characterization of ordered groups of rank one

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For an ordered group, having http://planetmath.org/IsolatedSubgrouprank one is to an Archimedean property. In this entry, we use multiplicative notation for groups.

Lemma An ordered group has rank one if and only if, for every two elements x and y such that $x < y < 1$, there exists an integer $n > 1$ such that $y^n < x$.

Proof Suppose that the Archimedean property is satisfied and that F is an isolated subgroup of G. We shall show that if F contains any element other than the identity, then $F = G$. First note that there must exist an $x \in F$ such that $x < 1$. By assumption, there must exist an element $x' \in F$ such that $x' \neq 1$. By conclusion 1 of the basic theorem on ordered groups, either $x' < 1$, or $x' > 1$ (since we assumed that the case $x' = 1$ is excluded). If $x' < 1$, set $x = x'$. If not, by conclusion 5, if $x' > 1$, then we will have $x'^{-1} < 0$ and therefore will set $x = x'^{-1}$ when $x > 1$.

Let y be any element of G. There are five possibilities:

1. $y = 1$
2. $x = y$
3. $x < y < 1$
4. $y < x < 1$
5. $1 < y$

We shall show that in each of these cases, $y \in F$.

1. Trivial — 1 is an element of every group.
2. Trivial — x is assumed to belong to F
3. Since F is an isolated subgroup, $y \in G$.
4. By the Archimedean property,there exists an integer n such that $x^n < y < 1$. Since $x^n \in F$ and F is http://planetmath.org/IsolatedSubgroupisolated, it follows that $y \in F$.
5. $1 < y$ By conclusion 5 of the basic theorem on ordered groups, $y^{-1} < 1$. By conclusion 1 of the same theorem, either $y^{-1} < x$ or $y^{-1} = 1$ or $x < y$. In each of these three cases, it follows that $y^{-1} \in F$ from what we have already shown. Since F is a group, $y^{-1} \in F$ implies $y \in F$.

This shows that the only isolated subgroups of G are the two trivial subgroups (i.e. the group $\{1\}$ and G itself), and hence G has rank one.

Next, suppose that G does not enjoy the Archimedean property. Then there must exist $x \in G$ and $y \in G$ such that $x < y^n < 1$ for all integers $n > 0$. Define the sets F_n as

$$F_n = \{z \in G \mid y^n \leqq z \leqq y^{-n}\}$$

and define $F = \bigcup_{n=1}^{\infty} F_n$.

We shall show that F is a subgroup of G. First, note that, by a corollary of the basic theorem on ordered groups, $y^n < 1 < y$, so $1 \in F_n$ for all n, hence $1 \in F$. Second, suppose that $z \in F_n$. Then $y^n \leqq z \leqq y^{-n}$. By conclusion 5 of the basic theorem, $y^n \leqq z$ implies $z^{-1} \leqq y^{-n}$ and $z \leqq y^{-n}$ implies $y^n \leqq z^{-1}$. Thus, $y^n \leqq z^{-1} \leqq y^{-n}$, so $z^{-1} \in F_n$. Hence, if $z \in F$, then $z^{-1} \in F$. Third, suppose that $z \in F$ and $w \in F$. Then there must exist integers m and n such that $z \in F_n$ and $w \in F_m$, so

$$y^n \leqq z \leqq y^{-n}$$

and

$$y^m \leqq w \leqq y^{-m}.$$

Using conclusion 4 of the main theorem repeatedly, we conclude that

$$y^{m+n} \leqq zw \leqq y^{-m-n}$$

so $zw \in F_{m+n}$. Hence, if $z \in F$ and $w \in F$, then $zw \in F$. this the proof that F is a subgroup of G.

Not only is F a subgroup of G, it is an isolated subgroup. Suppose that $f \in F$ and $g \in G$ and $f \leqq g \leqq 1$. Since $f \in F$, there must exist an n such that $f \in F_n$, hence $y^n \leqq f$. By conclusion 2 of the basic theorem on ordered groups, $y^n \leqq f$ and $f \leqq g$ imply $y^n \leqq g$. Combining this with the facts that $g \leqq 1$ and $1 \leqq y^{-n}$, we conclude that $y^n \leqq g \leqq y^{-n}$, so $g \in F_n$. Hence $g \in F$.

Note that F is not trivial since $y \notin F$. The reason for this is that $x \notin F_n$ for any n because we assumed that $x < y^n$ for all n. Hence, the order of the group G must be at least 2 because F and $\{1\}$ are two examples of isolated subgroups of F. Q.E.D.