purely periodic continued fractions

Canonical name	PurelyPeriodicContinuedFractions
Date of creation	2013-03-22 18:04:44
Last modified on	2013-03-22 18:04:44
Owner	rm50 (10146)
Last modified by	rm50 (10146)
Numerical id	6
Author	rm50 (10146)
Entry type	Theorem
Classification	msc 11Y65
Classification	msc 11A55

We know that periodic continued fractions represent quadratic irrationals; this article characterizes purely periodic continued fractions. We will use freely the results on convergents to a continued fraction.

Theorem 1. *(Galois) A quadratic irrational t is represented by a purely periodic simple continued fraction if and only if $t > 1$ and its conjugate s under the transformation $\sqrt{d} \mapsto -\sqrt{d}$ satisfies $-1 < s < 0$.*

Proof. Suppose first that t is represented by a purely periodic continued fraction

$$t = [\overline{a_0, a_1, \ldots, a_{r-1}}].$$

Note that $a_0 \geq 1$ since it appears again in the continued fraction. Thus $t > 1$. The r^{th} complete convergent is again t, so that we have

$$t = \frac{p_{r-2} + tp_{r-1}}{q_{r-2} + tq_{r-1}}$$

so that

$$q_{r-1}t^2 + (q_{r-2} - p_{r-1})t - p_{r-2} = 0$$

Consider the polynomial $f(x) = q_{r-1}x^2 + (q_{r-2} - p_{r-1})x - p_{r-2}$. $f(t) = 0$, so the other root of $f(x)$ is the conjugate s of t. But $f(-1) = (p_{r-1} - p_{r-2}) + (q_{r-1} - q_{r-2}) > 0$ since the p_i and the q_i are both strictly increasing sequences, while $f(0) = -p_{r-2} < 0$. Thus s lies between -1 and 0 and we are done.

Now suppose that $t > 1$ and $-1 < s < 0$, and let the continued fraction for t be $[a_0, a_1, \ldots]$. Let t_n be the n^{th} complete convergent of t, and $s_n = \overline{t_n}$. Thus $s_0 = s$. Then

$$t = t_0 = a_0 + \frac{1}{t_1}$$

so that

$$s_0 = \overline{t_0} = a_0 + \frac{1}{\overline{t_1}} = a_0 + \frac{1}{s_1}$$

and thus

$$\frac{1}{s_1} = -a_0 + s_0 < -a_0 \leq -1$$

so that $-1 < s_1 < 0$. Inductively, we have $-1 < s_n = \overline{t_n} < 0$ for all $n \geq 0$. Suppose now that the continued fraction for t is not purely periodic, but rather has the form

$$t = [a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+j-1}}]$$

for $k \geq 1$. Then $t_k = t_{k+j}$ and so

$$t_{k-1} - t_{k+j-1} = \left(a_{k-1} + \frac{1}{t_k}\right) - \left(a_{k+j-1} + \frac{1}{t_{k+j}}\right) = a_{k-1} - a_{k+j-1}$$

But $a_{k-1} \neq a_{k+j-1}$, otherwise a_{k-1} would have been the first element of the repeating period. Thus $t_{k-1} - t_{k+j-1}$ is a nonzero integer and thus $s_{k-1} - s_{k+j-1}$ is as well. But $-1 < s_{k-1} - s_{k+j-1} < 1$, which is a contradiction. Thus $k = 0$ and the continued fraction is purely periodic. $\square$

References

[1] A.M. Rockett & P. Szüsz, *Continued Fractions*, World Scientific Publishing, 1992.