proof of recurrences for derangement numbers

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The derangement numbers D_n satisfy two recurrence relations:

$$D_n = (n-1)[D_{n-1} + D_{n-2}] \tag{1}$$

$$D_n = nD_{n-1} + (-1)^n \tag{2}$$

These formulas can be derived algebraically (working from the explicit formula for the derangement numbers); there is also an enlightening combinatorial proof of (1).

The exponential generating function for the derangement numbers is

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

To derive the formulas algebraically, start with (2):

$$\begin{aligned} nD_{n-1} &= n(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} = n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} = n! \left(-\frac{(-1)^n}{n!} + \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right) \\ &= -(-1)^n + D_n \end{aligned}$$

and (2) follows. To derive (1), use (2) twice:

$$\begin{aligned} D_n &= nD_{n-1} + (-1)^n = (n-1)D_{n-1} + D_{n-1} + (-1)^n \\ &= (n-1)D_{n-1} + ((n-1)D_{n-2} + (-1)^{n-1}) + (-1)^n \\ &= (n-1)(D_{n-1} + D_{n-2}) \end{aligned}$$

Combinatorially, we can see (1) as follows. Write $[n]$ for $\{1, 2, \ldots, n\}$. Let π be any derangement of $[n-1]$, i.e. a permutation containing no 1-cycles. Adding n before any of the $n-1$ elements of π produces a derangement of $[n]$. For fixed π, these are clearly all distinct, since 1 has a different successor in each case; for distinct π, these are equally obviously distinct. Thus each derangement of $[n-1]$ corresponds to exactly $n-1$ derangements of $[n]$. Note also that since π had no 1-cycles that 1 is not a member of a transposition (a 2-cycle).

Now let π be a derangement of any $n-2$ elements chosen from $[n-1]$. There are clearly $(n-1)D_{n-2}$ such derangements. If the omitted element in π is x, then adding the transposition $(1\ x)$ to π produces a derangement of $[n]$, and all such derangements again are distinct from one another. Finally, since in this case 1 is a member of a transposition, these derangements are distinct from those in the first group. This proves (1).