a condition of algebraic extension

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Theorem. A field extension L/K is `http://planetmath.org/AlgebraicExtension`algebraic if and only if any subring of the extension field L containing the base field K is a field.

Proof. Assume first that L/K is algebraic. Let R be a subring of L containing K. For any non-zero element r of R, naturally $K[r] \subseteq R$, and since r is an algebraic element over K, the ring $K[r]$ coincides with the field $K(r)$. Therefore we have $r^{-1} \in K[r] \subseteq R$, and R must be a field.

Assume then that each subring of L which contains K is a field. Let a be any non-zero element of L. Accordingly, the subring $K[a]$ of L contains K and is a field. So we have $a^{-1} \in K[a]$. This means that there is a polynomial $f(x)$ in the polynomial ring $K[x]$ such that $a^{-1} = f(a)$. Because $af(a) - 1 = 0$, the element a is a zero of the polynomial $xf(x) - 1$ of $K[x]$, i.e. is algebraic over K. Thus every element of L is algebraic over K.

References

[1] David M. Burton: *A first course in rings and ideals*. Addison-Wesley Publishing Company. Reading, Menlo Park, London, Don Mills (1970).