free algebra

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Let $\mathcal{K}$ be a class of algebraic systems (of the same type τ). Consider an algebra $A \in \mathcal{K}$ `http://planetmath.org/SubalgebraOfAnAlgebraicSystem`generated by a set $X = \{x_i\}$ indexed by $i \in I$. A is said to be a *free algebra* over $\mathcal{K}$, with *free generating set* X, if for any algebra $B \in \mathcal{K}$ with any subset $\{y_i \mid i \in I\} \subseteq B$, there is a homomorphism $\phi : A \to B$ such that $\phi(x_i) = y_i$.

If we define $f : I \to A$ to be $f(i) = x_i$ and $g : I \to B$ to be $g(i) = y_i$, then freeness of A means the existence of $\phi : A \to B$ such that $\phi \circ f = g$.

Note that ϕ above is necessarily unique, since $\{x_i\}$ generates A. For any n-ary polynomial p over A, any $z_1, \ldots, z_n \in \{x_i \mid i \in I\}$, $\phi(p(z_1, \ldots, z_n)) = p(\phi(z_1), \ldots, \phi(z_n))$.

For example, any free group is a free algebra in the class of groups. In general, however, free algebras do not always exist in an arbitrary class of algebras.

Remarks.

- A is free over itself (meaning $\mathcal{K}$ consists of A only) iff A is free over some equational class.
- If $\mathcal{K}$ is an equational class, then free algebras exist in $\mathcal{K}$.
- Any term algebra of a given structure τ over some set X of variables is a free algebra with free generating set X.