examples of minimal polynomials

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Note that $\sqrt[4]{2}$ is algebraic over the fields $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$. The minimal polynomials for $\sqrt[4]{2}$ over these fields are $x^4 - 2$ and $x^2 - \sqrt{2}$, respectively. Note that $x^4 - 2$ is irreducible over $\mathbb{Q}$ by using Eisenstein's criterion and `http://planetmath.org/GausssLemmaII`Gauss's lemma (see `http://planetmath.org/Alternat`
entry for more details), and $x^2 - \sqrt{2}$ is irreducible over $\mathbb{Q}(\sqrt{2})$ since it is a quadratic polynomial and neither of its roots ($\sqrt[4]{2}$ and $-\sqrt[4]{2}$) are in $\mathbb{Q}(\sqrt{2})$.

A common method for constructing minimal polynomials for numbers that are expressible over $\mathbb{Q}$ is "backwards ": The number can be set equal to x, and the equation can be algebraically manipulated until a monic polynomial in $\mathbb{Q}[x]$ is equal to 0. Finally, if the monic polynomial is not irreducible, then it can be factored into irreducible polynomials $\mathbb{Q}[x]$, and the original number will be a root of one of these. A very example is $\sqrt[4]{2}$:

$$\begin{aligned} x &= \sqrt[4]{2} \\ x^4 &= 2 \\ x^4 - 2 &= 0 \end{aligned}$$

This method will be further demonstrated with three more examples: One for $\dfrac{1+\sqrt{5}}{2}$, one for $1 + \omega_5$ where ω_5 is a fifth root of unity, and one for $\sqrt[3]{2} + \sqrt[3]{3}$.

$$\begin{aligned} x &= \frac{1+\sqrt{5}}{2} \\ 2x &= 1 + \sqrt{5} \\ 2x - 1 &= \sqrt{5} \\ (2x-1)^2 &= 5 \\ 4x^2 - 4x + 1 &= 5 \\ 4x^2 - 4x - 4 &= 0 \\ x^2 - x - 1 &= 0 \end{aligned}$$

$$\begin{aligned} x &= 1 + \omega_5 \\ x - 1 &= \omega_5 \\ (x-1)^5 &= 1 \\ x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 &= 1 \\ x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 2 &= 0 \end{aligned}$$

$$
\begin{aligned}
x &= \sqrt[3]{2}+\sqrt[3]{3} \\
x^3 &= 2+3\sqrt[3]{2^2\cdot 3}+3\sqrt[3]{2\cdot 3^2}+3 \\
x^3-5 &= 3\sqrt[3]{6}(\sqrt[3]{2}+\sqrt[3]{3}) \\
x^3-5 &= 3\sqrt[3]{6}\,x \\
(x^3-5)^3 &= 27\cdot 6x^3 \\
x^9-3\cdot 5x^6+3\cdot 25x^3-125 &= 162x^3 \\
x^9-15x^6-87x^3-125 &= 0
\end{aligned}
$$

Since x^2-x-1 is a quadratic and has no roots in $\mathbb{Q}$, it is irreducible over $\mathbb{Q}$. Thus, it is the minimal polynomial over $\mathbb{Q}$ for $\dfrac{1+\sqrt{5}}{2}$.

On the other hand, $x^5-5x^4+10x^3-10x^2+5x-2$ factors over $\mathbb{Q}$ as $(x-2)(x^4-3x^3+4x^2-2x+1)$. Since $1+\omega_5$ is not a root of $x-2$, it must be a root of $x^4-3x^3+4x^2-2x+1$. Moreover, this polynomial must be irreducible. This fact can be proven in the following manner: Let $m(x)$ be the minimal polynomial for $1+\omega_5$ over $\mathbb{Q}$. Since $\mathbb{Q}(1+\omega_5)=\mathbb{Q}(\omega_5)$, $\deg m(x) = [\mathbb{Q}(1+\omega_5):\mathbb{Q}] = [\mathbb{Q}(\omega_5):\mathbb{Q}] = \varphi(5) = 4 = \deg(x^4-3x^3+4x^2-2x+1)$. (Here φ denotes the Euler totient function.) Since $m(x)$ divides $x^4-3x^3+4x^2-2x+1$ and they have the same degree, it follows that $m(x)=x^4-3x^3+4x^2-2x+1$.

It turns out that $x^9-15x^6-87x^3-125$ is irreducible over $\mathbb{Q}$. (This can be proven in a manner as above. Note that $[\mathbb{Q}(\sqrt[3]{2}+\sqrt[3]{3}):\mathbb{Q}]=9$.) Thus, it is the minimal polynomial over $\mathbb{Q}$ for $\sqrt[3]{2}+\sqrt[3]{3}$.