examples of totally real fields

Canonical name	ExamplesOfTotallyRealFields
Date of creation	2013-03-22 13:55:05
Last modified on	2013-03-22 13:55:05
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Last modified by	alozano (2414)
Numerical id	6
Author	alozano (2414)
Entry type	Example
Classification	msc 12D99
Related topic	TotallyRealAndImaginaryFields
Related topic	NumberField
Defines	examples of totally imaginary fields
Defines	examples of CM-fields

Here we present examples of totally real fields, totally imaginary fields and CM-fields.

Examples:

1. Let $K = \mathbb{Q}(\sqrt{d})$ with d a square-free **positive** integer. Then

$$\Sigma_K = \{\mathrm{Id}_K, \sigma\}$$

where $\mathrm{Id}_K \colon K \hookrightarrow \mathbb{C}$ is the identity map ($\mathrm{Id}_K(k) = k$, for all $k \in K$), whereas

$$\sigma \colon K \hookrightarrow \mathbb{C}, \quad \sigma(a + b\sqrt{d}) = a - b\sqrt{d}$$

Since $\sqrt{d} \in \mathbb{R}$ it follows that K is a totally real field.

2. Similarly, let $K = \mathbb{Q}(\sqrt{d})$ with d a square-free **negative** integer. Then

$$\Sigma_K = \{\mathrm{Id}_K, \sigma\}$$

where $\mathrm{Id}_K \colon K \hookrightarrow \mathbb{C}$ is the identity map ($\mathrm{Id}_K(k) = k$, for all $k \in K$), whereas

$$\sigma \colon K \hookrightarrow \mathbb{C}, \quad \sigma(a + b\sqrt{d}) = a - b\sqrt{d}$$

Since $\sqrt{d} \in \mathbb{C}$ and it is not in $\mathbb{R}$, it follows that K is a totally imaginary field.

3. Let $\zeta_n, n \geq 3$, be a primitive n^{th} root of unity and let $L = \mathbb{Q}(\zeta_n)$, a cyclotomic extension. Note that the only roots of unity that are real are ± 1. If $\psi \colon L \hookrightarrow \mathbb{C}$ is an embedding, then $\psi(\zeta_n)$ must be a conjugate of ζ_n, i.e. one of

$$\{\zeta_n^a \mid a \in (\mathbb{Z}/n\mathbb{Z})^\times\}$$

but those are all imaginary. Thus $\psi(L) \nsubseteq \mathbb{R}$. Hence L is a totally imaginary field.

4. In fact, L as in (3) is a CM-field. Indeed, the maximal real subfield of L is

$$F = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$$

Notice that the minimal polynomial of ζ_n over F is

$$X^2 - (\zeta_n + \zeta_n^{-1})X + 1$$

so we obtain L from F by adjoining the square root of the discriminant of this polynomial which is

$$\zeta_n^2 + \zeta_n^{-2} - 2 = 2\cos(\frac{4\pi}{n}) - 2 < 0$$

and any other conjugate is

$$\zeta_n^{2a} + \zeta_n^{-2a} - 2 = 2\cos(\frac{4a\pi}{n}) - 2 < 0, a \in (\mathbb{Z}/n\mathbb{Z})^\times$$

Hence, L is a CM-field.

5. Notice that any quadratic imaginary number field is obviously a CM-field.