divisibility of central binomial coefficient

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Owner	rspuzio (6075)
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Author	rspuzio (6075)
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In this entry, we shall prove two results about the divisibility of central binomial coefficients which were stated in the main entry.

Theorem 1. *If $n \geq 3$ is an integer and p is a prime number such that $n < p < 2n$, then p divides $\binom{2n}{n}$.*

Proof. We will examine the following expression for our binomial coefficient:

$$\binom{2n}{n} = \frac{2n(2n-1)\cdots(n+2)(n+1)}{n(n-1)\cdots 3 \cdot 2 \cdot 1}.$$

Since $n < p < 2n$, we find p appearing in the numerator. However, p cannot appear in the denominator because the terms there are all smaller than n. Hence, p cannot be cancelled, so it must divide $\binom{2n}{n}$. $\square$

Theorem 2. *If $n \geq 3$ is an integer and p is a prime number such that $2n/3 < p \leq n$, then p does not divide $\binom{2n}{n}$.*

Proof. We will again examine our expression for our binomial coefficient:

$$\binom{2n}{n} = \frac{2n(2n-1)\cdots(n+2)(n+1)}{n(n-1)\cdots 3 \cdot 2 \cdot 1}.$$

This time, because $2n/3 < p \leq n$, we find p appearing in the denominator and $2p$ appearing in the numerator. No other multiples will appear because, if $m > 2$, then $mp > 2n$. The two occurrences of p noted above cancel, hence p is not a prime factor of $\binom{2n}{n}$. $\square$