proof of chromatic number and girth

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Let $\alpha(G)$ denote the size of the largest independent set in G, and $\chi(G)$ the chromatic number of G. We want to show that there is a graph G with girth larger than ℓ and $\chi(G) > k$, for any $\ell, k > 0$.

We first prove the following claim.

Claim: Given a positive integer ℓ and a positive real number $t < 1/\ell$, for all sufficiently large n, there is a graph G on n vertices satisfying properties

1. the number of cycles of length at most ℓ is less than $n/2$,
2. $\alpha(G) < 3n^{1-t} \log n$.

Proof of claim: Let G be a random graph on n vertices, in which each pair of vertices joint by an edge independently with probability $p = n^{t-1}$. Let X be the number of cycles of length at most ℓ in G. The expected value of X is

$$\begin{aligned} E[X] &= \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i \\ &< \sum_{i=3}^{\ell} \frac{(np)^i}{2i} < \sum_{i=3}^{\ell} n^{ti} < \ell n^{t\ell} \end{aligned}$$

By Markov inequality,

$$\Pr(X \geq n/2) < 2\ell n^{t\ell-1},$$

we have

$$\Pr(X \geq n/2) \to 0$$

as $n \to \infty$, since $t\ell < 1$.

On the other hand, let $y = \lceil (3 \log n)/p \rceil$, and Y be the number of independent sets of size y in G. By Markov inequality again,

$$\Pr(\alpha(G) \geq y) = \Pr(Y \geq 1) \leq E[Y].$$

However,

$$E[Y] = \binom{n}{y} (1-p)^{y(y-1)/2}$$

Using the inequalities, $\binom{n}{y} < n^y$ and $(1-p) \le e^{-p}$, we get

$$\Pr(\alpha(G) \ge y) < (ne^{-p(y-1)/2})^y$$

Our choice of y guarantees that $ne^{-p(y-1)/2} < \beta < 1$ for some β, and $y \to \infty$ as n approaches infinity. Therefore,

$$\Pr(\alpha(G) \ge y) \to 0, \quad \text{as } n \to \infty.$$

We can thus find n_0 such that for all $n > n_0$, both $\Pr(X \ge n/2)$ and $\Pr(\alpha(G) \ge y)$ are strictly less than $1/2$. For all $n > n_0$,

$$\Pr(X < \ell \text{ and } \alpha(G) < y) > 1 - \Pr(X \ge \ell) - \Pr(\alpha(G) \ge y) > 1.$$

Therefore there exists a graph that satisfies the two properties in the claim. This ends the proof of the claim.

Let G be a graph that satisfies the two properties in the claim. Remove a vertex from each cycle of length at most ℓ in G. The resulting graph G' has girth larger than ℓ, more than $n/2$ vertices, and $\alpha(G') \le \alpha(G)$. Since `http://planetmath.org/node/6037`$\chi(G')\alpha(G') \ge |G'|$, we have

$$\chi(G') \ge \frac{n/2}{3n^{1-t}\log n} = \frac{n^t}{6\log n}$$

We can pick sufficiently large n such that $\chi(G')$ is larger than k. Then the chromatic number of G' is larger than k and girth is larger than ℓ.

Reference: N. Alon and J. Spencer, *The probabilistic method*, 2nd, John Wiley.