example of Boolean algebras

Canonical name	ExampleOfBooleanAlgebras
Date of creation	2013-03-22 17:52:33
Last modified on	2013-03-22 17:52:33
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	15
Author	CWoo (3771)
Entry type	Example
Classification	msc 06B20
Classification	msc 03G05
Classification	msc 06E05
Classification	msc 03G10

Below is a list of examples of Boolean algebras. Note that the phrase "usual set-theoretic operations" refers to the operations of union $\cup$, intersection $\cap$, and set complement $'$.

1. Let A be a set. The power set $P(A)$ of A, or the collection of all the subsets of A, together with the operations of union, intersection, and set complement, the empty set $\varnothing$ and A, is a Boolean algebra. This is the canonical example of a Boolean algebra.

2. In $P(A)$, let $F(A)$ be the collection of all finite subsets of A, and $cF(A)$ the collection of all cofinite subsets of A. Then $F(A) \cup cF(A)$ is a Boolean algebra.

3. More generally, any field of sets is a Boolean algebra. In particular, any sigma algebra σ in a set is a Boolean algebra.

4. (product of algebras) Let A and B be Boolean algebras. Then $A \times B$ is a Boolean algebra, where

$$
\begin{aligned}
(a,b) \vee (c,d) &:= (a \vee c, b \vee d), && (1)\\
(a,b) \wedge (c,d) &:= (a \wedge c, b \wedge d), && (2)\\
(a,b)' &:= (a', b'). && (3)
\end{aligned}
$$

5. More generally, if we have a collection of Boolean algebras A_i, indexed by a set I, then $\prod_{i \in I} A_i$ is a Boolean algebra, where the Boolean operations are defined componentwise.

6. In particular, if A is a Boolean algebra, then set of functions from some non-empty set I to A is also a Boolean algebra, since $A^I = \prod_{i \in I} A$.

7. (subalgebras) Let A be a Boolean algebra, any subset $B \subseteq A$ such that $0 \in B$, $a' \in B$ whenever $a \in B$, and $a \vee b \in B$ whenever $a, b \in B$ is a Boolean algebra. It is called a *Boolean subalgebra* of A. In particular, the homomorphic image of a Boolean algebra homomorphism is a Boolean algebra.

8. (quotient algebras) Let A be a Boolean algebra and I a Boolean ideal in A. View A as a Boolean ring and I an ideal in A. Then the quotient ring A/I is Boolean, and hence a Boolean algebra.

9. Let A be a set, and $R_n(A)$ be the set of all n-ary relations on A. Then $R_n(A)$ is a Boolean algebra under the usual set-theoretic operations. The easiest way to see this is to realize that $R_n(A) = P(A^n)$, the powerset of the n-fold power of A.

10. The set of all clopen sets in a topological space is a Boolean algebra.

11. Let X be a topological space and A be the collection of all regularly open sets in X. Then A has a Boolean algebraic structure. The meet and the constant operations follow the usual set-theoretic ones: $U \wedge V = U \cap V$, $0 = \varnothing$ and $1 = X$. However, the join $\wedge$ and the complementation $'$ on A are different. Instead, they are given by

$$U' := X - \overline{U}, \tag{4}$$
$$U \vee V := (U \cup V)''. \tag{5}$$