field homomorphisms fix prime subfields

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Theorem. *Let F and K be fields having the same prime subfield L and $\varphi\colon F \to K$ be a field homomorphism. Then φ fixes L.*

Proof. Without loss of generality, it will be assumed that L is either $\mathbb{Q}$ or $\mathbb{Z}/c\mathbb{Z}$.

Since φ is a field homomorphism, $\varphi(0) = 0$, $\varphi(1) = 1$, and, for every $x \in F$, $\varphi(-x) = -\varphi(x)$.

Let $n \in \mathbb{Z}$ and c be the characteristic of F. Then

$$\begin{aligned}
\varphi(n) \quad &\equiv \varphi(\operatorname{sign}(n)|n|) \bmod c, \text{ where sign denotes the signum function}\\
&\equiv \operatorname{sign}(n)\varphi(|n|) \bmod c\\
&\equiv \operatorname{sign}(n)\varphi\left(\sum_{j=1}^{|n|} 1\right) \bmod c\\
&\equiv \operatorname{sign}(n)\sum_{j=1}^{|n|} \varphi(1) \bmod c\\
&\equiv \operatorname{sign}(n)\sum_{j=1}^{|n|} 1 \bmod c\\
&\equiv \operatorname{sign}(n)|n| \bmod c\\
&\equiv n \bmod c.
\end{aligned}$$

This the proof in the case that c is prime.

Now consider $c = 0$. Let $x \in \mathbb{Q}$. Then there exist $a, b \in \mathbb{Z}$ with $b > 0$ such that $x = \dfrac{a}{b}$. Thus, $b\varphi(x) = \displaystyle\sum_{j=1}^{b} \varphi\left(\frac{a}{b}\right) = \varphi\left(\sum_{j=1}^{b} \frac{a}{b}\right) = \varphi(a) = a$. Therefore, $\varphi(x) = \dfrac{a}{b} = x$. Hence, φ fixes $\mathbb{Q}$. $\square$