proof that a Noetherian domain is Dedekind if it is locally a PID

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We show that for a http://planetmath.org/NoetherianNoetherian domain R with field of fractions k, the following are equivalent.

1. R is Dedekind. That is, it is integrally closed and every nonzero prime ideal is maximal.

2. for every maximal ideal $\mathfrak{m}$,

$$R_{\mathfrak{m}} \equiv \left\{s^{-1}x : s \in R \setminus \mathfrak{m}, x \in R\right\}$$

   is a principal ideal domain.

For a given maximal ideal $\mathfrak{m}$ and ideal $\mathfrak{a}$ of R, we shall write $\bar{\mathfrak{a}}$ for the ideal generated by $\mathfrak{a}$ in $R_{\mathfrak{m}}$, which consists of the elements of the form $s^{-1}a$ for $a \in \mathfrak{a}$ and $s \in R \setminus \mathfrak{m}$. It is then easily seen that $\mathfrak{p} \mapsto \bar{\mathfrak{p}}$ gives a bijection between the prime ideals of R contained in $\mathfrak{m}$ and the prime ideals of $R_{\mathfrak{m}}$, with inverse $\mathfrak{p} \mapsto R \cap \mathfrak{p}$. In particular $\bar{\mathfrak{m}}$ is the unique maximal ideal of $R_{\mathfrak{m}}$, which is therefore a local ring.

Now suppose that R is Dedekind, then the localization $R_{\mathfrak{m}}$ will be a Dedekind domain (localizations of Dedekind domains are Dedekind) with a unique maximal ideal, so it is a principal ideal domain (Dedekind domains with finitely many primes are PIDs).

Only the converse remains to be shown, so suppose that R is a Noetherian domain such that $R_{\mathfrak{m}}$ is a principal ideal domain for every maximal ideal $\mathfrak{m}$. In particular, $R_{\mathfrak{m}}$ is integrally closed and every nonzero prime ideal is maximal, so it contains a unique nonzero prime ideal $\bar{\mathfrak{m}}$.

We start by showing that every nonzero prime ideal $\mathfrak{p}$ of R is maximal. Choose a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$. Then, $\bar{\mathfrak{p}}$ is a nonzero prime ideal, so $\bar{\mathfrak{p}} = \bar{\mathfrak{m}}$ and therefore $\mathfrak{p} = \mathfrak{m}$ is maximal.

We finally show that R is integrally closed. So, choose any x integral over R and lying in its field of fractions. Let $\mathfrak{a}$ be the ideal

$$\mathfrak{a} = \{a \in R : ax \in R\}.$$

We use proof by contradiction to show that $\mathfrak{a}$ is the whole of R. So, supposing that this is not the case, there exists a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. Then x will be integral over the integrally closed ring $R_{\mathfrak{m}}$ and therefore $x \in R_{\mathfrak{m}}$. So, $x = s^{-1}y$ for some $s \in R \setminus \mathfrak{m}$ and $y \in R$. Then, $sx = y \in R$ so $s \in \mathfrak{a} \subseteq \mathfrak{m}$, which is the required contradiction. Therefore, $\mathfrak{a} = R$ and, in particular, $1 \in \mathfrak{a}$ and $x = 1x \in R$, showing that R is integrally closed.