complete ultrametric field

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A field K equipped with a non-archimedean valuation $|\cdot|$ is called a *non-archimedean field* or also an *ultrametric field*, since the valuation the ultrametric $d(x,\,y) := |x-y|$ of K.

Theorem. Let $(K,\,d)$ be a `http://planetmath.org/Complete`complete ultrametric field. A necessary and sufficient condition for the convergence of the *series*

$$a_1+a_2+a_3+\ldots \tag{1}$$

in K is that

$$\lim_{n\to\infty} a_n \;=\; 0. \tag{2}$$

Proof. Let ε be any positive number. When (1) converges, it satisfies the Cauchy condition and therefore exists a number m_ε such that surely

$$|a_{m+1}| \;=\; \left|\sum_{j=1}^{m+1}a_j-\sum_{j=1}^{m}a_j\right| \;<\; \varepsilon$$

for all $m \geqq m_\varepsilon$; thus (2) is necessary. On the contrary, suppose the validity of (2). Now one may determine such a great number n_ε that

$$|a_m| \;<\; \varepsilon \qquad \forall m \;\geqq\; n_\varepsilon.$$

No matter how great is the natural number n, the ultrametric then guarantees the inequality

$$|a_m+a_{m+1}+\ldots+a_{m+n}| \;\leqq\; \max\{|a_m|,\,|a_{m+1}|,\,\ldots,\,|a_{m+n}|\} \;<\; \varepsilon$$

always when $m \geqq n_\varepsilon$. Thus the partial sums of (1) form a Cauchy sequence, which converges in the complete field. Hence the series (1) converges, and (2) is sufficient.