Levi-Civita permutation symbol

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Definition 1. *Let* $k_i \in \{1, \cdots, n\}$ *for all* $i = 1, \cdots, n$. *The* Levi-Civita permutation symbols $\varepsilon_{k_1 \cdots k_n}$ *and* $\varepsilon^{k_1 \cdots k_n}$ *are defined as*

$$\varepsilon_{k_1 \cdots k_m} = \varepsilon^{k_1 \cdots k_m} = \begin{cases} +1 & when \{l \mapsto k_l\} \text{ is an even permutation (of } \{1, \cdots, n\}), \\ -1 & when \{l \mapsto k_l\} \text{ is an odd permutation,} \\ 0 & otherwise, \text{ i.e., } when\, k_i = k_j, \text{ for some } \; i \neq j. \end{cases}$$

The Levi-Civita permutation symbol is a special case of the generalized Kronecker delta symbol. Using this fact one can write the Levi-Civita permutation symbol as the determinant of an $n \times n$ matrix consisting of traditional delta symbols. See the entry on the generalized Kronecker symbol for details.

When using the Levi-Civita permutation symbol and the generalized Kronecker delta symbol, the Einstein summation convention is usually employed. In the below, we shall also use this convention.

Properties

- When $n = 2$, we have for all i, j, m, n in $\{1, 2\}$,

$$\varepsilon_{ij}\varepsilon^{mn} = \delta_i^m \delta_j^n - \delta_i^n \delta_j^m, \tag{1}$$

$$\varepsilon_{ij}\varepsilon^{in} = \delta_j^n, \tag{2}$$

$$\varepsilon_{ij}\varepsilon^{ij} = 2. \tag{3}$$

- When $n = 3$, we have for all i, j, k, m, n in $\{1, 2, 3\}$,

$$\varepsilon_{jmn}\varepsilon^{imn} = 2\delta_j^i, \tag{4}$$

$$\varepsilon_{ijk}\varepsilon^{ijk} = 6. \tag{5}$$

Let us prove these properties. The proofs are instructional since they demonstrate typical argumentation methods for manipulating the permutation symbols.

Proof. For equation **??**, let us first note that both sides are antisymmetric with respect of ij and mn. We therefore only need to consider the case $i \neq j$ and $m \neq n$. By substitution, we see that the equation holds for $\varepsilon_{12}\varepsilon^{12}$, i.e., for $i = m = 1$ and $j = n = 2$. (Both sides are then one). Since the equation is anti-symmetric in ij and mn, any set of values for these can be reduced the above case (which holds). The equation thus holds for all values of ij and mn. Using equation **??**, we have for equation **??**

$$\begin{aligned}\varepsilon_{ij}\varepsilon^{in} &= \delta^i_i\delta^n_j - \delta^n_i\delta^i_j \\ &= 2\delta^n_j - \delta^n_j \\ &= \delta^n_j.\end{aligned}$$

Here we used the Einstein summation convention with i going from 1 to 2. Equation **??** follows similarly from equation **??**. To establish equation **??**, let us first observe that both sides vanish when $i \neq j$. Indeed, if $i \neq j$, then one can not choose m and n such that both permutation symbols on the left are nonzero. Then, with $i = j$ fixed, there are only two ways to choose m and n from the remaining two indices. For any such indices, we have $\varepsilon_{jmn}\varepsilon^{imn} = (\varepsilon^{imn})^2 = 1$ (no summation), and the result follows. The last property follows since $3! = 6$ and for any distinct indices i, j, k in $\{1, 2, 3\}$, we have $\varepsilon_{ijk}\varepsilon^{ijk} = 1$ (no summation). $\square$

Examples and Applications.

- The determinant of an $n \times n$ matrix $A = (a_{ij})$ can be written as

  $$\det A = \varepsilon_{i_1 \cdots i_n} a_{1i_1} \cdots a_{ni_n},$$

  where each i_l should be summed over $1, \ldots, n$.

- If $A = (A^1, A^2, A^3)$ and $B = (B^1, B^2, B^3)$ are vectors in $\mathbb{R}^3$ (represented in some right hand oriented orthonormal basis), then the ith component of their cross product equals

  $$(A \times B)^i = \varepsilon^{ijk} A^j B^k.$$

  For instance, the first component of $A \times B$ is $A^2B^3 - A^3B^2$. From the above expression for the cross product, it is clear that $A \times B = -B \times A$. Further, if $C = (C^1, C^2, C^3)$ is a vector like A and B, then the triple scalar product equals

  $$A \cdot (B \times C) = \varepsilon^{ijk} A^i B^j C^k.$$

  From this expression, it can be seen that the triple scalar product is antisymmetric when exchanging any adjacent arguments. For example, $A \cdot (B \times C) = -B \cdot (A \times C)$.

- Suppose $F = (F^1, F^2, F^3)$ is a vector field defined on some open set of $\mathbb{R}^3$ with Cartesian coordinates $x = (x^1, x^2, x^3)$. Then the ith component of the curl of F equals

$$(\nabla \times F)^i(x) = \varepsilon^{ijk} \frac{\partial}{\partial x^j} F^k(x).$$