ideals contained in a union of ideals

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Assume that R is a commutative ring.

Lemma. Let A, B, C be ideals in R such that $A \subseteq B \cup C$. Then $A \subseteq B$ or $A \subseteq C$.

Proof. Assume that this is not true. Then there are $x, y \in A$ such that $x \in B$, $y \in C$ and $x \notin C$, $y \notin B$. Obviously $x + y \in A \subseteq B \cup C$ and without loss of generality we may assume that $x + y \in B$. Then $y = (x + y) - x \in B$. Contradiction. □

Remark. This lemma is also true if we exchange ring with a group and ideals with subgroups (because we didn't use multiplication and commutativity of addition in proof).

Proposition. Let I, $P_1, \ldots, P_n$ be ideals in R such that each P_i is prime. If $I \subseteq P_1 \cup \cdots \cup P_n$, then there exists $i \in \{1, \ldots, n\}$ such that $I \subseteq P_i$.

Proof. We will use the induction on n. For $n = 2$ our lemma applies. Let $n > 2$. Assume that $I \nsubseteq P_1 \cup \cdots \cup P_n$. For $i \in \{1, \ldots, n\}$ define

$$\overline{P_i} = P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_n.$$

By our assumption (and induction hypothesis) $I \nsubseteq \overline{P_i}$ for any $i \in \{1, \ldots, n\}$. Thus for any i there is $x_i \in I$ such that $x_i \notin \overline{P_i}$.

Now for any $i \in \{1, \ldots, n\}$ define $\overline{x_i} = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n \in I$. Then we have

$$\overline{x_1} + \cdots + \overline{x_n} \in I$$

and thus there is $j \in \{1, \ldots, n\}$ such that $\overline{x_1} + \cdots + \overline{x_n} \in P_j$. Since $\overline{x_i} \in P_j$ for any $i \neq j$, then we have that

$$\overline{x_j} \in P_j.$$

But P_j is prime, so there is $k \neq j$ such that $x_k \in P_j \subseteq \overline{P_k}$. Contradiction. □

Counterexample. We will show, that if P_i's are not prime, then the thesis no longer hold, even when $n = 3$. Consider the ring of polynomials in two variables over a simple field of order 2, i.e. $\mathbb{Z}_2[X, Y]$. Let $R = \mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$. For $W(X, Y) \in \mathbb{Z}_2[X, Y]$ we shall write $\overline{W(X, Y)} = W(X, Y) + (X^2, XY, Y^2) \in R$. Then it is easy to see, that

$$R = \{\overline{0}, \overline{1}, \overline{X}, \overline{Y}, \overline{X} + \overline{Y}, \overline{X} + \overline{1}, \overline{Y} + \overline{1}, \overline{X} + \overline{Y} + \overline{1}\}.$$

Let

$$I = \{\overline{0}, \overline{X}, \overline{Y}, \overline{X} + \overline{Y}\};$$

$$A_1 = \{\overline{0}, \overline{X}\};$$
$$A_2 = \{\overline{0}, \overline{Y}\};$$
$$A_3 = \{\overline{0}, \overline{X} + \overline{Y}\}.$$

It can be easily checked, that I, A_1, A_2, A_3 are all ideals and $I \subseteq A_1 \cup A_2 \cup A_3$ but obviously $I \not\subseteq A_i$ for any $i = 1, 2, 3$. $\square$