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equivalent characterizations of Dedekind domains

Canonical name	EquivalentCharacterizationsOfDedekindDomains
Date of creation	2013-03-22 18:34:27
Last modified on	2013-03-22 18:34:27
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	10
Author	gel (22282)
Entry type	Theorem
Classification	msc 13A15
Classification	msc 13F05
Related topic	DedekindDomain
Related topic	FinitelyGeneratedTorsionFreeModulesOverPruferDomains

Dedekind domains can be defined as integrally closed <http://planetmath.org/NoetherianNoe> domains in which every nonzero prime ideal is maximal. However, there are several alternative characterizations which can be used, and are listed in the following theorem.

**Theorem.** *Let  $R$  be an integral domain. Then, the following are equivalent.*

1.  *$R$  is Noetherian, integrally closed, and every prime ideal is maximal.*
2. *Every nonzero proper ideal is a product of maximal ideals.*
3. *Every nonzero proper ideal is product of prime ideals.*
4. *Every nonzero ideal is <http://planetmath.org/FractionalIdeal> invertible.*
5. *Every ideal is <http://planetmath.org/ProjectiveModule> projective as an  $R$ -module.*
6.  *$R$  is Noetherian and every finitely generated torsion-free  $R$ -module is projective.*
7.  *$R$  is Noetherian and the localization  $R_{\mathfrak{m}}$  is a principal ideal domain for each maximal ideal  $\mathfrak{m}$ .*

*Furthermore, if these properties are satisfied then the decomposition into primes in (??) and (??) is unique up to reordering of the factors.*

For example, if  $R$  is a principal ideal domain then ideals are clearly invertible and, by condition ?? it is Dedekind, so is integrally closed. In this case, factorization of ideals coincides with prime factorization in the ring. For the equivalence of ?? and ?? see proof that a domain is Dedekind if its ideals are invertible.

Once it is known that a ring is Dedekind then conditions ?? and ?? show that we get unique factorization of ideals in terms of prime or, equivalently, maximal ideals and conversely, Dedekind domains are the only integral domains in which such decompositions exist (see proof that a domain is Dedekind if its ideals are products of maximals and proof that a domain is Dedekind if its ideals are products of primes).

The equivalence of ?? and ?? is immediate once it is known that invertible ideals are projective. For rings which are not integral domains, the property that ideals are projective still makes sense. Such rings are called

<http://planetmath.org/HereditaryRing>hereditary, and give one possible generalization of the concept of Dedekind domains.

As ideals in a Noetherian domain are finitely generated torsion-free submodules of  $R$ , condition ?? clearly implies ?. Domains which are not necessarily Noetherian, but for which every finitely generated torsion-free module is projective are known as Prüfer domains. The equivalence of ? and ? then follows from the alternative characterization of Prüfer domains as integral domains in which every finitely generated ideal is projective.

Condition ? (see proof that a Noetherian domain is Dedekind if it is locally a PID) shows that for Noetherian rings, being a Dedekind domain is a <http://planetmath.org/Localization>local property and therefore the notion generalizes to apply to <http://planetmath.org/Variety>algebraic varieties and <http://planetmath.org/Scheme>schemes.

## References

- [1] P.M. Cohn, *Algebra. Vol 2*, Second edition. John Wiley & Sons Ltd, 1989.