representing a distributive lattice by ring of sets

Canonical name	RepresentingADistributiveLatticeByRingOfSets
Date of creation	2013-03-22 19:08:24
Last modified on	2013-03-22 19:08:24
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	7
Author	CWoo (3771)
Entry type	Theorem
Classification	msc 06D99
Classification	msc 06D05
Related topic	RingOfSets
Related topic	RepresentingABooleanLatticeByFieldOfSets

In this entry, we present the proof of a fundamental fact that every distributive lattice is lattice isomorphic to a ring of sets, originally proved by Birkhoff and Stone in the 1930's. The proof uses the http://planetmath.org/BirkhoffPrimeIdea ideal theorem of Birkhoff. First, a simple results from the prime ideal theorem:

Lemma 1. *Let L be a distributive lattice and $a, b \in L$ with $a \neq b$. Then there is a prime ideal containing one or the other.*

Proof. Let $I = \langle a \rangle$ and $J = \langle b \rangle$, the principal ideals generated by a, b respectively. If $I = J$, then $b \leq a$ and $a \leq b$, or $a = b$, contradicting the assumption. So $I \neq J$, which means either $a \notin J$ or $b \notin I$. In either case, apply the prime ideal theorem to obtain a prime ideal containing I (or J) not containing b (or a). $\square$

Before proving the theorem, we have one more concept to introduce:

Definition. Let L be a distributive lattice, and X the set of all prime ideals of L. Define $F : L \to P(X)$, the powerset of X, by

$$F(a) := \{P \mid a \notin P\}.$$

Proposition 1. *F is an injection.*

Proof. If $a \neq b$, then by the lemma there is a prime ideal P containing one but not another, say $a \in P$ and $b \notin P$. Then $P \notin F(a)$ and $P \in F(b)$, so that $F(a) \neq F(b)$. $\square$

Proposition 2. *F is a lattice homomorphism.*

Proof. There are two things to show:

- F preserves $\wedge$: If $P \in F(a \wedge b)$, then $a \wedge b \notin P$, so that $a \notin P$ and $b \notin P$, since P is a sublattice. So $P \in F(a)$ and $P \in F(b)$ as a result. On the other hand, if $P \in F(a) \cap F(b)$, then $a \notin P$ and $b \notin P$. Since P is prime, $a \wedge b \notin P$, so that $P \in F(a \wedge b)$. Therefore, $F(a \wedge b) = F(a) \cap F(b)$.

- F preserves $\vee$: If $P \in F(a \vee b)$, then $a \vee b \notin P$, which implies that $a \notin P$ or $b \notin P$, since P is a sublattice of L. So $P \in F(a) \cup F(b)$. On the other hand, if $P \in F(a) \cup F(b)$, then $a \vee b \notin P$, since P is a lattice ideal. Hence $F(a \vee b) = F(a) \cup F(b)$.

Therefore, F is a lattice homomorphism. $\square$

The function F is called the *canonical embedding* of L into $P(X)$.

Theorem 1. *Every distributive lattice is isomorphic to a ring of sets.*

Proof. Let L, X, F be as above. Since $F : L \to P(X)$ is an embedding, L is lattice isomorphic to $F(L)$, which is a ring of sets. $\square$

Remark. Using the result above, one can show that if L is a Boolean algebra, then L is isomorphic to a field of sets. See link below for more detail.