imaginaries

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Given an algebraic structure S to investigate, mathematicians consider substructures, restrictions of the structure, quotient structures and the like. A natural question for a mathematician to ask if he is to understand S is "What structures naturally live in S?" We can formalise this question in the following manner: Given some logic appropriate to the structure S, we say another structure T is *definable* in S iff there is some definable subset T' of S^n, a bijection $\sigma : T' \to T$ and a definable function (respectively relation) on T' for each function (resp. relation) on T so that σ is an isomorphism (of the relevant type for T).

For an example take some infinite group $(G, .)$. Consider the centre of G, $Z := \{x \in G : \forall y \in G(xy = yx)\}$. Then Z is a first order definable subset of G, which forms a group with the restriction of the multiplication, so $(Z, .)$ is a first order definable structure in $(G, .)$.

As another example consider the structure $(\mathbf{R}, +, ., 0, 1)$ as a field. Then the structure $(\mathbf{R}, <)$ is first order definable in the structure $(\mathbf{R}, +, ., 0, 1)$ as for all $x, y \in \mathbf{R}^2$ we have $x \leq y$ iff $\exists z(z^2 = y - x)$. Thus we know that $(\mathbf{R}, +, ., 0, 1)$ is unstable as it has a definable order on an infinite subset.

Returning to the first example, Z is normal in G, so the set of (left) cosets of Z form a factor group. The domain of the factor group is the quotient of G under the equivalence relation $x \equiv y$ iff $\exists z \in Z(xz = y)$. Therefore the factor group G/Z will not (in general) be a definable structure, but would seem to be a "natural" structure. We therefore weaken our formalisation of "natural" from definable to interpretable. Here we require that a structure is isomorphic to some definable structure on equivalence classes of definable equivalence relations. The equivalence classes of a $\emptyset$-definable equivalence relation are called *imaginaries*.

In [**?**] Poizat defined the property of *Elimination of Imaginaries*. This is equivalent to the following definition:

Definition 0.1 *A structure $\mathfrak{A}$ with at least two distinct $\emptyset$-definable elements admits* elimination of imaginaries *iff for every $n \in \mathbf{N}$ and $\emptyset$-definable equivalence relation $\sim$ on $\mathfrak{A}^n$ there is a $\emptyset$-definable function $f : \mathfrak{A}^n \to \mathfrak{A}^p$ (for some p) such that for all x and y from $\mathfrak{A}^n$ we have*

$$x \sim y \text{ iff } f(x) = f(y).$$

Given this property, we think of the function f as coding the equivalence classes of $\sim$, and we call $f(x)$ a code for $x/\sim$. If a structure has elimination of imaginaries then every interpretable structure is definable.

In [**?**] Shelah defined, for any structure $\mathfrak{A}$ a multi-sorted structure $\mathfrak{A}^{eq}$. This is done by adding a sort for every $\emptyset$-definable equivalence relation, so that the equivalence classes are elements (and code themselves). This is a closure operator i.e. $\mathfrak{A}^{eq}$ has elimination of imaginaries. See [**?**] chapter 4 for a good presentation of imaginaries and $\mathfrak{A}^{eq}$. The idea of passing to $\mathfrak{A}^{eq}$ is very useful for many purposes. Unfortunately $\mathfrak{A}^{eq}$ has an unwieldy language and theory. Also this approach does not answer the question above. We would like to show that our structure has elimination of imaginaries with just a small selection of sorts added, and perhaps in a simple language. This would allow us to describe the definable structures more easily, and as we have elimination of imaginaries this would also describe the interpretable structures.

References

[1] Wilfrid Hodges, *A shorter model theory* Cambridge University Press, 1997.

[2] Bruno Poizat, *Une théorie de Galois imaginaire*, Journal of Symbolic Logic, **48** (1983), pp. 1151-1170.

[3] Saharon Shelah, *Classification Theory and the Number of Non-isomorphic Models*, North Hollans, Amsterdam, 1978.