quantale

Canonical name	Quantale
Date of creation	2013-03-22 17:00:08
Last modified on	2013-03-22 17:00:08
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	12
Author	CWoo (3771)
Entry type	Definition
Classification	msc 06F07
Synonym	standard Kleene algebra
Defines	quantale homomorphism

A *quantale* Q is a set with three binary operations on it: $\wedge, \vee$, and $\cdot$, such that

1. $(Q, \wedge, \vee)$ is a complete lattice (with 0 as the bottom and 1 as the top), and
2. $(Q, \cdot)$ is a monoid (with $1'$ as the identity with respect to $\cdot$), such that
3. $\cdot$ distributes over arbitrary joins; that is, for any $a \in Q$ and any subset $S \subseteq Q$,

$$a \cdot \big(\bigvee S\big) = \bigvee\{a \cdot s \mid s \in S\} \quad \text{and} \quad \big(\bigvee S\big) \cdot a = \bigvee\{s \cdot a \mid s \in S\}.$$

It is sometimes convenient to drop the multiplication symbol, when there is no confusion. So instead of writing $a \cdot b$, we write ab.

The most obvious example of a quantale comes from ring theory. Let R be a commutative ring with 1. Then $L(R)$, the lattice of ideals of R, is a quantale.

Proof. In addition to being a (complete) lattice, $L(R)$ has an inherent multiplication operation induced by the multiplication on R, namely,

$$IJ := \{\sum_{i=1}^{n} r_i s_i \mid r_i \in I \text{ and } s_i \in J,\, n \in \mathbb{N}\},$$

making it into a semigroup under the multiplication.

Now, let $S = \{I_i \mid i \in N\}$ be a set of ideals of R and let $I = \bigvee S$. If J is any ideal of R, we want to show that $IJ = \bigvee\{I_iJ \mid i \in N\}$ and, since R is commutative, we would have the other equality $JI = \bigvee\{JI_i \mid i \in N\}$. To see this, let $a \in IJ$. Then $a = \sum r_i s_i$ with $r_i \in I$ and $s_i \in J$. Since each r_i is a finite sum of elements of $\bigcup S$, $r_i s_i$ is a finite sum of elements of $\bigcup\{I_iJ \mid i \in N\}$, so $a \in \bigvee\{I_iJ \mid i \in N\}$. This shows $IJ \subseteq \bigvee\{I_iJ \mid i \in N\}$. Conversely, if $a \in \bigvee\{I_iJ \mid i \in N\}$, then a can be written as a finite sum of elements of $\bigcup\{I_iJ \mid i \in N\}$. In turn, each of these additive components is a finite sum of products of the form $r_k s_k$, where $r_k \in I_i$ for some i, and $s_k \in J$. As a result, a is a finite sum of elements of the form $r_k s_k$, so $a \in IJ$ and we have the other inclusion $\bigvee\{I_iJ \mid i \in N\} \subseteq IJ$.

Finally, we observe that R is the multiplicative identity in $L(R)$, as $IR = RI = I$ for all $I \in L(R)$. This completes the proof. $\square$

Remark. In the above example, notice that $IJ \leq I$ and $IJ \leq J$, and we actually have $IJ \leq I \wedge J$. In particular, $I^2 \leq I$. With an added condition, this fact can be characterized in an arbitrary quantale (see below).

Properties. Let Q be a quantale.

1. Multiplication is monotone in each argument. This means that if $a, b \in Q$, then $a \leq b$ implies that $ac \leq bc$ and $ca \leq cb$ for all $c \in Q$. This is easily verified. For example, if $a \leq b$, then $ac \vee bc = (a \vee b)c = bc$, so $ac \leq bc$. So a quantale is a partially ordered semigroup, and in fact, an l-monoid (an l-semigroup and a monoid at the same time).

2. If $1 = 1'$, then $ab \leq a \wedge b$: since $a \leq 1$, then $ab \leq a1 = a1' = a$; similarly, $b \leq ab$. In particular, the bottom 0 is also the multiplicative zero: $a0 \leq a \wedge 0 = 0$, and $0a = 0$ similarly.

3. Actually, $a0 = 0a = 0$ is true even without $1 = 1'$: since $a\varnothing = \{ab \mid b \in \varnothing\} = \varnothing$ and $0 := \bigvee \varnothing$, we have $a0 = a \bigvee \varnothing = \bigvee a\varnothing = \bigvee \varnothing = 0$. Similarly $0a = 0$. So a quantale is a semiring, if $\vee$ is identified as $+$ (with 0 as the additive identity), and $\cdot$ is again $\cdot$ (with $1'$ the multiplicative identity).

4. Viewing quantale Q now as a semiring, we see in fact that Q is an idempotent semiring, since $a + a = a \vee a = a$.

5. Now, view Q as an i-semiring. For each $a \in Q$, let $S = \{1', a, a^2, \ldots\}$ and define $a^* = \bigvee S$. We observe some basic properties

   - $1' + aa^* = a^*$: since $1' \vee (a \bigvee S) = 1' \vee (\bigvee\{a1', aa, aa^2, \ldots\}) = \bigvee\{1', a, a^2, \ldots\} = \bigvee S = a^*$
   - $1' + a^*a = a^*$ as well
   - if $ab \leq b$, then $a^*b \leq b$: by induction on n, we have $a^nb \leq b$ whenever $a \leq b$, so that $a^*b = \bigvee\{a^nb \mid n \in \mathbb{N} \cup \{0\}\} \leq b$.
   - similarly, if $ba \leq b$, then $ba^* \leq b$

   All of the above properties satisfy the conditions for an i-semiring to be a Kleene algebra. For this reason, a quantale is sometimes called a *standard Kleene algebra*.

6. Call the multiplication *idempotent* if each element is an idempotent with respect to the multiplication: $aa = a$ for any $a \in Q$. If $\cdot$ is idempotent and $1 = 1'$, then $\cdot = \wedge$. In other words, $ab = a \wedge b$.

   Proof. As we have seen, $ab \leq a \wedge b$ in the 2 above. Now, suppose $c \leq a \wedge b$. Then $c \leq a$ and $c \leq b$, so $c = c^2 \leq cb \leq ab$. So ab is the greatest lower bound of a and b, i.e., $ab = a \wedge b$. This also means that $ba = b \wedge a = a \wedge b = ab$. $\square$

7. In fact, a locale is a quantale if we define $\cdot := \wedge$. Conversely, a quantale where $\cdot$ is idempotent and $1 = 1'$ is a locale.

   Proof. If Q is a locale with $\cdot = \wedge$, then $aa = a \wedge a = a$ and $a1 = a \wedge 1 = a = 1 \wedge a = 1a$, implying $1 = 1'$. The infinite distributivity of $\cdot$ over $\vee$ is just a restatement of the infinite distributivity of $\wedge$ over $\vee$ in a locale. Conversely, if $\cdot$ is idempotent and $1 = 1'$, then $\cdot = \wedge$ as shown previously, so $a \wedge (\bigvee S) = a(\bigvee S) = \bigvee\{as \mid s \in S\} = \bigvee\{a \wedge s \mid s \in S\}$. Similarly $(\bigvee S) \wedge a = \bigvee\{s \wedge a \mid s \in S\}$. Therefore, Q is a locale. $\square$

Remark. A *quantale homomorphism* between two quantales is a complete lattice homomorphism and a monoid homomorphism at the same time.

References

[1] S. Vickers, *Topology via Logic*, Cambridge University Press, Cambridge (1989).