upper set operation is a closure operator

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In this entry, we shall prove the assertion made in the `http://planetmath.org/UpperSet`main entry that $\uparrow$ is a closure operator. This will be done by checking that the defining properties are satisfied. To begin, recall the definition of our operation:

Definition 1. *Let P be a poset and A a subset of P. The* upper set *of A is defined to be the set*

$$\uparrow A = \{b \in P \mid (\exists\, a \in A)\, a \leq b\}$$

Now, we verify each of the properties which is required of a closure operator.

Theorem 1. $\uparrow\emptyset = \emptyset$

Proof. Any statement of the form "$(\exists\, a \in \emptyset) P(a)$" is identically false no matter what the predicate P (i.e. it is an antitautology) and the set of objects satsfying an identically false condition is empty, so $\uparrow\emptyset = \emptyset$. $\square$

Theorem 2. $A \subseteq \uparrow A$

Proof. This follows from reflexivity — for every $a \in A$, one has $a \leq a$, hence $a \in \uparrow A$. $\square$

Theorem 3. $\uparrow\uparrow A = \uparrow A$

Proof. By the previous result, $\uparrow A \subseteq \uparrow\uparrow A$. Hence, it only remains to show that $\uparrow\uparrow A \subseteq \uparrow A$. This follows from transitivity. In order for some a to be an element of $\uparrow\uparrow A$, there must exist b and c such that $a \geq b \geq C$ and $C \in A$. By transitivity, $A \geq C$, so $a \in \uparrow A$, hence $\uparrow\uparrow A \subseteq \uparrow A$ as well. $\square$

Theorem 4. *If A and B are subsets of a partially ordered set, then*

$$\uparrow(A \cup B) = (\uparrow A) \cup (\uparrow B)$$

Proof. On the one hand, if $a \in \uparrow(A \cup B)$, then $a \geq b$ for some $b \in A \cup B$. It then follows that either $b \in A$ or $b \in B$. In the former case, $a \in \uparrow A$, in the latter case, $a \in \uparrow B$ so, either way $a \in (\uparrow A) \cup (\uparrow B)$. Hence $\uparrow(A \cup B) \subseteq (\uparrow A) \cup (\uparrow B)$.

On the other hand, if $a \in (\uparrow A) \cup (\uparrow B)$, then either $a \in (\uparrow A)$ or $a \in (\uparrow B)$. In the former case, there exists b such that $a \geq b$ and $b \in A$. Since $A \subseteq A \cup B$, we also have $b \in A \cup B$, hence $a \in \uparrow(A \cup B)$. Likewise, in the second case, we also conclude that $a \in \uparrow(A \cup B)$. Therefore, we have $(\uparrow A) \cup (\uparrow B) \subseteq \uparrow(A \cup B)$. $\square$

Theorem 5. $\uparrow P = P$

Theorem 6. $A \subseteq B$*,* $\uparrow A \subseteq \uparrow B$