generalized Boolean algebra

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A lattice L is called a *generalized Boolean algebra* if

- L is distributive,
- L is relatively complemented, and
- L has 0 as the bottom.

Clearly, a Boolean algebra is a generalized Boolean algebra. Conversely, a generalized Boolean algebra L with a top 1 is a Boolean algebra, since $L = [0, 1]$ is a bounded distributive complemented lattice, so each element $a \in L$ has a unique complement a' by distributivity. So $'$ is a unary operator on L which makes L into a de Morgan algebra. A complemented de Morgan algebra is, as a result, a Boolean algebra.

As an example of a generalized Boolean algebra that is not Boolean, let A be an infinite set and let B be the set of all finite subsets of A. Then B is generalized Boolean: order B by inclusion, then B is a distributive as the operation is inherited from $P(A)$, the powerset of A. It is also relatively complemented: if $C \in [X, Y]$ where $C, X, Y \in B$, then $(Y - C) \cup X$ is the relative complement of C in $[X, Y]$. Finally, $\varnothing$ is, as usual, the bottom element in B. B is not a Boolean algebra, because the union of all the singletons (all in B) is A, which is infinite, thus not in B.

One property of a generalized Boolean algebra L is the following: if y and z are complements of $x \in [a, b]$, then $y = z$; in other words, relative complements are uniquely determined. This is true because in any distributive lattice, complents are uniquely determined. As L is distributive, so is each lattice interval $[a, b]$ in L.

In fact, because of the existence of 0, we can actually construct the relative complement. Let $b - x$ denote the unique complement of x in $[0, b]$. Then $(b-x) \vee a$ is the unique complement of $x \in [a, b]$: $x \wedge ((b-x) \vee a) = (x \wedge (b-x)) vee (x \wedge a) = 0 \vee a = a$ and $x \vee ((b-x) \vee a) = (x \vee (b-x)) \vee a = b \vee a = b$.

Conversely, if L is a distributive lattice with 0 such that any lattice interval $[0, a]$ is complemented, then L is a generalized Boolean algebra. Again, $(b - x) \vee a$ provides the necessary complement of x in $[a, b]$.