Euler's derivation of the quartic formula

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Let us consider the quartic equation

$$y^4 + py^2 + qy + r = 0, \tag{1}$$

where p, q, r are arbitrary known complex numbers. We substitute in the equation

$$y := u + v + w. \tag{2}$$

We get firstly
$y^2 = (u^2 + v^2 + w^2) + 2(vw + wu + uv)$,
$y^4 = (u^2 + v^2 + w^2)^2 + 4(u^2 + v^2 + w^2)(vw + wu + uv) + 4(v^2w^2 + w^2u^2 + u^2v^2) + 8uvw(u + v + w)$.

Thus (1) attains the form

$$4(v^2w^2 + w^2u^2 + u^2v^2) + (u^2 + v^2 + w^2)^2 + p(u^2 + v^2 + w^2) + r$$

$$+(vw + wu + uv)[4(u^2 + v^2 + w^2) + 2p] + (u + v + w)[8uvw + q] = 0.$$

When u, v, w are determined so that

$$u^2 + v^2 + w^2 = -\frac{p}{2}, \tag{3}$$

$$uvw = -\frac{q}{8}, \tag{4}$$

the expressions in the brackets vanish and our equation shrinks to the form

$$v^2w^2 + w^2u^2 + u^2v^2 = \frac{p^2 - 4r}{16}. \tag{5}$$

Squaring (4) gives

$$u^2v^2w^2 = \frac{q^2}{64}. \tag{6}$$

The left hand sides of (3), (5) and (6) are the elementary symmetric polynomials of u^2, v^2, w^2, whence these three squares are the roots z_1, z_2, z_3 of the so-called cubic resolvent equation

$$z^3 + \frac{p}{2}z^2 + \frac{p^2 - 4r}{16}z - \frac{q^2}{64} = 0. \tag{7}$$

Therefore we may write

$$u = \pm\sqrt{z_1}, \quad v = \pm\sqrt{z_2}, \quad w = \pm\sqrt{z_3}.$$

All 8 sign combinations of those square roots satisfy the equations (3), (5), (6). In order to satisfy also (4) the signs must be chosen suitably. If u_0, v_0, w_0 is some suitable combination of the values of the square roots, then all possible combinations are

$$u_0,\, v_0,\, w_0; \quad u_0,\, -v_0,\, -w_0; \quad -u_0,\, v_0,\, -w_0; \quad -u_0,\, -v_0,\, w_0.$$

Accordingly, we have the

Theorem (Euler 1739). The roots of the equation (1) are

$$\begin{cases} y_1 = u_0 + v_0 + w_0, \\ y_2 = u_0 - v_0 - w_0, \\ y_3 = -u_0 + v_0 - w_0, \\ y_4 = -u_0 - v_0 + w_0, \end{cases} \tag{8}$$

where u_0, v_0, w_0 are square roots of the roots of the cubic resolvent (7). The signs of the square roots must be chosen such that

$$u_0 v_0 w_0 = -\frac{q}{8}.$$

The equations (8) imply an important formula

$$\begin{aligned}(y_1-y_2)(y_1-y_3)(y_1-y_4)(y_2-y_3)(y_2-y_4)(y_3-y_4) &= -\,2^6(v_0^2-w_0^2)(w_0^2-u_0^2)(u_0^2-v_0^2) \\ &= -\,64(z_2-z_3)(z_3-z_1)(z_1-z_2),\end{aligned}$$

which yields the

Corollary. A quartic equation has a multiple root always and only when its cubic resolvent has such one.

References

[1] Ernst Lindelöf: *Johdatus korkeampaan analyysiin.* Fourth edition. Werner Söderström Osakeyhtiö, Porvoo ja Helsinki (1956).

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