proof of the weak Nullstellensatz

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Let K be an algebraically closed field, let $n \geq 0$, and let I be an ideal in the polynomial ring $K[x_1, \ldots, x_n]$. Suppose I is strictly smaller than $K[x_1, \ldots, x_n]$. Then I is contained in a maximal ideal M of $K[x_1, \ldots, x_n]$ (note that we don't have to accept Zorn's lemma to find such an M, since $K[x_1, \ldots, x_n]$ is Noetherian by Hilbert's basis theorem), and the quotient ring

$$L = K[x_1, \ldots, x_n]/M$$

is a field. We view K as a subfield of L via the natural homomorphism $K \hookrightarrow L$, and we denote the images of $x_1, \ldots, x_n$ in L by $\bar{x}_1, \ldots, \bar{x}_n$. Let $\{t_1, \ldots, t_m\}$ be a transcendence basis of L over K; it is finite since L is finitely generated as a K-algebra. Now L is an algebraic extension of $K(t_1, \ldots, t_m)$. By multiplying the minimal polynomial of $\bar{x}_i$ over $K(t_1, \ldots, t_m)$ by a suitable element of $K[t_1, \ldots, t_m]$ for each i, we obtain non-zero polynomials $f_i \in K[t_1, \ldots, t_m][X]$ with the property that $f_i(\bar{x}_i) = 0$ in L:

$$f_i = c_{i,0} + c_{i,1}X + \cdots + c_{i,d_i}X^{d_i} \qquad (1 \leq i \leq n)$$

for certain integers $d_i > 0$ and polynomials $c_{i,j} \in K[t_1, \ldots, t_m]$ with $c_{i,d_i} \neq 0$. Since K is algebraically closed (hence infinite), we can choose $u_1, \ldots, u_n \in K$ such that $c_{i,d_i}(u_1, \ldots, u_m) \neq 0$ for all i. We define a homomorphism

$$\phi\colon K[t_1, \ldots, t_m] \longrightarrow K$$

by taking ϕ to be the identity on K and sending t_j to u_j. Let N be the kernel of this homomorphism. Then ϕ can be extended to the localization $K[t_1, \ldots, t_m]_N$ of $K[t_1, \ldots, t_m]$. Since $c_{i,d_i} \notin N$ for all i, the $\bar{x}_i$ are integral over this ring. Since K is algebraically closed, the extension theorem for ring homomorphisms implies that ϕ can be extended to a homomorphism

$$\phi\colon (K[t_1, \ldots, t_m]_N)[\bar{x}_1, \ldots, \bar{x}_n] = L \longrightarrow K.$$

Because L is an extension field of K and ϕ is the identity on K, we see that ϕ is actually an isomorphism, that $m = 0$, and that N is the zero ideal of K. Now let $a_1 = \phi(\bar{x}_1), \ldots, a_n = \phi(\bar{x}_n)$. Then for all polynomials f in the ideal I we started with, the fact that $f \in M$ implies

$$f(a_1, \ldots, a_n) = \phi(f(x_1, \ldots, x_n) + M) = 0.$$

We conclude that the zero set $V(I)$ of I is not empty.