congruence on a partial algebra

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Definition

There are two types of congruences on a partial algebra $\boldsymbol{A}$, both are special types of a certain equivalence relation on A:

1. Θ is a *congruence relation* on $\boldsymbol{A}$ if, given that

   - $a_1 \equiv b_1 \pmod{\Theta}, \ldots, a_n \equiv b_n \pmod{\Theta}$,
   - both $f_{\boldsymbol{A}}(a_1, \ldots, a_n)$ and $f_{\boldsymbol{A}}(b_1, \ldots, b_n)$ are defined,

   then $f_{\boldsymbol{A}}(a_1, \ldots, a_n) \equiv f_{\boldsymbol{A}}(b_1, \ldots, b_n) \pmod{\Theta}$.

2. Θ is a *strong congruence relation* on $\boldsymbol{A}$ if it is a congruence relation on $\boldsymbol{A}$, and, given

   - $a_1 \equiv b_1 \pmod{\Theta}, \ldots, a_n \equiv b_n \pmod{\Theta}$,
   - $f_{\boldsymbol{A}}(a_1, \ldots, a_n)$ is defined,

   then $f_{\boldsymbol{A}}(b_1, \ldots, b_n)$ is defined.

Proposition 1. *If $\phi : \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then the equivalence relation E_ϕ induced by ϕ on A is a congruence relation. Furthermore, if ϕ is a strong, so is E_ϕ.*

Proof. Let $f \in \tau$ be an n-ary function symbol. Suppose $a_i \equiv b_i \pmod{E_\phi}$ and both $f_{\boldsymbol{A}}(a_1, \ldots, a_n)$ and $f_{\boldsymbol{A}}(b_1, \ldots, b_n)$ are defined. Then $\phi(a_i) = \phi(b_i)$, and therefore

$$\phi(f_{\boldsymbol{A}}(a_1, \ldots, a_n)) = f_{\boldsymbol{B}}(\phi(a_1), \ldots, \phi(a_n)) = f_{\boldsymbol{B}}(\phi(b_1), \ldots, \phi(b_n)) = \phi(f_{\boldsymbol{A}}(b_1, \ldots, b_n)),$$

so $f_{\boldsymbol{A}}(a_1, \ldots, a_n) \equiv f_{\boldsymbol{A}}(b_1, \ldots, b_n) \pmod{E_\phi}$. In other words, E_ϕ is a congruence relation.

Now, suppose in addition that ϕ is a strong homomorphism. Again, let $a_i \equiv b_i \pmod{E_\phi}$. Assume $f_{\boldsymbol{A}}(a_1, \ldots, a_n)$ is defined. Since $\phi(a_i) = \phi(b_i)$, we get

$$\phi(f_{\boldsymbol{A}}(a_1, \ldots, a_n)) = f_{\boldsymbol{B}}(\phi(a_1), \ldots, \phi(a_n)) = f_{\boldsymbol{B}}(\phi(b_1), \ldots, \phi(b_n)).$$

Since ϕ is strong, $f_{\boldsymbol{A}}(b_1, \ldots, b_n)$ is defined, which means that E_ϕ is strong. $\square$

Congruences as Subalgebras

If $\boldsymbol{A}$ is a partial algebra of type τ, then the direct power $\boldsymbol{A}^2$ is a partial algebra of type τ. A binary relation Θ on A may be viewed as a subset of A^2. For each n-ary operation $f_{\boldsymbol{A}^2}$ on $\boldsymbol{A}^2$, take the restriction on Θ, and call it $f_{\boldsymbol{\Theta}}$. For $a_i \in \Theta$, $f_{\boldsymbol{\Theta}}(a_1, \ldots, a_n)$ is defined in Θ iff $f_{\boldsymbol{A}^2}(a_1, \ldots, a_n)$ is defined at all, and its value is in Θ. When $f_{\boldsymbol{\Theta}}(a_1, \ldots, a_n)$ is defined in Θ, its value is set as $f_{\boldsymbol{A}^2}(a_1, \ldots, a_n)$. This turns $\boldsymbol{\Theta}$ into a partial algebra. However, the type of $\boldsymbol{\Theta}$ is τ only when $f_{\boldsymbol{\Theta}}$ is non-empty for each function symbol $f \in \tau$. In particular,

Proposition 2. *If Θ is reflexive, then $\boldsymbol{\Theta}$ is a relative subalgebra of $\boldsymbol{A}^2$.*

Proof. Pick any n-ary function symbol $f \in \tau$. Then $f_{\boldsymbol{A}}(a_1, \ldots, a_n)$ is defined for some $a_i \in A$. Then $f_{\boldsymbol{A}^2}((a_1, a_1), \ldots, (a_n, a_n))$ is defined and is equal to $(f_{\boldsymbol{A}}(a_1, \ldots, a_n), f_{\boldsymbol{A}}(a_1, \ldots, a_n))$, which is in Θ, since Θ is reflexive. This shows that $f_{\boldsymbol{\Theta}}((a_1, a_1), \ldots, (a_n, a_n))$ is defined. As a result, $\boldsymbol{\Theta}$ is a partial algebra of type τ. Furthermore, by virtue of the way $f_{\boldsymbol{\Theta}}$ is defined for each $f \in \tau$, $\boldsymbol{\Theta}$ is a relative subalgebra of $\boldsymbol{A}$. $\square$

Proposition 3. *An equivalence relation Θ on A is a congruence iff $\boldsymbol{\Theta}$ is a subalgebra of $\boldsymbol{A}^2$.*

Proof. First, assume that Θ is a congruence relation on A. Since Θ is reflexive, $\boldsymbol{\Theta}$ is a relative subalgebra of $\boldsymbol{A}^2$. Now, suppose $f_{\boldsymbol{A}^2}((a_1, b_1), \ldots, (a_n, b_n))$ exists, where $a_i \equiv b_i \pmod{\Theta}$. Then $f_{\boldsymbol{A}}(a_1, \ldots, a_n), f_{\boldsymbol{A}}(b_1, \ldots, b_n)$ both exist. Since Θ is a congruence, $f_{\boldsymbol{A}}(a_1, \ldots, a_n) \equiv f_{\boldsymbol{A}}(b_1, \ldots, b_n) \pmod{\Theta}$. In other words, $(f_{\boldsymbol{A}}(a_1, \ldots, a_n), f_{\boldsymbol{A}}(b_1, \ldots, b_n)) \in \Theta$. Hence $\boldsymbol{\Theta}$ is a subalgebra of $\boldsymbol{A}^2$.

Conversely, assume $\boldsymbol{\Theta}$ is a subalgebra of $\boldsymbol{A}^2$. Suppose $(a_i, b_i) \in \Theta$ and both $f_{\boldsymbol{A}}(a_1, \ldots, a_n)$ and $f_{\boldsymbol{A}}(b_1, \ldots, b_n)$ are defined. Then $f_{\boldsymbol{A}^2}((a_1, b_1), \ldots, (a_n, b_n))$ is defined. Since $\boldsymbol{\Theta}$ is a subalgebra of $\boldsymbol{A}^2$, $f_{\boldsymbol{\Theta}}((a_1, b_1), \ldots, (a_n, b_n))$ is also defined, and $(f_{\boldsymbol{A}}(a_1, \ldots, a_n), f_{\boldsymbol{A}}(b_1, \ldots, b_n)) = f_{\boldsymbol{A}^2}((a_1, b_1), \ldots, (a_n, b_n)) = f_{\boldsymbol{\Theta}}((a_1, b_1), \ldots, (a_n, b_n)) \in \Theta$. This shows that Θ is a congruence relation on A. $\square$

Quotient Partial Algebras

With congruence relations defined, one may then define quotient partial algebras: given a partial algebra $\boldsymbol{A}$ of type τ and a congruence relation Θ on

A, the *quotient partial algebra* of $\boldsymbol{A}$ by Θ is the partial algebra $\boldsymbol{A/\Theta}$ whose underlying set is A/Θ, the set of congruence classes, and for each n-ary function symbol $f \in \tau$, $f_{\boldsymbol{A/\Theta}}([a_1], \dots, [a_n])$ is defined iff there are $b_1, \dots, b_n \in A$ such that $[a_i] = [b_i]$ and $f_{\boldsymbol{A}}(b_1, \dots, b_n)$ is defined. When this is the case:

$$f_{\boldsymbol{A/\Theta}}([a_1], \dots, [a_n]) := [f_{\boldsymbol{A}}(b_1, \dots, b_n)].$$

Suppose there are $c_1, \dots, c_n \in A$ such that $[a_i] = [c_i]$, or $a_i \equiv c_i \pmod{\Theta}$, and $f_{\boldsymbol{A}}(c_1, \dots, c_n)$ is defined, then $b_i \equiv c_i \pmod{\Theta}$ and $f_{\boldsymbol{A}}(b_1, \dots, b_n) \equiv f_{\boldsymbol{A}}(c_1, \dots, c_n) \pmod{\Theta}$, or, equivalently, $[f_{\boldsymbol{A}}(b_1, \dots, b_n)] = [f_{\boldsymbol{A}}(c_1, \dots, c_n)]$, so that $f_{\boldsymbol{A/\Theta}}$ is a well-defined operation.

In addition, it is easy to see that $\boldsymbol{A/\Theta}$ is in fact a τ-algebra. For each n-ary $f \in \tau$, pick $a_1, \dots, a_n \in A$ such that $f_{\boldsymbol{A}}(a_1, \dots, a_n)$ is defined. Then $f_{\boldsymbol{A/\Theta}}([a_1], \dots, [a_n])$ is defined, and is equal to $[f_{\boldsymbol{A}}(a_1, \dots, a_n)]$.

Proposition 4. *Let $\boldsymbol{A}$ and Θ be defined as above. Then $[\cdot] : \boldsymbol{A} \to \boldsymbol{A/\Theta}$, given by $[\cdot](a) = [a]$, is a surjective full homomorphism, and $E_{[\cdot]} = \Theta$. Furthermore, $[\cdot]$ is a strong homomorphism iff Θ is a strong congruence relation.*

Proof. $[\cdot]$ is obviously surjective. The fact that $[\cdot]$ is a full homomorphism follows directly from the definition of $f_{\boldsymbol{A/\Theta}}$, for each $f \in \tau$. Next, $aE_{[\cdot]}b$ iff $[a] = [b]$ iff $a \equiv b \pmod{\Theta}$. This proves the first statement.

The next statement is proved as follows:

($\Rightarrow$). If $a_i \equiv b_i \pmod{\Theta}$ and $f_{\boldsymbol{A}}(a_1, \dots, a_n)$ is defined, then $f_{\boldsymbol{A/\Theta}}([a_1], \dots, [a_n])$ is defined, which is just $f_{\boldsymbol{A/\Theta}}([b_1], \dots, [b_n])$, and, as $[\cdot]$ is strong, $f_{\boldsymbol{A}}(b_1, \dots, b_n)$ is defined, showing that Θ is strong.

($\Leftarrow$). Suppose $f_{\boldsymbol{A/\Theta}}([a_1], \dots, [a_n])$ is defined. Then there are $b_1, \dots, b_n \in A$ with $a_i \equiv b_i \pmod{\Theta}$ such that $f_{\boldsymbol{A}}(b_1, \dots, b_n)$ is defined. Since Θ is strong, $f_{\boldsymbol{A}}(a_1, \dots, a_n)$ is defined as well, which shows that $[\cdot]$ is strong. $\square$

References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).