Ferrari-Cardano derivation of the quartic formula

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Given a quartic equation $x^4+ax^3+bx^2+cx+d=0$, apply the Tchirnhaus transformation $x \mapsto y - \frac{a}{4}$ to obtain

$$y^4 + py^2 + qy + r = 0 \tag{1}$$

where

$$\begin{aligned} p &= b - \frac{3a^2}{8} \\ q &= c - \frac{ab}{2} + \frac{a^3}{8} \\ r &= d - \frac{ac}{4} + \frac{a^2 b}{16} - \frac{3a^4}{256} \end{aligned}$$

Clearly a solution to Equation (**??**) solves the original, so we replace the original equation with Equation (**??**). Move $qy + r$ to the other side and complete the square on the left to get:

$$(y^2 + p)^2 = py^2 - qy + (p^2 - r).$$

We now wish to add the quantity $(y^2 + p + z)^2 - (y^2 + p)^2$ to both sides, for some unspecified value of z whose purpose will be made clear in what follows. Note that $(y^2 + p + z)^2 - (y^2 + p)^2$ is a quadratic in y. Carrying out this addition, we get

$$(y^2 + p + z)^2 = (p + 2z)y^2 - qy + (z^2 + 2pz + p^2 - r) \tag{2}$$

The goal is now to choose a value for z which makes the right hand side of Equation (**??**) a perfect square. The right hand side is a quadratic polynomial in y whose discriminant is

$$-8z^3 - 20pz^2 + (8r - 16p^2)z + q^2 + 4pr - 4p^3.$$

Our goal will be achieved if we can find a value for z which makes this discriminant zero. But the above polynomial is a cubic polynomial in z, so its roots can be found using the cubic formula. Choosing then such a value for z, we may rewrite Equation (**??**) as

$$(y^2 + p + z)^2 = (sy + t)^2$$

for some (complicated!) values s and t, and then taking the square root of both sides and solving the resulting quadratic equation in y provides a root of Equation (**??**).