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criterion for a module to be noetherian

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Theorem 1. *A module is noetherian if and only if all of its submodules and* `http://planetmath.org/QuotientModule`*quotients are noetherian.*

Proof. Suppose M is Noetherian (over a ring R), and $N \subseteq M$ a submodule. Since any submodule of M is finitely generated, any submodule of N, being a submodule of M, is finitely generated as well. Next, if A/N is a submodule of M/N, and if $a_1, \ldots, a_n$ is a generating set for $A \subseteq M$, then $a_1 + N, \ldots, a_n + N$ is a generating set for A/N. Conversely, if every submodule of M is Noetherian, then M, being a submodule itself, must be Noetherian. □

A weaker form of the converse is the following:

Theorem 2. *If $N \subseteq M$ is a submodule of M such that N and M/N are Noetherian, then M is Noetherian.*

Proof. Suppose $A_1 \subseteq A_2 \subseteq \cdots$ is an ascending chain of submodules of M. Let $B_i = A_i \cap N$, then $B_1 \subseteq B_2 \subseteq \cdots$ is an ascending chain of submodules of N. Since N is Noetherian, the chain terminates at, say B_n. Let $C_i = (A_i + N)/N$, then $C_1 \subseteq C_2 \subseteq \cdots$ is an ascending chain of submodules of M/N. Since M/N is Noetherian, the chain stops at, say C_m. Let $k = \max(m, n)$. Then we have $B_k = B_{k+1}$ and $C_k = C_{k+1}$. We want to show that $A_k = A_{k+1}$. Since $A_k \subseteq A_{k+1}$, we need the other inclusion. Pick $a \in A_{k+1}$. Then $a + N = b + N$, where $b \in A_k$. This means that $a - b \in N$. But $b \in A_{k+1}$ as well, so $a - b \in N \cap A_{k+1}$. Since $N \cap A_k = N \cap A_{k+1}$, this means that $a - b \in A_k$ or $a \in A_k$. □