# counting compositions of an integer



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A *composition* of a nonnegative integer $n$ is a sequence $(a_1, \ldots, a_k)$ of positive integers with $\sum a_i = n$. Denote by $C_n$ the number of compositions of $n$, and denote by $S_n$ the set of those compositions. (Note that this is a very different - and simpler - concept than the number of partitions of an integer; here the http://planetmath.org/PartialOrderorder matters).

For some small values of $n$, we have

$$\begin{aligned}
C_0 &= 1 \\
C_1 &= 1 \\
C_2 &= 2 \qquad (2), (1,1) \\
C_3 &= 4 \qquad (3), (1,2), (2,1), (1,1,1)
\end{aligned}$$

In fact, it is easy to see that $C_n = 2C_{n-1}$ for $n > 1$: each composition $(a_1, \ldots, a_k)$ of $n-1$ can be associated with two different compositions of $n$

$$\begin{gathered}
(a_1, a_2, \ldots, a_k, 1) \\
(a_1, a_2, \ldots, a_k + 1)
\end{gathered}$$

We thus get a map $\varphi : S_{n-1} \times \{0, 1\} \to S_n$ given by

$$\begin{aligned}
\varphi((a_1, \ldots, a_k), 0) &= (a_1, \ldots, a_k, 1) \\
\varphi((a_1, \ldots, a_k), 1) &= (a_1, \ldots, a_k + 1)
\end{aligned}$$

and this map is clearly injective. But it is also clearly surjective, for given $(a_1, \ldots, a_k) \in S_n$, if $a_k = 1$ then the composition is the image of $((a_1, \ldots, a_{k-1}), 0)$ while if $a_k > 1$, then it is the image of $((a_1, \ldots, a_{k-1}), 1)$. This proves that (for $n > 1$) $C_n = 2C_{n-1}$.

We can also figure out how many compositions there are of $n$ with $k$ parts. Think of a box with $n$ sections in it, with dividers between each pair of sections and a chip in each section; there are thus $n$ chips and $n-1$ dividers. If we leave $k-1$ of the dividers in place, the result is a composition of $n$ with $k$ parts; there are obviously $\binom{n-1}{k-1}$ ways to do this, so the number of compositions of $n$ into $k$ parts is simply $\binom{n-1}{k-1}$. Note that this gives even a simpler proof of the first result, since

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$$