planetmath.org

Math for the people, by the people.

Lagrange's identity

Canonical name	LagrangesIdentity
Date of creation	2013-03-22 13:18:01
Last modified on	2013-03-22 13:18:01
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	21
Author	mathcam (2727)
Entry type	Theorem
Classification	msc 13A99

Let R be a commutative ring, and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be arbitrary elements in R. Then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 = \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) - \sum_{1 \le k < i \le n} (x_k y_i - x_i y_k)^2.$$

Proof. Since R is commutative, we can apply the binomial formula.We start out with

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 = \sum_{i=1}^{n} (x_i^2 y_i^2) + \sum_{1 \le i < j \le n} 2 x_i y_j x_j y_i \tag{1}$$

Using the binomial formula, we see that

$$(x_i y_j - x_j y_i)^2 = x_i^2 y_j^2 - 2 x_i x_j y_i y_j + x_j^2 y_i^2.$$

So we get

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 + \sum_{1 \le i < j \le n}^{n} (x_i y_j - x_j y_i)^2 = \sum_{i=1}^{n} \left(x_i^2 y_i^2\right) + \sum_{1 \le i < j \le n}^{n} \left(x_i^2 y_j^2 + x_j^2 y_i^2\right) \tag{2}$$

$$= \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \tag{3}$$

Note that changing the roles of i and j in $x_i y_j - x_j y_i$, we get

$$x_j y_i - x_i y_j = -(x_i y_j - x_j y_i),$$

but the negative sign will disappear when we square. So we can rewrite the last equation to

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 + \sum_{1 \le i < j \le n} (x_i y_j - x_j y_i)^2 = \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right). \tag{4}$$

This is equivalent to the stated identity. $\square$