commutativity relation in an orthocomplemented lattice

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Let L be an orthocomplemented lattice with $a, b \in L$. We say that a *commutes* with b if $a = (a \wedge b) \vee (a \wedge b^{\perp})$. When a commutes with b, we write $a \, \mathrm{C} \, b$. Dualize everything, we have that a *dually commutes* with b, written $a \, \mathrm{D} \, b$, if $a = (a \vee b) \wedge (a \vee b^{\perp})$.

Some properties. Below are some properties of the commutativity relations C and D.

1. C is reflexive.
2. $a \, \mathrm{C} \, b$ iff $a \, \mathrm{C} \, b^{\perp}$.
3. $a \, \mathrm{C} \, b$ iff $a^{\perp} \, \mathrm{D} \, b^{\perp}$.
4. if $a \le b$ or $a \le b^{\perp}$, then $a \, \mathrm{C} \, b$.
5. a is said to *orthogonally commute* with b if $a \, \mathrm{C} \, b$ and $b \, \mathrm{C} \, a$. In this case, we write $a \, \mathrm{M} \, b$. The terminology comes from the following fact: $a \, \mathrm{M} \, b$ iff there are $x, y, z, t \in L$, with:
   (a) $x \perp y$ (x is orthogonal to y, or $x \le y^{\perp}$),
   (b) $z \perp t$,
   (c) $x \perp z$,
   (d) $a = x \vee y$, and
   (e) $b = z \vee t$.
6. C is symmetric iff $\mathrm{D} = \mathrm{C}(= \mathrm{M})$ iff L is an orthomodular lattice.
7. C is an equivalence relation iff $\mathrm{C} = L \times L$ iff L is a Boolean algebra.

Remark. More generally, one can define commutativity C on an orthomodular poset P: for $a, b \in P$, $a \, \mathrm{C} \, b$ iff $a \wedge b$, $a \wedge b^{\perp}$, and $(a \wedge b) \vee (a \wedge b^{\perp})$ exist, and $(a \wedge b) \vee (a \wedge b^{\perp}) = a$. Dual commutativity and mutual commutativity in an orthomodular poset are defined similarly (with the provision that the binary operations on the pair of elements are meaningful).

References

[1] L. Beran, *Orthomodular Lattices, Algebraic Approach*, Mathematics and Its Applications (East European Series), D. Reidel Publishing Company, Dordrecht, Holland (1985).