subdirect product of algebraic systems

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Entry type	Definition
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Defines	subdirect product
Defines	subdirect power
Defines	subdirectly irreducible
Defines	trivial subdirect product

In this entry, all algebraic systems are of the same type. For each algebraic system, we drop the associated operator set for simplicity.

Let A_i be algebraic systems indexed by $i \in I$. B is called a *subdirect product* of A_i if

1. B is a subalgebra of the direct product of A_i.
2. for each $i \in I$, $\pi_i(B) = A_i$.

In the second condition, π_i denotes the projection homomorphism $\prod A_i \to A_i$. By restriction, we may consider π_i as homomorphisms $B \to A_i$. When B is isomorphic to $\prod A_i$, then B is a *trivial subdirect product* of A_i.

This generalizes the notion of a direct product, since in many instances, an algebraic system can not be decomposed into a direct product of algebras.

When all $A_i = C$ for some algebraic system C of the same type, then B is called a *subdirect power* of C.

Remarks.

1. A very simple example of a subdirect product is the following: let $A_1 = A_2 = \{1, 2, 3\}$. Then the subset $B = \{(x, y) \in A_1 \times A_2 \mid x \leq y\}$ is a subdirect product of the sets A_1 and A_2 (considered as algebraic systems with no operators).
2. Let B is a subdirect product of A_i, and $p_i := (\pi_i)_B$, the restriction of π_i to B. Then $B/\ker(p_i) \cong A_i$. In addition,
$$\bigcap\{\ker(p_i) \mid i \in I\} = \Delta,$$
where Δ is the diagonal relation. To see the last equality, suppose $a, b \in B$ with $a \equiv b \pmod{p_i}$. Then $a(i) = \pi_i(a) = p_i(a) = p_i(b) = \pi_i(b) = b(i)$. Since this is true for every $i \in I$, $a = b$.
3. Conversely, if A is an algebraic system and $\{\mathfrak{C}_i \mid i \in I\}$ is a set of congruences on A such that
$$\bigcap\{\mathfrak{C}_i \mid i \in I\} = \Delta.$$
Then A is isomorphic to a subdirect product of $A/\mathfrak{C}_i$.
4. An algebraic system is said to be *subdirectly irreducible* if, whenever $\mathfrak{C}_i$ are congruences on A and $\bigcap\{\mathfrak{C}_i \mid i \in I\} = \Delta$, then one of $\mathfrak{C}_i = \Delta$.

5. **Birkhoff's Theorem on the Decomposition of an Algebraic System**. Every algebraic system is isomorphic to a subdirect product of subdirectly irreducible algebraic systems. This works only when the algebraic system is finitary.