commensurable numbers

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Two positive real numbers a and b are *commensurable*, iff there exists a positive real number u such that

$$a\;=\;mu,\quad b\;=\;nu \tag{1}$$

with some positive integers m and n. If the positive numbers a and b are not commensurable, they are *incommensurable*.

Theorem. The positive numbers a and b are commensurable if and only if their ratio is a rational number $\frac{m}{n}$ $(m,\,n \in \mathbb{Z})$.

Proof. The equations (1) imply the http://planetmath.org/ProportionEquationproportion

$$\frac{a}{b}\;=\;\frac{m}{n}. \tag{2}$$

Conversely, if (2) is valid with $m,\,n \in \mathbb{Z}$, then we can write

$$a\;=\;m\!\cdot\!\frac{b}{n},\quad b\;=\;n\!\cdot\!\frac{b}{n},$$

which means that a and b are multiples of $\frac{b}{n}$ and thus commensurable. Q.E.D.

Example. The lengths of the side and the diagonal of http://planetmath.org/node/1086squ are always incommensurable.

0.1 Commensurability as relation

- The commensurability is an equivalence relation in the set $\mathbb{R}_+$ of the positive reals: the reflexivity and the symmetry are trivial; if $a\!:\!b = r$ and $b\!:\!c = s$, then $a:c = (a:b)(b:c) = rs$, whence one obtains the transitivity.

- The equivalence classes of the commensurability are of the form

$$[\varrho]\;:=\;\{r\varrho\vdots\;\;r \in \mathbb{Q}_+\}.$$

- One of the equivalence classes is the set $[1] = \mathbb{Q}_+$ of the positive rationals, all others consist of positive irrational numbers.
- If one sets $[\varrho]\cdot[\sigma] := [\varrho\sigma],$ the equivalence classes form with respect to this binary operation an Abelian group.