cylindric algebra

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A *cylindric algebra* is a quadruple $(B, V, \exists, d)$, where B is a Boolean algebra, V is a set whose elements we call *variables*, $\exists$ and d are functions

$$\exists : V \to B^B \qquad \text{and} \qquad d : V \times V \to B$$

such that

1. $(B, \exists x)$ is a monadic algebra for each $x \in V$,
2. $\exists x \circ \exists y = \exists y \circ \exists x$ for any $x, y \in V$,
3. $d_{xx} = 1$ for all $x \in V$,
4. for any $x, y \in V$ with $x \neq y$, and any $a \in B$, we have the equality
$$\exists x(a \wedge d_{xy}) \wedge \exists x\big(a' \wedge d_{xy}\big) = 0$$
5. for any $x, y, z \in V$ with $x \neq y$ and $x \neq z$, we have the equality
$$\exists x(d_{xy} \wedge d_{xz}) = d_{yz}.$$

where $\exists x$ and d_{xy} are the abbreviations for $\exists(x)$ and $d(x, y)$ respectively.

Basically, the first two conditions say that the $(B, V, \exists)$ portion of the cylindric algebra is very similar to a quantifier algebra, except the domain is no longer the subsets of V, but the elements of V instead. The function d is the algebraic abstraction of equality. Condition 3 says that $x = x$ is always true, condition 4 says that the proposition a and its complement a', where any occurrences of the variable x are replaced by the variable y, distinct from x, is always false, while condition 5 says $y = z$ iff there is an x such that $x = y$ and $x = z$.

Below are some elementary properties of d:

- (symmetric property) $d_{xy} = d_{yx}$
- (transitive property) $d_{xy} \wedge d_{yz} \leq d_{xz}$
- $\exists x(d_{xy}) = 1$
- $\exists x(d_{yz}) = d_{yz}$ provided that $x \notin \{y, z\}$
- if $x \neq y$, then

1. $\exists x(d_{xy} \wedge a') = (\exists x(d_{xy} \wedge a))'$,
2. $d_{xy} \wedge a = d_{xy} \wedge \exists x(a \wedge d_{xy})$.

Remarks

1. The *dimension* of a cylindric algebra $(B, V, \exists, d)$ is the cardinality of V.

2. From the definition above, a cylindric algebra is a two-sorted structure, with B and V as the two distinct universes. However, it is often useful to view a cylindric algebra as a one-sorted structure. The way to do this is to dispense with V and identify each $\exists x$ as a unary operator on B, and each d_{xy} as a constant in B. As a result, the cylindric algebra $(B, V, \exists, d)$ becomes a Boolean algebra with possibly infinitely many operators:
$$(B, \exists x, d_{xy})_{x,y \in V}.$$

3. Let L be a the language of a first order logic, and S a set of sentences in L. Define $\equiv$ on L so that
$$\varphi \equiv \psi \quad \text{iff} \quad S \vdash (\varphi \leftrightarrow \psi).$$
Then $\equiv$ is an equivalence relation on L. For each formula $\varphi \in L$, let $[\varphi]$ be the equivalence class containing φ. Let V be a countably infinite set of variables available to L. Now, define operations $\vee, \wedge, ', \exists x, d_{xy}$ as follows:

$$\begin{aligned}
[\varphi] \vee [\psi] &:= [\varphi \vee \psi], && (1)\\
[\varphi] \wedge [\psi] &:= [\varphi \wedge \psi], && (2)\\
[\varphi]' &:= [\neg\varphi], && (3)\\
0 &:= [\neg x = x], && (4)\\
1 &:= [x = x], && (5)\\
\exists x[\varphi] &:= [\exists x\varphi], && (6)\\
d_{xy} &:= [x = y]. && (7)
\end{aligned}$$

Then it can be shown that $(L/\equiv, V, \exists, d)$ is a cylindric algebra. Thus a cylindric algebra can be thought of as an "algebraization" of first order logic (with equality), much the same way as a Boolean algebra (Lindenbaum-Tarski algebra) as the algebraic counterpart of propositional logic.

References

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