continuous poset

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Defines	continuous lattice

A poset P is said to be *continuous* if for every $a \in P$

1. the set $\mathrm{wb}(a) = \{u \in P \mid u \ll a\}$ is a directed set,
2. $\bigvee \mathrm{wb}(a)$ exists, and
3. $a = \bigvee \mathrm{wb}(a)$.

In the first condition, $\ll$ indicates the way below relation on P. It is true that in *any* poset, if $b := \bigvee \mathrm{wb}(a)$ exists, then $b \leq a$. So for a poset to be continuous, we require that $a \leq b$.

A *continuous lattice* is a complete lattice whose underlying poset is continuous. Note that if P is a complete lattice, condition 1 above is automatically satisfied: suppose $u, v \ll a$ and $D \subseteq P$ with $a \leq \bigvee D$, then there are finite subsets F, G of D with $u \leq \bigvee F$ and $v \leq \bigvee G$. Then $H := F \cup G \subseteq D$ is finite and $u \vee v \leq \left(\bigvee F\right) \vee \left(\bigvee G\right) = \bigvee H$, or $u \vee v \ll a$, implying that $\mathrm{wb}(a)$ is directed.

Examples.

1. Any finite poset is continuous, and so is any finite lattice (since it is complete).
2. A chain is continuous iff it is complete.
3. The lattice of ideals of a ring is continuous.
4. The set of all lower semicontinuous functions from a fixed compact topological space into the extended real numbers is a continuous lattice.
5. The set of all closed convex subsets of a compact convex subset of $\mathbb{R}^n$ ordered by reverse inclusion is a continuous lattice.

Remarks.

- Every algebraic lattice is continuous.
- Every continuous meet semilattice is meet continuous.

References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge (2003).