derangement

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Let S_n be the symmetric group of order $n \geq 1$. A permutation $\phi \in S_n$ without a fixed point (that is, $\phi(i) \neq i$ for any $i \in \{1, \ldots, n\}$) is called a *derangement*.

In combinatorial theory, one is usually interested in the number $d(n)$ of derangements in S_n. It is clear that $d(1) = 0, d(2) = 1$, and $d(3) = 2$. It is also not difficult to calculate $d(n)$ for small n. For general n, one can appeal to the principle of inclusion-exclusion. Let A_i denote the collection of permutations that fix i. Then the collection of A of derangments in S_n is just the complement of

$$A_1 \cup A_2 \cup \cdots \cup A_n$$

in S_n. Let $C = \{A_1, \ldots, A_n\}$ and I_k be the k-fold intersection of members of C (that is, each member of I_k has the form $A_{j_1} \cap \cdots \cap A_{j_k}$). We can interpret a member of I_k as a set of permutations that fix k elements from $\{1, \ldots, n\}$. The cardinality of each of these members is $(n-k)!$. Furthermore, there are $\binom{n}{k}$ members in I_k. Then,

$$\begin{aligned}
d(n) &= |A| = |S_n| - |A_1 \cup \cdots \cup A_k \cup \cdots \cup A_n| \\
&= n! - \Big[\sum_{S \in I_1} |S| - \cdots + (-1)^k \sum_{S \in I_k} |S| + \cdots + (-1)^n \sum_{S \in I_n} |S| \Big] \\
&= n! - \Big[\binom{n}{1}(n-1)! - \cdots + (-1)^k \binom{n}{k}(n-k)! + \cdots + (-1)^n \binom{n}{n}(n-n)! \Big] \\
&= n! - \Big[\frac{n!}{1!} - \cdots + (-1)^k \frac{n!}{k!} + \cdots + (-1)^n \frac{n!}{n!} \Big] \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\end{aligned}$$

With this equation, one can easily derive the recurrence relation

$$d(n) = nd(n-1) + (-1)^n.$$

Apply this formula twice, we are led to another recurrence relation

$$d(n) = (n-1)\big[d(n-1) + d(n-2)\big].$$

Application. A group of n men arrive at a party and check their hats. Upon departure, the hat-checker, being forgetful, randomly (meaning that the distribution of picking any hat out of all possible hats is a uniform distribution) hands back a hat to each man. What is the probability $p(n)$ that no man receives his own hat? Does this probability go to 0 as n gets larger and larger?

Answer: According to the above calculation,

$$p(n) = \frac{d(n)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Therefore,

$$\lim_{n \to \infty} p(n) = \frac{1}{e} \approx 0.368.$$