matroid independence axioms

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Owner	sgraves (6614)
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Author	sgraves (6614)
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Hassler Whitney's definition of a matroid was based upon the idea of independent sets and was given in terms of the following three axioms:

A finite set E along with a collection $\mathcal{I}$ of subsets of E is a matroid if $M = (E, \mathcal{I})$ meets the following criteria:
(I1) $\emptyset \in \mathcal{I}$;
(I2) If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$;
(I3) If $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, then there is some $e \in I_2$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The third axiom, (I3), is equivalent to the following alternative axiom:
(I3*) If $S, T \in \mathcal{I}$ and $S, T \subset U \subset E$ and S and T are both maximal subsets of U with the property that they are in $\mathcal{I}$, then $|S| = |T|$.

Proof. Suppose (I3) holds, and that S and T are maximal independent subsets of $A \subseteq E$. Also assume, without loss of generality, that $|S| < |T|$. Then there is some $e \in T$ such that $S \cup \{e\} \in \mathcal{I}$, but $S \subset S \cup \{e\} \subseteq A$, contradicting maximality of S.
Now suppose that (I3*) holds, and assume that $S, T \in \mathcal{I}$ with $|S| < |T|$. Let $A = S \cup T$. Then S cannot be maximal in A by (I3*), so there must be elements $e_i \in A$ such that $S \cup \{e_i\} \in \mathcal{I}$ is maximal, and by construction these $e_i \in T$. So (I3) holds. $\square$