periodic continued fractions represent quadratic irrationals

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This article shows that infinite simple continued fractions that are eventually periodic correspond precisely to quadratic irrationals.

Throughout, we will freely use results on convergents to a continued fraction; see that article for details.

Definition 1. A *periodic simple continued fraction* is a simple continued fraction

$$[a_0, a_1, a_2, \ldots]$$

such that for some $k \geq 0$ there is $m > 0$ such that whenever $r \geq k$, we have $a_r = a_{r+m}$. Informally, a periodic continued fraction is one that eventually repeats. A *purely periodic* simple continued fraction is one for which $k = 0$; that is, one whose repeating period starts with the initial element.

If

$$[a_0, a_1, \ldots, a_{k-1}, a_k, \ldots, a_{k+j-1}, a_k, \ldots, a_{k+j-1}, a_k, \ldots]$$

is a periodic continued fraction, we write it as

$$[a_0, a_1, \ldots, a_{k-1}, \overline{a_k, \ldots, a_{k+j-1}}].$$

Theorem 1. *If*

$$\alpha = [a_0, a_1, \ldots, a_r, \overline{b_1, \ldots, b_t}]$$

is a periodic simple continued fraction, then α is a quadratic irrational $p + q\sqrt{d}$ for p, q rational and d squarefree. Conversely, every such quadratic irrational is represented by such a continued fraction.

Proof. The forward direction is pretty straightforward. Given such a continued fraction, let β be the $(r+1)^{\text{st}}$ complete convergent, i.e.

$$\beta = [\overline{b_1, \ldots, b_t}]$$

Note first that β must be irrational since the continued fraction for any rational number terminates. Then the article on convergents to a continued fraction shows that

$$\beta = \frac{\beta p_t + p_{t-1}}{\beta q_t + q_{t-1}}$$

where the p_i, q_i are the convergents to the continued fraction for β. Thus

$$q_t\beta^2 + (q_{t-1} - p_t)\beta - p_{t-1} = 0$$

and thus β is irrational and satisfies a quadratic equation so is a quadratic irrational. A simple computation then shows that α is as well.

In the other direction, suppose that α is a quadratic irrational satisfying

$$a\alpha^2 + b\alpha + c = 0$$

and with continued fraction representation

$$\alpha = [a_0, a_1, \ldots]$$

Then for any $n > 0$, we have

$$\alpha = \frac{t_n p_{n-1} + p_{n-2}}{t_n q_{n-1} + q_{n-2}}$$

where the t_i are the complete convergents of the continued fraction, so that from the quadratic equation we have

$$A_n t_n^2 + B_n t_n + C_n = 0$$

where

$$\begin{gathered} A_n = a p_{n-1}^2 + b p_{n-1} q_{n-1} + c q_{n-1}^2 \\ B_n = 2a p_{n-1} p_{n-2} + b(p_{n-1} q_{n-2} + p_{n-2} q_{n-1}) + 2c q_{n-1} q_{n-2} \\ C_n = a p_{n-2}^2 + b p_{n-2} q_{n-2} + c q_{n-2}^2 = A_{n-1} \end{gathered}$$

Note that $A_n \neq 0$ for each $n > 0$ since otherwise

$$a p_{n-1}^2 + b p_{n-1} q_{n-1} + c q_{n-1}^2 = 0$$

so that

$$a \left(\frac{p_{n-1}}{q_{n-1}} \right)^2 + b \frac{p_{n-1}}{q_{n-1}} + c = 0$$

and the quadratic equation would have a rational root, contradicting the fact that α is irrational.

The remainder of the proof is an elaborate computation that shows we can bound each of A_n, B_n, C_n independent of n. Assuming that, it follows that there are only a finite number of possibilities for the triples (A_n, B_n, C_n), so we can choose n_1, n_2, n_3 such that

$$(A_{n_1}, B_{n_1}, C_{n_1}) = (A_{n_2}, B_{n_2}, C_{n_2}) = (A_{n_3}, B_{n_3}, C_{n_3})$$

Then each of $t_{n_1}, t_{n_2}, t_{n_3}$ is a root of (say)

$$A_{n_1}t^2 + B_{n_1}t + C_{n_1}$$

so that two of them must be equal. But then $t_{n_1} = t_{n_2}$ (say), and

$$\begin{aligned} t_{n_1} &= [a_{n_1}, a_{n_1+1}, \ldots] \\ t_{n_2} &= [a_{n_2}, a_{n_2+1}, \ldots] \end{aligned}$$

and the continued fraction is periodic.

We proceed to find the bounds. We know that

$$\left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{1}{q_{n-1}^2}$$

so that

$$\alpha - \frac{p_{n-1}}{q_{n-1}} = \frac{\epsilon}{q_{n-1}^2}$$

for some ϵ (depending on n) with $|\epsilon| < 1$. Thus

$$\begin{aligned} A_n &= ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2 \\ &= a\left(\alpha q_{n-1} + \frac{\epsilon}{q_{n-1}}\right)^2 + bq_{n-1}\left(\alpha q_{n-1} + \frac{\epsilon}{q_{n-1}}\right) + cq_{n-1}^2 \\ &= (a\alpha^2 + b\alpha + c)q_{n-1}^2 + 2a\alpha\epsilon + a\frac{\epsilon^2}{q_{n-1}^2} + b\epsilon \\ &= 2a\alpha\epsilon + a\frac{\epsilon^2}{q_{n-1}^2} + b\epsilon \end{aligned}$$

so that

$$|A_n| = \left|2a\alpha\epsilon + a\frac{\epsilon^2}{q_{n-1}^2} + b\epsilon\right| < 2\,|a\alpha| + |a| + |b|$$

and thus also

$$|C_n| < 2\,|a\alpha| + |a| + |b|$$

It remains to bound B_n. But

$$B_n^2 - 4A_nC_n = (2A_nt_n + B_n)^2$$

Substituting the values of A_n, B_n on the right, and using the fact that

$$t_n = -\frac{p_{n-2} - xq_{n-2}}{p_{n-1} - xq_{n-1}}$$

we get after a computation

$$
\begin{aligned}
\sqrt{B_n^2 - 4A_nC_n} &= (p_{n-1}q_{n-2} - p_{n-2}q_{n-1})\frac{2axp_{n-1} + bp_{n-1} + bxq_{n-1} + 2cq_{n-1}}{p_{n-1} - xq_{n-1}} \\
&= \pm\frac{(2ax+b)p_{n-1} + (2ax^2 + 2bx + 2c)q_{n-1} - (2ax^2 + bx)q_{n-1}}{p_{n-1} - xq_{n-1}} \\
&= \pm\frac{(p_{n-1} - xq_{n-1})(2ax+b)}{p_{n-1} - xq_{n-1}} = \pm(2ax+b)
\end{aligned}
$$

so that

$$B_n^2 - 4A_nC_n = (2ax+b)^2 = b^2 - 4ac$$

Thus

$$B_n^2 \leq 4\left|A_nC_n\right| + \left|b^2 - 4ac\right| < 4(2\left|a\alpha\right| + \left|a\right| + \left|b\right|)^2 + \left|b^2 - 4ac\right|$$

and we have thus bounded all of A_n, B_n, C_n independent of n. $\square$

References

[1] G.H. Hardy & E.M. Wright, *An Introduction to the Theory of Numbers*, Fifth Edition, Oxford Science Publications, 1979.