subformula

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Let L be a first order language and suppose φ is a formula of L. A *subformula* of φ is defined as any of the following:

1. φ is a subformula of φ;
2. if $\neg\psi$ is a subformula of φ for some L-formula ψ, then so is ψ;
3. if $\alpha \wedge \beta$ is a subformula of φ for some L-formulas α, β, then so are α and β;
4. if $\exists x(\psi)$ is a subformula of φ for some L-formula ψ, then so is $\psi[t/x]$ for any t free for x in ψ.

Remark. And if the language contains a modal connective, say $\Box$, then we also have

5. if $\Box\alpha$ is a subformula of φ for some L-formula α, then so is α.

The phrase "t is free for x in ψ" means that after substituting the term t for the variable x in the formula ψ, no free variables in t will become bound variables in $\psi[t/x]$.

For example, if $\varphi = \alpha \vee \beta$, then α and β are subformulas of φ. This is so because $\alpha \vee \beta = \neg(\neg\alpha \wedge \neg\beta)$, so that $\neg\alpha \wedge \neg\beta$ is a subformula of φ by applications of 1 followed by 2 above. By 3 above, $\neg\alpha$ and $\neg\beta$ are subformulas of φ. Therefore, by 2 again, α and β are subformulas of φ.

For another example, if $\varphi = \exists x(\exists y(x^2 + y^2 = 1))$, then $\exists y(t^2 + y^2 = 1)$ is a subformula of φ as long as t is a term that does not contain the variable y. Therefore, if $t = y + 2$, then $\exists y((y + 2)^2 + y^2 = 1)$ is not a subformula of φ. In fact, if $y \in \mathbb{R}$, the equation $(y + 2)^2 + y^2 = 1$ is never true.

Finally, it is easy to see (by induction) that if α is a subformula of ψ and ψ is a subformula of φ, then α is a subformula of φ. "Being a subformula of" is a reflexive transitive relation on L-formulas.

Remark. There is also the notion of a *literal subformula* of a formula φ. A formula ψ is a literal subformula of φ if it is a subformula of φ obtained in any one of the first three ways above, or if $\exists x(\psi)$ is a literal subformula of φ.

Note that any literal subformula of φ is a subformula of φ, for if $\varphi = \exists x(\psi)$, then x occurs free in ψ and $\psi = \psi[x/x]$.

In the second example above, $\exists y(x^2 + y^2 = 1)$ and $x^2 + y^2 = 1$ are both literal subformulas of $\varphi = \exists x(\exists y(x^2 + y^2 = 1))$.