alternative definition of algebraically closed

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Proposition 1. *If K is a field, the following are equivalent:*

(1) K is algebraically closed, i.e. every nonconstant polynomial f in $K[x]$ has a root in K.

(2) Every nonconstant polynomial f in $K[x]$ splits completely over K.

(3) If $L|K$ is an algebraic extension then $L = K$.

Proof. If (1) is true then we can prove by induction on degree of f that every nonconstant polynomial f splits completely over K. Conversely, (2)$\Rightarrow$ (1) is trivial.
(2)$\Rightarrow$ (3) If $L|K$ is algebraic and $\alpha \in L$, then α is a root of a polynomial $f \in K[x]$. By (2) f splits over K, which implies that $\alpha \in K$. It follows that $L = K$.
(3)$\Rightarrow$ (1) Let $f \in K[x]$ and α a root of f (in some extension of K). Then $K(\alpha)$ is an algebraic extension of K, hence $\alpha \in K$. □

Examples 1) The field of real numbers $\mathbb{R}$ is not algebraically closed. Consider the equation $x^2 + 1 = 0$. The square of a real number is always positive and cannot be -1 so the equation has no roots.
2) The p-adic field $\mathbb{Q}_p$ is not algebraically closed because the equation $x^2 - p = 0$ has no roots. Otherwise $x^2 = p$ implies $2v_p x = 1$, which is false.