subalgebra of an algebraic system

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Defines	restriction
Defines	proper subalgebra
Defines	lattice of subalgebras
Defines	spanning set
Defines	finitely generated
Defines	cyclic

Let (A, O) be an algebraic system ($A \neq \varnothing$ is the underlying set and O is the set of operators on A).

Subalgebras of an Algebra

Let B be a non-empty subset of A. B is *closed* under operators of A if for each n-ary operator ω_A on A, and any $b_1, \ldots, b_n \in B$, we have $\omega_A(b_1, \ldots, b_n) \in B$.

Suppose B is closed under operators of A. For each n-ary operator ω_A on A, we define $\omega_B : B^n \to B$ by $\omega_B(b_1, \ldots, b_n) := \omega_A(b_1, \ldots, b_n)$. Each of these operators is well-defined and is called a *restriction* (of the corresponding ω_A). Furthermore, (B, O) is a well-defined algebraic system, and is called the *subalgebra* of (A, O). When (B, O) is a subalgebra of (A, O), we also say that (A, O) is an *extension* of (B, O).

(A, O) is clearly a subalgebra of itself. Any other subalgebra of (A, O) is called a *proper subalgebra*.

Remark. If (A, O) contains constants, then any subalgebra of (A, O) must contain the exact same constants. For example, the ring $\mathbb{Z}$ of integers is an algebraic system with no proper subalgebras. Indeed, if R is a subring of $\mathbb{Z}$, $1 \in R$, so $R = \mathbb{Z}$.

Since we are operating under the same operator set, we can, for convenience, drop O and simply call A an algebra, B a subalgebra of A, etc... If B_1, B_2 are subalgebras of A, then $B_1 \cap B_2$ is also a subalgebra. In fact, given any set of subalgebras B_i of A, their intersection $\bigcap B_i$ is also a subalgebra.

Generating Set of an Algebra

Let C be any subset of an algebra A. Consider the collection $[C]$ of all subalgebras of A containing C. This collection is non-empty because $A \in [C]$. The intersection of all these subalgebras is again a subalgebra containing the set C. Denote this subalgebra by $\langle C \rangle$. $\langle C \rangle$ is called the subalgebra *spanned* by C, and C is called the *spanning set* of $\langle C \rangle$. Conversely, any subalgebra B of A has a spanning set, namely itself: $B = \langle B \rangle$.

Given a subalgebra B of A, a minimal spanning set X of B is called a *generating set* of B. By minimal we mean that the set obtained by deleting any element from X no longer spans B. When B has a generating set X, we also say that X *generates* B. If B can be generated by a finite set, we say that B is *finitely generated*. If B can be generated by a single element, we say that B is *cyclic*.

Remark. $\langle \varnothing \rangle =$ the subalgebra generated by the constants of A. If no such constants exist, $\langle \varnothing \rangle := \varnothing$.

From the discussion above, the set of subalgebras of an algebraic system forms a complete lattice. Given subalgebras A_i, $\bigvee A_i$ is the intersection of all A_i, and $\bigvee A_i$ is the subalgebra $\langle \bigcup A_i \rangle$. The lattice of all subalgebras of A is called the *subalgebra latttice* of A, and is denoted by $\mathrm{Sub}(A)$.