a finite extension of fields is an algebraic extension

Canonical name	AFiniteExtensionOfFieldsIsAnAlgebraicExtension
Date of creation	2013-03-22 13:57:30
Last modified on	2013-03-22 13:57:30
Owner	alozano (2414)
Last modified by	alozano (2414)
Numerical id	6
Author	alozano (2414)
Entry type	Theorem
Classification	msc 12F05
Related topic	Algebraic
Related topic	AlgebraicExtension
Related topic	ProofOfTranscendentalRootTheorem

Theorem 1. *Let L/K be a finite field extension. Then L/K is an algebraic extension.*

Proof. In order to prove that L/K is an algebraic extension, we need to show that any element $\alpha \in L$ is algebraic, i.e., there exists a non-zero polynomial $p(x) \in K[x]$ such that $p(\alpha) = 0$.

Recall that L/K is a finite extension of fields, by definition, it means that L is a finite dimensional vector space over K. Let the dimension be

$$[L\colon K] = n$$

for some $n \in \mathbb{N}$.

Consider the following set of "vectors" in L:

$$\mathcal{S} = \{1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^n\}$$

Note that the cardinality of S is $n+1$, one more than the dimension of the vector space. Therefore, the elements of S must be linearly dependent over K, otherwise the dimension of S would be greater than n. Hence, there exist $k_i \in K,\ 0 \leq i \leq n$, not all zero, such that

$$k_0 + k_1\alpha + k_2\alpha^2 + k_3\alpha^3 + \ldots + k_n\alpha^n = 0$$

Thus, if we define

$$p(X) = k_0 + k_1X + k_2X^2 + k_3X^3 + \ldots + k_nX^n$$

then $p(X) \in K[X]$ and $p(\alpha) = 0$, as desired. □

NOTE: The converse is not true. See the entry "algebraic extension" for details.