central binomial coefficient

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The nth *central binomial coefficient* is defined to be

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

where $\binom{2n}{n}$ is a binomial coefficient. These numbers have the generating function

$$\frac{1}{\sqrt{1-4x}} = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + 252x^5 + \cdots$$

They are closely related to the Catalan sequence, in that

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Alternate definition

A less frequently-encountered definition for the nth central binomial coefficient is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Note that the set of these numbers meeting this alternate criterion is a superset of those meeting the first criterion, since for $n = 2m$ we have

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{2m}{\lfloor \frac{2m}{2} \rfloor} = \binom{2m}{m}$$

By cancelling terms of one of the $n!$'s against terms of the $2n!$, one may rewrite the central binomial coefficient as follows:

$$\binom{2n}{n} = \frac{2n(2n-1)\cdots(n+2)(n+1)}{n(n-1)\cdots 3\cdot 2\cdot 1}.$$

Alternatively, one may cancel each term of the $n!$ against twice itself, leaving 2's in the numerator:

$$\binom{2n}{n} = 2^n \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{n(n-1)\cdots 3\cdot 2\cdot 1}$$

Doubling the terms in the denominator, we obtain an expression for the central binomial coeficient in terms of a quotient of successive odd numbers by successive even numbers:

$$\binom{2n}{n} = 4^n \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{2n(2n-2)\cdots 6\cdot 4\cdot 2}$$

By means of these formulae, one may derive some important properties of the central binomial coeficients. By examining the first two formulae, one may deduce results about the prime factors of central binomial coefficients (for proofs, please see the attachments to this entry):

Theorem 1 *If $n \geq 3$ is an integer and p is a prime number such that $n < p < 2n$, then p divides $\binom{2n}{n}$.*

Theorem 2 *If $n \geq 3$ is an integer and p is a prime number such that $2n/3 < p \leq n$, then p does not divide $\binom{2n}{n}$.*

In conjunction with Wallis' formula for π, the third formula for the central binomial coefficient may be used to derive an asymptotic expression, as is done in an attachment to this entry:

$$\binom{2n}{n} \approx \sqrt{\frac{2}{\pi}} \frac{4^n}{\sqrt{2n+1}}$$