splitting and ramification in number fields and Galois extensions

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Defines	totally ramified
Defines	totally split
Defines	wildly ramified
Defines	tamely ramified

Let F/K be an extension of number fields and let $\mathcal{O}_F$ and $\mathcal{O}_K$ be their respective rings of integers. The ring of integers of a number field is a Dedekind domain, and these enjoy the property that every ideal $\mathfrak{A}$ factors uniquely as a finite product of prime ideals (see the entry `http://planetmath.org/FractionalIdeal`fractional ideal). Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$. Then $\mathfrak{p}\mathcal{O}_F$ is an ideal of $\mathcal{O}_F$. Let us assume that the prime ideal factorization of $\mathfrak{p}\mathcal{O}_F$ into primes of $\mathcal{O}_F$ is as follows:

$$\mathfrak{p}\mathcal{O}_F = \prod_{i=1}^{r} \mathfrak{P}_i^{e_i} \tag{1}$$

We say that the primes $\mathfrak{P}_i$ lie above $\mathfrak{p}$ and $\mathfrak{P}_i|\mathfrak{p}$ (divides). The exponent e_i (commonly denoted as $e(\mathfrak{P}_i|\mathfrak{p})$) is the ramification index of $\mathfrak{P}_i$ over $\mathfrak{p}$. Notice that for each prime ideal $\mathfrak{P}_i$, the quotient ring $\mathcal{O}_F/\mathfrak{P}_i$ is a finite field extension of the finite field $\mathcal{O}_K/\mathfrak{p}$ (also called the residue field). The degree of this extension is called the inertial degree of $\mathfrak{P}_i$ over $\mathfrak{p}$ and it is usually denoted by:

$$f(\mathfrak{P}_i|\mathfrak{p}) = [\mathcal{O}_F/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{p}].$$

Notice that as it is pointed out in the entry "`http://planetmath.org/InertialDegree`inertial degree", the ramification index and the inertial degree are related by the formula:

$$\sum_{i=1}^{r} e(\mathfrak{P}_i|\mathfrak{p}) f(\mathfrak{P}_i|\mathfrak{p}) = [F : K] \tag{2}$$

where r is the number of prime ideals lying above $\mathfrak{p}$ (as in Eq. (**??**)). See the theorem below for an improvement of Eq. (**??**) in the case when F/K is Galois.

Definition 1. *Let F, K and $\mathfrak{P}_i, \mathfrak{p}$ be as above.*

1. *If $e_i > 1$ for some i, then we say that $\mathfrak{P}_i$ is* **ramified** *over $\mathfrak{p}$ and $\mathfrak{p}$ ramifies in F/K. If $e_i = 1$ for all i then we say that $\mathfrak{p}$ is unramified in F/K.*

2. *If there is a unique prime ideal $\mathfrak{P}$ lying above $\mathfrak{p}$ (so $r = 1$) and $f(\mathfrak{P}|\mathfrak{p}) = 1$ then we say that $\mathfrak{p}$ is* **totally ramified** *in F/K. In this case $e(\mathfrak{P}|\mathfrak{p}) = [F : K]$.*

3. *On the other hand, if* $e(\mathfrak{P}_i|\mathfrak{p}) = f(\mathfrak{P}_i|\mathfrak{p}) = 1$ *for all* i*, we say that* $\mathfrak{p}$ *is* **totally split** *(or splits completely) in* F/K*. Notice that there are exactly* $r = [F : K]$ *prime ideals of* $\mathcal{O}_F$ *lying above* $\mathfrak{p}$*.*

4. *Let* p *be the characteristic of the residue field* $\mathcal{O}_K/\mathfrak{p}$*. If* $e_i = e(\mathfrak{P}_i|\mathfrak{p}) > 1$ *and* e_i *and* p *are relatively prime, then we say that* $\mathfrak{P}_i$ *is* **tamely ramified***. If* $p|e_i$ *then we say that* $\mathfrak{P}_i$ *is* **strongly ramified** *(or wildly ramified).*

When the extension F/K is a Galois extension then Eq. (**??**) is quite more simple:

Theorem 1. *Assume that* F/K *is a Galois extension of number fields. Then all the ramification indices* $e_i = e(\mathfrak{P}_i|\mathfrak{p})$ *are equal to the same number* e*, all the inertial degrees* $f_i = f(\mathfrak{P}_i|\mathfrak{p})$ *are equal to the same number* f *and the ideal* $\mathfrak{p}\mathcal{O}_F$ *factors as:*

$$\mathfrak{p}\mathcal{O}_F = \prod_{i=1}^{r} \mathfrak{P}_i^e = (\mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdot \ldots \cdot \mathfrak{P}_r)^e$$

Moreover:

$$e \cdot f \cdot r = [F : K].$$