arithmetic derivative

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Author	PrimeFan (13766)
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The *arithmetic derivative* n' of a natural number n is defined by the following rules:

- $p' = 1$ for any prime p.
- $(ab)' = a'b + ab'$ for any $a, b \in \mathbb{N}$ (Leibniz rule).

To define the arithmetic derivative of a negative number, we first note that $1' = 0$ by the Leibniz rule ($1' = (1 \cdot 1)' = 1 \cdot 1' + 1' \cdot 1 = 2 \cdot 1'$, so $1' = 0$), and further that we must have

$$0 = 1' = ((-1) \cdot (-1))' = -2 \cdot (-1)',$$

so $(-1)' = 0$. The product rule now requires that we define $(-n)' = -(n') + n(-1)' = -(n')$.

Further, we can extend this definition to rational numbers by insisting that the quotient rule holds, i.e. for a prime p we should have

$$0 = 1' = \left(p \cdot \frac{1}{p}\right)' = \left(\frac{1}{p}\right)' p + \frac{1}{p},$$

giving us that

$$\left(\frac{1}{p}\right)' = -\frac{1}{p^2},$$

i.e. the usual quotient rule from calculus. We now complete the definition by extending multiplicatively (i.e. using the Leibniz rule).

The arithmetic derivatives for the first few positive integers are 0, 1, 1, 4, 1, 5, 1, 12, 6, 7, 1, 16, 1, 9, 8, 32, 1, 21, 1, 24, 10, 13, 1, 44, 10, etc.

As a consequence of $p' = 1$ for a prime p, the arithmetic derivative of a semiprime (whether squarefree or not) works out to $(pq)' = p'q + pq' = 1p + q1 = p + q$. For example, the arithmetic derivative of 10 is 7, which is 2 plus 5.

The only cases of $n' = n$ for $-1 < n < 1024$ are 0, 4, 27.

n	n'	n	n'	n	n'	n	n'	n	n'	n	n'	n	n'	n	n'	n	n'	n	n'
0	0	10	7	20	24	30	31	40	68	50	45	60	92	70	59	80	176	90	12
1	0	11	1	21	10	31	1	41	1	51	20	61	1	71	1	81	108	91	2
2	1	12	16	22	13	32	80	42	41	52	56	62	33	72	156	82	43	92	9
3	1	13	1	23	1	33	14	43	1	53	1	63	51	73	1	83	1	93	3
4	4	14	9	24	44	34	19	44	48	54	81	64	192	74	39	84	124	94	4
5	1	15	8	25	10	35	12	45	39	55	16	65	18	75	55	85	22	95	2
6	5	16	32	26	15	36	60	46	25	56	92	66	61	76	80	86	45	96	27
7	1	17	1	27	27	37	1	47	1	57	22	67	1	77	18	87	32	97	
8	12	18	21	28	32	38	21	48	112	58	31	68	72	78	71	88	140	98	7
9	6	19	1	29	1	39	16	49	14	59	1	69	26	79	1	89	1	99	7

References

[1] EJ Barbeau, “Remark on an arithmetic derivative”. *Can. Math. Bull.* **4** (1961): 117 - 122