proof of inverse function theorem

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Since $\det Df(a) \neq 0$ the Jacobian matrix $Df(a)$ is invertible: let $A = (Df(a))^{-1}$ be its inverse. Choose $r > 0$ and $\rho > 0$ such that

$$B = \overline{B_\rho(a)} \subset E,$$

$$\|Df(x) - Df(a)\| \leq \frac{1}{2n\|A\|} \quad \forall x \in B,$$

$$r \leq \frac{\rho}{2\|A\|}.$$

Let $y \in B_r(f(a))$ and consider the mapping

$$T_y \colon B \to \mathbb{R}^n$$

$$T_y(x) = x + A \cdot (y - f(x)).$$

If $x \in B$ we have

$$\|DT_y(x)\| = \|1 - A \cdot Df(x)\| \leq \|A\| \cdot \|Df(a) - Df(x)\| \leq \frac{1}{2n}.$$

Let us verify that T_y is a contraction mapping. Given $x_1, x_2 \in B$, by the Mean-value Theorem on $\mathbb{R}^n$ we have

$$|T_y(x_1) - T_y(x_2)| \leq \sup_{x \in [x_1, x_2]} n\|DT_y(x)\| \cdot |x_1 - x_2| \leq \frac{1}{2}|x_1 - x_2|.$$

Also notice that $T_y(B) \subset B$. In fact, given $x \in B$,

$$|T_y(x) - a| \leq |T_y(x) - T_y(a)| + |T_y(a) - a| \leq \frac{1}{2}|x - a| + |A \cdot (y - f(a))| \leq \frac{\rho}{2} + \|A\| r \leq \rho.$$

So $T_y \colon B \to B$ is a contraction mapping and hence by the contraction principle there exists one and only one solution to the equation

$$T_y(x) = x,$$

i.e. x is the only point in B such that $f(x) = y$.

Hence given any $y \in B_r(f(a))$ we can find $x \in B$ which solves $f(x) = y$. Let us call $g \colon B_r(f(a)) \to B$ the mapping which gives this solution, i.e.

$$f(g(y)) = y.$$

Let $V = B_r(f(a))$ and $U = g(V)$. Clearly $f : U \to V$ is one to one and the inverse of f is g. We have to prove that U is a neighbourhood of a. However since f is continuous in a we know that there exists a ball $B_\delta(a)$ such that $f(B_\delta(a)) \subset B_r(y_0)$ and hence we have $B_\delta(a) \subset U$.

We now want to study the differentiability of g. Let $y \in V$ be any point, take $w \in \mathbb{R}^n$ and $\epsilon > 0$ so small that $y + \epsilon w \in V$. Let $x = g(y)$ and define $v(\epsilon) = g(y + \epsilon w) - g(y)$.

First of all notice that being

$$|T_y(x + v(\epsilon)) - T_y(x)| \leq \frac{1}{2}|v(\epsilon)|$$

we have

$$\frac{1}{2}|v(\epsilon) \geq |v(\epsilon) - \epsilon A \cdot w| \geq |v(\epsilon)| - \epsilon \|A\| \cdot |w|$$

and hence

$$|v(\epsilon)| \leq 2\epsilon \|A\| \cdot |w|.$$

On the other hand we know that f is differentiable in x that is we know that for all v it holds

$$f(x + v) - f(x) = Df(x) \cdot v + h(v)$$

with $\lim_{v \to 0} h(v)/|v| = 0$. So we get

$$\frac{|h(v(\epsilon))|}{\epsilon} \leq \frac{2\|A\| \cdot |w| \cdot |h(v(\epsilon))|}{v(\epsilon)} \to 0 \qquad \text{when } \epsilon \to 0.$$

So

$$\lim_{\epsilon \to 0} \frac{g(y + \epsilon) - g(y)}{\epsilon} = \lim_{\epsilon \to 0} \frac{v(\epsilon)}{\epsilon} = \lim_{\epsilon \to 0} Df(x)^{-1} \cdot \frac{\epsilon w - h(v(\epsilon))}{\epsilon} = Df(x)^{-1} \cdot w$$

that is

$$Dg(y) = Df(x)^{-1}.$$