the inverse image commutes with set operations

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Theorem. Let f be a mapping from X to Y. If $\{B_i\}_{i\in I}$ is a (possibly uncountable) collection of subsets in Y, then the following relations hold for the inverse image:

(1) $f^{-1}\big(\bigcup_{i\in I} B_i\big) = \bigcup_{i\in I} f^{-1}\big(B_i\big)$

(2) $f^{-1}\big(\bigcap_{i\in I} B_i\big) = \bigcap_{i\in I} f^{-1}\big(B_i\big)$

If A and B are subsets in Y, then we also have:

(3) For the set complement,

$$\big(f^{-1}(A)\big)^{\complement} = f^{-1}(A^{\complement}).$$

(4) For the set difference,

$$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B).$$

(5) For the symmetric difference,

$$f^{-1}(A \triangle B) = f^{-1}(A) \triangle f^{-1}(B).$$

Proof. For part (1), we have

$$\begin{aligned}
f^{-1}\big(\bigcup_{i\in I} B_i\big) &= \Big\{x \in X \mid f(x) \in \bigcup_{i\in I} B_i\Big\} \\
&= \{x \in X \mid f(x) \in B_i \text{ for some } i \in I\} \\
&= \bigcup_{i\in I} \{x \in X \mid f(x) \in B_i\} \\
&= \bigcup_{i\in I} f^{-1}\big(B_i\big).
\end{aligned}$$

Similarly, for part (2), we have

$$\begin{aligned}
f^{-1}\big(\bigcap_{i\in I} B_i\big) &= \{x \in X \mid f(x) \in \bigcap_{i\in I} B_i\} \\
&= \{x \in X \mid f(x) \in B_i \text{ for all } i \in I\} \\
&= \bigcap_{i\in I} \{x \in X \mid f(x) \in B_i\} \\
&= \bigcap_{i\in I} f^{-1}\big(B_i\big).
\end{aligned}$$

For the set complement, suppose $x \notin f^{-1}(A)$. This is equivalent to $f(x) \notin A$, or $f(x) \in A^{\complement}$, which is equivalent to $x \in f^{-1}(A^{\complement})$. Since the set difference $A \setminus B$ can be written as $A \cap B^c$, part (4) follows from parts (2) and (3). Similarly, since $A \triangle B = (A \setminus B) \cup (B \setminus A)$, part (5) follows from parts (1) and (4). $\square$