proof of factor theorem

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Suppose that $f(x)$ is a polynomial with real or complex coefficients of degree $n-1$. Since f is a polynomial, it is infinitely differentiable. Therefore, f has a Taylor expansion about a. Since $f^{(n)}(x)=0$, the terminates after the $n-1^{\text{th}}$ term. Also, the n^{th} remainder of the Taylor series vanishes; `http://planetmath.org/I`ei.e., $R_n(x)=\dfrac{f^{(n)}(y)}{n!}x^n=0$. Thus, the function is equal to its Taylor series. Hence,

$$\begin{aligned}
f(x) &= \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a)+\sum_{k=1}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a)+(x-a)\sum_{k=1}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^{k-1} \\
&= f(a)+(x-a)\sum_{k=0}^{n-2}\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^k.
\end{aligned}$$

If $f(a)=0$, then $f(x)=(x-a)\displaystyle\sum_{k=0}^{n-2}\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^k$. Thus, $f(x)=(x-a)g(x)$, where $g(x)$ is the polynomial $\displaystyle\sum_{k=0}^{n-2}\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^k$. Hence, $x-a$ is a factor of $f(x)$.

Conversely, if $x-a$ is a factor of $f(x)$, then $f(x)=(x-a)g(x)$ for some polynomial $g(x)$. Hence, $f(a)=(a-a)g(a)=0$.

It follows that $x-a$ is a factor of $f(x)$ if and only if $f(a)=0$.