algebraic categories without free objects

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An initial object is always a free object. So in the context of algebraic systems with a trivial object, such as groups, or modules, there is always at least one free object. However, we usually dismiss this example as it does not lead to any useful results such as the existence of presentations.

However, there are many ways in which a cateogry of algebraic objects can fail to include non-trivial free objects.

1 Restriction to finite sets

The restriction of a category which naturally includes infinite objects can often be restricted to just the finite objects and in so doing often remove all non-trivial free objects.

- The category of finite groups has only the trivial free object. Indeed, even the rank 1 free group, the integers $\mathbb{Z}$ is already infinite.
- Similarly, finite modules of in a module category over an infinite ring are never free. For examples use the rings $\mathbb{Z}$, $\mathbb{Z}_m[x]$, etc.

However, this is not always the case. For example, if we consider finite $\mathbb{Z}_p$-modules (vector spaces) each of these are free.

2 Homomorphism restrictions

In the category of rings with 1 it is often beneficial to force all ring homomorphisms to be unital. However, this restriction can prevent the construction of free objects.

Suppose F is a free ring in the category of rings with positive characteristic. Then we ask, what is the characteristic of F?

If it is $m > 0$ then we choose another ring R of a different characteristic, a characteristic relatively prime to m, and then there can be no unital homomorphism from F to R. So F must have characteristic 0. In contrast to the above examples we have not excluded infinite objects in this restriction. This example is even more powerful than those above as it also exclude the existance of an initial object, so indeed NO free objects exist in this category.

If we return to the full category of unital rings we observe every ring is a $\mathbb{Z}$-algebra we can use the free associative algebras $\mathbb{Z}\langle X\rangle$ does exist here.