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field adjunction

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Let K be a field and E an extension field of K. If $\alpha \in E$, then the smallest subfield of E, that contains K and α, is denoted by $K(\alpha)$. We say that $K(\alpha)$ is obtained from the field K by *adjoining* the element α to K via *field adjunction*.

Theorem. $K(\alpha)$ is identical with the quotient field Q of $K[\alpha]$.

Proof. (1) Because $K[\alpha]$ is an integral domain (as a subring of the field E), all possible quotients of the elements of $K[\alpha]$ belong to E. So we have

$$K \cup \{\alpha\} \subseteq K[\alpha] \subseteq Q \subseteq E,$$

and because $K(\alpha)$ was the smallest, then $K(\alpha) \subseteq Q$.

(2) $K(\alpha)$ is a subring of E containing K and α, therefore also the whole ring $K[\alpha]$, that is, $K[\alpha] \subseteq K(\alpha)$. And because $K(\alpha)$ is a field, it must contain all possible quotients of the elements of $K[\alpha]$, i.e., $Q \subseteq K(\alpha)$.

In to the adjunction of one single element, we can adjoin to K an arbitrary subset S of E: the resulting field $K(S)$ is the smallest of such subfields of E, i.e. the intersection of such subfields of E, that contain both K and S. We say that $K(S)$ is obtained from K by adjoining the set S to it. Naturally,

$$K \subseteq K(S) \subseteq E.$$

The field $K(S)$ contains all elements of K and S, and being a field, also all such elements that can be formed via addition, subtraction, multiplication and division from the elements of K and S. But such elements constitute a field, which therefore must be equal with $K(S)$. Accordingly, we have the

Theorem. $K(S)$ constitutes of all rational expressions formed of the elements of the field K with the elements of the set S.

Notes.

1. $K(S)$ is the union of all fields $K(T)$ where T is a finite subset of S.
2. $K(S_1 \cup S_2) = K(S_1)(S_2)$.
3. If, especially, S also is a subfield of E, then one may denote $K(S) = KS$.

References

[1] B. L. van der Waerden: *Algebra. Erster Teil.* Siebte Auflage der *Modernen Algebra.* Springer-Verlag; Berlin, Heidelberg, New York (1966).