derivation of the generating series for the Stirling numbers of the second kind

Canonical name	DerivationOfTheGeneratingSeriesForTheStirlingNumbersOfTheSecondKind
Date of creation	2013-03-22 15:13:35
Last modified on	2013-03-22 15:13:35
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Last modified by	cgibbard (959)
Numerical id	6
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Entry type	Derivation
Classification	msc 05A15

The derivation of the generating series is much simpler if one makes use of the composition lemma for exponential generating series. We are looking for the generating series for sets of nonempty sets, so in the notation of Jackson and Goulden, we have the set decomposition:

$$\mathcal{A} \xrightarrow{\sim} \mathcal{U} \circledast (\mathcal{U} \backslash \{\emptyset\})$$

where $\mathcal{U}$ is the set of all canonical unordered sets, $\mathcal{A}$ is the set which we are interested in counting, and $\circledast$ is star-composition of sets of labelled combinatorial objects.

The set $\mathcal{U}$ has one object in it of each weight, and so has exponential generating series:

$$[(\mathcal{U}, \omega)]_e(x) = \sum_{n \geq 0} \frac{x^n}{n!} = e^x$$

The set $\mathcal{U}\backslash\{\emptyset\}$ then has generating series:

$$[(\mathcal{U}\backslash\{\emptyset\}, \omega)]_e(x) = e^x - 1$$

So, by the star composition lemma and the above decomposition,

$$\begin{aligned} [(\mathcal{A}, \omega)]_e(x) &= [(\mathcal{U} \circledast (\mathcal{U}\backslash\{\emptyset\}), \omega)]_e(x) \\ &= \left([(\mathcal{U}, \omega)]_e \circ [(\mathcal{U}\backslash\{\emptyset\}, \omega)]_e\right)(x) \\ &= e^{e^x - 1} \end{aligned}$$

By tensoring the weight function ω with a weight function λ counting the number of parts each set partition contains, we get

$$[(\mathcal{A}, \omega \otimes \lambda)]_{e,o}(x, t) = e^{t(e^x - 1)}$$

using a derivation similar to the one above.