# principle of inclusion-exclusion, proof of



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The proof is by induction. Consider a single set $A_1$. Then the principle of inclusion-exclusion states that $|A_1| = |A_1|$, which is trivially true.

Now consider a collection of exactly two sets $A_1$ and $A_2$. We know that

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$$

Furthermore, the three sets on the right-hand side of that equation must be disjoint. Therefore, by the addition principle, we have

$$\begin{aligned}
|A \cup B| &= |A \setminus B| + |B \setminus A| + |A \cap B| \\
&= |A \setminus B| + |A \cap B| + |B \setminus A| + |A \cap B| - |A \cap B| \\
&= |A| + |B| - |A \cap B|
\end{aligned}$$

So the principle of inclusion-exclusion holds for any two sets.

Now consider a collection of $N > 2$ finite sets $A_1, A_2, \ldots A_N$. We assume that the principle of inclusion-exclusion holds for any collection of $M$ sets where $1 \leq M < N$. Because the union of sets is associative, we may break up the union of all sets in the collection into a union of two sets:

$$\bigcup_{i=1}^{N} A_i = \left( \bigcup_{i=1}^{N-1} A_i \right) \cup A_N$$

By the principle of inclusion-exclusion for two sets, we have

$$\left| \bigcup_{i=1}^{N} A_i \right| = \left| \bigcup_{i=1}^{N-1} A_i \right| + |A_N| - \left| \left( \bigcup_{i=1}^{N-1} A_i \right) \cap A_N \right|$$

Now, let $I_k$ be the collection of all $k$-fold intersections of $A_1, A_2, \ldots A_{N-1}$, and let $I'_k$ be the collection of all $k$-fold intersections of $A_1, A_2, \ldots A_N$ that include $A_N$. Note that $A_N$ is included in every member of $I'_k$ and in no member of $I_k$, so the two sets do not duplicate one another.

We then have

$$\left| \bigcup_{i=1}^{N} A_i \right| = \sum_{j=1}^{N} \left( (-1)^{(j+1)} \sum_{S \in I_j} |S| \right) + |A_N| - \left| \left( \bigcup_{i=1}^{N-1} A_i \right) \cap A_N \right|$$

by the principle of inclusion-exclusion for a collection of $N-1$ sets. Then, we may distribute set intersection over set union to find that

$$\left|\bigcup_{i=1}^{N} A_i\right| = \sum_{j=1}^{N}\left((-1)^{(j+1)}\sum_{S\in I_j}|S|\right) + |A_N| - \left|\bigcup_{i=1}^{N-1}(A_i\cap A_N)\right|$$

Note, however, that

$$(A_x \cap A_N) \cup (A_y \cap A_N) = (A_x \cap A_y \cap A_N)$$

Hence we may again apply the principle of inclusion-exclusion for $N-1$ sets, revealing that

$$\begin{aligned}
\left|\bigcup_{i=1}^{N} A_i\right| &= \sum_{j=1}^{N-1}\left((-1)^{(j+1)}\sum_{S\in I_j}|S|\right) + |A_N| - \sum_{j=1}^{N-1}\left((-1)^{(j+1)}\sum_{S\in I_j}|S\cap A_N|\right)\\
&= \sum_{j=1}^{N-1}\left((-1)^{(j+1)}\sum_{S\in I_j}|S|\right) + |A_N| - \sum_{j=1}^{N-1}\left((-1)^{(j+1)}\sum_{S\in I'_{j+1}}|S|\right)\\
&= \sum_{j=1}^{N-1}\left((-1)^{(j+1)}\sum_{S\in I_j}|S|\right) + |A_N| - \sum_{j=2}^{N}\left((-1)^{(j)}\sum_{S\in I'_{j}}|S|\right)\\
&= \sum_{j=1}^{N-1}\left((-1)^{(j+1)}\sum_{S\in I_j}|S|\right) + |A_N| + \sum_{j=2}^{N}\left((-1)^{(j+1)}\sum_{S\in I'_{j}}|S|\right)
\end{aligned}$$

The second sum does not include $I'_1$. Note, however, that $I'_1 = \{A_N\}$, so we have

$$\begin{aligned}
\left|\bigcup_{i=1}^{N} A_i\right| &= \sum_{j=1}^{N-1}\left((-1)^{(j+1)}\sum_{S\in I_j}|S|\right) + \sum_{j=1}^{N}\left((-1)^{(j+1)}\sum_{S\in I'_{j}}|S|\right)\\
&= \sum_{j=1}^{N-1}\left[(-1)^{(j+1)}\left(\sum_{S\in I_j}|S| + \sum_{S\in I'_j}|S|\right)\right] + (-1)^{N+1}\left|\bigcap_{i=1}^{N} A_i\right|
\end{aligned}$$

Combining the two sums yields the principle of inclusion-exclusion for $N$ sets.