rules of calculus for derivative of formal power series

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In this entry, we will show that the rules of calculus hold for derivatives of formal power series. While this could be verified directly in a manner analogous to what was done for polynomials in the parent entry, we will take a different tack, deriving the results for power series from the corresponding results for polynomials. The basis for our approach is the observation that the ring of formal power series can be expressed as a limit of quotients of the ring of polynomials:

$$A[[x]] = \lim_{k \to \infty} A[x]/\langle x^k \rangle$$

Thus, we will proceed in two steps, first extending the derivative operation to the quotient rings and showing that its properties still hold there, then extending it to the limit and showing that its properties hold there as well.

We begin by noting that the derivative is well-defined as a map from $A[x]/\langle x^{k+1} \rangle$ to $A[x]/\langle x^k \rangle$ for all integers $k \geq 0$.

Theorem 1. *Suppose that A is a commutative ring, k is a non-negative integer, and that p and q are elements of $A[x]$ such that $p \equiv q$ modulo x^{k+1}. Then $p' \equiv q'$ modulo x^k.*

Proof. By definition of congruence, $p(x) = q(x) + x^{k+1} r(x)$ for some polynomial $r \in A[x]$. Taking derivaitves, $p'(x) = q'(x) + x^k (kr(x) + xr'(x))$, so p' and q' are equivalent modulo x^k. □

It is easy to verify that the sum and product rules hold in this new context:

Theorem 2. *If A is a commutative ring, k is a non-negative integer, and f, g are elements of $A[x]/\langle x^{k+1} \rangle$, then $(f + g)' = f' + g'$.*

Proof. Let p, q be representatives of the equivalence classes f, g. Then we have $(p + q)' = p' + q'$ by the corresponding theorem for polynomials. Hence, by definition of quotient, we have $(f + g)' = f' + g'$. □

Theorem 3. *If A is a commutative ring, k is a non-negative integer, and f, g are elements of $A[x]/\langle x^{k+1} \rangle$, then $(f \cdot g)' = f' \cdot g + f \cdot g'$.*

Proof. Let p, q be representatives of the equivalence classes f, g. Then we have $(p \cdot q)' = p' \cdot q + p \cdot q'$ by the corresponding theorem for polynomials. Hence, by definition of quotient, we have $(f \cdot g)' = f' \cdot g + f \cdot g'$. □

When considering the chain rule, we need to note that composition does not always pass to the quotient, so we need to restrict the operands to obtain a well-defined operation. In particular, we will consider the following two cases:

Theorem 4. *If A is a commutative ring, p, q, r is a n element of $A[x]$, and $q \equiv r$ modulo x^k for some integer $k > 0$, then $p \circ q \equiv p \circ r$ modulo x^k.*

Theorem 5. *If A is a commutative ring, k is a non-negative integer, and p, q, P, Q are elements of $A[x]$ such that $p \equiv P$ modulo x^k, $q \equiv Q$ modulo x^k and $p(0) = 0$, then $p \circ q \equiv P \circ Q$ modulo x^k.*

[More to come]