chromatic polynomial

Canonical name	ChromaticPolynomial
Date of creation	2013-03-22 13:58:25
Last modified on	2013-03-22 13:58:25
Owner	bbukh (348)
Last modified by	bbukh (348)
Numerical id	8
Author	bbukh (348)
Entry type	Definition
Classification	msc 05C15
Classification	msc 05B35
Related topic	Matroid
Related topic	ChromaticNumber
Defines	Tutte polynomial

Let G be a graph (in the sense of graph theory) whose set V of vertices is finite and nonempty, and which has no loops or multiple edges. For any natural number x, let $\chi(G,x)$, or just $\chi(x)$, denote the number of x-colorations of G, i.e. the number of mappings $f\colon V \to \{1,2,\ldots,x\}$ such that $f(a) \neq f(b)$ for any pair (a,b) of adjacent vertices. Let us prove that χ (which is called the *chromatic polynomial* of the graph G) is a polynomial function in x with coefficients in $\mathbb{Z}$. Write E for the set of edges in G. If $|E|$=0, then trivially $\chi(x) = x^{|V|}$ (where $|\cdot|$ denotes the number of elements of a finite set). If not, then we choose an edge e and construct two graphs having fewer edges than G: H is obtained from G by contracting the edge e, and K is obtained from G by omitting the edge e. We have

$$\chi(G,x) = \chi(K,x) - \chi(H,x) \tag{1}$$

for all $x \in \mathbb{N}$, because the polynomial $\chi(K,x)$ is the number of colorations of the vertices of G which might or might not be valid for the edge e, while $\chi(H,x)$ is the number which are not valid. By induction on $|E|$, (**??**) shows that $\chi(G,x)$ is a polynomial over $\mathbb{Z}$.

By refining the argument a little, one can show

$$\chi(x) = x^{|V|} - |E|x^{|V|-1} + \ldots \pm sx^k ,$$

for some nonzero integer s, where k is the number of connected components of G, and the coefficients alternate in sign.

With the help of the Möbius-Rota inversion formula (see Moebius Inversion), or directly by induction, one can prove

$$\chi(x) = \sum_{F \subset E} (-1)^{|F|} x^{|V|-r(F)}$$

where the sum is over all subsets F of E, and $r(F)$ denotes the rank of F in G, i.e. the number of elements of any maximal cycle-free subset of F. (Alternatively, the sum may be taken only over subsets F such that F is equal to the span of F; all other summands cancel out in pairs.)

The *chromatic number* of G is the smallest $x > 0$ such that $\chi(G,x) > 0$ or, equivalently, such that $\chi(G,x) \neq 0$.

The *Tutte polynomial* of a graph, or more generally of a matroid (E,r), is this function of two variables:

$$t(x,y) = \sum_{F \subset E} (x-1)^{r(E)-r(F)} (y-1)^{|F|-r(F)} .$$

Compared to the chromatic polynomial, the Tutte contains more information about the matroid. Still, two or more nonisomorphic matroids may have the same Tutte polynomial.