Bell number

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The *Bell number*, denoted $B(n)$ is the total number of partitions of a set with n elements. For $n = 0$, we have $B(0) = 1$. For $n \geq 1$, we have

$$B(n) = \sum_{k=0}^{n} S(n,k) \qquad \text{for } n \geq 1$$

where $S(n,k)$ are the Stirling numbers of the second kind.

Proposition 1.

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k)$$

Proof. We count the number of partitions of a set of $n+1$ elements, depending on the size of the block containing the $n+1$st element. If the block has size j for $1 \leq j \leq n+1$ then we have $\binom{n}{j-1}$ choices for the $j-1$ other elements of the block. The remaining $n+1-j$ elements can be partitioned in $B(n+1-j)$ ways. We have therefore that:

$$\begin{aligned} B(n+1) &= \sum_{j=1}^{n+1} \binom{n}{j-1} B(n+1-j) \\ &= \sum_{j=1}^{n+1} \binom{n}{n+1-j} B(n+1-j) \\ &= \sum_{k=0}^{n} \binom{n}{k} B(k) \end{aligned}$$

$\square$

Using the formula above, one can easily derive the first few Bell numbers. Starting with $n = 0$, the first ten Bell numbers are 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147.