quartic polynomial with Galois group D_8

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The polynomial $f(x) = x^4 - 2x^2 - 2$ is Eisenstein at 2 and thus irreducible over $\mathbb{Q}$. Solving $f(x)$ as a quadratic in x^2, we see that the roots of $f(x)$ are

$$\begin{array}{ll} \alpha_1 = \sqrt{1+\sqrt{3}} & \alpha_3 = -\sqrt{1+\sqrt{3}} \\ \alpha_2 = \sqrt{1-\sqrt{3}} & \alpha_4 = -\sqrt{1-\sqrt{3}} \end{array}$$

Note that the discriminant of $f(x)$ is $-4608 = -2^9 \cdot 3^2$, and that its resolvent cubic is

$$x^3 + 4x^2 + 12x = x(x^2 + 4x + 12) = 0$$

which factors over $\mathbb{Q}$ into a linear and an irreducible quadratic. Additionally, $f(x)$ remains irreducible over $\mathbb{Q}(\sqrt{-4608}) = \mathbb{Q}(\sqrt{-2})$, since none of the roots of $f(x)$ lie in this field and the discriminant of $f(x)$, regarded as a quadratic in x^2, does not lie in this field either, so $f(x)$ cannot factor as a product of two quadratics. So according to the article on the Galois group of a quartic polynomial, $f(x)$ should indeed have Galois group isomorphic to D_8. We show that this is the case by explicitly examining the structure of its splitting field.

Let K be the splitting field of $f(x)$ over $\mathbb{Q}$, and let $G = \mathrm{Gal}(K/\mathbb{Q})$.

Let $K_1 = \mathbb{Q}(\alpha_1) = \mathbb{Q}(\alpha_3)$ and $K_2 = \mathbb{Q}(\alpha_2) = \mathbb{Q}(\alpha_4)$. Clearly K contains both K_1 and K_2 and thus contains $K_1K_2 = \mathbb{Q}(\alpha_1, \alpha_2)$. But obviously $f(x)$ splits in K_1K_2, so that $K = K_1K_2$. We next determine the degree of K over $\mathbb{Q}$.

Note that $K_1 \neq K_2$ since K_1 is a real field while K_2 is not. Thus $K_1 \cap K_2 \subsetneq K_1, K_2$. Clearly $[K_1 : \mathbb{Q}] = [K_2 : \mathbb{Q}] = 4$, so $[K_1 \cap K_2 : \mathbb{Q}] \leq 2$. But

$$\sqrt{3} = \left(\sqrt{1+\sqrt{3}}\right)^2 - 1 = -\left(\sqrt{1-\sqrt{3}}\right)^2 + 1$$

so $\sqrt{3} \in K_1 \cap K_2$. Hence $K_1 \cap K_2 = \mathbb{Q}(\sqrt{3})$; call this field F.

Since $K_1 \neq K_2$, we also have $K = K_1K_2 \neq K_1$ and $K = K_1K_2 \neq K_2$; thus K is a quadratic extension of each and $[K : F] = 4$.

Putting these results together, we see that

$$[K : \mathbb{Q}] = [K : F][F : \mathbb{Q}] = 8$$

so that G has order 8.

Now, neither K_1 nor K_2 is Galois over $\mathbb{Q}$ (since the Galois closure of either one is K), so that the subgroup of G fixing (say) K_1 is a nonnormal subgroup of G. Thus G must be nonabelian, so must be isomorphic to either D_8 or Q_8 (the quaternions). But the subgroups of G corresponding to K_1 and K_2 are distinct subgroups of order 2 in G, and Q_8 has only one subgroup of order 2. Thus $G \cong D_8$. (Alternatively, note that all subgroups of Q_8 are normal, so $G \cong D_8$ since it has a nonnormal subgroup).