characterizations of integral

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Theorem. Let R be a subring of a field K, $1 \in R$ and let α be a non-zero element of K. The following conditions are equivalent:

1. α is integral over R.
2. α belongs to $R[\alpha^{-1}]$.
3. α is unit of $R[\alpha^{-1}]$.
4. $\alpha^{-1}R[\alpha^{-1}] = R[\alpha^{-1}]$.

Proof. Supposing the first condition that an equation

$$\alpha^n + a_1\alpha^{n-1} + \ldots + a_{n-1}\alpha + a_n = 0,$$

with a_j's belonging to R, holds. Dividing both by α^{n-1} gives

$$\alpha = -a_1 - a_2\alpha^{-1} - \ldots - a_n\alpha^{-n+1}.$$

One sees that α belongs to the ring $R[\alpha^{-1}]$ even being a unit of this (of course $\alpha^{-1} \in R[\alpha^{-1}]$). Therefore also the principal ideal $\alpha^{-1}R[\alpha^{-1}]$ of the ring $R[\alpha^{-1}]$ coincides with this ring. Conversely, the last circumstance implies that α is integral over R.

References

[1] Emil Artin: . Lecture notes. Mathematisches Institut, Göttingen (1959).