# Scott topology

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Let $P$ be a dcpo. A subset $U$ of $P$ is said to be *Scott open* if it satisfies the following two conditions:

1. $U$ an upper set: $\uparrow U = U$, and
2. if $D$ is a directed set with $\bigvee D \in U$, then there is a $y \in D$ such that $(\uparrow y) \cap D \subseteq U$.

Condition 2 is equivalent to saying that $U$ has non-empty intersection with $D$ whenever $D$ is directed and its supremum is in $U$.

For example, for any $x \in P$, the set $U(x) := P - (\downarrow x)$ is Scott open: if $y \in \uparrow U(x)$, then there is $z \in U(x)$ with $z \leq y$. Since $z \notin \downarrow x$, $y \notin \downarrow x$. So $y \in U(x)$, or that $U(x)$ is upper. If $D$ is directed and $e \leq x$ for all $e \in D$, then $d := \bigvee D \leq x$ as well. Therefore, $d \in U(x)$ implies $e \in U(x)$ for some $e \in D$. Hence $U(x)$ is Scott open.

The collection $\sigma(P)$ of all Scott open sets of $P$ is a topology, called the *Scott topology* of $P$, named after its inventor Dana Scott. Let us prove that $\sigma(P)$ is indeed a topology:

*Proof.* We verify each of the axioms of an open set:

- Clearly $P$ itself is Scott open, and $\varnothing$ is vacuously Scott open.
- Suppose $U$ and $V$ are Scott open. Let $W = U \cap V$ and $b \in \uparrow W$. Then for some $a \in W$, $a \leq b$. Since $a \in U \cap V$, $b \in \uparrow U = U$ and $b \in \uparrow V = V$. This means $b \in W$, so $W$ is an upper set. Next, if $D$ is directed with $\bigvee D \in W$, then, $\bigvee D \in U \cap V$. So there are $y, z \in D$ with $(\uparrow y) \cap D \subseteq U$ and $(\uparrow z) \cap D \subseteq V$. Since $D$ is directed, there is $t \in D$ such that $t \in (\uparrow y) \cap (\uparrow z)$. So $(\uparrow t) \cap D \subseteq (\uparrow y) \cap (\uparrow z) \cap D = \big((\uparrow y) \cap D\big) \cap \big((\uparrow z) \cap D\big) \subseteq U \cap V = W$. This means that $W$ is Scott open.
- Suppose $U_i$ are open and $i \in I$ an index set. Let $U = \bigcup\{U_i \mid i \in I\}$ and $b \in \uparrow U$. So $a \leq b$ for some $a \in U$. Since $a \in U_i$ for some $i \in I$, $b \in \uparrow U_i = U_i$ as $U_i$ is upper. Hence $b \in U_i \subseteq U$, or that $U$ is upper. Next, suppose $D$ is directed with $\bigvee D \in U$. Then $\bigvee D \in U_i$ for some $i \in I$. Since $U_i$ is Scott open, there is $y \in D$ with $(\uparrow y) \cap D \subseteq U_i \subseteq U$, so $U$ is Scott open.

Since the Scott open sets satisfy the axioms of a topology, $\sigma(P)$ is a topology on $P$. $\square$

**Examples**. If $P$ is the unit interval: $P = [0, 1]$, then $P$ is a complete chain, hence a dcpo. Any Scott open set has the form $(a, 1]$ if $0 < a \leq 1$, or $[0, 1]$. If $P = [0, 1] \times [0, 1]$, the unit square, then $P$ is a dcpo as it is already a continuous lattice. The Scott open sets of $P$ are any upper subset of $P$ that is also an open set in the usual sense.

# References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge (2003).