absolute value

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Let R be an ordered ring and let $a \in R$. The *absolute value* of a is defined to be the function $|\ |\colon R \to R$ given by

$$|a| := \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{otherwise.} \end{cases}$$

In particular, the usual absolute value $|\ |$ on the field $\mathbb{R}$ of real numbers is defined in this manner. An equivalent definition over the real numbers is $|a| := \max\{a, -a\}$.

Absolute value has a different meaning in the case of complex numbers: for a complex number $z \in \mathbb{C}$, the absolute value $|z|$ of z is defined to be $\sqrt{x^2 + y^2}$, where $z = x + yi$ and $x, y \in \mathbb{R}$ are real.

All absolute value functions satisfy the defining properties of a valuation, including:

- $|a| \geq 0$ for all $a \in R$, with equality if and only if $a = 0$
- $|ab| = |a| \cdot |b|$ for all $a, b \in R$
- $|a + b| \leq |a| + |b|$ for all $a, b \in R$ (triangle inequality)

However, in general they are not literally valuations, because valuations are required to be real valued. In the case of $\mathbb{R}$ and $\mathbb{C}$, the absolute value is a valuation, and it induces a metric in the usual way, with distance function defined by $d(x, y) := |x - y|$.