homomorphisms from fields are either injective or trivial

Canonical name	HomomorphismsFromFieldsAreEitherInjectiveOrTrivial
Date of creation	2013-03-22 14:39:07
Last modified on	2013-03-22 14:39:07
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	4
Author	mathcam (2727)
Entry type	Corollary
Classification	msc 12E99

Suppose F is a field, R is a ring, and $\phi\colon F \to R$ is a homomorphism of rings. Then ϕ is either trivial or injective.

Proof. We use the fact that kernels of ring homomorphism are ideals. Since F is a field, by the above result, we have that the kernel of ϕ is an ideal of the field F and hence either empty or all of F. If the kernel is empty, then since a ring homomorphism is injective iff the kernel is trivial, we get that ϕ is injective. If the kernel is all of F, then ϕ is the zero map from F to R. □

Finally, it is clear that both of these possibilities are in fact achieved:

- The map $\phi : \mathbb{Q} \to \mathbb{Q}$ given by $\phi(n) = 0$ is trivial (has all of $\mathbb{Q}$ as a kernel)
- The inclusion $\mathbb{Q} \to \mathbb{Q}[x]$ is injective (i.e. the kernel is trivial).