tournament

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Defines	paradoxical tournament
Defines	transitive tournament

A *tournament* is a directed graph obtained by choosing a direction for each edge in an undirected complete graph. For example, here is a tournament on 4 vertices:

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \\ 3 \arrow[u] \arrow[ur] & 4 \arrow[l] \end{tikzcd}$$

Any tournament on a finite number n of vertices contains a Hamiltonian path, i.e., directed path on all n vertices. This is easily shown by induction on n: suppose that the statement holds for n, and consider any tournament T on $n+1$ vertices. Choose a vertex v_0 of T and consider a directed path $v_1, v_2, \ldots, v_n$ in $T \backslash \{v_0\}$. Now let $i \in \{0, \ldots, n\}$ be maximal such that $v_j \to v_0$ for all j with $1 \leq j \leq i$. Then

$$v_1, \ldots, v_i, v_0, v_{i+1}, \ldots, v_n$$

is a directed path as desired.

The name "tournament" originates from such a graph's interpretation as the outcome of some sports competition in which every player encounters every other player exactly once, and in which no draws occur; let us say that an arrow points from the winner to the loser. A player who wins all games would naturally be the tournament's winner. However, as the above example shows, there might not be such a player; a tournament for which there isn't is called a 1-*paradoxical* tournament. More generally, a tournament $T = (V, E)$ is called k-*paradoxical* if for every k-subset V' of V there is a $v_0 \in V \setminus V'$ such that $v_0 \to v$ for all $v \in V'$. By means of the probabilistic method Erdős showed that if $|V|$ is sufficiently large, then almost every tournament on V is k-paradoxical.

A *transitive tournament* is a tournament in which, for all vertices v_0, v_1 and v_2, if there is an edge from v_0 to v_1 and an edge from v_1 to v_2 then there is also an edge from v_0 to v_2.