decomposition group

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1 Decomposition Group

Let A be a Noetherian integrally closed integral domain with field of fractions K. Let L be a Galois extension of K and denote by B the integral closure of A in L. Then, for any prime ideal $\mathfrak{p} \subset A$, the Galois group $G := \mathrm{Gal}(L/K)$ acts transitively on the set of all prime ideals $\mathfrak{P} \subset B$ containing $\mathfrak{p}$. If we fix a particular prime ideal $\mathfrak{P} \subset B$ lying over $\mathfrak{p}$, then the stabilizer of $\mathfrak{P}$ under this group action is a subgroup of G, called the *decomposition group* at $\mathfrak{P}$ and denoted $D(\mathfrak{P}/\mathfrak{p})$. In other words,

$$D(\mathfrak{P}/\mathfrak{p}) := \{\sigma \in G \mid \sigma(\mathfrak{P}) = (\mathfrak{P})\}.$$

If $\mathfrak{P}' \subset B$ is another prime ideal of B lying over $\mathfrak{p}$, then the decomposition groups $D(\mathfrak{P}/\mathfrak{p})$ and $D(\mathfrak{P}'/\mathfrak{p})$ are conjugate in G via any Galois automorphism mapping $\mathfrak{P}$ to $\mathfrak{P}'$.

2 Inertia Group

Write l for the residue field $B/\mathfrak{P}$ and k for the residue field $A/\mathfrak{p}$. Assume that the extension l/k is separable (if it is not, then this development is still possible, but considerably more complicated; see [**?**, p. 20]). Any element $\sigma \in D(\mathfrak{P}/\mathfrak{p})$, by definition, fixes $\mathfrak{P}$ and hence descends to a well defined automorphism of the field l. Since σ also fixes A by virtue of being in G, it induces an automorphism of the extension l/k fixing k. We therefore have a group homomorphism

$$D(\mathfrak{P}/\mathfrak{p}) \longrightarrow \mathrm{Gal}(l/k),$$

and the `http://planetmath.org/KernelOfAGroupHomomorphism`kernel of this homomorphism is called the *inertia group* of $\mathfrak{P}$, and written $T(\mathfrak{P}/\mathfrak{p})$. It turns out that this homomorphism is actually surjective, so there is an exact sequence

$$1 \longrightarrow T(\mathfrak{P}/\mathfrak{p}) \longrightarrow D(\mathfrak{P}/\mathfrak{p}) \longrightarrow \mathrm{Gal}(l/k) \longrightarrow 1 \tag{1}$$

3 Decomposition of Extensions

The decomposition group is so named because it can be used to decompose the field extension L/K into a series of intermediate extensions each of which has very simple factorization behavior at $\mathfrak{p}$. If we let L^D denote the fixed field of $D(\mathfrak{P}/\mathfrak{p})$ and L^T the fixed field of $T(\mathfrak{P}/\mathfrak{p})$, then the exact sequence (**??**) corresponds under Galois theory to the lattice of fields

$$\begin{tikzcd} L \arrow[d, dash, "e"] \\ L^T \arrow[d, dash, "f"] \\ L^D \arrow[d, dash, "g"] \\ K \end{tikzcd}$$

If we write e, f, g for the degrees of these intermediate extensions as in the diagram, then we have the following remarkable series of equalities:

1. The number e equals the ramification index $e(\mathfrak{P}/\mathfrak{p})$ of $\mathfrak{P}$ over $\mathfrak{p}$, which is independent of the choice of prime ideal $\mathfrak{P}$ lying over $\mathfrak{p}$ since L/K is Galois.

2. The number f equals the inertial degree $f(\mathfrak{P}/\mathfrak{p})$ of $\mathfrak{P}$ over $\mathfrak{p}$, which is also independent of the choice of prime ideal $\mathfrak{P}$ since L/K is Galois.

3. The number g is equal to the number of prime ideals $\mathfrak{P}$ of B that lie over $\mathfrak{p} \subset A$.

Furthermore, the fields L^D and L^T have the following independent characterizations:

- L^T is the smallest intermediate field F such that $\mathfrak{P}$ is totally ramified over $\mathfrak{P} \cap F$, and it is the largest intermediate field such that $e(\mathfrak{P} \cap F, \mathfrak{p}) = 1$.

- L^D is the smallest intermediate field F such that $\mathfrak{P}$ is the only prime of B lying over $\mathfrak{P} \cap F$, and it is the largest intermediate field such that $e(\mathfrak{P} \cap F, \mathfrak{p}) = f(\mathfrak{P} \cap F, \mathfrak{p}) = 1$.

Informally, this decomposition of the extension says that the extension L^D/K encapsulates all of the factorization of $\mathfrak{p}$ into distinct primes, while the extension L^T/L^D is the source of all the inertial degree in $\mathfrak{P}$ over $\mathfrak{p}$ and the extension L/L^T is responsible for all of the ramification that occurs over $\mathfrak{p}$.

4 Localization

The decomposition groups and inertia groups of $\mathfrak{P}$ behave well under localization. That is, the decomposition and inertia groups of $\mathfrak{P}B_{\mathfrak{P}} \subset B_{\mathfrak{P}}$ over the prime ideal $\mathfrak{p}A_{\mathfrak{p}}$ in the localization $A_{\mathfrak{p}}$ of A are identical to the ones obtained using A and B themselves. In fact, the same holds true even in the completions of the local rings $A_{\mathfrak{p}}$ and $B_{\mathfrak{P}}$ at $\mathfrak{p}$ and $\mathfrak{P}$.

References

[1] J.P. Serre, *Local Fields*, Springer–Verlag, 1979 (GTM **67**)