sets that do not have an infimum

Canonical name	SetsThatDoNotHaveAnInfimum
Date of creation	2013-03-22 13:09:51
Last modified on	2013-03-22 13:09:51
Owner	sleske (997)
Last modified by	sleske (997)
Numerical id	9
Author	sleske (997)
Entry type	Example
Classification	msc 06A06
Related topic	InfimumAndSupremumForRealNumbers

Some examples for sets that do not have an infimum:

- The set $M_1 := \mathbb{Q}$ (as a subset of $\mathbb{Q}$) does not have an infimum (nor a supremum). Intuitively this is clear, as the set is unbounded. The (easy) formal proof is left as an exercise for the reader.

- A more interesting example: The set $M_2 := \{x \in \mathbb{Q} : x^2 \geq 2, x > 0\}$ (again as a subset of $\mathbb{Q}$) .

  Proof. Clearly, $\inf(M_2) > 0$. Assume $i > 0$ is an infimum of M_2. Now we use the fact that $\sqrt{2}$ is not rational, and therefore $i < \sqrt{2}$ or $i > \sqrt{2}$.

  If $i < \sqrt{2}$, choose any $j \in \mathbb{Q}$ from the interval $(i, \sqrt{2}) \subset \mathbb{R}$ (this is a real interval, but as the rational numbers are `http://planetmath.org/Dense`dense in the real numbers, every nonempty interval in $\mathbb{R}$ contains a rational number, hence such a j exists).

  Then $j > i$, but $j < \sqrt{2}$, hence $j^2 < 2$ and therefore j is a lower bound for M_2, which is a contradiction.

  On the other hand, if $i > \sqrt{2}$, the argument is very similar: Choose any $j \in \mathbb{Q}$ from the interval $(\sqrt{2}, i) \subset \mathbb{R}$. Then $j < i$, but $j > \sqrt{2}$, hence $j^2 > 2$ and therefore $j \in M_2$. Thus M_2 contains an element j smaller than i, which is a contradiction to the assumption that $i = \inf(M_2)$ $\square$

  Intuitively speaking, this example exploits the fact that $\mathbb{Q}$ does not have "enough elements". More formally, $\mathbb{Q}$ as a metric space is not `http://planetmath.org/Complete`complete. The M_2 defined above is the real interval $M_2' := (\sqrt{2}, \infty) \subset \mathbb{R}$ intersected with $\mathbb{Q}$. M_2' as a subset of $\mathbb{R}$ does have an infimum (namely $\sqrt{2}$), but as that is not an element of $\mathbb{Q}$, M_2 does not have an infimum as a subset of $\mathbb{Q}$.

  This example also makes it clear that it is important to clearly state the superset one is working in when using the notion of infimum or supremum.

  It also illustrates that the infimum is a natural generalization of the minimum of a set, as a set that does not have a minimum may still have an infimum (such as M_2').

  Of course all the ideas expressed here equally apply to the supremum, as the two notions are completely analogous (just reverse all inequalities).