partially ordered algebraic system

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Let A be a poset. Recall a function f on A is said to be

- order-preserving (or isotone) provided that $f(a) \leq f(b)$, or
- order-reversing (or antitone) provided that $f(a) \geq f(b)$, or

whenever $a \leq b$. Furthermore, f is called monotone if f is either isotone or antitone.

For every function f on A, we denote it to be $\uparrow$, $\downarrow$, or $\updownarrow$ according to whether it is isotone, antitone, or both. The following are some easy consequences:

- $\uparrow \circ \downarrow = \downarrow \circ \uparrow = \downarrow$ (meaning that the composition of an isotone and an antitone maps is antitone),
- $\uparrow \circ \uparrow = \downarrow \circ \downarrow = \uparrow$ (meaning that the composition of two isotone or two antitone maps is isotone),
- f is $\updownarrow$ iff it is a constant on any chain in A, and if this is the case, for every $a \in A$, $f^{-1}(a)$ is a maximal chain in A.

The notion above can be generalized to n-ary operations on a poset A. An n-ary operation f on a poset A is said to be *isotone*, *antitone*, or *monotone* iff when f is isotone, antitone, or monotone with respect to each of its n variables. We continue to use to arrow notations above to denote n-ary monotone functions. For example, a ternary function that is $(\uparrow, \downarrow, \uparrow)$ is isotone with respect to its first and third variables, and antitone with respect to its second variable.

Definition. A *partially ordered algebraic system* is an algebraic system $\mathcal{A} = (A, O)$ such that A is a poset, and every operation $f \in O$ on A is monotone. A partially ordered algebraic system is also called a partially ordered algebra, or a po-algebra for short.

Examples of po-algebras are po-groups, po-rings, and po-semigroups. In all three cases, the multiplication operations are $(\uparrow, \uparrow)$, as well as the addition operation in a po-ring.. In the case of a po-group, the multiplicative inverse operation is $\downarrow$, as well as the additive inverse operation in a po-ring.

Another example is an ordered vector space V over a field k. The underlying universe is V (not k). Addition over V is, like the other examples above, isotone. Each element $r \in k$ acts as a unary operator on V, given by $r(v) = rv$, the scalar multiplication of r and v. As k is itself a poset, it can be partitioned into three sets: the positive cone $P(k)$ of k, the negative cone $-P(k)$, and $\{0\}$. Then $r \in P(k)$ iff it is $\uparrow$ as a unary operator, $r \in -P(k)$ iff it is $\downarrow$, and $r = 0$ iff it is $\updownarrow$.

Remarks

- A homomorphism from one po-algebra $\mathcal{A}$ to another $\mathcal{B}$ is an isotone map ϕ from posets A to B that is at the same time a homomorphism from the algebraic systems $\mathcal{A}$ to $\mathcal{B}$.
- A partially ordered subalgebra of a po-algebra $\mathcal{A}$ is just a subalgebra of $\mathcal{A}$ viewed as an algebra, where the partial ordering on the universe of the subalgebra is inherited from the partial ordering on A.

References

[1] L. Fuchs, *Partially Ordered Algebraic Systems*, Addison-Wesley, (1963).