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Stirling numbers of the second kind

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**Summary.** The Stirling numbers of the second kind,

$$S(n, k) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}, \quad k, n \in \mathbb{N}, \quad 1 \leq k \leq n,$$

are a doubly indexed sequence of natural numbers, enjoying a wealth of interesting combinatorial properties. There exist several logically equivalent characterizations, but the starting point of the present entry will be the following definition:

The Stirling number  $S(n, k)$  is the number of ways to partition a set of  $n$  objects into  $k$  groups.

For example,  $S(4, 2) = 7$  because there are seven ways to partition 4 objects — call them a, b, c, d — into two groups, namely:

$$(a)(bcd), (b)(acd), (c)(abd), (d)(abc), (ab)(cd), (ac)(bd), (ad)(bc)$$

Four additional characterizations will be discussed in this entry:

- a recurrence relation
- a generating function related to the falling factorial
- differential operators
- a double-index generating function

Each of these will be discussed below, and shown to be equivalent.

**A recurrence relation.** The Stirling numbers of the second kind can be characterized in terms of the following recurrence relation:

$$S(n, k) = kS(n-1, k) + S(n-1, k-1), \quad 1 \leq k < n,$$

subject to the following initial conditions:

$$S(n, n) = S(n, 1) = 1.$$

Let us now show that the recurrence formula follows from the enumerative definition. Evidently, there is only one way to partition  $n$  objects into 1 group (everything is in that group), and only one way to partition  $n$  objects into  $n$  groups (every object is a group all by itself). Proceeding recursively, a division of  $n$  objects  $a_1, \dots, a_{n-1}, a_n$  into  $k$  groups can be achieved by only one of two basic maneuvers:

- We could partition the first  $n - 1$  objects into  $k$  groups, and then add object  $a_n$  into one of those groups. There are  $kS(n - 1, k)$  ways to do this.
- We could partition the first  $n - 1$  objects into  $k - 1$  groups and then add object  $a_n$  as a new, 1 element group. This gives an additional  $S(n - 1, k - 1)$  ways to create the desired partition.

The recursive point of view, therefore explains the connection between the recurrence formula, and the original definition.

Using the recurrence formula we can easily obtain a table of the initial Stirling numbers:

$n \backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

**Falling Factorials.** Consider the vector space of polynomials in indeterminate  $x$ . The most obvious basis of this infinite-dimensional vector space is the sequence of monomial powers:  $x^n$ ,  $n \in \mathbb{N}$ . However, the sequence of falling factorials:

$$x^{\underline{n}} = x(x - 1)(x - 2) \dots (x - n + 1), \quad n \in \mathbb{N}$$

is also a basis, and hence can be used to generate the monomial basis. Indeed, the Stirling numbers of the second kind can be characterized as the coefficients involved in the corresponding change of basis matrix, i.e.

$$x^n = \sum_{k=1}^n S(n, k) x^{\underline{k}}.$$

So, for example,

$$x^4 = x + 7x(x - 1) + 6x(x - 1)(x - 2) + x(x - 1)(x - 2)(x - 3).$$

Arguing inductively, let us prove that this characterization follows from the recurrence relation. Evidently the formula is true for  $n = 1$ . Suppose then that the formula is true for a given  $n$ . We have

$$xx^{\underline{k}} = x^{\underline{k+1}} + kx^{\underline{k}},$$

and hence using the recurrence relation we deduce that

$$\begin{aligned}
x^{n+1} &= \sum_{k=1}^n S(n, k) x x^k \\
&= \sum_{k=1}^n (kS(n, k)x^k + S(n, k+1)) x^{k+1} \\
&= \sum_{k=1}^{n+1} S(n, k)x^k
\end{aligned}$$

**Differential operators.** Let  $D_x$  denote the ordinary derivative, applied to polynomials in indeterminate  $x$ , and let  $T_x$  denote the differential operator  $x D_x$ . We have the following characterization of the Stirling numbers of the second kind in terms of these two operators:

$$(T_x)^n = \sum_{k=1}^n S(n, k) x^k (D_x)^k,$$

where an exponentiated differential operator denotes the operator composed with itself the indicated number of times. Let us show that this follows from the recurrence relation. The proof is once again, inductive. Suppose that the characterization is true for a given  $n$ . We have

$$T_x(x^k(D_x)^k) = kx^k(D_x)^k + x^{k+1}(D_x)^{k+1},$$

and hence using the recurrence relation we deduce that

$$\begin{aligned}
(T_x)^{n+1} &= x D_x \left( \sum_{k=1}^n S(n, k) x^k (D_x)^k \right) \\
&= \sum_{k=1}^n S(n, k) (kx^k(D_x)^k + x^{k+1}(D_x)^{k+1}) \\
&= \sum_{k=1}^{n+1} S(n+1, k) x^k (D_x)^k
\end{aligned}$$

**Double index generating function.** One can also characterize the Stirling numbers of the second kind in terms of the following generating function:

$$e^{x(e^t-1)} = \sum_{n=1}^{\infty} \sum_{k=1}^n S(n, k) x^k \frac{t^n}{n!}.$$

Let us now prove this. Note that the differential equation

$$\frac{d\xi}{dt} = \xi,$$

admits the general solution

$$\xi = e^t x.$$

It follows that for any polynomial  $p(\xi)$  we have

$$\exp(tT_\xi)[p(\xi)] \Big|_{\xi=x} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (T_\xi)^n [p(\xi)] \Big|_{\xi=x} = p(e^t x).$$

The proof is simple: just take  $D_t$  of both sides. To be more explicit,

$$D_t [p(e^t x)] = p'(e^t x) e^t x = T_\xi [p(\xi)] \Big|_{\xi=xe^t},$$

and that is exactly equal to  $D_t$  of the left-hand side. Since this relation holds for all polynomials, it also holds for all formal power series. In particular if we apply the above relation to  $e^\xi$ , use the result of the preceding section, and note that

$$D_\xi [e^\xi] = e^\xi,$$

we obtain

$$\begin{aligned} e^{xe^t} &= \sum_{n=1}^{\infty} \frac{t^n}{n!} (T_\xi)^n [e^\xi] \Big|_{\xi=x} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^n S(n, k) \frac{t^n}{n!} \xi^k (D_\xi)^k [e^\xi] \Big|_{\xi=x} \\ &= e^x \sum_{n=1}^{\infty} \sum_{k=1}^n S(n, k) x^k \frac{t^n}{n!} \end{aligned}$$

Dividing both sides by  $e^x$  we obtain the desired generating function. Q.E.D.