definition of vector space needs no commutativity

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In the definition of http://planetmath.org/VectorSpacevector space one usually lists the needed properties of the vectoral addition and the multiplication of vectors by scalars as eight axioms, one of them the commutative law

$$u + v = v + u.$$

The latter is however not necessary, because it may be proved to be a consequence of the other seven axioms. The proof can be based on the fact that in defining the http://planetmath.org/Groupgroup, it suffices to postulate only the existence of a right identity element and the right inverses of the elements (see the article "http://planetmath.org/RedundancyOfTwoSidednessInDefinitionOfGro of two-sidedness in definition of group").

Now, suppose the validity of http://planetmath.org/VectorSpacethe seven other axioms, but not necessarily the above commutative law of addition. We will show that the commutative law is in force.

We need the identity $(-1)v = -v$ which is easily justified (we have $\vec{0} = 0v = (1 + (-1))v = \ldots$). Then we can calculate as follows:

$$\begin{aligned} v + u &= (v + u) + \vec{0} = (v + u) + [-(u + v) + (u + v)] \\ &= \ [(v + u) + (-(u + v))] + (u + v) = [(v + u) + (-1)(u + v)] + (u + v) \\ &= [(v + u) + ((-1)u + (-1)v)] + (u + v) = [((v + u) + (-u)) + (-v)] + (u + v) \\ &= [(v + (u + (-u))) + (-v)] + (u + v) = [(v + \vec{0}) + (-v)] + (u + v) \\ &= [v + (-v)] + (u + v) = \vec{0} + (u + v) \\ &= u + v \end{aligned}$$

Q.E.D.

This proof by Y. Chemiavsky and A. Mouftakhov is found in the 2012 March issue of *The American Mathematical Monthly.*