partition function

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Defines	partition generating function

The *partition function* $p(n)$ is defined to be the number of partitions of the integer n. The sequence of values $p(0), p(1), p(2), \dots$ is Sloane's A000041 and begins $1, 1, 2, 3, 5, 7, 11, 15, 22, 30, \dots$. This function grows very quickly, as we see in the following theorem due to Hardy and `http://planetmath.org/SrinivasaRamanuja`

Theorem 1 *As $n \to \infty$, the ratio of $p(n)$ and*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

approaches 1.

The generating function of $p(n)$ is called F: by definition

$$F(x) = \sum_{n=0}^{\infty} p(n)x^n.$$

F can be written as an infinite product:

$$F(x) = \prod_{i=1}^{\infty}(1 - x^i)^{-1}.$$

To see this, expand each term in the product as a power series:

$$\prod_{i=1}^{\infty}(1 + x^i + x^{2i} + x^{3i} + \cdots).$$

Now expand this as a power series. Given a partition of n with a_i parts of size $i \geq 1$, we get a term x^n in this expansion by choosing x^{a_1} from the first term in the product, x^{2a_2} from the second, x^{3a_3} from the third and so on. Clearly any term x^n in the expansion arises in this way from a partition of n.

One can prove in the same way that the generating function F_m for the number $p_m(n)$ of partitions of n into at most m parts (or equivalently into parts of size at most m) is

$$F_m(x) = \prod_{i=1}^{m}(1 - x^i)^{-1}.$$

References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 2003.