real and complex embeddings

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Let L be a subfield of $\mathbb{C}$.

Definition 1.

1. *A* real embedding *of L is an injective field homomorphism*
$$\sigma\colon L \hookrightarrow \mathbb{R}$$

2. *A (non-real)* complex embedding *of L is an injective field homomorphism*
$$\tau\colon L \hookrightarrow \mathbb{C}$$
such that $\tau(L) \nsubseteq \mathbb{R}$.

3. *We denote Σ_L the set of all embeddings, real and complex, of L in $\mathbb{C}$ (note that all of them must fix $\mathbb{Q}$, since they are field homomorphisms).*

Note that if σ is a real embedding then $\bar{\sigma} = \sigma$, where $\bar{\cdot}$ denotes the complex conjugation automorphism:

$$\bar{\cdot}\colon \mathbb{C} \to \mathbb{C}, \quad \overline{(a+bi)} = a - bi$$

On the other hand, if τ is a complex embedding, then $\bar{\tau}$ is another complex embedding, so the complex embeddings always come in pairs $\{\tau, \bar{\tau}\}$.

Let $K \subseteq L$ be another subfield of $\mathbb{C}$. Moreover, assume that $[L : K]$ is finite (this is the dimension of L as a vector space over K). We are interested in the embeddings of L that fix K pointwise, i.e. embeddings $\psi\colon L \hookrightarrow \mathbb{C}$ such that
$$\psi(k) = k, \quad \forall k \in K$$

Theorem 1. *For any embedding ψ of K in $\mathbb{C}$, there are exactly $[L : K]$ embeddings of L such that they extend ψ. In other words, if φ is one of them, then*
$$\varphi(k) = \psi(k), \quad \forall k \in K$$
Thus, by taking $\psi = \mathrm{Id}_K$, there are exactly $[L : K]$ embeddings of L which fix K pointwise.

Hence, by the theorem, we know that the order of Σ_L is $[L : \mathbb{Q}]$. The number $[L : \mathbb{Q}]$ is usually decomposed as

$$[L : \mathbb{Q}] = r_1 + 2r_2$$

where r_1 is the number of embeddings which are real, and $2r_2$ is the number of embeddings which are complex (non-real). Notice that by the remark above this number is always even, so r_2 is an integer.

Remark: Let ψ be an embedding of L in $\mathbb{C}$. Since ψ is injective, we have $\psi(L) \cong L$, so we can regard ψ as an automorphism of L. When $L/\mathbb{Q}$ is a Galois extension, we can prove that $\Sigma_L \cong \mathrm{Gal}(L/\mathbb{Q})$, and hence proving in a different way the fact that

$$| \Sigma_L | = [L : \mathbb{Q}] = | \mathrm{Gal}(L/\mathbb{Q}) |$$