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Multiple Recurrence Theorem

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Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $T_i : X \to X$ be measure-preserving transformations, for i between 1 and q. Assume that all the transformations T_i commute. If $E \subset X$ is a positive measure set $\mu(E) > 0$, then, there exists $n \in \mathbb{N}$ such that

$$\mu(E \cap T_1^{-n}(E) \cap \cdots \cap T_q^{-n}(E)) > 0$$

In other words there exist a certain time n such that the subset of E for which all elements return to E simultaneously for all transformations T_i is a subset of E with positive measure. Observe that the theorem may be applied again to the set $G = E \cap T_1^{-n}(E) \cap \cdots \cap T_q^{-n}(E)$, obtaining the existence of $m \in \mathbb{N}$ such that

$$\mu(G \cap T_1^{-m}(G) \cap \cdots \cap T_q^{-m}(G)) > 0$$

so that

$$\mu(E \cap T_1^{-(m+n)}(E) \cap \cdots \cap T_q^{-(m+n)}(E)) \geq \mu(G \cap T_1^{-m}(G) \cap \cdots \cap T_q^{-m}(G)) > 0$$

So we may conclude that, when E has positive measure, there are infinite times for which there is a simultaneous return for a subset of E with positive measure.

As a corollary, since the powers $T, T^2 \cdots T^q$ of a transformation T commute, we have that, for E with positive measure there exists $n \in \mathbb{N}$ such that

$$\mu(E \cap T^{-n}(E) \cap \cdots \cap T^{-qn}(E)) > 0$$

As a consequence of the multiple recurrence theorem one may prove Szemerdi's Theorem about arithmetic progressions.