criteria for a poset to be a complete lattice

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Proposition. Let L be a poset. Then the following are equivalent.

1. L is a complete lattice.
2. for every subset A of L, $\bigvee A$ exists.
3. for every finite subset F of L and every directed set D of L, $\bigvee F$ and $\bigvee D$ exist.

Proof. Implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. are clear. We will show 3. $\Rightarrow$ 2. $\Rightarrow$ 1.

(3. $\Rightarrow$ 2.) If $A = \varnothing$, then $\bigvee A = 0$ by definition. So assume A be a non-empty subset of L. Let A' be the set of all finite subsets of A and $B = \{\bigvee F \mid F \in A'\}$. By assumption, B is well-defined and $A \subseteq B$. Next, let B' be the set of all directed subsets of B, and $C = \{\bigvee D \mid D \in B'\}$. By assumption again, C is well-defined and $B \subseteq C$. Now, every chain in C has a maximal element in C (since a chain is a directed set), C itself has a maximal element d by Zorn's Lemma. We will show that d is the least upper bound of elments of A. It is clear that each $a \in A$ is bounded above by d ($A \subseteq B \subseteq C$). If t is an upper bound of elements of A, then it is an upper bound of elements of B, and hence an upper bound of elements of C, which means $d \leq t$.

(2. $\Rightarrow$ 1.) By assumption $\bigvee \varnothing$ exists ($= 0$), so that $\bigwedge L = 0$. Now suppose A is a proper subset of L. We want to show that $\bigwedge A$ exists. If $A = \varnothing$, then $\bigwedge A = \bigvee L = 1$ by definition of an arbitrary meet over the empty set. So assume $A \neq \varnothing$. Let A' be the set of lower bounds of A: $A' = \{x \in L \mid x \leq a \text{ for all } a \in A\}$ and let $b = \bigvee A'$, the least upper bound of A'. b exists by assumption. Since A is a set of upper bounds of A', $b \leq a$ for all $a \in A$. This means that b is a lower bound of elements of A, or $b \in A'$. If x is any lower bound of elements of A, then $x \leq b$, since x is bounded above by b ($b = \bigvee A'$). This shows that $\bigwedge A$ exists and is equal to b. $\square$

Remarks.

- Dually, a poset is a complete lattice iff every subset has an infimum iff infimum exists for every finite subset and every directed subset.
- The above proposition shows, for example, that every closure system is a complete lattice.