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Moore graphs of $d = 2$ are v-valent and order is $v^2 + 1$

Canonical name	MooreGraphsOfD2AreVvalentAndOrderIsV21
Date of creation	2013-03-22 15:11:30
Last modified on	2013-03-22 15:11:30
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Numerical id	6
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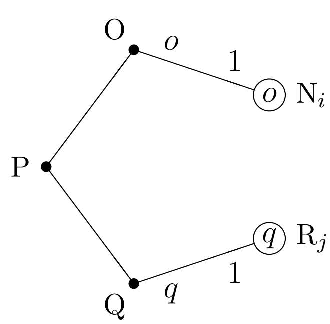
Entry type	Proof
Classification	msc 05C75

Given a Moore graph with diameter 2 and girth 5, which implies the existence of cycles. **To prove** every node (vertex) has the same valency, v say, and the number n of nodes (vertices) is $v^2 + 1$.

Lemma: a key feature of Moore graphs with diameter 2 is that any nodes X and that are not adjacent have exactly one shared neighbour Y. **Proof of lemma:** at least one Y_k because distance XY is not 1 so must be 2, at most one because $\mathrm{XY_0ZY_1}$ would be a 4-cycle. ∴

Lemma: every two adjacent nodes B and C lie together on some 5-cycle. **Proof of lemma:** every node (other than B) is at distance 1 or 2 from B, and every node (other than C) at distance 1 or 2 from C. There are no nodes X at distance 1 from both (BXC would be a 3-cycle). Suppose the graph only has nodes at distance 1 from B and 2 from C (call them A_i), and nodes at distance 1 from C and 2 from B (call them D_j). Now no cycle can exist (the only edges are A_iB, BC, CD_j; any edge of type AA, AC, , BD, DD would create a 3-or 4-cycle). But Moore graphs have cycles by definition. So there must be at least one node E at distance 2 from both B and C. Let A be the joint neighbour of B and E, and D that of C and E (note A $\neq$ D, otherwise BAC would be a 3-cycle). Now CDEAB is a 5-cycle with edge BC. ∴

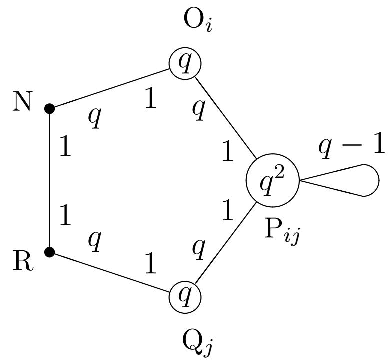
Lemma: two nodes O and Q at distance 2 have the same valency. **Proof of lemma:** let P be the unique joint neighbour. Let O have o other neighbours N_i and let Q have q other neighbours R_j.



No N_i can be adjacent to P (3-cycle PON_i) so the unique joint neighbours of any N_i and Q must be among the R_j. Different N_i and $\mathrm{N}_{i'}$ cannot use the same R_j (4-cycle $\mathrm{ON}_i\mathrm{R}_j\mathrm{N}_{i'}$) so we have $o \leqslant q$. By the same argument (swapping O and Q, N_i and R_j) also $q \leqslant o$ so we have $o = q$, both nodes have valency $q + 1$. ∴

Lemma: two adjacent nodes O and S have the same valency. **Proof of lemma:** let OPQRS be the 5-cycle through SO. Calling the valency of Q again $q+1$, both O and S have that same valency by the previous lemma. ∴

Proof of theorem: the graph is connected. Travel from any node to any other via adjacent ones, the valency stays the same by the last lemma (let's keep calling it $q+1$).



Now let N and R be any two adjacent nodes. N has q other neighbours O$_i$ and R has q other neighbours Q$_j$. Call the joint neighbour of O$_i$ and Q$_j$ now P$_{ij}$, these q^2 nodes are all distinct (4-cycles of type NOPO and/or RQPQ otherwise) and none of them coincide with N, R, the Os or Qs (3- or 4-cycles otherwise). On the other hand, there are no further nodes (distance >2 from N or R otherwise). Tally: $q^2+2q+2=(q+1)^2+1$, for valency $q+1$. ⁂