polynomial ring which is PID

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Theorem. If a polynomial ring $D[X]$ over an integral domain D is a principal ideal domain, then coefficient ring D is a field. (Cf. the corollary 4 in the entry polynomial ring over a field.)

Proof. Let a be any non-zero element of D. Then the ideal $(a,\,X)$ of $D[X]$ is a principal ideal $(f(X))$ with $f(X)$ a `http://planetmath.org/ZeroPolynomial2`non-zero polynomial. Therefore,

$$a\;=\;f(X)g(X),\quad X\;=\;f(X)h(X)$$

with $g(X)$ and $h(X)$ certain polynomials in $D[X]$. From these equations one infers that $f(X)$ is a polynomial c and $h(X)$ is a first degree polynomial $b_0{+}b_1X$ $(b_1\neq 0)$. Thus we obtain the equation

$$cb_0+cb_1X\;=\;X,$$

which shows that cb_1 is the unity 1 of D. Thus $c=f(X)$ is a unit of D, whence

$$(a,\,X)\;=\;(f(X))\;=\;(1)\;=\;D[X].$$

So we can write

$$1\;=\;a{\cdot}u(X)+X{\cdot}v(X),$$

where $u(X),\,v(X)\in D[X]$. This equation cannot be possible without that a times the constant term of $u(X)$ is the unity. Accordingly, a has a multiplicative inverse in D. Because a was arbitrary non-zero elenent of the integral domain D, D is a field.

References

[1] David M. Burton: *A first course in rings and ideals.* Addison-Wesley Publishing Company. Reading, Menlo Park, London, Don Mills (1970).