



Math for the people, by the people.

## Informal type theory

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One difficulty often encountered by the classical mathematician when faced with learning about type theory is that it is usually presented as a fully or partially formalized deductive system. This style, which is very useful for proof-theoretic investigations, is not particularly convenient for use in applied, informal reasoning. Nor is it even familiar to most working mathematicians, even those who might be interested in foundations of mathematics. One objective of the present work is to develop an informal style of doing mathematics in univalent foundations that is at once rigorous and precise, but is also closer to the language and style of presentation of everyday mathematics.

In present-day mathematics, one usually constructs and reasons about mathematical objects in a way that could in principle, one presumes, be formalized in a system of elementary set theory, such as ZFC — at least given enough ingenuity and patience. For the most part, one does not even need to be aware of this possibility, since it largely coincides with the condition that a proof be “fully rigorous” (in the sense that all mathematicians have come to understand intuitively through education and experience). But one does need to learn to be careful about a few aspects of “informal set theory”: the use of collections too large or inchoate to be sets; the axiom of choice and its equivalents; even (for undergraduates) the method of proof by contradiction; and so on. Adopting a new foundational system such as homotopy type theory as the *implicit formal basis* of informal reasoning will require adjusting some of one’s instincts and practices. The present text is intended to serve as an example of this “new kind of mathematics”, which is still informal, but could now in principle be formalized in homotopy type theory, rather than ZFC, again given enough ingenuity and patience.

It is worth emphasizing that, in this new system, such formalization can have real practical benefits. The formal system of type theory is suited to computer systems and has been implemented in existing proof assistants. A proof assistant is a computer program which guides the user in construction of a fully formal proof, only allowing valid steps of reasoning. It also provides some degree of automation, can search libraries for existing theorems, and can even extract numerical algorithms from the resulting (constructive) proofs.

We believe that this aspect of the univalent foundations program distinguishes it from other approaches to foundations, potentially providing a new practical utility for the working mathematician. Indeed, proof assistants based on older type theories have already been used to formalize substantial mathematical proofs, such as the four-color theorem and the Feit–Thompson

theorem. Computer implementations of univalent foundations are presently works in progress (like the theory itself). However, even its currently available implementations (which are mostly small modifications to existing proof assistants such as COQ and AGDA) have already demonstrated their worth, not only in the formalization of known proofs, but in the discovery of new ones. Indeed, many of the proofs described in this book were actually *first* done in a fully formalized form in a proof assistant, and are only now being “unformalized” for the first time — a reversal of the usual relation between formal and informal mathematics.

One can imagine a not-too-distant future when it will be possible for mathematicians to verify the correctness of their own papers by working within the system of univalent foundations, formalized in a proof assistant, and that doing so will become as natural as typesetting their own papers in T<sub>E</sub>X. In principle, this could be equally true for any other foundational system, but we believe it to be more practically attainable using univalent foundations, as witnessed by the present work and its formal counterpart.