Zariski lemma

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Proposition 1. *Let $R \subseteq S \subseteq T$ be commutative rings. If R is noetherian, and T finitely generated as an R-algebra and as an S-module, then S is finitely generated as an R-algebra.*

Lemma 1 (Zariski's lemma)**.** *Let $(L : K)$ be a field extension and $a_1, \dots, a_n \in L$ be such that $K(a_1, \dots, a_n) = K[a_1, \dots, a_n]$. Then the elements $a_1, \dots, a_n$ are algebraic over K.*

Proof. The case $n = 1$ is clear. Now suppose $n > 1$ and not all $a_i, 1 \leq i \leq n$ are algebraic over K.
Wlog we may assume $a_1, \dots, a_n$ are algebraically independent and each element $a_{r+1}, \dots, a_n$ is algebraic over $D := K(a_1, \dots, a_r)$. Hence $K[a_1, \dots, a_n]$ is a finite algebraic extension of D and therefore is a finitely generated D-module.
The above proposition applied to $K \subseteq D \subseteq K[a_1, \dots, a_n]$ shows that D is finitely generated as a K-algebra, i.e $D = K[d_1, \dots, d_n]$.

Let $d_i = \frac{p_i(a_1,\dots,a_n)}{q_i(a_1,\dots,a_n)}$, where $p_i, q_i \in K[x_1, \dots, x_n]$.
Now $a_1, \dots, a_n$ are algebraically independent so that $K[a_1, \dots, a_n] \cong K[x_1, \dots, x_n]$, which is a `http://planetmath.org/UFD`UFD.
Let h be a prime divisor of $q_1 \cdots q_r + 1$. Since q is relatively prime to each of q_i, the element $q(a_1, \dots, a_n)^{-1} \in D$ cannot be in $K[d_1, \dots, d_n]$. We obtain a contradiction. $\square$