Hilbert's Nullstellensatz

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Let K be an algebraically closed field, and let I be an ideal in $K[x_1, \ldots, x_n]$, the polynomial ring in n indeterminates.

Define $V(I)$, the *zero set* of I, by

$$V(I) = \{(a_1, \ldots, a_n) \in K^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}$$

Weak Nullstellensatz:
If $V(I) = \emptyset$, then $I = K[x_1, \ldots, x_n]$. In other words, the zero set of any proper ideal of $K[x_1, \ldots, x_n]$ is nonempty.

Hilbert's (Strong) Nullstellensatz:
Suppose $f \in K[x_1, \ldots, x_n]$ satisfies $f(a_1, \ldots, a_n) = 0$ for every $(a_1, \ldots, a_n) \in V(I)$. Then $f^r \in I$ for some integer $r > 0$.

In the of algebraic geometry, the latter result is equivalent to the statement that $\text{Rad}(I) = I(V(I))$, that is, the radical of I is equal to the ideal of $V(I)$.