polynomial function is a proper map

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Assume that $\mathbb{K}$ is either the field of real numbers or the field of complex numbers and let $W : \mathbb{K} \to \mathbb{K}$ be a polynomial function in one variable over $\mathbb{K}$ with positive degree.

Proposition. Polynomial function $W : \mathbb{K} \to \mathbb{K}$ is a proper map, i.e. for any compact subset $K \subseteq \mathbb{K}$ the preimage $W^{-1}(K)$ is compact.

Proof. Assume that

$$W(x) = \sum_{k=0}^{m} a_k \cdot x^k,$$

where $m = \deg(W) \geq 1$ is the degree of W.

Recall that $K \subseteq \mathbb{K}$ is compact if and only if K is closed and bounded. Since polynomial functions are continous, it is sufficient to show that preimage of a bounded set is bounded. So assume that K is bounded and $W^{-1}(K)$ is not bounded. Take a sequence $\{x_n\}_{n=1}^{\infty} \subseteq K$ such that

$$\lim_{n \to \infty} \|x_n\| = +\infty,$$

where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{K}$.

Recall that for any $x, y \in \mathbb{K}$ we have $\|x + y\| \geq \|x\| - \|y\|$. Thus we have:

$$\|W(x)\| = \|\sum_{k=0}^{m} a_k \cdot x^k\| \geq \|a_m \cdot x^m\| - \sum_{k=0}^{m-1} \|a_k \cdot x^k\| = \|a_m\| \cdot \|x\|^m - \sum_{k=0}^{m-1} \|a_k\| \cdot \|x\|^k.$$

Let

$$V(x) = \|a_m\| \cdot x^m - \sum_{k=0}^{m-1} \|a_k\| \cdot x^k.$$

Then V is a real polynomial of degree m and the leading coefficient of V is positive, which implies that

$$\lim_{x \to +\infty} V(x) = +\infty.$$

Now for each $n \in \mathbb{N}$ we have

$$\|W(x_n)\| \geq V(\|x_n\|),$$

but $V(\|x_n\|)$ tends to infinity, therefore $\|W(x_n)\|$ tends to infinty. Contradiction, since for each $n \in \mathbb{N}$ we have that $W(x_n) \in K$ and K is bounded. $\square$

Corollary 1. Polynomial functions on $\mathbb{K}$ are closed maps.

Proof. Note that $\mathbb{K}$ is compactly generated Hausdorff space and therefore every proper and continous map $f : \mathbb{K} \to \mathbb{K}$ is closed. Thus (due to proposition) polynomial functions are closed. $\square$

Corollary 2. Assume that $W : \mathbb{K} \to \mathbb{K}$ is a polynomial function such that $W(x) \neq 0$ for any $x \in \mathbb{K}$. Let $f : \mathbb{K} \to \mathbb{K}$ be a map defined by the formula

$$f(x) = \frac{1}{W(x)}.$$

Then f is bounded.

Proof. We wish to show that there exists $M > 0$ such that for all $x \in \mathbb{K}$ the inequality $\|f(x)\| \leq M$ holds. Since polynomial functions are closed maps, then the image $\mathrm{Im}(W)$ of W is a closed subset of $\mathbb{K}$. Therefore $\mathbb{K} \setminus \mathrm{Im}(W)$ is open and it contains 0, thus there exists $\epsilon > 0$ such that the ball around 0 with radius ϵ has empty intersection with $\mathrm{Im}(W)$. This means that for all $x \in \mathbb{K}$ we have that $\|W(x)\| \geq \epsilon > 0$. Now for $M = \epsilon^{-1}$ and for any $x \in \mathbb{K}$ we have:

$$\|f(x)\| = \left\| \frac{1}{W(x)} \right\| = \frac{1}{\|W(x)\|} \leq \frac{1}{\epsilon} = M$$

which completes the proof. $\square$