residuated lattice

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Entry type	Definition
Classification	msc 06B99
Defines	left residual
Defines	right residual
Defines	commutative residuated lattice

A *residuated lattice* is a lattice L with an additional binary operation $\cdot$ called *multiplication*, with a multiplicative identity $e \in L$, such that

- $(L, \cdot, e)$ is a monoid, and
- for each $x \in L$, the left and right multiplications by x are residuated.

The second condition says: for every $x, z \in L$, each of the sets

$$L(x,z) := \{y \in L \mid x \cdot y \leq z\}$$

and

$$R(x,z) := \{y \in L \mid y \cdot x \leq z\}$$

is a down set, and has a maximum.

Clearly, $\max L(x,z)$ and $\max R(x,z)$ are both unique. $\max L(x,z)$ is called the *right residual* of z by x, and is commonly denoted by $x \backslash z$, while $\max R(x,z)$ is called the *left residual* of z by x, denoted by x/z.

Residuated lattices are mostly found in algebraic structures associated with a variety of logical systems. For examples, Boolean algebras associated with classical propositional logic, and more generally Heyting algebras associated with the intuitionistic propositional logic are both residuated, with multiplication the same as the lattice meet operation. MV-algebras and BL-algebras associated with many-valued logics are further examples of residuated lattices.

Remark. A residuated lattice is said to be *commutative* if $\cdot$ is commutative. All of the examples cited above are commutative.

References

[1] T.S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer, New York (2005)

[2] M. Bergmann, *An Introduction to Many-Valued and Fuzzy Logic: Semantic, Algebras, and Derivation Systems*, Cambridge University Press (2008)

[3] R. P. Dilworth, M. Ward *Residuated Lattices*, Transaction of the American Mathematical Society 45, pp.335-354 (1939)