irreducible polynomials obtained from biquadratic fields

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Corollary. *Let m and n be distinct squarefree integers, neither of which is equal to 1. Then the polynomial*

$$x^4 - 2(m+n)x^2 + (m-n)^2$$

is ***http://planetmath.org/IrreduciblePolynomial2****irreducible (over $\mathbb{Q}$).*

Proof. By the theorem stated in the http://planetmath.org/PrimitiveElementOfBiquadraticF
entry, $\sqrt{m}+\sqrt{n}$ is an algebraic number of http://planetmath.org/DegreeOfAnAlgebraicNumberc
4. Thus, a polynomial of degree 4 that has $\sqrt{m}+\sqrt{n}$ as a root must be over $\mathbb{Q}$. We set out to construct such a polynomial.

$$\begin{aligned}
x &= \sqrt{m}+\sqrt{n} \\
x-\sqrt{m} &= \sqrt{n} \\
(x-\sqrt{m})^2 &= n \\
x^2 - 2\sqrt{m}\,x + m &= n \\
x^2 + m - n &= 2\sqrt{m}\,x \\
(x^2+m-n)^2 &= 4mx^2 \\
x^4 + (2m-2n)x^2 + (m-n)^2 &= 4mx^2 \\
x^4 + (2m-2n-4m)x^2 + (m-n)^2 &= 0 \\
x^4 - 2(m+n)x^2 + (m-n)^2 &= 0 \quad \square
\end{aligned}$$