Boolean lattice

Canonical name	BooleanLattice
Date of creation	2013-03-22 12:27:20
Last modified on	2013-03-22 12:27:20
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	19
Author	mathcam (2727)
Entry type	Definition
Classification	msc 06E05
Classification	msc 03G05
Classification	msc 06B20
Classification	msc 03G10
Classification	msc 06E20
Synonym	Boolean algebra
Related topic	BooleanRing

In this entry, the notions of a Boolean lattice, a Boolean algebra, and a Boolean ring are defined, compared and contrasted.

Boolean Lattices

A *Boolean lattice* B is a distributive lattice in which for each element $x \in B$ there exists a complement $x' \in B$ such that

$$\begin{aligned} x \wedge x' &= 0 \\ x \vee x' &= 1 \\ (x')' &= x \\ (x \wedge y)' &= x' \vee y' \\ (x \vee y)' &= x' \wedge y' \end{aligned}$$

In other words, a Boolean lattice is the same as a complemented distributive lattice. A morphism between two Boolean lattices is just a lattice homomorphism (so that $0, 1$ and $'$ may not be preserved).

Boolean Algebras

A Boolean algebra is a Boolean lattice such that $'$ and 0 are considered as operators (unary and nullary respectively) on the algebraic system. In other words, a morphism (or a Boolean algebra homomorphism) between two Boolean algebras must preserve $0, 1$ and $'$. As a result, the category of Boolean algebras and the category of Boolean lattices are not the same (and the former is a subcategory of the latter).

Boolean Rings

A *Boolean ring* is an (associative) unital ring R such that for any $r \in R$, $r^2 = r$. It is easy to see that

- any Boolean ring has characteristic 2, for $2r = (2r)^2 = 4r^2 = 4r$,
- and hence a commutative ring, for $a+b = (a+b)^2 = a^2+ab+ba+b^2 = a+ab+ba+b$, so $0 = ab+ba$, and therefore $ab = ab+ab+ba = ba$.

Boolean rings (with identity, but allowing 0=1) are equivalent to Boolean lattices. To view a Boolean ring as a Boolean lattice, define

$$x \wedge y = xy, \qquad x \vee y = x + y + xy, \qquad \text{and} \qquad x' = 1 + x.$$

To view a Boolean lattice as a Boolean ring, define

$$xy = x \wedge y \qquad \text{and} \qquad x + y = (x' \wedge y) \vee (x \wedge y').$$

The category of Boolean algebras is naturally equivalent to the category of Boolean rings.

References

[1] G. Grätzer, *General Lattice Theory*, 2nd Edition, Birkhäuser (1998).

[2] R. Sikorski, *Boolean Algebras*, 2nd Edition, Springer-Verlag, New York (1964).