proof of Euler four-square identity

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Using Lagrange's identity, we have

$$\left(\sum_{k=1}^{4} x_k y_k\right)^2 = \left(\sum_{k=1}^{4} x_k^2\right)\left(\sum_{k=1}^{4} y_k^2\right) - \sum_{1\le k<i\le 4}(x_k y_i - x_i y_k)^2. \tag{1}$$

We group the six squares into 3 groups of two squares and rewrite:

$$\begin{aligned}
& (x_1y_2 - x_2y_1)^2 + (x_3y_4 - x_4y_3)^2 & (2)\\
= \ & ((x_1y_2 - x_2y_1) + (x_3y_4 - x_4y_3))^2 - 2((x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3)) & \\
& (x_1y_3 - x_3y_1)^2 + (x_2y_4 - x_4y_2)^2 & (3)\\
= \ & ((x_1y_3 - x_3y_1) - (x_2y_4 - x_4y_2))^2 + 2(x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) & \\
& (x_1y_4 - x_4y_1)^2 + (x_2y_3 - x_3y_2)^2 & (4)\\
= \ & ((x_1y_4 - x_4y_1) + (x_2y_3 - x_3y_2))^2 - 2(x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2). & (5)
\end{aligned}$$

Using

$$\begin{aligned}
-2((x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3)) \quad & +2(x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) & (6)\\
-2(x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2) \quad & = 0 &
\end{aligned}$$

we get

$$\begin{aligned}
\sum_{1\le k<i\le 4}(x_ky_i - x_iy_k)^2 \quad & = ((x_1y_2 - x_2y_1) \quad +(x_3y_4 - x_4y_3))^2 & (7)\\
& +((x_1y_3 - x_3y_1) - (x_2y_4 - x_4y_2))^2 & (8)\\
& +((x_1y_4 - x_4y_1) + (x_2y_3 - x_3y_2))^2 &
\end{aligned}$$

by adding equations **??**-**??**. We put the result of equation **??** into **??** and get

$$\begin{aligned}
& \left(\sum_{k=1}^{4} x_k y_k\right)^2 & (9)\\
= & \left(\sum_{k=1}^{4} x_k^2\right)\left(\sum_{k=1}^{4} y_k^2\right) \quad -((x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)^2 & \\
& \quad -(x_1y_3 - x_3y_1 + x_4y_2 - x_2y_4)^2 \quad -(x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)^2 &
\end{aligned}$$

which is equivalent to the claimed identity.