proof of Vizing's theorem (for graphs)

Canonical name	ProofOfVizingsTheoremforGraphs
Date of creation	2013-03-22 15:10:42
Last modified on	2013-03-22 15:10:42
Owner	marijke (8873)
Last modified by	marijke (8873)
Numerical id	6
Author	marijke (8873)
Entry type	Proof
Classification	msc 05C15
Related topic	KempeChain

Proof of Vizing's theorem (for graphs)

We must prove, for any integer ρ, the following: if G is a graph all of whose vertices have valency $\leqslant \rho$, then its edges can be colored (with adjacent edges receiving different colors) in no more than $\rho + 1$ colors.

This form of stating the theorem allows us to do induction on the number of edges. A graph with zero edges, for instance, certainly doesn't need more than $\rho+1$ colors. Now assume (for any given ρ) that all graphs with m edges can be thus colored. For any graph G with $m + 1$ edges we choose an edge to remove, color the remaining m edges, and try putting the edge back.

Another way of looking at this: suppose there was (for a certain ρ) a graph G^* with m^* edges that could not be colored thus. There would then be a largest number m, bounded by $0 \leqslant m < m^*$, such that all graphs with m edges can still be colored thus. Then there would also be a G with $m + 1$ edges that cannot. Choose one of its edges and proceed as above. This time, succeeding in coloring it after all would prove by contradiction that there was no such m and hence no such G^*.

Either way we have our G where after picking an edge we can color everything except that edge, and fitting it in too proves the theorem. Let the edge to fit run between v and w_1. Some terms:

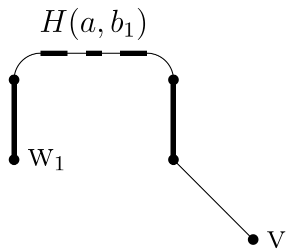
- With a palette of $\rho + 1$ edge colors, and no more than ρ edges at any vertex, each vertex has at least one **missing color**.

- Let a **Kempe chain** $H(a, b)$ be any connected component of the subgraph formed by all edges colored a or b. This is either a cycle of even length, or an open path terminating at two vertices where one of a or b is missing.

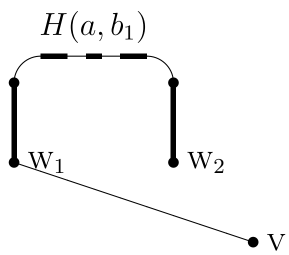
Step 1. If there is a color missing both at v and at w_1, use that color for the edge vw_1 and we're finished.

Else, let color a be missing at v (but present at w_1), and color b_1 be missing at w_1 (but present at v). If the Kempe chain of colors a and b_1 that terminates at v is disjoint from the one terminating at w_1, swap the colors in the latter chain, then use a for edge vw_1 and be done.

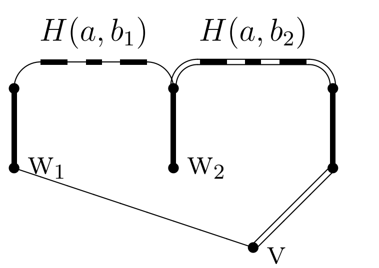
Else it's the same $H(a, b_1)$ chain, in which case we proceed as follows.



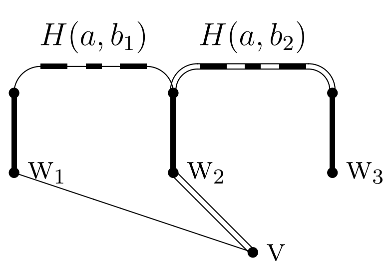
Find the b_1-colored edge from v, call it vw_2. Steal color b_1 from it and give it to edge vw_1, now vw_2 is the problem edge that needs a color.



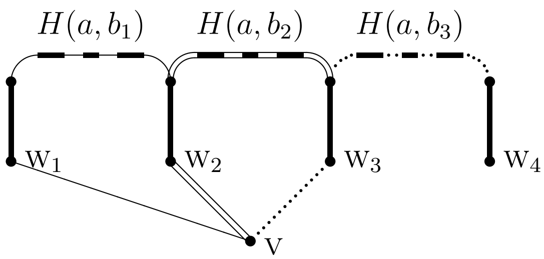
Go back to Step 1 calling it **Step 2**, replacing subscripts $_1$ by $_2$ in all text. This involves a color b_2 that was missing at w_2 (a remains the missing color at v). If v and w_2 are awkward too, being in the same $H(a, b_2)$ chain



find the b_2-colored edge vw_3, give its color to vw_2 making vw_3 the problem:

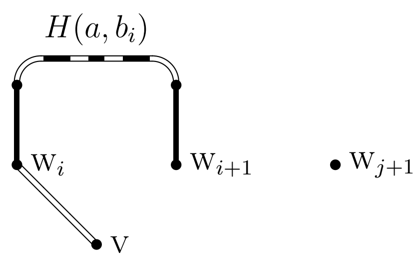


Next time round (**Step 3**) increase the subscripts again. And if still necessary, transfer the color again.

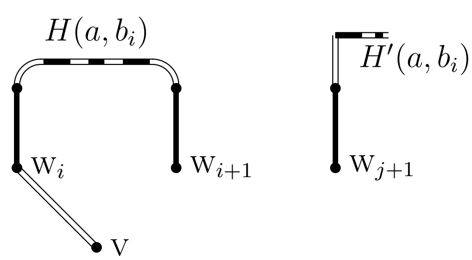


Sooner or later (if we don't get to bail out at one of the Steps) we will run out of colors, in the following sense: the color missing at some w_{j+1} is not some new color b_{j+1} but a color b_i we've seen before (it can't be a, that's present at w_{j+1}). And not only is $i < j+1$ but also $i < j$ because b_j is the color we just stole from w_{j+1}.

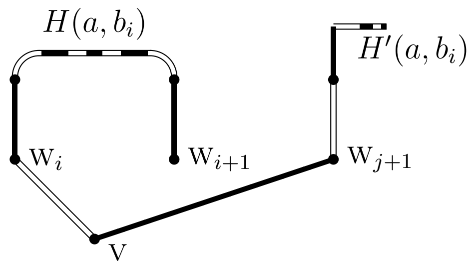
A chain $H(a, b_i)$ runs from v to w_{i+1} and all its vertices except w_{i+1} have color b_i. Now w_{j+1} isn't w_{i+1} (because $i < j$) nor one of the other vertices of the chain (because it doesn't have color b_i). So w_{j+1} isn't on this $H(a, b_i)$.



w_{k+1} does have color a so it's on some other $H'(a, b_i)$ disjoint from $H(a, b_i)$. This could be as short as a single a-colored edge or longer, it doesn't matter.



Swap the colors in that $H'(a, b_i)$. Now use a to finally color vw_{j+1} (merging $H(a, b_i)$ and $H'(a, b_i)$ in the process).



This is in essence the proof usually given, as taught to students and found for instance in references [FW77] and [Wil02] of the `http://planetmath.org/node/6930`parent entry.