properties of a function

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Owner	CWoo (3771)
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Let X, Y be sets and $f : X \to Y$ be a function. For any $A \subseteq X$, define

$$f(A) := \{f(x) \in Y \mid x \in A\}$$

and any $B \subseteq Y$, define

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

So $f(A)$ is a subset of Y and $f^{-1}(B)$ is a subset of X.

Let A, A_1, A_2, A_i be arbitrary subsets of X and B, B_1, B_2, B_j be arbitrary subsets of Y, where i belongs to the index set I and j to the index set J. We have the following properties:

1. If $A_1 \subset A_2$, then $f(A_1) \subseteq f(A_2)$. In particular, $f(A) \subseteq f(X)$.

2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. More generally, $f(\bigcup_i A_i) = \bigcup_i f(A_i)$.

3. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. The equality fails in the example where f is a real function defined by $f(x) = x^2$ and $A_1 = \{1\}$, $A_2 = \{-1\}$. Equality occurs iff f is one-to-one:

   > Suppose $f(x) = f(y) = z$. Pick $A_1 = \{x\}$ and $A_2 = \{y\}$. Then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2) = \{z\} \neq \varnothing$. This means that $A_1 \cap A_2 \neq \varnothing$. Since both A_1 and A_2 are singletons, $A_1 = A_2$, or $x = y$.
   >
   > Conversely, let's show that f is one-to-one then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. To do this, we only need to show the right hand side is included in the left, and this follows since if $x \in f(A_1) \cap f(A_2)$ then for some $a_1 \in A_1$ and $a_2 \in A_2$ we have $x = f(a_1) = f(a_2)$. As f is one-to-one, $a_1 = a_2$ and so a_1 lies in $A_1 \cap A_2$ and x is in $f(A_1 \cap A_2)$.

   More generally, $f(\bigcap_i A_i) \subseteq \bigcap_i f(A_i)$.

4. $f(A_1) - f(A_2) \subseteq f(A_1 - A_2)$: If $y \in f(A_1) - f(A_2)$, then $y = f(x)$ for some $x \in A_1$. If $x \in A_2$, then $y = f(x) \in f(A_2)$ as well, a contradiction. So $x \in A_1 - A_2$, and $y = f(x) \in f(A_1 - A_2)$. The inequality is strict in the case when $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 1$, and $A_1 = \mathbb{Z}$ and $A_2 = \{2\}$.

5. $A \subseteq f^{-1}f(A)$. Again, one finds that equality fails for the real function $f(x) = x^2$ by selecting $A = \{1\}$. Equality again holds iff f is injective:

   > Suppose $x \in f^{-1}f(A)$. By definition this means that $f(x) = f(a)$ for some $x \in A$, and since f is injective we have $x = a \in A$. It follows that $f^{-1}f(A) \subseteq A$. Convserly, if $f(x) = f(y) = z$, then $\{x, y\} = f^{-1}f(\{x, y\}) = f^{-1}(\{z\})$. On the other hand $\{x\} = f^{-1}f(\{x\}) = f^{-1}(\{z\})$. So $\{x, y\} = \{x\}$, $x = y$.

6. If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$. In particular, $f^{-1}(B) \subseteq f^{-1}(Y)$.

7. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. More generally, $f^{-1}(\bigcup_j B_j) = \bigcup_j f^{-1}(B_j)$.

8. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. More generally, $f^{-1}(\bigcap_j B_j) = \bigcap_j f^{-1}(B_j)$.

9. $f^{-1}(Y - B) = X - f^{-1}(B)$. As a result, $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$.

10. $ff^{-1}(B) \subseteq B$. Yet again, one finds that equality fails for the real function $f(x) = x^2$ by selecting $B = [-1, 1]$. Equality holds iff f is surjective:

    > Suppose f is onto. Pick any $y \in B \subset Y$. Then $y = f(x)$ for some $x \in X$. In other words, $x \in f^{-1}(B)$ and hence $y = f(x) \in ff^{-1}(B)$. Now suppose the convserse, then pick $B = Y$, and we have $Y = ff^{-1}(Y) = f(X)$.

11. Combining **??** and **??**, we have that $ff^{-1}f(A) = f(A)$ and $f^{-1}ff^{-1}(B) = f^{-1}(B)$. Let's show the first equality:

    > From **??**, $A \subseteq f^{-1}f(A)$, so that $f(A) \subseteq ff^{-1}f(A)$ (by 1). Set $B = f(A)$. Then by **??**, $ff^{-1}f(A) = ff^{-1}(B) \subseteq B = f(A)$.

Remarks.

- $f^{-1}f$ and ff^{-1} the compositions of the function and its inverse as defined at the beginning of the entry, so that $f^{-1}f(A) = f^{-1}(f(A))$ and $ff^{-1}(B) = f(f^{-1}(B))$.

- From the definition above, we see that a function $f : X \to Y$ induces two functions $[f]$ and $[f^{-1}]$ defined by

$$[f] : 2^X \to 2^Y \text{ such that } [f](A) := f(A) \text{ and}$$

$$[f^{-1}] : 2^Y \to 2^X \text{ such that } [f^{-1}](B) := f^{-1}(B).$$

  The last property **??** says that $[f]$ and $[f^{-1}]$ are quasi-inverses of each other.

- f is a bijection iff $[f]$ and $[f^{-1}]$ are inverses of one another.