closure of a subset under relations

Canonical name	ClosureOfASubsetUnderRelations
Date of creation	2013-03-22 16:21:26
Last modified on	2013-03-22 16:21:26
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	36
Author	CWoo (3771)
Entry type	Definition
Classification	msc 08A02
Defines	closed under
Defines	closure property

Let A be a set and R be an n-ary relation on A, $n \geq 1$. A subset B of A is said to be *closed under* R, or R-*closed*, if, whenever $b_1, \ldots, b_{n-1} \in B$, and $(b_1, \ldots, b_{n-1}, b_n) \in R$, then $b_n \in B$.

Note that if R is unary, then B is R-closed iff $R \subseteq B$.

More generally, let A be a set and $\mathcal{R}$ a set of (finitary) relations on A. A subset B is said to be $\mathcal{R}$-*closed* if B is R-closed for each $R \in \mathcal{R}$.

Given $B \subseteq A$ with a set of relations $\mathcal{R}$ on A, we say that $C \subseteq A$ is an $\mathcal{R}$-*closure* of B if

1. $B \subseteq C$,
2. C is $\mathcal{R}$-closed, and
3. if $D \subseteq A$ satisfies both 1 and 2, then $C \subseteq D$.

By condition 3, C, if exists, must be unique. Let us call it the $\mathcal{R}$-closure of B, and denote it by $\mathrm{Cl}_{\mathcal{R}}(B)$. If $\mathcal{R} = \{R\}$, then we call it the R-closure of B, and denote it by $\mathrm{Cl}_R(B)$ correspondingly.

Here are some examples.

1. Let $A = \mathbb{Z}$, $B = \{5\}$, and R be the relation that mRn whenever m divides n. Clearly B is not closed under R (for example, $(5, 10) \in R$ but $10 \notin B$). Then $\mathrm{Cl}_R(B) = 5\mathbb{Z}$. If R is instead the relation $\leq$, then $\mathrm{Cl}_{\leq}(B) = \{n \in A \mid n \geq 5\}$.

2. This is an example where R is in fact a function (operation). Suppose $A = \mathbb{Z}$ and R is the binary operation subtraction $-$. Suppose $B = \{3, 5\}$. Then $\mathrm{Cl}_-(B) = A$. To see this, set $C = \mathrm{Cl}_-(B)$. Note that $2 = 5 - 3 \in C$ so $1 = 3 - 2$ as well as $-1 = 2 - 3 \in C$. This means that if $n \in C$, both $n - 1$ and $n + 1 \in C$. By induction, $C = \mathbb{Z}$. In general, if $B = \{p, q\}$, where p, q are coprime, then $\mathrm{Cl}_-(B) = \mathbb{Z}$. This is essentially the result of the Chinese Remainder Theorem.

3. If R is unary, then the R-closure of $B \subseteq A$ is just $B \cup R$. When every $R \in \mathcal{R}$ is unary, then the $\mathcal{R}$-closure of B in A is $(\bigcup \mathcal{R}) \cup B$.

Proposition 1. *$\mathrm{Cl}_{\mathcal{R}}(B)$ exists for every $B \subseteq A$.*

Proof. Let $S_{\mathcal{R}}(B)$ be the set of subsets of A satisfying the defining conditions 1 and 2 of $\mathcal{R}$-closures above, partially ordered by $\subseteq$. If $\mathcal{C} \subseteq S_{\mathcal{R}}(B)$, then $\bigcap \mathcal{C} \in S_{\mathcal{R}}(B)$. To see this, we break the statement down into cases:

- In the case when $\mathcal{C} = \varnothing$, we have $\bigcap \mathcal{C} = A \in S_{\mathcal{R}}(B)$.
- When $C := \bigcap \mathcal{C} \neq \varnothing$, pick any n-ary relation $R \in \mathcal{R}$.

  1. If $n = 1$, then, since aach $D \in \mathcal{C}$ is R-closed, $R \subseteq D$. Therefore, $R \subseteq \bigcap \mathcal{C} = \bigcap \{D \mid D \in \mathcal{C}\} = C$. So C is R-closed.
  2. If $n > 1$, pick elements $c_1, \ldots, c_{n-1} \in C$ such that $(c_1, \ldots, c_n) \in R$. As each $c_i \in D$ for $i = 1, \ldots, n-1$, and D is R-closed, $c_n \in D$. Since $c_n \in D$ for every $D \in \mathcal{C}$, $c_n \in C$ as well. This shows that C is R-closed.

  In both cases, $B \subseteq C$ since $B \subseteq D$ for every $D \in \mathcal{C}$. Therefore, $C \in S_{\mathcal{R}}(B)$.

Hence, $S_{\mathcal{R}}(B)$ is a complete lattice by virtue of `http://planetmath.org/CriteriaForAPosetToBe` fact, which means that $S_{\mathcal{R}}(B)$ has a minimal element, which is none other than the $\mathcal{R}$-closure $\mathrm{Cl}_{\mathcal{R}}(B)$ of B. □

Remark. It is not hard to see $\mathrm{Cl}_{\mathcal{R}}$ has the following properties:

1. $B \subseteq \mathrm{Cl}_{\mathcal{R}}(B)$,
2. $\mathrm{Cl}_{\mathcal{R}}(\mathrm{Cl}_{\mathcal{R}}(B)) = \mathrm{Cl}_{\mathcal{R}}(B)$, and
3. if $B \subseteq C$, then $\mathrm{Cl}_{\mathcal{R}}(B) \subseteq \mathrm{Cl}_{\mathcal{R}}(C)$.

Next, assume that $\mathcal{S}$ is another set of finitary relations on A. Then

1. if $\mathcal{R} \subseteq \mathcal{S}$, then $\mathrm{Cl}_{\mathcal{S}}(B) \subseteq \mathrm{Cl}_{\mathcal{R}}(B)$,
2. $\mathrm{Cl}_{\mathcal{S}}(\mathrm{Cl}_{\mathcal{R}}(B)) \subseteq \mathrm{Cl}_{\mathcal{R} \cap \mathcal{S}}(B)$, and
3. $\mathrm{Cl}_{\mathcal{S}}(\mathrm{Cl}_{\mathcal{R}}(B)) = \mathrm{Cl}_{\mathcal{R} \cap \mathcal{S}}(B)$ if $\mathcal{R} \subseteq \mathcal{S}$ or $\mathcal{S} \subseteq \mathcal{R}$.