Hausdorff paradox

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Let S^2 be the unit sphere in the Euclidean space $\mathbb{R}^3$. Then it is possible to take "half" and "a third" of S^2 such that both of these parts are essentially congruent (we give a formal version in a minute). This sounds paradoxical: wouldn't that mean that half of the sphere's area is equal to only a third? The "paradox" resolves itself if one takes into account that one can choose non-measurable subsets of the sphere which ostensively are "half" and a "third" of it, using geometric congruence as means of comparison.

Let us now formally state the Theorem.

Theorem (Hausdorff paradox [**?**])**.** *There exists a disjoint of the unit sphere S^2 in the Euclidean space $\mathbb{R}^3$ into four subsets A, B, C, D, such that the following conditions are met:*

1. *Any two of the sets A, B, C and $B \cup C$ are congruent.*
2. *D is countable.*

A crucial ingredient to the proof is the `http://planetmath.org/node/310`axiom of choice, so the sets A, B and C are not constructible. The theorem itself is a crucial ingredient to the proof of the so-called Banach-Tarski paradox.

References

[H] F. Hausdorff, Bemerkung über den Inhalt von Punktmengen, *Math. Ann.* 75, 428–433, (1915), `http://dz-srv1.sub.uni-goettingen.de/sub/digbib/loader?did=D28919http://dz-srv` (in German).