continuous functions on the extended real numbers

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Within this entry, $\overline{\mathbb{R}}$ will be used to refer to the extended real numbers.

Theorem 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Then $\overline{f}\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ defined by*

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ A & \text{if } x = \infty \\ B & \text{if } x = -\infty \end{cases}$$

is continuous if and only if f is continuous such that $\lim\limits_{x \to \infty} f(x) = A$ and $\lim\limits_{x \to -\infty} f(x) = B$ for some $A, B \in \overline{\mathbb{R}}$.

Proof. Note that $\overline{f}$ is continuous if and only if $\lim\limits_{x \to c} \overline{f}(x) = \overline{f}(c)$ for all $c \in \overline{\mathbb{R}}$. By defintion of $\overline{f}$ and the topology of $\overline{\mathbb{R}}$, $\lim\limits_{x \to c} \overline{f}(x) = \lim\limits_{x \to c} f(x)$ for all $c \in \overline{\mathbb{R}}$. Thus, $\overline{f}$ is continuous if and only if $\lim\limits_{x \to c} f(x) = \overline{f}(c)$ for all $c \in \overline{\mathbb{R}}$. The latter condition is `http://planetmath.org/Equivalent3`equivalent to the hypotheses that f is continuous on $\mathbb{R}$, $\lim\limits_{x \to \infty} f(x) = A$, and $\lim\limits_{x \to -\infty} f(x) = B$. $\square$

Note that, without the universal assumption that f is a function from $\mathbb{R}$ to $\mathbb{R}$, necessity holds, but sufficiency does not. As a counterexample to sufficiency, consider the function $\overline{f}\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\overline{f}(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if } x = 0 \\ 0 & \text{if } x = \pm\infty. \end{cases}$$