example of polyadic algebra

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Recall that the canonical example of a monadic algebra is that of a functional monadic algebra, which is a pair $(B, \exists)$ such that B is the set of all functions from a non-empty set X to a Boolean algebra A such that, for each $f \in B$, the supremum and the infimum of $f(X)$ exist, and $\exists$ is a function on B that maps each element f to $f^{\exists}$, a constant element whose range is a singleton consisting of the supremum of $f(X)$.

The canonical example of a polyadic algebra is an extension (generalization) of a functional monadic algebra, known as the *functional polyadic algebra*. Instead of looking at functions from X to A, we look at functions from X^I (where I is some set), the I-fold cartesian power of X, to A. In this entry, an element $x \in X^I$ is written as a sequence of elements of A: $(x_i)_{i\in I}$ where $x_i \in A$, or (x_i) for short.

Before constructing *the* functional polyadic algebra based on the sets X, I and the Boolean algebra A, we first introduce the following notations:

- for any $J \subseteq I$ and $x \in X^I$, define the subset (of X^I)

$$[x]_J := \{y \in X^I \mid x_i = y_i \text{ for every } i \notin J\},$$

- for any function $\tau : I \to I$ and any $f : X^I \to A$, define the function f_τ from X^I to A, given by

$$f_\tau(x_i) := f(x_{\tau(i)}).$$

Now, let B be the set of all functions from X^I to A such that

1. for every $f \in B$, every $J \subseteq I$ and every $x \in X^I$, the arbitrary join

$$\bigvee f\left([x]_J\right)$$

   exists.

   Before stating the next condition, we introduce, for each $f \in B$, a function $f^{\exists J} : X^I \to A$ as follows:

$$f^{\exists J}(x) := \bigvee f\left([x]_J\right).$$

   Now, we are ready for the next condition:

2. if $f \in B$, then $f^{\exists J} \in B$,

3. if $f \in B$, then $f_\tau \in B$ for $\tau : I \to I$.

Note that if A were a complete Boolean algebra, we can take B to be A^{X^I}, the set of all functions from X^I to A.

Next, define $\exists : P(I) \to B^B$ by $\exists(J)(f) = f^{\exists J}$, and let S be the semigroup of functions on I (with functional compositions as multiplications), then we call the quadruple $(B, I, \exists, S)$ the *functional polyadic algebra* for the triple (A, X, I).

Remarks. Let $(B, I, \exists, S)$ be the functional polyadic algebra for (A, X, I).

- $(B, I, \exists, S)$ is a polyadic algebra. The proof of this is not difficult, but involved, and can be found in the reference below.
- If I is a singleton, then $(B, I, \exists, S)$ can be identified with the functional monadic algebra $(B, \exists)$ for (A, X), for S is just I, and X^I is just X.
- If I is $\varnothing$, then $(B, I, \exists, S)$ can be identified with the Boolean algebra A, for $S = \varnothing$ and X^I is a singleton, and hence the set of functions from X^I to A is identified with A.

References

[1] P. Halmos, *Algebraic Logic*, Chelsea Publishing Co. New York (1962).