invertible ideal is finitely generated

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Theorem. Let R be a commutative ring containing regular elements. Every http://planetmath.org/FractionalIdealOfCommutativeRinginvertible fractional ideal $\mathfrak{a}$ of R is finitely generated and http://planetmath.org/RegularIdealregular, i.e. regular elements.

Proof. Let T be the total ring of fractions of R and e the unity of T. We first show that the inverse ideal of $\mathfrak{a}$ has the unique http://planetmath.org/QuotientOfIdealsqu presentation $[R' : \mathfrak{a}]$ where $R' := R + \mathbb{Z}e$. If $\mathfrak{a}^{-1}$ is an inverse ideal of $\mathfrak{a}$, it means that $\mathfrak{a}\mathfrak{a}^{-1} = R'$. Therefore we have

$$\mathfrak{a}^{-1} \subseteq \{t \in T \vdots\ t\mathfrak{a} \subseteq R'\} = [R' : \mathfrak{a}],$$

so that

$$R' = \mathfrak{a}\mathfrak{a}^{-1} \subseteq \mathfrak{a}[R' : \mathfrak{a}] \subseteq R'.$$

This implies that $\mathfrak{a}\mathfrak{a}^{-1} = \mathfrak{a}[R' : \mathfrak{a}]$, and because $\mathfrak{a}$ is a cancellation ideal, it must that $\mathfrak{a}^{-1} = [R' : \mathfrak{a}]$, i.e. $[R' : \mathfrak{a}]$ is the unique inverse of the ideal $\mathfrak{a}$.

Since $\mathfrak{a}[R' : \mathfrak{a}] = R'$, there exist some elements $a_1, \ldots, a_n$ of $\mathfrak{a}$ and the elements $b_1, \ldots, b_n$ of $[R' : \mathfrak{a}]$ such that $a_1b_1 + \cdots + a_nb_n = e$. Then an arbitrary element a of $\mathfrak{a}$ satisfies

$$a = a_1(b_1a) + \cdots + a_n(b_na) \in (a_1, \ldots, a_n)$$

because every b_ia belongs to the ring R'. Accordingly, $\mathfrak{a} \subseteq (a_1, \ldots, a_n)$. Since the converse inclusion is apparent, we have seen that $\{a_1, \ldots, a_n\}$ is a finite of the invertible ideal $\mathfrak{a}$.

Since the elements b_i belong to the total ring of fractions of R, we can choose such a regular element d of R that each of the products b_id belongs to R. Then

$$d = a_1(b_1d) + \cdots + a_n(b_nd) \in (a_1, \ldots, a_n) = \mathfrak{a},$$

and thus the fractional ideal $\mathfrak{a}$ contains a regular element of R, which obviously is regular in T, too.

References

[1] R. Gilmer: *Multiplicative ideal theory*. Queens University Press. Kingston, Ontario (1968).