direct limit of algebraic systems

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An immediate generalization of the concept of the direct limit of a direct family of sets is the direct limit of a direct family of algebraic systems.

Direct Family of Algebraic Systems

The definition is almost identical to that of a direct family of sets, except that functions ϕ_{ij} are now homomorphisms. For completeness, we will spell out the definition in its entirety.

Let $\mathcal{A} = \{A_i \mid i \in I\}$ be a family of algebraic systems of the same type (say, they are all O-algebras), indexed by a non-empty set I. $\mathcal{A}$ is said to be a *direct family* if

1. I is a directed set,
2. whenever $i \leq j$ in I, there is a homomorphism $\phi_{ij} : A_i \to A_j$,
3. ϕ_{ii} is the identity on A_i,
4. if $i \leq j \leq k$, then $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$.

An example of this is a direct family of sets. A homomorphism between two sets is just a function between the sets.

Direct Limit of Algebraic Systems

Let $\mathcal{A}$ be a direct family of algebraic systems A_i, indexed by I $(i \in I)$. Take the disjoint union of the underlying sets of each algebraic system, and call it A. Next, a binary relation $\sim$ is defined on A as follows:

> given that $a \in A_i$ and $b \in A_j$, $a \sim b$ iff there is A_k such that $\phi_{ik}(a) = \phi_{jk}(b)$.

It is shown `http://planetmath.org/DirectLimitOfSets`here that $\sim$ is an equivalence relation on A, so we can take the quotient $A/\sim$, and denote it by A_∞. Elements of A_∞ are denoted by $[a]_I$ or $[a]$ when there is no confusion, where $a \in A$. So A_∞ is just the direct limit of A_i *considered as sets*.

Next, we want to turn A_∞ into an O-algebra. Corresponding to each set of n-ary operations ω_i defined on A_i for all $i \in I$, we define an n-ary operation ω on A_∞ as follows:

for $i = 1, \ldots, n$, pick $a_i \in A_{j(i)}$, $j(i) \in I$. Let $J := \{j(i) \mid i = 1, \ldots, n\}$. Since I is directed and J is finite, J has an upper bound $j \in I$. Let $\alpha_i = \phi_{j(i)j}(a_i)$. Define

$$\omega([a_1], \ldots, [a_n]) := [\omega_j(\alpha_1, \ldots, \alpha_n)].$$

Proposition 1. *ω is a well-defined n-ary operation on A_∞.*

Proof. Suppose $[b_1] = [a_1], \ldots, [b_n] = [a_n]$. Let α_i be defined as above, and let $a := \omega_j(\alpha_1, \ldots, \alpha_n) \in A_j$. Similarly, β_i are defined: $\beta_i := \phi_{k(i)k}(b_i) \in A_k$, where $b_i \in A_{k(i)}$. Let $b := \omega_k(\beta_1, \ldots, \beta_n) \in A_k$. We want to show that $a \sim b$.

Since $a_i \sim b_i$, $\alpha_i \sim \beta_i$. So there is $c_i := \phi_{j\ell(i)}(\alpha_i) = \phi_{k\ell(i)}(\beta_i) \in A_{\ell(i)}$. Let ℓ be the upper bound of the set $\{\ell(1), \ldots, \ell(n)\}$ and define $\gamma_i := \phi_{\ell(i)\ell}(c_i) \in A_\ell$. Then

$$\begin{aligned}
\phi_{j\ell}(a) &= \phi_{j\ell}\big(\omega_j(\alpha_1, \ldots, \alpha_n)\big) \\
&= \omega_\ell\big(\phi_{j\ell}(\alpha_1), \ldots, \phi_{j\ell}(\alpha_n)\big) \\
&= \omega_\ell\big(\phi_{\ell(1)\ell} \circ \phi_{j\ell(1)}(\alpha_1), \ldots, \phi_{\ell(n)\ell} \circ \phi_{j\ell(n)}(\alpha_n)\big) \\
&= \omega_\ell\big(\phi_{\ell(1)\ell}(c_1), \ldots, \phi_{\ell(n)\ell}(c_n)\big) \\
&= \omega_\ell\big(\phi_{\ell(1)\ell} \circ \phi_{k\ell(1)}(\beta_1), \ldots, \phi_{\ell(n)\ell} \circ \phi_{k\ell(n)}(\beta_n)\big) \\
&= \omega_\ell\big(\phi_{k\ell}(\beta_1), \ldots, \phi_{k\ell}(\beta_n)\big) \\
&= \phi_{k\ell}\big(\omega_k(\beta_1, \ldots, \beta_n)\big) \\
&= \phi_{k\ell}(b),
\end{aligned}$$

which shows that $a \sim b$. $\square$

Definition. Let $\mathcal{A}$ be a direct family of algebraic systems of the same type (say O) indexed by I. The O-algebra A_∞ constructed above is called the *direct limit* of $\mathcal{A}$. A_∞ is alternatively written $\varinjlim A_i$.

Remark. Dually, one can define an *inverse family of algebraic systems*, and its inverse limit. The inverse limit of an inverse family $\mathcal{A}$ is written A^∞ or $\varprojlim A_i$.