lattice of fields

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Let K be a field and $\overline{K}$ be its algebraic closure. The set $\mathrm{Latt}(K)$ of all intermediate fields E (where $K \subseteq E \subseteq \overline{K}$), ordered by set theoretic inclusion, is a poset. Furthermore, it is a complete lattice, where K is the bottom and $\overline{K}$ is the top.

This is the direct result of the fact that any topped intersection structure is a complete lattice, and $\mathrm{Latt}(K)$ is such a structure. However, it can be easily proved directly: for any collection of intermediate fields $\{E_i \mid i \in I\}$, the intersection is clearly an intermediate field, and is the infimum of the collection. The compositum of these fields, which is the smallest intermediate field E such that $E_i \subseteq E$, is the supremum of the collection.

It is not hard to see that $\mathrm{Latt}(K)$ is an algebraic lattice, since the union of any directed family of intermediate fields between K and $\overline{K}$ is an intermediate field. The compact elements in $\mathrm{Latt}(K)$ are the finite algebraic extensions of K. The set of all compact elements in $\mathrm{Latt}(K)$, denoted by $\mathrm{Latt}_F(K)$, is a lattice ideal, for any subfield of a finite algebraic extension of K is finite algebraic over K. However, $\mathrm{Latt}_F(K)$, as a sublattice, is usually not complete (take the compositum of all simple extensions $\mathbb{Q}(\sqrt{p})$, where $p \in \mathbb{Z}$ are rational primes).