De Morgan algebra

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A bounded distributive lattice L is called a *De Morgan algebra* if there exists a unary operator $\sim: L \to L$ such that

1. $\sim (\sim a) = a$ and
2. $\sim (a \vee b) = (\sim a) \wedge (\sim b)$.

From the definition, we have the following properties:

- $\sim$ is a bijection, since for any $a \in L$, $a = \sim (\sim a)$.
- $\sim (a \wedge b) = \sim [(\sim (\sim a)) \wedge (\sim (\sim b))] = \sim [\sim ((\sim a) \vee (\sim b))] = (\sim a) \vee (\sim b)$, which is the dual statement of (2) above. This, together with condition (2), are commonly known as the De Morgan's laws.
- $\sim 0 = \sim (0 \wedge (\sim a)) = (\sim 0) \vee a$ for all $a \in L$, so $\sim 0 = 1$. Dually, $\sim 1 = 0$. As a result, a De Morgan algebra is an Ockham algebra.
- $a \leq b$ iff $b = a \vee b$ iff $\sim b = \sim (a \vee b) = (\sim a) \wedge (\sim b)$ iff $\sim b \leq \sim a$.
- A Boolean algebra is always a De Morgan algebra, where the $\sim$ is the complementation operator $'$. The converse is not true. In general, $\sim a$ is not a complement of a (that is, $(\sim a) \wedge a \neq 0$ and $(\sim a) \vee a \neq 1$). Otherwise, L is a complemented lattice and consequently a Boolean algebra.

Furthermore, a Kleene algebra is, by definition, a De Morgan algebra. But the converse is false. For example, consider $L = \mathbf{n} \times \mathbf{n}$, where $\mathbf{n} = \{0, 1, \ldots, n\}$ is a chain with the usual ordering. Define $\sim$ on L by $\sim (a, b) = (n - b, n - a)$. Then $\sim^2 (a, b) = (a, b)$. The De Morgan's laws follow from the identity $n - (a \vee b) = (n - a) \wedge (n - b)$ applied to each of the two components. But L is not Kleene in general. Take $n = 3$, then $\sim (1, 2) = (1, 2)$ and $\sim (2, 1) = (2, 1)$. But $(1, 2) = \sim (1, 2) \wedge (1, 2)$ and $(2, 1) = \sim (2, 1) \vee (2, 1)$ are not comparable.

Next, for any $a, b \in L$, define $a - b := a \wedge (\sim b)$. Then $-$ is a binary operator. It has the following properties:

- $a - 0 = a \wedge (\sim 0) = a \wedge 1 = a$.
- $0 - a = 0 \wedge (\sim a) = 0$.
- $a - 1 = a \wedge (\sim 1) = a \wedge 0 = 0$.

- $1 - a = 1 \wedge (\sim a) = \sim a$.
- $(a - b) - c = (a \wedge (\sim b)) \wedge (\sim c) = a \wedge (\sim (b \vee c)) = a - (b \vee c)$.

Finally, we define for $a, b \in L$, $a + b = (a - b) \vee (b - a)$. This is again a binary operator, with the following properties:

- $a + b = b + a$. This is obvious by the symmetry in the definition of $+$.
- $a + 0 = a$. We have $a + 0 = (a - 0) \vee (0 - a) = a \vee 0 = a$.
- $a + 1 = \sim a$, since $a + 1 = (a - 1) \vee (1 - a) = 0 \vee (\sim a) = \sim a$. In particular $1 + 1 = 0$.
- $a + a = (a - a) \vee (a - a) = a - a = a \wedge (\sim a)$. If we define $2a := a + a$, then $a + 2a = (a - 2a) \vee (2a - a) = (a \wedge (a \vee (\sim a)) \vee ((a \wedge (\sim a) \wedge (\sim a)) = a \vee ((a \wedge (\sim a)) = a$.
- More generally, we have

$$\begin{aligned}
a + (a + b) &= (a - (a + b)) \vee ((a + b) - a) \\
&= (a - ((a - b) \vee (b - a))) \vee (((a - b) \vee (b - a)) - a) \\
&= (a \wedge (\sim a \vee b) \wedge (\sim b \vee a)) \vee (((\sim b \wedge a) \vee (\sim a \wedge b)) \wedge (\sim a)) \\
&= (a \wedge (\sim a \vee b)) \vee (((\sim b \wedge a) \wedge (\sim a)) \vee (\sim a \wedge b)) \\
&= ((a \wedge (\sim a \vee b)) \vee (\sim a \wedge b)) \vee (\sim b \wedge (\sim a \wedge a)) \\
&= ((\sim a \wedge a) \vee (a \wedge b) \vee (\sim a \wedge b)) \vee (\sim b \wedge 2a) \\
&= (2a \vee ((\sim a \wedge a) \vee b)) \vee (\sim b \wedge 2a) \\
&= (2a \vee (2a \vee b)) \vee (\sim b \wedge 2a) \\
&= (2a \vee b) \vee (\sim b \wedge 2a) \\
&= 2a \vee (\sim b \wedge 2a) \vee b \\
&= 2a \vee b.
\end{aligned}$$

Remark. Since a De Morgan algebra is an Ockham algebra, a morphism between any two objects in the category of De Morgan algebras behaves just like an Ockham algebra homomorphism: it preserves $\sim$.

References

[1] G. Grätzer, *General Lattice Theory*, 2nd Edition, Birkhäuser (1998)