decomposition of a module using orthogonal idempotents

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Let K be a field and let G be a finite abelian group. For simplicity, we will assume that the characteristic of K does not divide the order of G. Let $\varphi_1, \ldots, \varphi_n$ be a complete set (up to equivalence) of distinct `http://planetmath.org/GroupRepresentation`irreducible (linear) representations of G over K, so that φ_i is a homomorphism:

$$\varphi_i \colon G \longrightarrow \mathrm{GL}(n_i, K)$$

where n_i is the degree of the representation φ_i and $\sum_i n_i = |G|$. Let $\chi_1, \ldots, \chi_n$ be the irreducible characters attached to the φ_i, i.e. the function $\chi_i \colon G \to K$ is defined by

$$\chi_i(g) = \mathrm{Trace}(\varphi_i(g)).$$

Notice, however, that in general the map χ_i is not a homomorphism from the group into either the additive or multiplicative group of K. We define a system of primitive orthogonal idempotents of the group ring $K[G]$, one for each χ_i, by:

$$\mathbf{1}_{\chi_i} = \frac{1}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g \in K[G]$$

so that $\sum_i \mathbf{1}_{\chi_i} = 1 \in K$ and $\mathbf{1}_{\chi_i} \cdot \mathbf{1}_{\chi j} = \delta_{ij}$ where δ_{ij} is the Kronecker delta function. We define the χ_i component of $K[G]$to be the ideal $K[G]_{\chi_i} = \mathbf{1}_{\chi_i} \cdot K[G]$. Notice that $V_i = K[G]_{\chi_i}$ is a finite dimensional K-vector space, on which G acts. Thus, the representation of G afforded by the $K[G]$-module V_i, call it φ, must be one of the representations φ_j defined above. Comparing the trace, one concludes that $\varphi = \varphi_i$ and $V_i = K[G]_{\chi i}$ is a vector space of dimension n_i. In particular, there is a decomposition:

$$K[G] = \oplus_\chi K[G]_\chi.$$

If $k \in K[G]$ then by the previous decomposition, we can write:

$$k = \sum_\chi k_\chi$$

where $k_\chi \in K[G]_\chi$. Notice that the representations φ_i can be retrieved as:

$$\varphi_i \colon G \longrightarrow \mathrm{GL}(\mathrm{K[G]}_{\chi_i}).$$

Lemma. *Let M be a $K[G]$-module and define submodules $M_\chi = \mathbf{1}_\chi \cdot M$, for each irreducible character χ. Then:*

1. *There is a decomposition $M = \oplus_\chi M_\chi$.*
2. *The group $K[G]$ acts on M_χ via $K[G]_\chi$. In other words, if $k \in K[G]$, with $k = \sum_\chi k_\chi$ then:*

$$k \cdot m = k_\chi \cdot m, \text{ for all } m \in M_\chi.$$

3. *The representation φ of G afforded by the K-vector space M_{χ_i} is, up to equivalence, a number of copies of φ_i, i.e.*

$$\varphi = \varphi_i \oplus \ldots \oplus \varphi_i = \varphi_i^{\oplus r}$$

for some integer $r \geq 0$. In other words, M_{χ_i} is the submodule consisting of the sum of all $K[G]$-submodules of M isomorphic to $K[G]_{\chi_i}$.

4. *Suppose that M, N and R are $K[G]$-modules which fit in the short exact sequence:*

$$0 \longrightarrow R \longrightarrow M \longrightarrow N \longrightarrow 0$$

where every map above is a $K[G]$-module homomorphism, i.e. each map is a K-homomorphism which is compatible with the action of G. Then, the exact sequence above yields an exact sequence of χ components:

$$0 \longrightarrow R_\chi \longrightarrow M_\chi \longrightarrow N_\chi \longrightarrow 0$$

for every irreducible character χ.