subfunction

Canonical name	Subfunction
Date of creation	2013-03-22 18:41:54
Last modified on	2013-03-22 18:41:54
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	8
Author	CWoo (3771)
Entry type	Definition
Classification	msc 08A55
Classification	msc 03E20
Defines	restriction

Definition. Let $f : A \to B$ and $g : C \to D$ be partial functions. g is said to be a *subfunction* of f if

$$g \subseteq f \cap (C \times D).$$

In other words, g is a subfunction of f iff whenever $x \in C$ such that $g(x)$ is defined, then $x \in A$, $f(x)$ is defined, and $g(x) = f(x)$.

If we set $C' = A \cap C$ and $D' = B \cap D$, then $g \subseteq f \cap (C' \times D')$, so there is no harm in assuming that C and D are subsets of A and B respectively, which we will do for the rest of the discussion.

In practice, whenever g is a subfunction of f, we often assume that g and f have the same domain and codomain. Otherwise, we would specify that g is a subfunction of $f : A \to B$ with domain C and codomain D.

For example, $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sqrt{x^2 - 1}$$

is a partial function, whose domain of definition is $(-\infty, -1] \cup [1, \infty)$, and the partial function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \frac{x^2 - 1}{\sqrt{x^2 - 1}}$$

is a subfunction of f. The domain of definition of g is $(-\infty, -1) \cup (1, \infty)$.

Two immediate properties of a subfunction $g : C \to D$ of $f : A \to B$ are

- the range of g is a subset of the range of f:

$$g(C) \subseteq f(C),$$

- the domain of definition of g is a subset of the domain of definition of f:

$$g^{-1}(D) \subseteq f^{-1}(D).$$

Definition. A subfunction $g : C \to D$ of $f : A \to B$ is called a *restriction of* f *relative to* D, if $g(C) = f(C) \cap D$, and a *restriction of* f if $g(C) = f(C)$.

Every partial function $g : C \to D$ corresponds to a unique restriction $g' : C \to g(C)$ of g.

A restriction $g : C \to D$ of $f : A \to B$ is certainly a restriction of f relative to D, since $f(C) \cap D = g(C) \cap D = g(C)$, but not conversely. For example, let A be the set of all non-negative integers and $-_A : A^2 \to A$ the ordinary subtraction. $-_A$ is easily seen to be a partial function. Let B be the set of all positive integers. Then $-_B : B^2 \to B$ is a restriction of $-_A : A^2 \to A$, relative to B. However, $-_B$ is not a restriction of $-_A$, for $n -_B n$ is not defined, while $n -_A n = 0 \in A$.

References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).