kernel of a homomorphism between algebraic systems

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Defines	congruence induced by a homomorphism

Let $f:(A,O)\to(B,O)$ be a homomorphism between two algebraic systems A and B (with O as the operator set). Each element $b\in B$ corresponds to a subset $K(b):=f^{-1}(b)$ in A. Then $\{K(b)\mid b\in B\}$ forms a partition of A. The *kernel* $\ker(f)$ of f is defined to be

$$\ker(f):=\bigcup_{b\in B}K(b)\times K(b).$$

It is easy to see that $\ker(f)=\{(x,y)\in A\times A\mid f(x)=f(y)\}$. Since it is a subset of $A\times A$, it is relation on A. Furthermore, it is an equivalence relation on A: [1]

1. $\ker(f)$ is reflexive: for any $a\in A$, $a\in K(f(a))$, so that $(a,a)\in K(f(a))^2\subseteq\ker(f)$
2. $\ker(f)$ is symmetric: if $(a_1,a_2)\in\ker(f)$, then $f(a_1)=f(a_2)$, so that $(a_2,a_1)\in\ker(f)$
3. $\ker(f)$ is transitive: if $(a_1,a_2),(a_2,a_3)\in\ker(f)$, then $f(a_1)=f(a_2)=f(a_3)$, so $(a_1,a_3)\in\ker(f)$.

We write $a_1\equiv a_2\pmod{\ker(f)}$ to denote $(a_1,a_2)\in\ker(f)$.

In fact, $\ker(f)$ is a congruence relation: for any n-ary operator symbol $\omega\in O$, suppose $c_1,\dots,c_n$ and $d_1,\dots,d_n$ are two sets of elements in A with $c_i\equiv d_i \mod \ker(f)$. Then

$$f(\omega_A(c_1,\dots,c_n)=\omega_B(f(c_1),\dots,f(c_n))=\omega_B(f(d_1),\dots,f(d_n))=f(\omega_A(d_1,\dots,d_n)),$$

so $\omega_A(c_1,\dots,c_n)\equiv\omega_A(d_1,\dots,d_n)\pmod{\ker(f)}$. For this reason, $\ker(f)$ is also called the *congruence induced by* f.

Example. If A,B are groups and $f:A\to B$ is a group homomorphism. Then the kernel of f, using the definition above is just the union of the square of the cosets of

$$N=\{x\mid f(x)=e\},$$

the traditional definition of the kernel of a group homomorphism (where e is the identity of B).

Remark. The above can be generalized. See the `http://planetmath.org/KernelOfAHomomorp`
in model theory.

[1] In general, $\{N_i\}$ is a partition of a set A iff $\bigcup N_i^2$ is an equivalence relation on A.