bounded maximization

Canonical name	BoundedMaximization
Date of creation	2013-03-22 19:05:37
Last modified on	2013-03-22 19:05:37
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	8
Author	CWoo (3771)
Entry type	Definition
Classification	msc 03D20
Related topic	BoundedMinimization

The proof involved in showing that functions obtained via http://planetmath.org/BoundedMi minimizing primitive recursive functions are themselves primitive recursive can be used to show the primitive recursiveness of another family of functions derived from existing primitive recursive functions via a similar technique, called *bounded maximization*.

Definition. Let $f:\mathbb{N}^{m+1}\to\mathbb{N}$ be a function. For each $y\in\mathbb{N}$, set

$$A_f(\boldsymbol{x},y):=\{z\in\mathbb{N}\mid z\le y \text{ and } f(\boldsymbol{x},z)=0\}.$$

Define

$$f_{bmax}(\boldsymbol{x},y):=\begin{cases}\max A_f(\boldsymbol{x},y) & \text{if } A_f(\boldsymbol{x},y)\neq\varnothing,\\ 0 & \text{otherwise.}\end{cases}$$

f_{bmax} is called the function obtained from f by *bounded maximization*.

Proposition 1. *If $f:\mathbb{N}^{m+1}\to\mathbb{N}$ is primitive recursive, so is f_{bmax}.*

Proof. The proof of this is very similar to the proof that f_{bmin} is primitive recursive in the parent entry. The initial set up is the same: define $g:=\operatorname{sgn}\circ f$, where sgn is the sign function. So g is primitive recursive.

Next, define $h:\mathbb{N}^{m+2}\to\mathbb{N}$ by $h(\boldsymbol{x},y,z)=g(\boldsymbol{x},y\dot{-}z)$. So h, and therefore its bounded product:

$$h_p(\boldsymbol{x},y,z):=\prod_{i=0}^{z}h(\boldsymbol{x},y,i)$$

are primitive recursive. h_p has the following property: given y,

- if k is the largest number less than or equal to y such that $f(\boldsymbol{x},k)=0$, then
$$h_p(\boldsymbol{x},y,z):=\begin{cases}1 & \text{if } z<y-k,\\ 0 & \text{otherwise.}\end{cases}$$
- if no such k exists, then $h_p(\boldsymbol{x},y,z)=1$, for all $(\boldsymbol{x},y,z)\in\mathbb{N}^{m+2}$.

As a result, the bounded sum

$$(h_p)_s(\boldsymbol{x},y,z):=\sum_{i=0}^{z}h_p(\boldsymbol{x},y,i),$$

and in particular, the function $g^*(\boldsymbol{x},y) := (h_p)_s(\boldsymbol{x},y,y)$, are primitive recursive. After some simplification, g^* becomes

$$g^*(\boldsymbol{x},y) := \begin{cases} y-k & \text{if } k=\max A_f(\boldsymbol{x},y) \text{ exists,} \\ s(y) & \text{otherwise.} \end{cases}$$

Finally, the function $g^{**}(\boldsymbol{x},y) := y\dot{-}g^*(\boldsymbol{x},y)$, which is just $f_{bmax}(\boldsymbol{x},y)$, is primitive recursive too. □

Example. Using bounded maximization, we can show that $q(x,y)$, the quotient of $x \div y$, is primitive recursive. When $y=0$, we set $q(x,y)=0$

First note that $q(x,y)$ is the largest integer z less than or equal to x such that $zy \le x$. Let $A(y,x) = \{z \in \mathbb{N} \mid zy \le x\}$. Then $A(y,x)$ can be rewritten as

$$\{z \in \mathbb{N} \mid z \le x \text{ and } \operatorname{sgn}(yz\dot{-}x) = 0\}.$$

Define $f:\mathbb{N}^3 \to \mathbb{N}$ by $f(x,y,t) = \operatorname{sgn}(yt\dot{-}x)$. Then

$$A_f(x,y,t) = \{z \in \mathbb{N} \mid z \le t \text{ and } \operatorname{sgn}(yz\dot{-}x) = 0\}.$$

Since f is primitive recursive, so is $f_{bmax}(x,y,t)$. Since $A(x,y) = A_f(x,y,x)$, the quotient $q(x,y)$ may be defined as $f_{bmax}(x,y,x)$, and therefore is primitive recursive.

With $q(x,y)$, we may define $\operatorname{rem}(x,y)$, the remainder of $x \div y$, as $x\dot{-}yq(x,y)$, which is easily seen to be primitive recursive.

Remark. Please see `http://planetmath.org/ExamplesOfPrimitiveRecursiveFunctions`this entry for an alternative way of showing that $q(x,y)$ and $\operatorname{rem}(x,y)$ are primitive recursive without using bounded maximization. In the alternative method, one shows that $\operatorname{rem}(x,y)$ is primitive recursive first. In addition, $\operatorname{rem}(x,0) := 0$ in the alternative method, as opposed to x discussed here.