Archimedean ordered fields are real

Canonical name	ArchimedeanOrderedFieldsAreReal
Date of creation	2013-03-22 17:26:22
Last modified on	2013-03-22 17:26:22
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	12
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Entry type	Theorem
Classification	msc 12D99
Related topic	GelfandTornheimTheorem

In this entry, we shall show that every Archimedean ordered field is isomorphic to a subfield of the field of real numbers. To accomplish this, we shall construct an isomorphism using Dedekind cuts.

As a preliminary, we agree to some conventions. Let $\mathbb{F}$ denote an ordered field with ordering relation $>$. We will identify the integers with multiples of the multiplicative identity of fields. We further assume that $\mathbb{F}$ satisfies the Archimedean property: for every element x of $\mathbb{F}$, there exists an integer n such that $x < n$.

Since $\mathbb{F}$ is an ordered field, it must have characteristic zero. Hence, the subfield generated by the multiplicative identity is isomorphic to the field of rational numbers. Following the convention proposed above, we will identify this subfield with $\mathbb{Q}$. We use this subfield to construct a map ρ from $\mathbb{F}$ to $\mathbb{R}$:

Definition 1. *For every $x \in \mathbb{F}$, we define*

$$\rho(x) = (\{y \in \mathbb{Q} \mid y > x\}, \{y \in \mathbb{Q} \mid y \leq x\})$$

Theorem 1. *For every $x \in \mathbb{F}$, we find that $\rho(x)$ is a Dedekind cut.*

Proof. Because, for all $y \in \mathbb{Q}$, we have either $y > x$ or $y \leq x$, the two sets of $\rho(x)$ form a partition of $\mathbb{Q}$. Furthermore, every element of the latter set is less than every element of the former set. By the Archimedean property, there exists an integer n such that $x < n$; hence the former set is not empty. Likewise, there exists and integer m such that $-x < m$, or $x > -m$, so the latter set is also not empty. □

Having seen that ρ is a bona fide map into the real numbers, we now show that it is not just any old map, but a monomorphism of fields.

Theorem 2. *The map $\rho\colon \mathbb{F} \to \mathbb{R}$ is a monomorphism.*

Proof. Let p and q be elements of $\mathbb{F}$; set $(A, B) = \rho(p)$, set $(C, D) = \rho(q)$, and set $(E, F) = \rho(p + q)$. Since $a > p$ and $b > q$ implies $a + b > p + q$ for all rational numbers a and b, it follows that $a \in A$ and $b \in C$ implies that $a + b \in E$. Likewise, since $a \leq p$ and $b \leq q$ implies $a + b \leq p + q$ for all rational numbers a and b, it follows that $a \in B$ and $b \in D$ implies $a + b \in F$. Hence, $\rho(p) + \rho(q) = \rho(p + q)$.

Since a rational number is positive if and only if it is greater than 0, it follows that $\rho(0) = 0$. Together with the fact proven in the last paragraph, this implies that $\rho(-x) = -\rho(x)$ for all $x \in \mathbb{F}$.

Suppose that p and q are positive elements of $\mathbb{F}$. As before, set $(A, B) = \rho(p)$, set $(C, D) = \rho(q)$, and set $(E, F) = \rho(p \cdot q)$. Since $a > p$ and $b > q$ implies $a \cdot b > p \cdot q$ for all rational numbers a and b, it follows that $a \in A$ and $b \in C$ implies that $a \cdot b \in E$. Likewise, since $a \leq p$ and $b \leq q$ implies $a \cdot b \leq p \cdot q$ for all rational numbers a and b, it follows that $a \in B$ and $b \in D$ implies $a \cdot b \in F$. Hence, $\rho(p) \cdot \rho(q) = \rho(p \cdot q)$.

By using the fact demonstrated previously that $\rho(-x) = -\rho(x)$, we may extend what was shown above to the statement that $\rho(p \cdot q) = \rho(p) \cdot \rho(q)$ for all $p, q \in \mathbb{F}$. Thus, ρ is a morphism of fields. Since $\mathbb{F}$ is Archmiedean, if $p \neq q$, there must exist a rational number r between p and q, hence $\rho(p) \neq \rho(q)$, so ρ is a monomorphism. $\square$

Since ρ is a morphism of fields, its image is a subring of $\mathbb{R}$. Since ρ is a monomorphism, its restriction to this image is an isomorphism, hence $\mathbb{F}$ is isomorphic to a subfield of $\mathbb{R}$.