partial fractions for polynomials

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This entry precisely states and proves the existence and uniqueness of partial fraction decompositions of ratios of polynomials of a single variable, with coefficients over a field.

The theory is used for, for example, the method of `http://planetmath.org/ALectureOnThePar` fraction decomposition for integrating rational functions over the reals.

The proofs involve fairly elementary algebra only. Although we refer to Euclidean domains in our proofs, the reader who is not familiar with abstract algebra may simply read that as "set of polynomials" (which is one particular Euclidean domain).

Also note that the proofs themselves furnish a method for actually computing the partial fraction decomposition, as a finite-time algorithm, provided the irreducible factorization of the denominator is known. It is not an efficient way to find the partial fraction decomposition; usually one uses instead the method of making substitutions into the polynomials, to derive linear constraints on the coefficients. But what is important is that the existence proofs here *justify* the substitution method. The uniqueness property proved here might also simplify some calculations: it shows that we never have to consider multiple solutions for the coefficients in the decomposition.

Theorem 1. *Let p and $q \neq 0$ be polynomials over a field, and n be any positive integer. Then there exist unique polynomials $\alpha_1, \dots, \alpha_n, \beta$ such that*

$$\frac{p}{q^n} = \beta + \frac{\alpha_1}{q} + \frac{\alpha_2}{q^2} + \cdots + \frac{\alpha_n}{q^n}\,, \quad \deg \alpha_j < \deg q\,. \tag{1}$$

Proof. Existence has already been proven as a special case of partial fractions in Euclidean domains; we now prove uniqueness. Suppose equation (**??**) has been given. Multiplying by q^n and rearranging,

$$p = \beta q^n + r_1\,, \quad r_1 = \alpha_1 q^{n-1} + \cdots + \alpha_n\,, \quad \deg r_1 < \deg q^n\,.$$

But according to the division algorithm for polynomials (also known as long division), the quotient and remainder polynomial after a division (by q^n in this case) are unique. So β must be uniquely determined. Then we can rearrange:

$$p - \beta\, q^n = \alpha_1 q^{n-1} + r_2\,, \quad r_2 = \alpha_2 q^{n-2} + \cdots + \alpha_n\,, \quad \deg r_2 < \deg q^{n-1}\,.$$

By uniqueness of division again (by q^{n-1}), α_1 is determined. Repeating this process, we see that all the polynomials α_j and β are uniquely determined. □

Theorem 2. *Let p and $q \neq 0$ be polynomials over a field. Let $q = \phi_1^{n_1}\,\phi_2^{n_2}\,\cdots\,\phi_k^{n_k}$ be the factorization of q to irreducible factors ϕ_i (which is unique except for the ordering and constant factors). Then there exist unique polynomials α_{ij}, β such that*

$$\frac{p}{q} = \beta + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\alpha_{ij}}{\phi_i^j}\,, \quad \deg \alpha_{ij} < \deg \phi_i\,. \tag{2}$$

Proof. Existence has already been proven as a special case of partial fractions in Euclidean domains; we now prove uniqueness. Suppose equation (**??**) has been given. First, multiply the equation by q:

$$p = \beta\, q + \sum_{i,j} \alpha_{ij}\, \frac{q}{\phi_i^j}\,.$$

The polynomial sum on the far right of this equation has degree $< q$, because each summand has degree $\deg(\alpha_{ij}\, q/\phi_i^j) < \deg \phi_i + \deg q - j \cdot \deg \phi_i \leq \deg q$. So the polynomial sum is the remainder of a division of p by q. Then the quotient polynomial β is uniquely determined.

Now suppose s_i and s_i' are polynomials of degree $< \phi_i^{n_i}$, such that

$$\sum_{i=1}^{k} \frac{s_i}{\phi_i^{n_i}} = \sum_{i=1}^{k} \frac{s_i'}{\phi_i^{n_i}}\,. \tag{3}$$

We claim that $s_i = s_i'$. Let $q_1 = \phi_1^{n_1}$ and $q_2 = q/q_1$, and write

$$\frac{s_1}{q_1} + \frac{u}{q_2} = \sum_{i=1}^{k} \frac{s_i}{\phi_i^{n_i}} = \sum_{i=1}^{k} \frac{s_i'}{\phi_i^{n_i}} = \frac{s_1'}{q_1} + \frac{u'}{q_2}\,,$$

for some polynomials u and u'. Rearranging, we get:

$$(s_1 - s_1')\, q_2 = (u' - u)\, q_1\,.$$

In particular, q_1 divides the left side. Since $q_1 = \phi_1^{n_1}$ is relatively prime from q_2, it must divide the factor $(s_1 - s_1')$. But $\deg(s_1 - s_1') < \deg q_1$, hence $s_1 - s_1'$ must be the zero polynomial. That is, $s_1 = s_1'$.

So we can cancel the term $s_1/\phi_1^{n_1} = s_1'/\phi_1^{n_1}$ on both sides of equation (**??**). And we could repeat the argument, and show that s_2 and s_2' are the same, s_3 and s_3' are the same, and so on. Therefore, we have shown that the polynomials s_i in the following expression

$$\frac{p}{q} - \beta = \sum_{i=1}^{k} \frac{s_i}{\phi_i^{n_i}}\,, \quad \deg s_i < \deg \phi_i^{n_i}$$

are unique. In particular, s_i is the following numerator that results when the fractions α_{ij}/ϕ_i^j are put under a common denominator $\phi_{n_i}^i$:

$$s_i = \sum_{j=1}^{n_i} \alpha_{ij}\, \phi_i^{n_i - j}\,.$$

But by the uniqueness part of Theorem **??**, the decomposition

$$\frac{s_i}{\phi_i^{n_i}} = \beta_i + \sum_{j=1}^{n_i} \frac{\alpha_{ij}}{\phi_i^j}\,, \quad \deg \alpha_{ij} < \deg \phi_i$$

uniquely determines α_{ij}. (Note that the proof of Theorem **??** shows that $\beta_i = 0$, as $\deg s_i < \deg \phi_i^{n_i}$.) □