relational system

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A *relational system*, loosely speaking, is a pair (A, R) where A is a set and R is a set of finitary relations defined on A (a finitary relation is just an n-ary relation where $n \in \mathbb{N}$; when $n = 1$, it is called a property). Since an n-ary operator on a set is an $(n + 1)$-ary the set, a relational system can be thought of as a generalization of an algebraic system. We can formalize the notion of a relation system as follows:

> Call a set R a relation set, if there is a function $f : R \to \mathbb{N}$, the set of natural numbers. For each $r \in R$, call $f(r)$ the arity of r.
>
> Let A be a set and R a *relation set*. The pair (A, R) is called an R-relational system if there is a set R_A such that
>
> - R_A is a set of finitary relations on A, called the *relation set* of A, and
> - there is a one-to-one correspondence between R and R_A, given by $r \mapsto r_A$, such that the $f(r) =$ the arity of r_A.

Since operators and partial operators are special types of relations. algebraic systems and partial algebraic systems can be treated as relational systems.

Below are some exmamples of relational systems:

- any algebraic or partial algebraic system.
- a poset $(P, \{\leq_P\})$, where $\leq_P$ is a binary relation, called the partial ordering, on P. A lattice, generally considered an algebraic system, can also be considered as a relational system, because it is a poset, and that $\leq$ alone defines the algebraic operations ($\vee$ and $\wedge$).
- a pointed set $(A, \{a\})$ is also a relational system, where a unary relation, or property, is the singled-out element $a \in A$. A pointed set is also an algebraic system, if we treat a as the lone nullary operator (constant).
- a bounded poset $(P, \leq_P, 0, 1)$ is a relational system. It is a poset, with two unary relations $\{0\}$ and $\{1\}$.
- a Buekenhout-Tits geometry can be thought of as a relational system. It consists of a set Γ with two binary relations on it: one is an equivalence relation T called type, and the other is a symmetric reflexive relation # called incidence, such that if $a\#b$ and aTb, then $a = b$ (incident objects of the same type are identical).

- ordered algebraic structures, such as ordered groups $(G, \{\cdot, {}^{-1}, e, \leq_G\})$ and ordered rings $(R, \{+, -, \cdot, {}^{-1}, 0, \leq_R\})$ are also relational systems. They are not algebraic systems because of the additional ordering relations ($\leq_G$ and $\leq_R$) defined on these objects. Note that these orderings are generally considered total orders.

- ordered partial algebras such as ordered fields $(D, \{+, -, \cdot, {}^{-1}, 0, 1, \leq_F\})$, etc...

- structures that are not relational are `http://planetmath.org/CompleteLattice`complete lattices and topological spaces, because the operations involved are infinitary.

Remark. Relational systems and algebraic systems are both examples of structures in model theory. Although an algebraic system is a relational system in the sense discussed above, they are treated as distinct entities. A structure involves three objects, a set A, a set of function symbols F, and a set of relation symbols R, so a relational system is a structure where $F = \varnothing$ and an algebraic system is a structure where $R = \varnothing$.

References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).