falling factorial

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Entry type	Definition
Classification	msc 05A10
Defines	rising factorial
Defines	Pochhammer symbol

For $n \in \mathbb{N}$, the rising and falling factorials are n^{th} degree polynomial described, respectively, by

$$x^{\overline{n}} = x(x+1)\dots(x+n-1)$$
$$x^{\underline{n}} = x(x-1)\dots(x-n+1)$$

The two types of polynomials are related by:

$$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}}.$$

The rising factorial is often written as $(x)_n$, and referred to as the Pochhammer symbol (see hypergeometric series). Unfortunately, the falling factorial is also often denoted by $(x)_n$, so great care must be taken when encountering this notation.

Notes.

Unfortunately, the notational conventions for the rising and falling factorials lack a common standard, and are plagued with a fundamental inconsistency. An examination of reference works and textbooks reveals two fundamental sources of notation: works in combinatorics and works dealing with hypergeometric functions.

Works of combinatorics [1,2,3] give greater focus to the falling factorial because of its role in defining the Stirling numbers. The symbol $(x)_n$ almost always denotes the falling factorial. The notation for the rising factorial varies widely; we find $\langle x\rangle_n$ in [1] and $(x)^{(n)}$ in [3].

Works focusing on special functions [4,5] universally use $(x)_n$ to denote the rising factorial and use this symbol in the description of the various flavours of hypergeometric series. Watson [5] credits this notation to Pochhammer [6], and indeed the special functions literature eschews "falling factorial" in favour of "Pochhammer symbol". Curiously, according to Knuth [7], Pochhammer himself used $(x)_n$ to denote the binomial coefficient (Note: I haven't verified this.)

The notation featured in this entry is due to D. Knuth [7,8]. Given the fundamental inconsistency in the existing notations, it seems sensible to break with both traditions, and to adopt new and graphically suggestive notation for these two concepts. The traditional notation, especially in the hypergeometric camp, is so deeply entrenched that, realistically, one needs to be familiar with the traditional modes and to take care when encountering the symbol $(x)_n$.

References

1. Comtet, *Advanced combinatorics.*

2. Jordan, *Calculus of finite differences.*

3. Riordan, *Introduction to combinatorial analysis.*

4. Erdélyi, et. al., *Bateman manuscript project.*

5. Watson, *A treatise on the theory of Bessel functions.*

6. Pochhammer, "Ueber hypergeometrische Functionen n^{ter} Ordnung," *Journal für die reine und angewandte Mathematik* **71** (1870), 316–352.

7. Knuth, "Two notes on notation" `http://www-cs-faculty.stanford.edu/ knuth/papers/t`

8. Greene, Knuth, *Mathematics for the analysis of algorithms.*