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representing a complete atomic Boolean algebra by power set

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It is a known fact that every Boolean algebra is isomorphic to a field of sets (of some set) (proof `http://planetmath.org/RepresentingABooleanLatticeByFieldOfSets`here). In this entry, we show that, furthermore, if a Boolean algebra is atomic and complete, then it is isomorphic to *the* field of sets of some set, in other words, the powerset of some set, viewed as a Boolean algebra via the usual set-theoretic operations of union, intersection, and complement.

The proof is based on the following function, defined for any atomic Boolean algebra:

Definition. Let B be an atomic Boolean algebra, and X *the* set of its atoms. Define $f : B \to P(X)$ by

$$f(x) := \{a \mid a \leq x\}.$$

It is easy to see that $f(x) = \{x\}$ iff x is an atom of B.

Proposition 1. *$f(x)$ and $f(x')$ are complement of one another in X.*

Proof. For any $a \in X$, $a \leq 1 = x \vee x'$, so that $a \leq x$ or $a \leq x'$, or $a \in f(x)$ or $a \in f(x')$. This shows that $f(x) \cup f(x') = X$. If $a \in f(x) \cap f(x')$, then $a \leq x$ and $a \leq x'$, so that $a \leq x \wedge x' = 0$, which is impossible, since a is an atom, and by definition, must be greater than 0. □

Proposition 2. *f is a Boolean algebra homomorphism.*

Proof. First, $f(x') = X - f(x)$ by the last proposition.

Next, $f(x \vee y) = \{a \mid a \leq x \vee y\} = \{a \mid a \leq x \text{ or } a \leq y\}$ since a is an atom. But the right hand side equals $\{a \mid a \leq x\} \cup \{a \mid a \leq y\} = f(x) \cup f(y)$, we see that f preserves $\vee$.

Finally, $f(0) = \{a \mid a \leq 0\} = \varnothing$ since any atom must be greater than 0.

Hence, f is a Boolean algebra homomorphism. □

Proposition 3. *f is an injection.*

Proof. Suppose $f(x) = \varnothing$. If $x \neq 0$, then there must be some atom a such that $a \leq x$. But this implies that $f(x) \neq \varnothing$, a contradiction. Hence $x = 0$ and f is injective. □

Proposition 4. *f is conditionally complete, in the sense that if $\bigvee A$ is defined for any $A \subseteq B$, then*

$$f(\bigvee A) = \bigcup \{f(x) \mid x \in A\}.$$

Proof. Suppose $y = \bigvee A$ and $Y = f(y)$. Let $Z = \bigcup\{f(x) \mid x \in A\}$. We want to show that $Y = Z$. If $a \in Y$, then $a \leq y$, or $a \leq x$ for some $x \in A$, since a is an atom. So $a \in f(x) \subseteq Z$. Conversely, if $a \in Z$, then $a \in f(x)$, or $a \leq x$ for some $x \in A$. This means that $a \leq x \leq \bigvee A = y$, and therefore $a \in f(y) = Y$. $\square$

Proposition 5. *If B is complete, so is f. Moreover, f is surjective.*

Proof. The first sentence is a direct consequence of the previous proposition. For the second setnence, let $Y \in P(X)$. Let $y = \bigvee Y$. x exists because B is complete. So $f(y) = f(\bigvee Y) = \bigcup\{f(x) \mid x \in Y\} = \bigcup\{\{x\} \mid x \in Y\} = Y$, since each $x \in Y$ is an atom. $\square$

Rewording the above proposition, we have

Theorem 1. *Any complete atomic Boolean algebra is isomorphic (as complete Boolean algebras) to the powerset of some set, namely, the set of all of its atoms.*

A useful application of this representation theorem is the following:

Corollary 1. *The cardinality of a finite Boolean algebra is a power of* 2.

Proof. Every finite Boolean algebra is complete and atomic, and hence isomorphic to the powerset of a set, which is also finite, and the result follows. $\square$

Remark. The proof does not depend on the representation of a Boolean algebra by a field of sets.