planetmath.org

Math for the people, by the people.

complete semilattice

Canonical name	CompleteSemilattice
Date of creation	2013-03-22 17:44:49
Last modified on	2013-03-22 17:44:49
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	13
Author	CWoo (3771)
Entry type	Definition
Classification	msc 06A12
Classification	msc 06B23
Synonym	countably complete upper-semilattice
Synonym	countably complete lower-semilattice
Synonym	complete upper-semilattice homomorphism
Synonym	complete lower-semilattice homomorphism
Related topic	CompleteLattice
Related topic	Semilattice
Related topic	ArbitraryJoin
Defines	countably complete join-semilattice
Defines	countably complete meet-semilattice
Defines	complete join-semilattice homomorphism
Defines	complete meet-semilattice homomorphism

A complete join-semilattice is a join-semilattice L such that for any subset $A \subseteq L$, $\bigvee A$, the arbitrary join operation on A, exists. Dually, a complete meet-semilattice is a meet-semilattice such that $\bigwedge A$ exists for any $A \subseteq L$. Because there are no restrictions placed on the subset A, it turns out that a complete join-semilattice is a complete meet-semilattice, and therefore a complete lattice. In other words, by dropping the arbitrary join (meet) operation from a complete lattice, we end up with nothing new. For a proof of this, see `http://planetmath.org/CriteriaForAPosetToBeACompleteLattice`here. The crux of the matter lies in the fact that $\bigvee$ ($\bigwedge$) applies to *any* set, including L itself, and the empty set $\varnothing$, so that L always contains has a top and a bottom.

Variations. To obtain new objects, one looks for variations in the definition of "complete". For example, if we require that any $A \subseteq L$ to be countable, we get what is a called a *countably complete join-semilattice* (or dually, a *countably complete meet-semilattice*). More generally, if κ is any cardinal, then a κ-complete join-semilattice is a semilattice L such that for any set $A \subseteq L$ such that $|A| \leq \kappa$, $\bigvee A$ exists. If κ is finite, then L is just a join-semilattice. When $\kappa = \infty$, the only requirement on $A \subseteq L$ is that it be non-empty. In [1], a complete semilattice is defined to be a poset L such that for any non-empty $A \subseteq L$, $\bigwedge A$ exists, and any directed set $D \subseteq L$, $\bigvee D$ exists.

Example. Let A and B be two isomorphic complete chains (a chain that is a complete lattice) whose cardinality is κ. Combine the two chains to form a lattice L by joining the top of A with the top of B, and the bottom of A with the bottom of B, so that

- if $a \leq b$ in A, then $a \leq b$ in L
- if $c \leq d$ in B, then $c \leq d$ in L
- if $a \in A$, $c \in B$, then $a \leq c$ iff a is the bottom of A and c is the top of B
- if $a \in A$, $c \in B$, then $c \leq a$ iff a is the top of A and c is the bottom of B

Now, L can be easily seen to be a κ-complete lattice. Next, remove the bottom element of L to obtain L'. Since, the meet operation no longer works on all pairs of elements of L' while $\vee$ still works, L' is a join-semilattice that is not a lattice. In fact, $\bigvee$ works on all subsets of L'. Since $|L'| = \kappa$, we see that L' is a κ-complete join-semilattice.

Remark. Although a complete semilattice is the same as a complete lattice, a homomorphism f between, say, two complete join-semilattices L_1 and L_2, may fail to be a homomorphism between L_1 and L_2 as complete lattices. Formally, a *complete join-semilattice homomorphism* between two complete join-semilattices L_1 and L_2 is a function $f : L_1 \to L_2$ such that for any subset $A \subseteq L_1$, we have

$$f(\bigvee A) = \bigvee f(A)$$

where $f(A) = \{f(a) \mid a \in A\}$. Note that it is not required that $f(\bigwedge A) = \bigwedge f(A)$, so that f needs not be a complete lattice homomorphism.

To give a concrete example where a complete join-semilattice homomorphism f fails to be complete lattice homomorphism, take L from the example above, and define $f : L \to L$ by $f(a) = 1$ if $a \neq 0$ and $f(0) = 0$. Then for any $A \subseteq L$, it is evident that $f(\bigvee A) = \bigvee f(A)$. However, if we take two incomparable elements $a, b \in L$, then $f(a \wedge b) = f(0) = 0$, while $f(a) \wedge f(b) = 1 \wedge 1 = 1$.

References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge (2003).

[2] P. T. Johnstone, *Stone Spaces*, Cambridge University Press (1982).