proof of fundamental theorem of symmetric polynomials

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Let $P := P(x_1, x_2, \ldots, x_n)$ be an arbitrary symmetric polynomial in $x_1, x_2, \ldots, x_n$. We can assume that P is http://planetmath.org/Polynomialhomogeneous, because if $P = P_1+P_2+\ldots+P_m$ where each P_i is homogeneous and if the http://planetmath.org/FundamentalTheoremOfSymmetricPolynomialstheorem is true for each P_i, it is evidently true for the sum P, too.

Let the http://planetmath.org/Polynomialdegree of P be d. For any two terms

$$M := c_\mu x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}, \qquad N := c_\nu x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$$

of P, if the first of the differences

$$\mu_1-\nu_1, \quad \mu_2-\nu_2, \quad \ldots, \quad \mu_n-\nu_n,$$

which differs from 0, is positive, we say that M is *higher* than N. Since, of cource, the terms of P have been merged, always one of two arbitrary terms is higher than the other. The higherness is obviously http://planetmath.org/Transitive3transitiv
Thus there is a certain *highest* term

$$A := c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

in P. Then we have

$$\alpha_1+\alpha_2+\ldots+\alpha_n = d,$$
$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n.$$

In fact, if e.g. $\alpha_2 > \alpha_1$, then the term

$$c_\alpha x_2^{\alpha_1} x_1^{\alpha_2} x_3^{\alpha_3} \cdots x_n^{\alpha_n} = c_\alpha x_1^{\alpha_2} x_2^{\alpha_1} x_3^{\alpha_3} \cdots x_n^{\alpha_n},$$

which is obtained from A by changing x_1 and x_2 with each other, would be higher than A.

For proving the http://planetmath.org/FundamentalTheoremOfSymmetricPolynomialsfund
theorem, we form now the homogeneous polynomial

$$Q_\alpha := c_\alpha p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n}$$

and we will show that the http://planetmath.org/Exponentiationexponents i_j can be determined such that the highest term in Q_α is same as in P.

It is easily seen that the highest term of a product of homogeneous symmetric polynomials is equal to the product of the highest terms of the factors. Since the highest term of

$$p_1 \quad \text{is} \quad x_1,$$

$$p_2 \quad \text{is} \quad x_1x_2,$$
$$\ldots \qquad \ldots$$
$$p_n \quad \text{is} \quad x_1x_2\cdots x_n,$$

therefore the highest term of

$$p_1^{i_1} \quad \text{is} \quad x_1^{i_1},$$
$$p_2^{i_2} \quad \text{is} \quad x_1^{i_2}x_2^{i_2},$$
$$\ldots \qquad \ldots$$
$$p_n^{i_n} \quad \text{is} \quad x_1^{i_n}x_2^{i_n}\cdots x_n^{i_n}$$

and thus the highest term of Q_α is

$$c_\alpha x_1^{i_1+i_2+\cdots+i_n}x_2^{i_2+\cdots+i_n}\cdots x_n^{i_n}.$$

This term coincides with the highest term of P, when one determines the numbers i_j from the equations

$$\begin{cases} i_1+i_2+\ldots+i_n \;=\; \alpha_1 \\ i_2+\ldots+i_n \;=\; \alpha_2 \\ \qquad\quad \ldots \qquad \ldots \\ i_n \;=\; \alpha_n. \end{cases}$$

Subtracting here the second equation from the first, the third equation from the second and so on, the result is

$$i_1 \;=\; \alpha_1-\alpha_2, \qquad i_2 \;=\; \alpha_2-\alpha_3, \qquad \ldots, \qquad i_{n-1} \;=\; \alpha_{n-1}-\alpha_n, \quad i_n \;=\; \alpha_n,$$

which are nonnegative integers. Hence we get the homogeneous symmetric polynomial

$$Q_\alpha \;=\; c_\alpha p_1^{\alpha_1-\alpha_2}p_2^{\alpha_2-\alpha_3}\cdots p_{n-1}^{\alpha_{n-1}-\alpha_n}p_n^{\alpha_n}$$

having the same highest term as P, and consequently the difference

$$P-Q_\alpha \;:=\; P_\alpha$$

is a homogeneous symmetric polynomial of degree d having the highest term lower than in P. If then

$$c_\beta x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n}$$

is the highest term of P_α and one denotes

$$Q_\beta \;:=\; c_\beta p_1^{\beta_1-\beta_2} p_2^{\beta_2-\beta_3} \cdots p_{n-1}^{\beta_{n-1}-\beta_n} p_n^{\beta_n},$$

one infers as above that the difference

$$P_\alpha - Q_\beta \;:=\; P_\beta$$

is a homogeneous symmetric polynomial of degree d having the highest term lower than in P_α. Continuing similarly, one finally (after a finite amount of steps) shall come to a difference which is equal to 0. Accordingly one obtains

$$P \;=\; Q_\alpha + Q_\beta + \ldots + Q_\omega \;:=\; Q(p_1,\, p_2,\, \ldots,\, p_n).$$

The degree of Q_α with respect to the elementary symmetric polynomials is

$$(\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3) + \ldots + (\alpha_{n-1} - \alpha_n) + \alpha_n \;=\; \alpha_1.$$

Similarly, the degree of Q_β is β_1 which is $\leq \alpha_1$; thus one infers that the degree of Q is equal to α_1. This number is also the degree of the highest term of P and as well the degree of P itself, with respect to x_1.

The preceding construction implies immediately that the coefficients of G are elements of the ring determined by the coefficients of P. We have still to prove the uniqueness of Q. Let's make the antithesis that P may be represented also by another polynomial in $p_1,\, p_2,\, \ldots,\, p_n$ which differs from Q. Forming the difference of it and Q we get an equation of the form

$$0 \;=\; \sum_i g_i p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n},$$

where the coefficients are distinct from zero. The equation becomes identical if one expresses $p_1,\, p_2,\, \ldots,\, p_n$ in it via the indeterminates $x_1,\, x_2,\, \ldots,\, x_n$. The general term of the right hand side of the equation is a homogeneous symmetric polynomial in those indeterminates; if its highest term is $g_i p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_n^{\lambda_n}$, one infers as before that

$$i_1 \;=\; \lambda_1 - \lambda_2, \qquad i_2 \;=\; \lambda_2 - \lambda_3, \qquad \ldots, \qquad i_{n-1} \;=\; \lambda_{n-1} - \lambda_n, \qquad i_n \;=\; \lambda_n.$$

Thus, distinct addends of the sum cannot have equal highest terms. It means that the highest term of the sum appears only in one of the addends of the sum. This is, however, impossible, because after the substitution of x_is the equation would not be identical. Consequently, the antithesis is wrong and the whole fundamental theorem of symmetric polynomials has been proved.

References

[1] K. Väisälä: *Lukuteorian ja korkeamman algebran alkeet.* Tiedekirjasto No. 17. Kustannusosakeyhtiö Otava, Helsinki (1950).