partially ordered group

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Related topic	OrderedGroup
Defines	directed group
Defines	positive element
Defines	positive cone
Defines	lattice ordered group
Defines	Archimedean partially ordered group
Defines	integrally closed group
Defines	integrally closed partially ordered group
Defines	partially ordered semigroup
Defines	lattice ordered semigroup
Defines	Archimedean

A *partially ordered group* is a group G that is a poset at the same time, such that if $a, b \in G$ and $a \leq b$, then

1. $ac \leq bc$, and
2. $ca \leq cb$,

for any $c \in G$. The two conditions are equivalent to the one condition $cad \leq cbd$ for all $c, d \in G$. A partially ordered group is also called a *po-group* for short.

Remarks.

- One of the immediate properties of a po-group is this: if $a \leq b$, then $b^{-1} \leq a^{-1}$. To see this, left multiply by the first inequality by a^{-1} on both sides to obtain $e \leq a^{-1}b$. Then right multiply the resulting inequality on both sides by b^{-1} to obtain the desired inequality: $b^{-1} \leq a^{-1}$.
- If can be seen that for every $a \in G$, the automorphisms $L_a, R_a : G \to G$ also preserve order, and hence are order automorphisms as well. For instance, if $b \leq c$, then $L_a(b) = ab \leq ac = L_a(c)$.
- A element a in a po-group G is said to be *positive* if $e \leq a$, where e is the identity element of G. The set of positive elements in G is called the *positive cone* of G.
- (special po-groups)
  1. A po-group whose underlying poset is a directed set is called a *directed group*.
     - If G is a directed group, then G is also a filtered set: if $a, b \in G$, then there is a $c \in G$ such that $a \leq c$ and $b \leq c$, so that $ac^{-1}b \leq a$ and $ac^{-1}b \leq b$ as well.
     - Also, if G is directed, then $G = \langle G^+ \rangle$: for any $x \in G$, let a be the upper bound of $\{x, e\}$ and let $b = ax^{-1}$. Then $e \leq b$ and $x = a^{-1}b \in \langle G^+ \rangle$.
  2. A po-group whose underlying poset is a lattice is called a *lattice ordered group*, or an *l-group*.
  3. If the partial order on a po-group G is a linear order, then G is called a totally ordered group, or simply an ordered group.

4. A po-group is said to be *Archimedean* if $a^n \leq b$ for all $n \in \mathbb{Z}$, then $a = e$. Equivalently, if $a \neq e$, then for any $b \in G$, there is some $n \in \mathbb{Z}$ such that $b < a^n$. This is a generalization of the Archimedean property on the reals: if $r \in \mathbb{R}$, then there is some $n \in \mathbb{N}$ such that $r < n$. To see this, pick $b = r$, and $a = 1$.

5. A po-group is said to be *integrally closed* if $a^n \leq b$ for all $n \geq 1$, then $a \leq e$. An integrally closed group is Archimedean: if $a^n \leq b$ for all $n \in \mathbb{Z}$, then $a \leq e$ and $e \leq b$. Since we also have $(a^{-1})^{-n} \leq b$ for all $n < 0$, this implies $a^{-1} \leq e$, or $e \leq a$. Hence $a = e$. In fact, an directed integrally closed group is an Abelian po-group.

- Since the definition above does not involve any specific group axioms, one can more generally introduce partial ordering on a semigroup in the same fashion. The result is called a partially ordered semigroup, or a po-semigroup for short. A *lattice ordered semigroup* is defined similarly.