operations on relations

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Defines	section
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Recall that an n-ary relation ($n > 0$) is a subset of a product of some n sets. In this definition, any n-ary relation for which $n > 1$ is automatically an $(n-1)$-ary relation, and consequently a binary relation. On the other hand, a unary, or 1-ary relation, being the subset B of some set A, can be viewed as a binary relation (either realized as $B \times B$ or $\Delta_B := \{(b, b) \mid b \in B\}$) on A. Hence, we shall concentrate our discussion on the most important kind of relations, the binary relations, in this entry.

Compositions, Inverses, and Complements

Let $\rho \subseteq A \times B$ and $\sigma \subseteq B \times C$ be two binary relations. We define the *composition* of ρ and σ, the *inverse* (or *converse*) and the *complement* of ρ by

- $\rho \circ \sigma := \{(a, c) \mid (a, b) \in \rho \text{ and } (b, c) \in \sigma \text{ for some } b \in B\}$,
- $\rho^{-1} := \{(b, a) \mid (a, b) \in \rho\}$, and
- $\rho' := \{(a, b) \mid (a, b) \notin \rho\}$.

$\rho \circ \sigma, \rho^{-1}$ and ρ' are binary relations of $A \times C, B \times A$ and $A \times B$ respectively.

Properties. Suppose ρ, σ, τ are binary relations of $A \times B, B \times C, C \times D$ respectively.

1. Associativity of relational compositions: $(\rho \circ \sigma) \circ \tau = \rho \circ (\sigma \circ \tau)$.
2. $(\rho \circ \sigma)^{-1} = \sigma^{-1} \circ \rho^{-1}$.
3. For the rest of the remarks, we assume $A = B$. Define the *power* of ρ recursively by $\rho^1 = \rho$, and $\rho^{n+1} = \rho \circ \rho^n$. By 1 above, $\rho^{n+m} = \rho^n \circ \rho^m$ for $m, n \in \mathbb{N}$.
4. ρ^{-n} may be also be defined, in terms of ρ^{-1} for $n > 0$.
5. However, $\rho^{n+m} \neq \rho^n \circ \rho^m$ for *all* integers, since $\rho^{-1} \circ \rho \neq \rho \circ \rho^{-1}$ in general.
6. Nevertheless, we may define $\Delta := \{(a, a) \mid a \in A\}$. This is called the *identity* or *diagonal relation* on A. It has the property that $\Delta \circ \rho = \rho \circ \Delta = \rho$. For this, we may define, for every relation ρ on A, $\rho^0 := \Delta$.

7. Let $\mathcal{R}$ be the set of all binary relations on a set A. Then $(\mathcal{R}, \circ)$ is a http://planetmath.org/Monoidmonoid with Δ as the identity element.

Let ρ be a binary relation on a set A, below are some special relations definable from ρ:

- ρ is reflexive if $\Delta \subseteq \rho$;
- ρ is symmetric if $\rho = \rho^{-1}$;
- ρ is anti-symmetric if $\rho \cap \rho^{-1} \subseteq \Delta$;
- ρ is transitive if $\rho \circ \rho \subseteq \rho$;
- ρ is a pre-order if it is reflexive and transitive
- ρ is a partial order if it is a pre-order and is anti-symmetric
- ρ is an equivalence if it is a pre-order and is symmetric

Of these, only reflexivity is preserved by $\circ$ and both symmetry and anti-symmetry are preserved by the inverse operation.

Other Operations

Some operations on binary relations yield unary relations. The most common ones are the following:

Given a binary relation $\rho \in A \times B$, and an elements $a \in A$ and $b \in B$, define

$$\rho(a, \cdot) := \{y \mid (a, y) \in \rho\} \quad \text{and} \quad \rho(\cdot, b) := \{x \mid (x, b) \in \rho\}.$$

$\rho(a, \cdot) \subseteq B$ is called the *section of ρ in B based at a*, and $\rho(\cdot, b) \subseteq A$ is the *section of ρ in A (based) at b*. When the base points are not mentioned, we simply say a section of ρ in A or B, or an A-section or a B-section of ρ.

Finally, we define the domain and range of a binary relation $\rho \subseteq A \times B$, to be

- $\text{dom}(\rho) := \{a \mid (a, b) \in \rho \text{ for some } b \in B\}$, and
- $\text{ran}(\rho) := \{b \mid (a, b) \in \rho \text{ for some } a \in A\}$.

respectively. When ρ is a function, the domain and range of ρ coincide with the domain of range of ρ as a function.

Remarks. Given a binary relation $\rho \subseteq A \times B$.

1. $\rho(a, \cdot)$, realized as a binary relation $\{a\} \times \rho(a, \cdot)$ can be viewed as the composition of $\Delta_a \circ \rho$, where $\Delta_a = \{(a, a)\}$. Similarly, $\rho \circ \Delta_b = \rho(\cdot, b) \times \{b\}$.

2. $\mathrm{dom}(\rho)$ is the disjoint union of A-sections of ρ and $\mathrm{ran}(\rho)$ is the disjoint union of B-sections of ρ.

3. $\mathrm{dom}(\rho^{-1}) = \mathrm{ran}(\rho)$ and $\mathrm{ran}(\rho^{-1}) = \mathrm{dom}(\rho)$.

4. $\rho \circ \rho^{-1} = \mathrm{dom}(\rho) \times \mathrm{dom}(\rho)$ and $\rho^{-1} \circ \rho = \mathrm{ran}(\rho) \times \mathrm{ran}(\rho)$.

5. If $A = B$, then $\mathrm{dom}(\rho) \times A = \rho \circ A^2$ and $\mathrm{ran}(\rho) = A^2 \circ \rho$.

6. The composition of binary relations can be generalized: let R be a subset of $A_1 \times \cdots \times A_n$ and S be a subset of $B_1 \times \cdots \times B_m$, where m, n are positive integers. Further, we assume that $A_n = B_1 = C$. Define $R \circ S$, as a subset of $A_1 \times \cdots \times A_{n-1} \times B_2 \times \cdots B_m$, to be the set

   $$\{(a_1, \ldots, a_{n-1}, b_2, \ldots, b_m) \mid \exists c \in C \text{ with } (a_1, \ldots, a_{n-1}, c) \in R \text{ and } (c, b_2, \ldots, b_m) \in S\}.$$

   If $m = n = 2$, then we have the familiar composition of binary relations. If $m = 1$ and $n = 2$, then $R \circ S = \{a \mid (a, c) \in R \text{ and } c \in S\}$. The case where $m = 2$ and $n = 1$ is similar. If $m = n = 1$, then $R \circ S = \{ \text{ true } \}$ if $R \cap S \neq \varnothing$. Otherwise, it is set to $\{ \text{ false } \}$.