principle of finite induction

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The principle of finite induction, also known as *mathematical induction*, is commonly formulated in two ways. Both are equivalent. The first formulation is known as *weak* induction. It asserts that if a statement $P(n)$ holds for $n = 0$ and if $P(n) \Rightarrow P(n+1)$, then $P(n)$ holds for all natural numbers n. The case $n = 0$ is called the *base case* or *base step* and the implication $P(n) \Rightarrow P(n+1)$ is called the *inductive step*. In an inductive proof, one uses the term *induction hypothesis* or *inductive hypothesis* to refer back to the statement $P(n)$ when one is trying to prove $P(n+1)$ from it.

The second formulation is known as *strong* or *complete* induction. It asserts that if the implication $\forall n((\forall m < nP(m)) \Rightarrow P(n))$ is true, then $P(n)$ is true for all natural numbers n. (Here, the quantifiers range over all natural numbers.) As we have formulated it, strong induction does not require a separate base case. Note that the implication $\forall n((\forall m < nP(m)) \Rightarrow P(n)$ already entails $P(0)$ since the statement $\forall m < 0P(m)$ holds vacuously (there are no natural numbers less that zero).

A moment's thought will show that the first formulation (weak induction) is equivalent to the following:

> Let S be a set natural numbers such that
>
> 1. 0 belongs to S, and
> 2. if n belongs to S, so does $n+1$.
>
> Then S is the set of all natural numbers.

Similarly, strong induction can be stated:

> If S is a set of natural numbers such that n belongs to S whenever all numbers less than n belong to S, then S is the set of all natural numbers.

The principle of finite induction can be derived from the fact that every nonempty set of natural numbers has a smallest element. This fact is known as the *well-ordering principle for natural numbers*. (Note that this is not the same thing as the *well-ordering principle*, which is equivalent to the axiom of choice and has nothing to do with induction.)