partition lattice

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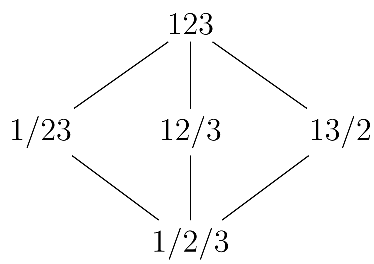
The *partition lattice* (or *lattice of partitions*) Π_n is the lattice of `http://planetmath.org/Parti` partitions of the set $[n] = \{1, \ldots, n\}$. The partial order on Π_n is defined by refinement, setting $x \leq y$ if any only if each cell of x is contained in a cell of y.

If $n < 3$, then Π_n is a chain. But Π_3 is not even a distributive lattice:



Moreover, the lattice Π_n is an interval in the lattice Π_{n+1}, so the lattice of partitions on $[n]$ is distributive only if $n < 3$. On the other hand, it is always a graded poset with rank function $\rho(x) = n - |x|$, where $|x|$ is the number of cells in x.

Each partition of $[n]$ has a corresponding Young tableau. To determine the Young tableau corresponding to a partition, we arrange the cells of the partition in order of decreasing size, breaking ties by allowing cells with smaller minimal elements to come first. The shape of the tableau is determined by the sizes of the cells, and the labels for the boxes come from the sets.

To illustrate the process of associating a partition with a tableau, we perform it for the partition $\{\{1\}, \{2, 3\}, \{4\}, \{5, 6, 7\}, \{8, 9\}\} = 1/23/4/567/89$ of $[9]$. There is one cell of size 3, namely, 567. There are two cells of size 2, 23 and 89. To order them we compare their minimal elements. Since $2 < 8$, we list 23 before 89. Similarly, we list 1 before 4. After sorting we have rewritten the partition as 567/23/89/1/4. Thus our tableau will have shape $(3, 2, 2, 1, 1)$. Labeling the shape gives us the following Young tableau.

5	6	7
2	3	
8	9	
1		
4		

References

[1] Stanley, R., *Enumerative Combinatorics*, vol. 1, 2nd ed., Cambridge University Press, Cambridge, 1996.