# directed graph

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| Defines | in-degree |
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| Defines | directed spanning tree |

A *directed graph* or *digraph* is a pair $G = (V, E)$ where $V$ is a set of *vertices* and $E$ is a subset of $V \times V$ called *edges* or *arcs.*

If $E$ is symmetric (i.e., $(u, v) \in E$ if and only if $(v, u) \in E$), then the digraph is isomorphic to an ordinary (that is, undirected) graph.

Digraphs are generally drawn in a similar manner to graphs with arrows on the edges to indicate a sense of direction. For example, the digraph

$$(\{a, b, c, d\}, \{(a, b), (b, d), (b, c), (c, b), (c, c), (c, d)\})$$

may be drawn as

$$\begin{tikzcd}
a \arrow[r] & b \arrow[d] \arrow[dl, shift left] \\
c \arrow[ur, shift left] \arrow[r] \arrow[loop below] & d
\end{tikzcd}$$

Since the graph is directed, one has the concept of the number of edges originating or terminating at a given vertex $v$. The out-degree, $d_{\text{out}}(v)$ of a vertex $v$ is the number of edges having $v$ as their originating vertex; similarly, the in-degree, $d_{\text{in}}(v)$ is the number of edges having $v$ as their terminating vertex.

If the graph has a finite number of vertices, say $v_1, \ldots, v_n$, then obviously

$$\sum_{i=1}^{n} d_{\text{in}}(v_i) = \sum_{i=1}^{n} d_{\text{out}}(v_i)$$

A *directed path* in a digraph $G$ is a sequence of edges $e_1, \ldots, e_k$ such that the end vertex of $e_i$ is the start vertex of $e_{i+1}$ for $i = 1, 2, \ldots, k-1$. Such a path is called a *directed circuit* if, in addition, the end vertex of $e_k$ is the start vertex of $e_1$.

A digraph is *connected* (or, sometimes, *strongly connected*) if for every pair of vertices $u$ and $v$ there is a directed path from $u$ to $v$. In addition, a digraph $G = (V, E)$ is said to have a *root* $r \in V$ if every vertex $v \in V$ is reachable from $r$, i.e. if there is a directed path from $r$ to $v$.

A digraph is called a *directed tree* if it has a root and if the underlying (undirected) graph is a tree. That is, it must morphologically look like a tree, and the structure imposed by the directional arrows must flow "away" from the root.

If $H$ is a subgraph of a digraph $G$, then $H$ is said to be a *directed spanning tree* of $G$ if $H$ is a directed tree and $H$ contains all vertices of $G$. This is a direct analog of the concept of spanning trees for undirected graphs. Note that if $r$ is the root of $H$, then $r$ is clearly a root of $G$. Also, if $r$ is any root of $G$, it is possible to construct a directed spanning tree of $G$ with root $r$: construct $H$ edge by edge starting from $r$. At each vertex, add any edge from $G$ whose terminus is a vertex not yet in $H$. Since $r$ is a root of $G$, this process is guaranteed to include each vertex in $G$; since we are choosing at each step only vertices not yet visited, we are guaranteed to end up with a tree.