enumerating algebras

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1 How many algebras are there?

Unlike categories of discrete objects, such as simple graphs with n vertices, (see `http://planetmath.org/EnumeratingGraphs`article on enumerating graphs) such a question is a little malposed as the quantity can be infinite. However the spirit of the question can be addressed by appealing to algebraic varieties and considering their dimension.

Let A be an non-associative algebra over a field k of dimension n. For example, A could be a Lie algebra, an associative algebra, or a commutative algebra.

From every basis $e_1, \ldots, e_n$ for A, the addition of the algebra is completely understood as all n-dimensional k-vector spaces are isomorphic. Thus we must consider only the multiplication. For this the structure constants of the algebra are considered. That is:

$$e_i e_j = \sum_{k=1}^{n} c_{ij}^k e_k$$

for $c_{ij}^k \in k$. These structure constants completely define the algebra A.

Due to the axioms of multiplication, the structure constants satisfy certain relations. For example, if A is a Lie algebra then multiplication is via the associated Lie bracket and we know

$$[e_i, e_i] = 0$$

Hence we find

$$c_{ii}^k = 0$$

for all $1 \leq i \leq n$. Likewise the Jacobi identity/associativity/commutative conditions each imply their particular relations. If one replaces the structure constants with variables x_{ijk} we find that each algebra A of a given type (Lie/Associative/Commutative/etc.) is a solution to the polynomial equations given by the relations of the algebra. Thus the algebras themselves are parameterized by the algebraic variety, in n^3-dimensional affine space, of these equations.

Theorem 1 (Neretin, 1987)**.** *The dimension of the algebraic variety for n-dimensional Lie algebras, associative algebras, and commutative algebras is respectively*

$$\frac{2}{27}n^3 + O(n^{8/3}), \quad \frac{4}{27}n^3 + O(n^{8/3}),$$

$$\text{and } \frac{2}{27}n^3 + O(n^{8/3}).$$

Lower bounds of $\frac{2}{27}n^3 + O(n^2)$ (and/or $\frac{4}{27} + O(n^2)$) are attainable by exhibiting large families of algebras. For example, class 2 nilpotent Lie algebras attain the lower bound.

As with the related problems for p-groups, it is also expected that the true upper bound has error term $O(n^2)$ [Neretin,Sims].

Neretin, Yu. A., *An estimate for the number of parameters defining an n-dimensional algebra*, Izv. Akad. Nauk SSSR Ser. Mat., vol. 51,1987, no. 2, pp. 306–318, 447.

Mann, Avinoam, *Some questions about p-groups*, J. Austral. Math. Soc. Ser. A, vol. 67, 1999, no. 3, pp. 356–379.