examples for Hensel's lemma

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Example 1. Let p be a prime number greater than 2. Are there solutions to $x^2+7=0$ in the field $\mathbb{Q}_p$ (the `http://planetmath.org/PAdicIntegers`p-adic numbers)? If there are, -7 must be a quadratic residue modulo p. Thus, let p be a prime such that

$$\left(\frac{-7}{p}\right)=1$$

where $(\frac{\cdot}{p})$ is the Legendre symbol. Hence, there exist $\alpha_0 \in \mathbb{Z}$ such that $\alpha_0^2 \equiv -7 \mod p$. We claim that $x^2+7=0$ has a solution in $\mathbb{Q}_p$ if and only if -7 is a quadratic residue modulo p. Indeed, if we let $f(x)=x^2+7$ (so $f'(x)=2x$), the element $\alpha_0 \in \mathbb{Z}_p$ satisfies the conditions of the (trivial case of) Hensel's lemma. Therefore there exist a root $\alpha \in \mathbb{Q}_p$ of $x^2+7=0$.

Example 2. Let $p=2$. Are there any solutions to $x^2+7=0$ in $\mathbb{Q}_2$? Notice that if we let $f(x)=x^2+7$, then $f'(x)=2x$ and for any $\alpha_0 \in \mathbb{Z}_2$, the number $f'(\alpha_0)=2\alpha_0$ is congruent to 0 modulo 2. Thus, we cannot use the trivial case of Hensel's lemma.

Let $\alpha_0=1 \in \mathbb{Z}_2$. Notice that $f(1)=8$ and $f'(1)=2$. Thus

$$|8|_2 < |2^2|_2$$

and the general case of Hensel's lemma applies. Hence, there exist a 2-adic solution to $x^2+7=0$. The following is the `http://planetmath.org/PAdicCanonicalForm`2-adic canonical form for one of the solutions:

$$\alpha = 1+1\cdot 2^3+1\cdot 2^4+\ldots = \ldots 11001$$