permutable congruences

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Let A be an algebraic system and Θ_1 and Θ_2 are two congruences on A. Θ_1 and Θ_2 are said to be *permutable* if $\Theta_1 \circ \Theta_2 = \Theta_2 \circ \Theta_1$, where $\circ$ is the composition of relations.

For example, let A be the direct product of A_1 and A_2. Define Θ_1 on A as follows:

$$(a,b) \equiv (c,d) \pmod{\Theta_1} \quad \text{iff} \quad a = c.$$

Then Θ_1 is clearly an equivalence relation on A. For any n-ary operator f on A, let f_1 and f_2 be the corresponding n-ary operators on A_1 and A_2 respectively: $f = (f_1, f_2)$. Suppose $(a_i, b_i) \equiv (c_i, d_i) \pmod{\Theta_1}$, $i = 1, \ldots, n$. Then

$$\begin{aligned} f((a_1,b_1),\ldots,(a_n,b_n)) &= (f_1(a_1,\ldots,a_n), f_2(b_1,\ldots,b_n)) && (1)\\ &\equiv (f_1(c_1,\ldots,c_n), f_2(d_1,\ldots,d_n)) && (2)\\ &= f((c_1,d_1),\ldots,(c_n,d_n)) \pmod{\Theta_1}. && (3) \end{aligned}$$

The equivalence of (1) and (2) follows from the assumption that $a_i = c_i$ for each $i = 1, \ldots, n$, so that $f_1(a_1, \ldots, a_n) = f_1(c_1, \ldots, c_n)$. Similarly define

$$(a,b) \equiv (c,d) \pmod{\Theta_2} \quad \text{iff} \quad b = d.$$

By a similar argument, Θ_2 is a congruence on A too. Pick any $(a,b),(c,d) \in A$. Then $(a,b) \equiv (a,d) \pmod{\Theta_1}$ and $(a,d) \equiv (c,d) \pmod{\Theta_2}$ so that $(a,b)(\Theta_1 \circ \Theta_2)(c,d)$. This implies that $\Theta_1 \circ \Theta_2 = A^2$. Similarly $\Theta_2 \circ \Theta_1 = A^2$. Therefore, Θ_1 and Θ_2 are permutable.

In fact, we have the following:

Proposition 1. *Let A be an algebraic system with congruenes Θ_1 and Θ_2. Then Θ_1 and Θ_2 are permutable iff $\Theta_1 \circ \Theta_2 = \Theta_1 \vee \Theta_2$, where $\vee$ is the join operation on,* $\mathrm{Con}(A)$*, the lattice of congruences on A.*

Proof. Clearly, if $\Theta_1 \circ \Theta_2 = \Theta_1 \vee \Theta_2$, then they are permutable. Conversely, suppose they are permutable. Let $C = \Theta_1 \circ \Theta_2$ and $D = \Theta_1 \vee \Theta_2$. We want to show that $C = D$. If $(a,b) \in C$, then there is $c \in A$ such that $(a,c) \in \Theta_1$ and $(c,b) \in \Theta_2$, so $a \equiv b \pmod{D}$. This shows $C \subseteq D$. If $a \equiv b \pmod{D}$, then there is $c \in A$ such that $a \equiv c \pmod{R}$ and $c \equiv b \pmod{S}$ with $R, S \in \{\Theta_1, \Theta_2\}$. If $R = S$, then we are done, since $\Theta_i \subseteq C$ (as an element (a,b) belonging to, say Θ_1, can be written as $(a,b) \circ (b,b) \in C$). If $R = \Theta_1$ and $S = \Theta_2$ then we are done too, since this is just the definition of C. If $R = \Theta_2$ and $S = \Theta_1$, then $(a,b) \in \Theta_2 \circ \Theta_1 = \Theta_1 \circ \Theta_2 = C$, by permutability. $\square$

Remark. From the example above, it is not hard to see that an algebraic system A is the direct product of two algebraic systems B, C iff there are two permutable congruences Θ and Φ on A such that $\Theta \vee \Phi = A^2$ and $\Theta \wedge \Phi = \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$ is the diagonal relation on A, and that $B \cong A/\Theta$ and $C \cong A/\Phi$. This result can be generalized to arbitrary direct products.