Stirling polynomial

Canonical name	StirlingPolynomial
Date of creation	2013-03-22 15:38:36
Last modified on	2013-03-22 15:38:36
Owner	kronos (12218)
Last modified by	kronos (12218)
Numerical id	9
Author	kronos (12218)
Entry type	Definition
Classification	msc 05A15

Stirling's polynomials $S_k(x)$ are defined by the generating function

$$\left(\frac{t}{1-e^{-t}}\right)^{x+1} = \sum_{k=0} \frac{S_k(x)}{k!} t^k.$$

The sequence $S_k(x-1)$ is of binomial type, since $S_k(x+y-1) = \sum_{i=0}^{k} \binom{k}{i} S_i(x-1) S_{k-i}(y-1)$. Moreover, this basic recursion holds: $S_k(x) = (x-k)\frac{S_k(x-1)}{x} + kS_{k-1}(x+1)$.

These are the first polynomials:

1. $S_0(x) = 1$;
2. $S_1(x) = \frac{1}{2}(x+1)$;
3. $S_2(x) = \frac{1}{12}(3x^2 + 5x + 2)$;
4. $S_3(x) = \frac{1}{8}(x^3 + 2x^2 + x)$;
5. $S_4(x) = \frac{1}{240}(15x^4 + 30x^3 + 5x^2 - 18x - 8)$.

In addition we have these special values:

1. $S_k(-m) = \frac{(-1)^k}{\binom{k+m-1}{k}} S_{k+m-1,m-1}$, where $S_{m,n}$ denotes Stirling numbers of the second kind. Conversely, $S_{n,m} = (-1)^{n-m}\binom{n}{m} S_{n-m}(-m-1)$;
2. $S_k(-1) = \delta_{k,0}$;
3. $S_k(0) = (-1)^k B_k$, where B_k are Bernoulli's numbers;
4. $S_k(1) = (-1)^{k+1}((k-1)B_k + kB_{k-1})$;
5. $S_k(2) = \frac{(-1)^k}{2}((k-1)(k-2)B_k + 3k(k-2)B_{k-1} + 2k(k-1)B_{k-2})$;
6. $S_k(k) = k!$;
7. $S_k(m) = \frac{(-1)^k}{\binom{m}{k}} s_{m+1,m+1-k}$, where $s_{m,n}$ are Stirling numbers of the first kind. They may be recovered by $s_{n,m} = (-1)^{n-m}\binom{n-1}{n-m} S_{n-m}(n-1)$.

Explicit representations involving Stirling numbers can be deduced with Lagrange's interpolation formula:

$$S_k(x) = \sum_{n=0}^{k} (-1)^{k-n} S_{k+n,n} \frac{\binom{x+n}{n}\binom{x+k+1}{k-n}}{\binom{k+n}{n}} = \sum_{n=0}^{k} (-1)^n s_{k+n+1,n+1} \frac{\binom{x-k}{n}\binom{x-k-n-1}{k-n}}{\binom{k+n}{k}}.$$

These following formulae hold as well:

$$\binom{k+m}{k} S_k(x-m) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k+m}{i} S_{k-i+m,m} S_i(x),$$

$$\binom{k-m}{k} S_k(x+m) = \sum_{i=0}^{k} \binom{k-m}{i} s_{m,m-k+i} S_i(x).$$