solid set

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Defines	vector lattice homomorphism
Defines	solid closure

Let V be a vector lattice and $|\cdot|$ be the absolute value defined on V. A subset $A \subseteq V$ is said to be *solid*, or *absolutely convex*, if, $|v| \leq |u|$ implies that $v \in A$, whenever $u \in A$ in the first place.

From this definition, one deduces immediately that 0 belongs to every non-empty solid set. Also, if a is in a solid set, so is a^+, since $|a^+| = a^+ \leq a^+ + a^- = |a|$. Similarly $a^- \in S$, and $|a| \in S$, as $||a|| = |a|$. Furthermore, we have

Proposition 1. *If S is a solid subspace of V, then S is a vector sublattice.*

Proof. Suppose $a, b \in S$. We want to show that $a \wedge b \in S$, from which we see that $a \vee b = a + b - (a \wedge b) \in S$ also since S is a vector subspace. Since both $a \wedge b, a \vee b \in S$, we have that S is a sublattice.

To show that $a \wedge b \in S$, we need to find $c \in S$ with $|a \wedge b| \leq |c|$. Let $c = |a| + |b|$. Since $a, b \in S$, $|a|, |b| \in S$, and so $c \in S$ as well. We also have that $|c| = c$. So to show $a \wedge b \in S$, it is enough to show that $|a \wedge b| \leq c$. To this end, note first that $a \leq |a|$ and $b \leq |b|$, so $a \wedge b \leq |a| \wedge |b| \leq |a| \vee |b|$. Also, since $-a \leq |a|$ and $-b \leq |b|$, $-(a \wedge b) = (-a) \vee (-b) \leq |a| \vee |b|$. As a result, $|a \wedge b| = -(a \wedge b) \vee (a \wedge b) \leq |a| \vee |b|$. But $|a| \vee |b| \leq |a| \vee |b| + |a| \wedge |b| = |a| + |b| = c$, we have that $|a \wedge b| \leq |a| \vee |b| \leq c$. $\square$

Examples Let V be a vector lattice.

- 0 and V itself are solid subspaces.
- If V is finite dimensional, the only solid subspaces are the improper ones.
- An example of a proper solid subspace of a vector lattice is found, when we take V to be the countably infinite direct product of $\mathbb{R}$, and S to be the countably infinite direct sum of $\mathbb{R}$.
- An example of a solid set that is not a subspace is the unit disk in $\mathbb{R}^2$, where the ordering is defined componentwise.
- Given any set A, the smallest solid set containing A is called the *solid closure* of A. For example, if $A = \{a\}$, then its solid closure is $\{v \in V \mid |v| \leq |a|\}$. In $\mathbb{R}^2$, the solid closure of any point p is the disk centered at O whose radius is $|p|$.
- The solid closure of V^+, the positive cone, is V.

Proposition 2. *If V is a vector lattice and S is a solid subspace of V, then V/S is a vector lattice.*

Proof. Since S is a subspace V/S has the structure of a vector space, whose vector space operations are inherited from the operations on V. Since S is solid, it is a sublattice, so that V/S has the structure of a lattice, whose lattice operations are inherited from those on V. It remains to show that the partial ordering is "compatible" with the vector operatons. We break this down into two steps:

- for any $u+S, v+S, w+S \in V/S$, if $(u+S) \leq (v+S)$, then $(u+S)+(w+S) \leq (v+S)+(w+S)$. This is a disguised form of the following: if $u-v \leq a \in S$, then $(u+w)-(v+w) \leq b \in S$ for some b. This is obvious: just pick $b=a$.

- if $0+S \leq u+S \in V/S$, then for any $0 < \lambda \in k$ (k an ordered field), $0+S \leq \lambda(u+S)$. This is the same as saying: if $c \leq u$ for some $b \in S$, then $d \leq \lambda u$ for some $d \in S$. This is also obvious: pick $d = \lambda c$.

The proof is now complete. □