dimension of a poset

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Let P be a finite poset and $\mathcal{R}$ be the family of all realizers of P. The *dimension* of P, written $\dim(P)$, is the cardinality of a member $E \in \mathcal{R}$ with the smallest cardinality. In other words, the dimension n of P is the least number of linear extensions $L_1, \ldots, L_n$ of P such that $P = L_1 \cap \cdots \cap L_n$. ($E$ can be chosen to be $\{L_1, \ldots, L_n\}$).

If P is a chain, then $\dim(P) = 1$. The converse is clearly true too. An example of a poset with dimension 2 is an antichain with at least 2 elements. For if $P = \{a_1, \ldots, a_m\}$ is an antichain, then one way to linearly extend P is to simply put $a_i \leq a_j$ iff $i \leq j$. Called this extension L_1. Another way to order P is to reverse L_1, by $a_i \leq a_j$ iff $j \leq i$. Call this L_2. Note that L_1 and L_2 are duals of each other. Let $L = L_1 \cap L_2$. As both L_1 and L_2 are linear extensions of P, $P \subseteq L$. On the other hand, if $(a_i, a_j) \in L$, then $a_i \leq a_j$ in both L_1 and L_2, so that $i \leq j$ and $j \leq i$, or $i = j$ and whence $a_i = a_j$, which implies $(a_i, a_j) = (a_i, a_i) \in P$. $L \subseteq P$ and thus $\dim(P) = 2$.

Remark. Let P be a finite poset. A theorem of Dushnik and Miller states that the smallest n such that P can be embedded in $\mathbb{R}^n$, considered as the n-fold product of posets, or chains of real numbers $\mathbb{R}$, is the dimension of P.

References

[1] W. T. Trotter, *Combinatorics and Partially Ordered Sets*, Johns-Hopkins University Press, Baltimore (1992).