Scott continuous

Canonical name	ScottContinuous
Date of creation	2013-03-22 16:49:53
Last modified on	2013-03-22 16:49:53
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	6
Author	CWoo (3771)
Entry type	Definition
Classification	msc 06B35

Let P_1, P_2 be two http://planetmath.org/Dcpodcpos. A function $f: P_1 \to P_2$ is said to be *Scott continuous* if for any directed set $D \subseteq P_1$, $f(\bigvee D) = \bigvee f(D)$.

First, observe that f is monotone. If $a \leq b$, then $f(b) = f(\bigvee\{a,b\}) = \bigvee\{f(a), f(b)\}$, so that $f(a) \leq f(b)$. As a result, if D is directed, so is $f(D)$.

Proposition 1. *$f: P_1 \to P_2$ is Scott continuous iff it is continuous when P_1 and P_2 are equipped with the Scott topologies.*

Before proving this, let's make one additional observation:

Lemma 1. *If f is continuous (under Scott topologies), then f is monotone.*

Proof. Suppose $a \leq b \in P_1$. We wish to show that $f(a) \leq f(b)$, or $f(a) \in \downarrow f(b)$. Assume the contrary. Consider $U = P_2 - \downarrow f(b)$. Then $f(a) \in U$ and U is Scott open, hence $a \in f^{-1}(U)$ is Scott open also. Since $a \leq b$ and $f^{-1}(U)$ is upper, $b \in f^{-1}(U)$, which implies $f(b) \in U = P_2 - \downarrow f(b)$, a contradiction. Therefore, $f(a) \leq f(b)$. □

Now the proof of the proposition.

Proof. Suppose first that f is Scott continuous. Take an open set $U \in P_2$. We want to show that $V := f^{-1}(U)$ is open in P_1. In other words, V is upper and that V has non-empty intersection with any directed set $D \in P_1$ whenever its supremum $\bigvee D$ lies in V. If $a \in \uparrow V$, then some $b \in V$ with $b \leq a$, which implies $f(b) \leq f(a)$. Since $f(b) \in U$, $f(a) \in \uparrow U = U$, so $a \in f^{-1}(U) = V$, V is upper. Now, suppose $\bigvee D \in V$. So $\bigvee f(D) = f(\bigvee D) \in U$. Since $f(D)$ is directed, there is $y \in f(D) \cap U$, which means there is $x \in P_1$ such that $f(x) = y$ and $x \in D \cap V$. This shows that V is Scott open.

Conversely, suppose f is continuous (inverse of a Scott open set is Scott open). Let D be a directed subset of P_1 and let $d = \bigvee D$. We want to show that $f(d) = \bigvee f(D)$. First, for any $e \in D$, we have that $e \leq d$ so that $f(e) \leq f(d)$ since f is monotone. This shows $\bigvee f(D) \leq f(d)$. Now suppose r is any upper bound of $f(D)$. We want to show that $f(d) \leq r$, or $f(d) \in \downarrow r$. Assume not. Then $f(d)$ lies in $U := P_2 - \downarrow r$, a Scott open set. So $\bigvee D = d \in f^{-1}(U)$, also Scott open, which implies some $e \in D$ with $e \in f^{-1}(U)$, or $f(e) \in U$. This means $f(e) \not\leq r$, a contradiction. Thus $f(d) \leq r$, and the proof is complete. □

Remark. This notion of continuity is attributed to Dana Scott when he was trying to come up with a model for the formal system of untyped lambda calculus.

References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge (2003).