properties of well-ordered sets

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The purpose of this entry is to collect properties of well-ordered sets. We denote all orderings uniformly by $\leq$. If you are interested in history, have a look at [**?**].

The following properties are easy to see:

- If A is a totally ordered set such that for every subset $B \subseteq A$ the set of all elements of A strictly greater than the elements of B,
$$B_> := \{a \in A \setminus B \mid b \leq a \text{ for all } b \in B\},$$
is either empty or has a least element, then A is well-ordered.

- Every subset of a well-ordered set is well-ordered.

- If A is well-ordered and B is a poset such that there is a bijective order morphism $\varphi\colon A \to B$, then B is well-ordered.

Now we define an important ingredient for understanding the structure of well-ordered sets.

Definition 1 (section)**.** Let A be well-ordered. Then for every $a \in A$ we define the *section* of a:
$$\widehat{a} := \{b \in A \mid b \leq a\}.$$

A section is also known as an *initial segment*. We denote the set of all sections of A by $\widehat{A}$. This set is ordered by inclusion.

Theorem 1. *Let A be a well-ordered set. Then the mapping $\widehat{\cdot}\colon A \to \widehat{A}$ defined by $a \mapsto \widehat{a}$ is a bijective order morphism. In particular, $\widehat{A}$ is well-ordered.*

Proof. Let $a, b \in A$ with $a \leq b$. Then $\widehat{a} \subseteq \widehat{b}$, so $\widehat{\cdot}$ is an order morphism. Now assume that $\widehat{a} = \widehat{b}$. If a didn't equal b, we would have $b \notin \widehat{a}$, leading to a contradiction. Therefore $\widehat{\cdot}$ is injective. Now let $C \in \widehat{A}$, then there exists a $c \in A$ such that $\widehat{c} = C$, so $\widehat{\cdot}$ is surjective. $\square$

Theorem 2. *Let A and B be well-ordered sets and $\varphi\colon A \to B$ a bijective order morphism. Then there exists a bijective order morphism $\widehat{\varphi}\colon \widehat{A} \to \widehat{B}$ such that for all $a \in A$*
$$\widehat{\varphi}(\widehat{a}) = \widehat{\varphi(a)}.$$

Proof. Setting $\widehat{\varphi}(\widehat{a}) := \widehat{\varphi(a)}$ is well-defined by Theorem **??**. The rest of the theorem follows since $\widehat{\cdot}$ and φ are bijective order morphisms. $\square$

Theorem 3. *Let A be a well-ordered set and $a \in A$ such that there is an injective order morphism $\varphi\colon A \to \widehat{a}$. Then $A = \widehat{a}$.*

Proof. The image of a section of A under φ has a maximal element which in turn defines a smaller section of A. We may therefore define the following two monotonically decreasing sequences of sets:

$$\begin{aligned} B_0 &:= \varphi(\widehat{a}), & A_0 &:= \text{section defined by maximal element of } B_0, \\ B_n &:= \varphi(A_{n-1}), & A_n &:= \text{section defined by maximal element of } B_n. \end{aligned}$$

Now $\widehat{A}$ is well-ordered, so the set defined by the elements of the sequence (A_n) has a minimal element, that is $A_N = A_{N+1}$ and hence $B_N = B_{N+1}$ for some sufficiently large N. Applying φ^{-1} $N+1$ times to the latter equation yields $\widehat{a} = A_0$, that is a is the maximal element of B, and thus of A. $\square$

Theorem 4. *Let A and B be well-ordered sets. Then there exists at most one bijective order morphism $\varphi\colon A \to B$.*

Proof. Let $\varphi, \psi\colon A \to B$ be two bijective order morphisms. Let $a \in A$ and set $b := \varphi(a)$ and $c := \psi(a)$. Then the restrictions $\varphi|_{\widehat{a}} : \widehat{a} \to \widehat{b}$ and $\psi|_{\widehat{a}} : \widehat{a} \to \widehat{c}$ are bijective order morphisms, so the restriction of $\psi\varphi^{-1}$ to $\widehat{b}$ is a bijective order morphism to $\widehat{c}$. Now either $\widehat{b} \subseteq \widehat{c}$ or $\widehat{c} \subseteq \widehat{b}$, so by Theorem **??** $\widehat{b} = \widehat{c}$, hence $b = c$ and thus $\varphi = \psi$. $\square$

Theorem 5. *Let A and B be well-ordered sets such that for every section $\widehat{a} \in \widehat{A}$ there is a bijective order morphism to a section $\widehat{b} \in \widehat{B}$ and vice-versa, then there is a bijective order morphism $\varphi\colon A \to B$.*

Proof. Let $a \in A$ and let $\widehat{b} \in \widehat{B}$ be a section such that there is a bijective order morphism $\psi_a\colon \widehat{a} \to \widehat{b}$. By Theorem **??**, $\widehat{b}$ is unique, and so is ψ_a by Theorem **??**. Defining $\varphi\colon A \to B$ by setting $\varphi(a) = b$ gives therefore a well-defined (by Theorem **??**) and injective order morphism. But φ is also surjective, since any $b \in B$ maps uniquely to A via $\widehat{b} \to \widehat{a}$, and back again by φ. $\square$

Theorem 6. *Let A and B be well-ordered sets. Then there is an injective order morphism $\iota\colon A \to B$ or $\iota\colon B \to A$. If ι cannot be chosen bijective, then it can at least be chosen such that its image is a section.*

Proof. Let $\widehat{A}_0$ be the set of sections of A from which there is an injective order morphism to B. If $\widehat{A}_0$ is the empty set, then B must be empty, since otherwise we could map the least element of A to B. If $\widehat{A}_0$ is not empty, we may consider the set $A_0 := \cap\widehat{A}_0$. If $A_0 = A$, nothing remains to be shown. Otherwise the set $A \setminus A_0$ is nonempty an hence has a least element $a_0 \in A$. By construction, there is no injective order morphism from $\widehat{a}_0$ to B, but there is an injective order morphism from $\varphi_a \colon \widehat{a} \to B$ for every element $a \in A$ which is strictly smaller than a_0. Now assume there is an element $b \in B$ such that there is no injective order morphism from $\widehat{b} \to A$. Then we can similarly construct a least element $b_0 \in B$ for which there is no injective order morphism $\widehat{b}_0 \to A$. Surely, b_0 is greater than all the elements from the images of the functions φ_a, but then there is a bijective order morphism from $\widehat{a}_0$ to $\widehat{b}_0$ by Theorem **??** which is a contradiction. Therefore, all sections of B and B itself map injectively and order-preserving to A. □

Theorem 7. *Let A be a well-ordered set and $B \subseteq A$ a nonempty subset. Then there is a bijective order morphism from B to one of the sets in $\widehat{A} \cup \{A\}$.*

Proof. The set B is well-ordered with respect to the order induced by A. Assume a bijective order morphism as stated by the theorem does not exist. Then, by virtue of Theorem **??**, there is an injective but not surjective order morphism $\iota \colon A \to B$ whose image is a section $\widehat{b} \in \widehat{B}$. The element b defines a section in $\widehat{A}$ which is identical to A by Theorem **??**. Thus ι is surjective which is a contradiction. □

References

[C] G. Cantor, Beiträge zur Begründung der transfiniten Mengenlehre (Zweiter Artikel), *Math. Ann.* 49, 207–246 (1897).