proof of upper and lower bounds to binomial coefficient

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Let $2 \leq k \leq n$ be natural numbers. We'll first prove the inequality

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k.$$

We rewrite $\binom{n}{k}$ as

$$\begin{aligned}\binom{n}{k} &= \frac{n(n-1)\cdots(n-k+1)}{k!} \\ &= \left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\cdot\frac{n^k}{k!}\end{aligned}$$

Since each of the parenthesized factors lies between 0 and 1, we have

$$\binom{n}{k} < \frac{n^k}{k!}$$

Since all the terms of the series $e^k = \sum_{n=0}^{\infty} k^n/n!$ are positive when k is a positive real number, each term must be smaller than the whole sum; in particular, this implies that, for any non-negative integer k, we have $e^k > k^k/k!$. Rearranging this slightly,

$$1 < \frac{k!e^k}{k^k}$$

Multiplying this inequality by the previous inequality for the binomial coefficient yields

$$\binom{n}{k} < \frac{n^k}{k!}\cdot\frac{k!e^k}{k^k} = \left(\frac{ne}{k}\right)^k$$

To conclude the proof we show that

$$\prod_{i=1}^{n-1}\left(1+\frac{1}{i}\right)^i = \frac{n^n}{n!}\ \forall\, n \geq 2 \in \mathbb{N}. \tag{1}$$

$$\begin{aligned}\prod_{i=1}^{n-1}\left(1+\frac{1}{i}\right)^i &= \prod_{i=1}^{n-1}\frac{(i+1)^i}{i^i} \\ &= \frac{\prod\limits_{i=2}^{n} i^{i-1}}{\left(\prod\limits_{i=1}^{n-1} i^{i-1}\right)\cdot(n-1)!}\end{aligned}$$

Since each left-hand factor in (**??**) is $< e$, we have $\frac{n^n}{n!} < e^{n-1}$. Since $n - i < n \;\forall\; 1 \leq i \leq k-1$, we immediately get

$$\binom{n}{k} \quad = \quad \frac{\prod_{i=2}^{k-1} \left(1 - \frac{1}{i}\right)}{k!} < \frac{n^k}{k!}.$$

And from

$$k \leq n \quad \Leftrightarrow \quad (n-i) \cdot k \geq (k-i) \cdot n \;\forall\; 1 \leq i \leq k-1$$

we obtain

$$\begin{aligned} \binom{n}{k} \quad &= \quad \frac{n}{k} \cdot \prod_{i=1}^{k-1} \frac{n-i}{k-i} \\ &\geq \left(\tfrac{n}{k}\right)^k. \end{aligned}$$