cardinality of algebraic closure

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Theorem 1. *If a field is finite, then its algebraic closure is countably infinite.*

Proof. Because a finite field cannot be algebraically closed, the algebraic closure of a finite field must be infinite. Hence, it only remains to show that the algebraic closure is countable. Every element of the algebraic closure is the root of some polynomial. Furthermore, every polynomial has a finite number of roots (the number is bounded by its degree) and there are a countable number of polynomials whose coefficients belong to a given finite set. Since the union of a countable family of finite sets is countable, the number of elements of the algebraic closure is countable. □

Theorem 2. *If a field is infinite, then its algebraic closure has the same cardinality as the original field.*

Proof. Since a field is isomorphic to a subset of its algebraic closure, it follows that the cardinality of the closure is at least the cardinality of the original field. The number of polynomials of degree n with coefficients in a given set is the same as the number of n- tuplets of elements of S, which is the cardinality of the set raised to the n-th power. Since an infinite cardinal raised to an finite power equals itself, the number of polynomials of a given degree equals the the cardinality the original field. Since the cardinality of the union of a countable number of sets each of which has the same infinite number of elements equals the common cardinality of the sets, the total number of polynomials with coefficients in the field equals the cardinality of the field. Since every element of the algebraic closure of a field is the root of some polynomial with elements of the field for coefficients and a polynomial has a finite number of roots, it follows that the cardinality of the algebraic closure is bounded by the cardinality of the original field. □

Theorem 3. *For every transfinite cardinal number N, there exists an algebrically closed field with exactly N elements.*

Proof. Let F be the field of rational functions with integer coefficients in variables x_i, where the index i ranges over an index set I whose cardinality is N. We claim that the cardinality of F is N. The cardinality is at least N becasue we have the N rational functions x_i, so it only remains to show that the cardinality is not greater than N. To do this, we first show that the number of polynomials in the x_i with integer coefficients equals N. A polynomial is determined by a finite set of coefficients and a finite set of monomials. The number of possible sets of coefficients is the number of finite tuplets of integers, which is $\aleph_0$. Since a monomial may be determined by a mapping of a finite set into the set $\{x_i \mid i \in I\}$, the number of possible monomials of degree n is bounded by N^n. Since N is transfinite and n is finite, we have $N^n = N$. Thus the number of possible monomials is bounded by $N\aleph_0 = N$. So the number of polynomials is bounded by the product of $\aleph_0$ and N, which is N and the number of rational functions is bounded by N^2, which equals N. $\square$