Thomassen's theorem on 3-connected graphs

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Every http://planetmath.org/KConnectedGraph3-connected graph G with more than 4 vertices has an edge e such that http://planetmath.org/EdgeContractionG/e is also 3-connected.

Note: G/e denotes the graph obtained from G by contracting the edge e. If $e = xy$ we also use the notation G/xy.

Suppose such an edge doesn't exist. Then, for every edge $e = xy$, the graph G/e isn't 3-connected and can be made disconnected by removing 2 vertices. Since $\kappa(G) \geq 3$, our contracted vertex v_{xy} has to be one of these two. So for every edge e, G has a vertex $z \neq x, y$ such that $\{v_{xy}, z\}$ separates G/e. Any 2 vertices separated by $\{v_{xy}, z\}$ in G/e are separated in G by $S := \{x, y, z\}$. Since the minimal size of a separating set is 3, every vertex in S has an adjacent vertex in every component of $G - S$.

Now we choose the edge e, the vertex z and the component C such that $|C|$ is minimal. We also choose a vertex v adjacent to z in C.

By construction G/zv is not 3-connected since removing xy disconnects $C - v$ from G/zv. So there is a vertex w such that $\{z, v, w\}$ separates G and as above every vertex in $\{z, v, w\}$ has an adjacent vertex in every component of $G - \{z, v, w\}$. We now consider a component D of $G - \{z, v, w\}$ that doesn't contain x or y. Such a component exists since x and y belong to the same component and $G - \{z, v, w\}$ isn't connected. Any vertex adjacent to v in D is also an element of C since v is an element of C. This means D is a proper subset of C which contradicts our assumption that $|C|$ was minimal.