ultra-universal

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Let T be a first order theory. A model M of T is said to be an *ultra-universal model* of T iff for every model A of T there exists and ultra-power of M into which A can be embedded. [?, ?]

If T has an ultra-universal model it is referred to as an *ultra-universal theory*. The class of models of an ultra-universal theory is called an *ultra-universal class*. If T is an ultra-universal theory with elementary class K and ultra-universal model M then M is said to be *ultra-universal in* K. [?]

0.0.1 Characterizations

Ultra-universal classes are precisely the non-empty elementary classes having the joint embedding property. [?]

Ultra-universal models can be characterized in terms of universal or existential sentences:

Let T be theory and let M be a model of T. The following are equivalent: [?]

1. M is an ultra-universal model of T
2. Every universal sentence holding in M holds in all models of T
3. Every existential sentence holding in some model of T holds in M

A theory T is ultra-universal iff it is consistent and for all universal sentences ϕ and ψ, $T \vdash \phi \vee \psi$ implies $T \vdash \phi$ or $T \vdash \psi$. [?]

A complete consistent theory is always ultra-universal. More generally the set of universal sentences Σ of a complete consistent theory T is always an ultra-universal theory - a model of T is an ultra-universal model of Σ. Ultra-universal theories are precisely those theories T which are consistent and can be extended to a complete consistent theory without introducing any universal sentences that are not deducible from T. [?]

In terms of the Lindenbaum-Tarski algebra for a first order language L, a theory T in L is ultra-universal iff the filter F that it generates in the Lindenbaum-Tarski algebra is proper and can be extended to an ultrafilter U such that $F \cap A = U \cap A$ where A is the sub-lattice of universal sentences. Moreover T is ultra-universal iff $F \cap A$ is a prime proper filter in A. Thus ultra-universal theories correspond to prime proper filters in the bounded distributive lattice of universal sentences. [?]

0.0.2 Examples

- Any infinite partition lattice is ultra-universal in the variety of lattices [?]
- Any infinite symmetric group is ultra-universal in the variety of groups [?]
- The monoid of functions defined on an infinite set is ultra-universal in the variety of monoids [?]
- The semigroup reduct of the monoid of functions defined on an infinite set is ultra-universal in the varierty of semigroups [?]
- The power set interior (or closure) algebra on Cantor's discontinuum or on a denumerable co-finite topological space is ultra-universal in the variety of interior (or closure) algebras [?]
- The product of all fintely generated substructures (up to isomorphism) of members of a factor embeddable universal Horn class (in particular a factor embeddable variety of algebraic structures) is ultra-universal in the class. [?]
- More generally, any model in an elementary class having the property that all fintely generated substructures of the class are embeddable in it, is ultra-universal in the class. [?]

References

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[3] Colin Naturman, Henry Rose: *Ultra-universal models*, Quaestiones Mathematicae, 15(2), 1992, 189-195