complete ring of quotients

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Consider a commutative unitary ring R and set

$$\mathcal{S} := \{\mathrm{Hom}_R(I, R) : I \text{ is dense in } R\}$$

(here $\mathrm{Hom}_R(I, R)$ is the set of R-module morphisms from I to R) and define $A := \bigcup_{B \in \mathcal{S}} B$.

Now we shall assign a ring structure to A by defining its addition and multiplication. Given two dense ideals $I_1, I_2 \subset R$ and two elements $f_i \in \mathrm{Hom}_R(I_i, R)$ for $i \in \{1, 2\}$, one can easily check that $I_1 \cap I_2$ and $f_2^{-1}(I_1)$ are nontrivial (i.e. they aren't $\{0\}$) and in fact also dense ideals so we define

$f_1 + f_2 \in \mathrm{Hom}_R(I_1 \cap I_2, R)$ by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$f_1 * f_2 \in \mathrm{Hom}_R(f_2^{-1}(I_1), R)$ by $(f_1 * f_2)(x) = f_1(f_2(x))$

It is easy to check that A is in fact a commutative ring with unity. The elements of A are called .

There is also an equivalence relation that one can define on A. Given $f_i \in \mathrm{Hom}_R(I_i, R)$ for $i \in \{1, 2\}$, we write

$$f_1 \sim f_2 \Leftrightarrow f_1|I_1 \cap I_2 = f_2|I_1 \cap I_2$$

(i.e. f_1 and f_2 belong to the same equivalence class iff they agree on the intersection of the dense ideal where they are defined).

The factor ring $Q(R) := A/\sim$ is then called the *complete ring of quotients.*

Remark. *$R \subset T(R) \subset Q(R)$, where $T(R)$ is the total quotient ring. One can also in general define complete ring of quotients on noncommutative rings.*

References

[Huckaba] **J.A. Huckaba**, "Commutative rings with zero divisors", Marcel Dekker 1988