irreducibility of binomials with unity coefficients

Canonical name	IrreducibilityOfBinomialsWithUnityCoefficients
Date of creation	2013-03-22 15:13:08
Last modified on	2013-03-22 15:13:08
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	14
Author	pahio (2872)
Entry type	Result
Classification	msc 12D05
Classification	msc 13F15
Related topic	FactoringASumOrDifferenceOfTwoCubes
Related topic	PrimeFaxtorsOfXn1
Related topic	PrimeFactorsOfXn1
Related topic	ExpressibleInClosedForm

Let n be a positive integer. We consider the possible factorization of the binomial x^n+1.

- If n has no odd prime factors, then the binomial x^n+1 is http://planetmath.org/IrreduciblPolynomialirreducible. Thus, $x+1$, x^2+1, x^4+1, x^8+1 and so on are irreducible polynomials (i.e. in the field $\mathbb{Q}$ of their coefficients). N.B., only $x+1$ and x^2+1 are in the field $\mathbb{R}$; e.g. one has $x^4+1 = (x^2-x\sqrt{2}+1)(x^2+x\sqrt{2}+1)$.

- If n is an odd number, then x^n+1 is always divisible by $x+1$:

$$x^n + 1 = (x+1)(x^{n-1} - x^{n-2} + x^{n-3} - + \cdots - x + 1) \qquad (1)$$

  This is usable when n is an odd prime number, e.g.

$$x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1).$$

- When n is not a prime number but has an odd prime factor p, say $n = mp$, then we write $x^n+1 = (x^m)^p+1$ and apply the idea of (1); for example:

$$x^{12} + 1 = (x^4)^3 + 1 = (x^4+1)[(x^4)^2 - x^4 + 1] = (x^4+1)(x^8 - x^4 + 1)$$

There are similar results for the binomial x^n+y^n, and the corresponding to (1) is

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - + \cdots - xy^{n-2} + y^n), \qquad (2)$$

which may be verified by performing the multiplication on the right hand .