splitting field of a finite set of polynomials

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Lemma 1. *(Cauchy,Kronecker) Let K be a field. For any irreducible polynomial f in $K[X]$ there is an extension field of K in which f has a root.*

Proof. If I is the ideal generated by f in $K[X]$, since f is irreducible, I is a maximal ideal of $K[X]$, and consequently $K[X]/I$ is a field.
We can construct a canonical monomorphism v from K to $K[X]$. By tracking back the field operation on $K[X]/I$, v can be extended to an isomorphism w from an extension field L of K to $K[X]/I$.
We show that $\alpha = w^{-1}(X + I)$ is a root of f.
If we write $f = \sum_{i=1}^{n} f_i X^i$ then $f + I = 0$ implies:

$$\begin{aligned} w(f(\alpha)) &= w(\sum_{i=1}^{n} f_i \alpha^i) \\ &= \sum_{i=1}^{n} w(f_i) w(\alpha)^i \\ &= \sum_{i=1}^{n} v(f_i) w(\alpha)^i \\ &= \sum_{i=1}^{n} (f_i + I)(X + I)^i \\ &= (\sum_{i=1}^{n} f_i X^i) + I \\ &= f + I = 0, \end{aligned}$$

which means that $f(\alpha) = 0$. □

Theorem 1. *Let K be a field and let M be a finite set of nonconstant polynomials in $K[X]$. Then there exists an extension field L of K such that every polynomial in M splits in $L[X]$*

Proof. If L is a field extension of K then the nonconstant polynomials $f_1, f_2, ..., f_n$ split in $L[X]$ iff the polynomial $\prod_{i=1}^{n} f_i$ splits in $L[X]$. Now the proof easily follows from the above lemma. □