derivation of properties of regular open set

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Owner	CWoo (3771)
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Recall that a subset A of a topological space X is regular open if it is equal to the interior of the closure of itself.

To facilitate further analysis of regular open sets, define the operation $^\perp$ as follows:

$$A^\perp := X - \overline{A}.$$

Some of the properties of $^\perp$ and regular openness are listed and derived:

1. For any $A \subseteq X$, $A^\perp$ is open. This is obvious.

2. $^\perp$ reverses inclusion. This is also obvious.

3. $\varnothing^\perp = X$ and $X^\perp = \varnothing$. This too is clear.

4. $A \cap A^\perp = \varnothing$, because $A \cap A^\perp \subseteq A \cap (X - A) = \varnothing$.

5. $A \cup A^\perp$ is dense in X, because $X = \overline{A} \cup A^\perp \subseteq \overline{A} \cup \overline{A^\perp} = \overline{A \cup A^\perp}$.

6. $A^\perp \cup B^\perp \subseteq (A \cap B)^\perp$. To see this, first note that $A \cap B \subseteq A$, so that $A^\perp \subseteq (A \cap B)^\perp$. Similarly, $A^\perp \subseteq (A \cap B)^\perp$. Take the union of the two inclusions and the result follows.

7. $A^\perp \cap B^\perp = (A \cup B)^\perp$. This can be verified by direct calculation:

   $$A^\perp \cap B^\perp = (X - \overline{A}) \cap (X - \overline{B}) = X - (\overline{A} \cup \overline{B}) = X - \overline{A \cup B} = (A \cup B)^\perp.$$

8. A is regular open iff $A = A^{\perp\perp}$. See the remark at the end of `http://planetmath.org/Derivat` entry.

9. If A is open, then $A^\perp$ is regular open.

   Proof. By the previous property, we want to show that $A^{\perp\perp\perp} = A^\perp$ if A is open. For notational convenience, let us write A^- for the closure of A and A^c for the complement of A. As $^\perp = ^{-c}$, the equation now becomes $A^{-c-c-c} = A^{-c}$ for any open set A.

   Since $A \subseteq A^-$ for any set, $A^{-c} \subseteq A^c$. This means $A^{-c-} \subseteq A^{c-}$. Since A is open, A^c is closed, so that $A^{c-} = A^c$. The last inclusion becomes $A^{-c-} \subseteq A^c$. Taking complement again, we have

   $$A \subseteq A^{-c-c}. \tag{1}$$

Since ${}^{\perp} = {}^{-c}$ reverses inclusion, we have $A^{-c-c-c} \subseteq A^{-c}$, which is one of the inclusions. On the other hand, the inclusion (1) above applies to *any* open set, and because A^{-c} is open, $A^{-c} \subseteq A^{-c-c-c}$, which is the other inclusion. $\square$

10. If A and B are regular open, then so is $A \cap B$.

    Proof. Since A, B are regular open, $(A \cap B)^{\perp\perp} = (A^{\perp\perp} \cap B^{\perp\perp})^{\perp\perp}$, which is equal to $(A^{\perp} \cup B^{\perp})^{\perp\perp\perp}$ by property 7 above. Since $A^{\perp} \cup B^{\perp}$ is open, the last expression becomes $(A^{\perp} \cup B^{\perp})^{\perp}$ by property 9, or $A \cap B$ by property 7 again. $\square$

Remark. All of the properties above can be dualized for regular closed sets. If fact, proving a property about regular closedness can be easily accomplished once we have the following:

$(*)$ A is regular open iff $X - A$ is regular closed.

Proof. Suppose first that A is regular open. Then $\overline{\mathrm{int}(X - A)} = \overline{X - \overline{A}} = X - \mathrm{int}(\overline{A}) = X - A$. The converse is proved similarly. $\square$

As a corollary, for example, we have: if A is closed, then $\overline{X - A}$ is regular closed.

Proof. If A is closed, then $X - A$ is open, so that $(X - A)^{\perp} = X - \overline{X - A}$ is regular open by property 9 above, which implies that $X - (X - A)^{\perp} = \overline{X - A}$ is regular closed by $(*)$. $\square$