proof of rational root theorem

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Owner	Wkbj79 (1863)
Last modified by	Wkbj79 (1863)
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Author	Wkbj79 (1863)
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Let $p(x) \in \mathbb{Z}[x]$. Let n be a positive integer with $\deg p(x) = n$. Let $c_0, \ldots, c_n \in \mathbb{Z}$ such that $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$.

Let $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and $b > 0$ such that $\dfrac{a}{b}$ is a root of $p(x)$. Then

$$\begin{aligned}
0 &= p\left(\frac{a}{b}\right) \\
&= c_n \left(\frac{a}{b}\right)^n + c_{n-1}\left(\frac{a}{b}\right)^{n-1} + \cdots + c_1 \cdot \frac{a}{b} + c_0 \\
&= c_n \cdot \frac{a^n}{b^n} + c_{n-1} \cdot \frac{a^{n-1}}{b^{n-1}} + \cdots + c_1 \cdot \frac{a}{b} + c_0.
\end{aligned}$$

Multiplying through by b^n and rearranging yields:

$$\begin{aligned}
c_n a^n + c_{n-1}a^{n-1}b + \cdots + c_1 a b^{n-1} + c_0 b^n &= 0 \\
c_0 b^n &= -c_n a^n - c_{n-1}a^{n-1}b - \cdots - c_1 a b^{n-1} \\
c_0 b^n &= a\left(-c_n a^{n-1} - c_{n-1}a^{n-2}b - \cdots - c_1 b^{n-1}\right)
\end{aligned}$$

Thus, $a | c_0 b^n$ and, by hypothesis, $\gcd(a, b) = 1$. This implies that $a | c_0$.

Similarly:

$$\begin{aligned}
c_n a^n + c_{n-1}a^{n-1}b + \cdots + c_1 a b^{n-1} + c_0 b^n &= 0 \\
c_n a^n &= -c_{n-1}a^{n-1}b - \cdots - c_1 a b^{n-1} - c_0 b^n \\
c_n a^n &= b\left(-c_{n-1}a^{n-1} - \cdots - c_1 a b^{n-1} - c_0 b^{n-1}\right)
\end{aligned}$$

Therefore, $b | c_n a^n$ and $b | c_n$.