examples of integrally closed extensions

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Example. $\mathbb{Z}[\sqrt{5}]$ is not integrally closed, for $u = \frac{1+\sqrt{5}}{2} \in \mathbb{Q}[\sqrt{5}]$ is integral over $\mathbb{Z}[\sqrt{5}]$ since $u^2 - u - 1 = 0$, but $u \notin \mathbb{Z}[\sqrt{5}]$.

Example. $R = \mathbb{Z}[\sqrt{2}, \sqrt{3}]$ is not integrally closed. Note that $(\sqrt{6} + \sqrt{2})/2 \notin R$, but that

$$\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 = 2 + \sqrt{3}$$

and so $(\sqrt{6} + \sqrt{2})/2$ is integral over $\mathbb{Z}$ since it satisfies the polynomial $(z^2 - 2)^2 - 3 = 0$.

Example. $\mathcal{O}_K$ is integrally closed when $[K : \mathbb{Q}] < \infty$. For if $u \in K$ is integral over $\mathcal{O}_K$, then $\mathbb{Z} \subset \mathcal{O}_K \subset \mathcal{O}_K[u]$ are all integral extensions, so u is integral over $\mathbb{Z}$, so $u \in \mathcal{O}_K$ by definition. In fact, $\mathcal{O}_K$ can be defined as the integral closure of $\mathbb{Z}$ in K.

Example. $\mathbb{C}[x, y]/(y^2 - x^3)$. This is a domain because $y^2 - x^3$ is irreducible hence a prime ideal. But this quotient ring is not integrally closed. To see this, parameterize $\mathbb{C}[x, y] \to \mathbb{C}[t]$ by

$$\begin{aligned} x &\mapsto t^2 \\ y &\mapsto t^3 \end{aligned}$$

The kernel of this map is $(y^2 - x^3)$, and its image is $\mathbb{C}[t^2, t^3]$. Hence

$$\mathbb{C}[x, y]/(y^2 - x^3) \cong \mathbb{C}[t^2, t^3]$$

and the field of fractions of the latter ring is obviously $\mathbb{C}(t)$. Now, t is integral over $\mathbb{C}[t^2, t^3]$ ($z^2 - t^2$ is its polynomial), but is not in $\mathbb{C}[t^2, t^3]$. t corresponds to $\frac{y}{x}$ in the original ring $\mathbb{C}[x, y]/(y^2 - x^3)$, which is thus not integrally closed (the minimal polynomial of $\frac{y}{x}$ is $z^2 - x$ since $(\frac{y}{x})^2 - x = \frac{y^2}{x^2} - x = \frac{x^3}{x^2} - x = 0$). The failure of integral closure in this coordinate ring is due to a codimension 1 singularity of $y^2 - x^3$ at 0.

Example. $A = \mathbb{C}[x, y, z]/(z^2 - xy)$ is integrally closed. For again, parameterize $A \to \mathbb{C}[u, v]$ by

$$\begin{aligned} x &\mapsto u^2 \\ y &\mapsto v^2 \\ z &\mapsto uv \end{aligned}$$

The kernel of this map is $z^2 - xy$ and its image is $B = \mathbb{C}[u^2, v^2, uv]$. Claim B is integrally closed. We prove this by showing that the integral closure of

$\mathbb{C}[x,y]$ in $\mathbb{C}(x,y,\sqrt{xy})$ is $\mathbb{C}[x,y,\sqrt{xy}]$. Choose $r+s\sqrt{xy} \in \mathbb{C}(x,y,\sqrt{xy}), r,s \in \mathbb{C}(x,y)$ such that $r+s\sqrt{xy}$ is integral over $\mathbb{C}[x,y]$. Then $r-s\sqrt{xy}$ is also integral over $\mathbb{C}[x,y]$, so their sum is. Hence $2r$ is integral over $\mathbb{C}[x,y]$. But $\mathbb{C}[x,y]$ is a UFD, hence integrally closed, so $2r \in \mathbb{C}[x,y]$ and thus $r \in \mathbb{C}[x,y]$. Similarly, $s\sqrt{xy}$ is integral over $\mathbb{C}[x,y]$, hence $s^2xy \in \mathbb{C}[x,y], s \in \mathbb{C}(x,y)$. Clearly, then, s can have no denominator, so $s \in \mathbb{C}[x,y]$. Hence $r+s\sqrt{xy} \in \mathbb{C}[x,y,\sqrt{xy}]$.