proof of algebraic independence of elementary symmetric polynomials

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Geometric proof, works when R is a division ring.

Consider the quotient field Q of R and then the algebraic closure K of Q.

Consider the substitution map that associates to values $t_1, \ldots, t_n \in K^n$ the symmetric functions in these variables $s_1, \ldots, s_n$.

$$\begin{array}{cccc} \phi : & K^n & \to & K^n \\ & (t_i) & \mapsto & (s_i) \end{array}$$

Because K is algebraic closed this map is surjective. Indeed, fix values v_i, then on an algebraic closed field there are roots t_i such that

$$X^n + \sum_i v_i X^i = \Pi_i (X + t_i)$$

And by developing the right-hand side we get $v_i = s_i$.

Then we consider the transposition morphism of algebras ϕ^* :

$$\begin{array}{cccc} \phi^* : & R[S_1, \ldots, S_n] & \to & R[T_1, \ldots, T_n] \\ & f & \mapsto & f \circ \phi \end{array}$$

The capital letters are there to emphasize the S_i and T_i are variables and $R[S_1, \ldots, S_n]$ and $R[T_1, \ldots, T_n]$ are regarded as function algebras over K^n.

The theorem stating that the symmetric functions are algebraically independent is no more than saying that this morphism is injective. As a matter of fact, $\phi^*(S_i)$ is the i^{th} symmetric function in the T_i, and ϕ^* is clearly a morphism of algebras.

The conclusion is then straightforward from the surjectivity of ϕ because if $f \circ \phi = 0$ for some f, then by surjectivity of ϕ it means that f was zero in the first place. In other words the kernel of ϕ^* is reduced to 0.