Newtonian coalgebra

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Let R be a commutative ring. A *Newtonian coalgebra* over R is an R-module C which is simultaneously a coalgebra with comultiplication $\Delta\colon C \to C \otimes C$ and an algebra with multiplication $\cdot\colon C \otimes C \to C$ such that Δ is a derivation over $\cdot$, that is, such that the identity

$$\Delta(u \cdot v) = \Delta(u) \cdot v + u \cdot \Delta(v)$$

holds for any u and v in C. Newtonian coalgebras were introduced by Joni and Rota in [**?**], where they were called *infinitesimal coalgebras*. They reserved the term "Newtonian coalgebra" for the special case of the coalgebra of divided differences. This example was studied in more detail by Hirschhorn and Raphael [**?**]. Joni and Rota also showed that Newtonian coalgebras provide a language which can explain iterated differentiation of trigonometric functions as well as Faà di Bruno's formula. See also the paper of Nichols and Sweedler [**?**] for more on trigonometric coalgebras.

A Newtonian coalgebra cannot have both a unit and a counit, so no Newtonian coalgebra is a Hopf algebra. However, Aguiar [**?**] developed a notion of antipode that makes sense for Newtonian coalgebras, leading to what he calls an infinitesimal Hopf algebra. Ehrenborg and Readdy [**?**] used Newtonian coalgebras to give an algebraic structure to the `http://planetmath.org/CdIndex`**cd**-index, a poset invariant generalizing the f-vector of polytopes.

One example of a Newtonian coalgebra is the free associative algebra $R\langle \mathbf{a}, \mathbf{b} \rangle$ of polynomials on the noncommuting variables $\mathbf{a}$ and $\mathbf{b}$ with coefficients in R. The product is the ordinary noncommutative polynomial product, and the comultiplication is defined by setting

$$\Delta(u_1 \cdots u_n) = \sum_{j \in [n]} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n$$

for each monomial and extending by linearity.

References

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