convergence in the mean

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Entry type	Definition

Let

$$b_n := \frac{a_1 + a_2 + \ldots + a_n}{n} \quad (n = 1, 2, 3, \ldots)$$

be the arithmetic mean of the numbers $a_1, a_2, \ldots, a_n$. The sequence

$$a_1, a_2, a_3, \ldots \tag{1}$$

is said to `http://planetmath.org/ConvergenceInTheMean`*converge in the mean* iff the sequence

$$b_1, b_2, b_3, \ldots \tag{2}$$

converges.
On has the

Theorem. If the sequence (1) is convergent having the limit A, then also the sequence (2) converges to the limit A. Thus, a convergent sequence is always convergent in the mean.

Proof. Let ε be an arbitrary positive number. We may write

$$\begin{aligned}
|A - b_n| &= |A - \frac{1}{n}(a_1 + \ldots + a_k) - \frac{1}{n}(a_{k+1} + \ldots + a_n)| \\
&= |\frac{1}{n}[(A - a_1) + \ldots + (A - a_k)] + \frac{1}{n}[(A - a_{k+1}) + \ldots + (A - a_n)]| \\
&\leqq \frac{|(A - a_1) + \ldots + (A - a_k)|}{n} + \frac{|A - a_{k+1}| + \ldots + |A - a_n|}{n}.
\end{aligned}$$

The supposition implies that there is a positive integer k such that

$$|A - a_i| < \frac{\varepsilon}{2} \quad \text{for all } i > k.$$

Let's fix the integer k. Choose the number l so great that

$$\frac{|(A - a_1) + \ldots + (A - a_k)|}{n} < \frac{\varepsilon}{2} \quad \text{for } n > l.$$

Let now $n > \max\{k, l\}$. The three above inequalities yield

$$|A - b_n| \;<\; \frac{\varepsilon}{2} + \frac{1}{n}(n-k)\frac{\varepsilon}{2} \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whence we have

$$\lim_{n \to \infty} b_n = A.$$

Note. The `http://planetmath.org/Converse`converse of the theorem is not true. For example, if

$$a_n := \frac{1+(-1)^n}{2}$$

i.e. if the sequence (1) has the form $0, 1, 0, 1, 0, 1, \ldots,$ then it is divergent but converges in the mean to the limit $\frac{1}{2}$; the corresponding sequence (2) is $0, \frac{1}{2}, \frac{1}{3}, \frac{2}{4}, \frac{2}{5}, \frac{3}{6}, \frac{3}{7}, \frac{4}{8}, \frac{4}{9}, \ldots$