proof of Alon-Chung lemma

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Let the vertices of G be labeled by $\{1, 2, \ldots, n\}$, and $\mathbf{x}$ be the column vector defined by

$$x_i = \begin{cases} 1 & \text{if vertex } i \text{ is in } X \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots, n$.

Let $\mathbf{A}$ denote the adjacency matrix of G. The number of edges in the subgraph induced by X equals $\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}$, and we are going to show the following equivalent inequality,

$$\mathbf{x}^T\mathbf{A}\mathbf{x} \leq \frac{1}{n}\Big(d|X|^2 + \lambda|X|(n - |X|)\Big).$$

We label the eigenvalues of $\mathbf{A}$ in decreasing order as

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$

The largest eigenvalue λ_1 is equal to the degree d, and we let $\mathbf{u}_1$ be the corresponding normalized eigenvector,

$$\mathbf{u}_1 := \frac{1}{\sqrt{n}}[1, 1, \ldots, 1]^T.$$

As $\mathbf{A}$ is symmetric, there is a unitary matrix $\mathbf{U}$ that diagonalizes $\mathbf{A}$,

$$\mathbf{U}^T\mathbf{A}\mathbf{U} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

The first column of $\mathbf{U}$ is the column vector $\mathbf{u}_1$. We obtain

$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} &= \sum_{k=1}^{n} \lambda_k (\mathbf{u}_k^T\mathbf{x})^2 \\ &\leq d(\mathbf{u}_1^T\mathbf{x})^2 + \lambda_2 \sum_{k=2}^{n} (\mathbf{u}_k^T\mathbf{x})^2. \end{aligned}$$

In the line above, the first term is

$$d(\mathbf{u}_1^T\mathbf{x})^2 = \frac{d|X|^2}{n},$$

while the summation is equal to

$$\sum_{k=2}^{n}(\mathbf{u}_k^T\mathbf{x})^2 = \|\mathbf{x}\|^2 - (\mathbf{u}_1^T\mathbf{x})^2 = |X| - \frac{|X|^2}{n}.$$

Hence

$$\begin{aligned}\mathbf{x}^T\mathbf{A}\mathbf{x} &\leq \frac{d|X|^2}{n} + \lambda_2\Big(|X| - \frac{|X|^2}{n}\Big)\\ &= \frac{1}{n}\Big(d|X|^2 + \lambda_2|X|(n - |X|)\Big).\end{aligned}$$