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examples of fields

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http://planetmath.org/FieldFields are typically sets of "numbers" in which the arithmetic operations of addition, subtraction, multiplication and division are defined. The following is a list of examples of fields.

- The set of all rational numbers $\mathbb{Q}$, all real numbers $\mathbb{R}$ and all complex numbers $\mathbb{C}$ are the most familiar examples of fields.
- Slightly more exotic, the hyperreal numbers and the surreal numbers are fields containing infinitesimal and infinitely large numbers. (The surreal numbers aren't a field in the strict sense since they form a proper class and not a set.)
- The *algebraic numbers* form a field; this is the algebraic closure of $\mathbb{Q}$. In general, every field has an (essentially unique) algebraic closure.
- The computable complex numbers (those whose digit sequence can be produced by a Turing machine) form a field. The definable complex numbers (those which can be precisely specified using a logical formula) form a field containing the computable numbers; arguably, this field contains all the numbers we can ever talk about. It is countable.
- The so-called *algebraic number fields* (sometimes just called number fields) arise from $\mathbb{Q}$ by adjoining some (finite number of) algebraic numbers. For instance $\mathbb{Q}(\sqrt{2}) = \{u + v\sqrt{2} \mid u, v \in \mathbb{Q}\}$ and $\mathbb{Q}(\sqrt[3]{2}, i) = \{u + vi + w\sqrt[3]{2} + xi\sqrt[3]{2} + y\sqrt[3]{4} + zi\sqrt[3]{4} \mid u, v, w, x, y, z \in \mathbb{Q}\} = \mathbb{Q}(i\sqrt[3]{2})$ (every separable finite field extension is simple).
- If p is a prime number, then the *p-adic numbers* form a field $\mathbb{Q}_p$ which is the completion of the field $\mathbb{Q}$ with respect to the p-adic valuation.
- If p is a prime number, then the integers modulo p form a finite field with p elements, typically denoted by $\mathbb{F}_p$. More generally, for every http://planetmath.org/Primeprime power p^n there is one and only one finite field $\mathbb{F}_{p^n}$ with p^n elements.
- If K is a field, we can form the field of rational functions over K, denoted by $K(X)$. It consists of quotients of polynomials in X with coefficients in K.

- If V is a http://planetmath.org/AffineVarietyvariety over the field K, then the function field of V, denoted by $K(V)$, consists of all quotients of polynomial functions defined on V.

- If U is a domain (= connected open set) in $\mathbb{C}$, then the set of all meromorphic functions on U is a field. More generally, the meromorphic functions on any Riemann surface form a field.

- If X is a variety (or scheme) then the rational functions on X form a field. At each point of X, there is also a residue field which contains information about that point.

- The field of formal Laurent series over the field K in the variable X consists of all expressions of the form

$$\sum_{j=-M}^{\infty} a_j X^j$$

  where M is some integer and the coefficients a_j come from K.

- More generally, whenever R is an integral domain, we can form its field of fractions, a field whose elements are the fractions of elements of R.

Many of the fields described above have some sort of additional structure, for example a topology (yielding a topological field), a total order, or a canonical absolute value.