quadratic closure

Canonical name	QuadraticClosure
Date of creation	2013-03-22 15:42:43
Last modified on	2013-03-22 15:42:43
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	8
Author	CWoo (3771)
Entry type	Definition
Classification	msc 12F05
Defines	quadratically closed

A field K is said to be *quadratically closed* if it has no quadratic extensions. In other words, every element of K is a square. Two obvious examples are $\mathbb{C}$ and $\mathbb{F}_2$.

A field K is said to be a *quadratic closure* of another field k if

1. K is quadratically closed, and
2. among all quadratically closed subfields of the algebraic closure $\overline{k}$ of k, K is the smallest one.

By the second condition, a quadratic closure of a field is unique up to field isomorphism. So we say *the* quadratic closure of a field k, and we denote it by $\widetilde{k}$ Alternatively, the second condition on K can be replaced by the following:

> K is the smallest field extension over k such that, if L is any field extension over k obtained by a finite number of quadratic extensions starting with k, then L is a subfield of K.

Examples.

- $\mathbb{C} = \widetilde{\mathbb{R}}$.
- If $\mathbb{E}$ is the Euclidean field in $\mathbb{R}$, then the quadratic extension $\mathbb{E}(\sqrt{-1})$ is the quadratic closure $\widetilde{\mathbb{Q}}$ of the rational numbers $\mathbb{Q}$.
- If $k = \mathbb{F}_5$, consider the chain of fields

$$k \leq k(\sqrt{2}) \leq k(\sqrt[4]{2}) \leq \cdots \leq k(\sqrt[2^n]{2}) \leq \cdots$$

  Take the union of all these fields to obtain a field K. Then it can be shown that $K = \widetilde{k}$.