partitions form a lattice

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Let S be a set. Let $\mathrm{Part}(S)$ be the set of all partitions on S. Since each partition is a cover of S, $\mathrm{Part}(S)$ is partially ordered by covering refinement relation, so that $P_1 \preceq P_2$ if for every $a \in P_1$, there is a $b \in P_2$ such that $a \subseteq b$. We say that a partition P is finer than a partition Q if $P \preceq Q$, and coarser than Q if $Q \preceq P$.

Proposition 1. $\mathrm{Part}(S)$ *is a complete lattice*

Proof. For any set $\mathcal{P}$ of partitions P_i of S, the intersection $\bigcap \mathcal{P}$ is a partition of S. Take the meet of P_i to be this intersection. Next, let $\mathcal{Q}$ be the set of those partitions of S such that each $Q \in \mathcal{Q}$ is coarser than each P_i. This set is non-empty because $S \times S \in \mathcal{Q}$. Take the meet P' of all these partitions which is again coarser than all partitions P_i. Define the join of P_i to be P' and the proof is complete. □

Remarks.

- The top element of $\mathrm{Part}(S)$ is $S \times S$ and the bottom is the diagonal relation on S.
- Correspondingly, the partition lattice of S also defines the *lattice of equivalence relations* Δ on S.
- Given a family $\{E_i \mid i \in I\}$ of equvialence relations on S, we can explicitly describe the join $E := \bigvee E_i$ of E_i, as follows: $a \equiv b \pmod{E}$ iff there is a finite sequence $a = c_1, \ldots, c_n = b$ such that

$$c_k \equiv c_{k+1} \pmod{E_{i(k)}} \qquad \text{for } k = 1, \ldots, n-1. \tag{1}$$

  It is easy to see this definition makes E an equivalence relation. To see that E is the supremum of the E_i, first note that each $E_i \leq E$. Suppose now F is an equivalence relation on S such that $E_i \leq F$ and $a \equiv b \pmod{E}$. Then we get a finite sequence c_k as described by (1) above, so $c_k \equiv c_{k+1} \pmod{F}$ for each $k \in \{1, \ldots, n-1\}$. Hence $a \equiv b \pmod{F}$ also.