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inductive proof of binomial theorem

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We prove the theorem for a ring. We do not assume a unit for the ring. We do not need commutativity of the ring, but only that a and b commute.

When $n = 1$, the result is clear.

For the inductive step, assume it holds for m. Then for $n = m + 1$,

$$\begin{aligned}
(a+b)^{m+1} &= (a+b)(a+b)^m \\
&= (a+b)\left(a^m + b^m + \sum_{k=1}^{m-1} \binom{m}{k} a^{m-k} b^k\right) \text{ by the inductive hypothesis} \\
&= a^{m+1} + b^{m+1} + ab^m + ba^m + \sum_{k=1}^{m-1} \binom{m}{k} a^{m-k+1} b^k + \sum_{k=1}^{m-1} \binom{m}{k} a^{m-k} b^{k+1} \\
&= a^{m+1} + b^{m+1} + \sum_{k=1}^{m} \binom{m}{k} a^{m-k+1} b^k + \sum_{k=0}^{m-1} \binom{m}{k} a^{m-k} b^{k+1} \text{ by combining terms} \\
&= a^{m+1} + b^{m+1} + \sum_{k=1}^{m} \binom{m}{k} a^{m-k+1} b^k + \sum_{j=1}^{m} \binom{m}{j-1} a^{m+1-j} b^j \text{ let j=k+1 in second sum} \\
&= a^{m+1} + b^{m+1} + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1}\right] a^{m+1-k} b^k \text{ by combining the sums} \\
&= a^{m+1} + b^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} a^{m+1-k} b^k \text{ from Pascal's rule}
\end{aligned}$$

as desired.