Boolean prime ideal theorem

Canonical name	BooleanPrimeIdealTheorem
Date of creation	2013-03-22 18:45:52
Last modified on	2013-03-22 18:45:52
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Theorem
Classification	msc 06E05
Classification	msc 03G05
Classification	msc 03E25
Synonym	BPI

Let A be a Boolean algebra. Recall that an ideal I of A if it is closed under $\vee$, and for any $a \in I$ and $b \in A$, $a \wedge b \in I$. I is proper if $I \neq A$ and non-trivial if $I \neq (0)$, and I is prime if it is proper, and, given $a \wedge b \in I$, either $a \in I$ or $b \in I$.

Theorem 1 (Boolean prime ideal theorem)**.** *Every Boolean algebra contains a prime ideal.*

Proof. Let A be a Boolean algebra. If A is trivial (the two-element algebra), then (0) is the prime ideal we want. Otherwise, pick $a \in A$, where $0 \neq a \neq 1$, and let be the trivial ideal. By Birkhoff's prime ideal theorem for distributive lattices, A, considered as a distributive lattice, has a prime ideal P (containing (0) obviously) such that $a \notin P$. Then P is also a prime ideal of A considered as a Boolean algebra. $\square$

There are several equivalent versions of the Boolean prime ideal theorem, some are listed below:

1. Every Boolean algebra has a prime ideal.
2. Every ideal in a Boolean algebra can be enlarged to a prime ideal.
3. Given a set S in a Boolean algebra A, and an ideal I disjoint from S, then there is a prime ideal P containing I and disjoint from S.
4. An ideal and a filter in a Boolean algebra, disjoint from one another, can be enlarged to an ideal and a filter that are complement (as sets) of one another.

Remarks.

1. Because the Boolean prime ideal theorem has been extensively studied, it is often abbreviated in the literature as BPI. Since the prime ideal theorem for distributive lattices uses the axiom of choice, ZF+AC implies BPI. However, there are models of ZF+BPI where AC fails.
2. It can be shown (see John Bell's online article `http://plato.stanford.edu/entries/axiom-` that BPI is equivalent, under ZF, to some of the well known theorems in mathematics:
   - Tychonoff's theorem for Hausdorff spaces: the product of compact Hausdorff spaces is compact under the product topology,

- the Stone representation theorem,
- the http://planetmath.org/CompactnessTheoremForFirstOrderLogiccompactness theorem for first order logic, and
- the completeness theorem for first order logic.

References

[1] G. Grätzer, *General Lattice Theory*, 2nd Edition, Birkhäuser (1998).

[2] T. J. Jech, *The Axiom of Choice*, North-Holland Pub. Co., Amsterdam, (1973).

[3] R. Sikorski, *Boolean Algebras*, 2nd Edition, Springer-Verlag, New York (1964).