example of an extension that is not normal

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In this entry, $\sqrt[3]{2}$ indicates the real cube root of 2.

Consider the extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. The minimal polynomial for $\sqrt[3]{2}$ over $\mathbb{Q}$ is x^3-2. This polynomial factors in $\mathbb{Q}(\sqrt[3]{2})$ as $x^3-2=(x-\sqrt[3]{2})(x^2+x\sqrt[3]{2}+\sqrt[3]{4})$. Let $f(x)=x^2+x\sqrt[3]{2}+\sqrt[3]{4}$. Note that $\operatorname{disc}(f(x))=(\sqrt[3]{2})^2-4\sqrt[3]{4}=\sqrt[3]{4}-4\sqrt[3]{4}=-3\sqrt[3]{4}<0$. Thus, $f(x)$ has no real roots. Therefore, $f(x)$ has no roots in $\mathbb{Q}(\sqrt[3]{2})$ since $\mathbb{Q}(\sqrt[3]{2})\subseteq\mathbb{R}$. Hence, x^3-2 has a root in $\mathbb{Q}(\sqrt[3]{2})$ but does not split in $\mathbb{Q}(\sqrt[3]{2})$. It follows that the extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal.