alternate proof of Mantel's theorem

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Let G be a triangle-free graph of order n. For each edge xy of G we consider the http://planetmath.org/NeighborhoodOfAVertexneighbourhoods $\Gamma(x)$ and $\Gamma(y)$ of G. Since G is triangle-free, these are disjoint.

This is only possible if the sum of the degrees of x and y is less than or equal to n. So for each edge xy we get the inequality

$$\delta(x) + \delta(y) \leq n$$

Summing these inequalities for all edges of G gives us

$$\Sigma_{x \in V(G)}(\delta(x))^2 \leq n|E(G)|$$

(The left hand side is a sum of $\delta(x)$ where each edge incident with x contributes one term and their are $\delta(x)$ such edges.)

Since $\Sigma_{x \in V(G)}\delta(x) = 2|E(G)|$, we get $4|E(G)|^2 = (\Sigma_{x \in V(G)}\delta(x))^2$ and applying the Cauchy-Schwarz inequality gives $4|E(G)|^2 \leq n\Sigma_{x \in V(G)}(\delta(x))^2 \leq n^2|E(G)|$.

So we conclude that for a triangle-free graph G,

$$|E(G)| \leq \frac{n^2}{4}$$