Krasner's lemma

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Krasner's lemma (along with Hensel's lemma) connects valuations on fields to the algebraic structure of the fields, and in particular to polynomial roots.

Lemma 1. *(Krasner's Lemma) Let K be a field of characteristic 0 complete with respect to a nontrivial nonarchimedean absolute value. Assume $\alpha, \beta \in \bar{K}$ (where $\bar{K}$ is some algebraic closure of K) are such that for all nonidentity embeddings $\sigma \in \mathrm{Hom}_K(K(\alpha), \bar{K})$ we have $|\alpha - \beta| < |\sigma(\alpha) - \alpha|$. Then $K(\alpha) \subset K(\beta)$.*

This says that for any $\alpha \in \bar{K}$, there is a neighborhood of α each of whose elements generates at least the same field as α does.

Proof. It suffices to show that for every $\sigma \in \mathrm{Hom}_{K(\beta)}(K(\alpha, \beta), \bar{K})$, we have $\sigma(\alpha) = \alpha$, for then α is in the fixed field of every embedding of $K(\beta)$, so $\alpha \in K(\beta)$. Note that

$$|\sigma(\alpha) - \beta| = |\sigma(\alpha) - \sigma(\beta)| = |\sigma(\alpha - \beta)| = |\alpha - \beta|$$

where the final equality follows since $|\sigma(\cdot)|$ is another absolute value extending $|\cdot|_K$ to $K(\alpha, \beta)$ and thus must be equal to $|\cdot|$. But then

$$|\sigma(\alpha) - \alpha| = |(\sigma(\alpha) - \beta) + (\beta - \alpha)| \leq \max(|\sigma(\alpha) - \beta|, |\alpha - \beta|) = |\alpha - \beta|$$

But this is impossible by the bounds on α, β unless $\sigma(\alpha) = \alpha$. $\square$

The first application of Krasner's lemma is to show that splitting fields are "locally constant" in the sense that sufficiently close polynomials in $K[X]$ have the same splitting fields.

Proposition 2. *With K as above, let $P(X) \in K[X]$ be a monic irreducible polynomial of degree n with (distinct) roots $\alpha_1, \ldots, \alpha_n$. Then any monic polynomial $Q(X) \in K[X]$ of degree n that is "sufficiently close" to $P(X)$ will be irreducible over K with roots $\beta_1, \ldots \beta_n$, and (after renumbering) $K(\alpha_i) = K(\beta_i)$.*

Here "sufficiently close" means the following: consider the space of degree n polynomials over K as homeomorphic to K^n as a topological space; close then means close in the obvious metric induced by $|\cdot|$.

Proof. Since $P(X)$ has distinct roots, we may choose $0 < \gamma < \min(|\alpha_i - \alpha_j|)$ for $i \neq j \leq n$. Since the roots of a polynomial vary continuously with its coefficients, we say that a degree n polynomial $Q(X) \in K[X]$ is sufficiently close to $P(X)$ if $Q(X)$ has roots $\beta_1, \ldots, \beta_n$ with $|\alpha_i - \beta_i| < \gamma$. But $\{\alpha_j\}_{j \neq i}$ are all the Galois conjugates of α_i, and $|\alpha_i - \beta_i| < \gamma < |\alpha_i - \alpha_j|$ by construction, so by Krasner's lemma, $K(\alpha_i) \subset K(\beta_i)$. But

$$[K(\beta_i) : K] \leq \deg Q = \deg P = [K(\alpha_i) : K]$$

so that $K(\beta_i) = K(\alpha_i)$. In addition, we see that $\deg Q = [K(\beta_i) : K]$ and thus that $Q(X)$ is irreducible. □

We use this fact to show that every finite extension of $\mathbb{Q}_p$ arises as a completion of some number field.

Corollary 3. *Let K be a finite extension of $\mathbb{Q}_p$ of degree n. Then there is a number field E and an absolute value $|\cdot|$ on E such that $\hat{E} \cong K$.*

Proof. Let $K = \mathbb{Q}_p(\alpha)$ and let P be the minimal polynomial for α over $\mathbb{Q}_p$. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, we can choose $Q(X) \in \mathbb{Q}[X]$ (note: in $\mathbb{Q}[X]$, not $\mathbb{Q}_p[X]$), and β a root of $Q(X)$, as in the proposition, so that $\mathbb{Q}_p(\alpha) = \mathbb{Q}_p(\beta)$. Let $E = \mathbb{Q}(\beta)$. Clearly E is a number field which, when regarded as embedded in $\mathbb{Q}_p(\beta)$, has absolute value $|\cdot|_E$, the restriction of the absolute value on $\mathbb{Q}_p(\alpha) = \mathbb{Q}_p(\beta)$. Then $\hat{E}$ is a complete field with respect to that absolute value; $\mathbb{Q}_p(\beta)$ is as well, and E is dense in both, so we must have $\hat{E} = \mathbb{Q}_p(\beta) = \mathbb{Q}_p(\alpha) = K$. □

Finally, we can prove the following generalization of Krasner's Lemma, which is also given that name in the literature:

Lemma 4. *Let K be a field of characteristic 0 complete with respect to a nontrivial nonarchimedean absolute value, and $\bar{K}$ an algebraic closure of K. Extend the absolute value on K to $\bar{K}$; this extension is unique. Let $\hat{\bar{K}}$ be the completion of $\bar{K}$ with respect to this absolute value. Then $\hat{\bar{K}}$ is algebraically closed.*

Proof. Let α be algebraic over $\hat{\bar{K}}$ and $P(X)$ its monic irreducible polynomial in $\hat{\bar{K}}[X]$. Since $\bar{K}$ is dense in $\hat{\bar{K}}$, by proposition **??** we may choose $Q(x) \in \bar{K}[X]$ with a root $\beta \in \hat{\bar{K}}$ such that $\hat{\bar{K}}(\alpha) = \hat{\bar{K}}(\beta)$. But $\hat{\bar{K}}(\beta) = \hat{\bar{K}}$ so that $\alpha \in \hat{\bar{K}}$. □