Lasker-Noether theorem

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Theorem 1 (Lasker-Noether)**.** *Let R be a commutative Noetherian ring with 1. Every ideal in R is* `http://planetmath.org/DecomposableIdeal`*decomposable.*

The theorem can be proved in two steps:

Proposition 1. *Every ideal in R can be written as a finite intersection of irreducible ideals*

Proof. Let S be the set of all ideals of a Noetherian ring R which can not be written as a finite intersection of irreducible ideals. Suppose $S \neq \varnothing$. Then any chain $I_1 \subseteq I_2 \subseteq \cdots$ in S must terminate in a finite number of steps, as R is Noetherian. Say $I = I_n$ is the maxmimal element of this chain. Since $I \in S$, I itself can not be irreducible, so that $I = J \cap K$ where J and K are ideals strictly containing I. Now, if $J \in S$, then then I would not be maximal in the chain $I_1 \subseteq I_2 \subseteq \cdots$. Therefore, $J \notin S$. Similarly, $K \notin S$. By the definition of S, J and K are both finite intersections of irreducible ideals. But this would imply that $I \notin S$, a contradiction. So $S = \varnothing$ and we are done. $\square$

Proposition 2. *Every irreducible ideal in R is primary*

Proof. Suppose I is irreducible and $ab \in I$. We want to show that either $a \in I$, or some power n of b is in I. Define $J_i = I:(b^i)$, the quotient of ideals I and (b^i). Since

$$\cdots \subseteq (b^n) \subseteq \cdots \subseteq (b^2) \subseteq (b),$$

we have, by one of the rules on quotients of ideals, an ascending chain of ideals

$$J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$$

Since R is Noetherian, $J := J_n = J_m$ for all $m > n$. Next, define $K = (b^n) + I$, the sum of ideals (b^n) and I. We want to show that $I = J \cap K$.

First, it is clear that $I \subseteq J$ and $I \subseteq K$, which takes care of one of the inclusions. Now, suppose $r \in J \cap K$. Then $r = s + tb^n$, where $s \in I$ and $t \in R$, and $rb^n \in I$. So, $rb^n = sb^n + tb^{2n}$. Now, $t \in I:(b^{2n})$, so $t \in I:(b^n)$. But this means that $r = s + tb^n \in I$, and this proves the other inclusion.

Since I is irreducible, either $I = J$ or $I = K$. We analyze the two cases below:

- If $I = J = I:(b^n)$, then $I = I:(b)$ in particular, since $I \subseteq I:(b) \subseteq I:(b^n)$. As $ab \in I$ by assumption, $a \in I:(b) = I$.

- If $I = K = (b^n) + I$, then $b^n \in I$.

This completes the proof. $\square$

Remarks.

- The above theorem can be generalized to any submodule of a finitely generated module over a commutative Noetherian ring with 1.
- A ring is said to be *Lasker* if every ideal is decomposable. The theorem above says that every commutative Noetherian ring with 1 is Lasker. There are Lasker rings that are not Noetherian.