compact element

Canonical name	CompactElement
Date of creation	2013-03-22 15:52:50
Last modified on	2013-03-22 15:52:50
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	17
Author	CWoo (3771)
Entry type	Definition
Classification	msc 06B23
Synonym	finite element

Let X be a set and $\mathcal{T}$ be a topology on X, a well-known concept is that of a compact set: a set A is compact if every open cover of A has a finite subcover. Another way of putting this, symbolically, is that if

$$A \subseteq \bigcup \mathcal{S},$$

where $\mathcal{S} \subset \mathcal{T}$, then there is a finite subset $\mathcal{F}$ of $\mathcal{S}$, such that

$$A \subseteq \bigcup \mathcal{F}.$$

A more general concept, derived from above, is that of a *compact element* in a lattice. Let L be a lattice and $a \in L$. Then a is said to be *compact* if

> *whenever a subset S of L such that $\bigvee S$ exists and $a \leq \bigvee S$, then there is a finite subset $F \subset S$ such that $a \leq \bigvee F$.*

If we let $\mathcal{D}$ to be the collection of closed subsets of X, and partial order $\mathcal{D}$ by inclusion, then $\mathcal{D}$ becomes a lattice with meet and join defined by set theoretic intersection and union. It is easy to see that an element $A \in \mathcal{D}$ is a compact element iff D is a compact closed subset in X.

Here are some other common examples:

1. Let C be a set and 2^C the subset lattice (power set) of C. The compact elements of 2^C are the finite subsets of C.

2. Let V be a vector space and $L(V)$ be the subspace lattice of V. Then the compact elements of $L(V)$ are exactly the finite dimensional subspaces of V.

3. Let G be a group and $L(G)$ the subgroup lattice of G. Then the compact elements are the finitely generated subgroups of G.

4. Note in all of the above examples, atoms are compact. However, this is not true in general. Let's construct one such example. Adjoin the symbol ∞ to the lattice $\mathbb{N}$ of natural numbers (with linear order), so that $n < \infty$ for all $n \in \mathbb{N}$. So ∞ is the top element of $\mathbb{N} \cup \{\infty\}$ (and 1 is the bottom element!). Next, adjoin a symbol a to $\mathbb{N} \cup \{\infty\}$, and define the meet and join properties with a by

   - $a \vee n = \infty$, $a \wedge n = 1$ for all $n \in \mathbb{N}$, and

- $a \vee \infty = \infty$, $a \wedge \infty = a$.

The resulting set $L = \mathbb{N} \cup \{\infty, a\}$ is a lattice where a is a non-compact atom.

Remarks.

- As we have seen from the examples above, compactness is closely associated with the concept of finiteness, a compact element is sometimes called a *finite element*.
- Any finite join of compact elements is compact.
- An element a in a lattice L is compact iff for any `http://planetmath.org/DirectedSet`direct subset D of L such that $\bigvee D$ exists and $a \le \bigvee D$, then there is an element $d \in D$ such that $a \le d$.
- As the last example indicates, not all atoms are compact. However, in an algebraic lattice, atoms are compact. The first three examples are all instances of algebraic lattices.
- A compact element may be defined in an arbitrary poset P: $a \in P$ is compact iff a is way below itself: $a \ll a$.

References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge (2003).