definition of prime ideal by Artin

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Lemma. Let R be a commutative ring and S a multiplicative semigroup consisting of a subset of R. If there exist `http://planetmath.org/node/371`ideals of R which are disjoint with S, then the set $\mathfrak{S}$ of all such ideals has a maximal element with respect to the set inclusion.

Proof. Let C be an arbitrary chain in $\mathfrak{S}$. Then the union

$$\mathfrak{b} := \bigcup_{\mathfrak{a} \in C} \mathfrak{a},$$

which belongs to $\mathfrak{S}$, may be taken for the upper bound of C, since it clearly is an ideal of R and disjoint with S. Because $\mathfrak{S}$ thus is inductively ordered with respect to "$\subseteq$", our assertion follows from Zorn's lemma.

Definition. The maximal elements in the Lemma are *prime ideals* of the commutative ring.

The ring R itself is always a prime ideal ($S = \varnothing$). If R has no zero divisors, the zero ideal (0) is a prime ideal ($S = R \smallsetminus \{0\}$).

If the ring R has a non-zero unity element 1, the prime ideals corresponding the semigroup $S = \{1\}$ are the maximal ideals of R.

References

[1] Emil Artin: *Theory of Algebraic Numbers.* Lecture notes. Mathematisches Institut, Göttingen (1959).