multinomial theorem (proof)

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Owner	Koro (127)
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Author	Koro (127)
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Proof. The below proof of the multinomial theorem uses the binomial theorem and induction on k. In addition, we shall use multi-index notation.

First, for $k = 1$, both sides equal x_1^n. For the induction step, suppose the multinomial theorem holds for k. Then the binomial theorem and the induction assumption yield

$$\begin{aligned}
(x_1 + \cdots + x_k \, + \, x_{k+1})^n &= \sum_{l=0}^{n} \binom{n}{l} (x_1 + \cdots + x_k)^l x_{k+1}^{n-l} \\
&= \sum_{l=0}^{n} \binom{n}{l} l! \sum_{|i|=l} \frac{x^i}{i!} x_{k+1}^{n-l} \\
&= n! \sum_{l=0}^{n} \sum_{|i|=l} \frac{x^i x_{k+1}^{n-l}}{i!(n-l)!}
\end{aligned}$$

where $x = (x_1, \ldots, x_k)$ and i is a multi-index in I_+^k. To complete the proof, we need to show that the sets

$$\begin{aligned}
A &= \{(i_1, \ldots, i_k, n-l) \in I_+^{k+1} \mid l = 0, \ldots, n, \, |(i_1, \ldots, i_k)| = l\}, \\
B &= \{j \in I_+^{k+1} \mid |j| = n\}
\end{aligned}$$

are equal. The inclusion $A \subset B$ is clear since

$$|(i_1, \ldots, i_k, n-l)| = l + n - l = n.$$

For $B \subset A$, suppose $j = (j_1, \ldots, j_{k+1}) \in I_+^{k+1}$, and $|j| = n$. Let $l = |(j_1, \ldots, j_k)|$. Then $l = n - j_{k+1}$, so $j_{k+1} = n - l$ for some $l = 0, \ldots, n$. It follows that that $A = B$.

Let us define $y = (x_1, \cdots, x_{k+1})$ and let $j = (j_1, \ldots, j_{k+1})$ be a multi-index in I_+^{k+1}. Then

$$\begin{aligned}
(x_1 + \cdots + x_{k+1})^n &= n! \sum_{|j|=n} \frac{x^{(j_1, \ldots, j_k)} x_{k+1}^{j_{k+1}}}{(j_1, \ldots, j_k)! j_{k+1}!} \\
&= n! \sum_{|j|=n} \frac{y^j}{j!}.
\end{aligned}$$

This completes the proof. $\square$