topological vector lattice

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A *topological vector lattice* V over $\mathbb{R}$ is

- a Hausdorff topological vector space over $\mathbb{R}$,
- a vector lattice, and
- *locally solid*. This means that there is a neighborhood base of 0 consisting of solid sets.

Proposition 1. *A topological vector lattice V is a topological lattice.*

Before proving this, we show the following equivalence on the continuity of various operations on a vector lattice V that is also a topological vector space.

Lemma 1. *Let V be a vector lattice and a topological vector space. The following are equivalent:*

1. *$\vee : V^2 \to V$ is continuous (simultaneously in both arguments)*
2. *$\wedge : V^2 \to V$ is continuous (simultaneously in both arguments)*
3. *${}^+ : V \to V$ given by $x^+ := x \vee 0$ is continuous*
4. *${}^- : V \to V$ given by $x^- := -x \vee 0$ is continuous*
5. *$|\cdot| : V \to V$ given by $|x| := -x \vee x$ is continuous*

Proof. ($1 \Leftrightarrow 2$). If $\vee$ is continuous, then $x \wedge y = x + y - x \vee y$ is continuous too, as $+$ and $-$ are both continuous under a topological vector space. This proof works in reverse too. ($1 \Rightarrow 3$), ($1 \Rightarrow 4$), and ($3 \Leftrightarrow 4$) are obvious. To see ($4 \Rightarrow 5$), we see that $|x| = x^+ + x^-$, since ${}^-$ is continuous, ${}^+$ is continuous also, so that $|\cdot|$ is continuous. To see ($5 \Rightarrow 4$), we use the identity $x = x^+ - x^-$, so that $|x| = (x + x^-) + x^-$, which implies $x^- = \frac{1}{2}(|x| - x)$ is continuous. Finally, ($3 \Rightarrow 1$) is given by $x \vee y = (x - y + y) \vee (0 + y) = (x - y) \vee 0 + y = (x - y)^+ + y$, which is continuous. □

In addition, we show an important inequality that is true on any vector lattice:

Lemma 2. *Let V be a vector lattice. Then $|a^+ - b^+| \leq |a - b|$ for any $a, b \in V$.*

Proof. $|a^+ - b^+| = (b^+ - a^+) \vee (a^+ - b^+) = (b \vee 0 - a \vee 0) \vee (a \vee 0 - b \vee 0)$. Next, $a \vee 0 - b \vee 0 = (b + (-a \wedge 0) \vee (-a \wedge 0) = ((b-a) \wedge b) \vee (-a \wedge 0)$ so that $|a^+ - b^+| = ((b-a) \wedge b) \vee (-a \wedge 0) \vee ((a-b) \wedge a) \vee (-b \wedge 0) \leq (b-a) \vee (-a \wedge 0) \vee (a-b) \vee (-b \wedge 0)$. Since $(b-a) \vee (a-b) = |a-b|$ and $a \vee 0$ are both in the positive cone of V, so is their sum, so that $0 \leq (b-a) \vee (a-b) + (a \vee 0) = (b-a) \vee (a-b) - (-a \wedge 0)$, which means that $(-a \wedge 0) \leq (b-a) \vee (a-b)$. Similarly, $(-b \wedge 0) \leq (b-a) \vee (a-b)$. Combining these two inequalities, we see that $|a^+ - b^+| \leq (b-a) \vee (-a \wedge 0) \vee (a-b) \vee (-b \wedge 0) \leq (b-a) \vee (a-b) = |a-b|$. □

We are now ready to prove the main assertion.

Proof. To show that V is a topological lattice, we need to show that the lattice operations meet $\wedge$ and join $\vee$ are continuous, which, by Lemma 1, is equivalent in showing, say, that $^+$ is continuous. Suppose N is a neighborhood base of 0 consisting of solid sets. We prove that $^+$ is continuous. This amounts to showing that if x is close to x_0, then x^+ is close to x_0^+, which is the same as saying that if $x - x_0$ is in a solid neighborhood U of 0 ($U \in N$), then so is $x^+ - x_0^+$ in U. Since $x - x_0 \in U$, $|x - x_0| \in U$. But $|x^+ - x_0^+| \leq |x - x_0|$ by Lemma 2, and U is solid, $x^+ - x_0^+ \in U$ as well, and therefore $^+$ is continuous. □

As a corollary, we have

Proposition 2. *A topological vector lattice is an ordered topological vector space.*

Proof. All we need to show is that the positive cone is a closed set. But the positive cone is defined as $\{x \mid 0 \leq x\} = \{x \mid x^- = 0\}$, which is closed since $^-$ is continuous, and the positive cone is the inverse image of a singleton, a closed set in $\mathbb{R}$. □