eventual property

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Let X be a set and P a property on the elements of X. Let $(x_i)_{i\in D}$ be a net (D a directed set) in X (that is, $x_i \in X$). As each $x_i \in X$, x_i either has or does not have property P. We say that the net (x_i) has property P *above* $j \in D$ if x_i has property P for all $i \geq j$. Furthermore, we say that (x_i) *eventually* has property P if it has property P above some $j \in D$.

Examples.

1. Let A and B be non-empty sets. For $x \in A$, let $P(x)$ be the property that $x \in B$. So P is nothing more than the property of elements being in the intersection of A and B. A net $(x_i)_{i\in D}$ eventually has P means that for some $j \in D$, the set $\{x_i \mid i \in A,\, i \geq j\} \subseteq B$. If $D = \mathbb{Z}$, then we have that A and B eventually coincide.

2. Now, suppose A is a topological space, and B is an open neighborhood of a point $x \in A$. For $y \in A$, let $P_B(y)$ be the property that $y \in B$. Then a net (x_i) has P_B eventually for every neighborhood B of x is a characterization of convergence (to the point x, and x is the accumulation point of (x_i)).

3. If A is a poset and $B = \{x\} \subseteq A$. For $y \in A$, let $P(y)$ again be the property that $y = x$. Let (x_i) be a net that eventually has property P. In other words, (x_i) is *eventually constant*. In particular, if for every chain D, the net $(x_i)_{i\in D}$ is eventually constant in A, then we have a characterization of the ascending chain condition in A.

4. **directed net**. Let R be a preorder and let $(x_i)_{i\in D}$ be a net in R. Let $x(D)$ be the image of the net: $x(D) = \{x_i \in R \mid i \in D\}$. Given a fixed $k \in D$ and some $y \in x(D)$, let $P_k(y)$ be the property (on $x(D)$) that $x_k \leq y$. Let
$$S = \{k \in D \mid (x_i) \text{ eventually has } P_k\}.$$
If $S = D$, then we say that the net (x_i) is *directed*, or that (x_i) is a *directed net*. In other words, a directed net is a net $(x_i)_{i\in D}$ such that for *every* $i \in D$, there is a $k(i) \in D$, such that $x_i \leq x_j$ for all $j \geq k(i)$.

   If $(x_i)_{i\in D}$ is a directed net, then $x(D)$ is a directed set: Pick $x_i, x_j \in x(D)$, then there are $k(i), k(j) \in D$ such that $x_i \leq x_m$ for all $m \geq k(i)$ and $x_j \leq x_n$ for all $n \geq k(j)$. Since D is directed, there is a $t \in D$ such that $t \geq k(i)$ and $t \geq k(j)$. So $x_t \geq x_{k(i)} \geq x_i$ and $x_t \geq x_{k(j)} \geq x_j$.

However, if $(x_i)_{i \in D}$ is a net such that $x(D)$ is directed, (x_i) need not be a directed net. For example, let $D = \{p, q, r\}$ such that $p \leq q \leq r$, and $R = \{a, b\}$ such that $a \leq b$. Define a net $x : D \to R$ by $x(p) = x(r) = b$ and $x(q) = a$. Then x is not a directed net.

Remark. The eventual property is a property on the class of nets (on a given set X and a given property P). We can write $\text{Eventually}(P, X)$ to denote its dependence on X and P.