image ideal of divisor

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Theorem. If an integral domain $\mathcal{O}$ has a divisor theory $\mathcal{O}^* \to \mathfrak{D}$, then the subset $[\mathfrak{a}]$ of $\mathcal{O}$, consisting of 0 and all elements divisible by a divisor $\mathfrak{a}$, is an ideal of $\mathcal{O}$. The mapping

$$\mathfrak{a} \mapsto [\mathfrak{a}]$$

from the set $\mathfrak{D}$ of divisors into the set of ideals of $\mathcal{O}$ is injective and maps any principal divisor (α) to the principal ideal (α).

Proof. Let $\alpha, \beta \in [\mathfrak{a}]$ and $\lambda \in \mathcal{O}$. Then, by the postulate 2 of `http://planetmath.org/DivisorTheory`divisor theory, $\alpha - \beta$ is divisible by $\mathfrak{a}$ or is 0, and in both cases belongs to $[\mathfrak{a}]$. When $\lambda\alpha \neq 0$, we can write $(\alpha) = \mathfrak{a}\mathfrak{c}$ with $\mathfrak{c}$ a divisor. According to the homomorphicity of the mapping $\mathcal{O}^* \to \mathfrak{D}$, we have

$$(\lambda\alpha) = (\lambda)(\alpha) = (\lambda)\mathfrak{a}\mathfrak{c},$$

and therefore the element $\lambda\alpha$ is divisible by $\mathfrak{a}$, i.e. $\lambda\alpha \in [\mathfrak{a}]$. Thus, $[\mathfrak{a}]$ is an ideal of $\mathcal{O}$.

The injectivity of the mapping $\mathfrak{a} \mapsto [\mathfrak{a}]$ follows from the postulate 3 of `http://planetmath.org/DivisorTheory`divisor theory.

The ideal $[\mathfrak{a}]$ may be called the *image ideal* of $\mathfrak{a}$ or the *ideal determined by the divisor* $\mathfrak{a}$.

Remark. There are integral domains $\mathcal{O}$ having a divisor theory but also having ideals which are not of the form $[\mathfrak{a}]$ (for example a polynomial ring in two indeterminates and its ideal formed by the polynomials without constant term). Such rings have "too many ideals". On the other hand, in some integral domains the monoid of principal ideals cannot be embedded into a free monoid; thus those rings cannot have a divisor theory.

References

[1] М. М. Постников: *Введение в теорию алгебраических чисел.* Издательство "Наука". Москва (1982).