formally real field

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A field F is called *formally real* if -1 can not be expressed as a sum of squares (of elements of F).

Given a field F, let S_F be the set of all sums of squares in F. The following are equivalent conditions that F is formally real:

1. $-1 \notin S_F$
2. $S_F \neq F$ and $\mathrm{char}(F) \neq 2$
3. $\sum {a_i}^2 = 0$ implies each $a_i = 0$, where $a_i \in F$
4. F can be ordered (There is a total order $<$ which makes F into an ordered field)

Some Examples:

- $\mathbb{R}$ and $\mathbb{Q}$ are both formally real fields.
- If F is formally real, so is $F(\alpha)$, where α is a root of an irreducible polynomial of odd degree in $F[x]$. As an example, $\mathbb{Q}(\sqrt[3]{2}\omega)$ is formally real, where $\omega \neq 1$ is a third root of unity.
- $\mathbb{C}$ is not formally real since $-1 = i^2$.
- Any field of characteristic non-zero is not formally real; it is not orderable.

A formally real field is said to be *real closed* if any of its algebraic extension which is also formally real is itself. For any formally real field k, a formally real field K is said to be a *real closure* of k if K is an algebraic extension of k and is real closed.

In the example above, $\mathbb{R}$ is real closed, and $\mathbb{Q}$ is not, whose real closure is $\tilde{\mathbb{Q}}$. Furthermore, it can be shown that the real closure of a countable formally real field is countable, so that $\tilde{\mathbb{Q}} \neq \mathbb{R}$.