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extension of Krull valuation

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The Krull valuation $|\cdot|_K$ of the field K is called the *of the Krull valuation* $|\cdot|_k$ of the field k, if k is a subfield of K and $|\cdot|_k$ is the restriction of $|\cdot|_K$ on k.

Theorem 1. The trivial valuation is the only of the trivial valuation of k to an algebraic extension field K of k.

Proof. Let's denote by $|\cdot|$ the trivial valuation of k and also its arbitrary Krull to K. Suppose that there is an element α of K such that $|\alpha| > 1$. This element satisfies an algebraic equation

$$\alpha^n + a_1\alpha^{n-1} + ... + a_n = 0,$$

where $a_1, ..., a_n \in k$. Since $|a_j| \leqq 1$ for all j's, we get the impossibility

$$0 = |\alpha^n + a_1\alpha^{n-1} + ... + a_n| = \max\{|\alpha|^n, |a_1|\cdot|\alpha|^{n-1}, ..., |a_n|\} = |\alpha|^n > 1$$

(cf. the sharpening of the ultrametric triangle inequality). Therefore we must have $|\xi| \leqq 1$ for all $\xi \in K$, and because the condition $0 < |\xi| < 1$ would imply that $|\xi^{-1}| > 1$, we see that

$$|\xi| = 1 \quad \forall \xi \in K \setminus \{0\},$$

which that the valuation is trivial.

The proof (in [1]) of the next "extension theorem" is much longer (one must utilize the extension theorem concerning the place of field):

Theorem 2. Every Krull valuation of a field k can be extended to a Krull valuation of any field of k.

References

[1] Emil Artin: . Lecture notes. Mathematisches Institut, Göttingen (1959).