proof of additive form of Hilbert's theorem 90

Canonical name	ProofOfAdditiveFormOfHilbertsTheorem90
Date of creation	2013-03-22 15:21:25
Last modified on	2013-03-22 15:21:25
Owner	mathwizard (128)
Last modified by	mathwizard (128)
Numerical id	4
Author	mathwizard (128)
Entry type	Proof
Classification	msc 12F10
Classification	msc 11R32

Set $n = [L : K]$.

First, let there be $x \in L$ such that $y = x - \sigma(x)$. Then

$$\mathrm{Tr}(y) = (x - \sigma(x)) + (\sigma(x) - \sigma^2(x)) + \cdots + (\sigma^{n-1}(x) - \sigma^n(x)) = 0$$

because $x = \sigma^n(x)$.

Now, let $\mathrm{Tr}(y) = 0$. Choose $z \in L$ with $\mathrm{Tr}(z) \neq 0$. Then there exists $x \in L$ with

$$x\,\mathrm{Tr}(z) = y\sigma(z) + (y + \sigma(y))\sigma^2(z) + \cdots + (y + \sigma(y) + \cdots + \sigma^{n-1}(y))\sigma^{n-1}(z).$$

Since $\mathrm{Tr}(z) \in K$ we have

$$\sigma(x)\,\mathrm{Tr}(z) = \sigma(y)\sigma^2(z) + (\sigma(y) + \sigma^2(y))\sigma^3(z) + \cdots + (\sigma(y) + \cdots + \sigma^{n-2})\sigma^{n-1}(z) + (\sigma(y) + \cdots + \sigma^{n-1}(y))\sigma^n$$

Now remember that $\mathrm{Tr}(y) = 0$. We obtain

$$\begin{aligned}(x - \sigma(x))\,\mathrm{Tr}(z) &= y\sigma(z) + (y + \sigma(y))\sigma^2(z) + \cdots + (y + \sigma(y) + \cdots + \sigma^{n-1}(y))\sigma^{n-1}(z) \\ &\quad - \sigma(y)\sigma^2(z) - (\sigma(y) + \sigma^2(y))\sigma^3(z) - \cdots - (\sigma(y) + \cdots + \sigma^{n-2})\sigma^{n-1}(z) + yz \\ &= y\,\mathrm{Tr}(z),\end{aligned}$$

so $y = x - \sigma(x)$, as we wanted to show.