integer harmonic means

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Let u and v be positive integers. As is seen in the http://planetmath.org/IntegerContraharm entry, there exist nontrivial cases ($u \neq v$) where their contraharmonic mean

$$c \;:=\; \frac{u^2+v^2}{u+v} \;=\; u+v-\frac{2uv}{u+v} \tag{1}$$

is an integer. Because the subtrahend of the last is the harmonic mean of u and v, the equation means that the contraharmonic mean c and the harmonic mean

$$h \;:=\; \frac{2uv}{u+v} \tag{2}$$

of u and v are simultaneously integers.

The integer contraharmonic mean of two distinct positive integers ranges exactly the set of hypotenuses of Pythagorean triples (see contraharmonic integers), but the integer harmonic mean of two distinct positive integers the wider set $\{3,\,4,\,5,\,6,\,\ldots\}$. As a matter of fact, one cathetus of a right triangle is the harmonic mean of the same positive integers u and v the contraharmonic mean of which is the hypotenuse of the triangle (see http://planetmath.org/PythagoreanT triangle).

The following table allows to compare the values of u, v, c, h when $1 < u < v$.

u	2	3	3	4	4	5	5	6	6	6	6	7	7	8	8	8	9	9	...
v	6	6	15	12	28	20	45	12	18	30	66	42	91	24	56	120	18	45	...
c	5	5	13	10	25	17	41	10	15	26	61	37	85	20	50	113	15	39	...
h	3	4	5	6	7	8	9	8	9	10	11	12	13	12	14	15	12	15	...

Some of the propositions concerning the integer contraharmonic means directly imply corresponding propositions of the integer harmonic means:

Proposition 1. For any value of $u > 2$, there are at least two **greater** values

$$v_1 \;:=\; (u-1)u, \quad v_2 \;:=\; (2u-1)u \tag{3}$$

of v such that h in (2) is an integer.

Proposition 2. For all $u > 1$, a necessary condition for $h \in \mathbb{Z}$ is that

$$\gcd(u,\, v) \;>\; 1.$$

Proposition 3. If u is an odd prime number, then the values (3) are the only possibilities for $v > u$ enabling integer harmonic means with u.

Proposition 5. When the harmonic mean of two different positive integers u and v is an integer, their sum is never squarefree.

Proposition 6. For each integer $u > 0$ there are only a finite number of solutions $(u,\, v,\, h)$ of the Diophantine equation (2).

Proposition 6 follows also from the inequality

$$\frac{1}{h} \;=\; \frac{1}{2}\left(\frac{1}{u}+\frac{1}{v}\right) \;>\; \frac{1}{2u}$$

which yields the estimation

$$0 \;<\; h \;<\; 2u \tag{4}$$

(cf. the above table). This is of course true for any harmonic means h of positive numbers u and v. The difference of $2u$ and h is $\frac{2u^2}{u+v}$.

The estimation (4) implies that the number of solutions is less than $2u$. From the proof of the corresponding proposition in the `http://planetmath.org/node/11241`parent entry one can see that the number in fact does not exceed $u-1$.