proof of the fundamental theorem of algebra (Liouville's theorem)

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Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial, and suppose f has no root in $\mathbb{C}$. We will show f is constant.

Let $g = \frac{1}{f}$. Since f is never zero, g is defined and holomorphic on $\mathbb{C}$ (ie. it is entire). Moreover, since f is a polynomial, $|f(z)| \to \infty$ as $|z| \to \infty$, and so $|g(z)| \to 0$ as $|z| \to \infty$. Then there is some $M > 0$ such that $|g(z)| < 1$ whenever $|z| > M$, and g is continuous and so bounded on the compact set $\{z \in \mathbb{C} : |z| \leq M\}$.

So g is bounded and entire, and therefore by Liouville's theorem g is constant. So f is constant as required.$\square$