equivalent conditions for normality of a field extension

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Theorem. *If K/F is an algebraic extension of fields, then the following are equivalent:*

1. *K is normal over F;*
2. *K is the splitting field over F of a set of polynomials in $F[X]$;*
3. *if $\overline{F}$ is an algebraic closure F containing K and $\sigma : K \to \overline{F}$ is an F-monomorphism, then $\sigma(K) = K$.*

Proof. (1)$\Rightarrow$(2) Let X be an F-basis for K, and for each $x \in X$, let f_x be the irreducible polynomial of x over F. By hypothesis, each f_x splits over K, and because we evidently have $K = F(X)$, it follows that K is a splitting field of $\{f_x : x \in X\}$ over F.
(2)$\Rightarrow$(3) Assume that K is a splitting field over F of $S \subseteq F[X]$. Given $f \in S$, we may write $f(X) = u \prod_{i=1}^{n}(X - u_i)$ for some $u, u_1, \ldots, u_n \in K$; because σ fixes F pointwise, we have $\sigma(u_i) \in \{u_1, \ldots, u_n\}$ for $1 \leq i \leq n$, and since σ is injective, it must simply permute the roots of f. Thus $u_1, \ldots, u_n \in \sigma(K)$. As K is generated over F by the roots of the polynomials in S, we obtain $K = \sigma(K)$.
(3)$\Rightarrow$(1) Let $\overline{K}$ be an algebraic closure of K, noting that, since K is algebraic over F, that same is true of $\overline{K}$, and consequently $\overline{K}$ is an algebraic closure of F containing K. Now suppose $f \in F[X]$ is irreducible and that $u \in K$ is a root of f, and let v be any root of F in $\overline{K}$. There exists an F-isomorphism $\tau : F(u) \to F(v)$ such that $\tau(u) = v$. Because $\overline{K}$ is a splitting field over both $F(u)$ and $F(v)$ of the set of irreducible polynomials in $F[X]$, τ extends to an F-isomorphism $\sigma : \overline{K} \to \overline{K}$. It follows that $\sigma|_K : K \to \overline{K}$ is an F-monomorphism, so that, by hypothesis, $\sigma(K) = K$, hence that $v = \sigma(u) \in K$. Thus f splits over K, and therefore K/F is normal. $\square$