Jordan-Banach and Jordan-Lie algebras

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0.0.1 Definitions of Jordan-Banach, Jordan-Lie, and Jordan-Banach-Lie algebras

Firstly, a specific *algebra* consists of a vector space E over a ground field (typically $\mathbb{R}$ or $\mathbb{C}$) equipped with a bilinear and distributive multiplication $\circ$. Note that E is not necessarily commutative or associative.

A *Jordan algebra* (over $\mathbb{R}$), is an algebra over $\mathbb{R}$ for which:

$$S \circ T = T \circ S \ ,$$
$$S \circ (T \circ S^2) = (S \circ T) \circ S^{2}{}^{,}$$

for all elements S, T of the algebra.

It is worthwhile noting now that in the algebraic theory of Jordan algebras, an important role is played by the *Jordan triple product* $\{STW\}$ as defined by:
$\{STW\} = (S \circ T) \circ W + (T \circ W) \circ S - (S \circ W) \circ T$,

which is linear in each factor and for which $\{STW\} = \{WTS\}$. Certain examples entail setting $\{STW\} = \frac{1}{2}\{STW + WTS\}$.

A *Jordan Lie algebra* is a real vector space $\mathfrak{A}_{\mathbb{R}}$ together with a *Jordan product* $\circ$ and *Poisson bracket*

$\{\ ,\ \}$, satisfying :

1. for all $S, T \in \mathfrak{A}_{\mathbb{R}}$, $\begin{array}{c} S \circ T = T \circ S \\ \{S, T\} = -\{T, S\} \end{array}$

2. the *Leibniz rule* holds

   $\{S, T \circ W\} = \{S, T\} \circ W + T \circ \{S, W\}$ for all $S, T, W \in \mathfrak{A}_{\mathbb{R}}$, along with

3. the *Jacobi identity* :

   $\{S, \{T, W\}\} = \{\{S, T\}, W\} + \{T, \{S, W\}\}$

4. for some $\hbar^2 \in \mathbb{R}$, there is the *associator identity* :

   $(S \circ T) \circ W - S \circ (T \circ W) = \frac{1}{4}\hbar^2\{\{S, W\}, T\}$.

0.0.2 Poisson algebra

By a *Poisson algebra* we mean a Jordan algebra in which $\circ$ is associative. The usual algebraic types of morphisms automorphism, isomorphism, etc.) apply to Jordan-Lie (Poisson) algebras (see Landsman, 2003).

Consider the classical configuration space $Q = \mathbb{R}^3$ of a moving particle whose phase space is the cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^6$, and for which the space of (classical) observables is taken to be the real vector space of smooth functions

$$\mathfrak{A}^0_{\mathbb{R}} = C^\infty(T^*R^3, \mathbb{R})$$

. The usual pointwise multiplication of functions fg defines a bilinear map on $\mathfrak{A}^0_{\mathbb{R}}$, which is seen to be commutative and associative. Further, the Poisson bracket on functions

$$\{f, g\} := \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} \ ,$$

which can be easily seen to satisfy the Liebniz rule above. The axioms above then set the stage of passage to quantum mechanical systems which the parameter k^2 suggests.

0.0.3 C*–algebras (C*–A), JLB and JBW Algebras

An *involution* on a complex algebra $\mathfrak{A}$ is a real–linear map $T \mapsto T^*$ such that for all

$S, T \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, we have $T^{**} = T \ , \ (ST)^* = T^*S^* \ , \ (\lambda T)^* = \bar{\lambda} T^*$.

A **–algebra* is said to be a complex associative algebra together with an involution $*$.

A *C*–algebra* is a simultaneously a *–algebra and a Banach space $\mathfrak{A}$, satisfying for all $S, T \in \mathfrak{A}$:

$\|S \circ T\| \leq \|S\| \ \|T\|$,
$\|T^*T\|^2 = \|T\|^2$.

We can easily see that $\|A^*\| = \|A\|$. By the above axioms a C*–algebra is a special case of a Banach algebra where the latter requires the above norm property but not the involution (*) property. Given Banach spaces E, F the space $\mathcal{L}(E, F)$ of (bounded) linear operators from E to F forms a Banach space, where for $E = F$, the space $\mathcal{L}(E) = \mathcal{L}(E, E)$ is a Banach algebra with respect to the norm

$\|T\| := \sup\{\|Tu\| : u \in E \ , \ \|u\| = 1\}$.

In quantum field theory one may start with a Hilbert space H, and consider the Banach algebra of bounded linear operators $\mathcal{L}(H)$ which given to be closed under the usual algebraic operations and taking adjoints, forms a $*$–algebra of bounded operators, where the adjoint operation functions as the involution, and for $T \in \mathcal{L}(H)$ we have :

$\|T\| := \sup\{(Tu, Tu) : u \in H \ , \ (u, u) = 1\}$, and $\|Tu\|^2 = (Tu, Tu) = (u, T^*Tu) \leq \|T^*T\| \ \|u\|^2$.

By a morphism between C*–algebras $\mathfrak{A}, \mathfrak{B}$ we mean a linear map $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$, such that for all $S, T \in \mathfrak{A}$, the following hold :

$\phi(ST) = \phi(S)\phi(T) \ , \ \phi(T^*) = \phi(T)^*$, where a bijective morphism is said to be an isomorphism (in which case it is then an isometry). A fundamental relation is that any norm-closed $*$–algebra $\mathcal{A}$ in $\mathcal{L}(H)$ is a C*–algebra, and conversely, any C*–algebra is isomorphic to a norm–closed $*$–algebra in $\mathcal{L}(H)$ for some Hilbert space H .

For a C*–algebra $\mathfrak{A}$, we say that $T \in \mathfrak{A}$ is *self–adjoint* if $T = T^*$. Accordingly, the self–adjoint part $\mathfrak{A}^{sa}$ of $\mathfrak{A}$ is a real vector space since we can decompose $T \in \mathfrak{A}^{sa}$ as :

$$T = T' + T'' := \frac{1}{2}(T + T^*) + \iota(\frac{-\iota}{2})(T - T^*) \ .$$

A *commutative* C*–algebra is one for which the associative multiplication is commutative. Given a commutative C*–algebra $\mathfrak{A}$, we have $\mathfrak{A} \cong C(Y)$, the algebra of continuous functions on a compact Hausdorff space Y .

A *Jordan–Banach algebra* (a JB–algebra for short) is both a real Jordan algebra and a Banach space, where for all $S, T \in \mathfrak{A}_{\mathbb{R}}$, we have

$$\|S \circ T\| \leq \|S\| \ \|T\| \ ,$$
$$\|T\|^2 \leq \|S^2 + T^2\| \ .$$

A *JLB–algebra* is a JB–algebra $\mathfrak{A}_{\mathbb{R}}$ together with a Poisson bracket for which it becomes a Jordan–Lie algebra for some $\hbar^2 \geq 0$. Such JLB–algebras often constitute the real part of several widely studied complex associative algebras.

For the purpose of quantization, there are fundamental relations between $\mathfrak{A}^{sa}$, JLB and Poisson algebras.

For further details see Landsman (2003) (Thm. 1.1.9).

A JB–algebra which is monotone complete and admits a separating set of normal sets is called a *JBW-algebra*. These appeared in the work of von Neumann who developed a (orthomodular) lattice theory of projections on $\mathcal{L}(H)$ on which to study quantum logic (see later). BW-algebras have the following property: whereas $\mathfrak{A}^{sa}$ is a J(L)B–algebra, the self adjoint part of a von Neumann algebra is a JBW–algebra.

A *JC–algebra* is a norm closed real linear subspace of $\mathcal{L}(H)^{sa}$ which is closed under the bilinear product $S \circ T = \frac{1}{2}(ST + TS)$ (non–commutative and nonassociative). Since any norm closed Jordan subalgebra of $\mathcal{L}(H)^{sa}$ is a JB–algebra, it is natural to specify the exact relationship between JB and JC–algebras, at least in finite dimensions. In order to do this, one introduces the 'exceptional' algebra $H_3(\mathbb{O})$, the algebra of 3×3 Hermitian matrices with values in the octonians $\mathbb{O}$. Then a finite dimensional JB–algebra is a JC–algebra if and only if it does not contain $H_3(\mathbb{O})$ as a (direct) summand [**?**].

The above definitions and constructions follow the approach of Alfsen and Schultz (2003) and Landsman (1998).

References

[1] Alfsen, E.M. and F. W. Schultz: Geometry of State Spaces of Operator Algebras, Birkhäuser, Boston-Basel-Berlin.(2003).