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regular open algebra

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A *regular open algebra* is an algebraic system $\mathcal{A}$ whose universe is the set of all regular open sets in a topological space X, and whose operations are given by

1. a constant 1 such that $1 := X$,
2. a unary operation $'$ such that for any U, $U' := U^{\perp}$, where $U^{\perp}$ is the complement of the closure of U in X,
3. a binary operation $\wedge$ such that for any $U, V \in \mathcal{A}$, $U \wedge V := U \cap V$, and
4. a binary operation $\vee$ such that for any $U, V \in \mathcal{A}$, $U \vee V := (U \cup V)^{\perp\perp}$.

From the parent entry, all of the operations above are well-defined (that the result sets are regular open). Also, we have the following:

Theorem 1. *$\mathcal{A}$ is a Boolean algebra*

Proof. We break down the proof into steps:

1. $\mathcal{A}$ is a lattice. This amounts to verifying various laws on the operations:
   - (idempotency of $\vee$ and $\wedge$): Clearly, $U \wedge U = U$. Also, $U \vee U = (U \cup U)^{\perp\perp} = U^{\perp\perp} = U$, since U is regular open.
   - Commutativity of the binary operations are obvious.
   - The associativity of $\wedge$ is also obvious. The associativity of $\vee$ goes as follows: $U \vee (V \vee W) = (U \cup (V \cup W)^{\perp\perp})^{\perp\perp} = U^{\perp} \cap (V \cup W)^{\perp\perp\perp} = U^{\perp} \cap (V \cup W)^{\perp}$, since $V \cup W$ is open (which implies that $(V \cup W)^{\perp}$ is regular open). The last expression is equal to $U^{\perp} \cap (V^{\perp} \cap W^{\perp})$. Interchanging the roles of U and W, we obtain the equation $W \vee (V \vee U) = W^{\perp} \cap (V^{\perp} \cap U^{\perp})$, which is just $U^{\perp} \cap (V^{\perp} \cap W^{\perp})$, or $U \vee (V \vee W)$. The commutativity of $\vee$ completes the proof of the associativity of $\vee$.
   - Finally, we verify the absorption laws. First, $U \wedge (U \vee V) = U \cap (U \cup V)^{\perp\perp} = U^{\perp\perp} \cap (U \cup V)^{\perp\perp} = (U^{\perp} \cup (U \cup V)^{\perp})^{\perp} = (U^{\perp} \cup (U^{\perp} \cap V^{\perp})^{\perp} = (U^{\perp})^{\perp} = U$. Second, $U \vee (U \wedge V) = (U \cup (U \vee W)^{\perp\perp} = U^{\perp\perp} = U$.
2. $\mathcal{A}$ is complemented. First, it is easy to see that $\varnothing$ and X are the bottom and top elements of $\mathcal{A}$. Furthermore, for any $U \in \mathcal{A}$, $U \wedge U' = U \cap U^{\perp} = U \cap (X \setminus \overline{U}) \subseteq \overline{U} \cap (X \setminus \overline{U}) = \varnothing$. Finally, $U \vee U' = (U \cup U^{\perp})^{\perp\perp} = (U^{\perp} \cap U^{\perp\perp})^{\perp} = (U^{\perp} \cap U)^{\perp} = \varnothing^{\perp} = X$.

3. $\mathcal{A}$ is distributive. This can be easily proved once we show the following: for any open sets U, V:

$$(*) \qquad U^{\perp\perp} \cap V^{\perp\perp} = (U \cap V)^{\perp\perp}.$$

To begin, note that since $U \cap V \subseteq U$, and $^\perp$ is order reversing, $(U \cap V)^{\perp\perp} \subseteq U^{\perp\perp}$ by applying $^\perp$ twice. Do the same with V and take the intersection, we get one of the inclusions: $(U \cap V)^{\perp\perp} \subseteq U^{\perp\perp} \cap V^{\perp\perp}$. For the other inclusion, we first observe that

$$U \cap \overline{V} \subseteq \overline{U \cap V}.$$

If $x \in$ LHS, then $x \in U$ and for any open set W with $x \in W$, we have that $W \cap V \neq \varnothing$. In particular, $U \cap W$ is such an open set (for $x \in U \cap W$), so that $(U \cap W) \cap V \neq \varnothing$, or $W \cap (U \cap V) \neq \varnothing$. Since W is arbitrary, $x \in$ RHS. Now, apply the set complement, we have $(U \cap V)^\perp \subseteq U^{\complement} \cup V^\perp$. Applying $^\perp$ next we get $(U \cap V)^{\perp\perp}$ for the LHS, and $(U^{\complement} \cup V^\perp)^\perp = U^{\complement - \complement} \cap V^{\perp\perp} = U^{\complement\complement} \cap V^{\perp\perp} = U \cap V^{\perp\perp}$ for RHS, since $U^{\complement}$ is closed. As $^\perp$ reverses order, the new inclusion is

$$(**) \qquad U \cap V^{\perp\perp} \subseteq (U \cap V)^{\perp\perp}.$$

From this, a direct calculation shows $U^{\perp\perp} \cap V^{\perp\perp} \subseteq (U^{\perp\perp} \cap V)^{\perp\perp} \subseteq (U \cap V)^{\perp\perp\perp\perp} = (U \cap V)^{\perp\perp}$, noticing that the first and second inclusions use $(**)$ above (and the fact that $^{\perp\perp}$ preserves order), and the last equation uses the fact that for any open set W, $W^\perp$ is regular open. This proves the $(*)$.

Finally, to finish the proof, we only need to show one of two distributive laws, say, $U \wedge (V \vee W) = (U \wedge V) \vee (U \wedge W)$, for the other one follows from the use of the distributive inequalities. This we do be direct computation: $U \wedge (V \vee W) = U \cap (V \cup W)^{\perp\perp} = U^{\perp\perp} \cap (V \cup W)^{\perp\perp} = (U \cap (V \cup W))^{\perp\perp} = ((U \cap V) \cup (U \cap W))^{\perp\perp} = ((U \wedge V) \cup (U \wedge W))^{\perp\perp} = (U \wedge V) \vee (U \wedge W)$.

Since a complemented distributive lattice is Boolean, the proof is complete. $\square$

Theorem 2. *The subset $\mathcal{B}$ of all clopen sets in X forms a Boolean subalgebra of $\mathcal{A}$.*

Proof. Clearly, every clopen set is regular open. In addition, $1 \in \mathcal{B}$. If U is clopen, so is the complement of its closure, and hence $U' \in \mathcal{B}$. If U, V are clopen, so is their intersection $U \wedge V$. Similarly, $U \cup V$ is clopen, so that $U \vee V = U \cup V$ is clopen also. $\square$

Theorem 3. *In fact, $\mathcal{A}$ is a complete Boolean algebra. For an arbitrary subset $\mathcal{K}$ of A, the meet and join of $\mathcal{K}$ are $(\bigcap\{U \mid U \in \mathcal{K}\})^{\perp\perp}$ and $(\bigcup\{U \mid U \in \mathcal{K}\})^{\perp\perp}$ respectively.*

Proof. Let $V = (\bigcup\{U \mid U \in \mathcal{K}\})^{\perp\perp}$. For any $U \in \mathcal{K}$, $U \subseteq \bigcup\{U \mid U \in \mathcal{K}\}$ so that $U = U^{\perp\perp} = (\bigcup\{U \mid U \in \mathcal{K}\})^{\perp\perp} = V$. This shows that V is an upper bound of elements of $\mathcal{K}$. If W is another such upper bound, then $U \subseteq W$, so that $\bigcup\{U \mid U \in \mathcal{K}\} \subseteq W$, whence $V = (\bigcup\{U \mid U \in \mathcal{K}\})^{\perp\perp} \subseteq W^{\perp\perp} = W$. The infimum is proved similarly. $\square$

Theorem 4. *$\mathcal{A}$ is the smallest complete Boolean subalgebra of $P(X)$ extending $\mathcal{B}$.*

More to come...