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projective basis

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In the parent entry, we see how one may define dimension of a projective space inductively, from its subspaces starting with a point, then a line, and working its way up. Another way to define dimension start with defining dimensions of the empty set, a point, a line, and a plane to be $-1, 0, 1$, and 2, and then use the fact that any other projective space is isomorphic to the projective space $P(V)$ associated with a vector space V, and then define the dimension to be the dimension of V, minus 1. In this entry, we introduce a more natural way of defining dimensions, via the concept of a basis.

Throughout the discussion, $\mathbf{P}$ is a projective space (as in any model satisfying the axioms of projective geometry).

Given a subset S of $\mathbf{P}$, the *span* of S, written $\langle S\rangle$, is the smallest subspace of $\mathbf{P}$ containing S. In other words, $\langle S\rangle$ is the intersection of all subspaces of $\mathbf{P}$ containing S. Thus, if S is itself a subspace of $\mathbf{P}$, $\langle S\rangle = S$. We also say that S spans $\langle S\rangle$.

One may think of $\langle\cdot\rangle$ as an operation on the powerset of $\mathbf{P}$. It is easy to verify that this operation is a closure operator. In addition, $\langle\cdot\rangle$ is *algebraic*, in the sense that any point in $\langle S\rangle$ is in the span of a finite subset of S. In other words,

$$\langle S\rangle = \{P \mid P \in \langle F\rangle \text{ for some finite } F \subseteq S\}.$$

Another property of $\langle\cdot\rangle$ is the exchange property: for any subspace U, if $P \notin U$, then for any point Q, $\langle U \cup \{P\}\rangle = \langle U \cup \{Q\}\rangle$ iff $Q \in \langle U \cup \{P\}\rangle - U$.

A subset S of $\mathbf{P}$ is said to be *projectively independent*, or simply *independent*, if, for any proper subset S' of S, the span of S' is a proper subset of the span of S: $\langle S'\rangle \subset \langle S\rangle$. This is the same as saying that S is a *minimal* spanning set for $\langle S\rangle$, in the sense that no proper subset of S spans $\langle S\rangle$. Equivalently, S is independent iff for any $x \in S$, $\langle S - \{x\}\rangle \neq \langle S\rangle$.

S is called a *projective basis*, or simply *basis* for $\mathbf{P}$, if S is independent and spans $\mathbf{P}$.

All of the properties about spanning sets, independent sets, and bases for vector spaces have their projective counterparts. We list some of them here:

1. Every projective space has a basis.
2. If S_1, S_2 are independent, then $\langle S_1 \cap S_2\rangle = \langle S_1\rangle \cap \langle S_2\rangle$.
3. If S is independent and $P \in \langle S\rangle$, then there is $Q \in S$ such that $(\{P\} \cup S) - \{Q\}$ spans $\langle S\rangle$.

4. Let B be a basis for $\mathbf{P}$. If S spans $\mathbf{P}$, then $|B| \leq |S|$. If S is independent, then $|S| \leq |B|$. As a result, all bases for $\mathbf{P}$ have the same cardinality.

5. Every independent subset in $\mathbf{P}$ may be extended to a basis for $\mathbf{P}$.

6. Every spanning set for $\mathbf{P}$ may be reduced to a basis for $\mathbf{P}$.

In light of items 1 and 4 above, we may define the *dimension* of $\mathbf{P}$ to be the cardinality of its basis.

One of the main result on dimension is the dimension formula: if U, V are subspaces of $\mathbf{P}$, then

$$\dim(U) + \dim(V) = \dim(U \cup V) + \dim(U \cap V),$$

which is the counterpart of the same formula for vector subspaces of a vector space (see `http://planetmath.org/DimensionFormulaeForVectorSpaces`this entry).

References

[1] A. Beutelspacher, U. Rosenbaum *Projective Geometry, From Foundations to Applications*, Cambridge University Press (2000)