equivalent statements of Lindemann-Weierstrass theorem

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Proposition 1. *The following versions of the Lindemann-Weierstrass Theorem are equivalent:*

1. *If $\alpha_1, \ldots, \alpha_n$ are linearly independent algebraic numbers over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

2. *If $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\mathbb{Q}$.*

Proof. ($1 \Longrightarrow 2$). Write $A_i = e^{\alpha_i}$ for each $i = 1, \ldots, n$. Suppose $0 = r_1 A_1 + \cdots + r_n A_n$, where $r_i \in \mathbb{Q}$. Moving $r_1 A$ to the LHS and by multiplying a common denominator we can assume that $r_1 A_1 = r_2 A_2 + \cdots + r_n A_n$ where $r_i \in \mathbb{Z}$. We want to show that $r_1 = \cdots = r_n = 0$. We induct on n. The case when $n = 1$ is trivial because A_1 is never 0 and therefore $0 = r_1 A_1$ forces $r_1 = 0$.

By induction hypothesis, suppose the statement is true when $n < k$. Now suppose $n = k$. If $\alpha_1, \ldots, \alpha_k$ are linearly independent over $\mathbb{Q}$, $A_1, \ldots, A_k$ are algebraically independent and certainly linearly independent over $\mathbb{Q}$. So suppose $\alpha_1, \ldots, \alpha_k$ are not linearly independent over $\mathbb{Q}$. Without loss of generality, we can assume $s_1\alpha_1 = s_2\alpha_2 + \cdots + s_k\alpha_k$, where $s_i \in \mathbb{Q}$ and $s_1 \neq 0$. By multiplying a common denominator we can further assume that $s_i \in \mathbb{Z}$ and $s_1 > 0$. Then

$$A_1{}^{s_1} = e^{s_1\alpha_1} = e^{s_2\alpha_2 + \cdots + s_k\alpha_k} = e^{s_2\alpha_2} \cdots e^{s_k\alpha_k} = A_2{}^{s_2} \cdots A_k{}^{s_k}.$$

Since $r_1 A_1 = r_2 A_2 + \cdots + r_n A_n$, we get

$$\begin{aligned}
(r_1^{s_1})(A_2{}^{s_2} \cdots A_k{}^{s_k}) &= (r_1^{s_1}) A_1{}^{s_1} \\
&= (r_1 A_1)^{s_1} \\
&= (r_2 A_2 + \cdots + r_k A_k)^{s_1} \\
&= g(A_2, \ldots, A_k),
\end{aligned}$$

where $g(x_2, \ldots, x_k) = (r_2 x_2 + \cdots + r_k x_k)^{s_1} \in \mathbb{Q}[x_2, \ldots, x_k]$. Partition the numbers $s_i' s$ into non-negative and negative ones, so that, say, $s_{c(1)}, \ldots, s_{c(l)}$ are non-negative and $s_{d(1)}, \ldots, s_{d(m)}$ are negative, then

$$(r_1^{s_1}) A_{c(1)}{}^{s_{c(1)}} \cdots A_{c(i)}{}^{s_{c(i)}} = g(A_2, \ldots, A_k) A_{d(1)}{}^{s_{d(1)}} \cdots A_{d(j)}{}^{s_{d(j)}}.$$

If we define

$$f(x_2, \ldots, x_k) = r_1^{s_1} x_{c(1)}{}^{s_{c(1)}} \cdots x_{c(i)}{}^{s_{c(i)}} - g(x_2, \ldots, x_k) x_{d(1)}{}^{s_{d(1)}} \cdots x_{d(j)}{}^{s_{d(j)}},$$

then $f(x_2,\ldots,x_k) \in \mathbb{Q}[x_2,\ldots,x_k]$ and $f(A_2,\ldots,A_k) = 0$. By the induction hypothesis, $f = 0$. It is not hard to see that $r_1 = \cdots = r_k = 0$ and therefore $A_1,\ldots,A_k$ are linearly independent.

$(1 \Longleftarrow 2)$. We first need two lemmas:

Lemma 1. Given 2., if $\alpha \neq 0$ is algebraic over $\mathbb{Q}$, then e^α is transcendental over $\mathbb{Q}$.

Proof. Suppose $f(e^\alpha) = 0$ where $f(x) = r_0 + r_1 x + \cdots + r_n x^n \in \mathbb{Q}[x]$. Then we have

$$\begin{aligned} 0 &= r_0 + r_1 e^\alpha + \cdots + r_n (e^\alpha)^n \\ &= r_0 e^0 + r_1 e^\alpha + \cdots + r_n e^{n\alpha}. \end{aligned}$$

Since $\alpha \neq 0$, $0, \alpha, \ldots, n\alpha$ are all distinct, $1, e^\alpha, \ldots, e^{n\alpha}$ are linearly independent by the hypothesis. Thus, $r_0 = r_1 = \ldots = r_n = 0$ and we have $f(x) = 0$, which means that e^α is transcendental over $\mathbb{Q}$. $\square$

Lemma 2. Given 2., if α and β are linearly independent and algebraic over $\mathbb{Q}$, then e^α is transcendental over $\mathbb{Q}(e^\beta)$.

Proof. Let $A = e^\alpha$ and $B = e^\beta$. Suppose $f(A) = 0$ where $f(x) \in \mathbb{Q}(B)[x]$. We want to show that $f(x) = 0$. Write

$$f(x) = r_0(B) + r_1(B)x + \cdots + r_n(B)x^n,$$

where each $r_i(x) = p_i(x)/q_i(x)$ with $p_i(x)$, $q_i(x) \neq 0 \in \mathbb{Q}[x]$. Let $Q(x) = q_1(x)\cdots q_n(x)$. So $Q(B)$, being the product of the denominators $q_i(B) \neq 0$, is non-zero. Multiply $f(x)$ by $Q(B)$ we get a new polynomial $g(x)$ such that

$$g(x) = R_0(B) + R_1(B)x + \cdots + R_n(B)x^n,$$

where each $R_i(x) = r_i(x)Q(x) = p_i(x)Q(x)/q_i(x) \in \mathbb{Q}[x]$. Now, $g(A) = f(A)Q(B) = 0$. So

$$\begin{aligned} 0 &= R_0(B) + R_1(B)A + \cdots + r_n(B)A^n \\ &= \sum_{j=0}^{m_0} a_{0j}B^j + \sum_{j=0}^{m_1} a_{1j}B^jA + \cdots + \sum_{j=0}^{m_n} a_{nj}B^jA^n \\ &= \sum_{j=0}^{m_0} a_{0j}e^{j\beta} + \sum_{j=0}^{m_1} a_{1j}e^{j\beta+\alpha} + \cdots + \sum_{j=0}^{m_n} a_{nj}e^{j\beta+n\alpha}, \end{aligned}$$

where each $a_{ij} \in \mathbb{Q}$. Now, the exponents in the above equation are all distinct, or else we would end up with α and β being linearly dependent, contrary to the assumption. Therefore, by 2 (Lindemann-Weierstrass Version 2), all $e^{i\beta+j\alpha}$ are linearly independent, which means each $a_{ij} = 0$. This implies that $g(x) = 0$. But $g(x) = f(x)Q(B)$ and $Q(B) \neq 0$, we must have $f(x) = 0$. $\square$

Now onto the main problem. We proceed by induction on the number of linearly independent algebraic elements over $\mathbb{Q}$. The case when $n = 1$ is covered in Lemma 1, since a linearly independent singleton is necessarily non-zero. So suppose $\alpha_1, \ldots, \alpha_k$ are linearly independent and algebraic over $\mathbb{Q}$. Then each pair α_k, α_i are independent and algebraic over $\mathbb{Q}$, $i \neq k$. Thus e^{α_k} is transcendental over $\mathbb{Q}(e^{\alpha_i})$ for all $i \neq k$. This means that e^{α_k} is transcendental over $\mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_{k-1}})$.

Now let $A_i = e^{\alpha_i}$ for all $i = 1, \ldots, k$. Suppose $f(A_1, \ldots, A_k) = 0$ where $f \in \mathbb{Q}[x_1, \ldots, x_k]$. To show the algebraic independence of the $A_i's$, we need to show that $f = 0$. Rearranging terms of f and we have

$$0 = f(A_1, \ldots, A_k) = \sum_{j=0}^{m} g_j(A_1, \ldots, A_{k-1})(A_k)^j.$$

If we let $g(x) = f(A_1, \ldots, A_{k-1}, x)$, we see that $g(x) \in \mathbb{Q}(A_1, \ldots, A_{k-1})[x]$ and $g(A_k) = f(A_1, \ldots, A_{k-1}, A_k) = 0$. Since e^{α_k} is transcendental over $\mathbb{Q}(A_1, \ldots, A_{k-1})$, we must have $g(x) = 0$. This implies that each $g_j(A_1, \ldots, A_{k-1}) = 0$. But then $A_1, \ldots, A_{k-1}$ are algebraically independent by the induction hypothesis, we must have each $g_j = 0$. This means that $f = 0$. $\square$