planetmath.org

Math for the people, by the people.

height of an element in a poset

Canonical name	HeightOfAnElementInAPoset
Date of creation	2013-03-22 16:31:32
Last modified on	2013-03-22 16:31:32
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Definition
Classification	msc 06A06
Related topic	GradedPoset
Defines	Jordan-Dedekind chain condition

Let P be a poset. Given any $a \in P$, the lower set $\downarrow a$ of a is a subposet of P. Call the height of $\downarrow a$ less 1 the *height* of a. Let's denote $h(a)$ the height of a, so

$$h(a) = \text{height}(\downarrow a) - 1.$$

From this definition, we see that $h(a) = 0$ iff a is minimal and $h(a) = 1$ iff a is an atom. Also, h partitions P into equivalence classes, so that a is equivalent to b in P iff $h(a) = h(b)$. Two distinct elements in the same equivalence class are necessarily incomparable. In other words, the equivalence classes are antichains. Furthermore, given any two equivalence classes $[a], [b]$, set $[a] \leq [b]$ iff $h(a) \leq h(b)$, then the set of equivalence classes form a chain.

The height function of a poset P looks remarkably like the rank function of a graded poset: h is constant on the set of all minimal elements, and h is isotone (preserves order). When is h a rank function (the additional condition being the preservation of the covering relation)? The answer is given by a chain condition imposed on P, called the *Jordan-Dedekind chain condition*:

> (*) In a poset, the cardinalities of two maximal chains between common end points must be the same.

Suppose for each $a \in P$, $h(a)$ is finite and P has a unique minimal element 0. Then P can be graded by h iff (*) is satisfied. More generally, if we drop the assumption of the uniqueness of a minimal element, then P can be graded by h iff any two maximal chains ending at the same end point have the same length.