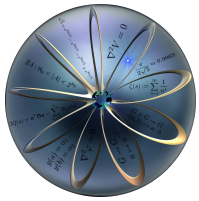

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distributive inequalities

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Let L be a lattice. Then for $a, b, c \in L$, we have the following inequalities:

1. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$,
2. $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$.

Proof. Since $a \leq a \vee b$ and $a \leq a \vee c$, $a \leq (a \vee b) \wedge (a \vee c)$. Similarly, $b \wedge c \leq b \leq a \vee b$ and $b \wedge c \leq c \leq a \vee c$ imply $b \wedge c \leq (a \vee b) \wedge (a \vee c)$. Together, we have $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

The second inequality is the dual of the first one. □

The two inequalities above are called the *distributive inequalities*.

Proposition A lattice L is a distributive lattice if one of the following inequalities holds:

1. $(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c)$,
2. $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$.

Proof. By the distributive inequalities, all we need to show is that 1. implies 2. (that 2. implies 1. is just the dual statement). So suppose 1. holds. Then

$$\begin{aligned}
(a \wedge b) \vee (a \wedge c) &\geq ((a \wedge b) \vee a) \wedge ((a \wedge b) \vee c) && \text{by assumption} \\
&= a \wedge ((a \wedge b) \vee c) && \text{by absorption} \\
&\geq a \wedge ((c \vee a) \wedge (c \vee b)) && \text{by assumption} \\
&= (a \wedge (c \vee a)) \wedge (c \vee b) && \text{meet associativity} \\
&= a \wedge (c \vee b). && \text{by absorption}
\end{aligned}$$

□