alternative definition of Krull valuation

Canonical name	AlternativeDefinitionOfKrullValuation
Date of creation	2013-03-22 17:02:08
Last modified on	2013-03-22 17:02:08
Owner	polarbear (3475)
Last modified by	polarbear (3475)
Numerical id	10
Author	polarbear (3475)
Entry type	Definition
Classification	msc 13F30
Classification	msc 13A18
Classification	msc 12J20
Classification	msc 11R99
Related topic	OrderValuation
Related topic	Valuation
Related topic	Krullvaluation
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Let G be an abelian totally ordered group, denoted additively. We adjoin to G a new element ∞ such that $g < +\infty$, for all $g \in G$ and we extend the addition on $G_\infty = G \cup \{+\infty\}$ by declaring $g + (+\infty) = (+\infty) + (+\infty) = +\infty$.

Definition 1. *Let R be an unital ring, a* valuation *of R with values in G is a function from R to G_∞ such that , for all $x, y \in R$:*
1) $v(xy) = v(x) + v(y)$,
2) $v(x + y) \geq \min\{v(x), v(y)\}$,
3) $v(x) = +\infty$ iff $v(x) = 0$.

Remarks a) The condition 1) means that v is a homomorhism of $R \setminus \{0\}$ with multiplication in the group G. In particular, $v(1) = 0$ and $v(-x) = v(x)$, for all $x \in G$. If x is invertible then $0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1})$, so $v(x^{-1}) = -v(x)$.
b) If 3) is replaced by the condition $v(0) = +\infty$ then the set $P = v^{-1}\{+\infty\}$ is a prime ideal of R and v is on the integral domain R/P.
c) In particular, conditions 1) and 3) that R is an integral domain and let K be its quotient field. There is a unique valuation of K with values in G that extends v, namely $v(x/y) = v(x) - v(y)$, for all $x \in R$ and $y \in R \setminus \{0\}$.
d) The element $v(x)$ is sometimes denoted by vx.