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polyadic algebra

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A *polyadic algebra* is a quadruple $(B, V, \exists, S)$, where $(B, V, \exists)$ is a quantifier algebra, and S is a function from the set of functions on V to the set of endomorphisms on the Boolean algebra B, in other words

$$S : V^V \to \mathrm{End}(B)$$

such that

1. $S(1_V) = 1_B$,
2. $S(f \circ g) = S(f) \circ S(g)$,
3. $S(f) \circ \exists(I) = S(g) \circ \exists(I)$ if $f(V - I) = g(V - I)$,
4. $\exists(I) \circ S(f) = S(f) \circ \exists(f^{-1}(I))$ if f is one-to-one when restricted to $f^{-1}(I)$.

Explanation of notations: $1_V, 1_B$ are identity functions on V, B respectively; f, g are functions on V, and I is a subset of V. The circle $\circ$ represents functional compositions.

The *degree* and *local finiteness* of a polyadic algebra are defined as the degree and local finiteness of the underlying quantifier algebra.

Heuristically, the function S can be thought of as changes to propositional functions due to a "substitution" of variables (elements of V). Let us see some examples. Let $V = \{x_0, x_1, \ldots\}$ be a countably indexed set of variables. For any propositional function φ, define $S(f)(\varphi)$ to be the propositional function φ_1 obtained by replacing each variable x that occurs in it by $f(x)$. Below are two examples illustrating how $S(f)$ changes propositional functions:

- Let $f : V \to V$ be the function given by $f(x_0) = f(x_1) = x_0$ and $f(x_i) = x_{i+1}$ for all $i > 1$. If φ is the propositional function $x_0^2 - x_1 + x_2/x_3$, then $S(f)(\varphi)$ is the propositional function $x_0^2 - x_0 + x_3/x_4$.
- Let $f : V \to V$ be the function given by $f(x_0) = x_2$, and $f(x_i) = x_i$ for all $i \neq 0$. Then the propositional function "$\exists x_0, x_1, x_2 (x_0 \neq x_1 \wedge x_1 \neq x_2 \wedge x_2 \neq x_0)$" becomes "$\exists x_2, x_1, x_2 (x_2 \neq x_1 \wedge x_1 \neq x_2 \wedge x_2 \neq x_2)$" under $S(f)$.

It is not hard to see from the examples above that $S(f)$ respects Boolean operations $\wedge$ and $'$, which is why we want to make $S(f)$ an endomorphism on B. Furthermore, the four conditions above can be interpreted as

1. if there are no substitutions of variables, then there should be no corresponding changes to the propositional functions

2. applying substitutions $f \circ g$ of varaibles in a propositional function φ should have the same effect as applying substitutions g of variables in φ, followed by substitutions f of variables in $S(g)(\varphi)$

3. a substitution f of variables should have no effect to a propositional function beginning with $\exists$ if every variable bound by $\exists$ is fixed by f. For example, if we replace f in the second example above by $f(x_3) = x_2$ and $f(x_i) = x_i$ otherwise, then

$$\text{“}\exists x_0, x_1, x_2(x_0 \neq x_1 \wedge x_1 \neq x_2 \wedge x_2 \neq x_0)\text{”}$$

   is unchanged by $S(f)$, since x_0, x_1, x_2 are all fixed by f.

4. Let $\varphi = \exists I \psi(I, J)$ be a propositional function, where I, J are sets of variables with I bound by $\exists$ and J free. If no two variables I get mapped to the same variable, and no free variable (in J) becomes bound (in $f(I)$) under the substitution, then $\exists I \psi(f(I), f(J))$ and $\exists f(I) \psi(f(I), f(J))$ are semantically the same, which is exactly the statement in the condition.

Remarks.

- Paul Halmos first introduced the notion of polyadic algebras. In his *Algebraic Logic*, a compilation of articles on the subject, he called a function on the set V of variables a transformation, and the triple (B, V, S) satisfying the first two conditions above a *transformation algebra*. Therefore, a polyadic algebra is a quadruple $(B, V, \exists, S)$ where $(B, V, \exists)$ is a quantifier algebra and (B, V, S) is a transformation algebra, such that conditions 3 and 4 above are satisfied.

- The notion of polyadic algebras generalizes the notion of monadic algebras. Indeed, a monadic algebra is a polyadic algebra where V is a singleton.

- Just as a Lindenbaum-Tarski algebra is the algebraic counterpart of a classical propositional logic, a polyadic algebra is the algebraic counterpart of a classical first order logic without equality. A variant of the polyadic algebra is what is known as a cylindric algebra, which algebratizes a classical first order logic with equality.

References

[1] P. Halmos, *Algebraic Logic*, Chelsea Publishing Co. New York (1962).

[2] B. Plotkin, *Universal Algebra, Algebraic Logic, and Databases*, Kluwer Academic Publishers (1994).