ordered group

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Defines	ordered group equipped with zero

Definition 1. We say that the subsemigroup S of the group G (with the operation denoted multiplicatively) defines an G, if

- $a^{-1}Sa \subseteq S \quad \forall a \in G$,
- $G = S \cup \{1\} \cup S^{-1}$ where $S^{-1} = \{s^{-1} : s \in S\}$ and the members of the union are pairwise disjoint.

The order “$<$” of the group G is explicitly given by setting in G:

$$a < b \;\Leftrightarrow\; ab^{-1} \in S$$

Then we speak of the *ordered group* $(G, <)$, or simply G.

Theorem 1. The order “$<$” defined by the subsemigroup S of the group G has the following properties.

1. For all $a, b \in G$, exactly one of the conditions $a < b$, $a = b$, $b < a$ holds.
2. $a < b \wedge b < c \;\Rightarrow\; a < c$
3. $a < b \;\Rightarrow\; ac < bc \wedge ca < cb$
4. $a < b \wedge c < d \;\Rightarrow\; ac < bd$
5. $a < b \;\Leftrightarrow\; b^{-1} < a^{-1}$
6. $a < 1 \;\Leftrightarrow\; a \in S$

Definition 2. The set G is an *ordered group equipped with zero* 0, if the set G^* of its elements distinct from its element 0 forms an ordered group $(G^*, <)$ and if

- $0a = a0 = 0 \quad \forall a \in G$,
- $0 < a \quad \forall a \in G^*$.

Cf. 7 in examples of semigroups.

References

[1] Emil Artin: *Theory of Algebraic Numbers.* Lecture notes. Mathematisches Institut, Göttingen (1959).

[2] Paul Jaffard: *Les systèmes d'idéaux.* Dunod, Paris (1960).