# simple field extension

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| Canonical name | SimpleFieldExtension |
| Date of creation | 2013-03-22 14:23:06 |
| Last modified on | 2013-03-22 14:23:06 |
| Owner | pahio (2872) |
| Last modified by | pahio (2872) |
| Numerical id | 25 |
| Author | pahio (2872) |
| Entry type | Definition |
| Classification | msc 12F99 |
| Related topic | PrimitiveElementTheorem |
| Related topic | CanonicalFormOfElementOfNumberField |
| Defines | primitive element |

Let $K(\alpha)$ be obtained from the field $K$ via the  of the element $\alpha$, which is called the *primitive element* of the field extension $K(\alpha)/K$. We shall settle the  of the field $K(\alpha)$.

We consider the substitution homomorphism $\varphi : K[X] \to K[\alpha]$, where

$$\sum a_\nu X^\nu \mapsto \sum a_\nu \alpha^\nu.$$

According to the ring homomorphism theorem, the image ring $K[\alpha]$ is isomorphic with the residue class ring $K[X]/\mathfrak{p}$, where $\mathfrak{p}$ is the ideal of polynomials having $\alpha$ as their zero. Because $K[\alpha]$ is, as subring of the field $K(\alpha)$, an integral domain, then also $K[X]/\mathfrak{p}$ has no zero divisors, and hence $\mathfrak{p}$ is a prime ideal. It must be principal, for $K[X]$ is a principal ideal ring.

There are two possibilities:

1. $\mathfrak{p} = (p(X))$, where $p(X)$ is an irreducible polynomial with $p(\alpha) = 0$. Because every non-zero prime ideal of $K[X]$ is maximal, the isomorphic image $K[X]/(p(X))$ of $K[\alpha]$ is a field, and it must give the  of $K(\alpha) = K[\alpha]$. We say that $\alpha$ is *algebraic with respect to* $K$ (or *over* $K$). In this case, we have a finite field extension $K(\alpha)/K$.

2. $\mathfrak{p} = (0)$. This means that the homomorphism $\varphi$ is an isomorphism between $K[X]$ and $K[\alpha]$, i.e. all expressions $\sum a_\nu \alpha^\nu$ behave as the polynomials $\sum a_\nu X^\nu$. Now, $K[\alpha]$ is no field because $K[X]$ is not such, but the isomorphy of the rings implies the isomorphy of the corresponding fields of fractions. Thus the simple extension field $K(\alpha)$ is isomorphic with the field $K(X)$ of rational functions in one indeterminate $X$. We say that $\alpha$ is *http://planetmath.org/Algebraictranscendental with respect to* $K$ (or *over* $K$). This time we have a simple infinite field extension $K(\alpha)/K$.