derived Boolean operations

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Defines	symmetric difference
Defines	conditional
Defines	biconditional

Recall that a Boolean algebra is an algebraic system A consisting of five operations:

1. two binary operations: the meet $\wedge$ and the join $\vee$,
2. one unary operation: the complementation $'$, and
3. two nullary operations (constants): 0 and 1.

From these operations, define the following "derived" operations (on A): for $a, b \in A$

1. (subtraction) $a - b := a \wedge b'$,
2. (symmetric difference or addition) $a \Delta b$ (or $a + b$):$= (a - b) \vee (b - a)$,
3. (conditional) $a \to b := (a - b)'$,
4. (biconditional) $a \leftrightarrow b := (a \to b) \wedge (b \to a)$, and
5. (Sheffer stroke) $a|b := a' \wedge b'$.

Notice that the operators $\to$ and $\leftrightarrow$ are dual of $-$ and Δ respectively.

It is evident that these derived operations (and indeed the entire theory of Boolean algebras) owe their existence to those operations and connectives that are found in logic and set theory, as the following table illustrates:

symbol \ operation	Boolean	Logic	
$\vee$ or $\cup$	join	logical or	
$\wedge$ or $\cap$	meet	logical and	int
$'$ or $\neg$ or $^\complement$	complement	logical not	cor
0	bottom element	falsity	en
1	top element	truth	u
$-$ or $\setminus$	subtraction		set
Δ or $+$	symmetric difference		`http://planetmath.org/Symm`
$\to$	conditional	implication	
$\leftrightarrow$	biconditional	logical equivalence	
$\mid$	Sheffer stroke	Sheffer stroke	

Some of the elementary properties of these derived Boolean operators are:

1. $a - 0 = a$ and $a - a = 0 - a = a - 1 = 0$,
2. $(A, +, \wedge, 0, 1)$ is a ring (a Boolean ring),
3. all Boolean operations can be defined in terms of the Sheffer stroke $|$.

The proofs of these properties mimic the proofs for the properties of the corresponding operators found in naive set theory and propositional logic, such as `http://planetmath.org/LogicalConnective`this entry.