basic facts about ordered rings

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Throughout this entry, $(R, \leq)$ is an ordered ring.

Lemma 1. *If $a, b, c \in R$ with $a < b$, then $a + c < b + c$.*

Proof. The contrapositive will be proven.

Let $a, b, c \in R$ with $a + c \geq b + c$. Note that $-c \in R$. Thus,

$$\begin{aligned} b &= b + 0 \\ &= b + c + (-c) \\ &\leq a + c + (-c) \\ &= a + 0 \\ &= a. \quad \square \end{aligned}$$

Lemma 2. *If $|R| \neq 1$ and R has a characteristic, then it must be 0.*

Proof. Suppose not. Let n be a positive integer such that $\operatorname{char} R = n$. Since $|R| \neq 1$, it must be the case that $n > 1$.

Let $r \in R$ with $r > 0$. By the previous lemma, $0 < r \leq \ldots \leq \displaystyle\sum_{j=1}^{n-1} r \leq \sum_{j=1}^{n} r = 0$, a contradiction. $\square$

Lemma 3. *If $a, b \in R$ with $a \leq b$ and $c \in R$ with $c < 0$, then $ac \geq bc$.*

Proof. Note that $-c \in R$ and $0 = c + (-c) < 0 + (-c) = -c$. Since $a \leq b$, $-(ac) = a(-c) \leq b(-c) = -(bc)$. Thus,

$$\begin{aligned} bc &= bc + 0 \\ &= bc + (ac + (-(ac))) \\ &= (bc + ac) + (-(ac)) \\ &\leq (bc + ac) + (-(bc)) \\ &= -(bc) + (bc + ac) \\ &= (-(bc) + bc) + ac \\ &= 0 + ac \\ &= ac. \quad \square \end{aligned}$$

Lemma 4. *Suppose further that R is a ring with multiplicative identity $1 \neq 0$. Then $0 < 1$.*

Proof. Suppose that $0 \not< 1$. Since R is an ordered ring, it must be the case that $1 < 0$. By the previous lemma, $1 \cdot 1 \geq 0 \cdot 1$. Thus, $1 \geq 0$, a contradiction. $\square$