derivation of generating function for the reciprocal central binomial coefficients

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According to the article, the ordinary generating function for $\binom{2n}{n}^{-1}$ is

$$\frac{4\left(\sqrt{4-x}+\sqrt{x}\arcsin\left(\frac{\sqrt{x}}{2}\right)\right)}{(4-x)^{3/2}}$$

To see this, let $C_n = \binom{2n}{n}^{-1}$, and $C(x) = \sum_{n\geq 0} C_n x^n$ its ordinary generating function. Then

$$\begin{aligned}
C_{n+1} = \binom{2n+2}{n+1}^{-1} &= \frac{(n+1)!(n+1)!}{(2n+2)!} \\
&= \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \cdot \frac{n!n!}{(2n)!} \\
&= \frac{n+1}{2(2n+1)} \cdot C_n
\end{aligned}$$

Thus

$$(4n+2)C_{n+1} = (n+1)C_n$$

so that

$$\sum_{n\geq 0}(4n+2)C_{n+1}x^n = \sum_{n\geq 0}(n+1)C_n x^n$$

A little algebra gives

$$4\sum_{n\geq 0}(n+1)C_{n+1}x^n - 2\sum_{n\geq 0}C_{n+1}x^n = \sum_{n\geq 0}nC_n x^n + \sum_{n\geq 0}C_n x^n$$

so that

$$4C'(x) - \frac{2}{x}(C(x)-1) = xC'(x) + C(x)$$

and, collecting terms,

$$(4x - x^2)C'(x) = (x+2)C(x) - 2$$

We now have a first-order linear ODE to solve. Put it in the form

$$C'(x) + \frac{-x-2}{x(4-x)}C(x) = \frac{-2}{4x-x^2}$$

and we must now integrate the coefficient of $C(x)$. Expand by partial fractions and integrate to get

$$\int \frac{-x-2}{x(4-x)}dx = \ln\left(\frac{(4-x)^{3/2}}{\sqrt{x}}\right)$$

Thus the solution to the equation is

$$\begin{aligned}
C(x) &= \frac{\sqrt{x}}{(4-x)^{3/2}}\left(k + \int \frac{(4-x)^{3/2}}{\sqrt{x}} \cdot \frac{-2}{x(4-x)}dx\right) \\
&= \frac{k\sqrt{x}}{(4-x)^{3/2}} - \frac{2\sqrt{x}}{(4-x)^{3/2}} \int \frac{\sqrt{4-x}}{x^{3/2}}dx \\
&= \frac{k\sqrt{x}}{(4-x)^{3/2}} - \frac{2\sqrt{x}}{(4-x)^{3/2}} \left(\frac{-2(4-x)}{\sqrt{x(4-x)}} - \arcsin\left(\frac{x}{2}-1\right)\right) \\
&= \frac{4}{4-x} + \frac{\sqrt{x}}{(4-x)^{3/2}}\left(k + 2\arcsin\left(\frac{x}{2}-1\right)\right)
\end{aligned}$$

To determine the constant k, note that we should have $C'(x)\big|_{x=0} = \frac{1}{2}$; looking at $\lim_{x\to 0} C'(x)$ we see that for $k = \pi$ this equation holds. Thus

$$C(x) = \frac{4}{4-x} + \frac{\sqrt{x}}{(4-x)^{3/2}}\left(\pi + 2\arcsin\left(\frac{x}{2}-1\right)\right)$$

We show below that the following is an identity:

$$\sqrt{\frac{z+1}{2}} = \sin\left(\frac{\pi}{4} + \frac{1}{2}\arcsin(z)\right)$$

Assuming that result, substitute $\frac{x}{2} - 1$ for z and simplify to get

$$\frac{\sqrt{x}}{2} = \sin\left(\frac{\pi}{4} + \frac{1}{2}\arcsin\left(\frac{x}{2}-1\right)\right)$$

so that

$$4\arcsin\left(\frac{\sqrt{x}}{2}\right) = \pi + 2\arcsin\left(\frac{x}{2}-1\right)$$

and then

$$\begin{aligned}
C(x) &= \frac{4}{4-x} + \frac{\sqrt{x}}{(4-x)^{3/2}}\left(4\arcsin\left(\frac{\sqrt{x}}{2}\right)\right) \\
&= \frac{4\left(\sqrt{4-x} + \sqrt{x}\arcsin\left(\frac{\sqrt{x}}{2}\right)\right)}{(4-x)^{3/2}}
\end{aligned}$$

as desired.

Finally, to prove the identity, first expand the right-hand using the formula for $\sin(a+b)$, and then apply the half-angle formulas:

$$\begin{aligned}
\sin\left(\frac{\pi}{4}+\frac{1}{2}\arcsin(z)\right) &= \frac{\sqrt{2}}{2}\left(\cos\left(\frac{1}{2}\arcsin(z)\right)+\sin\left(\frac{1}{2}\arcsin(z)\right)\right) \\
&= \frac{\sqrt{2}}{2}\left(\sqrt{\frac{1+\cos(\arcsin(z))}{2}}+\sqrt{\frac{1-\cos(\arcsin(z))}{2}}\right) \\
&= \frac{\sqrt{2}}{2}\left(\sqrt{\frac{1+\sqrt{1-z^2}}{2}}+\sqrt{\frac{1-\sqrt{1-z^2}}{2}}\right) \\
&= \frac{1}{2}\left(\sqrt{1+\sqrt{1-z^2}}+\sqrt{1-\sqrt{1-z^2}}\right)
\end{aligned}$$

Now square this expression to get

$$\frac{1}{4}\left(2+2\sqrt{1-1+z^2}\right) = \frac{|z|+1}{2}$$

Thus the identity holds for $0 \leq z \leq 1$; an almost identical computation using $-z$ in of z shows that it also holds for $-1 \leq z \leq 0$.