MacNeille completion

Canonical name	MacNeilleCompletion
Date of creation	2013-03-22 16:05:27
Last modified on	2013-03-22 16:05:27
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	8
Author	CWoo (3771)
Entry type	Definition
Classification	msc 06B23
Synonym	Dedekind-MacNeille completion
Synonym	normal completion
Related topic	DedekindCuts

In a first course on real analysis, one is generally introduced to the concept of a Dedekind cut. It is a way of constructing the set of real numbers from the rationals. This is a process commonly known as the completion of the rationals. Three key features of this completion are:

- the rationals can be embedded in its completion (the reals)
- every subset with an upper bound has a least upper bound
- every subset with a lower bound has a greatest lower bound

If we extend the reals by adjoining $+\infty$ and $-\infty$ and define the appropriate ordering relations on this new extended set (the extended real numbers), then it is a set where every subset has a least upper bound and a greatest lower bound.

When we deal with the rationals and the reals (and extended reals), we are working with linearly ordered sets. So the next question is: can the procedure of a completion be generalized to an arbitrary poset? In other words, if P is a poset ordered by $\leq$, does there exist another poset Q ordered by $\leq_Q$ such that

1. P can be embedded in Q as a poset (so that $\leq$ is compatible with $\leq_Q$), and
2. every subset of Q has both a least upper bound and a greatest lower bound

In 1937, MacNeille answered this question in the affirmative by the following construction:

> Given a poset P with order $\leq$, define for every subset A of P, two subsets of P as follows:
>
> $A^u = \{p \in P \mid a \leq p \text{ for all } a \in A\}$ and $A^\ell = \{q \in P \mid q \leq a \text{ for all } a \in A\}$.
>
> Then $M(P) := \{A \in 2^P \mid (A^u)^\ell = A\}$ ordered by the usual set inclusion is a poset satisfying conditions (1) and (2) above.

This is known as the *MacNeille completion* $M(P)$ of a poset P. In $M(P)$, since lub and glb exist for any subset, $M(P)$ is a complete lattice. So this process can be readily applied to any lattice, if we define a completion of a lattice to follow the two conditions above.

References

[1] H. M. MacNeille, *Partially Ordered Sets*. Trans. Amer. Math. Soc. 42 (1937), pp 416-460

[2] B. A. Davey, H. A. Priestley, *Introduction to Lattices and Order*, 2nd edition, Cambridge (2003)