convex subgroup

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We begin this article with something more general. Let P be a poset. A subset $A \subseteq P$ is said to be *convex* if for any $a, b \in A$ with $a \leq b$, the poset interval $[a, b] \subseteq A$ also. In other words, $c \in A$ for any $c \in P$ such that $a \leq c$ and $c \leq b$. Examples of convex subsets are intervals themselves, antichains, whose intervals are singletons, and the empty set.

One encounters convex sets most often in the study of partially ordered groups. A *convex subgroup* H of a po-group G is a subgroup of G that is a convex subset of the poset G at the same time. Since $e \in H$, we have that $[e, a] \subseteq H$ for any $e \leq a \in H$. Conversely, if a subgroup H satisfies the property that $[e, a] \subseteq H$ whenever $a \in H$, then H is a convex subgroup: if $a, b \in H$, then $a^{-1}b \in H$, so that $[e, a^{-1}b] \subseteq H$, which implies that $[a, b] = a[e, a^{-1}b] \subseteq H$ as well.

For example, let $G = \mathbb{R}^2$ be the po-group under the usual Cartesian ordering. G and 0 are both convex, but these are trivial examples. Let us see what other convex subgroups H there are. Suppose $P = (a, b) \in H$ with $(a, b) \neq (0, 0) = O$. We divide this into several cases:

1. $ab > 0$. If $a > 0$, then $b > 0$ (P in the first quadrant), so that $O \leq P$, which means $[O, P] \subseteq H$. If $a < 0$, then $b < 0$ (P in the third quandrant), so that $O \leq -P$. In either case, H contains a rectangle ($[O, P]$ or $[O, -P]$) that generates G, so $H = G$.

2. One of a or b is 0. Suppose $a = 0$ for now. Then either $0 < b$ so that $[O, P] \subseteq H$ or $b < 0$ so that $[O, -P] \subseteq H$. In either case, H contains a line segment on the y-axis. But this line segment generates the y-axis. So y-axis $\subseteq H$. If H is a subgroup of the y-axis, then H=y-axis.

   Otherwise, another point $Q = (c, d) \in H$ not on the y-axis. We have the following subcases:

   (a) If $cd > 0$, then $H = G$ as in the previous case.

   (b) If $cd < 0$, say $d < 0$ (or $0 < c$), then for some positive integer n, $0 < d + nb$, so that $O \leq Q + nP$, and $H = G$ as well. On the other hand, if $c < 0$ (or $0 < d$), then $-Q$ returns us to the previous argument and $H = G$ again.

   (c) If $d = 0$ (so $c \neq 0$), then either $O \leq P + Q$ (when $0 < c$) or $O \leq P - Q$ (when $c < 0$), so that $H = G$ once more.

A similar set of arguments shows that if H contains a segment of the x-axis, then either H is the x-axis or $H = G$. In conclusion, in the case when $ab = 0$, H is either one of the two axes, or the entire group.

3. $ab < 0$. It is enough to assume that $0 < a$ and $b < 0$ (that P lies in the fourth quadrant), for if P lies in the second quadrant, $-P$ lies in the fourth.

   Since $O, P \in H$, H could be a subgroup of the line group L containing O and P. No two points on L are comparable, for if $(r, s) < (t, u)$ on L, then the slope of L is positive

   $$0 < \frac{u - s}{t - r},$$

   a contradiction. So L, and hence H, is an antichaine. This means that H is convex.

   Suppose now H contains a point $Q = (c, d)$ not on L. We again break this down into subcases:

   (a) Q is in the first or third quandrant. Then $H = G$ as in the very first case above.

   (b) Q is on either of the axes. Then $H = G$ also, as in case 2(b) above.

   (c) Q is in the second or fourth quadrant. It is enough to assume that Q is in the same quadrant as P (fourth). So we have $0 < c$ and $d < 0$. Since L passes through P and not Q, we have that

   $$\frac{a}{c} \neq \frac{b}{d}.$$

   Let $0 < r = a/c$ and $0 < s = b/d$ and assume $r < s$. Then there is a rational number m/n (with $0 < m, n$) such that

   $$r < \frac{m}{n} < s.$$

   This means that $na < mc$ and $nb < md$, or $nP < mQ$. But $nP, mQ \in H$, so is $R = mQ - nP \in H$, which is in the first quadrant. This implies that $H = G$ too.

In summary, if H contains a point in the second or fourth quadrant, then either H is a subgroup of a line with slope < 0, or $H = G$.

The three main cases above exhaust all convex subgroups of $\mathbb{R}^2$ under the Cartesian ordering.

If the Euclidean plane is equipped with the lexicographic ordering, then the story is quite different, but simpler. If H is non-trivial, say $P = (a, b) \in H$, $P \neq O$. If $0 < a$, then $(c, d) \leq (a, b)$ for any $c < a$ regardless of d. Choose $Q = (c, d)$ to be in the first quadrant. Then $[O, Q] \subseteq H$, so that $H = G$. If $a < 0$, then $-P$ takes us back to the previous argument. If $a = 0$, then either $[O, P]$ (when $0 < b$), or $[O, -P]$ (when $b < 0$) is a positive interval on the y-axis. This implies that H is at least the y-axis. If H contains no other points, then $H = y$-axis. In summary, the po-group $\mathbb{R}^2$ with lexicographic order has the y-axis as the only non-trivial proper convex subgroup.

References

[1] G. Birkhoff *Lattice Theory*, 3rd Edition, AMS Volume XXV, (1967).