Hensel's lemma

Canonical name	HenselsLemma
Date of creation	2013-03-22 15:08:30
Last modified on	2013-03-22 15:08:30
Owner	alozano (2414)
Last modified by	alozano (2414)
Numerical id	5
Author	alozano (2414)
Entry type	Theorem
Classification	msc 13H99
Classification	msc 12J99
Classification	msc 11S99

The following results are used to show the existence of a solution to polynomial equations over local fields. Notice the similarities with Newton's method.

Theorem (Hensel's Lemma)**.** *Let K be a* `http://planetmath.org/LocalField`*local field, complete with respect to a valuation $|\cdot|$. Let $\mathcal{O}_K$ be the ring of integers in K (i.e. the set of elements of K with $|k| \leq 1$). Let $f(x)$ be a polynomial with coefficients in $\mathcal{O}_K$ and suppose there exist $\alpha_0 \in \mathcal{O}_K$ such that*

$$|f(\alpha_0)| < |f'(\alpha_0)^2|.$$

Then there exist a root $\alpha \in K$ of $f(x)$. Moreover, the sequence:

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)}{f'(\alpha_i)}$$

converges to α. Furthermore:

$$|\alpha - \alpha_0| \leq \left| \frac{f(\alpha_i)}{f'(\alpha_i)} \right| < 1.$$

Corollary (Trivial case of Hensel's lemma)**.** *Let K be a number field and let $\mathfrak{p}$ be a prime ideal in the ring of integers $\mathcal{O}_K$. Let $K_\mathfrak{p}$ be the completion of K at the finite place $\mathfrak{p}$ and let $\mathcal{O}_\mathfrak{p}$ be the ring of integers in $K_\mathfrak{p}$. Let $f(x)$ be a polynomial with coefficients in $\mathcal{O}_\mathfrak{p}$ and suppose there exist $\alpha_0 \in \mathcal{O}_\mathfrak{p}$ such that*

$$f(\alpha_0) \equiv 0 \mod \mathfrak{p}, \quad f'(\alpha_0) \neq 0 \mod \mathfrak{p}.$$

Then there exist a root $\alpha \in K_\mathfrak{p}$ of $f(x)$, i.e. $f(\alpha) = 0$.