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rank-selected poset

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Defines	alpha invariant
Defines	beta invariant
Defines	rank-selected Möbius invariant
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Let P be a graded poset of rank $n+1$ with rank function ρ. For any $S \subset \{0, 1, \ldots, n+1\}$ let P_S denote the subset

$$P_S = \{x \in P \colon \rho(x) \in S\} = \rho^{-1}(S).$$

Each such subset inherits a poset structure from P as an induced poset. So we call P_S the *rank-selected poset* of P induced by S, or more briefly the S-rank-selected subposet of P.

The rank-selected posets of a poset P can be used to define two special arithmetic invariants of P. First for each S, the *alpha invariant* $\alpha_S(P)$ is the number of saturated chains in P_S. Then define $\beta_S(P)$ by

$$\beta_S(P) = \sum_{T \subset S} (-1)^{|S|-|T|} \alpha_T(P).$$

The invariant β is called the *rank-selected Möbius invariant* of P.

For example, let L be the face poset of a convex polytope P of dimension n, including the special elements $\widehat{0}$ (representing the empty face) and $\widehat{1}$ (representing the interior of the polytope). For any $i \in \{0, \ldots, n-1\}$, the alpha invariant $\alpha_{\{i+1\}}(L)$ counts the number of faces of P of dimension i. For arbitrary $S \subset \{1, \ldots, n\}$, the numbers $\alpha_S(L)$ are entries in the flag f-vector of P and thus count flags of faces in P, while the $\beta_S(L)$ are entries in the flag h-vector of P.

While the alpha invariant is by construction always nonnegative, the Möbius invariant is not guaranteed to be nonnegative. Posets for which the Möbius invariant is always nonnegative (and therefore counts something) are of special interest to combinatorialists. In particular, the Möbius invariant is nonnegative for face posets of convex polytopes.

References

[1] Stanley, R., *Enumerative Combinatorics*, vol. 1, 2nd ed., Cambridge University Press, Cambridge, 1996.