ideal completion of a poset

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Let P be a poset. Consider the set $\mathrm{Id}(P)$ of all order ideals of P.

Theorem 1. $\mathrm{Id}(P)$ *is an algebraic dcpo, such that* P *can be embedded in.*

Proof. We shall list, and when necessary, prove the following series of facts which ultimately prove the main assertion. For convenience, write $P' = \mathrm{Id}(P)$.

1. P' is a poset with $\le$ defined by set theoretic inclusion.

2. For any $x \in P$, $\downarrow\!x \in P'$.

3. P can be embedded in P'. The function $f : P \to P'$ defined by $f(x) = \downarrow\!x$ is order preserving and one-to-one. If $x \le y$, and $a \le x$, then $a \le y$, hence $\downarrow\!x \subseteq \downarrow\!y$. If $\downarrow\!x = \downarrow\!y$, we have that $x \le y$ and $y \le x$, so $x = y$, since $\le$ is antisymmetric.

4. P' is a dcpo. Suppose D is a directed set in P'. Let $E = \bigcup D$. For any $x, y \in E$, $x \in I$ and $y \in J$ for some ideals $I, J \in D$. As D is directed, there is $K \in D$ such that $I \subseteq K$ and $J \subseteq K$. So $x, y \in K$ and hence there is $z \in K \subseteq E$ such that $x \le z$ and $y \le z$. This shows that E is directed. Next, suppose $x \in E$ and $y \le x$. Then $x \in I$ for some $I \in D$, so $y \in I \subseteq E$ as well. This shows that E is a down set. So E is an ideal of P: $\bigvee D = E \in P'$.

5. For every $x \in P$, $\downarrow\!x$ is a compact element of P'. If $\downarrow\!x \le \bigvee D$, where D is directed in P', then $\downarrow\!x \subseteq \bigcup D$, or $x \in \bigcup D$, which implies $x \in I$ for some ideal $I \in D$. Therefore $\downarrow\!x \subseteq I$, and $\downarrow\!x$ is way below itself: $\downarrow\!x$ is compact.

6. P' is an algebraic dcpo. Let $I \in P'$. Let $C = \{\downarrow\!x \mid x \in I\}$. For any $x, y \in I$, there is $z \in I$ such that $x \le z$ and $y \le z$. This shows that $\downarrow\!x \le \downarrow\!z$ and $\downarrow\!y \le \downarrow\!z$ in C, so that C is directed. It is easy to see that $I = \bigvee C$. Since I is a join of a directed set consisting of compact elements, P' is algebraic.

This completes the proof. $\square$

Definition. $\mathrm{Id}(P)$ is called the *ideal completion* of P.
Remarks.

- In general, the ideal completion of a poset is not a complete lattice. It is complete in the sense of being directed complete. This is different from another type of completion, called the MacNeille completion of P, which is a complete lattice.

- If P is an upper semilattice, then so is $\mathrm{Id}(P)$. In fact, the join of any non-empty family of ideals exists. Furthermore, if P has a bottom element 0, then $\mathrm{Id}(P)$ is a complete lattice.

  Proof. Let S be a non-empty family of ideals in P. Let A be the set of P consisting of all finite joins of elements of those ideals in S, and $B =\downarrow A$. Clearly, B is a lower set. For every $a, b \in B$, we have $c, d \in A$ such that $a \leq c$ and $b \leq d$. Since c and d are both finite joins of elements of those ideals in S, so is $c \vee d$. Since $a \leq c \vee d$ and $b \leq c \vee d$, B is directed. If I is any ideal larger than any of the ideals in S, clearly $A \subseteq I$, since I is directed. So $B =\downarrow A \subseteq \downarrow I = I$. Therefore, $B = \bigvee S$.

  If $0 \in P$, then $\langle 0 \rangle$, the bottom of $\mathrm{Id}(P)$, is the join of the empty family of ideals in P. By `http://planetmath.org/CriteriaForAPosetToBeACompleteLattice`this entry, $\mathrm{Id}(P)$ is a complete lattice. □

- If P is a lower semilattice, then so is $\mathrm{Id}(P)$.

  Proof. Let I, J be two ideals in P and $K = I \cap J$. By definition, I and J are non-empty, so let $a \in I$ and $b \in J$. As P is a lower semilattice, $c := a \wedge b$ exists and $c \leq a$ and $c \leq b$. So $c \in I \cap J$, and that $K = I \cap J$ is non-empty. If $x \leq y \in K$, then $x \leq y \in I$ or $x \in I$. Similarly $x \in J$. Therefore $x \in I \cap J = K$ and K is a lower set. If $r, s \in K$, then there is $u \in I$ and $v \in J$ such that $r, s \leq u, v$. So $r, s \leq u \wedge v$ and K is directed. This means that $I \cap J \in \mathrm{Id}(P)$. □

References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge (2003).