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lattice filter

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Synonym	ultra-filter
Synonym	maximal filter
Related topic	Ultrafilter
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Defines	filter
Defines	prime filter
Defines	ultrafilter
Defines	filter generated by
Defines	principal filter

Let L be a lattice. A *filter* (of L) is the dual concept of an http://planetmath.org/LatticeIde
Specifically, a filter F of L is a non-empty subset of L such that

1. F is a sublattice of L, and
2. for any $a \in F$ and $b \in L$, $a \vee b \in F$.

The first condition can be replaced by a weaker one: for any $a, b \in F$, $a \wedge b \in F$.

An equivalent characterization of a filter I in a lattice L is

1. for any $a, b \in F$, $a \wedge b \in F$, and
2. for any $a \in F$, if $a \leq b$, then $b \in F$.

Note that the dualization switches the meet and join operations, as well as reversing the ordering relationship.

Special Filters. Let F be a filter of a lattice L. Some of the common types of filters are defined below.

- F is a *proper filter* if $F \neq L$, and, if L contains 0, $F \neq 0$.
- F is a *prime filter* if it is proper, and $a \vee b \in F$ implies that either $a \in F$ or $b \in F$.
- F is an *ultrafilter* (or *maximal filter*) of L if F is proper and the only filter properly contains F is L.
- **filter generated by a set**. Let X be a subset of a lattice L. Let T be the set of all filters of L containing X. Since $T \neq \varnothing$ ($L \in T$), the intersection N of all elements in T, is also a filter of L that contains X. N is called the *filter generated by* X, written $[X)$. If X is a singleton $\{x\}$, then N is said to be a *principal filter* generated by x, written $[x)$.

Examples.

1. Consider the positive integers, with meet and join defined by the greatest common divisor and the least common multiple operations. Then the positive even numbers form a filter, generated by 2. If we toss in 3 as an additional element, then $1 = 2 \wedge 3 \in [\{2, 3\})$ and consequently any positive integer $i \in [\{2, 3\})$, since $1 \leq i$. In general, if p, q are relatively prime, then $[\{p, q\}) = \mathbb{Z}^+$. In fact, any proper filter in $\mathbb{Z}^+$ is principal. When the generator is prime, the filter is prime, which is also maximal. So prime filters and ultrafilters coincide in $\mathbb{Z}^+$.

2. Let A be a set and 2^A the power set of A. If the set inclusion is the ordering defined on 2^A, then the definition of a filter here coincides with the ususal definition of a http://planetmath.org/Filterfilter on a set in general.

Remark. If F is both a filter and an ideal of a lattice L, then $F = L$.