Ramsey numbers

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Define $R(a, b)$ to be the least integer N such that, in any red-blue 2-coloring of the edges of a N-vertex complete graph K_N, there must exist either an all-red K_a or an all-blue K_b.

Frank Ramsey proved these numbers always exist. He famously pointed out that among any 6 people, some three are mutual friends or some three mutual non-friends. That is, $R(3, 3) \leq 6$. Since a red pentagon with a blue pentagram drawn inside it has no monochromatic triangle, $R(3, 3) \geq 6$. So $R(3, 3) = 6$.

Special attention is usually paid to the diagonal $R(k, k)$, which is often just written $R(k)$.

One can also generalize this in various ways, e.g. consider $R(a, b, c)$ for *three*-colorings of edges, etc (any number of arguments), and allow general sets of graphs, not just pairs of complete ones.

Ramsey numbers are very difficult to determine. To prove lower bounds, construct good edge-colorings of some K_N and, use a clique-finder to find the largest mono-colored cliques. To prove upper bounds, the main tool has been $R(a, b) \leq R(a-1, b) + R(b-1, a)$ which implies $R(a, b) \leq \binom{a+b-2}{a-1}$ and then $R(k) \leq [1 + o(1)]4^k/(4\sqrt{\pi k}$ when $k \to \infty$. From considering random colorings and using a probabilistic nonconstructive existence argument, one may show $R(k) \geq k2^{k/2}[o(1) + \sqrt{2}/e]$. It is known that $R(1) = 1$, $R(2) = 2$, $R(3) = 6$, $R(4) = 18$, and $43 \leq R(5) \leq 49$. For a survey of the best upper and lower bounds available on small Ramsey numbers, see `http://www.combinatorics.org/Surveys/ds1.pdf`Radziszowski's survey (`http://www.cs.rit.edu/ spr/`alternate link). Another kind of Ramsey-like number which has not gotten as much attention as it deserves, are Ramsey numbers for *directed* graphs. Let $\vec{R}(n)$ denote the least integer N so that any tournament (complete directed graph with singly-directed arcs) with $\geq N$ vertices contains an acyclic (also called "transitive") n-node tournament. (Analogies: 2-color the edges $\to$ two directions for arcs. Monochromatic $\to$ acyclic, i.e. all arcs "point one way.")

Again, to prove lower bounds, construct good tournaments and apply something like a clique-finder (but instead aimed at trying to find the largest acyclic induced subgraph). To prove upper bounds, the main tool is $\vec{R}(n+1) \leq 2\vec{R}(n)$. That can be used to show the upper bound, and random-orientation arguments combined with a nonconstructive probabilistic existence argument show the lower bound, in $[1+o(1)]2^{n+1)/2} \leq \vec{R}(n) \leq 55 \cdot 2^{n-7}$. It is known that $\vec{R}(1) = 1$, $\vec{R}(2) = 2$, $\vec{R}(3) = 4$, $\vec{R}(4) = 8$, $\vec{R}(5) = 14$, $\vec{R}(6) =$

28, and $32 \leq \vec{R}(7) \leq 55$. For a full survey of directed graph Ramsey numbers includng proofs and refererences, see http://www.rangevoting.org/PuzzRamsey.htmlSmith's survey.