rational integers in ideals

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Any non-zero ideal of an algebraic number field K, i.e. of the maximal order $\mathcal{O}_K$ of K, contains positive rational integers.

Proof. Let $\mathfrak{a} \neq (0)$ be any ideal of $\mathcal{O}_K$. Take a nonzero element α of $\mathfrak{a}$. The `http://planetmath.org/NormInNumberField`norm of α is the product

$$\mathrm{N}(\alpha) \;=\; \alpha^{(1)}\,\underbrace{\alpha^{(2)}\cdots\alpha^{(n)}}_{\gamma}$$

where n is the degree of the number field and $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(n)}$ is the set of the `http://planetmath.org/node/12046`K-conjugates of $\alpha = \alpha^{(1)}$. The number

$$\gamma = \frac{\mathrm{N}(\alpha)}{\alpha}$$

belongs to the field K and it is an algebraic integer, since $\alpha^{(2)}, \cdots, \alpha^{(n)}$ are, as algebraic conjugates of α, also algebraic integers. Thus $\gamma \in \mathcal{O}_K$. Consequently, the non-zero integer

$$\mathrm{N}(\alpha) \;=\; \alpha\gamma$$

belongs to the ideal $\mathfrak{a}$, and similarly its opposite number. So, $\mathfrak{a}$ contains positive integers, in fact infinitely many.