p-adic exponential and p-adic logarithm

Canonical name	PadicExponentialAndPadicLogarithm
Date of creation	2013-03-22 15:13:50
Last modified on	2013-03-22 15:13:50
Owner	alozano (2414)
Last modified by	alozano (2414)
Numerical id	6
Author	alozano (2414)
Entry type	Definition
Classification	msc 12J12
Classification	msc 11S99
Classification	msc 11S80
Synonym	p-adic exponential
Synonym	p-adic logarithm
Related topic	PAdicRegulator
Related topic	PAdicAnalytic
Related topic	GeneralPower
Defines	general p-adic power

Let p be a prime number and let $\mathbb{C}_p$ be the field of http://planetmath.org/ComplexPAdicNumbe p-adic numbers.

Definition 1. *The p-adic exponential is a function* $\exp_p\colon R \to \mathbb{C}_p$ *defined by*

$$\exp_p(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!}$$

where

$$R = \{s \in \mathbb{C}_p : |s|_p < \frac{1}{p^{1/(p-1)}}\}.$$

The domain of $\exp_p$ is restricted because the radius of convergence of the series $\sum_{n=0}^{\infty} z^n/n!$ over $\mathbb{C}_p$ is precisely $r = p^{-1/(p-1)}$. Recall that, for $z \in \mathbb{Q}_p$, we define

$$|z|_p = \frac{1}{p^{\nu_p(z)}}$$

where $\nu_p(z)$ is the largest exponent ν such that p^ν divides z. For example, if $p \geq 3$, then $\exp_p$ is defined over $p\mathbb{Z}_p$. However, $e = \exp_p(1)$ is never defined, but $\exp_p(p)$ is well-defined over $\mathbb{C}_p$ (when $p = 2$, the number $e^4 \in \mathbb{C}_2$ because $|4|_2 = 0.25 < 0.5 = r$).

Definition 2. *The p-adic logarithm is a function* $\log_p\colon S \to \mathbb{C}_p$ *defined by*

$$\log_p(1+s) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{s^n}{n}$$

where

$$S = \{s \in \mathbb{C}_p : |s|_p < 1\}.$$

We extend the p-adic logarithm to the entire p-adic complex field $\mathbb{C}_p$ as follows. One can show that:

$$\mathbb{C}_p = \{p^t \cdot w \cdot u : t \in \mathbb{Q},\ w \in W,\ u \in U\} = p^{\mathbb{Q}} \times W \times U$$

where W is the group of all roots of unity of order prime to p in $\mathbb{C}_p^\times$ and U is the open circle of radius centered at $z = 1$:

$$U = \{s \in \mathbb{C}_p : |s-1|_p < 1\}.$$

We define $\log_p\colon \mathbb{C}_p \to \mathbb{C}_p$ *by:*

$$\log_p(s) = log_p(u)$$

where $s = p^r \cdot w \cdot u$, with $w \in W$ and $u \in U$.

Proposition (Properties of $\exp_p$ and $\log_p$). *With* $\exp_p$ *and* $\log_p$ *defined as above:*

1. *If* $\exp_p(s)$ *and* $\exp_p(t)$ *are defined then* $\exp_p(s+t) = \exp_p(s)\exp_p(t)$.
2. $\log_p(s) = 0$ *if and only if* s *is a rational power of* p *times a root of unity.*
3. $\log_p(xy) = \log_p(x) + \log_p(y)$, *for all* x *and* y.
4. *If* $|s|_p < p^{-1/(p-1)}$ *then*

$$\exp_p(\log_p(1+s)) = 1+s, \quad \log_p(\exp_p(s)) = s.$$

In a similar way one defines the general p-adic power by:

$$s^z = \exp_p(z \log_p(s))$$

where it makes sense.