examples of continuous functions on the extended real numbers

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Within this entry, $\overline{\mathbb{R}}$ will be used to refer to the extended real numbers. Examples of continuous functions on $\overline{\mathbb{R}}$ include:

- Polynomial functions: Let $f \in \mathbb{R}[x]$ with $f(x) = \displaystyle\sum_{j=0}^{n} a_n x^n$ for some $n \in \mathbb{N}$ and $a_0, \dots, a_n \in \mathbb{R}$ with $a_n \neq 0$ if $n \neq 0$. Then $\overline{f}$ is defined in the following manner:

  1. If $n = 0$, then $\overline{f}(x) = a_0$ for all $x \in \overline{\mathbb{R}}$.
  2. If n is odd and $a_n > 0$, then $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ x & \text{if } x \notin \mathbb{R}. \end{cases}$
  3. If n is odd and $a_n < 0$, then $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ -x & \text{if } x \notin \mathbb{R}. \end{cases}$
  4. If $n \neq 0$ is even and $a_n > 0$, then $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ \infty & \text{if } x \notin \mathbb{R}. \end{cases}$
  5. If $n \neq 0$ is even and $a_n < 0$, then $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ -\infty & \text{if } x \notin \mathbb{R}. \end{cases}$

- Exponential functions: Let $f(x) = a^x$ for some $a \in \mathbb{R}$ with $a > 0$ and $a \neq 1$. Then $\overline{f}$ is defined in the following manner:

  1. If $a < 1$, then $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ 0 & \text{if } x = \infty \\ \infty & \text{if } x = -\infty. \end{cases}$
  2. If $a > 1$, then $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ \infty & \text{if } x = \infty \\ 0 & \text{if } x = -\infty. \end{cases}$

- Miscellaneous

1. Let $f(x) = \arctan x$. Then $\overline{f}$ is defined by $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ \dfrac{\pi}{2} & \text{if } x = \infty \\ -\dfrac{\pi}{2} & \text{if } x = -\infty. \end{cases}$

2. Let $f(x) = \tanh x$. Then $\overline{f}$ is defined by $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ 1 & \text{if } x = \infty \\ -1 & \text{if } x = -\infty. \end{cases}$

Of course, not every function f that is continuous on $\mathbb{R}$ extends to a continuous function on $\overline{\mathbb{R}}$. Common examples of these include the real functions $x \mapsto \sin x$ and $x \mapsto \cos x$. (It is proven that these are continuous on $\mathbb{R}$ in the entry continuity of sine and cosine.)

On the other hand, there are some continuous functions $\overline{f} \colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ that have no analogous function $f \colon \mathbb{R} \to \mathbb{R}$. For example, consider

$$\overline{f}(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if } x = 0 \\ 0 & \text{if } x = \pm\infty. \end{cases}$$