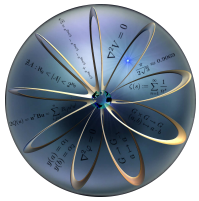

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Pythagorean field

Canonical name	PythagoreanField
Date of creation	2013-03-22 14:22:36
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Entry type	Definition
Classification	msc 12D15
Defines	Pythagorean extension
Defines	Pythagorean closure

Let F be a field. A field extension K of F is called a *Pythagorean extension* if $K = F(\sqrt{1+\alpha^2})$ for some α in F, where $\sqrt{1+\alpha^2}$ denotes a root of the polynomial $x^2 - (1+\alpha^2)$ in the algebraic closure $\overline{F}$ of F. A field F is *Pythagorean* if every Pythagorean extension of F is F itself.

The following are equivalent:

1. F is Pythagorean
2. Every sum of two squares in F is a square
3. Every sum of (finite number of) squares in F is a square

Examples:

- $\mathbb{R}$ and $\mathbb{C}$ are Pythagorean.
- $\mathbb{Q}$ is not Pythagorean.

Remark. Every field is contained in some Pythagorean field. The smallest Pythagorean field over a field F is called the *Pythagorean closure* of F, and is written F_{py}. Given a field F, one way to construct its Pythagorean closure is as follows: let K be an extension over F such that there is a tower

$$F = K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$$

of fields with $K_{i+1} = K_i(\sqrt{1+\alpha_i^2})$ for some $\alpha_i \in K_i$, where $i = 1, \ldots, n-1$. Take the compositum L of the family $\mathcal{K}$ of all such K's. Then $L = F_{py}$.