infinite Galois theory

Canonical name	InfiniteGaloisTheory
Date of creation	2013-03-22 12:39:06
Last modified on	2013-03-22 12:39:06
Owner	djao (24)
Last modified by	djao (24)
Numerical id	7
Author	djao (24)
Entry type	Topic
Classification	msc 12F10
Classification	msc 13B05
Related topic	FundamentalTheoremOfGaloisTheory
Related topic	GaloisGroup
Related topic	InverseLimit
Defines	Krull topology

Let L/F be a Galois extension, not necessarily finite dimensional.

1 Topology on the Galois group

Recall that the *Galois group* $G := \mathrm{Gal}(L/F)$ of L/F is the group of all field automorphisms $\sigma : L \longrightarrow L$ that restrict to the identity map on F, under the group operation of composition. In the case where the extension L/F is infinite dimensional, the group G comes equipped with a natural topology, which plays a key role in the statement of the Galois correspondence.

We define a subset U of G to be open if, for each $\sigma \in U$, there exists an intermediate field $K \subset L$ such that

- The degree $[K : F]$ is finite,
- If σ' is another element of G, and the restrictions $\sigma|_K$ and $\sigma'|_K$ are equal, then $\sigma' \in U$.

The resulting collection of open sets forms a topology on G, called the *Krull topology*, and G is a topological group under the Krull topology. Another way to define the topology is to state that the subgroups $\mathrm{Gal}(L/K)$ for finite extensions K/F form a neighborhood basis for $\mathrm{Gal}(L/F)$ at the identity.

2 Inverse limit structure

In this section we exhibit the group G as a projective limit of an inverse system of finite groups. This construction shows that the Galois group G is actually a profinite group.

Let $\mathcal{A}$ denote the set of finite normal extensions K of F which are contained in L. The set $\mathcal{A}$ is a partially ordered set under the inclusion relation. Form the inverse limit

$$\Gamma := \varprojlim \mathrm{Gal}(K/F) \subset \prod_{K \in \mathcal{A}} \mathrm{Gal}(K/F)$$

consisting, as usual, of the set of all $(\sigma_K) \in \prod_K \mathrm{Gal}(K/F)$ such that $\sigma_{K'}|_K = \sigma_K$ for all $K, K' \in \mathcal{A}$ with $K \subset K'$. We make Γ into a topological space by putting the discrete topology on each finite set $\mathrm{Gal}(K/F)$ and giving Γ

the subspace topology induced by the product topology on $\prod_K \mathrm{Gal}(K/F)$. The group Γ is a closed subset of the compact group $\prod_K \mathrm{Gal}(K/F)$, and is therefore compact.

Let

$$\phi : G \longrightarrow \prod_{K \in \mathcal{A}} \mathrm{Gal}(K/F)$$

be the group homomorphism which sends an element $\sigma \in G$ to the element (σ_K) of $\prod_K \mathrm{Gal}(K/F)$ whose K–th coordinate is the automorphism $\sigma|_K \in \mathrm{Gal}(K/F)$. Then the function ϕ has image equal to Γ and in fact is a homeomorphism between G and Γ. Since Γ is profinite, it follows that G is profinite as well.

3 The Galois correspondence

Theorem 1 (Galois correspondence for infinite extensions)**.** *Let G, L, F be as before. For every closed subgroup H of G, let L^H denote the fixed field of H. The correspondence*

$$K \mapsto \mathrm{Gal}(L/K),$$

defined for all intermediate field extensions $F \subset K \subset L$, is an inclusion reversing bijection between the set of all intermediate extensions K and the set of all closed subgroups of G. Its inverse is the correspondence

$$H \mapsto L^H,$$

defined for all closed subgroups H of G. The extension K/F is normal if and only if $\mathrm{Gal}(L/K)$ is a normal subgroup of G, and in this case the restriction map

$$G \longrightarrow \mathrm{Gal}(K/F)$$

has kernel $\mathrm{Gal}(L/K)$.

Theorem 2 (Galois correspondence for finite subextensions)**.** *Let G, L, F be as before.*

- *Every open subgroup $H \subset G$ is closed and has finite index in G.*
- *If $H \subset G$ is an open subgroup, then the field extension L^H/F is finite.*
- *For every intermediate field K with $[K : F]$ finite, the Galois group $\mathrm{Gal}(L/K)$ is an open subgroup of G.*