totally real and imaginary fields

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Related topic	RealAndComplexEmbeddings
Related topic	TotallyImaginaryExamplesOfTotallyReal
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Defines	totally real field
Defines	totally imaginary field
Defines	CM-field
Defines	maximal real subfield

For this entry, we follow the notation of the entry real and complex embeddings.

Let K be a subfield of the complex numbers, $\mathbb{C}$, and let Σ_K be the set of all embeddings of K in $\mathbb{C}$.

Definition 1. *With K as above:*

1. *K is a* totally real field *if all embeddings $\psi \in \Sigma_K$ are real embeddings.*

2. *K is a* totally imaginary field *if all embeddings $\psi \in \Sigma_K$ are (non-real) complex embeddings.*

3. *K is a* CM-field *or* complex multiplication field *if K is a totally imaginary quadratic extension of a totally real field.*

Note that, for example, one can obtain a CM-field K from a totally real number field F by adjoining the square root of a number all of whose conjugates are negative.

Note: A complex number ω is real if and only if $\bar{\omega}$, the complex conjugate of ω, equals ω:

$$\omega \in \mathbb{R} \Leftrightarrow \omega = \bar{\omega}$$

Thus, a field K which is fixed *pointwise* by complex conjugation is real (i.e. strictly contained in $\mathbb{R}$). However, K might not be *totally real*. For example, let α be the unique real third root of 2. Then $\mathbb{Q}(\alpha)$ is real but not totally real.

Given a field L, the subfield of L fixed pointwise by complex conjugation is called the *maximal real subfield of L*.

For examples (of (1), (2) and (3)), see examples of totally real fields.