proof of factor theorem due to Fermat

Canonical name	ProofOfFactorTheoremDueToFermat
Date of creation	2013-03-22 15:40:12
Last modified on	2013-03-22 15:40:12
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Last modified by	pahio (2872)
Numerical id	10
Author	pahio (2872)
Entry type	Proof
Classification	msc 12D10
Classification	msc 12D05
Synonym	proof of factor theorem without division

Lemma (cf. factor theorem). If the polynomial

$$f(x) := a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

vanishes at $x = c$, then it is divisible by the difference $x - c$, i.e. there is valid the identic equation

$$f(x) \equiv (x-c)q(x) \tag{1}$$

where $q(x)$ is a polynomial of degree $n-1$, beginning with the a_0x^{n-1}.

The lemma is here proved by using only the properties of the multiplication and addition, not the division.

Proof. If we denote $x-c = y$, we may write the given polynomial in the form

$$f(x) = a_0(y+c)^n + a_1(y+c)^{n-1} + \cdots + a_{n-1}(y+c) + a_n.$$

It's clear that every $(y+c)^k$ is a polynomial of degree k with respect to y, where y^k has the coefficient 1 and the is c^k. This implies that $f(x)$ may be written as a polynomial of degree n with respect to y, where y^n has the coefficient a_0 and the on y is equal to $a_0c^n + a_1c^{n-1} + \cdots + a_{n-1}c + a_n$, i.e. $f(c)$. So we have

$$f(x) = a_0y^n + b_1y^{n-1} + b_2y^{n-2} + \cdots + b_{n-1}y + f(c) = f(c) + y\cdot(a_0y^{n-1} + b_1y^{n-2} + \cdots + b_{n-1} + a_n),$$

where $b_1,\, b_2,\, \ldots,\, b_{n-1}$ are certain coefficients. If we return to the indeterminate x by substituting in the last identic equation $x-c$ for y, we get

$$f(x) \equiv f(c) + (x-c)[a_0(x-c)^{n-1} + b_1(x-c)^{n-2} + \cdots + b_{n-1}].$$

When the powers $(x-c)^k$ are expanded to polynomials, we see that the expression in the brackets is a polynomial $q(x)$ of degree $n-1$ with respect to x and with the coefficient a_0 of x^{n-1}. Thus we obtain

$$f(x) \equiv f(c) + (x-c)q(x). \tag{2}$$

This result is true independently on the value of c. If this value is chosen such that $f(c) = 0$, then (2) reduces to (1), Q. E. D.

References

[1] Ernst Lindelöf: *Johdatus korkeampaan analyysiin* ('Introduction to Higher Analysis'). Fourth edition. WSOY, Helsinki (1956).