field extension with Galois group Q_8

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Let $\alpha = \sqrt{(2+\sqrt{2})(3+\sqrt{3})}, E = \mathbb{Q}(\alpha)$. We will show that E is Galois over $\mathbb{Q}$ and that $G = Gal(E/\mathbb{Q}) \cong Q_8$ (the group of quaternions).

We begin by showing that $[E : \mathbb{Q}] = 8$. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Claim that $F \subsetneq E$. To show that they are not equal, we show that $\alpha \notin F$, i.e. that $(2+\sqrt{2})(3+\sqrt{3})$ is not a square in F. If it were, say $(2+\sqrt{2})(3+\sqrt{3}) = c^2, c \in F$, take $\sigma \in Gal(F/\mathbb{Q})$ to be the element

$$\sigma : \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \end{cases}$$

Then $(2+\sqrt{2})(3+\sqrt{3})\sigma((2+\sqrt{2})(3+\sqrt{3})) = (c\sigma(c))^2$, so

$$(2+\sqrt{2})^2(3+\sqrt{3})(3-\sqrt{3}) = 6(2+\sqrt{2})^2 = (c\sigma(c))^2$$

But $c\sigma(c) = \mathrm{Tr}_{F/\mathbb{Q}(\sqrt{2})}(c) \in \mathbb{Q}(\sqrt{2})$, and thus $6 = \left(\frac{c\sigma(c)}{2+\sqrt{2}}\right)^2$, so $\sqrt{6} \in \mathbb{Q}(\sqrt{2})$, a contradiction. Thus $F \neq E$. We show that $F \subset E$ by showing that $\sqrt{2} + \sqrt{3} \in E$.

$$\begin{gathered}
(2+\sqrt{2})(3+\sqrt{3}) = 6 + 3\sqrt{2} + 2\sqrt{3} + \sqrt{6} \in E, \text{ so} \\
3\sqrt{2} + 2\sqrt{3} + \sqrt{6} \in E, \text{ so} \\
(3\sqrt{2} + 2\sqrt{3} + \sqrt{6})^2 = 36 + 12(\sqrt{2} + \sqrt{3} + \sqrt{6}) \in E, \text{ so} \\
\sqrt{2} + \sqrt{3} + \sqrt{6} \in E, \text{ so} \\
3\sqrt{2} + 2\sqrt{3} + \sqrt{6} - \sqrt{2} - \sqrt{3} - \sqrt{6} = \sqrt{3} + 2\sqrt{2} \in E, \text{ so} \\
(\sqrt{3} + 2\sqrt{2})^2 = 11 + 4\sqrt{6} \in E, \text{ so} \\
\sqrt{6} \in E, \text{ so} \\
\sqrt{2} + \sqrt{3} + \sqrt{6} - \sqrt{6} = \sqrt{2} + \sqrt{3} \in E
\end{gathered}$$

So $F \subsetneq E$ and thus $[E : F] = 2$. Then $[E : \mathbb{Q}] = [E : F][F : \mathbb{Q}] = 8$.

Now, the irreducible polynomial $f(x)$ for $(2+\sqrt{2})(3+\sqrt{3})$ over $\mathbb{Q}$ is the product of $x - \tau((2+\sqrt{2})(3+\sqrt{3}))$ as τ ranges over $\mathrm{Gal}(F/\mathbb{Q})$:

$$f(x) = (x-(2+\sqrt{2})(3+\sqrt{3}))(x-(2-\sqrt{2})(3+\sqrt{3}))(x-(2+\sqrt{2})(3-\sqrt{3}))(x-(2-\sqrt{2})(3-\sqrt{3}))$$

so that $f(x^2)$ is a degree 8 polynomial with α as a root. In fact,

$$f(x^2) = x^8 - 24x^6 + 48x^4 - 288x^2 + 144$$

This polynomial must be irreducible since α is of degree 8, so $f(x^2)$ is the minimal polynomial for α over $\mathbb{Q}$. The roots of $f(x^2)$ are obviously

$$\pm\sqrt{(2\pm\sqrt{2})(3\pm\sqrt{3})}$$

Furthermore, it is easy to see that each of these roots lies in E, for

$$\begin{aligned}
\alpha\sqrt{(2-\sqrt{2})(3+\sqrt{3})} &= \sqrt{2}(3+\sqrt{3}) \in F \\
\alpha\sqrt{(2+\sqrt{2})(3-\sqrt{3})} &= \sqrt{6}(2+\sqrt{2}) \in F \\
\alpha\sqrt{(2-\sqrt{2})(3-\sqrt{3})} &= \sqrt{2}\sqrt{6} = 2\sqrt{3} \in F
\end{aligned}$$

so dividing through by α we see that

$$\sqrt{(2-\sqrt{2})(3+\sqrt{3})}, \sqrt{(2+\sqrt{2})(3-\sqrt{3})}, \sqrt{(2-\sqrt{2})(3-\sqrt{3})} \in E$$

Thus E is in fact Galois over $\mathbb{Q}$, is the splitting field for $f(x^2)$, and has Galois group $G = \mathrm{Gal}(E/\mathbb{Q})$ of `http://planetmath.org/OrderGroup`order 8.

G acts transitively on the roots of $f(x^2)$, and $E = \mathbb{Q}(\alpha)$, so an element of G is determined by the image of α. Thus the elements of G are the automorphisms of E that map α to any of the eight roots of $f(x^2)$. Let

$$\begin{aligned}
\alpha &= \sqrt{(2+\sqrt{2})(3+\sqrt{3})} \qquad & \beta &= \sqrt{(2-\sqrt{2})(3+\sqrt{3})} \\
\gamma &= \sqrt{(2+\sqrt{2})(3-\sqrt{3})} \qquad & \delta &= \sqrt{(2-\sqrt{2})(3-\sqrt{3})}
\end{aligned}$$

and let $\sigma : \alpha \mapsto \beta, \tau : \alpha \mapsto \gamma$ be elements of G.

$\sigma(\alpha^2) = \beta^2$, so $\sigma(2+\sqrt{2})\sigma(3+\sqrt{3}) = (2-\sqrt{2})(3+\sqrt{3})$. This is an equation in F, so regarding σ as an automorphism of $F/\mathbb{Q}$, it must be the automorphism $\sqrt{2} \mapsto -\sqrt{2}, \sqrt{3} \mapsto \sqrt{3}$. Since $\alpha\beta = \sqrt{2}(3+\sqrt{3})$, we have $\sigma(\alpha\beta) = -\alpha\beta$ and thus that $\sigma(\beta) = -\alpha$. It follows that σ is an element of `http://planetmath.org/OrderGroup`order 4 in G.

Similarly, $\tau(\alpha^2) = \gamma^2$, so $\tau(2+\sqrt{2})\tau(3+\sqrt{3}) = (2+\sqrt{2})(3-\sqrt{3})$, so that τ, regarded as an automorphism of $F/\mathbb{Q}$, must be $\sqrt{2} \mapsto \sqrt{2}, \sqrt{3} \mapsto -\sqrt{3}$. Since $\alpha\gamma = \sqrt{6}(2+\sqrt{2})$, we have $\tau(\alpha\gamma) = -\alpha\gamma$, so that $\tau(\gamma) = -\alpha$, and τ is also an element of `http://planetmath.org/OrderGroup`order 4 in G. Note also that $\sigma^2(\alpha) = -\alpha = \tau^2(\alpha)$, so that $\sigma^2 = \tau^2 \neq 1$.

Looking at $\sigma\tau$,

$$\sigma\tau(\alpha) = \sigma(\gamma) = \sigma\left(\frac{\alpha\gamma}{\alpha}\right) = \frac{\sigma(\sqrt{6}(2+\sqrt{2}))}{\sigma(\alpha)} = \frac{-\sqrt{6}(2-\sqrt{2})}{\beta} = -\frac{\beta\delta}{\beta} = -\delta$$

while

$$\tau\sigma(\alpha) = \tau(\beta) = \tau\left(\frac{\alpha\beta}{\alpha}\right) = \frac{\tau(\sqrt{2}(3+\sqrt{3}))}{\tau(\alpha)} = \frac{\sqrt{2}(3-\sqrt{3})}{\gamma} = \frac{\gamma\delta}{\gamma} = \delta$$

and thus $\tau\sigma^3(\alpha) = \tau\sigma\sigma^2(\alpha) = -\tau\sigma(\alpha) = -\delta = \sigma\tau(\alpha)$. So $\sigma\tau = \tau\sigma^3$.

Putting this all together, we see that G is generated by σ, τ, and that the generators satisfy the relations

$$\sigma^4 = \tau^4 = 1, \qquad \sigma^2 = \tau^2 \neq 1, \qquad \sigma\tau = \tau\sigma^3$$

Define $\varphi : G \to Q_8$ by $\varphi(\sigma) = i, \varphi(\tau) = j$. This is easily seen to be a homomorphism, and $\varphi(\sigma\tau) = ij = k$, so φ is surjective and is thus an isomorphism since both groups have `http://planetmath.org/OrderGroup`order 8. Thus $\mathrm{Gal}(E/\mathbb{Q}) \cong Q_8$.