Ramsey-theoretic proof of the Erdős-Szekeres theorem

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Let $n \geq 3$ be an integer. By the finite Ramsey theorem, there is a positive integer N such that the arrows relation

$$N \to (n)^3_2$$

holds. Let X be a planar point set in general position with $|X| \geq N$. Define a red-blue colouring of the triangles in X as follows. Let $T = \{a, b, c\}$ be a triangle of X with a, b, c having monotonically increasing x-coordinates. (If two points have the same x-coordinate, break the tie by placing the point with smaller y-coordinate first.) If b lies above the line determined by a and c (the triangle "points up"), then colour the triangle blue. Otherwise, b lies below the line (the triangle "points down"); in this case colour the triangle red.

Now there must be a homogeneous subset $Y \subset X$ with $|Y| \geq n$. Without loss of generality, every triangle in Y is coloured blue. To see that this implies that the points of Y are the vertices of a convex n-gon, suppose there exist a, b, c, d in Y such that $d \in \mathrm{conv}\{a, b, c\}$ and such that a, b, c have monotonically increasing x-coordinates (breaking ties as before). Since every triangle in Y is coloured blue, the triangle $\{a, b, c\}$ points up. If the x-coordinate of d is less than or equal to that of b, then the triangle $\{a, d, b\}$ points down. But if the x-coordinate of d is greater than that of b, the triangle $\{b, d, c\}$ points down. In either case there is a red triangle in the homogeneously blue Y, a contradiction. Hence Y is a convex n-gon. This shows that $g(n) \leq N < \infty$.

References

[1] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.

[2] W. Morris and V. Soltan, The Erdős–Szekeres problem on points in convex position – a survey, *Bull. Amer. Math. Soc.* **37** (2000), 437–458.