order ideal

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Order Ideals and Filters

Let P be a poset. A subset I of P is said to be an *order ideal* if

- I is a lower set: $\downarrow I = I$, and
- I is a directed set: I is non-empty, and every pair of elements in I has an upper bound in I.

An order ideal is also called an ideal for short. An ideal is said to be *principal* if it has the form $\downarrow x$ for some $x \in P$.

Given a subset A of a poset P, we say that B is the ideal generated by A if B is the smallest order ideal (of P) containing A. B is denoted by $\langle A \rangle$. Note that $\langle A \rangle$ exists iff A is a directed set. In particular, for any $x \in P$, $\downarrow x$ is the ideal generated by x. Also, if P is an upper semilattice, then for any $A \subseteq P$, let A' be the set of finite joins of elements of A, then A' is a directed set, and $\langle A \rangle = \downarrow A'$.

Dually, an *order filter* (or simply a *filter*) in P is a non-empty subset F which is both an upper set and a filtered set (every pair of elements in F has a lower bound in F). A *principal filter* is a filter of the form $\uparrow x$ for some $x \in P$.

Remark. This is a generalization of the notion of a `http://planetmath.org/Filter`filter in a set. In fact, both ideals and filters are generalizations of ideals and filters in semilattices and lattices.

Examples in a Semilattice

A subset I in an upper semilattice P is a *semilattice ideal* if

1. if $a, b \in I$, then $a \vee b \in I$ (condition for being an upper subsemilattice)
2. if $a \in I$ and $b \leq a$, then $b \in I$

Then the two definitions are equivalent: if P is an upper semilattice, then $I \subseteq P$ is a semilattice ideal iff I is an order ideal of P: if I is a semilattice ideal, then I is clearly a lower and directed (since $a \vee b$ is an upper bound of a and b); if I is an order ideal, then condition 2 of a semilattice ideal is satisfied. If $a, b \in I$, then there is a $c \in I$ that is an upper bound of a and b. Since I is lower, and $a \vee b \leq c$, we have $a \vee b \in I$.

Going one step further, we see that if P is a lattice, then a lattice ideal is exactly an order ideal: if I is a lattice ideal, then it is clearly an upper subsemilattice, and if $b \leq a \in I$, then $b = a \wedge b \in I$ also, so that I is a semilattice ideal. On the other hand, if I is a semilattice ideal, then I is an upper subsemilattice, as well as a lower subsemilattice, for if $a \in I$, then $a \wedge b \in I$ as well since $a \wedge b \leq a$. This shows that I is a lattice ideal.

Dually, we can define a *filter* in a lower semilattice, which is equivalent to an order filter of the underly poset. Going one step futher, we also see that a lattice filter in a lattice is an order filter of the underlying poset.

Remark. An alternative but equivalent characterization of a semilattice ideal I in an upper semilattice P is the following: $a, b \in I$ iff $a \vee b \in I$.