lattice polynomial

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Defines	lattice polynomial function
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Defines	weight of a lattice polynomial
Defines	arity of a lattice polynomial
Defines	Boolean polynomial

A *lattice polynomial*, informally, is an expression involving a finite number of variables $x, y, z, \ldots$, two symbols $\vee, \wedge$, and sometimes the parentheses (,) in a *meaningful manner*. Loosely speaking, whenever p and q are lattice polynomials, the only lattice polynomials that can be formed from $p, q, \vee, \wedge$ are $p \vee q$ and $p \wedge q$. We will explain formally what *meaningful manner* is a little later. Some examples of lattice polynomials are $x \vee (x \vee x)$, $(y \wedge x) \vee x$, and $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, while $\vee \wedge x$, $xy \wedge yz$, $z \vee)($ are not lattice polynomials.

To formally define what a lattice polynomial is, we resort to model theory. To begin with, we have a countable set of variables $V = \{x, y, z, \ldots\}$, a set of binary function symbols $F = \{\vee, \wedge\}$. We define pairwise disjoint sets $S_0, S_1, \ldots, S_n, \ldots$ recursively, as follows:

- $S_0 = V$,
- $S_{k+1} = \{(p \vee q), (p \wedge q) \mid p, q \in S_k\} \cup S_k$.

Then we set $S = \bigcup_{i=0}^{\infty} S_i$. An element of S is called a *lattice polynomial*.

Note that in the above definition, $((x \vee y) \vee z)$ is a lattice polynomial while $(x \vee y \vee z)$ is not, for any variables $x, y, z \in V$. To reduce the number of parentheses in a lattice polynomial, we typically identify $(p \vee q)$ with $p \vee q$ and $(p \wedge q)$ with $p \wedge q$. In addition, since the meet and join operations are associative in any lattice, it is a common practice to further reduce the number of parentheses in a lattice polynomial by identifying both $(p \vee (q \vee r))$ and $((p \vee q) \vee r)$ with $p \vee q \vee r$, and $(p \wedge (q \wedge r))$ and $((p \wedge q) \wedge r)$ with $p \wedge q \wedge r$.

Another thing that can be said about the above construction of is that any given lattice polynomial can be constructed from S_0 in a finite number of steps. If $p \in S_n - S_{n-1}$, $n \geq 1$, then p can be constructed in exactly n steps. The minimum number of variables (in S_0) that is required to construct p is called the *arity* of p. For example, if $p = ((x \vee y) \wedge x)$ then the arity of p is 2. If an n-ary lattice polynomial p can be constructed from $x_1, \ldots, x_n$, we often write $p = p(x_1, \ldots, x_n)$.

One more important number associated with a lattice polynomial p is its *weight*, defined recursively as $w(p) = 1$ if $p \in S_0$, and $w(p \vee q) = w(p \wedge q) = w(p) + w(q)$.

Given any n-ary lattice polynomial p and any lattice L, we can associate p with an m-ary *lattice polynomial function* $f : L^m \to L$ defined by

$$f(a_1, \ldots, a_m) := p(a_1, \ldots, a_n), \text{ where } m \geq n \text{ and } a_i \in L.$$

The expression $p(a_1, \ldots, a_n)$ is the *evaluation* of p at $(a_1, \ldots, a_n)$. That is, we substitute each x_i for a_i, and we interpret $\vee$ and $\wedge$ in p as the join and meet operations in L.

Two lattice polynomials p, q of arities m, n, where $m \geq n$, are said to be *equivalent* if their corresponding m-ary lattice polynomial functions evaluate to the same values in any lattice. For example, $x \vee y$ and $y \vee x$ are equivalent. Similarly, x, $x \vee x$, $x \wedge x$, and $x \vee (y \wedge x)$ are equivalent lattice polynomials.

Remark. Similarly, one can define a *Boolean polynomial* by enlarging the set F of function symbols to include the unary operator $'$, and $S_{k+1} = \{(p \vee q), (p \wedge q), (p') \mid p, q \in S_k\} \cup S_k$. Then a Boolean polynomial is just an element of $S = \cup_{i=0}^{\infty} S_i$. The weight of a Boolean polynomial is similarly defined, with the additional $w(p') = w(p)$.