unity plus nilpotent is unit

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Theorem. If x is a nilpotent element of a ring with unity 1 (which may be 0), then the sum $1+x$ is a unit of the ring.

Proof. If $x=0$, then $1+x=1$, which is a unit. Thus, we may assume that $x \neq 0$.

Since x is nilpotent, there is a positive integer n such that $x^n = 0$. We multiply $1+x$ by another ring element:

$$\begin{aligned}
(1+x)\cdot\sum_{j=0}^{n-1}(-1)^j x^j &= \sum_{j=0}^{n-1}(-1)^j x^j + \sum_{k=0}^{n-1}(-1)^k x^{k+1} \\
&= \sum_{j=0}^{n-1}(-1)^j x^j - \sum_{k=1}^{n}(-1)^k x^k \\
&= 1+\sum_{j=1}^{n-1}(-1)^j x^j - \sum_{k=1}^{n-1}(-1)^k x^k - (-1)^n x^n \\
&= 1+0+0 \\
&= 1
\end{aligned}$$

(Note that the summations include the term $(-1)^0 x^0$, which is why $x=0$ is excluded from this case.)

The reversed multiplication gives the same result. Therefore, $1+x$ has a multiplicative inverse and thus is a unit. $\square$

Note that there is a this proof and geometric series: The goal was to produce a multiplicative inverse of $1+x$, and geometric series yields that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n,$$

provided that the summation `http://planetmath.org/AbsoluteConvergence`converges. Since x is nilpotent, the summation has a finite number of nonzero terms and thus .