# constructible numbers

| | |
|---|---|
| Canonical name | ConstructibleNumbers |
| Date of creation | 2013-03-22 17:15:01 |
| Last modified on | 2013-03-22 17:15:01 |
| Owner | CWoo (3771) |
| Last modified by | CWoo (3771) |
| Numerical id | 17 |
| Author | CWoo (3771) |
| Entry type | Definition |
| Classification | msc 12D15 |
| Related topic | EuclideanField |
| Related topic | CompassAndStraightedgeConstruction |
| Related topic | TheoremOnConstructibleAngles |
| Related topic | TheoremOnConstructibleNumbers |
| Defines | ruler and compass operation |
| Defines | compass and ruler operation |
| Defines | compass and straightedge operation |
| Defines | straightedge and compass operation |
| Defines | constructible number |
| Defines | constructible from |
| Defines | constructible |
| Defines | field of constructible numbers |
| Defines | field of real constructible numbers |

The smallest subfield $\mathbb{E}$ of $\mathbb{R}$ over $\mathbb{Q}$ such that $\mathbb{E}$ is Euclidean is called the *field of real constructible numbers.* First, note that $\mathbb{E}$ has the following properties:

1. $0, 1 \in \mathbb{E}$;

2. If $a, b \in \mathbb{E}$, then also $a \pm b$, $ab$, and $a/b \in \mathbb{E}$, the last of which is meaningful only when $b \neq 0$;

3. If $r \in \mathbb{E}$ and $r > 0$, then $\sqrt{r} \in \mathbb{E}$.

The field $\mathbb{E}$ can be extended in a natural manner to a subfield of $\mathbb{C}$ that is not a subfield of $\mathbb{R}$. Let $\mathbb{F}$ be a subset of $\mathbb{C}$ that has the following properties:

1. $0, 1 \in \mathbb{F}$;

2. If $a, b \in \mathbb{F}$, then also $a \pm b$, $ab$, and $a/b \in \mathbb{F}$, the last of which is meaningful only when $b \neq 0$;

3. If $z \in \mathbb{F} \setminus \{0\}$ and $\arg(z) = \theta$ where $0 \leq \theta < 2\pi$, then $\sqrt{|z|}e^{\frac{i\theta}{2}} \in \mathbb{F}$.

Then $\mathbb{F}$ is the *field of constructible numbers.*

Note that $\mathbb{E} \subset \mathbb{F}$. Moreover, $\mathbb{F} \cap \mathbb{R} = \mathbb{E}$.

An element of $\mathbb{F}$ is called a *constructible number.* These numbers can be "constructed" by a process that will be described shortly.

Conversely, let us start with a subset $S$ of $\mathbb{C}$ such that $S$ contains a non-zero complex number. Call any of the binary operations in condition 2 as well as the square root unary operation in condition 3 a *ruler and compass operation.* Call a complex number *constructible from* $S$ if it can be obtained from elements of $S$ by a finite sequence of ruler and compass operations. Note that $1 \in S$. If $S'$ is the set of numbers constructible from $S$ using only the binary ruler and compass operations (those in condition 2), then $S'$ is a subfield of $\mathbb{C}$, and is the smallest field containing $S$. Next, denote $\hat{S}$ the set of all constructible numbers from $S$. It is not hard to see that $\hat{S}$ is also a subfield of $\mathbb{C}$, but an extension of $S'$. Furthermore, it is not hard to show that $\hat{S}$ is Euclidean. The general process (algorithm) of elements in $\hat{S}$ from elements in $S$ using finite sequences of ruler and compass operations is called a ruler and compass construction. These are so called because, given two points, one of which is 0, the other of which is a non-zero real number in $S$, one can use a ruler and compass to construct these elements of $\hat{S}$.

If $S = \{1\}$ (or any rational number), we see that $\hat{S} = \mathbb{F}$ is *the* field of constructible numbers.

Note that the lengths of `http://planetmath.org/Constructible2`constructible line segments on the Euclidean plane are exactly the positive elements of $\mathbb{E}$. Note also that the set $\mathbb{F}$ is in one-to-one correspondence with the set of `http://planetmath.org/Constructible2`constructible points on the Euclidean plane. These facts provide a  between abstract algebra and compass and straightedge constructions.