real closed fields are o-miminal

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It is clear that the axioms for a structure to be an ordered field can be written in L, the first order language of ordered rings. It is also true that the condition

for each odd degree polynomial $p \in K[x]$, p has a root

can be written in a schema of first order sentences in this language.

Let A be all these sentences together with one that states that all positive elements have a square root. Then one can show that the consequences of A are a complete theory T. It is clear that this theory is the theory of the real numbers. We call any L structure a *real closed field* (which can be defined purely algebraically also, see `http://planetmath.org/RealClosed`here).

The *semi algebraic sets* on a real closed field are Boolean combinations of solution sets of polynomial equalities and inequalities. Tarski showed that T has quantifier elimination, which is equivalent to the class of semi algebraic sets being closed under projection.

Let K be a real closed field. Consider the definable subsets of K. By quantifier elimination, each is definable by a quantifier free formula, i.e. a boolean combination of atomic formulas. An atomic formula in one variable has one of the following forms:

- $f(x) > g(x)$ for some $f, g \in K[x]$
- $f(x) = g(x)$ for some $f, g \in K[x]$.

The first defines a finite union of intervals, the second defines a finite union of points. Every definable subset of K is a finite union of these kinds of sets, so is a finite union of intervals and points. Thus any real closed field is o-minimal.