realization of a formula by a truth function

Canonical name	RealizationOfAFormulaByATruthFunction
Date of creation	2013-03-22 18:52:53
Last modified on	2013-03-22 18:52:53
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	8
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03B05

Fix a countable set $V = \{v_1, v_2, \ldots\}$ of propositional variables. Let p be a well-formed formula over V constructed by a set F of logical connectives. Let $S := \{v_{k_1}, \ldots, v_{k_n}\}$ be the set of variables occurring in p (S is finite as p is a string of finite length). Fix the n-tuple $\boldsymbol{v} := (v_{k_1}, \ldots, v_{k_n})$. Every valuation ν on V, when restricted to S, determines an n-tupe of zeros and ones: $\nu(\boldsymbol{v}) := (\nu(v_{k_1}), \ldots, \nu(v_{k_n})) \in \{0,1\}^n$. For this $\nu(\boldsymbol{v})$, we associate the interpretation $\overline{\nu}(p) \in \{0,1\}$.

Two valuations on V determine the same $a \in \{0,1\}^n$ iff they agree on every v_{k_i}. If we set $\nu_1 \sim \nu_2$ iff they determine the same $a \in \{0,1\}^n$, then $\sim$ is an equivalence relation on the set of all valuations on V. As there are 2^n elements in $\{0,1\}^n$, there are 2^n equivalence classes.

From the two paragraphs above, we see that there is a truth function $\phi : \{0,1\}^n \to \{0,1\}$ such that

$$\phi(\nu(\boldsymbol{v})) = \overline{\nu}(p)$$

for any valuation ν on V. This function is called a *realization* of the wff p. Since p is arbitrary, it is easy to see that every wff admits a realization. It is also not hard to see that a realization of p is unique up to the order of the variables in the n-tuple $\boldsymbol{v}$. From now only, we make the assumption that every n-tuple $(v_{k_1}, \ldots, v_{k_n})$ has the property that $k_1 < \cdots < k_n$. Let us write ϕ_p *the* realization of p.

Realizations of wffs are closely related to semantical implications and equivalences:

1. $p \models q$ (p semantically implies q, or p entails q) iff $\phi_p \leq \phi_q$;
2. $p \equiv q$ iff $\phi_p = \phi_q$, where $\equiv$ denotes semantical equivalence;
3. p is a tautology iff $\phi_p = 1$, the constant function whose value is $1 \in \{0,1\}$.

If $F = \{\neg, \vee, \wedge\}$, then every wff p over V corresponds to a realization $[p]$ that "looks" exactly likes p. We do this by induction:

- if p is a propositional variable v_i, let $[v_i]$ be the identity function on $\{0,1\}$;
- if p has the form $\neg q$, define $[p] := \neg[q]$;
- if p has the form $q \vee r$, define $[p] := [q] \vee [r]$;

- if p has the form $q \wedge r$, define $[p] := [q] \wedge [r]$;

where the $\neg, \vee$, and $\wedge$ on the right hand side of the definitions are the Boolean complementation, join and meet operations on the Boolean algebra $\{0, 1\}$. Again by an easy induction, for each wff p, the function $[p]$ is the realization of p (a function written in terms of symbols in F is called a polynomial).

Conversely, every n-ary truth function $\phi : \{0, 1\}^n \to \{0, 1\}$ is the realization of some wff p. This is true because every n-ary operation on $\{0, 1\}$ has a conjunctive normal form. Suppose ϕ is a function in variables $x_1, \ldots, x_n$, with the form $\alpha_1 \wedge \cdots \wedge \alpha_m$, where each α_i is the join of the variables in x_i. If α_i is a function in $x_{k_1}, \ldots, x_{k_m}$ (each $k_j \in \{1, \ldots, n\}$), then let p_i be the disjunction of propositional variables $v_{k_1}, \ldots, v_{k_m}$. Then ϕ is the realization of wff $p := p_1 \wedge \cdots \wedge p_n$. Notice that we have omitted parenthesis, and $p_1 \wedge \cdots \wedge p_n$ is an abbreviation of $(\cdots (p_1 \wedge p_2) \wedge \cdots) \wedge p_n)$.

Since every wff, regardless of logical connectives, has a realization, what we have just proved in fact is the following:

Theorem 1. $\{\neg, \vee, \wedge\}$ *is functionally complete.*

References

[1] H. Enderton: *A Mathematical Introduction to Logic*, Academic Press, San Diego (1972).