divisibility of prime-power binomial coefficients

Canonical name	DivisibilityOfPrimepowerBinomialCoefficients
Date of creation	2013-03-22 18:42:29
Last modified on	2013-03-22 18:42:29
Owner	rm50 (10146)
Last modified by	rm50 (10146)
Numerical id	4
Author	rm50 (10146)
Entry type	Theorem
Classification	msc 05A10
Classification	msc 11B65
Related topic	OrderValuation

For p a prime, n a nonzero integer, define $\operatorname{ord}_p(n)$ to be the largest integer r such that $p^r \mid n$.

An easy consequence of Kummer's theorem is:

Theorem 1. *Let p be a prime, $n \geq 1$ an integer. If $1 \leq rp^s \leq p^n$ where r, s are nonnegative integers with $p \nmid r$, then $\operatorname{ord}_p \binom{p^n}{rp^s} = n - s$.*

Proof. The result is clearly true for $r = 1, s = n$, so assume that $s < n$. By Kummer's theorem, $\operatorname{ord}_p \binom{p^n}{rp^s}$ is the number of carries when adding rp^s to $p^n - rp^s$ in base p. Consider the base p representations of rp^s and $p^n - rp^s$. They each have n digits (possibly with leading zeros) when represented in base p, and they each have s trailing zeros. If the rightmost nonzero digit in rp^s is k, then the rightmost nonzero digit in $p^n - rp^s$ is in the same "decimal" place and has value $p - k$. Each pair of corresponding digits (one from rp^s and one from $p^n - rp^s$) to the left of that point sum to $p - 1$ (it may help to think about how you subtract a decimal number from a power of 10, and what the result looks like).

It is then clear that adding those two numbers together will result in no carries in the rightmost s places, but there will be a carry out of the $s + 1^{\text{st}}$ place and out of each successive place up to and including the n^{th} place, for a total of $n - s$ carries. $\square$

A couple of examples may help to make this proof more transparent. Take $p = 3$. Then

$$\binom{27}{4} = 17550 = 2 \cdot 3^3 \cdot 5^2 \cdot 13$$

so that $\operatorname{ord}_3 \binom{27}{4} = 3$. Now, $27_{10} = 1000_3$ and $4_{10} = 11_3$, so that $27 - 4 = 23_{10}$ is 212_3. Adding $212_3 + 11_3$ indeed results in carries out of all three places since there are no trailing zeros.

$$\binom{27}{6} = 296010 = 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 23$$

so that $\operatorname{ord}_3 \binom{27}{6} = 2$. Now, $6_{10} = 20_3$ so that $27 - 6 = 21_{10}$ is 210_3. When adding $20_3 + 210_3$, there are two carries, out of the 3's place and out of the 9's place. There is no carry out of the ones place since both numbers have a 0 there.