example of non-complete lattice homomorphism

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The real number line $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$ is complete in its usual ordering of numbers. Furthermore, the meet of a subset S of $\mathbb{R}$ is the infimum of the set S.

Now define the map $f : [-\infty, \infty] \to [-\infty, \infty]$ as

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

First notice that if $x \leq y$ then $f(x) \leq f(y)$, for either $x \leq y \leq 0$ in which case $f(x) = 0 = f(y)$, or $x \leq 0 < y$ which gives $f(x) = 0 < 1 = f(y)$ or $0 < x \leq y$ so $f(x) = 1 = f(y)$.

In the second place, if S is a finite subset of $\mathbb{R}$ then S contains a minimum element $s \in S$. So $f(s) \in f(S)$ and $f(s) \leq f(t)$ for all $t \in S$, so $f(\min S) = f(s) = \min f(S)$. Hence f is a lattice homomorphism.

However, f is not a complete lattice homomorphism. To see this let $S = \{x \in \mathbb{R} : 0 < x\}$. Then $\inf S = 0$. However, $f(\inf S) = f(0) = 0$ while $\inf f(S) = \inf\{1\} = 1$.