proof of factor theorem using division

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Lemma (cf. factor theorem)**.** *Let R be a commutative ring with identity and let $p(x) \in R[x]$ be a polynomial with coefficients in R. The element $a \in R$ is a root of $p(x)$ if and only if $(x-a)$ divides $p(x)$.*

Proof. Let $p(x)$ be a polynomial in $R[x]$ and let a be an element of R.

1. First we assume that $(x-a)$ divides $p(x)$. Therefore, there is a polynomial $q(x) \in R[x]$ such that $p(x) = (x-a) \cdot q(x)$. Hence, $p(a) = (a-a) \cdot q(a) = 0$ and a is a root of $p(x)$.

2. Assume that a is a root of $p(x)$, i.e. $p(a) = 0$. Since $x-a$ is a monic polynomial, we can perform the `http://planetmath.org/LongDivision`polynomial long division of $p(x)$ by $(x-a)$. Thus, there exist polynomials $q(x)$ and $r(x)$ such that:
$$p(x) = (x-a) \cdot q(x) + r(x)$$
and the degree of $r(x)$ is less than the degree of $x-a$ (so $r(x)$ is just a constant). Moreover, $0 = p(a) = 0 + r(a) = r(a) = r(x)$. Therefore $p(x) = (x-a) \cdot q(x)$ and $(x-a)$ divides $p(x)$.

□