dual space of a Boolean algebra

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Let B be a Boolean algebra, and B^* the set of all maximal ideals of B. In this entry, we will equip B^* with a topology so it is a Boolean space.

Definition. For any $a \in B$, define $M(a) := \{M \in B^* \mid a \notin M\}$, and $\mathcal{B} := \{M(a) \mid a \in B\}$.

It is know that in a Boolean algebra, maximal ideals and prime ideals coincide. From `http://planetmath.org/RepresentingABooleanLatticeByFieldOfSets`this entry, we have the three following properties concerning $M(a)$:

$$M(a) \cap M(b) = M(a \wedge b), \qquad M(a) \cup M(b) = M(a \vee b), \qquad B^* - M(a) = M(a').$$

Furthermore, if $M(a) = M(b)$, then $a = b$.

From these properties, we see that $M(0) = \varnothing$ and $M(1) = B^*$. As a result, we see that

Proposition 1. *B^* is a topological space, whose topology $\mathcal{T}$ is generated by the basis $\mathcal{B}$.*

Proof. $\varnothing$ and B^* are both open, as they are $M(0)$ and $M(1)$ respectively. Also, the intersection of open sets $M(a)$ and $M(b)$ is again open, since it is $M(a \wedge b)$. □

We may in fact treat $\mathcal{B}$ as a subbasis for $\mathcal{T}$, since finite intersections of elements of $\mathcal{B}$ remain in $\mathcal{B}$.

Proposition 2. *Each member of $\mathcal{B}$ is closed, hence $\mathcal{T}$ is generated by a basis of clopen sets. In other words, B^* is zero-dimensional.*

Proof. Each $M(a)$ is open, by definition, and closed, since it is the complement of the open set $M(a')$. □

Proposition 3. *B^* is Hausdorff.*

Proof. If $M, N \in B^*$ such that $M \neq N$, then there is some $a \in B$ such that $a \in M$ and $a \notin N$. This means that $N \in M(a)$ and $M \notin M(a)$, which means that $M \in B^* - M(a) = M(a')$. Since $M(a)$ and $M(a')$ are open and disjoint, with $N \in M(a)$ and $M \in M(a')$, we see that B^* is Hausdorff. □

Now, based on a topological fact, every zero-dimensional Hausdorff space is totally disconnected. Hence B^* is totally disconnected.

Proposition 4. *B^* is compact.*

Proof. Suppose $\{U_i \mid i \in I\}$ is a collection of open sets whose union is B^*. Since each U_i is a union of elements of $\mathcal{B}$, we might as well assume that B^* is covered by elements of $\mathcal{B}$. In other words, we may assume that each U_i is some $M(a_i) \in \mathcal{B}$.

Let J be the ideal generated by the set $\{a_i \mid i \in I\}$. If $J \neq B$, then J can be extended to a maximal ideal M. Since each $a_i \in M$, we see that $M \notin M(a_i)$, so that $M \notin \bigcup\{M(a_i) \mid i \in I\} = B^*$, which is a contradiction. Therefore, $J = B$. In particular, $1 \in J$, which means that 1 can be expressed as the join of a finite number of the a_i's:

$$1 = \bigvee\{a_i \mid i \in K\},$$

where K is a finite subset of J. As a result, we have

$$\bigcup\{M(a_i) \mid i \in K\} = M(\bigvee\{a_i \mid i \in K\}) = M(1) = B^*.$$

So B^* has a finite subcover, and hence is compact. $\square$

Collecting the last three results, we see that B^* is a Boolean space.

Remark. It can be shown that B is isomorphic to the Boolean algebra of clopen sets in B^*. This is the famous Stone representation theorem.