supremum over closure

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Theorem 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and $A \subseteq \mathbb{R}$. Then $\sup\limits_{x\in A} f(x) = \sup\limits_{x\in \overline{A}} f(x)$, where $\overline{A}$ denotes the closure of A.*

Proof. The theorem is clearly true for $A = \emptyset$. Thus, it will be assumed that $A \neq \emptyset$.

Since $A \subseteq \overline{A}$, we have $\sup\limits_{x\in A} f(x) \leq \sup\limits_{x\in \overline{A}} f(x)$.

Suppose first that $\sup\limits_{x\in \overline{A}} f(x) = \infty$. Let $r \in \mathbb{R}$. Then there exists $x_0 \in \overline{A}$ with $f(x_0) \geq r+1$. Since f is continuous, there exists $\delta > 0$ such that, for any $x \in \mathbb{R}$ with $-\delta < x - x_0 < \delta$, we have $-1 < f(x) - f(x_0) < 1$. Since $x_0 \in \overline{A}$, there exists $x_1 \in A$ with $-\delta < x_1 - x_0 < \delta$. (Recall that $x \in \overline{A}$ if and only if every neighborhood of x intersects A.) Thus, $f(x_1) - f(x_0) > -1$. Therefore, $f(x_1) > f(x_0) - 1 \geq r + 1 - 1 = r$. Hence, $\sup\limits_{x\in A} f(x) = \infty$.

Now suppose that $\sup\limits_{x\in \overline{A}} f(x) = R$ for some $R \in \mathbb{R}$. Let $\varepsilon > 0$. Then there exists $x_2 \in \overline{A}$ with $f(x_2) \geq R - \frac{\varepsilon}{2}$. Since f is continuous, there exists $\delta' > 0$ such that, for any $x \in \mathbb{R}$ with $-\delta' < x - x_0 < \delta'$, we have $\frac{-\varepsilon}{2} < f(x) - f(x_2) < \frac{\varepsilon}{2}$. Since $x_2 \in \overline{A}$, there exists $x_3 \in A$ with $-\delta' < x_3 - x_2 < \delta'$. Thus, $f(x_3) - f(x_2) > \frac{-\varepsilon}{2}$. Therefore, $f(x_3) > f(x_2) - \frac{\varepsilon}{2} \geq R - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = R - \varepsilon$. Hence, $\sup\limits_{x\in A} f(x) \geq R$.

In either case, it follows that $\sup\limits_{x\in A} f(x) = \sup\limits_{x\in \overline{A}} f(x)$. $\square$

Note that this theorem also holds for continuous functions $f\colon X \to \mathbb{R}$, where X is an arbitrary topological space. To prove this fact, one would need to slightly adjust the proof supplied here.