Boolean ring

Canonical name	BooleanRing
Date of creation	2013-03-22 12:27:28
Last modified on	2013-03-22 12:27:28
Owner	yark (2760)
Last modified by	yark (2760)
Numerical id	24
Author	yark (2760)
Entry type	Definition
Classification	msc 06E99
Classification	msc 03G05
Related topic	Idempotency
Related topic	BooleanLattice
Related topic	BooleanIdeal

A *Boolean ring* is a ring R that has a multiplicative identity, and in which every element is idempotent, that is,

$$x^2 = x \text{ for all } x \in R.$$

Boolean rings are necessarily http://planetmath.org/CommutativeRingcommutative. Also, if R is a Boolean ring, then $x = -x$ for each $x \in R$.

Boolean rings are equivalent to Boolean algebras (or http://planetmath.org/BooleanLattice lattices). Given a Boolean ring R, define $x \wedge y = xy$ and $x \vee y = x + y + xy$ and $x' = x + 1$ for all $x, y \in R$, then $(R, \wedge, \vee, ', 0, 1)$ is a Boolean algebra. Given a Boolean algebra $(L, \wedge, \vee, ', 0, 1)$, define $x \cdot y = x \wedge y$ and $x + y = (x' \wedge y) \vee (x \wedge y')$, then $(L, \cdot, +)$ is a Boolean ring. In particular, the category of Boolean rings is isomorphic to the category of Boolean lattices.

Examples

As mentioned above, every Boolean algebra can be considered as a Boolean ring. In particular, if X is any set, then the power set $\mathcal{P}(X)$ forms a Boolean ring, with intersection as multiplication and symmetric difference as addition.

Let R be the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the operations being coordinate-wise. Then we can check:

$$\begin{aligned}
(1,1) \times (1,1) &= (1,1) \\
(1,0) \times (1,0) &= (1,0) \\
(0,1) \times (0,1) &= (0,1) \\
(0,0) \times (0,0) &= (0,0)
\end{aligned}$$

the four elements that form the ring are idempotent. So R is Boolean.