algebraic system

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Synonym	algebraic structure
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Related topic	RelationalSystem
Related topic	Model
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Related topic	PartiallyOrderedAlgebraicSystem
Defines	n-ary operator
Defines	finitary operator
Defines	infinitary operator
Defines	operator set
Defines	constant operator
Defines	operator symbol
Defines	nullary operator
Defines	type
Defines	trivial algebraic system
Defines	finite algebra

An algebraic system, loosely speaking, is a set, together with some operations on the set. Before formally defining what an algebraic system is, let us recall that a n-ary operation (or operator) on a set A is a function whose domain is A^n and whose range is a subset of A. Here, n is a non-negative integer. When $n = 0$, the operation is usually called a nullary operation, or a constant, since one element of A is singled out to be the (sole) value of this operation. A finitary operation on A is just an n-ary operation for some non-negative integer n.

Definition. An *algebraic system* is an ordered pair (A, O), where A is a set, called the underlying set of the algebraic system, and O is a set, called the operator set, of finitary operations on A.

We usually write $\boldsymbol{A}$, instead of (A, O), for brevity.

A prototypical example of an algebraic system is a group, which consists of the underlying set G, and a set O consisting of three operators: a constant e called the multiplicative identity, a unary operator called the multiplicative inverse, and a binary operator called the multiplication.

For a more comprehensive listing of examples, please see this `http://planetmath.org/Example`

Remarks.

- An algebraic system is also called an algebra for short. Some authors require that A be non-empty. Note that A is automatically non-empty if O contains constants. A *finite algebra* is an algebra whose underlying set is finite.
- By definition, all operators in an algebraic system are finitary. If we allow O to contain infinitary operations, we have an *infinitary algebraic system*. Other generalizations are possible. For example, if the operations are allowed to be multivalued, the algebra is said to be a *multialgebra*. If the operations are not everywhere defined, we get a *partial algebra*. Finally, if more than one underlying set is involved, then the algebra is said to be *many-sorted*.

The study of algebraic systems is called the theory of universal algebra. The first important thing in studying algebraic system is to compare systems that are of the same "type". Two algebras are said to have the same *type* if there is a one-to-one correspondence between their operator sets such that an n-ary operator in one algebra is mapped to an n-ary operator in the other algebra. A more formal way of doing this is to define what a *type* is:

Definition. A *type* is a set τ, whose elements are called operator symbols, such that there is a function $a : \tau \to \mathbb{N} \cup \{0\}$. Given an operator symbol f, its image $a(f)$ is called the arity of f.

Remark. It is often the practice to well-order τ, and write τ as a sequence of non-negative integers $\langle a(f_1), a(f_2), \ldots, \rangle$. When τ is finite, the convention is to order the sequence in non-increasing order: $a(f_1) \geq a(f_2) \geq \cdots \geq a(f_n)$.

Definition. An algebraic system $\boldsymbol{A}$ is said to be of type τ if there is a bijection between O and τ so that every operator symbol f in τ corresponds to an operator $f_{\boldsymbol{A}}$ of arity $a(f)$ in O. When the algebra $\boldsymbol{A}$ is said to be of type τ, we also say that $\boldsymbol{A}$ is a τ-algebra.

For example, a group is an algebraic system of type $\langle 2, 1, 0\rangle$, where 2 is the arity of the group multiplication, 1 is the arity of the group inverse, and 0 is the arity of the group multiplicative identity.

References

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[4] P. Jipsen: `http://math.chapman.edu/cgi-bin/structures?HomePage`*Mathematical Structures: Homepage*