regular open set

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Synonym	regularly closed set
Defines	regular open
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Let X be a topological space. A subset A of X is called a *regular open* set if A is equal to the interior of the closure of itself:

$$A = \mathrm{int}(\overline{A}).$$

Clearly, every regular open set is open, and every clopen set is regular open.

Examples. Let $\mathbb{R}$ be the real line with the usual topology (generated by open intervals).

- (a, b) is regular open whenever $-\infty < a \leq b < \infty$.
- $(a, b) \cup (b, c)$ is not regular open for $-\infty < a \leq b \leq c < \infty$ and $a \neq c$. The interior of the closure of $(a, b) \cup (b, c)$ is (a, c).

If we examine the structure of $\mathrm{int}(\overline{A})$ a little more closely, we see that if we define

$$A^{\perp} := X - \overline{A},$$

then

$$A^{\perp\perp} = \mathrm{int}(\overline{A}).$$

So an alternative definition of a regular open set is an open set A such that $A^{\perp\perp} = A$.

Remarks.

- For any $A \subseteq X$, $A^{\perp}$ is always open.
- $\varnothing^{\perp} = X$ and $X^{\perp} = \varnothing$.
- $A \cap A^{\perp} = \varnothing$ and $A \cup A^{\perp}$ is dense in X.
- $A^{\perp} \cup B^{\perp} \subseteq (A \cap B)^{\perp}$ and $A^{\perp} \cap B^{\perp} = (A \cup B)^{\perp}$.
- It can be shown that if A is open, then $A^{\perp}$ is regular open. As a result, following from the first property, $\mathrm{int}(\overline{A})$, being $A^{\perp\perp}$, is regular open for any subset A of X.
- In addition, if both A and B are regular open, then $A \cap B$ is regular open.
- It is not true, however, that the union of two regular open sets is regular open, as illustrated by the second example above.

- It can also be shown that the set of all regular open sets of a topological space X forms a Boolean algebra under the following set of operations:

  1. $1 = X$ and $0 = \varnothing$,
  2. $a \wedge b = a \cap b$,
  3. $a \vee b = (a \cup b)^{\perp\perp}$, and
  4. $a' = a^{\perp}$.

  This is an example of a Boolean algebra coming from a collection of subsets of a set that is not formed by the standard set operations union $\cup$, intersection $\cap$, and complementation $'$.

The definition of a regular open set can be dualized. A closed set A in a topological space is called a *regular closed set* if $A = \overline{\mathrm{int}(A)}$.

References

[1] P. Halmos (1970). *Lectures on Boolean Algebras*, Springer.

[2] S. Willard (1970). *General Topology*, Addison-Wesley Publishing Company.