closure of a relation with respect to a property

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Introduction

Fix a set A. A *property* $\mathcal{P}$ of n-ary relations on a set A may be thought of as some subset of the set of all n-ary relations on A. Since an n-ary relation is just a subset of A^n, $\mathcal{P} \subseteq P(A^n)$, the powerset of A^n. An n-ary relation is said to have property $\mathcal{P}$ if $R \in \mathcal{P}$.

For example, the transitive property is a property of binary relations on A; it consists of all transitive binary relations on A. Reflexive and symmetric properties are sets of reflexive and symmetric binary relations on A correspondingly.

Let R be an n-ary relation on A. By the *closure* of an n-ary relation R with respect to property $\mathcal{P}$, or the $\mathcal{P}$-closure of R for short, we mean the smallest relation $S \in \mathcal{P}$ such that $R \subseteq S$. In other words, if $T \in \mathcal{P}$ and $R \subseteq T$, then $S \subseteq T$. We write $\mathrm{Cl}_{\mathcal{P}}(R)$ for the $\mathcal{P}$-closure of R.

Given an n-ary relation R on A, and a property $\mathcal{P}$ on n-ary relations on A, does $\mathrm{Cl}_{\mathcal{P}}(R)$ always exist? The answer is no. For example, let $\mathcal{P}$ be the anti-symmetric property of binary relations on A, and $R = A^2$. For another example, take $\mathcal{P}$ to be the irreflexive property, and $R = \Delta$, the diagonal relation on A.

However, if $A^n \in \mathcal{P}$ and $\mathcal{P}$ is closed under arbitrary intersections, then $\mathcal{P}$ is a complete lattice according to `http://planetmath.org/CriteriaForAPosetToBeACompleteLat` fact, and, as a result, $\mathrm{Cl}_{\mathcal{P}}(R)$ exists for any $R \subseteq A^n$.

Reflexive, Symmetric, and Transitive Closures

From now on, we concentrate on binary relations on a set A. In particular, we fix a binary relation R on A, and let $\mathcal{X}$ the reflexive property, $\mathcal{S}$ the symmetric property, and $\mathcal{T}$ be the transitive property on the binary relations on A.

Proposition 1. *Arbitrary intersections are closed in $\mathcal{X}$, $\mathcal{S}$, and $\mathcal{T}$. Furthermore, if R is any binary relation on A, then*

- $R^{=} := \mathrm{Cl}_{\mathcal{X}}(R) = R \cup \Delta$, *where Δ is the diagonal relation on A,*
- $R^{\leftrightarrow} := \mathrm{Cl}_{\mathcal{S}}(R) = R \cup R^{-1}$, *where R^{-1} is the converse of R, and*
- $R^{+} := \mathrm{Cl}_{\mathcal{T}}(R)$ *is given by*

$$\bigcup_{n \in \mathbb{N}} R^n = R \cup (R \circ R) \cup \cdots \cup \underbrace{(R \circ \cdots \circ R)}_{n\text{-fold power}} \cup \cdots ,$$

where $\circ$ is the relational composition operator.

- $R^* := R^{=+} = R^{+=}$.

$R^=$, $R^\leftrightarrow$, R^+, and R^* are called the *reflexive closure*, the *symmetric closure*, the *transitive closure*, and the *reflexive transitive closure* of R respectively. The last item in the proposition permits us to call R^* the *transitive reflexive closure* of R as well (there is no difference to the order of taking closures). This is true because Δ is transitive.

Remark. In general, however, the order of taking closures of a relation is important. For example, let $A = \{a, b\}$, and $R = \{(a, b)\}$. Then $R^{\leftrightarrow +} = A^2 \neq \{(a, b), (b, a)\} = R^{+\leftrightarrow}$.