example of normal extension

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Let $F = \mathbb{Q}(\sqrt{2})$. Then the extension $F/\mathbb{Q}$ is normal because F is clearly the splitting field of the polynomial $f(x) = x^2 - 2$. Furthermore $F/\mathbb{Q}$ is a Galois extension with $\mathrm{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.

Now, let $2^{1/4}$ denote the positive real fourth root of 2 and define $K = F(2^{1/4})$. Then the extension K/F is normal because K is the splitting field of $k(x) = x^2 - \sqrt{2}$, and as before K/F is a Galois extension with $\mathrm{Gal}(K/F) \cong \mathbb{Z}/2\mathbb{Z}$.

However, the extension $K/\mathbb{Q}$ is neither normal nor Galois. Indeed, the polynomial $g(x) = x^4 - 2$ has one root in K (actually two), namely $2^{1/4}$, and yet $g(x)$ does not split in K into linear factors.

$$g(x) = x^4 - 2 = (x^2 - \sqrt{2}) \cdot (x^2 + \sqrt{2}) = (x - 2^{1/4}) \cdot (x + 2^{1/4}) \cdot (x^2 + \sqrt{2})$$

The Galois closure of K over $\mathbb{Q}$ is $L = \mathbb{Q}(2^{1/4}, i)$.