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upper set

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Let P be a poset and A a subset of P. The *upper set* of A is defined to be the set

$$\{b \in P \mid a \leq b \text{ for some } a \in A\},$$

and is denoted by $\uparrow\!A$. In other words, $\uparrow\!A$ is the set of all upper bounds of elements of A.

$\uparrow$ can be viewed as a unary operator on the power set 2^P sending $A \in 2^P$ to $\uparrow\!A \in 2^P$. $\uparrow$ has the following properties

1. $\uparrow\!\varnothing = \varnothing$,

2. $A \subseteq \uparrow\!A$,

3. $\uparrow\uparrow\!A = \uparrow\!A$, and

4. if $A \subseteq B$, $\uparrow\!A \subseteq \uparrow\!B$.

So $\uparrow$ is a closure operator.

An *upper set* in P is a subset A such that its upper set is itself: $\uparrow\!A = A$. In other words, A is closed with respect to $\leq$ in the sense that if $a \in A$ and $a \leq b$, then $b \in A$. An upper set is also said to be *upper closed*. For this reason, for any subset A of P, the $\uparrow\!A$ is also called the *upper closure* of A.

Dually, the *lower set* (or *lower closure*) of A is the set of all lower bounds of elements of A. The lower set of A is denoted by $\downarrow\!A$. If the lower set of A is A itself, then A is a called a *lower set*, or a *lower closed set*.

Remarks.

- $\uparrow\!A$ is *not* the same as the set of upper bounds of A, commonly denoted by A^u, which is defined as the set $\{b \in P \mid a \leq b \text{ for } \textit{all } a \in A\}$. Similarly, $\downarrow\!A \neq A^\ell$ in general, where A^ℓ is the set of lower bounds of A.

- When $A = \{x\}$, we write $\uparrow\!x$ for $\uparrow\!A$ and $\downarrow\!x$ for $\downarrow\!A$. $\uparrow\!x = \{x\}^u$ and $\downarrow\!x = \{x\}^d$.

- If P is a lattice and $x \in P$, then $\uparrow\!x$ is the principal filter generated by x, and $\downarrow\!x$ is the principal ideal generated by x.

- If A is a lower set of P, then its set complement $A^\complement$ is an upper set: if $a \in A^\complement$ and $a \leq b$, then $b \in A^\complement$ by a contrapositive argument.

- Let P be a poset. The set of all lower sets of P is denoted by $\mathcal{O}(P)$. It is easy to see that $\mathcal{O}(P)$ is a poset (ordered by inclusion), and $\mathcal{O}(P)^\partial = \mathcal{O}(P^\partial)$, where $^\partial$ is the dualization operation (meaning that P^∂ is the dual poset of P).