measure on a Boolean algebra

Canonical name	MeasureOnABooleanAlgebra
Date of creation	2013-03-22 17:59:16
Last modified on	2013-03-22 17:59:16
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	8
Author	CWoo (3771)
Entry type	Definition
Classification	msc 06B99
Related topic	Measure
Defines	measure
Defines	two-valued measure
Defines	finitely additive
Defines	countably additive

Let A be a Boolean algebra. A *measure* on A is a non-negative extended real-valued function m defined on A such that

1. there is an $a \in A$ such that $m(a)$ is a real number (not ∞),
2. if $a \wedge b = 0$, then $m(a \vee b) = m(a) + m(b)$.

For example, a sigma algebra $\mathcal{B}$ over a set E is a Boolean algebra, and a `http://planetmath.org/Measure`measure μ on the measurable space $(\mathcal{B}, E)$ is a measure on the Boolean algebra $\mathcal{B}$.

The following are some of the elementary properties of m:

- $m(0) = 0$.

  By condition 1, suppose $m(a) = r \in \mathbb{R}$, then $m(a) = m(0 \vee a) = m(0) + m(a)$, so that $m(0) = 0$.

- m is non-decreasing: $m(a) \leq m(b)$ for $a \leq b$

  If $a \leq b$, then $c = b - a$ and a are disjoint ($c \wedge a = 0$) and $b = c \vee a$. So $m(b) = m(c \vee a) = m(c) + m(a)$. As a result, $m(a) \leq m(b)$.

- m is subadditive: $m(a \vee b) \leq m(a) + m(b)$.

  Since $a \vee b = (a - b) \vee b$, and $a - b$ and b are disjoint, we have that $m(a \vee b) = m((a - b) \vee b) = m(a - b) + m(b)$. Since $a - b \leq a$, the result follows.

From the three properties above, one readily deduces that $I := \{a \in A \mid m(a) = 0\}$ is a Boolean ideal of A.

A measure on A is called a *two-valued measure* if m maps onto the two-element set $\{0, 1\}$. Because of the existence of an element $a \in A$ with $m(a) = 1$, it follows that $m(1) = 1$. Consequently, the set $F := \{a \in A \mid m(a) = 1\}$ is a Boolean filter. In fact, because m is two-valued, F is an ultrafilter (and correspondingly, the set $\{a \mid m(a) = 0\}$ is a maximal ideal).

Conversely, given an ultrafilter F of A, the function $m : A \to \{0, 1\}$, defined by $m(a) = 1$ iff $a \in F$, is a two-valued measure on A. To see this, suppose $a \wedge b = 0$. Then at least one of them, say a, can not be in F (or else $0 = a \wedge b \in F$). This means that $m(a) = 0$. If $b \in F$, then $a \vee b \in F$, so that $m(a \vee b) = 1 = m(b) = m(b) + m(a)$. On the other hand, if $b \notin F$, then $a', b' \in F$, so $a' \wedge b' \in F$, or $a \vee b \notin F$. This means that $m(a \vee b) = 0 = m(a) + m(b)$.

Remark. A measure (on a Boolean algebra) is sometimes called *finitely additive* to emphasize the defining condition 2 above. In addition, this terminology is used when there is a need to contrast a stronger form of additivity: *countable additivity*. A measure is said to be *countably additive* if whenever K is a countable set of pairwise disjoint elements in A such that $\bigvee K$ exists, then

$$m(\bigvee K) = \sum_{a \in K} m(a).$$