direct product of partial algebras

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Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two partial algebraic systems of type τ. The *direct product* of $\boldsymbol{A}$ and $\boldsymbol{B}$, written $\boldsymbol{A} \times \boldsymbol{B}$, is a partial algebra of type τ, defined as follows:

- the underlying set of $\boldsymbol{A} \times \boldsymbol{B}$ is $A \times B$,
- for each n-ary function symbol $f \in \tau$, the operation $f_{\boldsymbol{A}\times\boldsymbol{B}}$ is given by:

  for $(a_1, b_1), \ldots, (a_n, b_n) \in A \times B$, $f_{\boldsymbol{A}\times\boldsymbol{B}}((a_1, b_1) \ldots, (a_n, b_n))$ is defined iff both $f_{\boldsymbol{A}}(a_1, \ldots, a_n)$ and $f_{\boldsymbol{B}}(b_1, \ldots, b_n)$ are, and when this is the case,

  $$f_{\boldsymbol{A}\times\boldsymbol{B}}((a_1, b_1), \ldots, (a_n, b_n)) := (f_{\boldsymbol{A}}(a_1, \ldots, a_n), f_{\boldsymbol{B}}(b_1, \ldots, b_n)).$$

It is easy to see that the type of $\boldsymbol{A} \times \boldsymbol{B}$ is indeed τ: pick $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$ such that $f_{\boldsymbol{A}}(a_1, \ldots, a_n)$ and $f_{\boldsymbol{B}}(b_1, \ldots, b_n))$ are defined, then $f_{\boldsymbol{A}\times\boldsymbol{B}}((a_1, b_1), \ldots, (a_n, b_n))$ is defined, so that $f_{\boldsymbol{A}\times\boldsymbol{B}}$ is non-empty, where all operations are defined componentwise, and the two constants are $(0, 0)$ and $(1, 1)$.

For example, suppose k_1 and k_2 are fields. They are both partial algebras of type $\langle 2, 2, 1, 1, 0, 0\rangle$, where the two 2's are the arity of addition and multiplication, the two 1's are the arity of additive and multiplicative inverses, and the two 0's are the constants 0 and 1. Then $k_1 \times k_2$, while no longer a field, is still an algebra of the same type.

Let $\boldsymbol{A}, \boldsymbol{B}$ be partial algebras of type τ. Can we embed $\boldsymbol{A}$ into $\boldsymbol{A} \times \boldsymbol{B}$ so that $\boldsymbol{A}$ is some type of a subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$?

For example, if we fix an element $b \in B$, then the injection $i_b : \boldsymbol{A} \to \boldsymbol{A} \times \boldsymbol{B}$, given by $i_b(a) = (a, b)$ is in general not a homomorphism only unless b is an idempotent with respect to every operation $f_{\boldsymbol{B}}$ on B (that is, $f_{\boldsymbol{B}}(b, \ldots, b) = b$). In addition, b would have to be *the* constant for every constant symbol in τ. Following from the example above, if we pick any $r \in k_2$, then r would have to be 0, since, $(s_1 + s_2, 2r) = i_r(s_1) + i_r(s_2) = i_r(s_1 + s_2) = (s_1 + s_2, r)$, so that $2r = r$, or $r = 0$. But, on the other hand, $i_r(s^{-1}) = (s, r)^{-1} = (s^{-1}, r^{-1})$, forcing r to be invertible, a contradiction!

Now, suppose we have a homomorphism $\sigma : \boldsymbol{A} \to \boldsymbol{B}$, then we may embed $\boldsymbol{A}$ into $\boldsymbol{A} \times \boldsymbol{B}$, so that $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$. The embedding is given by $\phi(a) = (a, \sigma(a))$.

Proof. Suppose $f_{\boldsymbol{A}}(a_1, \ldots, a_n)$ is defined. Since σ is a homomorphism, $f_{\boldsymbol{B}}(\sigma(a_1), \ldots, \sigma(a_n))$ is defined, which means $f_{\boldsymbol{A}\times\boldsymbol{B}}((a_1, \sigma(a_1)), \ldots, (a_n, \sigma(a_n))) = f_{\boldsymbol{A}\times\boldsymbol{B}}(\phi(a_1), \ldots, \phi(a_n))$

is defined. Furthermore, we have that

$$\begin{aligned} f_{\boldsymbol{A}\times\boldsymbol{B}}((a_1,\sigma(a_1)),\ldots,(a_n,\sigma(a_n))) &= (f_{\boldsymbol{A}}(a_1,\ldots,a_n), f_{\boldsymbol{B}}(\sigma(a_1),\ldots,\sigma(a_n))) \\ &= (f_{\boldsymbol{A}}(a_1,\ldots,a_n), \sigma(f_{\boldsymbol{A}}(a_1,\ldots,a_n))) \\ &= \phi(f_{\boldsymbol{A}}(a_1,\ldots,a_n)), \end{aligned}$$

showing that ϕ is a homomorphism. In addition, if $f_{\boldsymbol{A}\times\boldsymbol{B}}(\phi(a_1),\ldots,\phi(a_n))$ is defined, then it is clear that $f_{\boldsymbol{A}}(a_1,\ldots,a_n)$ is defined, so that ϕ is a strong homomorphism. So $\phi(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{A}\times\boldsymbol{B}$. Clearly, ϕ is one-to-one, and therefore an embedding, so that $\boldsymbol{A}$ is isomorphic to $\phi(\boldsymbol{A})$, and we may view $\boldsymbol{A}$ as a subalgebra of $\boldsymbol{A}\times\boldsymbol{B}$. $\square$

Remark. Moving to the general case, let $\{\boldsymbol{A_i} \mid i \in I\}$ be a set of partial algebras of type τ, indexed by set I. The *direct product* of these algebras is a partial algebra $\boldsymbol{A}$ of type τ, defined as follows:

- the underlying set of $\boldsymbol{A}$ is $A := \prod\{A_i \mid i \in I\}$,
- for each n-ary function symbol $f \in \tau$, the operation $f_{\boldsymbol{A}}$ is given by: for $a \in A$, $f_{\boldsymbol{A}}(a)$ is defined iff $f_{\boldsymbol{A_i}}(a(i))$ is defined for each $i \in I$, and when this is the case,
$$f_{\boldsymbol{A}}(a)(i) := f_{\boldsymbol{A_i}}(a(i)).$$

Again, it is easy to verify that $\boldsymbol{A}$ is indeed a τ-algebra: for each symbol $f \in \tau$, the domain of definition $\mathrm{dom}(f_{\boldsymbol{A_i}})$ is non-empty for each $i \in I$, and therefore the domain of definition $\mathrm{dom}(f_{\boldsymbol{A}})$, being $\prod\{\mathrm{dom}(f_{\boldsymbol{A_i}}) \mid i \in I\}$, is non-empty as well, by the axiom of choice.

References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).