conjugate fields

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If ϑ_1, ϑ_2, $\ldots$, ϑ_n are the algebraic conjugates of the algebraic number ϑ_1, then the algebraic number fields $\mathbb{Q}(\vartheta_1)$, $\mathbb{Q}(\vartheta_2)$, $\ldots$, $\mathbb{Q}(\vartheta_n)$ are the *conjugate fields* of $\mathbb{Q}(\vartheta_1)$.

Notice that the conjugate fields of $\mathbb{Q}(\vartheta_1)$ are always isomorphic but not necessarily distinct.

All conjugate fields are equal, `http://planetmath.org/Iei`.e. $\mathbb{Q}(\vartheta_1) = \mathbb{Q}(\vartheta_2) = \ldots = \mathbb{Q}(\vartheta_n)$, or equivalently $\vartheta_1, \ldots, \vartheta_n$ belong to $\mathbb{Q}(\vartheta_1)$, if and only if the extension $\mathbb{Q}(\vartheta_1)/\mathbb{Q}$ is a Galois extension of fields. The reason for this is that if ϑ_1 is an algebraic number and $m(x)$ is the minimal polynomial of ϑ_1 then the roots of $m(x)$ are precisely the algebraic conjugates of ϑ_1.

For example, let $\vartheta_1 = \sqrt{2}$. Then its only conjugate is $\vartheta_2 = -\sqrt{2}$ and $\mathbb{Q}(\sqrt{2})$ is Galois and contains both ϑ_1 and ϑ_2. Similarly, let p be a prime and let $\vartheta_1 = \zeta$ be a `http://planetmath.org/PrimitiveRootOfUnity`primitive pth root of unity. Then the algebraic conjugates of ζ are $\zeta^2, \ldots, \zeta^{p-1}$ and so all conjugate fields are equal to $\mathbb{Q}(\zeta)$ and the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ is Galois. It is a cyclotomic extension of $\mathbb{Q}$.

Now let $\vartheta_1 = \sqrt[3]{2}$ and let ζ be a primitive 3rd root of unity (i.e. ζ is a root of $x^2 + x + 1$, so we can pick $\zeta = \frac{-1+\sqrt{-3}}{2}$). Then the conjugates of ϑ_1 are ϑ_1, $\vartheta_2 = \zeta\sqrt[3]{2}$, and $\vartheta_3 = \zeta^2\sqrt[3]{2}$. The three conjugate fields $\mathbb{Q}(\vartheta_1)$, $\mathbb{Q}(\vartheta_2)$, and $\mathbb{Q}(\vartheta_3)$ are distinct in this case. The Galois closure of each of these fields is $\mathbb{Q}(\zeta, \sqrt[3]{2})$.