Levy collapse

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Given any cardinals κ and λ in $\mathfrak{M}$, we can use the *Levy collapse* to give a new model $\mathfrak{M}[G]$ where $\lambda = \kappa$. Let $P = \mathrm{Levy}(\kappa, \lambda)$ be the set of partial functions $f : \kappa \to \lambda$ with $|\,\mathrm{dom}(f)| < \kappa$. These functions each give partial information about a function F which collapses λ onto κ.

Given any generic subset G of P, $\mathfrak{M}[G]$ has a set G, so let $F = \bigcup G$. Each element of G is a partial function, and they are all compatible, so F is a function. $\mathrm{dom}(G) = \kappa$ since for each $\alpha < \kappa$ the set of $f \in P$ such that $\alpha \in \mathrm{dom}(f)$ is dense (given any function without α, it is trivial to add $(\alpha, 0)$, giving a stronger function which includes α). Also $\mathrm{range}(G) = \lambda$ since the set of $f \in P$ such that $\alpha < \lambda$ is in the range of f is again dense (the domain of each f is bounded, so if β is larger than any element of $\mathrm{dom}(f)$, $f \cup \{(\beta, \alpha)\}$ is stronger than f and includes λ in its domain).

So F is a surjective function from κ to λ, and λ is collapsed in $\mathfrak{M}[G]$. In addition, $|\,\mathrm{Levy}(\kappa, \lambda)| = \lambda$, so it satisfies the λ^+ chain condition, and therefore λ^+ is not collapsed, and becomes κ^+ (since for any ordinal between λ and λ^+ there is already a surjective function to it from λ).

We can generalize this by forcing with $P = \mathrm{Levy}(\kappa, < \lambda)$ with κ regular, the set of partial functions $f : \lambda \times \kappa \to \lambda$ such that $f(0, \alpha) = 0$, $|\,\mathrm{dom}(f)| < \kappa$ and if $\alpha > 0$ then $f(\alpha, i) < \alpha$. In essence, this is the product of $\mathrm{Levy}(\kappa, \eta)$ for each $\eta < \lambda$.

In $\mathfrak{M}[G]$, define $F = \bigcup G$ and $F_\alpha(\beta) = F(\alpha, \beta)$. Each F_α is a function from κ to α, and by the same argument as above F_α is both total and surjective. Moreover, it can be shown that P satisfies the λ chain condition, so λ does not collapse and $\lambda = \kappa^+$.