lattice interval

Canonical name	LatticeInterval
Date of creation	2013-03-22 15:44:56
Last modified on	2013-03-22 15:44:56
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	11
Author	CWoo (3771)
Entry type	Definition
Classification	msc 06B99
Classification	msc 06A06
Defines	prime interval
Defines	poset interval
Defines	locally finite lattice

Definition. Let L be a lattice. A subset I of L is called a *lattice interval*, or simply an if there exist elements $a, b \in L$ such that

$$I = \{t \in L \mid a \leq t \leq b\} := [a, b].$$

The elements a, b are called the endpoints of I. Clearly $a, b \in I$. Also, the endpoints of a lattice interval are unique: if $[a, b] = [c, d]$, then $a = c$ and $b = d$.

Remarks.

- It is easy to see that the name is derived from that of an interval on a number line. From this analogy, one can easily define lattice intervals without one or both endpoints. Whereas an interval on a number line is linearly ordered, a lattice interval in general is not. Nevertheless, a lattice interval I of a lattice L is a sublattice of L.
- A bounded lattice is itself a lattice interval: $[0, 1]$.
- A *prime interval* is a lattice interval that contains its endpoints and nothing else. In other words, if $[a, b]$ is prime, then any $c \in [a, b]$ implies that either $c = a$ or $c = b$. Simply put, b covers a. If a lattice L contains 0, then for any $a \in L$, $[0, a]$ is a prime interval iff a is an atom.
- Since no operations of meet and join are used, all of the above discussion can be generalized to define an interval in a poset.
- Given a lattice L, let $\mathcal{B}$ be the collection of all lattice intervals without endpoints, we can form a topolgy on L with $\mathcal{B}$ as the subbasis. This does not insure that $\wedge$ and $\vee$ are continuous, so that L with this topological structure may not be a topological lattice.
- **Locally Finite Lattice**. A lattice that is derived based on the concept of lattice interval is that of a locally finite lattice. A lattice L is locally finite iff every one of its interval is finite. Unless the lattice is finite, a locally finite lattice, if infinite, is either topless or bottomless.