# Birkhoff prime ideal theorem

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**Birkhoff Prime Ideal Theorem**. Let $L$ be a distributive lattice and $I$ a proper lattice ideal of $L$. Pick any element $a \notin I$. Then there is a prime ideal $P$ in $L$ such that $I \subseteq P$ and $a \notin P$.

*Proof.* If $I$ is prime, then we are done. Let $S := \{J \mid J \text{ is an ideal in } L, \text{ and } a \notin J\}$. Then $I \in S$. Order $S$ by inclusion. This turns $S$ into a poset. Let $C$ be a chain in $S$. Let $K = \bigcup C$. If $x, y \in K$, then $x \in J_1$ and $y \in J_2$ for some ideals $J_1, J_2 \in C$. Since $C$ is a chain, we may assume that $J_1 \subseteq J_2$, so that $x \in J_2$ as well. This means $x \vee y \in J_2 \subseteq K$. Next, assume $x \in K$ and $y \leq x$. Then $x \in J$ for some ideal $J \in C$, so that $y \in J \subseteq K$ also. This shows that $K$ is an ideal. If $a \in K$, then $a \in J$ for some $J \in C \subseteq S$, contradicting the definition of $S$. So $a \notin K$ and $K \in S$ also. This shows that every chain in $S$ has an upper bound. We can now appeal to Zorn's lemma, and conclude that $S$ has a maximal element, say $P$.

We now want to show that $P$ is the candidate that we are seeking: $P$ is a prime ideal in $L$ and $a \notin P$. Since $P \in S$, $P$ is an ideal such that $a \notin P$. So the only thing left to prove is that $P$ is prime. This amounts to showing that if $x \wedge y \in P$, then $x \in P$ or $y \in P$. Suppose not: $x, y \notin P$. Let $Q_1$ be the ideal generated by elements of $P$ and $x$, and $Q_2$ the ideal generated by $P$ and $y$. Since $Q_1$ and $Q_2$ properly contain $P$, $a \in Q_1$ and $a \in Q_2$. Write $a \leq p_1 \vee x$ and $a \leq p_2 \vee y$, where $p_1, p_2 \in P$. Then $a \vee p_2 \leq (p_1 \vee p_2) \vee x$ and $a \vee p_1 \leq (p_1 \vee p_2) \vee y$. Take the meet of these two expressions, and we obtain $(a \vee p_2) \wedge (a \vee p_1) \leq ((p_1 \vee p_2) \vee x) \wedge ((p_1 \vee p_2) \vee y)$. Since $L$ is distributive, on the left hand side, we get $a \vee (p_1 \wedge p_2)$. On the right hand side, we have $(p_1 \vee p_2) \vee (x \wedge y) \in P$. As the left hand side is less than or equal to the right hand side, we get that $a \vee (p_1 \wedge p_2) \in P$. Since $a \leq a \vee (p_1 \wedge p_2) \in P$, $a \in P$, a contradiction. Therefore, $P$ is prime and the proof is complete. □

In the proof, we use the fact that, an element $a \in L$ belongs to the ideal generated by ideals $I_k$ iff $a$ is less than or equal to a finite join of elements, each of which belongs to some $I_k$.

**Remarks**.

1. The theorem can be generalized: if we use a subset $S \cap I = \varnothing$ instead of an element $a \notin I$, there is a prime ideal $P$ containing $I$ but excluding $S$.

2. Birkhoff's prime ideal theorem has been shown to be equivalent to the axiom of choice, under ZF.