complete Boolean algebra

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Defines	κ-complete Boolean algebra
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A Boolean algebra A is a *complete Boolean algebra* if for every subset C of A, the arbitrary join and arbitrary meet of C exist.

By de Morgan's laws, it is easy to see that a Boolean algebra is complete iff the arbitrary join of any subset exists iff the arbitrary meet of any subset exists. For a proof of this, see `http://planetmath.org/PropertiesOfArbitraryJoinsAndMeets`this link.

For an example of a complete Boolean algebra, let S be any set. Then the powerset $P(S)$ with the usual set theoretic operations is a complete Boolean algebra.

In a complete Boolean algebra, the infinite distributive and infinite de-Morgan's laws hold:

- $x \wedge \bigvee A = \bigvee (x \wedge A)$ and $x \vee \bigwedge A = \bigwedge (x \vee A)$
- $(\bigvee A)^* = \bigwedge A^*$ and $(\bigwedge A)^* = \bigvee A^*$, where $A^* := \{a^* \mid a \in A\}$.

In the category of complete Boolean algebras, a morphism between two objects is a Boolean algebra homomorphism that preserves arbitrary joins (equivalently, arbitrary meets), and is called a *complete Boolean algebra homomorphism*.

Remark There are infinitely many algebras between Boolean algebras and complete Boolean algebras. Let κ be a cardinal. A Boolean algebra A is said to be κ-complete if for every subset C of A with $|C| \leq \kappa$, $\bigvee C$ (and equivalently $\bigwedge C$) exists. A κ-complete Boolean algebra is usually called a κ-algebra. If $\kappa = \aleph_0$, the first aleph number, then it is called a *countably complete Boolean algebra*.

Any complete Boolean algebra is κ-complete, and any κ-complete is λ-complete for any $\lambda \leq \kappa$. An example of a κ-complete algebra that is not complete, take a set S with $\kappa < |S|$, then the collection $A \subseteq P(S)$ consisting of any subset T such that either $|T| \leq \kappa$ or $|S - T| \leq \kappa$ is κ-complete but not complete.

A Boolean algebra homomorphism f between two κ-algebras A, B is said to be κ-complete if

$$f(\bigvee \{a \mid a \in C\}) = \bigvee \{f(a) \mid a \in C\}$$

for any $C \subseteq A$ with $|C| \leq \kappa$.