restricted direct product of algebraic systems

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Defines	restricted direct product
Defines	weak direct product

Let $\{A_i \mid i \in I\}$ be a family of algebraic systems indexed by a set I. Let J be a Boolean ideal in $P(I)$, the Boolean algebra over the power set of I. A subset B of the direct product $\prod\{A_i \mid i \in I\}$ is called a *restricted direct product* of A_i if

1. B is a subalgebra of $\prod\{A_i \mid i \in I\}$, and
2. given any $(a_i) \in B$, we have that $(b_i) \in B$ iff $\{i \in I \mid a_i \neq b_i\} \in J$.

If it is necessary to distinguish the different restricted direct products of A_i, we often specify the "restriction", hence we say that B is a J-restricted direct product of A_i, or that B is restricted to J.

Here are some special restricted direct products:

- If $J = P(I)$ above, then B is the direct product $\prod A_i$, for if $(b_i) \in \prod A_i$, then clearly $\{i \in I \mid a_i \neq b_i\} \in P(I)$, where $(a_i) \in B$ (B is non-empty since it is a subalgebra). Therefore $(b_i) \in B$.

  This justifies calling the direct product the "unrestricted direct product" by some people.

- If J is the ideal consisting of all finite subsets of I, then B is called the *weak direct product* of A_i.

- If J is the singleton $\{\varnothing\}$, then B is also a singleton: pick $a, b \in B$, then $\{i \mid a_i \neq b_i\} = \varnothing$, which is equivalent to saying that $(a_i) = (b_i)$.

Remark. While the direct product of A_i always exists, restricted direct products may not. For example, in the last case above, A $\varnothing$-restricted direct product exists only when there is an element $a \in \prod A_i$ that is fixed by all operations on it: that is, if f is an n-ary operation on $\prod A_i$, then $f(a, \dots, a) = a$. In this case, $\{a\}$ is a $\varnothing$-restricted direct product of $\prod A_i$.

References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).