explicit definition of polynomial rings in arbitrarily many variables

Canonical name	ExplicitDefinitionOfPolynomialRingsInArbitrarlyManyVariables
Date of creation	2013-03-22 19:18:10
Last modified on	2013-03-22 19:18:10
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	8
Author	joking (16130)
Entry type	Definition
Classification	msc 12E05
Classification	msc 13P05
Classification	msc 11C08

Let R be a ring and let $\mathbb{X}$ be any set (possibly empty). We wish to give an explicit and formal definition of the polynomial ring $R[\mathbb{X}]$.

We start with the set

$$\mathcal{F}(\mathbb{X}) = \{f : \mathbb{X} \to \mathbb{N} \mid f(x) = 0 \text{ for almost all } x\}.$$

If $\mathbb{X} = \{X_1, \ldots, X_n\}$ then the elements of $\mathcal{F}(\mathbb{X})$ can be interpreted as monomials

$$X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

Now define

$$R[\mathbb{X}] = \{F : \mathcal{F}(\mathbb{X}) \to R \mid F(f) = 0 \text{ for almost all } f\}.$$

The addition in $R[\mathbb{X}]$ is defined as pointwise addition.

Now we will define multiplication. First note that we have a multiplication on $\mathcal{F}(\mathbb{X})$. For any $f, g : \mathbb{X} \to \mathbb{N}$ put

$$(fg)(x) = f(x) + g(x).$$

This is the same as multiplying $x^a \cdot x^b = x^{a+b}$.

Now for any $f \in \mathcal{F}(\mathbb{X})$ define

$$M(h) = \{(f, g) \in \mathcal{F}(\mathbb{X})^2 \mid h = fg\},$$

Now if $F, G \in R[\mathbb{X}]$ then we define the multiplication

$$FG : \mathcal{F}(\mathbb{X}) \to R$$

by putting

$$(FG)(h) = \sum_{(f,g))\in M(h)} F(f)G(g).$$

Note that all of this well-defined, since both F and G vanish almost everywhere.

It can be shown that $R[\mathbb{X}]$ with these operations is a ring, even an R-algebra. This algebra is commutative if and only if R is. Furthermore we have an algebra homomorphism

$$E : R \to R[\mathbb{X}]$$

which is defined as follows: for any $r \in R$ let $F_r : \mathcal{F}(\mathbb{X}) \to R$ be the function such that if $f : \mathbb{X} \to \mathbb{N}$ is such that $f(x) = 0$ for any $x \in \mathbb{X}$, then put $F_r(f) = r$ and for any other function $f \in \mathcal{F}(\mathbb{X})$ put $F_r(f) = 0$. Then

$$E(r) = F_r$$

is our function, which is a monomorphism. Furthermore if R is unital with the identity 1, then

$$E(1)$$

is the identity in $R[\mathbb{X}]$. Anyway we can always interpret R as a subset of $R[\mathbb{X}]$ if put $r = F_r$ for $r \in R$.

Note, that if $\mathbb{X} = \emptyset$, then $R[\emptyset]$ is still defined and $E : R \to R[\mathbb{X}]$ is an isomorphism of rings (it is ,,onto”). Actually these two conditions are equivalent.

Also note, that $\mathbb{X}$ itself can be interpreted as a subset of $R[\mathbb{X}]$. Indeed, for any $x \in \mathbb{X}$ define

$$f_x : \mathbb{X} \to \mathbb{N}$$

by $f_x(x) = 1$ and $f_x(y) = 0$ for any $y \neq x$. Then define

$$F_x : \mathcal{F}(\mathbb{X}) \to R$$

by putting $F_x(f_x) = 1$ and $F_x(f) = 0$ for any $f \neq f_x$. It can be easily seen that $F_x = F_y$ if and only if $x = y$. Thus we will use convention $x = F_x$.

With these notations (i.e. $R, \mathbb{X} \subseteq R[\mathbb{X}]$) we have that elements of $R[\mathbb{X}]$ are exactly polynomials in the set of variables $\mathbb{X}$ with coefficients in R.