reduced direct product

Canonical name	ReducedDirectProduct
Date of creation	2013-03-22 17:10:11
Last modified on	2013-03-22 17:10:11
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Definition
Classification	msc 08B25
Synonym	ultraproduct
Defines	prime product

Let $\{A_i \mid i \in I\}$ be a set of algebraic systems of the same type, indexed by I. Let A be the direct product of the A_i's. For any $a, b \in A$, set

$$\operatorname{supp}(a,b) := \{k \in I \mid a(k) \neq b(k)\}.$$

Consider a Boolean ideal L of the Boolean algebra $P(I)$ of I. Define a binary relation Θ_L on A as follows:

$$(a,b) \in \Theta_L \quad \text{iff} \quad \operatorname{supp}(a,b) \in L.$$

Lemma 1. Θ_L *defined above is a congruence relation on* A.

Proof. Since L is an ideal $\varnothing \in L$. Therefore, $(a,a) \in \Theta_L$, since $\{k \in I \mid a(k) \neq a(k)\} = \varnothing$. Clearly, Θ_L is symmetric. For transitivity, suppose $(a,b,(b,c) \in \Theta_L$. If $a(k) \neq c(k)$ for some $k \in I$, then either $a(k) \neq b(k)$ or $b(k) \neq c(k)$ (a contrapositive argument). So

$$\operatorname{supp}(a,c) \subseteq \operatorname{supp}(a,b) \cup \operatorname{supp}(b,c).$$

Since L is an ideal, $\operatorname{supp}(a,c) \in L$, so $(a,c) \in \Theta_L$, and Θ_L is an equivalence relation on A.

Next, let ω be an n-ary operator on A and $a_j \equiv b_j \pmod{\Theta_L}$, where $j = 1, \ldots, n$. We want to show that $\omega(a_1, \ldots, a_n) \equiv \omega(b_1, \ldots, b_n) \pmod{\Theta_L}$. Let ω_i be the associated n-ary operators on A_i. If $\omega(a_1, \ldots, a_n)(k) \neq \omega(b_1, \ldots, b_n)(k)$, then $\omega_k(a_1(k), \ldots, a_n(k)) \neq \omega_k(b_1(k), \ldots, b_n(k))$, which implies that $a_j(k) \neq b_j(k)$ for some $j = 1, \ldots, n$. This implies that

$$\operatorname{supp}(\omega(a_1, \ldots, a_n), \omega(b_1, \ldots, b_n)) \subseteq \bigcup_{j=1}^{n} \operatorname{supp}(a_j, b_j).$$

Since L is an ideal, and each $\operatorname{supp}(a_j, b_j) \in L$, we have that $\operatorname{supp}(\omega(a_1, \ldots, a_n), \omega(b_1, \ldots, b_n)) \in L$ as well, this means that $\omega(a_1, \ldots, a_n) \equiv \omega(b_1, \ldots, b_n) \pmod{\Theta_L}$. □

Definition. Let $A = \prod\{A_i \mid i \in I\}$, L be a Boolean ideal of $P(I)$ and Θ_L be defined as above. The quotient algebra A/Θ_L is called the L-*reduced direct product* of A_i. The L-reduced direct product of A_i is denoted by $\prod_L\{A_i \mid i \in I\}$. Given any element $a \in A$, its image in the reduced direct product $\prod_L\{A_i \mid i \in I\}$ is given by $[a]\Theta_L$, or $[a]$ for short.

Example. Let $A = A_1 \times \cdots \times A_n$, and let L be the principal ideal generated by 1. Then $L = \{\varnothing, \{1\}\}$. The congruence Θ_L is given by

$(a_1, \dots, a_n) \equiv (b_1, \dots, b_n) \pmod{\Theta_L}$ iff $\{i \mid a_i \neq b_i\} = \varnothing$ or $\{1\}$. This implies that $a_i = b_i$ for all $i = 2, \dots, n$. In other words, Θ_L is isomorphic to the direct product of $A_2 \times \cdots \times A_n$. Therefore, the L-reduced direct product of A_i is isomorphic to A_1.

The example above can be generalized: if $J \subseteq I$, then

$$\prod{}_{P(J)}\{A_i \mid i \in I\} \cong \prod\{A_i \mid i \in I - J\}.$$

For $a \in A = \prod\{A_i \mid i \in I\}$, write $a = (a_i)_{i \in I}$. It is not hard to see that the map $f : \prod_{P(J)}\{A_i \mid i \in I\} \to \prod\{A_i \mid i \in I - J\}$ given by $f([a]) = (a_i)_{i \in I - J}$ is the required isomorphism.

Remark. The definition of a reduced direct product in terms of a Boolean ideal can be equivalently stated in terms of a Boolean filter F. All there is to do is to replace $\operatorname{supp}(a, b)$ by its complement: $\operatorname{supp}(a, b)^c := \{k \in I \mid a(k) = b(k)\}$. The congruence relation is now $\Theta_{F'}$, where $F' = \{I - J \mid J \in F\}$ is the ideal complement of F. When F is prime, the F'-reduced direct product is called a *prime product*, or an *ultraproduct*, since any prime filter is also called an ultrafilter. Ultraproducts can be more generally defined over arbitrary structures.

References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).