congruence relation on an algebraic system

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Defines	congruence
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Defines	extension of a subalgebra by a congruence
Defines	principal congruence
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Let (A,O) be an algebraic system. A *congruence relation*, or simply a *congruence* $\mathfrak{C}$ on A

1. is an equivalence relation on A; if $(a,b)\in\mathfrak{C}$ we write $a\equiv b \pmod{\mathfrak{C}}$, and

2. respects every n-ary operator on A: if ω_A is an n-ary operator on A ($\omega\in O$), and for any $a_i,b_i\in A$, $i=1,\dots,n$, we have

$$a_i\equiv b_i \pmod{\mathfrak{C}} \qquad \text{implies} \qquad \omega_A(a_1,\dots,a_n)\equiv\omega_A(b_1,\dots,b_n) \pmod{\mathfrak{C}}.$$

For example, A^2 and $\Delta_A:=\{(a,a)\mid a\in A\}$ are both congruence relations on A. Δ_A is called the *trivial congruence* (on A). A *proper* congruence relation is one not equal to A^2.

Remarks.

- $\mathfrak{C}$ is a congruence relation on A if and only if $\mathfrak{C}$ is an equivalence relation on A and a subalgebra of the `http://planetmath.org/DirectProductOfAlgebras`product $A\times A$.

- The set of congruences of an algebraic system is a complete lattice. The meet is the usual set intersection. The join (of an arbitrary number of congruences) is the `http://planetmath.org/PartitionsFormALattice`join of the underlying equivalence relations. This join corresponds to the subalgebra (of $A\times A$) generated by the union of the underlying sets of the congruences. The *lattice of congruences* on A is denoted by $\mathrm{Con}(A)$.

- **(restriction)** If $\mathfrak{C}$ is a congruence on A and B is a subalgebra of A, then $\mathfrak{C}_B$ defined by $\mathfrak{C}\cap(B\times B)$ is a congruence on B. The equivalence of $\mathfrak{C}_B$ is obvious. For any n-ary operator ω_B inherited from A's ω_A, if $a_i\equiv b_i \pmod{\mathfrak{C}_B}$, then $\omega_B(a_1,\dots,a_n)=\omega_A(a_1,\dots,a_n)\equiv\omega_A(b_1,\dots,b_n)=\omega_B(b_1,\dots,b_n) \pmod{\mathfrak{C}}$. Since both $\omega_B(a_1,\dots,a_n)$ and $\omega_B(b_1,\dots,b_n)$ are in B, $\omega_B(a_1,\dots,a_n)\equiv\omega_B(b_1,\dots,b_n) \pmod{\mathfrak{C}_B}$ as well. $\mathfrak{C}_B$ is the congruence *restricted* to B.

- **(extension)** Again, let $\mathfrak{C}$ be a congruence on A and B a subalgebra of A. Define $B^{\mathfrak{C}}$ by $\{a\in A\mid (a,b)\in\mathfrak{C} \text{ and } b\in B\}$. In other words, $a\in B^{\mathfrak{C}}$ iff $a\equiv b \pmod{\mathfrak{C}}$ for some $b\in B$. We assert that $B^{\mathfrak{C}}$ is a subalgebra of A. If ω_A is an n-ary operator on A and $a_1,\dots,a_n\in$

$B^{\mathfrak{C}}$, then $a_i \equiv b_i \pmod{\mathfrak{C}}$, so $\omega_A(a_1, \ldots, a_n) \equiv \omega_A(b_1, \ldots, b_n) \pmod{\mathfrak{C}}$. Since $\omega_A(b_1, \ldots, b_n) \in B$, $\omega_A(a_1, \ldots, a_n) \in B^{\mathfrak{C}}$. Therefore, $B^{\mathfrak{C}}$ is a subalgebra. Because $B \subseteq B^{\mathfrak{C}}$, we call it the *extension* of B by $\mathfrak{C}$.

- Let B be a subset of $A \times A$. The smallest congruence $\mathfrak{C}$ on A such that $a \equiv b \pmod{\mathfrak{C}}$ for all $a, b \in B$ is called the *congruence generated by* B. $\mathfrak{C}$ is often written $\langle B \rangle$. When B is a singleton $\{(a, b)\}$, then we call $\langle B \rangle$ a *principal congruence*, and denote it by $\langle (a, b) \rangle$.

Quotient algebra

Given an algebraic structure (A, O) and a congruence relation $\mathfrak{C}$ on A, we can construct a new O-algebra $(A/\mathfrak{C}, O)$, as follows: elements of $A/\mathfrak{C}$ are of the form $[a]$, where $a \in A$. We set

$$[a] = [b] \text{ iff } a \equiv b \pmod{\mathfrak{C}}.$$

Furthermore, for each n-ary operator ω_A on A, define $\omega_{A/\mathfrak{C}}$ by

$$\omega_{A/\mathfrak{C}}\big([a_1], \ldots, [a_n]\big) := \big[\omega_A(a_1, \ldots, a_n)\big].$$

It is easy to see that $\omega_{A/\mathfrak{C}}$ is a well-defined operator on $A/\mathfrak{C}$. The O-algebra thus constructed is called the *quotient algebra* of A over $\mathfrak{C}$.

Remark. The bracket $[\cdot] : A \to A/\mathfrak{C}$ is in fact an epimorphism, with `http://planetmath.org/KernelOfAHomomorphismBetweenAlgebraicSystems`kernel $\ker([\cdot]) = \mathfrak{C}$. This means that every congruence of an algebraic system A is the kernel of some homomorphism from A. $[\cdot]$ is usually written $[\cdot]_{\mathfrak{C}}$ to signify its association with $\mathfrak{C}$.

References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).