every prime ideal is radical

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Let $\mathcal{R}$ be a commutative ring and let $\mathfrak{P}$ be a prime ideal of $\mathcal{R}$.

Proposition 1. *Every prime ideal $\mathfrak{P}$ of $\mathcal{R}$ is a radical ideal, i.e.*

$$\mathfrak{P} = \mathrm{Rad}(\mathfrak{P})$$

Proof. Recall that $\mathfrak{P} \subsetneq \mathcal{R}$ is a prime ideal if and only if for any $a, b \in \mathcal{R}$

$$a \cdot b \in \mathfrak{P} \Rightarrow a \in \mathfrak{P} \text{ or } b \in \mathfrak{P}$$

Also, recall that

$$\mathrm{Rad}(\mathfrak{P}) = \{r \in \mathcal{R} \mid \exists n \in \mathbb{N} \text{ such that } r^n \in \mathfrak{P}\}$$

Obviously, we have $\mathfrak{P} \subseteq \mathrm{Rad}(\mathfrak{P})$ (just take $n = 1$), so it remains to show the reverse inclusion.

Suppose $r \in \mathrm{Rad}(\mathfrak{P})$, so there exists some $n \in \mathbb{N}$ such that $r^n \in \mathfrak{P}$. We want to prove that r must be an element of the prime ideal $\mathfrak{P}$. For this, we use induction on n to prove the following proposition:

For all $n \in \mathbb{N}$, for all $r \in \mathcal{R}$, $r^n \in \mathfrak{P} \Rightarrow r \in \mathfrak{P}$.

Case $n = 1$**:** This is clear, $r \in \mathfrak{P} \Rightarrow r \in \mathfrak{P}$.

Case $n \Rightarrow$ **Case** $n+1$**:** Suppose we have proved the proposition for the case n, so our induction hypothesis is

$$\forall r \in \mathcal{R}, \quad r^n \in \mathfrak{P} \Rightarrow r \in \mathfrak{P}$$

and suppose $r^{n+1} \in \mathfrak{P}$. Then

$$r \cdot r^n = r^{n+1} \in \mathfrak{P}$$

and since $\mathfrak{P}$ is a prime ideal we have

$$r \in \mathfrak{P} \text{ or } r^n \in \mathfrak{P}$$

Thus we conclude, either directly or using the induction hypothesis, that $r \in \mathfrak{P}$ as desired.

$\square$