maximal matching/minimal edge covering theorem

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Theorem Let G be a graph. If M is a matching on G, and C is an edge covering for G, then $|M| \leq |C|$.

Proof Consider an arbitrary matching M on G and an arbitrary edge covering C on G. We will attempt to construct a one-to-one function $f : M \to C$.

Consider some edge $e \in M$. At least one of the vertices that e joins must be in C, because C is an edge covering and hence every edge is incident with some vertex in C. Call this vertex v_e, and let $f(e) = v_e$.

Now we will show that f one-to-one. Suppose we have two edges $e_1, e_2 \in M$ where $f(e_1) = f(e_2) = v$. By the definition of f, e_1 and e_2 must both be incident with v. Since M is a matching, however, no more than one edge in M can be incident with any given vertex in G. Therefore $e_1 = e_2$, so f is one-to-one.

Hence we now have that $|M| \leq |C|$.

Corollary Let G be a graph. Let M and C be a matching and an edge covering on G, respectively. If $|M| = |C|$, then M is a maximal matching and C is a minimal edge covering.

Proof Suppose M is not a maximal matching. Then, by definition, there exists another matching M' where $|M| < |M'|$. But then $|M'| > |C|$, which violates the above theorem.

Likewise, suppose C is not a minimal edge covering. Then, by definition, there exists another covering C' where $|C'| < |C|$. But then $|C'| < |M|$, which violates the above theorem.