ramification of archimedean places

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Defines	decomposition and inertia group for archimedean places
Defines	archimedean place
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Throughout this entry, if α is a complex number, we denote the complex conjugate of α by $\overline{\alpha}$.

Definition 1. *Let K be a number field.*

1. *An* **archimedean place** *of K is either a real embedding $\phi\colon K \to \mathbb{R}$ or a pair of complex-conjugate embeddings $(\psi, \overline{\psi})$, with $\overline{\psi} \neq \psi$ and $\psi\colon K \to \mathbb{C}$. The archimedean places are sometimes called the infinite places (cf. place of field).*

2. *The* **non-archimedean places** *of K are the prime ideals in $\mathcal{O}_K$, the ring of integers of K (see* `http://planetmath.org/Valuation`*non-archimedean valuation). The non-archimedean places are sometimes called the finite places.*

Notice that any archimedean place $\phi\colon K \to \mathbb{C}$ can be extended to an embedding $\hat{\phi}\colon \overline{\mathbb{Q}} \to \mathbb{C}$, where $\overline{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$ (in order to prove this, one uses the fact that $\mathbb{C}$ is algebraically closed and also Zorn's Lemma). See also `http://planetmath.org/PlaceAsExtensionOfHomomorphism`this entry. In particular, if F is a finite extension of K then ϕ can be extended to an archimidean place $\hat{\phi}\colon F \to \mathbb{C}$ of F.

Next, we define the decomposition and inertia group associated to archimedean places. For the case of non-archimedean places (i.e. prime ideals) see the entries decomposition group and ramification.

Let F/K be a finite Galois extension of number fields and let ϕ be a (real or a pair of complex) archimedean place of K. Let ϕ_1 and ϕ_2 be two archimedean places of F which extend ϕ. Notice that, since F/K is Galois, the image of ϕ_1 and ϕ_2 are equal, in other words:

$$\phi_1(F) = \phi_2(F) \subset \mathbb{C}.$$

Hence, the composition $\phi_1^{-1} \circ \phi_2$ is an automorphism of F (here ϕ_1^{-1} denotes the inverse map of ϕ_1, restricted to $\phi_1(F)$). Thus, $\phi_1^{-1} \circ \phi_2 = \sigma \in \mathrm{Gal}(F/K)$ and

$$\phi_2 = \phi_1 \circ \sigma$$

so ϕ_1 and ϕ_2 differ by an element of the Galois group. Similarly, if $(\psi_1, \overline{\psi_1})$ and $(\psi_2, \overline{\psi_2})$ are complex embeddings which extend ϕ, then there is $\sigma \in \mathrm{Gal}(F/K)$ such that

$$(\psi_2, \overline{\psi_2}) = (\psi_1, \overline{\psi_1}) \circ \sigma$$

meaning that either $\psi_2 = \psi_1 \circ \sigma$ (and thus $\overline{\psi_2} = \overline{\psi_1} \circ \sigma$) or $\overline{\psi_2} = \psi_1 \circ \sigma$ (and thus $\psi_2 = \overline{\psi_1} \circ \sigma$). We are ready now to make the definitions.

Definition 2. *Let F/K be a Galois extension of number fields and let w be an archimedean place of F lying above a place v of K. The decomposition and inertia subgroups for the pair $w|v$ are equal and are defined by:*

$$D(w|v) = T(w|v) = \{\sigma \in \mathrm{Gal}(F/K) : w \circ \sigma = w\}.$$

Let $e = e(w|v) = |T(w|v)|$ be the size of the inertia subgroup. If $e > 1$ then we say that the **archimedean place v is ramified** *in the extension F/K.*

The ramification in the archimedean case is much simpler than the non-archimedean analogue. One readily proves the following proposition:

Proposition 1. *The inertia subgroup $T(w|v)$ is nontrivial only when v is real, $w = (\psi, \overline{\psi})$ is a complex archimedean place of F and σ is the "complex conjugation" map which has order 2. Therefore $e(w|v) = 1$ or 2 and ramification of archimedean places occurs if and only if there is a complex place of F lying above a real place of K.*

Proof. Suppose first that $w = \phi \colon F \to \mathbb{R}$ is a real embedding. Then ϕ is injective and $\phi \circ \sigma = \phi$ implies that σ is the identity automorphism and $T(w|v)$ would be trivial. So let us assume that $w = (\psi, \overline{\psi})$ is a complex archimedean place and let $\sigma \in \mathrm{Gal}(F/K)$ such that

$$(\psi, \overline{\psi}) = (\psi, \overline{\psi}) \circ \sigma.$$

Therefore, either $\psi = \psi \circ \sigma$ (which implies that σ is the identity by the injectivity of ψ) or $\psi = \overline{\psi} \circ \sigma$. The latter implies that $\sigma = \overline{\psi^{-1}} \circ \psi$, which is simply complex conjugation:

$$\overline{\psi^{-1}} \circ \psi(k) = \overline{\psi^{-1}(\psi(k))} = \overline{k}.$$

Finally, since w is an extension of v, the equation $w \circ \sigma = w$ restricts to $\overline{v} = v$, thus v must be real. □

Corollary 1. *Suppose L/K is an extension of number fields and assume that K is a* `http://planetmath.org/TotallyRealAndImaginaryFields`*totally imaginary number field. Then the extension L/K is unramified at all archimedean places.*

Proof. Since K is totally imaginary none of the embeddings of K are real. By the proposition, only real places can ramify. □