Boolean quotient algebra

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Quotient Algebras via Congruences

Let A be a Boolean algebra. A congruence on A is an equivalence relation Q on A such that Q respects the Boolean operations:

- if aQb and cQd, then $(a \vee c)Q(b \vee d)$
- if aQb, then $a'Qb'$

By de Morgan's laws, we also have aQb and cQd implying $(a \wedge c)Q(b \wedge d)$.

When a is congruent to b, we usually write $a \equiv b \pmod{Q}$.

Let B be the set of congruence classes: $B = A/Q$, and write $[a]Q$, or simply $[a]$ for the congruence class containing the element $a \in A$. Define on B the following operations:

- $[a] \vee [b] := [a \vee b]$
- $[a]' := [a']$

Because Q respects join and complementation, it is clear that these are well-defined operations on B. Furthermore, we may define $[a] \wedge [b] := ([a]' \vee [b]')' = ([a'] \vee [b'])' = [a' \vee b']' = [(a' \vee b')'] = [a \wedge b]$. It is also easy to see that $[1]$ and $[0]$ are the top and bottom elements of B. Finally, it is straightforward to verify that B is a Boolean algebra. The algebra B is called the *Boolean quotient algebra* of A via the congruence Q.

Quotient Algebras via Ideals and Filters

It is also possible to define quotient algebras via Boolean ideals and Boolean filters. Let A be a Boolean algebra and I an ideal of A. Define binary relation $\sim$ on A as follows:

$$a \sim b \qquad \text{if and only if} \qquad a \Delta b \in I,$$

where Δ is the symmetric difference operator on A. Then

1. $\sim$ is an equivalence on A, because
   - $a \Delta a = 0 \in I$, so $\sim$ is reflexive
   - $b \Delta a = a \Delta b$, so $\sim$ is symmetric, and

- if $a \sim b$ and $b \sim c$, then $a \sim c$; to see this, note that $(a-b) \vee (b-c) = ((a-b) \vee b) \wedge ((a-b) \vee c') = (a \vee b) \wedge ((a-b) \vee c')$. Since the LHS (and hence the RHS) is in I, and that $a \leq a \vee b$ and $c' \leq (a-b) \vee c'$, RHS $\geq a \wedge c' = a - c \in I$ too. Similarly $c - a \in I$ so that $a \sim c$.

2. $\sim$ respects $\vee$ and $'$, because

   - if $a \sim b$ and $c \sim d$, then $(a \vee c) - (b \vee d) = (a \vee c) \wedge (b \vee d)' = (a \vee c) \wedge (b' \wedge d') = (a \wedge (b' \wedge d')) \vee (c \wedge (b' \wedge d')) \leq (a \wedge b') \vee (c \wedge d') \in I$, so that $(a \vee c) - (b \vee d) \in I$ as well. That $(b \vee d) - (a \vee c) \in I$ is proved similarly. Hence $(a \vee c) \sim (b \vee d)$.
   - $a' \Delta b' = a \Delta b$, so $\sim$ preserves $'$.

Thus, $\sim$ is a congruence on A. The quotient algebra $A/\sim$ is called the *quotient algebra* of A via the ideal I, and is often denoted by A/I.

From this congruence $\sim$, one can re-capture the ideal: $I = [0]$.

Dually, one can obtain a quotient algebra from a Boolean filter. Specifically, if F is a filter of a Boolean algebra A, define $\sim$ on A as follows:

$$a \sim b \qquad \text{if and only if} \qquad a \leftrightarrow b \in F,$$

where $\leftrightarrow$ is the biconditional operator on A. Then it is easy to show that $\sim$ too is a congruence on A, so that one forms the *quotient algebra* of A via the filter F, denoted by A/F. Of course, an easier approach to this is to realize that F is a filter of A iff $F' := \{a' \mid a \in F\}$ is an ideal of A, and the process of forming A/F' turns out to be identical to A/F.

From $\sim$, the filter F can be recovered: $F = [1]$.

In fact, given a congruence Q, the congruence class $[0]Q$ is a Boolean ideal and the congruence class $[1]Q$ is a Boolean filter, and that the quotient algebras derived from Q, $[0]Q$ and $[1]Q$ are all the same:

$$A/Q = A/[0]Q = A/[1]Q.$$