multi-index derivative of a power

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Theorem If i, k are multi-indices in $\mathbb{N}^n$, and $x = (x_1, \ldots, x_n)$, then

$$\partial^i x^k = \begin{cases} \frac{k!}{(k-i)!} x^{k-i} & \text{if } i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof follows from the corresponding rule for the ordinary derivative; if i, k are in $0, 1, 2, \ldots$, then

$$\frac{d^i}{dx^i} x^k = \begin{cases} \frac{k!}{(k-i)!} x^{k-i} & \text{if } i \leq k, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Suppose $i = (i_1, \ldots, i_n)$, $k = (k_1, \ldots, k_n)$, and $x = (x_1, \ldots, x_n)$. Then we have that

$$\begin{aligned} \partial^i x^k &= \frac{\partial^{|i|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} x_1^{k_1} \cdots x_n^{k_n} \\ &= \frac{\partial^{i_1}}{\partial x_1^{i_1}} x_1^{k_1} \cdot \cdots \cdot \frac{\partial^{i_n}}{\partial x_n^{i_n}} x_n^{k_n}. \end{aligned}$$

For each $r = 1, \ldots, n$, the function $x_r^{k_r}$ only depends on x_r. In the above, each partial differentiation $\partial / \partial x_r$ therefore reduces to the corresponding ordinary differentiation d/dx_r. Hence, from equation **??**, it follows that $\partial^i x^k$ vanishes if $i_r > k_r$ for any $r = 1, \ldots, n$. If this is not the case, i.e., if $i \leq k$ as multi-indices, then for each r,

$$\frac{d^{i_r}}{dx_r^{i_r}} x_r^{k_r} = \frac{k_r!}{(k_r - i_r)!} x_r^{k_r - i_r},$$

and the theorem follows. $\square$