ring of exponent

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Defines	ring of an exponent
Defines	ring of the exponent
Defines	integral with respect to an exponent

Definition. Let ν be an exponent valuation of the field K. The subring

$$\mathcal{O}_\nu \;:=\; \{\alpha \in K \,\vdots\; \nu(\alpha) \geqq 0\}$$

of K is called the ν. It is, naturally, an integral domain. Its elements are called ν.

Theorem 1. The ring of the exponent ν of the field K is integrally closed in K.

Theorem 2. The ring $\mathcal{O}_\nu$ only one prime element π, when one does not regard associated elements as different. Any non-zero element α can be represented uniquely with a π in the form

$$\alpha \;=\; \varepsilon\pi^m,$$

where ε is a unit of $\mathcal{O}_\nu$ and $m = \nu(\alpha) \geqq 0$. This means that $\mathcal{O}$ is a UFD.

Remark 1. The prime elements π of the ring $\mathcal{O}_\nu$ are characterised by the equation $\nu(\pi) = 1$ and the units ε the equation $\nu(\varepsilon) = 0$.

Remark 2. In an algebraically closed field Ω, there are no `http://planetmath.org/ExponentV` In fact, if there were an exponent ν of Ω and if π were a prime element of the ring of the exponent, then, since the equation $x^2 - \pi = 0$ would have a `http://planetmath.org/Equation`root ϱ in Ω, we would obtain $2\nu(\varrho) = \nu(\varrho^2) = \nu(\pi) = 1$; this is however impossible, because an exponent attains only integer values.

Theorem 3. Let $\mathfrak{O}_1, \ldots, \mathfrak{O}_r$ be the rings of the different exponent valuations $\nu_1, \ldots, \nu_r$ of the field K. Then also the intersection

$$\mathfrak{O} \;:=\; \bigcap_{i=1}^{r} \mathfrak{O}_i$$

is a subring of K with `http://planetmath.org/UFD`unique factorisation. To be precise, any non-zero element α of $\mathfrak{O}$ may be uniquely represented in the form

$$\alpha \;=\; \varepsilon\pi_1^{n_1}\cdots\pi_r^{n_r},$$

in which ε is a unit of $\mathfrak{O}$, the integers $n_1, \ldots, n_r$ are nonnegative and $\pi_1, \ldots, \pi_r$ are coprime prime elements of $\mathfrak{O}$ satisfying

$$\nu_i(\pi_j) \;=\; \delta_{ij} \;=\; \begin{cases} 1 \text{ for } i = j, \\ 0 \text{ for } i \neq j. \end{cases}$$