p-adic canonical form

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Defines	2-adic fractional number
Defines	2-adic integer
Defines	2-adic valuation

Every non-zero p-adic number (p is a positive rational prime number) can be uniquely written in *canonical form*, formally as a Laurent series,

$$\xi = a_{-m}p^{-m} + a_{-m+1}p^{-m+1} + \cdots + a_0 + a_1p + a_2p^2 + \cdots$$

where $m \in \mathbb{N},\ 0 \leqq a_k \leqq p-1$ for all k's, and at least one of the integers a_k is positive. In addition, we can write: $0 = 0 + 0p + 0p^2 + \cdots$

The field $\mathbb{Q}_p$ of the p-adic numbers is the completion of the field $\mathbb{Q}$ with respect to its `http://planetmath.org/PAdicValuation`p-adic valuation; thus $\mathbb{Q}$ may be thought the subfield (prime subfield) of $\mathbb{Q}_p$. We can call the elements of $\mathbb{Q}_p \setminus \mathbb{Q}$ the *proper p-adic numbers*.

If, e.g., $p = 2$, we have the 2-*adic* or, according to G. W. Leibniz, *dyadic numbers*, for which every a_k is 0 or 1. In this case we can write the sum expression for ξ in the reverse and use the ordinary `http://planetmath.org/Base3`positional (i.e., *dyadic*) `http://planetmath.org/Base3`figure system. Then, for example, we have the rational numbers

$$-1 = ...111111,$$

$$1 = ...0001,$$

$$6.5 = ...000110.1,$$

$$\frac{1}{5} = ...00110011001101.$$

(You may check the first by adding 1, and the last by multiplying by $5 = ...000101.$) All 2-adic rational numbers have periodic binary `http://planetmath.org/DecimalExpa` Similarly as the `http://planetmath.org/DecimalExpansion`decimal (according to Leibniz: *decadic*) expansions of irrational real numbers are aperiodic, the proper 2-adic numbers also have aperiodic binary expansion, for example the 2-*adic fractional number*

$$\alpha = ...1000010001001011.10111.$$

The *2-adic fractional numbers* have some bits "1" after the *dyadic point* "." (in continental Europe: comma ","), the *2-adic integers* have none. The 2-adic integers form a subring of the 2-adic field $\mathbb{Q}_2$ such that $\mathbb{Q}_2$ is the quotient field of this ring.

Every such 2-adic integer ε whose last bit is "1", as $-3/7 = ...11011011011$, is a unit of this ring, because the division $1 : \varepsilon$ clearly gives as quotient a integer (by the way, the divisions of the binary expansions in practice go from right to left and are very comfortable!).

Those integers ending in a "0" are non-units of the ring, and they apparently form the only maximal ideal in the ring (which thus is a local ring). This is a principal ideal $\mathfrak{p}$, the generator of which may be taken ...00010 = 10 (i.e., two). Indeed, two is the only prime number of the ring, but it has infinitely many associates, a kind of copies, namely all expansions of the form $...10 = \varepsilon \cdot 10$. The only non-trivial ideals in the ring of 2-adic integers are $\mathfrak{p}$, $\mathfrak{p}^2$, $\mathfrak{p}^3$, $\ldots$ They have only 0 as common element.

All 2-adic non-zero integers are of the form $\varepsilon \cdot 2^n$ where $n = 0, 1, 2, \ldots$. The values $n = -1, -2, -3, \ldots$ here would give non-integral, i.e. fractional 2-adic numbers.

If in the binary of an arbitrary 2-adic number, the last non-zero digit "1" corresponds to the power 2^n, then the *2-adic valuation* of the 2-adic number ξ is given by

$$|\xi|_2 = 2^{-n}.$$