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implicational class

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In this entry, we extend the notion of an equational class (or a variety) to a more general notion known as an implicational class (or a quasivariety). Recall that an equational class K is a class of algebraic systems satisfying a set Σ of "equations" and that K is the smallest class satisfying Σ. Typical examples are the varieties of groups, rings, or lattices.

An implicational class, loosely speaking, is the *smallest* class of algebraic systems satisfying a set of "implications", where an implication has the form $P \to Q$, where P and Q are some sentences. Formally, we define an *equational implication* in an algebraic system to be a sentence of the form

$$(\forall x_1) \cdots (\forall x_n)(e_1 \wedge \cdots \wedge e_p \to e_q),$$

where each e_i is an identity of the form $f_i(x_1, \ldots, x_n) = g_i(x_1, \ldots, x_n)$ for some n-ary polynomials f_i and g_i, and $i = 1, \ldots, p, q$.

Definition. A class K of algebraic systems of the same type (signature) is called an *implicational class* if there is a set Σ of equational implications such that

$$K = \{A \text{ is a structure } \mid A \text{ is a model in } \Sigma\} = \{A \mid (\forall q \in \Sigma) \to (A \models q)\}.$$

Examples

1. Any equational class is implicational. Each identity $p = q$ can be thought of as an equational implication $(p = p) \to (p = q)$. In other words, every algebra satisfying the identity also satisfies the corresponding equational implication, and vice versa.

2. The class of all Dedekind-finite rings. In addition to satisfying the identities for being a (unital) ring, each ring also satisfies the equational implication
$$(\forall x)(\forall y)(xy = 1) \to (yx = 1).$$

3. The class of all cancellation semigroups. In addition to satisfying the identities for being a semigroup, each semigroup also satisfies the implications
$$(\forall x)(\forall y)(\forall z)(xy = xz) \to (y = z) \quad \text{and} \quad (\forall x)(\forall y)(\forall z)(yx = zx) \to (y = z).$$

4. The class K of all torsion free abelian groups. In addition to satisfying the identities for being abelian groups, each group also satisfies the set of all implications
$$\{\forall x(nx = 0) \to (x = 0) \mid n \text{ is a positive integer}\}.$$

There is an equivalent formulation of an implicational class. Again, let K be a class of algebraic systems of the same type (signature) τ. Define the following four "operations" on *the* classes of algebraic systems of type τ:

1. $I(K)$ is the class of all isomorphic copies of algebras in K,
2. $S(K)$ is the class of all subalgebras of algebras in K,
3. $P(K)$ is the class of all product of algebras in K (including the empty products, which means $P(K)$ includes the trivial algebra), and
4. $U(K)$ is the class of all ultraproducts of algebras in K.

Suppose X is any one of the operations above, we say that K is *closed* under operation X if $X(K) \subseteq K$.

Definition. K is said to be an *algebraic class* if K is closed under I, and K is said to be a *quasivariety* if it is algebraic and is closed under S, P, U.

It can be shown that a class K of algebraic systems of the same type is implicational iff it is a quasivariety. Therefore, we may use the two terms interchangeably.

As we have seen earlier, a variety is a quasivariety. However, the converse is not true, as can be readily seen in the last example above, since a homomorphic image of a torsion free abelian is in general not torsion free: the homomorphic image of $\phi : \mathbb{Z} \to \mathbb{Z}_n$ is a subgroup of $\mathbb{Z}_n$, hence not torsion free.