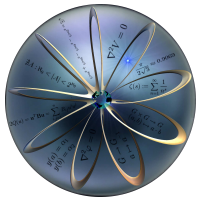

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rational rank of a group

Canonical name	RationalRankOfAGroup
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Entry type	Definition
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Defines	rationally independent
Defines	rational rank
Defines	divisible hull

In the following, G is an abelian group.

Definition 1. *The group $\mathbb{Q} \otimes_{\mathbb{Z}} G$ is called the divisible hull of G.*

It is a $\mathbb{Q}$-vector space such that the scalar $\mathbb{Z}$-multiplication of G is extended to $\mathbb{Q}$.

Definition 2. *The elements $g_1, g_2, ..., g_r \in G$ are called rationally independent if they are linearly independent over $\mathbb{Z}$, i.e. for all $n_1, ..., n_r \in Z$:*

$$n_1 g_1 + ... + n_r g_r = 0 \Rightarrow n_1 = ... = n_r = 0.$$

Definition 3. *The dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} G$ over $\mathbb{Q}$ is called the rational rank of G.*

We denote the rational rank of G by $r(G)$.
Example:
$r(\mathbb{Z} \times \mathbb{Z}) = 2$ because $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}) = (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}) \times (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}) = \mathbb{Q} \times \mathbb{Q}$.
Properties:

- If H is a subgroup of G then we have:

  $$r(G) = r(H) + r(G/H).$$

  It results from the fact that ${}_{\mathbb{Z}}\mathbb{Q}$ is a flat module.

- The rational rank of the group G can be defined as the least upper bound (finite or infinite) of the cardinals r such that there exist r rationally independent elements in G.