proof of Mantel's theorem

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Let G be a triangle-free graph. We may assume that G has at least three vertices and at least one edge; otherwise, there is nothing to prove. Consider the set P of all functions $c\colon V(G) \to \mathbb{R}_+$ such that $\sum_{v \in V(G)} c(v) = 1$. Define the total weight $W(c)$ of such a function by

$$W(c) = \sum_{uv \in E(G)} c(u) \cdot c(v).$$

By declaring that $c \leq c^*$ if and only if $W(c) \leq W(c^*)$ we make P into a poset.

Consider the function $c_0 \in P$ which takes the constant value $\frac{1}{|V(G)|}$ on each vertex. The total weight of this function is

$$W(c_0) = \sum_{uv \in E(G)} \frac{1}{|V(G)|} \cdot \frac{1}{|V(G)|} = \frac{|E(G)|}{|V(G)|^2},$$

which is positive because G has an edge. So if $c \geq c_0$ in P, then c has support on an induced subgraph of G with at least one edge.

We claim that a maximal element of P above c_0 is supported on a copy of K_2 inside G. To see this, suppose $c \geq c_0$ in P. If c has support on a subgraph larger than K_2, then there are nonadjacent vertices u and v such that $c(u)$ and $c(v)$ are both positive. Without loss of generality, suppose that

$$\sum_{uw \in E(G)} c(w) \geq \sum_{vw \in E(G)} c(w). \tag{*}$$

Now we push the function off v. To do this, define a function $c^*\colon V(G) \to \mathbb{R}_+$ by

$$c^*(w) = \begin{cases} c(u) + c(v) & w = u \\ 0 & w = v \\ c(w) & \text{otherwise.} \end{cases}$$

Observe that $\sum_{w \in V(G)} c^*(w) = 1$, so c^* is still in the poset P. Furthermore, by inequality (*) and the definition of c^*,

$$\begin{aligned}
W(c^*) &= \sum_{uw\in E(G)} c^*(u)\cdot c^*(w) + \sum_{vw\in E(G)} c^*(v)\cdot c^*(w) + \sum_{wz\in E(G)} c^*(w)\cdot c^*(z)\\
&= \sum_{uw\in E(G)} [c(u)+c(v)]\cdot c(w) + 0 + \sum_{wz\in E(G)} c(w)\cdot c(z)\\
&= \sum_{uw\in E(G)} c(u)\cdot c(w) + \sum_{vw} c(v)\cdot c(w) + \sum_{wz\in E(G)} c(w)\cdot c(z)\\
&= W(c).
\end{aligned}$$

Thus $c^* \geq c$ in G and is supported on one less vertex than c is. So let c be a maximal element of P above c_0. We have just seen that c must be supported on adjacent vertices u and v. The weight $W(c)$ is just $c(u)\cdot c(v)$; since $c(u)+c(v)=1$ and c has maximal weight, it must be that $c(u)=c(v)=\frac{1}{2}$. Hence

$$\frac{1}{4} = W(c) \geq W(c_0) = \frac{|E(G)|}{|V(G)|^2},$$

which gives us the desired inequality: $|E(G)| \leq \frac{|V(G)|^2}{4}$.