bilinearity and commutative rings

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Owner	Algeboy (12884)
Last modified by	Algeboy (12884)
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Author	Algeboy (12884)
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We show that a bilinear map $b : U \times V \to W$ is almost always definable only for commutative rings. The exceptions lie only where non-trivial commutators act trivially on one of the three modules.

Lemma 1. *Let R be a ring and U, V and W be R-modules. If $b : U \times V \to W$ is R-bilinear then b is also R-middle linear.*

Proof. Given $r \in R$, $u \in U$ and $v \in V$ then $b(ru, v) = rb(u, v)$ and $b(u, rv) = rb(u, v)$ so $b(ru, v) = b(u, rv)$. □

Theorem 2. *Let R be a ring and U, V and W be faithful R-modules. If $b : U \times V \to W$ is R-bilinear and (left or right) non-degenerate, then R must be commutative.*

Proof. We may assume that b is left non-degenerate. Let $r, s \in R$. Then for all $u \in U$ and $v \in V$ it follows that

$$b((sr)u, v) = sb(ru, v) = sb(u, rv) = b(su, rv) = b((rs)u, v).$$

Therefore $b([s, r]u, v) = 0$, where $[s, r] = sr - rs$. This makes $[s, r]u$ an element of the left radical of b as it is true for all $v \in V$. However b is non-degenerate so the radical is trivial and so $[s, r]u = 0$ for all $u \in U$. Since U is a faithful R-module this makes $[s, r] = 0$ for all $s, r \in R$. That is, R is commutative. □

Alternatively we can interpret the result in a weaker fashion as:

Corollary 3. *Let R be a ring and U, V and W be R-modules. If $b : U \times V \to W$ is R-bilinear with $W = \langle b(U, V)\rangle$ then every element $[R, R]$ acts trivially on one of the three modules U, V or W.*

Proof. Suppose $[r, s] \in [R, R]$, $[r, s]U \neq 0$ and $[r, s]V \neq 0$. Then we have shown $0 = b([r, s]u, v) = [r, s]b(u, v)$ for all $u \in U$ and $v \in V$. As $W = \langle b(U, V)\rangle$ it follows that $[r, s]W = 0$. □

Whenever a non-commutative ring is required for a biadditive map $U \times V \to W$ it is therefore often preferable to use a scalar map instead.