positivity in ordered ring

Canonical name	PositivityInOrderedRing
Date of creation	2013-03-22 14:46:40
Last modified on	2013-03-22 14:46:40
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	12
Author	pahio (2872)
Entry type	Theorem
Classification	msc 06F25
Classification	msc 12J15
Classification	msc 13J25
Related topic	PositiveCone
Related topic	TopicEntryOnRealNumbers

Theorem. If $(R, \leq)$ is an ordered ring, then it contains a subset R_+ having the following :

- R_+ is under ring addition and, supposing that the ring contains no zero divisors, also under ring multiplication.
- Every element r of R satisfies exactly one of the conditions (1) $r = 0$, (2) $r \in R_+$, (3) $-r \in R_+$.

Proof. We take $R_+ = \{r \in R : 0 < r\} = \{r \in R : 0 \leq r \wedge 0 \neq r\}$. Let $a, b \in R_+$. Then $0 < a$, $0 < b$, and therefore we have $0 < a+0 < a+b$, i.e. $a+b \in R_+$. If R has no zero-divisors, then also $ab \neq 0$ and $0 = a0 < ab$, i.e. $ab \in R_+$. Let r be an arbitrary non-zero element of R. Then we must have either $0 < r$ or $r < 0$ (not both) because R is totally ordered. The latter alternative gives that $0 = -r+r < -r+0 = -r$. The both cases that either $r \in R_+$ or $-r \in R_+$.