distributivity in po-groups

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Let G be a po-group and A be a set of elements of G. Denote the supremum of elements of A, if it exists, by $\bigvee A$. Similarly, denote the infimum of elements of A, if it exists, by $\bigwedge A$. Furthermore, let $A^{-1} = \{a^{-1} \mid a \in A\}$, and for any $g \in G$, let $gA = \{ga \mid a \in A\}$ and $Ag = \{ag \mid a \in A\}$.

1. If $\bigvee A$ exists, so do $\bigvee gA$ and $\bigvee Ag$.
2. If 1. is true, then $g \bigvee A = \bigvee gA = \bigvee Ag$.
3. $\bigvee A$ exists iff $\bigwedge A^{-1}$ exists; when this is the case, $\bigwedge A^{-1} = (\bigvee A)^{-1}$.
4. If $\bigwedge A$ exists, so do $\bigwedge gA$, and $\bigwedge Ag$.
5. If 4. is true, then $g \bigwedge A = \bigwedge gA = \bigwedge Ag$.
6. If 1. is true and $A = \{a, b\}$, then $a \wedge b$ exists and is equal to $a(a \vee b)^{-1}b$.

Proof. Suppose $\bigvee A$ exists.

- (1. and 2.) Clearly, for each $a \in A$, $a \leq \bigvee A$, so that $ga \leq g \bigvee A$, and therefore elements of gA are bounded from above by $g \bigvee A$. To show that $g \bigvee A$ is the least upper bound of elements of gA, suppose b is the upper bound of elements of gA, that is, $ga \leq b$ for all $a \in A$, this means that $a \leq g^{-1}b$ for all $a \in A$. Since $\bigvee A$ is the least upper bound of the a's, $\bigvee A \leq g^{-1}b$, so that $g \bigvee A \leq b$. This shows that $g \bigvee A$ is the supremum of elements of gA; in other words, $g \bigvee A = \bigvee gA$. Similarly, $\bigvee Ag$ exists and $g \bigvee A = \bigvee Ag$ as well.

- (3.) Write $c = \bigvee A$. Then $a \leq c$ for each $a \in A$. This means $c^{-1} \leq a^{-1}$. If $b \leq a^{-1}$ for all $a \in A$, then $a \leq b^{-1}$ for all $a \in A$, so that $c \leq b^{-1}$, or $b \leq c^{-1}$. This shows that c^{-1} is the greatest lower bound of elements of A^{-1}, or $(\bigvee A)^{-1} = \bigwedge A^{-1}$. The converse is proved likewise.

- (4. and 5.) This is just the dual of 1. and 2., so the proof is omitted.

- (6.) If $A = \{a, b\}$, then $aA^{-1}b = A$, and the existence of $\bigwedge A$ is the same as the existence of $\bigwedge(aA^{-1}b)$, which is the same as the existence of $a(\bigwedge A^{-1})b$ by 4 and 5 above. Since $\bigvee A$ exists, so does $\bigwedge A^{-1}$, and hence $a(\bigwedge A^{-1})b$, by 3 above. Also by 3, we have the equality $a(\bigwedge A^{-1})b = a(\bigvee A)^{-1}b$. Putting everything together, we have the result: $a \wedge b = a(a \vee b)^{-1}b$.

This completes the proof. $\square$

Remark. From the above result, we see that group multiplication distributes over arbitrary joins and meets, if these joins and meets exist.

One can use this result to prove the following: every Dedekind complete po-group is an Archimedean po-group.

Proof. Suppose $a^n \leq b$ for all integers n. Let $A = \{a^n \mid n \in \mathbb{Z}\}$. Then A is bounded from above by b so has least upper bound $\bigvee A$. Then $a \bigvee A = \bigvee aA = \bigvee A$, since $aA = A$. As a result, multiplying both sides by $(\bigvee A)^{-1}$, we get $a = e$. $\square$

Remark. The above is a generalization of a famous property of the real numbers: $\mathbb{R}$ has the Archimedean property.