properties of certain monotone functions

Canonical name	PropertiesOfCertainMonotoneFunctions
Date of creation	2013-03-22 19:03:31
Last modified on	2013-03-22 19:03:31
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	9
Author	CWoo (3771)
Entry type	Result
Classification	msc 06F99
Classification	msc 08C99
Classification	msc 08A99

In the definitions of some partially ordered algebraic systems such as po-groups and po-rings, the multiplication is set to be compatible with the partial ordering on the universe in the following sense:

$$ab \le ac \quad \text{iff} \quad b \le c \qquad \text{and} \qquad ab \le cb \quad \text{iff} \quad a \le c$$

This is no coincidence. In fact, these "definitions" are actually consequences of properties concerning monotone functions satisfying certain algebraic rules, which is the focus of this entry.

Recall that an n-ary function f on a set A is said to be monotone if it is monotone in each of its variables. In other words, for every $i = 1, 2, \ldots, n$, the function $f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ is monotone in x, where each of the a_j is a fixed but arbitrary element of A. We use the notation $\uparrow, \downarrow, \updownarrow$ to denote the monotonicity of each variable in f. For example, $(\uparrow, \uparrow, \uparrow)$ denotes a ternary isotone function, whereas $(\downarrow, \updownarrow)$ denotes a binary function which is antitone with respect to its first variable, and both isotone/antitone with respect to the second.

Proposition 1. *Let f be an n-ary commutative monotone operation on a set A. Then f is either isotone or antitone.*

Proof. Suppose f is isotone (or antitone) in its first variable. Since $f(x, a_1, \ldots, a_{n-1}) = f(a_1, x, \ldots, a_{n-1}) = \cdots = f(a_1, \ldots, a_{n-1}, x)$, f is isotone (or antitone) in each of its remaining variables. □

Proposition 2. *Let f be an n-ary monotone operation on a set A with an identity element $e \in A$. In other words, $f(x, e, \ldots, e) = f(e, x, \ldots, e) = \cdots = f(e, e, \ldots, x) = x$. Then f is either strictly isotone or strictly antitone.*

Proof. The proof is the same as the one before. Furthermore, if f is isotone and $a < b$, then $f(a, e, \ldots, e) = a < b = f(b, e, \ldots, e)$, so the strict ordering is preserved. The same holds true if f is antitone. □

Proposition 3. *Let f be a binary monotone operation on a set A such that it is isotone (antitone) with respect to its first variable. Suppose g is a unary operation on A such that $f(x, g(x))$ is a fixed element of A. Then g is antitone (isotone).*

Proposition 4. *Let f be an n-ary associative monotone operation on a set A. Then*

- *f is isotone if n is even*
- *f is either isotone, or is* $(\underbrace{\uparrow,\ldots,\uparrow}_{m},\downarrow,\underbrace{\uparrow,\ldots,\uparrow}_{m})$, *if n is odd, say $n = 2m+1$.*

Proof. Suppose first that $n = 2m$, $i \leq m$, and $g(x) = f(a_1,\ldots,a_{i-1},x,\ldots,a_{m+1},\ldots,a_{2m})$ is antitone. Then $g(g(x))$ is isotone. By the associativity of f, $g(g(x))$ is

$$\begin{aligned} & f(a_1,\ldots,a_{i-1},f(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_m,\ldots,a_{2m}),\ldots,a_{m+1},\ldots,a_{2m}) \\ = \; & f(a_1,\ldots,a_{i-1},a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,f(a_{2m-i+1},\ldots,a_{2m},a_{i+1},\ldots,a_{m+1},\ldots,a_{2m})). \end{aligned}$$

In the second expression, the position of x is $2i-2 \leq 2m-1 < 2m$, therefore implying that $g(g(x))$ is antitone, which is a contradiction! Therefore, $g(x)$ is isotone. Now, if $i > m$, and $h(x) = f(b_1,\ldots,b_m,\ldots,b_{i-1},x,b_{i+1},\ldots,b_{2m})$ is antitone, then $h(h(x))$ is isotone. But

$$\begin{aligned} & f(b_1,\ldots,b_m,\ldots,b_{i-1},f(b_1,\ldots,b_m,\ldots,b_{i-1},x,b_{i+1},\ldots,b_{2m}),b_{i+1},\ldots,b_{2m}) \\ = \; & f(f(b_1,\ldots,b_m,\ldots,b_{i-1},b_1,\ldots,b_{2m-i+1}),\ldots,b_{i-1},x,b_{i+1},\ldots,b_{2m},b_{i+1},\ldots,b_{2m})), \end{aligned}$$

and the position of x is the second expression is $(i-1)-(2m-i+1)+2 = 2i-2m > 1$, therefore implying that $h(h(x))$ is antitone, again a contradiction. As a result, f is isotone for all $i = 1,\ldots,n$.

The argument above also works when n is odd, say $n = 2m+1$ and $i \neq m+1$. Finally, since f is monotone, it is monotone with respect to the i-th variable when $i = m+1$, so f is one of the following three forms:

$$(\underbrace{\uparrow,\ldots,\uparrow}_{2m+1}), \qquad (\underbrace{\uparrow,\ldots,\uparrow}_{m},\updownarrow,\underbrace{\uparrow,\ldots,\uparrow}_{m}), \qquad (\underbrace{\uparrow,\ldots,\uparrow}_{m},\downarrow,\underbrace{\uparrow,\ldots,\uparrow}_{m}),$$

the first two of which imply that f is isotone. $\square$

An example of an associative function that is, say $(\uparrow,\downarrow,\uparrow)$, is given by

$$f : \mathbb{Z}^3 \to \mathbb{Z} \qquad \text{where} \qquad f(x,y,z) = x - y + z.$$

f is associative since $f(f(r,s,t),u,v) = f(r,f(s,t,u),v) = f(r,s,f(t,u,v)) = r - s + t - u + v$.