Stirling numbers of the first kind

Canonical name	StirlingNumbersOfTheFirstKind
Date of creation	2013-03-22 12:33:44
Last modified on	2013-03-22 12:33:44
Owner	rmilson (146)
Last modified by	rmilson (146)
Numerical id	6
Author	rmilson (146)
Entry type	Definition
Classification	msc 05A15
Synonym	Stirling numbers
Related topic	StirlingNumbersSecondKind

Introduction. The Stirling numbers of the first kind, frequently denoted as

$$s(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}, \quad k,n \in \mathbb{N}, \quad 1 \le k \le n,$$

are the integer coefficients of the falling factorial polynomials. To be more precise, the defining relation for the Stirling numbers of the first kind is:

$$x^{\underline{n}} = x(x-1)(x-2)\dots(x-n+1) = \sum_{k=1}^{n} s(n,k)x^k.$$

Here is the table of some initial values.

$n\backslash k$	1	2	3	4	5
1	1				
2	-1	1			
3	2	-3	1		
4	-6	11	-6	1	
5	24	-50	35	-10	1

Recurrence Relation. The evident observation that

$$x^{\underline{n+1}} = x x^{\underline{n}} - n x^{\underline{n}}.$$

leads to the following equivalent characterization of the $s(n,k)$, in terms of a 2-place recurrence formula:

$$s(n+1,k) = s(n,k-1) - ns(n,k), \qquad 1 \le k < n,$$

subject to the following initial conditions:

$$s(n,0) = 0, \qquad s(1,1) = 1.$$

Generating Function. There is also a strong connection with the generalized binomial formula, which furnishes us with the following generating function:

$$(1+t)^x = \sum_{n=0}^{\infty} \sum_{k=1}^{n} s(n,k) x^k \frac{t^n}{n!}.$$

This generating function implies a number of identities. Taking the derivative of both sides with respect to t and equating powers, leads to the recurrence relation described above. Taking the derivative of both sides with respect to x gives

$$(k+1)s(n,k+1) = \sum_{j=k}^{n} (-1)^{n-j}(n-j)!\binom{n+1}{j}s(j,k)$$

This is because the derivative of the left side of the generating funcion equation with respect to x is

$$(1+t)^x \ln(1+t) = (1+t)^x \sum_{k=1}^{\infty} (-1)^{k-1}\frac{t^k}{k}.$$

The relation

$$(1+t)^{x_1}(1+t)^{x_2} = (1+t)^{x_1+x_2}$$

yields the following family of summation identities. For any given $k_1, k_2, d \geq 1$ we have

$$\binom{k_1+k_2}{k_1}s(d+k_1+k_2, k_1+k_2) = \sum_{d_1+d_2=d} \binom{d+k_1+k_2}{k_1+d_1}s(d_1+k_1, k_1)s(d_2+k_2, k_2).$$

Enumerative interpretation. The absolute value of the Stirling number of the first kind, $s(n,k)$, counts the number of permutations of n objects with exactly k orbits (equivalently, with exactly k cycles). For example, $s(4,2) = 11$, corresponds to the fact that the symmetric group on 4 objects has 3 permutations of the form

$$(**)(**) \quad \text{— 2 orbits of size 2 each,}$$

and 8 permutations of the form

$$(***) \quad \text{— 1 orbit of size 3, and 1 orbit of size 1,}$$

(see the entry on cycle notation for the meaning of the above expressions.)

Let us prove this. First, we can remark that the unsigned Stirling numbers of the first are characterized by the following recurrence relation:

$$|s(n+1,k)| = |s(n,k-1)| + n|s(n,k)|, \qquad 1 \leq k < n.$$

To see why the above recurrence relation matches the count of permutations with k cycles, consider forming a permutation of $n+1$ objects from a permutation of n objects by adding a distinguished object. There are exactly two ways in which this can be accomplished. We could do this by forming a singleton cycle, i.e. leaving the extra object alone. This accounts for the $s(n, k-1)$ term in the recurrence formula. We could also insert the new object into one of the existing cycles. Consider an arbitrary permutation of n object with k cycles, and label the objects $a_1, \ldots, a_n$, so that the permutation is represented by

$$\underbrace{(a_1 \ldots a_{j_1})(a_{j_1+1} \ldots a_{j_2}) \ldots (a_{j_{k-1}+1} \ldots a_n)}_{k \text{ cycles}}.$$

To form a new permutation of $n+1$ objects and k cycles one must insert the new object into this array. There are, evidently n ways to perform this insertion. This explains the $n\, s(n, k)$ term of the recurrence relation. Q.E.D.