# Sikorski's extension theorem

| | |
|---|---|
| Canonical name | SikorskisExtensionTheorem |
| Date of creation | 2013-03-22 18:01:31 |
| Last modified on | 2013-03-22 18:01:31 |
| Owner | CWoo (3771) |
| Last modified by | CWoo (3771) |
| Numerical id | 21 |
| Author | CWoo (3771) |
| Entry type | Theorem |
| Classification | msc 06E10 |
| Synonym | Sikorski extension theorem |

**Theorem 1** (Sikorski's ). *Let $A$ be a Boolean subalgebra of a Boolean algebra $B$, and $f : A \to C$ a Boolean algebra homomorphism from $A$ to a complete Boolean algebra $C$. Then $f$ can be extended to a Boolean algebra homomorphism $g : B \to C$.*

**Remark**. In the category of Boolean algebras and Boolean algebra homomorphisms, this theorem says that every complete Boolean algebra is an injective object.

*Proof.* We prove this using Zorn's lemma. Let $M$ be the set of all pairs $(h, D)$ such that $D$ is a subalgebra of $B$ containing $A$, and $h : D \to C$ is an algebra homomorphism extending $f$. Note that $M$ is not empty because $(f, A) \in M$. Also, if we define $(h_1, D_1) \le (h_2, D_2)$ by requiring that $D_1 \subseteq D_2$ and that $h_2$ extending $h_1$, then $(M, \le)$ becomes a poset. Notice that for every chain $\mathcal{C}$ in $M$,

$$(\bigcup\{h \mid (h, D) \in \mathcal{C}\}, \bigcup\{D \mid (h, D) \in \mathcal{C}\})$$

is an upper bound of $\mathcal{C}$ (in fact, the least upper bound). So $M$ has a maximal element, say $(g, E)$, by Zorn's lemma. We want to show that $E = B$.

If $E \neq B$, pick $a \in B - E$. Let $r$ be the join of all elements of the form $g(x)$ where $x \in E$ and $x \le a$, and $t$ the meet of all elements of the form $g(y)$ where $y \in E$ and $a \le y$. $r$ and $t$ exist because $C$ is complete. Since $g$ preserves order, it is evident that $r \le t$. Pick an element $s \in C$ such that $r \le s \le t$.

Let $F = \langle E, a\rangle$. Every element in $F$ has the form $(e_1 \wedge a) \vee (e_2 \wedge a')$, with $e_1, e_2 \in E$. Define $h : F \to C$ by setting $h(b) = (g(e_1) \wedge s) \vee (g(e_2) \wedge s')$, where $b = (e_1 \wedge a) \vee (e_2 \wedge a')$. We now want to show that $h$ is a Boolean algebra homomorphism extending $g$. There are three steps to showing this:

1. $h$ is a function. Suppose $(e_1 \wedge a) \vee (e_2 \wedge a') = (e_3 \wedge a) \vee (e_4 \wedge a')$. Then, by the last remark of `http://planetmath.org/BooleanSubalgebra`this entry, $e_2 \Delta e_4 \le a \le e_1 \leftrightarrow e_3$, so that $g(e_2)\Delta g(e_4) = g(e_2 \Delta e_4) \le s \le g(e_1 \leftrightarrow e_3) = g(e_1) \leftrightarrow g(e_3)$, which in turn implies that $(g(e_1) \wedge s) \vee (g(e_2) \wedge s') = (g(e_3) \wedge s) \vee (g(e_4) \wedge s')$. Hence $h$ is well-defined.

2. $h$ is a Boolean homomorphism. All we need to show is that $h$ respects $\vee$ and $'$. Let $x = (e_1 \wedge a) \vee (e_2 \wedge a')$ and $y = (e_3 \wedge a) \vee (e_4 \wedge a')$. Then

$x \vee y = (e_5 \wedge a) \vee (e_6 \wedge a')$, where $e_5 = e_1 \vee e_3$ and $e_6 = e_2 \vee e_4$. So

$$\begin{aligned}
h(x \vee y) &= (g(e_5) \wedge s) \vee (g(e_6) \wedge s') \\
&= ((g(e_1) \vee g(e_3)) \wedge s) \vee ((g(e_2) \vee g(e_4)) \wedge s') \\
&= (g(e_1) \wedge s) \vee (g(e_2) \wedge s') \vee (g(e_3) \wedge s) \vee (g(e_4) \wedge s') \\
&= h(x) \vee h(y),
\end{aligned}$$

so $h$ respects $\vee$. In addition, $h$ respects $'$, as $x' = (e_2' \wedge a) \vee (e_1 \wedge a')$, so that

$$\begin{aligned}
h(x') &= h((e_2' \wedge a) \vee (e_1 \wedge a')) = (g(e_2') \wedge s) \vee (g(e_1) \wedge s') \\
&= (g(e_2)' \wedge s) \vee (g(e_1) \wedge s') = ((g(e_1) \wedge s) \vee (g(e_2) \wedge s'))' \\
&= h(x)'.
\end{aligned}$$

3. $h$ extends $g$. If $x \in E$, write $x = (x \wedge a) \vee (x \wedge a')$. Then

$$h(x) = (g(x) \wedge s) \vee (g(x) \wedge s') = g(x).$$

This implies that $(g, E) < (h, F)$, and with this, we have a contradiction that $(g, E)$ is maximal. This completes the proof. $\square$

One of the consequences of this theorem is the following variant of the Boolean prime ideal theorem:

**Corollary 1.** *Every Boolean ideal of a Boolean algebra is contained in a maximal ideal.*

*Proof.* Let $I$ be an ideal of a Boolean algebra $A$. Let $B = \langle I \rangle$, the Boolean subalgebra generated by $I$. The function $f : B \to \{0, 1\}$ given by $f(a) = 0$ iff $a \in I$ is a Boolean homomorphism. First, notice that $f(a) = 0$ iff $a \in I$ iff $a' \notin I$ iff $f(a') = 1$. Next, if at least one of $a, b$ is in $I$, $a \wedge b \in I$, so that $f(a \wedge b) = 0 = f(a) \wedge f(b)$. If neither are in $I$, then $a', b' \in I$, so $(a \wedge b)' = a' \vee b' \in I$, or $a \wedge b \notin I$. This means that $f(a \wedge b) = 1 = f(a) \wedge f(b)$.

Now, by Sikorski's extension theorem, $f$ can be extended to a homomorphism $g : A \to \{0, 1\}$. The kernel of $g$ clearly contains $I$, and is in addition maximal (either $a$ or $a'$ is in the kernel of $g$). $\square$

**Remarks**.

- As the proof of the theorem shows, ZF+AC (the axiom of choice) implies Sikorski's extension theorem (SET). It is still an open question whether the ZF+SET implies AC.
- Next, comparing with the Boolean prime ideal theorem (BPI), the proof of the corollary above shows that ZF+SET implies BPI. However, it was proven by John Bell in 1983 that SET is independent from ZF+BPI: there is a model satisfying all axioms of ZF, as well as BPI (considered as an axiom, not as a consequence of AC), such that SET fails.

# References


[1] R. Sikorski, *Boolean Algebras*, 2nd Edition, Springer-Verlag, New York (1964).

[2] J. L. Bell, *http://plato.stanford.edu/entries/axiom-choice/The Axiom of Choice*, Stanford Encyclopedia of Philosophy (2008).