representing a Boolean algebra by field of sets

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In this entry, we show that every Boolean algebra is isomorphic to a field of sets, originally noted by Stone in 1936. The bulk of the proof has actually been carried out in `http://planetmath.org/RepresentingADistributiveLatticeByRingOfSets` entry, which we briefly state:

> if L is a distributive lattice, and X the set of all prime ideals of L, then the map $F : L \to P(X)$ defined by $F(a) = \{P \mid a \notin P\}$ is an embedding.

Now, if L is a Boolean lattice, then every element $a \in L$ has a complement $a' \in L$. a' is in fact uniquely determined by a.

Proposition 1. *The embedding F above preserves $'$ in the following sense:*

$$F(a') = X - F(a).$$

Proof. $P \in F(a')$ iff $a' \notin P$ iff $a \in P$ iff $P \notin F(a)$ iff $P \in X - F(a)$. □

Theorem 1. *Every Boolean algebra is isomorphic to a field of sets.*

Proof. From what has been discussed so far, F is a Boolean algebra isomorphism between L and $F(L)$, which is a ring of sets first of all, and a field of sets, because $X - F(a) = F(a')$. □

Remark. There are at least two other ways to characterize a Boolean algebra as a field of sets: let L be a Boolean algebra:

- Every prime ideal is the kernel of a homomorphism into $\mathbf{2} := \{0, 1\}$, and vice versa. So for an element a to be not in a prime ideal P is the same as saying that $\phi(a) = 1$ for some homomorphism $\phi : L \to \mathbf{2}$. If we take Y to be the set of all homomorphisms from L to $\mathbf{2}$, and define $G : L \to P(Y)$ by $G(a) = \{\phi \mid \phi(a) = 1\}$, then it is easy to see that G is an embedding of L into $P(Y)$.

- Every prime ideal is a maximal ideal, and vice versa. Furthermore, P is maximal iff P' is an ultrafilter. So if we define Z to be the set of all ultrafilters of L, and set $H : L \to P(Z)$ by $H(a) = \{U \mid a \in U\}$, then it is easy to see that H is an embedding of L into $P(Z)$.

If we appropriately topologize the sets X, Y, or Z, then we have the content of the Stone representation theorem.