example of solving a cubic equation

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Let us use Cardano's formulae for solving algebraically the cubic equation

$$x^3+3x^2-1 \;=\; 0. \tag{1}$$

First apply the `http://planetmath.org/CardanosDerivationOfTheCubicFormula`Tschirnhaus transformation $x := y-1$ for removing the quadratic term; from $(y-1)^3 + 3(y-1)^2 - 1 = 0$ we get the simplified equation

$$y^3+3y-2 \;=\; 0. \tag{2}$$

We now suppose that $y := u+v$. Substituting this into (2) and rewriting the equation in the form

$$(u^3+v^3-2) + 3(uv+1)(u+v) \;=\; 0,$$

one can determine u and v such that $u^3+v^3-2 = 0$ and $uv+1 = 0$, i.e.

$$\begin{cases} u^3+v^3 \;=\; 2, \\ u^3v^3 \;=\; -1. \end{cases}$$

Using the properties of quadratic equation, we infer that u^3 and v^3 are the roots of the resolvent equation

$$z^2-2z-1 \;=\; 0.$$

Therefore, u and v satisfy the binomial equations

$$u^3 \;=\; 1+\sqrt{2}, \qquad v^3 \;=\; 1-\sqrt{2}, \tag{3}$$

respectively. If we choose the real radicals $u = u_0 = \sqrt[3]{1+\sqrt{2}}$ and $v = v_0 = \sqrt[3]{1-\sqrt{2}}$, the other solutions u, v of (3) are

$$\zeta u_0,\;\; \zeta^2 v_0; \qquad \zeta^2 u_0,\;\; \zeta v_0, \tag{4}$$

where $\zeta = \frac{-1+i\sqrt{3}}{2}$, $\zeta^2 = \frac{-1-i\sqrt{3}}{2}$ are the primitive third roots of unity. One must combine the pairs $(u,\, v)$ of (4) so that

$$uv \;=\; \sqrt[3]{u^3v^3} \;=\; -1.$$

Accordingly, all three roots of the cubic equation (2) are

$$\begin{cases} y_1 \;=\; u_0+v_0 \;=\; \sqrt[3]{1+\sqrt{2}}+\sqrt[3]{1-\sqrt{2}}, \\ y_2 \;=\; \zeta u_0+\zeta^2 v_0 \;=\; \frac{-1+i\sqrt{3}}{2}\sqrt[3]{1+\sqrt{2}}+\frac{-1-i\sqrt{3}}{2}\sqrt[3]{1-\sqrt{2}}, \\ y_3 \;=\; \zeta^2 u_0+\zeta v_0 \;=\; \frac{-1-i\sqrt{3}}{2}\sqrt[3]{1+\sqrt{2}}+\frac{-1+i\sqrt{3}}{2}\sqrt[3]{1-\sqrt{2}}. \end{cases}$$

The roots of the original equation (1) are gotten via the used substitution equation $x := y-1$, i.e. adding -1 to the values of y. If we also separate the `http://planetmath.org/RealPart`real and imaginary parts, we have the following solution of (1):

$$\begin{cases} x_1 \;=\; -1+\sqrt[3]{1+\sqrt{2}}+\sqrt[3]{1-\sqrt{2}}, \\ x_2 \;=\; -1-\frac{1}{2}\left(\sqrt[3]{1+\sqrt{2}}+\sqrt[3]{1-\sqrt{2}}\right)+i\frac{\sqrt{3}}{2}\left(\sqrt[3]{1+\sqrt{2}}-\sqrt[3]{1-\sqrt{2}}\right), \\ x_3 \;=\; -1-\frac{1}{2}\left(\sqrt[3]{1+\sqrt{2}}+\sqrt[3]{1-\sqrt{2}}\right)-i\frac{\sqrt{3}}{2}\left(\sqrt[3]{1+\sqrt{2}}-\sqrt[3]{1-\sqrt{2}}\right). \end{cases}$$

One of the roots is a real number, but the other two are (i.e. non-real) complex conjugates of each other.