Tutte theorem

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Let $G(V, E)$ be any finite graph. G has a complete matching if and only if for all $X \subseteq V(G)$ the inequality $c_p(G - X) \leq |X|$ holds, where $c_p(H)$ is the number of the components of the H graph with odd number of vertices. This is called the Tutte condition.

Proof. In a complete matching there is at least one edge between X and the odd components of $G - X$, and different edges have distinct endvertices. So $c_p(G - X) \leq |X|$. This proves the necessity.

The sufficiency is proved in a few steps indirectly.

1. It is very easy to see that if $|V(G)|$ is even, then $c_p(G - X) \neq |X| - 1$ and $c_p(G - X) \neq |X| + 1$. This will be used later. And if the Tutte condition holds, applying it to $X = \emptyset$ gives $|V(G)|$ is even.

2. Now assume there exists a graph satisfying the Tutte condition but without complete matching ,and let G be such a counterexample with the lowest number of vertices. There exist a set $Y \subseteq V(G)$ such that $c_p(G - Y) = |Y|$, examples are the empty set and every vertex. Let Y_0 be such set with the highest number of vertices.

3. If G_s is an even component in $G - Y_0$, then adding its vertex t to Y_0 gives $c_p(G - Y_0 - \{t\}) \geq |Y_0| + 1$, because it creates at least one odd component. But this with the Tutte condition gives $c_p(G - Y_0 - \{t\}) = |Y_0 \cup \{t\}|$, which contradicts the maximality of Y_0. So there are no even components in $G - Y_0$.

4. Let G_p be an odd component of $G - Y_0$, and let t any of its vertices. Assume there is no complete matching in $G_p - \{t\}$. Since G is minimal counterexample, there exits a set $X \subseteq G_p - \{t\}$ such that $c_p(G_p - \{t\} - X) > |X|$, but because of (1) it is at least $|X| + 2$. Then $c_p(G - X - Y_0 - \{t\}) \geq |Y_0| - 1 + |X| + 2 = |X \cup Y_0 \cup \{t\}|$, which contradicts the maximality of Y_0. So by removing any vertex from an odd component the remaining part has a complete matching.

5. Represent each odd components by one vertex, and remove the edges in Y_0. From the definition of Y_0 this is a bipartite graph. If we choose p vertices from the representatives of the odd components, they are together adjacent to at least p vertices in Y_0, otherwise we could remove less than p vertices from Y_0 thus from $V(G)$, and still get p odd

component, which violates the Tutte condition. So there is complete matching in this reduced graph because of Hall's marriage theorem.

6. The complete matching in the reduced graph covers Y_0 and exactly one vertex in each odd component. But there is also a matching in the rest of each odd component covering the rest as proven in (4), and from (3) there are no even components. Combining these gives a complete matching of G, which is a contradiction.

$\square$