Archimedean property

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Let x be any real number. Then there exists a natural number n such that $n > x$.

This theorem is known as the *Archimedean property* of real numbers. It is also sometimes called the axiom of Archimedes, although this name is doubly deceptive: it is neither an axiom (it is rather a consequence of the least upper bound property) nor attributed to Archimedes (in fact, Archimedes credits it to Eudoxus).

Proof. Let x be a real number, and let $S = \{a \in \mathbb{N} : a \leq x\}$. If S is empty, let $n = 1$; note that $x < n$ (otherwise $1 \in S$).

Assume S is nonempty. Since S has an upper bound, S must have a least upper bound; call it b. Now consider $b - 1$. Since b is the least upper bound, $b - 1$ cannot be an upper bound of S; therefore, there exists some $y \in S$ such that $y > b - 1$. Let $n = y + 1$; then $n > b$. But y is a natural, so n must also be a natural. Since $n > b$, we know $n \notin S$; since $n \notin S$, we know $n > x$. Thus we have a natural greater than x. $\square$

Corollary 1. *If x and y are real numbers with $x > 0$, there exists a natural n such that $nx > y$.*

Proof. Since x and y are reals, and $x \neq 0$, y/x is a real. By the Archimedean property, we can choose an $n \in \mathbb{N}$ such that $n > y/x$. Then $nx > y$. $\square$

Corollary 2. *If w is a real number greater than 0, there exists a natural n such that $0 < 1/n < w$.*

Proof. Using Corollary 1, choose $n \in \mathbb{N}$ satisfying $nw > 1$. Then $0 < 1/n < w$. $\square$

Corollary 3. *If x and y are real numbers with $x < y$, there exists a rational number a such that $x < a < y$.*

Proof. First examine the case where $0 \leq x$. Using Corollary 2, find a natural n satisfying $0 < 1/n < (y - x)$. Let $S = \{m \in \mathbb{N} : m/n \geq y\}$. By Corollary 1 S is non-empty, so let m_0 be the least element of S and let $a = (m_0 - 1)/n$. Then $a < y$. Furthermore, since $y \leq m_0/n$, we have $y - 1/n < a$; and $x < y - 1/n < a$. Thus a satisfies $x < a < y$.

Now examine the case where $x < 0 < y$. Take $a = 0$.

Finally consider the case where $x < y \leq 0$. Using the first case, let b be a rational satisfying $-y < b < -x$. Then let $a = -b$. $\square$