relation

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Related topic	WellFoundedRelation
Related topic	Property2
Related topic	GroundedRelation
Related topic	RelationBetweenObjects
Defines	unary relation
Defines	binary relation
Defines	ternary relation
Defines	n-ary relation
Defines	domain
Defines	range
Defines	nullary relation
Defines	field

Binary Relations

Before describing what a *relation* is generally, let us define a more specific kind of a relation: a *binary relation*. Basically, a binary relation R involves objects coming from two collections A, B, where the objects are paired up so that each pair consists of an object from A, and an object from B.

More formally, a *binary relation* is a subset R of the `http://planetmath.org/CartesianProduc` product of two sets A and B. One may write

$$a \, R \, b$$

to indicate that the ordered pair (a, b) is an element of R. A subset of $A \times A$ is simply called a binary relation on A. If R is a binary relation on A, then we write

$$a_1 \, R \, a_2 \, R \, a_3 \, \cdots \, a_{n-1} \, R \, a_n$$

to mean $a_1 \, R \, a_2, a_2 \, R \, a_3, \ldots,$ and $a_{n-1} \, R \, a_n$.

Given a binary relation $R \subseteq A \times B$, the *domain* $\operatorname{dom}(R)$ of R is the set of elements in A forming parts of the pairs in R. In other words,

$$\operatorname{dom}(R) := \{x \in A \mid (x, y) \in R \text{ for some } y \in B\}$$

and the *range* $\operatorname{ran}(R)$ of R is the set of parts of pairs of R coming from B:

$$\operatorname{ran}(R) := \{y \in B \mid (x, y) \in R \text{ for some } x \in A\}.$$

An example of a binary relation is the less-than relation on the integers, i.e., $< \subseteq \mathbb{Z} \times \mathbb{Z}$. $(1, 2) \in <$, but $(2, 1) \notin <$.

Remarks.

1. In set theory, a binary relation is simply a set of ordered pairs (of sets or classes, depending on the axiom system used). Notice that, unlike the previous definition, sets (or classes) A and B are not specified in advance. Given a (binary) relation R, the domain of R is the set (or class) of elements a such that aRb for some b, and the range of R is the set (or class) or elements b such that aRb for some a. The union and the domain and the range of R is called the *field* of R.

2. It may be possible to define a relation over a class. For example, if $\mathcal{C}$ is the class of all sets, then $\in$ can be thought of as a binary relation on $\mathcal{C}$.

3. In term rewriting theory, a binary relation on a set is sometimes called a *reduction*, and is written $\to$. This is to signify that $a \to b$ means that the element a is being "reduced" to b via $\to$.

Arbitrary Relations

From the definition of a binary relation, we can easily generalize it to that of an arbitrary relation. Since a binary relation involves two sets, an arbitrary relation involves an arbitrary collection of sets. More specifically, a *relation* R is a subset of some `http://planetmath.org/GeneralizedCartesianProduct`Cartesian product of a collection of sets. In symbol, this is

$$R \subseteq \prod_{i \in I} A_i$$

where each A_i is a set, indexed by some set I.

From this more general definition, we see that a binary relation is just a relation where I has two elements. In addition, an *n-ary relation* is a relation where the cardinality of I is n (n finite). In symbol, we have

$$R \subseteq \prod_{i=1}^{n} A_i.$$

It is not hard to see that any n-ary relation where $n > 1$ can be viewed as a binary relation in $n-1$ different ways, for

$$R \subseteq A_1 \times A_2 \times \cdots \times A_n = \prod_{i=1}^{j} A_i \times \prod_{i=j+1}^{n} A_i,$$

where j ranges from 1 through $n-1$.

A common name for a 3-ary relation is a *ternary relation*. It is also possible to have a 1-*ary relation*, or commonly known as a *unary relation*, which is nothing but a subset of some set.

Remarks.

1. Following from the first remark from the previous section, relations of higher arity can be inductively defined: for $n > 1$, an $(n+1)$-ary relation is a binary relation whose domain is an n-ary relation. In this setting, a "unary relation" and relations whose arity is of "arbitrary" cardinality are not defined.

2. A relation can also be viewed as a function (which itself is a relation). Let $R \subseteq A := \prod_{i \in I} A_i$. As a subset of A, R can be identified with the characteristic function

$$\chi_R : A \to \{0, 1\},$$

where $\chi_R(x) = 1$ iff $x \in R$ and $\chi_R(x) = 0$ otherwise. Therefore, an n-ary relation is equivalent to an $(n+1)$-ary characteristic function. From this, one may say that a 0-ary, or a *nullary relation* is a unary characteristic function. In other words, a nullary relation is just a an element in $\{0, 1\}$ (or truth/falsity).