integral element

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An element a of a field K is an *integral element of the field* K, iff

$$|a| \leq 1$$

for every non-archimedean valuation $|\cdot|$ of this field.

The set $\mathcal{O}$ of all integral elements of K is a subring (in fact, an integral domain) of K, because it is the intersection of all valuation rings in K.

Examples

1. $K = \mathbb{Q}$. The only non-archimedean valuations of $\mathbb{Q}$ are the p-adic valuations $|\cdot|_p$ (where p is a rational prime) and the trivial valuation (all values are 1 except the value of 0). The valuation ring $\mathcal{O}_p$ of $|\cdot|_p$ consists of all so-called *p-integral rational numbers* whose denominators are not divisible by p. The valuation ring of the trivial valuation is, generally, the whole field. Thus, $\mathcal{O}$ is, by definition, the intersection of the $\mathcal{O}_p$'s for all p; this is the set of rationals whose denominators are not divisible by any prime, which is exactly the set $\mathbb{Z}$ of ordinary integers.

2. If K is a finite field, it has only the trivial valuation. In fact, if $|\cdot|$ is a valuation and a any non-zero element of K, then there is a positive integer m such that $a^m = 1$, and we have $|a|^m = |a^m| = |1| = 1$, and therefore $|a| = 1$. Thus, $|\cdot|$ is trivial and $\mathcal{O} = K$. This means that all elements of the field are integral elements.

3. If K is the field $\mathbb{Q}_p$ of the `http://planetmath.org/NonIsomorphicCompletionsOfMathbbQ`p-adic numbers, it has only one non-trivial valuation, the p-adic valuation, and now the ring $\mathcal{O}$ is its valuation ring, which is the ring of `http://planetmath.org/PAdicIntegers`*p-adic integers*; this is visualized in the 2-adic (*dyadic*) case in the article "p-adic canonical form".