division in group

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In any group $(G, \cdot)$ one can introduce a *division operation* ":" by setting

$$x : y = x \cdot y^{-1}$$

for all elements x, y of G. On the contrary, the group operation and the unary inverse forming operation may be expressed via the division by

$$x \cdot y = x : ((y : y) : y), \quad x^{-1} = (x : x) : x. \tag{1}$$

The division, which of course is not associative, has the properties

1. $(x : z) : (y : z) = x : y$,

2. $x : (y : y) = x$,

3. $(x : x) : (y : z) = z : y$.

The above result may be conversed:

Theorem. If the operation ":" of the non-empty groupoid G has the properties 1, 2, and 3, then G equipped with the "multiplication" and inverse forming by (1) is a group.

Proof. Here we prove only the associativity of "$\cdot$". First we derive some auxiliary results. Using definitions and the properties 1 and 2 we obtain

$$(x : y) : y^{-1} = (x : y) : ((y : y) : y) = x : (y : y) = x,$$

$$(x : y^{-1}) : y = (x : y^{-1}) : ((y : y) : y^{-1}) = x : (y : y) = x$$

and using the property 3,

$$(x : y)^{-1} = ((x : y) : (x : y)) : (x : y) = y : x.$$

Then we get:

$$(xy)z = (x : y^{-1}) : z^{-1} = ((x : y^{-1}) : y) : (z^{-1} : y) = x : (z^{-1} : y) = x : (y : z^{-1})^{-1} = x(yz)$$

References

[1] А. И. Мальцев: *Алгебраические системы*. Издательство "Наука". Москва (1970).