residuated

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Let P, Q be two posets. Recall that a function $f : P \to Q$ is order preserving iff $x \leq y$ implies $f(x) \leq f(y)$. This is equivalent to saying that the preimage of any down set under f is a down set.

Definition A function $f : P \to Q$ between ordered sets P, Q is said to be *residuated* if the preimage of any principal down set is a principal down set. In other words, for any $y \in Q$, there is some $x \in P$ such that

$$f^{-1}(\downarrow y) = \downarrow x,$$

where $f^{-1}(A)$ for any $A \subseteq Q$ is the set $\{a \in X \mid f(a) \in A\}$, and $\downarrow b = \{c \mid c \leq b\}$.

Let us make some observations on a residuated function $f : P \to Q$:

1. f is order preserving.

   Proof. Suppose $a \leq b$ in P. Then there is some $c \in P$ such that $\downarrow c = f^{-1}(\downarrow f(b))$. Since $f(b) \in \downarrow f(b)$, $b \in \downarrow c$, or $b \leq c$, which means $a \leq c$, or $a \in \downarrow c$. But this implies $a \in f^{-1}(\downarrow f(b))$, so that $f(a) \in \downarrow f(b)$, or $f(a) \leq f(b)$. □

2. The x in the definition above is unique.

   Proof. If $\downarrow x = \downarrow z$, then $x \in \downarrow z$, or $x \leq z$. Similarly, $z \leq x$, so that $x = z$. □

3. $g : Q \to P$ given by $g(y) = x$ is a well-defined function, with the following properties:

   (a) g is order preserving,

   (b) $1_P \leq g \circ f$,

   (c) $f \circ g \leq 1_Q$.

   Proof. g is order preserving: if $r \leq s$ in Q, then $\downarrow r \subseteq \downarrow s$, so that $\downarrow u = f^{-1}(\downarrow r) \subseteq f^{-1}(\downarrow s) = \downarrow v$, where $u = g(r)$ and $v = g(s)$. But $\downarrow u \subseteq \downarrow v$ implies $u \leq v$, or $g(r) \leq g(s)$.

   $1_P \leq g \circ f$: let $x \in P$, $y = f(x)$, and $z = g(y)$. Then $x \in f^{-1}(y) \subseteq f^{-1}(\downarrow y) = \downarrow z$, which implies $x \leq z = g(f(x))$ as desired.

   $f \circ g \leq 1_Q$: pick $y \in Q$ and let $x = g(y)$. Then $f^{-1}(\downarrow y) = \downarrow x$, so that $f(x) \in f(\downarrow x) = f(f^{-1}(\downarrow y)) = \downarrow y$, or $f(x) \leq y$, or $f(g(y)) = f(x) \leq y$ as a result. □

Actually, the third observation above characterizes f being residuated:

Proposition 1. *If f is order preserving, and g satisfies* $3(a)$ *through* $3(c)$*, then f is residuated. In fact, such a g is unique.*

Proof. Suppose $y \in Q$, and let $x = g(y)$. We want to show that $\downarrow x = f^{-1}(\downarrow y)$. First, suppose $a \in f^{-1}(\downarrow y)$. Then $f(a) \leq y$, which means $g(f(a)) \leq g(y) = x$. But then $a \leq g(f(a))$, and we get $a \leq x$, or $a \in \downarrow x$. Next, suppose $a \leq x = g(y)$. Then $f(a) \leq f(x) = f(g(y)) \leq y$, so $a \in f^{-1}(f(a)) \subseteq f^{-1}(\downarrow y)$.

To see uniqueness, suppose $h : Q \to P$ is order preserving such that $1_P \leq h \circ f$ and $f \circ h \leq 1_Q$. Then $g = 1_P \circ g \leq (h \circ f) \circ g = h \circ (f \circ g) \leq = h \circ 1_Q = h$ and $h = 1_P \circ h \leq (g \circ f) \circ h = g \circ (f \circ h) \leq g \circ 1_Q = g$. □

Definition. Given a residuated function $f : P \to Q$, the unique function $g : Q \to P$ defined above is called the *residual* of f, and is denoted by f^+.

For example, given any function $f : A \to B$, the induced function $f : P(A) \to P(B)$ (by abuse of notation, we use the same notation as original function f), given by $f(S) = \{f(a) \mid a \in S\}$ is residuated. Its residual is the function $f^{-1} : P(B) \to P(A)$, given by $f^{-1}(T) = \{a \in A \mid f(a) \in T\}$.

Here are some properties of residuated functions and their residuals:

- A bijective residuated function is an order isomorphism, and conversely. Furthermore, the residual is residuated, and is its inverse.
- If $f : P \to Q$ is residuated, then $f \circ f^+ \circ f = f$ and $f^+ \circ f \circ f^+ = f^+$.
- If $f : P \to Q$ and $g : Q \to R$ are residuated, so is $g \circ f$ and $(g \circ f)^+ = f^+ \circ g^+$.
- If $f : P \to Q$ is residuated, then $f^+ \circ f : P \to P$ is a closure map on P. Conversely, any closure function can be decomposed as the functional composition of a residuated function and its residual.

Remark. Residuated functions and their residuals are closely related to Galois connections. If $f : P \to Q$ is residuated, then (f, f^+) forms a Galois connection between P and Q. On the other hand, if (f, g) is a Galois connection between P and Q, then $f : P \to Q$ is residuated, and $g : Q \to P$ is f^+. Note that PM defines a Galois connection as a pair of order-preserving maps, where as in Blyth, they are order reversing.

References

[1] T.S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer, New York (2005).

[2] G. Grätzer, *General Lattice Theory*, 2nd Edition, Birkhäuser (1998)