lattice of ideals

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Let R be a ring. Consider the set $L(R)$ of all left ideals of R. Order this set by inclusion, and we have a partially ordered set. In fact, we have the following:

Proposition 1. *$L(R)$ is a complete lattice.*

Proof. For any collection $S = \{J_i \mid i \in I\}$ of (left) ideals of R (I is an index set), define

$$\bigwedge S := \bigcap S \qquad \text{and} \qquad \bigvee S = \sum_i J_i,$$

the sum of ideals J_i. We assert that $\bigwedge S$ is the greatest lower bound of the J_i, and $\bigvee S$ the least upper bound of the J_i, and we show these facts separately

- First, $\bigwedge S$ is a left ideal of R: if $a, b \in \bigwedge S$, then $a, b \in J_i$ for all $i \in I$. Consequently, $a - b \in J_i$ and so $a - b \in \bigwedge S$. Furthermore, if $r \in R$, then $ra \in J_i$ for any $i \in I$, so $ra \in \bigwedge S$ also. Hence $\bigwedge S$ is a left ideal. By construction, $\bigwedge S$ is clearly contained in all of J_i, and is clearly the largest such ideal.

- For the second part, we want to show that $\bigvee S$ actually exists for arbitrary S. We know the existence of $\bigvee S$ if S is finite. Suppose now S is infinite. Define J to be the set of finite sums of elements of $\bigcup_i J_i$. If $a, b \in J$, then $a + b$, being a finite sum itself, clearly belongs to J. Also, $-a \in J$ as well, since the additive inverse of each of the additive components of a is an element of $\bigcup_i J_i$. Now, if $r \in R$, then $ra \in J$ too, since multiplying each additive component of a by r (on the left) lands back in $\bigcup_i J_i$. So J is a left ideal. It is evident that $J_i \subseteq J$. Also, if M is a left ideal containing each J_i, then any finite sum of elements of J_i must also be in M, hence $J \subseteq M$. This implies that J is the smallest ideal containing each of the J_i. Therefore S exists and is equal to J.

In summary, both $\bigvee S$ and $\bigwedge S$ are well-defined, and exist for finite S, so $L(R)$ is a lattice. Additionally, both operations work for arbitrary S, so $L(R)$ is complete. □

From the above proof, we see that the sum S of ideals J_i can be equivalently interpreted as

- the "ideal" of finite sums of the elements of J_i, or

- the "ideal" generated by (elements of) J_i, or

- the join of ideals J_i.

A special sublattice of $L(R)$ is the lattice of finitely generated ideals of R. It is not hard to see that this sublattice comprises precisely the compact elements in $L(R)$.

Looking more closely at the above proof, we also have the following:

Corollary 1. *$L(R)$ is an algebraic lattice.*

Proof. As we have already shown, $L(R)$ is a complete lattice. If J is any (left) ideal of R, by the previous remark, each J is the sum (or join) of ideals generated by individual elements of J. Since these ideals are principal ideals (generated by a single element), they are compact, and therefore $L(R)$ is algebraic. □

Remarks.

- One can easily reconstruct all of the above, if $L(R)$ is the set of *right ideals*, or even *two-sided ideals* of R. We may distinguish the three notions: $l.L(R), r.L(R)$, and $L(R)$ as the lattices of left, right, and two-sided ideals of R.

- When R is commutative, $l.L(R) = r.L(R) = L(R)$. Furthermore, it can also be shown that $L(R)$ has the additional structure of a quantale.

- There is also a related result on lattice theory: the set $\mathrm{Id}(L)$ of lattice ideals in a upper semilattice L with bottom 0 forms a complete lattice. For a proof of this, see `http://planetmath.org/IdealCompletionOfAPoset`this entry.

- However, the more general case is not true: the set of order ideals in a poset is a dcpo.