proof of Schroeder-Bernstein theorem using Tarski-Knaster theorem

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The Tarski-Knaster theorem can be used to give a short, elegant proof of the Schroeder-Bernstein theorem.

Proof. Suppose $f : S \to T$ and $g : T \to S$ are injective. Define a function $\varphi\colon P(S) \to P(S)$ by $\varphi(X) = S \setminus g(T \setminus f(X))$.

If $X \subseteq Y \subseteq S$, then $S \setminus g(T \setminus f(X)) \subseteq S \setminus g(T \setminus f(Y))$, and so φ is monotone. Since $P(S)$ is a complete lattice, we may apply the Tarski-Knaster theorem to conclude that the set of fixed points of φ is a complete lattice and thus nonempty.

Let C be a fixed point of φ. We have

$$S \setminus C = g(T \setminus f(C)).$$

Hence $g|_{T\setminus f(C)}\colon T \setminus f(C) \to S \setminus C$ and $f|_C : C \to f(C)$ are bijections. We can therefore construct the desired bijection $h\colon S \to T$ by defining

$$h(x) = \begin{cases} f(x) & \text{if } x \in C \\ (g|_{T\setminus f(C)})^{-1}(x) & \text{if } x \notin C. \end{cases} \qquad \square$$

The usual proof of Schroeder-Bernstein theorem explicitly constructs a fixed point of φ.

References

[1] Thomas Forster, *Logic, induction and sets*, Cambridge University Press, Cambridge, 2003.

[2] M. Kolibiar, A. Legéň, T. Šalát, and Š. Znám, *Algebra a príbuzné disciplíny*, Alfa, Bratislava, 1992 (Slovak).