second proof of Wedderburn's theorem

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We can prove Wedderburn's theorem,without using Zsigmondy's theorem on the conjugacy class formula of the first proof; let G_n set of n-th roots of unity and P_n set of n-th primitive roots of unity and $\Phi_d(q)$ the d-th cyclotomic polynomial.
It results

- $\Phi_n(q) = \prod_{\xi \in P_n}(q-\xi)$
- $p(q) = q^n - 1 = \prod_{\xi \in G_n}(q-\xi) = \prod_{d|n} \Phi_d(q)$
- $\Phi_n(q) \in \mathbb{Z}[q]$, it has multiplicative identity and $\Phi_n(q) \mid q^n - 1$
- $\Phi_n(q) \mid \frac{q^n-1}{q^d-1}$ with $d \mid n, d < n$

by conjugacy class formula, we have:

$$q^n - 1 = q - 1 + \sum_x \frac{q^n-1}{q^{n_x}-1}$$

by last two previous properties, it results:

$$\Phi_n(q) \mid q^n - 1 \ , \ \Phi_n(q) \mid \frac{q^n-1}{q^{n_x}-1} \Rightarrow \Phi_n(q) \mid q - 1$$

because $\Phi_n(q)$ divides the left and each addend of $\sum_x \frac{q^n-1}{q^{n_x}-1}$ of the right member of the conjugacy class formula.
By third property

$$q > 1 \ , \ \Phi_n(x) \in \mathbb{Z}[x] \Rightarrow \Phi_n(q) \in \mathbb{Z} \Rightarrow |\Phi_n(q)| \mid q - 1 \Rightarrow |\Phi_n(q)| \leqslant q - 1$$

If, for $n > 1$,we have $|\Phi_n(q)| > q - 1$, then $n = 1$ and the theorem is proved.
We know that

$$|\Phi_n(q)| = \prod_{\xi \in P_n} |q - \xi| \ , \ with \ q - \xi \in \mathbb{C}$$

by the triangle inequality in $\mathbb{C}$

$$|q - \xi| \geqslant ||q| - |\xi|| = |q - 1|$$

as ξ is a primitive root of unity, besides

$$|q-\xi| = |q-1| \Leftrightarrow \xi = 1$$

but

$$n > 1 \Rightarrow \xi \neq 1$$

therefore, we have

$$|q-\xi| > |q-1| = q-1 \Rightarrow |\Phi_n(q)| > q-1$$