Cardano's formulae

Canonical name	CardanosFormulae
Date of creation	2014-11-27 15:52:49
Last modified on	2014-11-27 15:52:49
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	26
Author	pahio (2872)
Entry type	Topic
Classification	msc 12D10
Synonym	solution of cubic equation
Synonym	Cardanic formulae
Related topic	CubicFormula
Related topic	ATrigonometricCubicFormula
Related topic	Complex
Related topic	GaloisGroupOfTheCubic
Related topic	CasusIrreducibilis
Related topic	QuadraticResolvent
Related topic	SimpleAnalyticDiscussionOfTheCubicEquation
Related topic	GoniometricSolutionOfCubicEquation

The http://planetmath.org/Equationroots of the (for the reducing via a Tschirnhaus transformation, see the http://planetmath.org/CardanosDerivationOfTheCubic entry) cubic equation

$$y^3+py+q=0, \tag{1}$$

with p and q any complex numbers, are

$$y_1=u+v, \qquad y_2=u\zeta+v\zeta^2, \qquad y_3=u\zeta^2+v\zeta, \tag{2}$$

where ζ is a http://planetmath.org/RootOfUnityprimitive third root of unity (e.g. $\frac{-1+i\sqrt{3}}{2}$) and

$$u\,:=\,\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}}, \qquad v\,:=\,\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}}. \tag{3}$$

The values of the cube roots must be chosen such that

$$uv=-\frac{p}{3}. \tag{4}$$

Cardano's formulae, essentially (2) and (3), were first published in 1545 in Geronimo Cardano's book *"Ars magna"*. The idea of (2) and (3) is illustrated in the entry example of solving a cubic equation.

Let's now assume that the coefficients p and q are real. The number of the real http://planetmath.org/Equationroots of (1) depends on the sign of the radicand $R:=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2$ of the above square root. Instead of R we may use the discriminant $D:=-108R$ of the equation. As in examining the number of real roots of a http://planetmath.org/QuadraticFormulaquadratic equation, we get three different cases also for the cubic (1):

1. $D=0$. This is possible only when either $p<0$ or $p=q=0$. Then we get the real roots $y_1=-2\sqrt[3]{q/2}$, $y_2=y_3=\sqrt[3]{q/2}$.

2. $D<0$. The square root $\sqrt{R}$ is real, and one can choose for u and v the real values of the cube roots (3); these satisfy (4). Thus the root $y_1=u+v$ is real, and since

$$y_{2,\,3}=-\frac{u+v}{2}\pm i\sqrt{3}\cdot\frac{u-v}{2},$$

with $u\neq v$, the roots y_2 and y_3 are non-real complex conjugates of each other.

3. $D > 0$. This requires that p is negative. The radicands of the cube roots (3) are non-real complex conjugates. Using the argument φ of $u^3 = -\frac{q}{2}+i\sqrt{-R}$ as auxiliary angle one is able to http://planetmath.org/CalculatingTheN the cube roots, obtaining the trigonometric

$$y_1 = 2\sqrt{-\frac{p}{3}}\cos\frac{\varphi}{3}, \qquad y_2 = 2\sqrt{-\frac{p}{3}}\cos\frac{\varphi+2\pi}{3}, \qquad y_3 = 2\sqrt{-\frac{p}{3}}\cos\frac{\varphi+4\pi}{3}.$$

This shows that the roots of (1) are three distinct real numbers. O. L. Hölder has proved in the end of the 19^{th} century that in this case one can not with algebraic means eliminate the imaginarity from the Cardano's formulae (2), but "the real roots must be calculated via the non-real numbers". This fact has been known already much earlier and called the *casus irreducibilis*. It actually coerced the mathematicians to begin to use non-real numbers, i.e. to introduce the complex numbers.

References

[1] K. Väisälä: *Lukuteorian ja korkeamman algebran alkeet.* Tiedekirjasto No. 17. Kustannusosakeyhtiö Otava, Helsinki (1950).