multi-index notation

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Defines	multi-index
Defines	multi-indices

Multi-indices form a powerful notational device for keeping track of multiple derivatives or multiple powers. In many respects these resemble natural numbers. For example, one can define the factorial, binomial coefficients, and derivatives for multi-indices. Using these one can state traditional results such as the multinomial theorem, Leibniz' rule, Taylor's formula, etc. very concisely. In fact, the multi-dimensional results are more or less obtained simply by replacing usual indices in $\mathbb{N}$ with multi-indices. See below for examples.

Definition A *multi-index* is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers $\alpha_1, \ldots, \alpha_n$. In other words, $\alpha \in \mathbb{N}^n$. Usually, n is the dimension of the underlying space. Therefore, when dealing with multi-indices, n is usually assumed clear from the context.

Operations on multi-indices

For a multi-index α, we define the *length* (or *order*) as

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

and the *factorial* as

$$\alpha! = \prod_{k=1}^{n} \alpha_k!.$$

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are two multi-indices, their sum and difference is defined component-wise as

$$\begin{aligned} \alpha + \beta &= (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n), \\ \alpha - \beta &= (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n). \end{aligned}$$

Thus $|\alpha \pm \beta| = |\alpha| \pm |\beta|$. Also, if $\beta_k \leq \alpha_k$ for all $k = 1, \ldots, n$, then we write $\beta \leq \alpha$. For multi-indices α, β, with $\beta \leq \alpha$, we define

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha - \beta)!\beta!}.$$

For a point $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ (with standard coordinates) we define

$$x^\alpha = \prod_{k=1}^{n} x_k^{\alpha_k}.$$

Also, if $f\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, we define

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} e_1 \cdots \partial^{\alpha_n} e_n} f,$$

where $e_1, \ldots, e_n$ are the standard unit vectors of $\mathbb{R}^n$. Since f is sufficiently smooth, the order in which the derivations are performed is irrelevant. For multi-indices α and β, we thus have

$$\partial^\alpha \partial^\beta = \partial^{\alpha+\beta} = \partial^{\beta+\alpha} = \partial^\beta \partial^\alpha.$$

Examples

1. If n is a positive integer, and $x_1, \ldots, x_k$ are complex numbers, the multinomial expansion states that

$$(x_1 + \cdots + x_k)^n = n! \sum_{|\alpha|=n} \frac{x^\alpha}{\alpha!},$$

   where $x = (x_1, \ldots, x_k)$ and α is a multi-index. (`http://planetmath.org/MultinomialTheor`

2. Leibniz' rule: If $f, g\colon \mathbb{R}^n \to \mathbb{R}$ are smooth functions, and β is a multi-index, then

$$\partial^\beta(fg) = \sum_{\alpha \le \beta} \binom{\beta}{\alpha} \partial^\alpha(f)\, \partial^{\beta-\alpha}(g),$$

   where α is a multi-index.

References

[1] M. Reed, B. Simon, *Methods of Mathematical Physics, I - Functional Analysis*, Academic Press, 1980.