continuous geometry

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Let V be a finite dimensional vector space (over some field) with dimension n. Let $PG(V)$ be its lattice of subspaces, also known as the projective geometry of V. It is well-known that we can associate each element $a \in PG(V)$ a unique integer $\dim(a)$, namely, the dimension of the a as a subspace of V. dim can be seen as a function from $PG(V)$ to $\mathbb{Z}$. One property of dim is that for every i between 0 and n, there is an $a \in PG(V)$ such that $\dim(a) = i$. If we normalize dim by dividing its values by n, then we get a function $d : PG(V) \to [0, 1]$. As n (the dimension of V) increases, the range of d begins to "fill up" $[0, 1]$. Of course, we know this is impossible as long as V is finite dimensional.

Question: is there a "geometry" on which a "dimension function" is defined so that it is onto the closed unit interval $[0, 1]$?

The answer is yes, and the geometry is the so-called "continuous geometry". However, like projective geometries, it is really just a lattice (with some special conditions). A continuous geometry L is a generalization of a projective geometry so that a "continuous" dimension function d can be defined on L such that for every real number $r \in [0, 1]$ there is an $a \in L$ such that $d(a) = r$. Furthermore, d takes infinite independent joins to infinite sums:

$$d(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} d(a_i) \text{ whenever } a_{j+1} \wedge (\bigvee_{i=1}^{j} a_i) = 0 \text{ for } j \geq 1.$$

Definition. A *continuous geometry* is a lattice L that is complemented, modular, meet continuous, and join continuous.

From a continuous geometry L, it can be shown that the `http://planetmath.org/Complemente` relation $\sim$ on elements of L is a transitive relation (Von Neumann). Since $\sim$ is also reflexive and symmetric, it is an equivalence relation. In a projective geometry, perspective elements are exactly subspaces having the same dimension. From this equivalence relation, one can proceed to define a "dimension" function from L into $[0, 1]$.

Continuous geometry was introduced by Von Neumann in the 1930's when he was working on the theory of operator algebras in Hilbert spaces. Write $PG(n-1)$ the projective geometry of dimension $n-1$ over D (lattice of left (right) subspaces of left (right) n-dimensional vector space over D). Von Neumann found that $PG(n-1)$ can be embedded into $PG(2n-1)$ in such a way that not only the lattice operations are preserved, but the values of the "normalized dimension function" d described above are also preserved.

In other words, if $\phi : PG(n-1) \to PG(2n-1)$ is the embedding, and d_n is the dimension function on $PG(n-1)$ and d_{2n} is the dimension function on $PG(2n-1)$, then $d_n(a) = d_{2n}(\phi(a))$. As a result, we get a chain of embeddings

$$PG(1) \hookrightarrow PG(3) \hookrightarrow \cdots \hookrightarrow PG(2^n - 1) \hookrightarrow \cdots .$$

Taking the union of these lattices, we get a lattice $PG(\infty)$, which is complemented and modular, which has a "normalized dimension function" d into $[0,1]$ whose values take the form $p/2^m$ (p, m positive integers). This d is also a valuation on $PG(\infty)$, turning it into a metric lattice, which in turn can be completed to a lattice $CG(D)$. This $CG(D)$ is the first example of a continuous geometry having a "continuous" dimension function.

Remarks.

- Any continuous geometry is a complete lattice and a topological lattice if order convergence is used to define a topology on it.
- An *irreducible continuous geometry* is a continuous geometry whose center is trivial (consisting of just 0 and 1). It turns out that an irreducible continuous geometry is just $CG(D)$ for some division ring D.
- (Kaplansky) Any orthocomplemented complete modular lattice is a continuous geometry.

References

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