de Moivre identity, proof of

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To prove the de Moivre identity, we will first prove by induction on n that the identity holds for all natural numbers.

For the case $n = 0$, observe that

$$\cos(0\theta) + i \sin(0\theta) = 1 + i0 = (\cos(\theta) + i \sin(\theta))^0 .$$

Assume that the identity holds for a certain value of n:

$$\cos(n\theta) + i \sin(n\theta) = (\cos(\theta) + i \sin(\theta))^n .$$

Multiply both sides of this identity by $\cos(\theta) + i \sin(\theta)$ and expand the left side to obtain

$$\begin{aligned} \cos(\theta)\cos(n\theta) - \sin(\theta)\sin(n\theta) + i\cos(\theta)\sin(n\theta) + i\sin(\theta)\cos(n\theta) &= (\cos(\theta) + i\sin(\theta))\,(\cos(n\theta) + i \\ &= (\cos(\theta) + i\sin(\theta))^{n+1} . \end{aligned}$$

By the angle sum identities,

$$\begin{aligned} \cos(\theta)\cos(n\theta) - \sin(\theta)\sin(n\theta) &= \cos(n\theta + \theta) \\ \cos(\theta)\sin(n\theta) + \sin(\theta)\cos(n\theta) &= \sin(n\theta + \theta) \end{aligned}$$

Therefore,

$$\cos((n+1)\theta) + i\sin((n+1)\theta) = (\cos(\theta) + i\sin(\theta))^{n+1} .$$

Hence by induction de Moivre's identity holds for all natural n.

Now let $-n$ be any negative integer. Then using the fact that cos is an even and sin an odd function, we obtain that

$$\begin{aligned} \cos(-n\theta) + i\sin(-n\theta) &= \cos(n\theta) - i\sin(n\theta) \\ &= \frac{\cos(n\theta) - i\sin(n\theta)}{\cos^2(n\theta) + \sin^2(n\theta)} \\ &= \frac{1}{\cos(n\theta) + i\sin(n\theta)} \cdot \frac{\cos(n\theta) - i\sin(n\theta)}{\cos(n\theta) - i\sin(n\theta)} \\ &= \frac{1}{\cos(n\theta) + i\sin(n\theta)}, \end{aligned}$$

the denominator of which is $(\cos(n\theta) + i\sin(n\theta))^n$. Hence

$$\cos(-n\theta) + i\sin(-n\theta) = (\cos(\theta) + i\sin(\theta))^{-n} .$$