algebraically solvable

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An equation

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0, \tag{1}$$

with coefficients a_j in a field K, is *algebraically solvable*, if some of its `http://planetmath.org/Equation`roots may be expressed with the elements of K by using rational operations (addition, subtraction, multiplication, division) and root extractions. I.e., a root of (1) is in a field $K(\xi_1,\, \xi_2,\, \ldots,\, \xi_m)$ which is obtained of K by `http://planetmath.org/FieldAdjunction`adjoining to it in succession certain suitable radicals $\xi_1,\, \xi_2,\, \ldots,\, \xi_m$. Each radical may under the root sign one or more of the previous radicals,

$$\begin{cases} \xi_1 = \sqrt[p_1]{r_1}, \\ \xi_2 = \sqrt[p_2]{r_2(\xi_1)}, \\ \xi_3 = \sqrt[p_3]{r_3(\xi_1,\, \xi_2)}, \\ \cdots \qquad \cdots \\ \xi_m = \sqrt[p_m]{r_m(\xi_1,\, \xi_2,\, \ldots,\, \xi_{m-1})}, \end{cases}$$

where generally $r_k(\xi_1,\, \xi_2,\, \ldots,\, \xi_{k-1})$ is an element of the field $K(\xi_1,\, \xi_2,\, \ldots,\, \xi_{k-1})$ but no p_k'th power of an element of this field. Because of the formula

$$\sqrt[jk]{r} = \sqrt[j]{\sqrt[k]{r}}$$

one can, without hurting the generality, suppose that the `http://planetmath.org/Root`indices $p_1,\, p_2,\, \ldots,\, p_m$ are prime numbers.

Example. Cardano's formulae show that all roots of the cubic equation $y^3 + py + q = 0$ are in the algebraic number field which is obtained by adjoining to the field $\mathbb{Q}(p,\, q)$ successively the radicals

$$\xi_1 = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}, \qquad \xi_2 = \sqrt[3]{-\frac{q}{2} + \xi_1}, \qquad \xi_3 = \sqrt{-3}.$$

In fact, as we consider also the equation (4), the roots may be expressed as

$$\begin{cases} y_1 = \xi_2 - \dfrac{p}{3\xi_2} \\ y_2 = \dfrac{-1+\xi_3}{2} \cdot \xi_2 - \dfrac{-1-\xi_3}{2} \cdot \dfrac{p}{3\xi_2} \\ y_3 = \dfrac{-1-\xi_3}{2} \cdot \xi_2 - \dfrac{-1+\xi_3}{2} \cdot \dfrac{p}{3\xi_2} \end{cases}$$

References

[1] K. Väisälä: *Lukuteorian ja korkeamman algebran alkeet.* Tiedekirjasto No. 17. Kustannusosakeyhtiö Otava, Helsinki (1950).