# subalgebra of a partial algebra

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| Canonical name | SubalgebraOfAPartialAlgebra |
| Date of creation | 2013-03-22 18:42:54 |
| Last modified on | 2013-03-22 18:42:54 |
| Owner | CWoo (3771) |
| Last modified by | CWoo (3771) |
| Numerical id | 10 |
| Author | CWoo (3771) |
| Entry type | Definition |
| Classification | msc 08A55 |
| Classification | msc 03E99 |
| Classification | msc 08A62 |
| Defines | weak subalgebra |
| Defines | relative subalgebra |
| Defines | subalgebra |

Unlike an algebraic system, where there is only one way to define a subalgebra, there are several ways to define a subalgebra of a partial algebra.

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are partial algebras of type $\tau$:

1. $\boldsymbol{B}$ is a *weak subalgebra* of $\boldsymbol{A}$ if $B \subseteq A$, and $f_{\boldsymbol{B}}$ is a subfunction of $f_{\boldsymbol{A}}$ for every operator symbol $f \in \tau$.

   In words, $\boldsymbol{B}$ is a weak subalgebra of $\boldsymbol{A}$ iff $B \subseteq A$, and for each $n$-ary symbol $f \in \tau$, if $b_1, \ldots, b_n \in B$ such that $f_B(b_1, \ldots, b_n)$ is defined, then $f_A(b_1, \ldots, b_n)$ is also defined, and is equal to $f_B(b_1, \ldots, b_n)$.

2. $\boldsymbol{B}$ is a *relative subalgebra* of $\boldsymbol{A}$ if $B \subseteq A$, and $f_{\boldsymbol{B}}$ is a `http://planetmath.org/Subfunction`re of $f_{\boldsymbol{A}}$ relative to $B$ for every operator symbol $f \in \tau$.

   In words, $\boldsymbol{B}$ is a relative subalgebra of $\boldsymbol{A}$ iff $B \subseteq A$, and for each $n$-ary symbol $f \in \tau$, given $b_1, \ldots, b_n \in B$, $f_B(b_1, \ldots, b_n)$ is defined iff $f_A(b_1, \ldots, b_n)$ is and belongs to $B$, and they are equal.

3. $\boldsymbol{B}$ is a *subalgebra* of $\boldsymbol{A}$ if $B \subseteq A$, and $f_{\boldsymbol{B}}$ is a `http://planetmath.org/Subfunction`restriction of $f_{\boldsymbol{A}}$ for every operator symbol $f \in \tau$.

   In words, $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ iff $B \subseteq A$, and for each $n$-ary symbol $f \in \tau$, given $b_1, \ldots, b_n \in B$, $f_B(b_1, \ldots, b_n)$ is defined iff $f_A(b_1, \ldots, b_n)$ is, and they are equal.

Notice that if $\boldsymbol{B}$ is a weak subalgebra of $\boldsymbol{A}$, then every constant of $\boldsymbol{B}$ is a constant of $\boldsymbol{A}$, and vice versa.

Every subalgebra is a relative subalgebra, and every relative subalgebra is a weak subalgebra. But the converse is false for both statements. Below are two examples.

1. Let $F$ be a field. Then every subalgebra of $F$ is a subfield, and every relative subalgebra of $F$ is a subring.

2. Let $A$ be the set of all non-negative integers, and $-_A$ the ordinary subtraction on integers. Consider the partial algebra $(A, -_A)$.

   - Let $B = A$ and $-_B$ the usual subtraction on integers, but $x -_B y$ is only defined when $x, y \in B$ have the same parity. Then $(B, -_B)$ is a weak subalgebra of $(A, -_A)$.
   - Let $C$ be the set of all positive integers, and $-_C$ the ordinary subtraction. Then $(C, -_C)$ is a relative subalgebra of $(A, -_A)$.

- Let $D$ be the set $\{0, 1, \dots, n\}$ and $-_D$ the ordinary subtraction. Then $(D, -_D)$ is a subalgebra of $(A, -_A)$.

Notice that $(B, -_B)$ is not a relative subalgebra of $(A, -_A)$, since $7 -_B 6$ is not defined, even though $7 - A6 = 1 \in B$, and and $(C, -_C)$ is not a subalgebra of $(A, -_A)$, since $1 -_C 1$ is not defined in $C$, even though $1 - A1$ is defined in $A$.

**Remarks**.

1. A weak subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ is a relative subalgebra iff given $b_1, \dots, b_n \in B$ such that $f_A(b_1, \dots, b_n)$ is defined and is in $B$, then $f_B(b_1, \dots, b_n)$ is defined. A relative subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ is a subalgebra iff whenever $f_A(b_1, \dots, b_n)$ is defined for $b_i \in B$, it is in $B$.

2. Let $\boldsymbol{A}$ be a partial algebra of type $\tau$, and $B \subseteq A$. For each $n$-ary function symbol $f \in \tau$, define $f_{\boldsymbol{B}}$ on $B$ as follows: $f_{\boldsymbol{B}}(b_1, \dots, b_n)$ is defined in $B$ iff $f_{\boldsymbol{A}}(b_1, \dots, b_n)$ is defined in $A$ and $f_{\boldsymbol{A}}(b_1, \dots, b_n) \in B$. This turns $\boldsymbol{B}$ into a partial algebra. However, $\boldsymbol{B}$ may not be of type $\tau$, since $f_{\boldsymbol{B}}$ may not be defined at all on $B$. When $\boldsymbol{B}$ is a partial algebra of type $\tau$, it is a relative subalgebra of $\boldsymbol{A}$.

3. When $\boldsymbol{A}$ is an algebra, all three notions of subalgebras are equivalent (assuming that the partial operations on a weak subalgebra are all total).

# References

[1] G. Grätzer: *Universal Algebra*, 2nd Edition, Springer, New York (1978).