Galois group of a quartic polynomial

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Consider a general (monic) quartic polynomial over $\mathbb{Q}$

$$f(x) = x^4 + ax^3 + bx^2 + cx + d$$

and denote the Galois group of $f(x)$ by G.

The Galois group G is isomorphic to a subgroup of S_4 (see the article on the Galois group of a cubic polynomial for a discussion of this question).

If the quartic splits into a linear factor and an irreducible cubic, then its Galois group is simply the Galois group of the cubic portion and thus is isomorphic to a subgroup of S_3 (embedded in S_4) - again, see the article on the Galois group of a cubic polynomial.

If it factors as two irreducible quadratics, then the splitting field of $f(x)$ is the compositum of $\mathbb{Q}(\sqrt{D_1})$ and $\mathbb{Q}(\sqrt{D_2})$, where D_1 and D_2 are the discriminants of the two quadratics. This is either a biquadratic extension and thus has Galois group isomorphic to V_4, or else $D_1 D_2$ is a square, and $\mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}) = \mathbb{Q}(\sqrt{D_1})$ and the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

This leaves us with the most interesting case, where $f(x)$ is irreducible. In this case, the Galois group acts transitively on the roots of $f(x)$, so it must be isomorphic to a `http://planetmath.org/GroupAction`transitive subgroup of S_4. The transitive subgroups of S_4 are

$$\begin{aligned}
&S_4\\
&A_4\\
&D_8 \cong \{e,\ (1234),\ (13)(24),\ (1432),\ (12)(34),\ (14)(23),\ (13),\ (24)\} \text{ and its conjugates}\\
&V_4 \cong \{e,\ (12)(34),\ (13)(24),\ (14)(23)\}\\
\mathbb{Z}/4\mathbb{Z} &\cong \{e,\ (1234),\ (13)(24),\ (1432)\} \text{ and its conjugates}
\end{aligned}$$

We will see that each of these transitive subgroups actually appears as the Galois group of some class of irreducible quartics.

The resolvent cubic of $f(x)$ is

$$C(x) = x^3 - 2bx^2 + (b^2 + ac - 4d)x + (c^2 + a^2 d - abc)$$

and has roots

$$\begin{aligned}
r_1 &= (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)\\
r_2 &= (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)\\
r_3 &= (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3)
\end{aligned}$$

But then a short computation shows that the discriminant D of $C(x)$ is the same as the discriminant of $f(x)$. Also, since $r_1, r_2, r_3 \in \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, it follows that the splitting field of $C(x)$ is a subfield of the splitting field of $f(x)$ and thus that the Galois group of $C(x)$ is a quotient of the Galois group of $f(x)$. There are four cases:

- If $C(x)$ is irreducible, and D is not a rational square, then G does not fix D and thus is not contained in A_4. But in this case, where D is not a square, the Galois group of $C(x)$ is S_3, which has order 6. The only subgroup of S_4 not contained in A_4 with order a multiple of 6 (and thus capable of having a subgroup of index 6) is S_4 itself, so in this case $G \cong S_4$.

- If $C(x)$ is irreducible but D is a rational square, then G fixes D, so $G \leq A_4$. In addition, the Galois group of $C(x)$ is A_3, so 3 divides the order of a transitive subgroup of A_4, which means that $G \cong A_4$ itself.

- If $C(x)$ is reducible, suppose first that it splits completely in $\mathbb{Q}$. Then each of $r_1, r_2, r_3 \in \mathbb{Q}$ and thus each element of G fixes each r_i. Thus $G \cong V_4$.

- Finally, if $C(x)$ splits into a linear factor and an irreducible quadratic, then one of the r_i, say r_2, is in $\mathbb{Q}$. Then G fixes $r_2 = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)$ but not r_1 or r_3. The only possibilities from among the transitive groups are then that $G \cong D_8$ or $G \cong \mathbb{Z}/4\mathbb{Z}$. In this case, the discriminant of the quadratic is not a rational square, but it is a rational square times D.

  Now, $G \cap A_4$ fixes $\mathbb{Q}(\sqrt{D})$, since G fixes $\sqrt{D}$ up to sign and A_4 restricts our attention to even permutations. But $|G : G \cap A_4| = 2$, so the fixed field of $G \cap A_4$ has dimension 2 over $\mathbb{Q}$ and thus is exactly $\mathbb{Q}(\sqrt{D})$. If $G \cong D_8$, then $G \cap A_4 \cong V_4$, while if $G \cong \mathbb{Z}/4\mathbb{Z}$, then $G \cap A_4 \cong \mathbb{Z}/2\mathbb{Z}$; in the first case only, $G \cap A_4$ acts transitively on the roots of $f(x)$. Thus $G \cap A_4 \cong V_4$ if and only if $f(x)$ is irreducible over $\mathbb{Q}(\sqrt{D})$.

So, in summary, for $f(x)$ irreducible, we have the following:

Condition	Galois group
$C(x)$ irreducible, D not a rational square	S_4
$C(x)$ irreducible, D a rational square	A_4
$C(x)$ splits completely	V_4
$C(x)$ factors as linear times irreducible quadratic, $f(x)$ irreducible over $\mathbb{Q}(\sqrt{D})$	D_8
$C(x)$ factors as linear times irreducible quadratic, $f(x)$ reducible over $\mathbb{Q}(\sqrt{D})$	$\mathbb{Z}/4\mathbb{Z}$

References

[1] D.S. Dummit, R.M. Foote, *Abstract Algebra*, Wiley and Sons, 2004.