field homomorphism

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Let F and K be fields.

Definition. *A* field homomorphism *is a function* $\psi\colon F \to K$ *such that:*

1. $\psi(a+b) = \psi(a) + \psi(b)$ *for all* $a, b \in F$
2. $\psi(a \cdot b) = \psi(a) \cdot \psi(b)$ *for all* $a, b \in F$
3. $\psi(1) = 1, \quad \psi(0) = 0$

If ψ *is injective and surjective, then we say that* ψ *is a* field isomorphism*.*

Lemma. *Let* $\psi\colon F \to K$ *be a field homomorphism. Then* ψ *is injective.*

Proof. Indeed, if ψ is a field homomorphism, in particular it is a ring homomorphism. Note that the kernel of a ring homomorphism is an ideal and a field F only has two ideals, namely $\{0\}, F$. Moreover, by the definition of field homomorphism, $\psi(1) = 1$, hence 1 is not in the kernel of the map, so the kernel must be equal to $\{0\}$. □

Remark: For this reason the terms "field homomorphism" and "field monomorphism" are synonymous. Also note that if ψ is a field monomorphism, then

$$\psi(F) \cong F, \quad \psi(F) \subseteq K$$

so there is a "copy" of F in K. In other words, if

$$\psi\colon F \to K$$

is a field homomorphism then there exist a subfield H of K such that $H \cong F$. Conversely, suppose there exists $H \subset K$ with H isomorphic to F. Then there is an isomorphism

$$\chi\colon F \to H$$

and we also have the inclusion homomorphism

$$\iota\colon H \hookrightarrow K$$

Thus the composition

$$\iota \circ \chi\colon F \to K$$

is a field homomorphism.

Remark: Let $\psi : F \to K$ be a field homomorphism. We claim that the characteristic of F and K must be the same. Indeed, since $\psi(1_F) = 1_K$

and $\psi(0_F) = 0_K$ then $\psi(n \cdot 1_F) = n \cdot 1_K$ for all natural numbers n. If the characteristic of F is $p > 0$ then $0 = \psi(p \cdot 1) = p \cdot 1$ in K, and so the characteristic of K is also p. If the characteristic of F is 0, then the characteristic of K must be 0 as well. For if $p \cdot 1 = 0$ in K then $\psi(p \cdot 1) = 0$, and since ψ is injective by the lemma, we would have $p \cdot 1 = 0$ in F as well.