planetmath.org

Math for the people, by the people.

valuation

Canonical name	Valuation
Date of creation	2013-03-22 12:35:07
Last modified on	2013-03-22 12:35:07
Owner	djao (24)
Last modified by	djao (24)
Numerical id	17
Author	djao (24)
Entry type	Definition
Classification	msc 13F30
Classification	msc 13A18
Classification	msc 12J20
Classification	msc 11R99
Synonym	absolute value
Related topic	DiscreteValuationRing
Related topic	DiscreteValuation
Related topic	Ultrametric
Related topic	HenselianField
Defines	infinite prime
Defines	finite prime
Defines	archimedean
Defines	non-archimedean
Defines	real prime
Defines	complex prime
Defines	prime

Let K be a field. A *valuation* or *absolute value* on K is a function $|\cdot|\colon K \to \mathbb{R}$ satisfying the properties:

1. $|x| \geq 0$ for all $x \in K$, with equality if and only if $x = 0$
2. $|xy| = |x| \cdot |y|$ for all $x, y \in K$
3. $|x + y| \leq |x| + |y|$

If a valuation satisfies $|x + y| \leq \max(|x|, |y|)$, then we say that it is a *non-archimedean valuation*. Otherwise we say that it is an *archimedean valuation*.

Every valuation on K defines a metric on K, given by $d(x, y) := |x - y|$. This metric is an ultrametric if and only if the valuation is non-archimedean. Two valuations are *equivalent* if their corresponding metrics induce the same topology on K. An equivalence class v of valuations on K is called a *prime* of K. If v consists of archimedean valuations, we say that v is an *infinite prime*, or *archimedean prime*. Otherwise, we say that v is a *finite prime*, or *non-archimedean prime*.

In the case where K is a number field, primes as defined above generalize the notion of prime ideals in the following way. Let $\mathfrak{p} \subset K$ be a nonzero prime ideal[1], considered as a fractional ideal. For every nonzero element $x \in K$, let r be the unique integer such that $x \in \mathfrak{p}^r$ but $x \notin \mathfrak{p}^{r+1}$. Define

$$|x|_{\mathfrak{p}} := \begin{cases} 1/N(\mathfrak{p})^r & x \neq 0, \\ 0 & x = 0, \end{cases}$$

where $N(\mathfrak{p})$ denotes the absolute norm of $\mathfrak{p}$. Then $|\cdot|_{\mathfrak{p}}$ is a non–archimedean valuation on K, and furthermore every non-archimedean valuation on K is equivalent to $|\cdot|_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$. Hence, the prime ideals of K correspond bijectively with the finite primes of K, and it is in this sense that the notion of primes as valuations generalizes that of a prime ideal.

As for the archimedean valuations, when K is a number field every embedding of K into $\mathbb{R}$ or $\mathbb{C}$ yields a valuation of K by way of the standard absolute value on $\mathbb{R}$ or $\mathbb{C}$, and one can show that every archimedean valuation of K is equivalent to one arising in this way. Thus the infinite primes of K correspond to embeddings of K into $\mathbb{R}$ or $\mathbb{C}$. Such a prime is called real or complex according to whether the valuations comprising it arise from real or complex embeddings.

[1] By "prime ideal" we mean "prime fractional ideal of K" or equivalently "prime ideal of the ring of integers of K". We do not mean literally a prime ideal of the ring K, which would be the zero ideal.