homomorphism between algebraic systems

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Entry type	Definition
Classification	msc 08A05
Defines	compatible function
Defines	homomorphism
Defines	monomorphism
Defines	epimorphism
Defines	endomorphism
Defines	isomorphism
Defines	automorphism
Defines	homomorphic image

Let $(A, O), (B, O)$ be two algebraic systems with operator set O. Given operators ω_A on A and ω_B on B, with $\omega \in O$ and $n =$ arity of ω, a function $f : A \to B$ is said to be *compatible* with ω if

$$f(\omega_A(a_1, \ldots, a_n)) = \omega_B(f(a_1), \ldots, f(a_n)).$$

Dropping the subscript, we now simply identify $\omega \in O$ as an operator for both algebras A and B. If a function $f : A \to B$ is compatible with every operator $\omega \in O$, then we say that f is a *homomorphism* from A to B. If O contains a constant operator ω such that $a \in A$ and $b \in B$ are two constants assigned by ω, then any homomorphism f from A to B maps a to b.

Examples.

1. When O is the empty set, any function from A to B is a homomorphism.
2. When O is a singleton consisting of a constant operator, a homomorphism is then a function f from one pointed set (A, p) to another (B, q), such that $f(p) = q$.
3. A homomorphism defined in any one of the well known algebraic systems, such as groups, modules, rings, and `http://planetmath.org/Lattice`lattices is consistent with the more general definition given here. The essential thing to remember is that a homomorphism preserves constants, so that between two rings with 1, both the additive identity 0 and the multiplicative identity 1 are preserved by this homomorphism. Similarly, a homomorphism between two `http://planetmath.org/BoundedLattice`bounded lattices is called a $\{0, 1\}$-`http://planetmath.org/LatticeHomomorphism`lattice homomorphism because it preserves both 0 and 1, the bottom and top elements of the lattices.

Remarks.

- Like the familiar algebras, once a homomorphism is defined, special types of homomorphisms can now be named:
  - a homomorphism that is one-to-one is a *monomorphism*;
  - an onto homomorphism is an *epimorphism*;
  - an *isomorphism* is both a monomorphism and an epimorphism;
  - a homomorphism such that its codomain is its domain is called an *endomorphism*;

- finally, an *automorphism* is an endomorphism that is also an isomorphism.

- All trivial algebraic systems (of the same type) are isomorphic.
- If $f : A \to B$ is a homomorphism, then the image $f(A)$ is a subalgebra of B. If ω_B is an n-ary operator on B, and $c_1, \ldots, c_n \in f(A)$, then $\omega_B(c_1, \ldots, c_n) = \omega_B(f(a_1), \ldots, f(a_n)) = f(\omega_A(a_1, \ldots, a_n)) \in f(A)$. $f(A)$ is sometimes called the *homomorphic image* of f in B to emphasize the fact that f is a homomorphism.