differential field

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Defines	differential ring
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Defines	partial differential ring
Defines	field of constants
Defines	ring of constants

Let F be a field (ring) together with a derivation $(\cdot)'\colon F \to F$. The derivation must satisfy two properties:

Additivity $(a+b)' = a' + b'$;

Leibniz' Rule $(ab)' = a'b + ab'$.

A derivation is the algebraic abstraction of a derivative from ordinary calculus. Thus the terms *derivation*, *derivative*, and *differential* are often used interchangeably.

Together, $(F, ')$ is referred to as a *differential field (ring)*. The subfield (subring) of all elements with vanishing derivative, $K = \{a \in F \mid a' = 0\}$, is called the *field (ring) of constants*. Clearly, $(\cdot)'$ is K-linear.

There are many notations for the derivation symbol, for example a' may also be denoted as da, δa, ∂a, etc. When there is more than one derivation ∂_i, $(F, \{\partial_i\})$ is referred to as a *partial differential field (ring)*.

1 Examples

Differential fields and rings (together under the name of differential algebra) are a natural setting for the study of algebraic properties of derivatives and anti-derivatives (indefinite integrals), as well as ordinary and partial differential equations and their solutions. There is an abundance of examples drawn from these areas.

- The trivial example is a field F with $a' = 0$ for each $a \in F$. Here, nothing new is gained by introducing the derivation.
- The most common example is the field of rational functions $\mathbb{R}(z)$ over an indeterminant satisfying $z' = 1$. The field of constants is $\mathbb{R}$. This is the setting for ordinary calculus.
- Another example is $\mathbb{R}(x, y)$ with two derivations ∂_x and ∂_y. The field of constants is $\mathbb{R}$ and the derivations are extended to all elements from the properties $\partial_x x = 1$, $\partial_y y = 1$, and $\partial_x y = \partial_y x = 0$.
- Consider the set of smooth functions $C^\infty(M)$ on a manifold M. They form a ring (or a field if we allow formal inversion of functions vanishing in some places). Vector fields on M act naturally as derivations on $C^\infty(M)$.

- Let A be an algebra and $U_t = \exp(tu)$ be a one-parameter subgroup of automorphisms of A. Here u is the infinitesimal generator of these automorphisms. From the properties of U_t, u must be a linear operator on A that satisfies the Leibniz rule $u(ab) = u(a)b + au(b)$. So (A, u) can be considered a differential ring.