complex p-adic numbers

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First, we review a possible construction of the complex numbers. We start from the rational numbers, $\mathbb{Q}$, which we consider as a metric space, where the distance is given by the usual absolute value $|\cdot|$, e.g. $|-3/2| = 3/2$. As we know, the field of rational numbers is not an algebraically closed field (e.g. $i = \sqrt{-1} \notin \mathbb{Q}$). Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$. The absolute value in $\mathbb{Q}$ extends uniquely to $\overline{\mathbb{Q}}$. However, $\overline{\mathbb{Q}}$ is not complete with respect to $|\cdot|$ (e.g. $e = \sum_{n\geq 0} 1/n! \notin \overline{\mathbb{Q}}$ because e is transcendental). The completion of $\overline{\mathbb{Q}}$ with respect to $|\cdot|$ is $\mathbb{C}$, the field of complex numbers.

Construction of $\mathbb{C}_p$

We follow the construction of $\mathbb{C}$ above to build $\mathbb{C}_p$. Let p be a prime number and let $\mathbb{Q}_p$ be the `http://planetmath.org/PAdicIntegers`p-adic rationals or (p-adic numbers). The p-adics, $\mathbb{Q}_p$, are the completion of $\mathbb{Q}$ with respect to the usual `http://planetmath.org/PAdicValuation`p-adic valuation $|\cdot|_p$. Thus, we regard $(\mathbb{Q}_p, |\cdot|_p)$ as a complete metric space. However, the field $\mathbb{Q}_p$ is not algebraically closed (e.g. $i = \sqrt{-1} \in \mathbb{Q}_p$ if and only if $p \equiv 1 \mod 4$). Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$. The p-adic valuation $|\cdot|_p$ extends uniquely to $\overline{\mathbb{Q}}_p$. However:

Proposition. *The field $\overline{\mathbb{Q}}_p$ is not complete with respect to $|\cdot|_p$.*

Proof. Let β_n be defined as:

$$\beta_n = \begin{cases} e^{2\pi i/n}, \text{ if } (n,p) = 1; \\ 1, \text{ otherwise.} \end{cases}$$

One can prove that if we define:

$$\alpha = \sum_{n=1}^{\infty} \beta_n p^n$$

then $\alpha \notin \overline{\mathbb{Q}}_p$, although $\sum_{n=m}^{\infty} \beta_n p^n \to 0$ as $m \to \infty$ (see [**?**], p. 48, for details). Thus, $\overline{\mathbb{Q}}_p$ is not complete with respect to $|\cdot|_p$. □

Definition. *The field of complex p-adic numbers is defined to be the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic absolute value $|\cdot|_p$.*

Proposition (Properties of $\mathbb{C}_p$). *The field $\mathbb{C}_p$ enjoys the following properties:*

1. *$\mathbb{C}_p$ is algebraically closed.*
2. *The absolute value $|\cdot|_p$ extends uniquely to $\mathbb{C}_p$, which becomes an algebraically closed, complete metric space.*
3. *$\mathbb{C}_p$ is a complete ultrametric field.*
4. *$\overline{\mathbb{Q}}_p$ is dense in $\mathbb{C}_p$.*
5. *$\mathbb{C}_p$ is isomorphic to $\mathbb{C}$ as fields, although they are not isomorphic as topological spaces.*

References

[1] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer-Verlag, New York.