simple algebraic system

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An algebraic system A is *simple* if the only congruences on it are $A \times A$ and Δ, the diagonal relation.

For example, let's find out what are the simple algebras in the class of groups. Let G be a group that is simple in the sense defined above.

First, what are the congruences on G? A congruence C on G is a subgroup of $G \times G$ and an equivalence relation on G at the same time. As an equivalence relation, C corresponds to a partition of G in the following manner: $G = \bigcup_{i \in I} N_i$ and $C = \bigcup_{i \in I} N_i^2$, where $N_i \cap N_j = \varnothing$ for $i \neq j$. Each of the N_i is an equivalence class of C. Let N be the equivalence class containing 1. If $a, b \in N$, then $[a] = [b] = [1]$, so that $[ab] = [a][b] = [1][1] = [1]$, or $ab \in N$. In addition, $[a^{-1}] = [1][a^{-1}] = [a][a^{-1}] = [aa^{-1}] = [1]$, so $a^1 \in N$. N is a subgroup of G. Furthermore, if $c \in G$, $[cac^{-1}] = [c][a][c^{-1}] = [c][1][c^{-1}] = [cc^{-1}] = [1]$, so that $cac^{-1} \in N$, N is a normal subgroup of G. Conversely, given a normal subgroup N of G, forming left (right) cosets N_i of N, and taking $C = \bigcup_{i \in I} N_i^2$ gives us the congruence C on G.

Now, if G is simple, then this says that the only congruences on G are $G \times G$ and Δ, which corresponds to G having G and $\langle 1 \rangle$ as the only normal subgroups. So, G as a simple algebra is just a simple group. Conversely, if G is a simple group, then the only congruences on G are those corresponding to G and $\langle 1 \rangle$, the only normal subgroups of G. Therefore, a simple group is a simple algebra.

Remark. Any simple algebraic system is subdirectly irreducible.