# Heyting algebra

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A *Heyting lattice* $L$ is a Brouwerian lattice with a bottom element 0. Equivalently, $L$ is Heyting iff it is relatively pseudocomplemented and pseudocomplemented iff it is bounded and relatively pseudocomplemented.

Let $a^*$ denote the pseudocomplement of $a$ and $a \to b$ the pseudocomplement of $a$ relative to $b$. Then we have the following properties:

1. $a^* = a \to 0$ (equivalence of definitions)
2. $1^* = 0$ (if $c = 1 \to 0$, then $c = c \wedge 1 \leq 0$ by the definition of $\to$.)
3. $a^* = 1$ iff $a = 0$ ($1 = a \to 0$ implies that $c \wedge a \leq 0$ whenever $c \leq 1$. In particular $a \leq 1$, so $a = a \wedge a \leq 0$ or $a = 0$. On the other hand, if $a = 0$, then $a^* = 0^* = 0 \to 0 = 1$.)
4. $a \leq a^{**}$ and $a^* = a^{***}$ (already true in any pseudocomplemented lattice)
5. $a^* \leq a \to b$ (since $a^* \wedge a = 0 \leq b$)
6. $(a \to b) \wedge (a \to b^*) = a^*$

   *Proof.* If $c \wedge a = 0$, then $c \wedge a \leq b$ so $c \leq (a \to b)$, and $c \leq (a \to b^*)$ likewise, so $c \leq (a \to b) \wedge (a \to b^*)$. This means precisely that $a^* = (a \to b) \wedge (a \to b^*)$. □
7. $a \to b \leq b^* \to a^*$ (since $(a \to b) \wedge b^* \leq (a \to b) \wedge (a \to b^*) = a^*$)
8. $a^* \vee b \leq a \to b$ (since $b \wedge a \leq b$ and $a^* \wedge a = 0 \leq b$)

Note that in property 4, $a \leq a^{**}$, whereas $a^{**} \leq a$ is in general not true, contrasting with the equality $a = a''$ in a Boolean lattice, where $'$ is the complement operator. It is easy to see that if $a^{**} \leq a$ for all $a$ in a Heyting lattice $L$, then $L$ is a Boolean lattice. In this case, the pseudocomplement coincides with the complement of an element $a^* = a'$, and we have the equality in property 7: $a^* \vee b = a \to b$, meaning that the concept of `http://planetmath.org/RelativelyPseudocomplemented`relative pseudocomplementation coincides with the material implication in classical propositional logic.

A *Heyting algebra* is a Heyting lattice $H$ such that $\to$ is a binary operator on $H$. A Heyting algebra homomorphism between two Heyting algebras is a lattice homomorphism that preserves $0, 1$, and $\to$. In addition, if $f$

is a Heyting algebra homomorphism, $f$ preserves psudocomplementation: $f(a^*) = f(a \to 0) = f(a) \to f(0) = f(a) \to 0 = f(a)^*$.

**Remarks**.

- In the literature, the assumption that a Heyting algebra contains 0 is sometimes dropped. Here, we call it a Brouwerian lattice instead.
- Heyting algebras are useful in modeling intuitionistic logic. Every intuitionistic propositional logic can be modelled by a Heyting algebra, and every intuitionistic predicate logic can be modelled by a complete Heyting algebra.