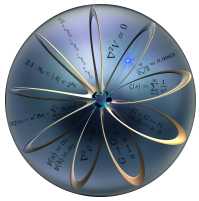

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König's lemma

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Theorem (König's lemma)**.** *Let T be a rooted directed tree. If each vertex has finite degree but there are arbitrarily long rooted paths in T, then T contains an infinite path.*

Proof. For each $n \geq 1$, let P_n be a rooted path in T of length n, and let c_n be the child of the root appearing in P_n. By assumption, the set $\{c_n \mid n \geq 1\}$ is finite. Since the set $\{P_n \mid n \geq 1\}$ is infinite, the pigeonhole principle implies that there is a child c of the root such that $c = c_n$ for infinitely many n.

Now let us look at the subtree T_c of T rooted at c. Each vertex has finite degree, and since there are paths P_n of arbitrarily long length in T passing through c, there are arbitrarily long paths in T_c rooted at c. Hence if T satisfies the hypothesis of the lemma, the root has a child c such that T_c also satisfies the hypothesis of the lemma. Hence we may inductively build up a path in T of infinite length, at each stage selecting a child so that the subtree rooted at that vertex still has arbitrarily long paths. □

References

[1] Kleene, Stephen., *Mathematical Logic*, New York: Wiley, 1967.