characterization of field

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Proposition 1. *Let $\mathcal{R} \neq 0$ be a commutative ring with identity. The ring $\mathcal{R}$ (as above) is a field if and only if $\mathcal{R}$ has exactly two ideals: $(0), \mathcal{R}$.*

Proof. ($\Rightarrow$) Suppose $\mathcal{R}$ is a field and let $\mathcal{A}$ be a non-zero ideal of $\mathcal{R}$. Then there exists $r \in \mathcal{A} \subseteq \mathcal{R}$ with $r \neq 0$. Since $\mathcal{R}$ is a field and r is a non-zero element, there exists $s \in \mathcal{R}$ such that

$$s \cdot r = 1 \in \mathcal{R}$$

Moreover, $\mathcal{A}$ is an ideal, $r \in \mathcal{A}, s \in \mathcal{S}$, so $s \cdot r = 1 \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{R}$. We have proved that the only ideals of $\mathcal{R}$ are (0) and $\mathcal{R}$ as desired.

($\Leftarrow$) Suppose the ring $\mathcal{R}$ has only two ideals, namely $(0), \mathcal{R}$. Let $a \in \mathcal{R}$ be a non-zero element; we would like to prove the existence of a multiplicative inverse for a in $\mathcal{R}$. Define the following set:

$$\mathcal{A} = (a) = \{r \in \mathcal{R} \mid r = s \cdot a, \text{ for some } s \in \mathcal{R}\}$$

This is clearly an ideal, the ideal generated by the element a. Moreover, this ideal is not the zero ideal because $a \in \mathcal{A}$ and a was assumed to be non-zero. Thus, since there are only two ideals, we conclude $\mathcal{A} = \mathcal{R}$. Therefore $1 \in \mathcal{A} = \mathcal{R}$ so there exists an element $s \in \mathcal{R}$ such that

$$s \cdot a = 1 \in \mathcal{R}$$

Hence for all non-zero $a \in \mathcal{R}$, a has a multiplicative inverse in $\mathcal{R}$, so $\mathcal{R}$ is, in fact, a field. $\square$