Galois connection

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The notion of a Galois connection has its root in Galois theory. By the http://planetmath.org/FundamentalTheoremOfGaloisTheoryfundamental theorem of Galois theory, there is a one-to-one correspondence between the intermediate fields between a field L and its subfield F (with appropriate conditions imposed on the extension L/F), and the subgroups of the Galois group $\operatorname{Gal}(L/F)$ such that the bijection is inclusion-reversing:

$$\operatorname{Gal}(L/F) \supseteq H \supseteq \langle e \rangle \quad \text{iff} \quad F \subseteq L^H \subseteq L, \text{ and}$$

$$F \subseteq K \subseteq L \quad \text{iff} \quad \operatorname{Gal}(L/F) \supseteq \operatorname{Gal}(L/K) \supseteq \langle e \rangle.$$

If the language of Galois theory is distilled from the above paragraph, what remains reduces to a more basic and general concept in the theory of ordered-sets:

Definition. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets. A *Galois connection* between $(P, \leq_P)$ and $(Q, \leq_Q)$ is a pair of functions $f := (f^*, f_*)$ with $f^* \colon P \to Q$ and $f_* \colon Q \to P$, such that, for all $p \in P$ and $q \in Q$, we have

$$f^*(p) \leq_Q q \quad \text{iff} \quad p \leq_P f_*(q).$$

We denote a Galois connection between P and Q by $P \overset{f}{\multimap} Q$, or simply $P \multimap Q$.

If we define $\leq'_P$ on P by $a \leq'_P b$ iff $b \leq_P a$, and define $\leq'_Q$ on Q by $c \leq'_Q d$ iff $d \leq_Q c$, then $(P, \leq'_P)$ and $(Q, \leq'_Q)$ are posets, (the duals of $(P, \leq_P)$ and $(Q, \leq_Q)$). The existence of a Galois connection between $(P, \leq_P)$ and $(Q, \leq_Q)$ is the same as the existence of a Galois connection between $(Q, \leq'_Q)$ and $(P, \leq'_P)$. In short, we say that there is a Galois connection between P and Q if there is a Galois connection between two posets S and T where P and Q are the underlying sets (of S and T respectively). With this, we may say without confusion that "a Galois connection exists between P and Q iff a Galois connection exists between Q and P".

Remarks.

1. Since $f^*(p) \leq_Q f^*(p)$ for all $p \in P$, then by definition, $p \leq_P f_* f^*(p)$. Alternatively, we can write

$$1_P \leq_P f_* f^*, \tag{1}$$

where 1_P stands for the identity map on P. Similarly, if 1_Q is the identity map on Q, then

$$f^* f_* \leq_Q 1_Q. \tag{2}$$

2. Suppose $a \leq_P b$. Since $b \leq_P f_* f^*(b)$ by the remark above, $a \leq_P f_* f^*(b)$ and so by definition, $f^*(a) \leq_Q f^*(b)$. This shows that f^* is monotone. Likewise, f_* is also monotone.

3. Now back to Inequality (1), $1_P \leq_P f_* f^*$ in the first remark. Applying the second remark, we obtain

$$f^* \leq_Q f^* f_* f^*. \tag{3}$$

   Next, according to Inequality (2), $f^* f_*(q) \leq_Q q$ for any $q \in Q$, it is true, in particular, when $q = f^*(p)$. Therefore, we also have

$$f^* f_* f^* \leq_Q f^*. \tag{4}$$

   Putting Inequalities (3) and (4) together we have

$$f^* f_* f^* = f^*. \tag{5}$$

   Similarly,

$$f_* f^* f_* = f_*. \tag{6}$$

4. If (f, g) and (f, h) are Galois connections between $(P, \leq_P)$ and $(Q, \leq_Q)$, then $g = h$. To see this, observe that $p \leq_P g(q)$ iff $f(p) \leq_Q q$ iff $p \leq_P h(q)$, for any $p \in P$ and $q \in Q$. In particular, setting $p = g(q)$, we get $g(q) \leq_P h(q)$ since $g(q) \leq_P g(q)$. Similarly, $h(q) \leq_P g(q)$, and therefore $g = h$. By a similarly argument, if (g, f) and (h, f) are Galois connections between $(P, \leq_P)$ and $(Q, \leq_Q)$, then $g = h$. Because of this uniqueness property, in a Galois connection $f = (f^*, f_*)$, f^* is called *the upper adjoint* of f_* and f_* *the lower adjoint* of f^*.

Examples.

- The most famous example is already mentioned in the first paragraph above: let L is a finite-dimensional Galois extension of a field F, and $G := \mathrm{Gal}(L/F)$ is the Galois group of L over F. If we define

   a. $P = \{K \mid K \text{ is a field such that } F \subseteq K \subseteq L\}$, with $\leq_P = \subseteq$,

   b. $Q = \{H \mid H \text{ is a subgroup of } G\}$, with $\leq_Q = \supseteq$,

   c. $f^* : P \to Q$ by $f^*(K) = \mathrm{Gal}(L/K)$, and

d. $f_* : Q \to P$ by $f_*(H) = L^H$, the fixed field of H in L.

Then, by the fundamental theorem of Galois theory, f^* and f_* are bijections, and (f^*, f_*) is a Galois connection between P and Q.

- Let X be a topological space. Define P be the set of all open subsets of X and Q the set of all closed subsets of X. Turn P and Q into posets with the usual set-theoretic inclusion. Next, define $f^* : P \to Q$ by $f^*(U) = \overline{U}$, the closure of U, and $f_* : Q \to P$ by $f_*(V) = \text{int}(V)$, the interior of V. Then (f^*, f_*) is a Galois connection between P and Q. Incidentally, those elements fixed by f_*f^* are precisely the regular open sets of X, and those fixed by f^*f_* are the regular closed sets.

Remark. The pair of functions in a Galois connection are order preserving as shown above. One may also define a Galois connection as a pair of maps $f^* : P \to Q$ and $f_* : Q \to P$ such that $f^*(p) \leq_Q q$ iff $f_*(q) \leq_P p$, so that the pair f^*, f_* are order reversing. In any case, the two definitions are equivalent in that one may go from one definition to another, (simply exchange Q with Q^∂, the `http://planetmath.org/DualPoset`dual of Q).

References

[1] T.S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer, New York (2005).

[2] B. A. Davey, H. A. Priestley, *Introduction to Lattices and Order*, 2nd Edition, Cambridge (2003)