# Hall's marriage theorem, proof of

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We prove Hall's marriage theorem by induction on $|S|$, the size of $S$.

The theorem is trivially true for $|S| = 0$.

Assuming the theorem true for all $|S| < n$, we prove it for $|S| = n$.

First suppose that we have the stronger condition

$$|\cup T| \geq |T| + 1$$

for all $\emptyset \neq T \subset S$. Pick any $x \in S_n$ as the representative of $S_n$; we must choose an SDR from

$$S' = \{S_1 \setminus \{x\}, \ldots, S_{n-1} \setminus \{x\}\}.$$

But if

$$\{S_{j_1} \setminus \{x\}, ..., S_{j_k} \setminus \{x\}\} = T' \subseteq S'$$

then, by our assumption,

$$|\cup T'| \geq \left|\cup_{i=1}^{k} S_{j_i}\right| - 1 \geq k.$$

By the already-proven case of the theorem for $S'$ we see that we can indeed pick an SDR for $S'$.

Otherwise, for some $\emptyset \neq T \subset S$ we have the "exact" size

$$|\cup T| = |T|.$$

Inside $T$ itself, for any $T' \subseteq T \subset S$ we have

$$|\cup T'| \geq |T'|,$$

so by an already-proven case of the theorem we can pick an SDR for $T$.

It remains to pick an SDR for $S \setminus T$ which avoids all elements of $\cup T$ (these elements are in the SDR for $T$). To use the already-proven case of the theorem (again) and do this, we must show that for any $T' \subseteq S \setminus T$, even after discarding elements of $\cup T$ there remain enough elements in $\cup T'$: we must prove

$$|\cup T' \setminus \cup T| \geq |T'|.$$

But

$$\begin{aligned} |\cup T' \setminus \cup T| &= |\bigcup(T \cup T')| - |\cup T| \geq && (1) \\ &\geq |T \cup T'| - |T| = && (2) \\ &= |T| + |T'| - |T| = |T'|, && (3) \end{aligned}$$

using the disjointness of $T$ and $T'$. So by an already-proven case of the theorem, $S \setminus T$ does indeed have an SDR which avoids all elements of $\cup T$.

This  the proof.