proof of Turan's theorem

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If the graph G has $n \leq p-1$ vertices it cannot contain any p–clique and thus has at most $\binom{n}{2}$ edges. So in this case we only have to prove that

$$\frac{n(n-1)}{2} \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}.$$

Dividing by n^2 we get

$$\frac{n-1}{n} = 1 - \frac{1}{n} \leq 1 - \frac{1}{p-1},$$

which is true since $n \leq p-1$.

Now we assume that $n \geq p$ and the set of vertices of G is denoted by V. If G has the maximum number of edges possible without containing a p–clique it contains a $p-1$–clique, since otherwise we might add edges to get one. So we denote one such clique by A and define $B := G \backslash A$.

So A has $\binom{p-1}{2}$ edges. We are now interested in the number of edges in B, which we will call e_B, and in the number of edges connecting A and B, which will be called $e_{A,B}$. By induction we get:

$$e_B \leq \frac{1}{2}\left(1 - \frac{1}{p-1}\right)(n-p+1)^2.$$

Since G does not contain any p–clique every vertice of B is connected to at most $p-2$ vertices in A and thus we get:

$$e_{A,B} \leq (p-2)(n-p+1).$$

Putting this together we get for the number of edges $|E|$ of G:

$$|E| \leq \binom{p-1}{2} + \frac{1}{2}\left(1 - \frac{1}{p-1}\right)(n-p+1)^2 + (p-2)(n-p+1).$$

And thus we get:

$$|E| \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}.$$