proof of finitely generated torsion-free modules over Prüfer domains

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Let M be a finitely generated torsion-free module over a Prüfer domain R with field of fractions k. We show that M is isomorphic to a http://planetmath.org/DirectSumdirect sum of finitely generated ideals in R.

We shall write $k\otimes M$ for the vector space over k generated by M. This is just the http://planetmath.org/LocalizationOfAModulelocalization of M at $R\setminus\{0\}$ and, as M is torsion-free, the natural map $M\rightarrow k\otimes M$ is one-to-one and we can regard M as a subset of $k\otimes M$.

As M is finitely generated, the vector space $k\otimes M$ will http://planetmath.org/Dimension2fini dimensional, and we use induction on its dimension n. Supposing that $n>0$, choose any basis $e_1,\ldots,e_n$ and define the linear map $f\colon k\otimes M\rightarrow k$ by http://planetmath.org/Projectionprojection onto the first component,

$$f(x_1e_1+\cdots+x_ne_n)=x_1.$$

Restricting to M, this gives a nonzero map $M\rightarrow k$. Furthermore, as M is finitely generated, $f(M)$ will be a finitely generated fractional ideal in k. Choosing any nonzero $c\in R$ such that $\mathfrak{a}\equiv cf(M)\subseteq R$,

$$g\colon M\rightarrow\mathfrak{a},\ g(u)=cf(u)$$

defines a homorphism from M onto the nonzero and finitely generated ideal $\mathfrak{a}$. As R is Prüfer and invertible ideals are projective, g has a right-inverse $h\colon\mathfrak{a}\rightarrow M$. Then h has the left-inverse g and is one-to-one, so defines an isomorphism between $\mathfrak{a}$ and its http://planetmath.org/ImageOfALinearTransformationimage. We decompose M as the direct sum of the kernel of g and the image of h,

$$M=\ker(g)\oplus\mathrm{Im}(h)\cong\ker(g)\oplus\mathfrak{a}.$$

Projection from the finitely generated module M onto $\ker(g)$ shows that it is finitely generated and,

$$\dim(k\otimes\ker(g))=\dim(k\otimes M)-\dim(k\otimes\mathfrak{a})=n-1.$$

So, the result follows from applying the induction hypothesis to $\ker(g)$.