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more on division in groups

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In the parent entry, it is shown that a non-empty set G equipped a binary operation "/" called "division" satisfying three identities has the structure of a group. In this entry, we show that two identities are enough. Associated with every $x, y \in G$, we set

1. $E(x) := x/x$,
2. (inverse) $x^{-1} := E(x)/x$, and
3. (multiplication) $x \cdot y := x/y^{-1}$ (we also write xy for $x \cdot y$ for simplicity)

Theorem 1. *Let G be a non-empty set with a binary operation / on it such that*

1. $(x/z)/(y/z) = x/y$
2. $(x/x)/((y/y)/y) = y$

hold for all $x, y, z \in G$. Then G has the structure of a group

Proof. From 1, we have $E(x/z) = (x/z)/(x/z) = x/x = E(x)$, so $E(E(x)) = E(x/x) = E(x)$. From 2, we have $y = E(x)/y^{-1}$, so $E(y) = E(E(x)/y^{-1}) = E(E(x)) = E(x)$. This shows that $E : G \to G$ is a constant function, whose value we denote by e.

Note that $x = e/x^{-1}$ by rewriting condition 2. This implies that $e \cdot x = e/x^{-1} = x$. In addition, $x^{-1} = e/x$ by rewriting the definition of the inverse. In particular, $e^{-1} = e/e = e$. Furthermore, since $x/e = (e/x^{-1})/(x^{-1}/x^{-1}) = e/x^{-1} = x$, this implies that $x \cdot e = x/e^{-1} = x/e = x$. So e is the "identity" in G with respect to $\cdot$.

Next, $x^{-1} \cdot x = x^{-1}/x^{-1} = e$. To see that $x \cdot x^{-1} = e$, first observe that $(x^{-1})^{-1} = e/x^{-1} = x$, so $x \cdot x^{-1} = x/(x^{-1})^{-1} = x/e = x$. This shows that x^{-1} is the "inverse" of x in G with respect to $\cdot$.

Finally, we need to verify $(xy)z = x(yz)$. To see this, first note that

1. $(xy)/y = (x/y^{-1})/y = (x/y^{-1})/(e/y^{-1}) = x/e = x$, and
2. $(xy)^{-1} = e/(xy) = e/(x/y^{-1}) = (y^{-1}/y^{-1})/(x/y^{-1}) = y^{-1}/x = y^{-1}x^{-1}$.

From the two identities above, we deduce

$$\begin{aligned}(xy)z &= (xy)/z^{-1} = (x/y^{-1})/z^{-1} = (x/y^{-1})/((z^{-1}y^{-1})/y^{-1}) \\ &= x/(z^{-1}y^{-1}) = x/(yz)^{-1} = x(yz),\end{aligned}$$

completing the proof. $\square$

There is also a companion theorem for abelian groups:

Theorem 2. *Let G be a non-empty set with a binary operation $/$ on it such that*

1. $x/(x/y) = y$
2. $(x/y)/z = (x/z)/y$

hold for all $x, y, z \in G$. Then G has the structure of an abelian group

Proof. First, note that $E(x/y) = (x/y)/(x/y) = (x/(x/y))/y = y/y = E(y)$, so $E(x) = E((x/y)/x) = E((x/x)/y) = E(y)$, implying that E is a constant function on G. Again, denote its value by e. Below are some simple consequences:

1. $x/e = x/(x/x) = x$
2. $e^{-1} = e/e = e$
3. $(x^{-1})^{-1} = (e/x)^{-1} = e/(e/x) = x$

So, $xe = x/e^{-1} = x/e = x$. Also, $ex = e/x^{-1} = e/(e/x) = x$. This shows that e is the "identity" of G with respect to $\cdot$. In addition, $x^{-1}x = x^{-1}/x^{-1} = e$ and $xx^{-1} = x/(x^{-1})^{-1} = x/x = e$, showing that x^{-1} is the "inverse" of x in G with respect to $\cdot$.

Finally, we show that $\cdot$ is commutative and associative. For commutativity, we have $xy = (ex)y = (e/x^{-1})/y^{-1} = (e/y^{-1})/x^{-1} = (ey)x = yx$, and associativity is shown by $x(yz) = (yz)x = (y/z^{-1})/x^{-1} = (y/x^{-1})/z^{-1} = (yx)z = (xy)z$. $\square$

Remark. Remarkably, it can be shown (see reference) that a non-empty set G with binary operation / satisfying a single identity:

$$x/((((x/x)/y)/z)/(((x/x)/x)/z)) = y$$

has the structure of a group, and satisfying

$$x/((y/z)/(y/x)) = z$$

has the structure of an abelian group.

References

[1] G. Higman, B. H. Neumann *Groups as groupoids with one law*. Publ. Math. Debrecen 2 pp. 215-221, (1952).