non-constant element of rational function field

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Let K be a field. Every http://planetmath.org/SimpleFieldExtensionsimple transcendent field extension $K(\alpha)/K$ may be represented by the extension $K(X)/K$, where $K(X)$ is the field of fractions of the polynomial ring $K[X]$ in one indeterminate X. The elements of $K(X)$ are rational functions, i.e. rational expressions

$$\varrho = \frac{f(X)}{g(X)} \tag{1}$$

with $f(X)$ and $g(X)$ polynomials in $K[X]$.

Theorem. Let the non-constant rational function (1) be reduced to lowest terms and let the greater of the degrees of its numerator and denominator be n. This element ϱ is transcendental with respect to the base field K. The field extension $K(X)/K(\varrho)$ is algebraic and of degree n.

Proof. The element X satisfies the equation

$$\varrho\, g(X) - f(X) = 0, \tag{2}$$

the coefficients of which are in the field $K(\varrho)$, actually in the ring $K[\varrho]$. If all these coefficients were zero, we could take one non-zero coefficient b_ν in $g(X)$ and the coefficient a_ν of the same power of X in $f(X)$, and then we would have especially $\varrho b_\nu - a_\nu = 0$; this would mean that $\varrho = \frac{a_\nu}{b_\nu} = \text{constant}$, contrary to the supposition. Thus at least one coefficient in (2) differs from zero, and we conclude that X is algebraic with respect to $K(\varrho)$. If $K(\varrho)$ were algebraic with respect to K, then also X should be algebraic with respect to K. This is not true, and therefore we see that $K(\varrho)$ is transcendental, Q.E.D.

Further, X is a zero of the n^{th} degree polynomial

$$h(Y) = \varrho\, g(Y) - f(Y)$$

of the ring $K(\varrho)[Y]$, actually of the ring $K[\varrho][Y]$, i.e. of $K[\varrho, \mathrm{Y}]$. The polynomial is irreducible in this ring, since otherwise it would have there two factors, and because $h(Y)$ is linear in ϱ, the other factor should depend only on Y; but there can not be such a factor, for the polynomials $f(Z)$ and $g(Z)$ are relatively prime. The conclusion is that X is an algebraic element over $K(\varrho)$ of degree n and therefore also

$$(K(X) : K(\varrho)) = n,$$

Q.E.D.

References

[1] B. L. van der Waerden: *Algebra*. Siebte Auflage der *Modernen Algebra*. Erster Teil.
— Springer-Verlag. Berlin, Heidelberg (1966).