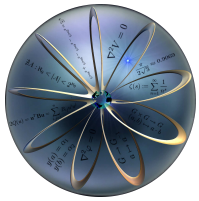

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order of six means

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The of the six usual means of two positive numbers (a and b) is from the least to the greatest one

1. harmonic mean,
2. geometric mean,
3. Heronian mean,
4. arithmetic mean,
5. quadratic mean,
6. contraharmonic mean,

i. e.

$$\frac{2ab}{a+b} \leqq \sqrt{ab} \leqq \frac{a+\sqrt{ab}+b}{3} \leqq \frac{a+b}{2} \leqq \sqrt{\frac{a^2+b^2}{2}} \leqq \frac{a^2+b^2}{a+b}.$$

The equality signs are valid iff $a = b$.

Proof. If $x^2 - y^2 \geqq 0$ for nonnegative x and y, then $x \geqq y$.

"$1 \leqq 2$":

$$\left(\sqrt{ab}\right)^2 - \left(\frac{a+b}{2}\right)^2 = ab - \frac{4a^2b^2}{(a+b)^2} = ab\left(1 - \frac{4ab}{(a+b)^2}\right) = ab \cdot \frac{(a+b)^2 - 4ab}{(a+b)^2} = \frac{ab(a-b)^2}{(a+b)^2} \geqq 0$$

"$2 \leqq 3$" and "$3 \leqq 4$": proven in Heronian mean is between geometric and arithmetic mean

"$4 \leqq 5$":

$$\left(\sqrt{\frac{a^2+b^2}{2}}\right)^2 - \left(\frac{a+b}{2}\right)^2 = \frac{2a^2+2b^2-a^2-2ab-b^2}{4} = \left(\frac{a-b}{2}\right)^2 \geqq 0$$

"$5 \leqq 6$":

$$\left(\frac{a^2+b^2}{a+b}\right)^2 - \left(\sqrt{\frac{a^2+b^2}{2}}\right)^2 = \frac{2(a^2+b^2)^2 - (a^2+b^2)(a+b)^2}{2(a+b)^2} = \frac{(a^2+b^2)(2a^2+2b^2-a^2-2ab-b^2)}{2(a+b)^2}$$
$$= \frac{(a^2+b^2)(a-b)^2}{2(a+b)^2} \geqq 0$$