Boolean algebra homomorphism

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Let A and B be Boolean algebras. A function $f : A \to B$ is called a *Boolean algebra homomorphism*, or homomorphism for short, if f is a $\{0,1\}$-`http://planetmath.org/LatticeHomomorphism`lattice homomorphism such that f respects $'$: $f(a') = f(a)'$.

Typically, to show that a function between two Boolean algebras is a Boolean algebra homomorphism, it is not necessary to check every defining condition. In fact, we have the following:

1. if f respects $'$, then f respects $\vee$ iff it respects $\wedge$;
2. if f is a lattice homomorphism, then f respects 0 and 1 iff it respects $'$.

The first assertion can be shown by de Morgan's laws. For example, to see the LHS implies RHS, $f(a \wedge b) = f((a' \vee b')') = f(a' \vee b')' = ((f(a') \vee f(b'))' = f(a')' \wedge f(b')' = f(a)'' \wedge f(b)'' = f(a) \wedge f(b)$. The second assertion can also be easily proved. For example, to see that the LHS implies RHS, we have that $f(a') \vee f(a) = f(a' \vee a) = f(1) = 1$ and $f(a') \wedge f(a) = f(a' \wedge a) = f(0) = 0$. Together, this implies that $f(a')$ is the complement of $f(a)$, which is $f(a)'$.

If a function satisfies one, and hence all, of the above conditions also satisfies the property that $f(0) = 0$, for $f(0) = f(a \wedge a') = f(a) \wedge f(a') = f(a) \wedge f(a)' = 0$. Dually, $f(1) = 1$.

As a Boolean algebra is an algebraic system, the definition of a Boolean algebra homormphism is just a special case of an algebra homomorphism between two algebraic systems. Therefore, one may similarly define a Boolean algebra monomorphism, epimorphism, endormophism, automorphism, and isomorphism.

Let $f : A \to B$ be a Boolean algebra homomorphism. Then the *kernel* of f is the set $\{a \in A \mid f(a) = 0\}$, and is written $\ker(f)$. Observe that $\ker(f)$ is a Boolean ideal of A.

Let κ be a cardinal. A Boolean algebra homomorphism $f : A \to B$ is said to be κ-complete if for any subset $C \subseteq A$ such that

1. $|C| \le \kappa$, and
2. $\bigvee C$ exists,

then $\bigvee f(C)$ exists and is equal to $f(\bigvee C)$. Here, $f(C)$ is the set $\{f(c) \mid c \in C\}$. Note that again, by de Morgan's laws, if $\bigwedge C$ exists, then $\bigwedge f(C)$ exists and is equal to $f(\bigwedge C)$. If we place no restrictions on the cardinality of C (i.e., drop condition 1), then $f : A \to B$ is said to be a *complete Boolean* *algebra homomorphism*. In the categories of κ-complete Boolean algebras and complete Boolean algebras, the morphisms are κ-complete homomorphisms and complete homomorphisms respectively.