connected poset

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Let P be a poset. Write $a \perp b$ if either $a \leq b$ or $b \leq a$. In other words, $a \perp b$ if a and b are comparable. A poset P is said to be *connected* if for every pair $a, b \in P$, there is a finite sequence $a = c_1, c_2, \ldots, c_n = b$, with each $c_i \in P$, such that $c_i \perp c_{i+1}$ for each $i = 1, 2, \ldots, n-1$.

For example, a poset with the property that any two elements are either bounded from above or bounded from below is a connected poset. In particular, every semilattice is connected. A fence is always connected. If P has more than one element and contains an element that is both maximal and minimal, then it is not connected. A *connected component* in a poset P is a maximal connected subposet. In the last example, the maximal-minimal point is a component in P. Any poset can be written as a disjoint union of its components.

It is true that a poset is connected if its corresponding Hasse diagram is a connected graph. However, the converse is not true. Before we see an example of this, let us recall how to construct a Hasse diagram from a poset P. The diagram so constructed is going to be an undirected graph (since this is all we need in our discussion). Draw an edge between $a, b \in P$ if either a covers b or b covers a. Let us denote this relation between a and b by $a \asymp b$. Let E be the collection of all these edges. Then $G = (P, E)$ is a graph where elements of P serve as vertices and E as the constructed edges. From this construction, one sees that a finite path exists between $a, b \in V(G) = P$ if there is a finite sequence $a = d_0, d_1, \ldots, d_m = b$, with each $d_i \in V(G)$, such that $d_i \asymp d_{i+1}$ for $i = 1, \ldots, m-1$. In other words, a and b can be "joined" by a finite number of edges, such that a is a vertex on the first edge and b is the vertex on the last edge.

As promised, here is an example of a connected poset whose underlying Hasse diagram is not connected. take the real line $\mathbb{R}$ with ∞ adjoined to the right (meaning every element $r \in \mathbb{R}$ is less than or equal to ∞). Then the resulting poset is connected, but its underlying Hasse diagram is not, since no element in $\mathbb{R}$ can be joined to ∞ by a finite path.