direct product of algebras

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Defines	direct product
Defines	direct factor
Defines	direct power
Defines	projection
Defines	empty direct product

In this entry, let O be a fixed operator set. All algebraic systems have the same type (they are all O-algebras).

Let $\{A_i \mid i \in I\}$ be a set of algebraic systems of the same type (O) indexed by I. Let us form the Cartesian product of the underlying sets and call it A:

$$A := \prod_{i \in I} A_i.$$

Recall that element a of A is a function from I to $\bigcup A_i$ such that for each $i \in I$, $a(i) \in A_i$.

For each $\omega \in O$ with arity n, let ω_{A_i} be the corresponding n-ary operator on A_i. Define $\omega_A : A^n \to A$ by

$$\omega_A(a_1, \dots, a_n)(i) = \omega_{A_i}(a_1(i), \dots, a_n(i)) \quad \text{for all } i \in I.$$

One readily checks that ω_A is a well-defined n-ary operator on A. A equipped with all ω_A on A is an O-algebra, and is called the *direct product* of A_i. Each A_i is called a *direct factor* of A.

If each $A_i = B$, where B is an O-algebra, then we call A the direct power of B and we write A as B^I (keep in mind the isomorphic identifications).

If A is the direct product of A_i, then for each $i \in I$ we can associate a homomorphism $\pi_i : A \to A_i$ called a *projection* given by $\pi_i(a) = a(i)$. It is a homomorphism because $\pi_i(\omega_A(a_1, \dots, a_n)) = \omega_A(a_1, \dots, a_n)(i) = \omega_{A_i}(a_1(i), \dots, a_n(i)) = \omega_{A_i}(\pi_i(a_1), \dots, \pi_i(a_n))$.

Remark. The direct product of a single algebraic system is the algebraic system itself. An *empty direct product* is defined to be a trivial algebraic system (one-element algebra).