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proof that Sylvester's matrix equals the resultant

Canonical name	ProofThatSylvestersMatrixEqualsTheResultant
Date of creation	2013-03-22 14:36:50
Last modified on	2013-03-22 14:36:50
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Last modified by	rspuzio (6075)
Numerical id	8
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Entry type	Definition
Classification	msc 13P10

In the derivation of Sylvester's matrix for the resultant, it was seen that if two polynomials have a common root, then Sylvester's determinant will equal zero. Since two polynomials have a common root if and only if their resultant is zero, it follows that if the resultant is zero, then Sylvester's determinant equals zero. In this entry, we shall use this fact to show that Sylvester's determinant equals the resultant.

The secret is to view both Sylvester's determinant and the resultant as functions of the roots. A more precise way of saying what this means is that we will study polnomials in the indeterminates $a_0, r_1, r_2, \ldots, r_m, b_0, s_1, s_2, \ldots, s_n$. We will regard $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$ as polynomials in these variables using the expression of coefficients of a polynomial as symmetric functions of its roots, e.g.

$$a_1 = a_0(r_1 + r_2 + \cdots)$$

$$a_2 = a_0(r_1 r_2 + r_1 r_3 + \cdots)$$

Note that a_k is a k^{th} order polynomial in the r_i's and b_k is a k^{th} order polynomial in the s_i's.

Let R be the polynomial

$$R = a_0^n b_0^m \prod_{i=1}^{m} \prod_{j=1}^{n} (r_i - s_j)$$

and let D be the polynomial which is gotten by replacing occurrences of $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$ in Sylvester's matrix by their expressions in of $a_0, r_1, r_2, \ldots, r_m, b_0, s_1, s_2, \ldots, s_n$. We want to show that $R = D$.

First, note that, in each row of Sylvester's matrix, every entry is multiplied by either an a_0 or a b_0. By a fundamental property of determinants, this means that we may pull all those factors of a_0 and b_0 outside the determinant. Since there are n rows containing a_0 and m rows containing b_0, this means that $D = a_0^m b_0^n D'(r_1, \ldots, r_m, s_1, \ldots, s_n)$. Note that these factors correspond to the powers of a_0 and b_0 in the definition of R. Hence, to show that $D = R$ it only remains to show that $D' = R'$, where

$$R' = \prod_{i=1}^{m} \prod_{j=1}^{n} (r_i - s_j)$$

Second, note that the degree of D' is not greater than the degree of R'. From the definition, it is obvious that R' is a polynomial of degree mn. By examining Sylvester's determinant and keeping in mind that a_k and b_k are of degree k, it is not hard to see that the degree of D' cannot exceed mn.

Third, we will show that R' divides D'. In the derivation of the Sylvester determinant, we saw that if $r_i = s_j$ for some choice of i and j, then $D = 0$, and hence $D' = 0$. The only way for a non-zero polynomial to to equal zero when $r_i = s_j$ is for $r_i - s_i$ to be a factor of the polynomial. It is easy to see that D' is not the zero polynomial, and hence, $s_i - s_j$ must be a factor of D'. This means that every factor of R' is also a factor of D'. Since all the factors of R' occur with multiplicity one, it follows that D' is a multiple of R'.

Combining the of the last two paragraphs, we come to the conclusion that D' must be a constant multiple of R'. To determine the constant of proportionality, all one needs to do is to compare the values of the two polynomials for a set of value of the variables for which they to not vanish. For instance, one could try $r_1 = r_2 = \cdots = r_m = 1$ and $s_1 = s_2 = \cdots = s_n = 0$. Both R' and D' equal 1 for this special set of values, and hence $R' = D'$.