Cardano's derivation of the cubic formula

Canonical name	CardanosDerivationOfTheCubicFormula
Date of creation	2013-03-22 12:10:28
Last modified on	2013-03-22 12:10:28
Owner	djao (24)
Last modified by	djao (24)
Numerical id	12
Author	djao (24)
Entry type	Proof
Classification	msc 12D10
Related topic	FerrariCardanoDerivationOfTheQuarticFormula

To solve the cubic polynomial equation $x^3 + ax^2 + bx + c = 0$ for x, the first step is to apply the Tchirnhaus transformation $x = y - \frac{a}{3}$. This reduces the equation to $y^3 + py + q = 0$, where

$$\begin{aligned} p &= b - \frac{a^2}{3} \\ q &= c - \frac{ab}{3} + \frac{2a^3}{27} \end{aligned}$$

The next step is to substitute $y = u - v$, to obtain

$$(u - v)^3 + p(u - v) + q = 0 \tag{1}$$

or, with the terms collected,

$$(q - (v^3 - u^3)) + (u - v)(p - 3uv) = 0 \tag{2}$$

From equation (**??**), we see that if u and v are chosen so that $q = v^3 - u^3$ and $p = 3uv$, then $y = u - v$ will satisfy equation (**??**), and the cubic equation will be solved!

There remains the matter of solving $q = v^3 - u^3$ and $p = 3uv$ for u and v. From the second equation, we get $v = p/(3u)$, and substituting this v into the first equation yields

$$q = \frac{p^3}{(3u)^3} - u^3$$

which is a quadratic equation in u^3. Solving for u^3 using the quadratic formula, we get

$$\begin{aligned} u^3 &= \frac{-27q + \sqrt{108p^3 + 729q^2}}{54} = \frac{-9q + \sqrt{12p^3 + 81q^2}}{18} \\ v^3 &= \frac{27q + \sqrt{108p^3 + 729q^2}}{54} = \frac{9q + \sqrt{12p^3 + 81q^2}}{18} \end{aligned}$$

Using these values for u and v, you can back–substitute $y = u - v$, $p = b - a^2/3$, $q = c - ab/3 + 2a^3/27$, and $x = y - a/3$ to get the expression for the first root r_1 in the cubic formula. The second and third roots r_2 and r_3 are obtained by performing synthetic division using r_1, and using the quadratic formula on the remaining quadratic factor.