meet continuous

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Let L be a meet semilattice. We say that L is *meet continuous* if

1. for any monotone net $D = \{x_i \mid i \in I\}$ in L, its supremum $\bigvee D$ exists, and
2. for any $a \in L$ and any monotone net $\{x_i \mid i \in I\}$,
$$a \wedge \bigvee\{x_i \mid i \in I\} = \bigvee\{a \wedge x_i \mid i \in I\}.$$

A monotone net $\{x_i \mid i \in I\}$ is a net $x : I \to L$ such that x is a non-decreasing function; that is, for any $i \leq j$ in I, $x_i \leq x_j$ in L.

Note that we could have replaced the first condition by saying simply that $D \subseteq L$ is a directed set. (A monotone net is a directed set, and a directed set is a trivially a monotone net, by considering the identity function as the net). It's not hard to see that if D is a directed subset of L, then $a \wedge D := \{a \wedge x \mid x \in D\}$ is also directed, so that the right hand side of the second condition makes sense.

Dually, a join semilattice L is *join continuous* if its dual (as a meet semilattice) is meet continuous. In other words, for any antitone net $D = \{x_i \mid i \in I\}$, its infimum $\bigwedge D$ exists and that
$$a \vee \bigwedge\{x_i \mid i \in I\} = \bigwedge\{a \vee x_i \mid i \in I\}.$$
An antitone net is just a net $x : I \to L$ such that for $i \leq j$ in I, $x_j \leq x_i$ in L.

Remarks.

- A meet continuous lattice is a complete lattice, since a poset such that finite joins and directed joins exist is a complete lattice (see the link below for a proof of this).
- Let a lattice L be both meet continuous and join continuous. Let $\{x_i \mid i \in I\}$ be any net in L. We define the following:
$$\overline{\lim}\ x_i = \bigwedge_{j \in I}\{\bigvee_{j \leq i} x_i\} \qquad \text{and} \qquad \underline{\lim}\ x_i = \bigvee_{j \in I}\{\bigwedge_{i \leq j} x_i\}$$
If there is an $x \in L$ such that $\overline{\lim}\ x_i = x = \underline{\lim}\ x_i$, then we say that the net $\{x_i\}$ *order converges* to x, and we write $x_i \to x$, or $x = \lim\ x_i$. Now, define a subset $C \subseteq L$ to be *closed* (in L) if for any net $\{x_i\}$ in C such that $x_i \to x$ implies that $x \in C$, and *open* if its set complement is closed, then L becomes a topological lattice. With respect to this topology, meet $\wedge$ and join $\vee$ are easily seen to be continuous.

References

[1] G. Birkhoff, *Lattice Theory*, 3rd Edition, Volume 25, AMS, Providence (1967).

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[3] G. Grätzer, *General Lattice Theory*, 2nd Edition, Birkhäuser (1998).