algebraic function

Canonical name	AlgebraicFunction
Date of creation	2013-03-22 15:19:24
Last modified on	2013-03-22 15:19:24
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	15
Author	CWoo (3771)
Entry type	Definition
Classification	msc 08A40
Classification	msc 26A09
Related topic	ElementaryFunction
Related topic	PropertiesOfEntireFunctions
Related topic	PolynomialsInAlgebraicSystems
Defines	transcendental function
Defines	transcendental
Defines	algebraic curve

A function of one variable is said to be *algebraic* if it satisfies a polynomial equation whose coefficients are polynomials in the same variable. Namely, the function $f(x)$ is algebraic if $y = f(x)$ is a solution of an equation of the form

$$p_n(x)y^n + \cdots + p_1(x)y + p_0(x) = 0,$$

where the $p_0(x), p_1(x), \ldots, p_n(x)$ are polynomials in x. A function that satisfies no such equation is said to be *transcendental*.

The graph of an algebraic function is an *algebraic curve*, which is, loosely speaking, the zero set of a polynomial in two variables.

Examples

Any rational function $f(x) = P(x)/Q(x)$ is algebraic, since $y = f(x)$ is a solution to $Q(x)y - P(x) = 0$.

The function $f(x) = \sqrt{x}$ is algebraic, since $y = f(x)$ is a solution to $y^2 - x = 0$. The same is true for any power function $x^{n/m}$, with n and m integers, it satisfies the equation $y^m - x^n = 0$.

It is known that the functions e^x and $\ln x$ are transcendental. Many special functions, such as Bessel functions, elliptic integrals, and others are known to be transcendental.

Remark. There is also a version of an algebraic function defined on algebraic systems. Given an algebraic system A, an *n-ary algebraic function* on A is an n-ary operator $f(x_1, \ldots, x_n)$ on A such that there is an `http://planetmath.org/PolynomialsInAlgebraicSystems`$(n+m)$-ary polynomial $p(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ on A for some non-negative integer m, and elements $a_1, \ldots, a_m \in A$ such that

$$f(x_1, \ldots, x_n) = p(x_1, \ldots, x_n, a_1, \ldots, a_m).$$

Equivalently, given an algebraic system A, if we associate each element a of A a corresponding symbol, also written a, we may form an algebraic system A' from A by adjoining every symbol a to the type of A considered as a unary operator symbol, and leaving everything else the same. Then an algebraic function on A is just a polynomial on A' (and vice versa).

For example, in a ring R, a function f on R given by $f(x) = a_n x^n + \cdots + a_1 x + a_0$ where $a_i \in R$ is a unary algebraic function on R, as $f(x) = p(x, a_0, \ldots, a_n)$, where p is an $(n+2)$-ary polynomial on R given by $p(x, x_0, \ldots, x_n) = x_n x^n + \cdots + x_1 x + x_0$.

References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).

[2] S. Burris, H.P. Sankappanavar: *A Course in Universal Algebra*, Springer, New York (1981).