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homeomorphism between Boolean spaces

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In this entry, we derive a test for deciding when a bijection between two Boolean spaces is a homeomorphism.

We start with two general remarks.

Lemma 1. *If Y is zero-dimensional, then $f : X \to Y$ is continuous provided that $f^{-1}(U)$ is open for every clopen set U in Y.*

Proof. Since Y is zero-dimensional, Y has a basis of clopen sets. To check the continuity of f, it is enough to check that $f^{-1}(U)$ is open for each member of the basis, which is true by assumption. Hence f is continuous. $\square$

Lemma 2. *If X is compact and Y is Hausdorff, and f is a bijection, then f is a homeomorphism iff it is continuous.*

Proof. One direction is obvious. We want to show that f^{-1} is continuous, or equivalently, for any closed set U in X, $f(U)$ is closed in Y. Since X is compact, U is compact, and therefore $f(U)$ is compact since f is continuous. But Y is Hausdorff, so $f(U)$ is closed. $\square$

Proposition 1. *If X, Y are Boolean spaces, then a bijection $f : X \to Y$ is homeomorphism iff it maps clopen sets to clopen sets.*

Proof. Once more, one direction is clear. Now, suppose f maps clopen sets to clopen sets. Since X is zero-dimensional, $f^{-1} : Y \to X$ is continuous by the first proposition. Since Y is compact and X Hausdorff, f^{-1} is a homeomorphism by the second proposition. $\square$