planetmath.org

Math for the people, by the people.

quotient ring modulo prime ideal

Canonical name	QuotientRingModuloPrimeIdeal
Date of creation	2013-03-22 17:37:09
Last modified on	2013-03-22 17:37:09
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	7
Author	pahio (2872)
Entry type	Theorem
Classification	msc 13C99
Related topic	CharacterisationOfPrimeIdeals
Related topic	QuotientRing

Theorem. Let R be a commutative ring with non-zero unity 1 and $\mathfrak{p}$ an ideal of R. The quotient ring $R/\mathfrak{p}$ is an integral domain if and only if $\mathfrak{p}$ is a prime ideal.

Proof. 1º. First, let $\mathfrak{p}$ be a prime ideal of R. Then $R/\mathfrak{p}$ is of course a commutative ring and has the unity $1+\mathfrak{p}$. If the product $(r+\mathfrak{p})(s+\mathfrak{p})$ of two residue classes vanishes, i.e. equals $\mathfrak{p}$, then we have $rs+\mathfrak{p} = \mathfrak{p}$, and therefore rs must belong to $\mathfrak{p}$. Since $\mathfrak{p}$ is , either r or s belongs to $\mathfrak{p}$, i.e. $r+\mathfrak{p} = \mathfrak{p}$ or $s+\mathfrak{p} = \mathfrak{p}$. Accordingly, $R/\mathfrak{p}$ has no zero divisors and is an integral domain.

2º. Conversely, let $R/\mathfrak{p}$ be an integral domain and let the product rs of two elements of R belong to $\mathfrak{p}$. It follows that $(r+\mathfrak{p})(s+\mathfrak{p}) = rs+\mathfrak{p} = \mathfrak{p}$. Since $R/\mathfrak{p}$ has no zero divisors, $r+\mathfrak{p} = \mathfrak{p}$ or $s+\mathfrak{p} = \mathfrak{p}$. Thus, r or s belongs to $\mathfrak{p}$, i.e. $\mathfrak{p}$ is a prime ideal.