. *A subset F of a Heyting algebra H is an ultrafilter iff there is a Heyting algebra homomorphism $f : H \to \{0, 1\}$ with $F = f^{-1}(1)$.*

Proof. First, assume $f : H \to \{0, 1\}$ is a Heyting algebra homomorphism, and $F = f^{-1}(1)$. Clearly, F is a filter. Suppose $0 \neq a \notin F$, then $f(a) = 0$. Now, $f(a^*) = f(a)^* = 0^* = 1$, so $a^* \in F$. If F is not maximal, let G be a proper filter containing F and a, then $a^* \in G$, so that $0 \in a \wedge a^* \in G$, and hence $G = H$, contradicting the fact that G is proper. So F is maximal.

Conversely, suppose F is an ultrafilter of H. Define $f : H \to \{0, 1\}$ by $f(x) = 1$ iff $x \in F$. Let $a, b \in H$. We first show that f is a lattice homomorphism:

- First, $f(a \wedge b) = 1$ iff $a \wedge b \in F$ iff $a, b \in F$ (since F is a filter) iff $f(a) = f(b) = 1$. So f respects $\wedge$.
- Next, if $f(a \vee b) = 0$, then $a \vee b \notin F$, which means neither a nor b is in F, or that $f(a) = f(b) = 0$. On the other hand, if $f(a) = f(b) = 0$, then neither a nor b is in F, since F is an ultrafilter. As a result, neither is $a \vee b \in F$, which means $f(a \vee b) = 0$. So f respects $\vee$.

So f is a lattice homomorphism. Next, we show that f is a Heyting algebra homomorphism, which means showing that f respects $\to$: $f(a \to b) = f(a) \to f(b)$. It suffices to show $f(a \to b) = 0$ iff $f(a) = 1$ and $f(b) = 0$.

- First, if $f(a) = 1$ and $f(b) = 0$ then $a \in F$ and $b \notin F$. If $a \to b \in F$, then $(a \to b) \wedge a \in F$. Since $(a \to b) \wedge a \leq b$, $b \in F$, a contradiction. So $a \to b \notin F$.

- On the other hand, suppose $f(a \to b) = 0$. So $a \to b \notin F$. Now, since $b \leq a \to b$, $b \notin F$, or $f(b) = 0$. If $f(a) = 0$, then $a \notin F$, so there is some $c \in F$ with $0 = a \wedge c$. But this means $c \leq a^*$, or $a^* \in F$. Since $a^* \leq a \to b$, we would have $a \to b \in F$, a contradiction. Hence $f(a) = 1$.

Therefore f is a Heyting algebra homomorphism. $\square$

In the proof above, we use the fact that, for any ultrafilter F in a bounded lattice L, if $x \notin F$, then there is $y \in F$ such that $0 = x \wedge y$ (for otherwise, the filter generated by x and F would be proper and properly contains F, contradicting the maximality of F). If in addition L were distributive, then $a \vee b \in F$ implies that either $a \in F$ or $b \in F$. To see this, suppose $a \notin F$. Then there is $c \in F$ such that $0 = a \wedge c$. Similarly, if $b \notin F$, there is $d \in F$ such that $0 = b \wedge d$. Let $e = c \wedge d \in F$. So $e \neq 0$, and $a \wedge e = 0 = b \wedge e$. Furthermore, $0 = (a \wedge e) \vee (b \wedge e) = (a \vee b) \wedge e$. If $a \vee b \in F$, so would $0 \in F$, a contradiction. Hence $a \vee b \notin F$.