example of a probabilistic proof

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Our example is the existence of k-paradoxical tournaments. The proof hinges upon the following basic probabilistic inequality, for any events A and B,

$$P\left(A \cup B\right) \leq P(A) + P(B)$$

Theorem 1. *For every k, there exists a k-paradoxical tournament.*

Proof. We will construct a tournament T on n vertices. We will show that for n large enough, a k-paradoxical tournament must exist. The probability space in question is all possible directions of the arrows of T, where each arrow can point in either direction with probability $1/2$, independently of any other arrow.

We say that a set K of k vertices is *arrowed* by a vertex v_0 outside the set if every arrow between v_0 to $w_i \in K$ points from v_0 to w_i, for $i = 1, \ \ldots, k$. Consider a fixed set K of k vertices and a fixed vertex v_0 outside K. Thus, there are k arrows from v_0 to K, and only one arrangement of these arrows permits K to be arrowed by v_0, thus

$$P(K \ \text{ is arrowed by } \ v_0) = \frac{1}{2^k}.$$

The complementary event, is therefore,

$$P(K \ \text{ is } \textit{not} \text{ arrowed by } \ v_0) = 1 - \frac{1}{2^k}.$$

By independence, and because there are $n - k$ vertices outside of K,

$$P(K \ \text{ is not arrowed by } \textit{any} \text{ vertex}) = \left(1 - \frac{1}{2^k}\right)^{n-k}. \tag{1}$$

Lastly, since there are $\binom{n}{k}$ sets of cardinality k in T, we employ the inequality mentioned above to obtain that for the union of all events of the form in equation (**??**)

$$P(\text{Some set of } k \text{ vertices is not arrowed by any vertex}) \leq \binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k}.$$

If the probability of this last event is less than 1 for some n, then there must exist a k-paradoxical tournament of n vertices. Indeed there is such an n, since

$$\begin{aligned}\binom{n}{k}\left(1-\frac{1}{2^k}\right)^{n-k} &= \frac{1}{k!}n(n-1)\cdots(n-k+1)\left(1-\frac{1}{2^k}\right)^{n-k} \\ &< \frac{1}{k!}n^k\left(1-\frac{1}{2^k}\right)^{n-k}\end{aligned}$$

Therefore, regarding k as fixed while n tends to infinity, the right-hand-side above tends to zero. In particular, for some n it is less than 1, and the result follows. □