identity in a class

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Defines	identity

Let K be a class of algebraic systems of the same type. An *identity* on K is an expression of the form $p = q$, where p and q are n-ary polynomial symbols of K, such that, for every algebra $A \in K$, we have

$$p_A(a_1, \ldots, a_n) = q_A(a_1, \ldots, a_n) \qquad \text{for all } a_1, \ldots, a_n \in A,$$

where p_A and q_A denote the induced polynomials of A by the corresponding polynomial symbols. An identity is also known sometimes as an *equation*.

Examples.

- Let K be a class of algebras of the type $\{e, ^{-1}, \cdot\}$, where e is nullary, $^{-1}$ unary, and $\cdot$ binary. Then

  1. $x \cdot e = x$,
  2. $e \cdot x = e$,
  3. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
  4. $x \cdot x^{-1} = e$,
  5. $x^{-1} \cdot x = e$, and
  6. $x \cdot y = y \cdot x$.

  can all be considered identities on K. For example, in the fourth equation, the right hand side is the unary polynomial $q(x) = e$. Any algebraic system satisfying the first three identities is a monoid. If a monoid also satisfies identities 4 and 5, then it is a group. A group satisfying the last identity is an abelian group.

- Let L be a class of algebras of the type $\{\vee, \wedge\}$ where $\vee$ and $\wedge$ are both binary. Consider the following possible identities

  1. $x \vee x = x$,
  2. $x \vee y = y \vee x$,
  3. $x \vee (y \vee z) = (x \vee y) \vee z$,
  4. $x \wedge x = x$,
  5. $x \wedge y = y \wedge x$,
  6. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
  7. $x \vee (y \wedge x) = x$,

8. $x \wedge (y \vee x) = x$,
9. $x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$,
10. $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$,
11. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, and
12. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

If algebras of K satisfy identities 1-8, then K is a class of lattices. If 9 and 10 are satisfied as well, then K is a class of modular lattices. If every identity is satisisfied by algebras of K, then K is a class of distributive lattices.