example of resultant (2)

Canonical name	ExampleOfResultant2
Date of creation	2013-03-22 14:36:36
Last modified on	2013-03-22 14:36:36
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	7
Author	rspuzio (6075)
Entry type	Example
Classification	msc 13P10

This example shows how resultants can be used to solve simultaneous algebraic equations in two variables. We shall compute the intersection of two ellipses.

Consider the system of equations $f(x, y) = 0, g(x, y) = 0$ where

$$f(x, y) = 3x^2 + 2xy + 3y^2 - 2$$

$$g(x, y) = 3x^2 - 2xy + 3y^2 - 2$$

We will consider f and g as polynomials in x whose coefficients are functions of x. What this means can be seen by writing f and g as

$$(3)x^2 + (2y)x + (3y^2 - 2)$$

$$(3)x^2 + (-2y)x + (3y^2 - 2)$$

We will now construct the resultant by computing Sylvester's determinant. In the notation of the main article, the coefficients of the various powers of x may be notated as

$$a_0 = 3 \quad a_1 = 2y \quad a_2 = 3y^2 - 2$$

$$b_0 = 3 \quad b_1 = -2y \quad b_2 = 3y^2 - 2$$

The determinant is

$$\begin{vmatrix} 3 & 2y & 3y^2 - 2 & 0 \\ 0 & 3 & 2y & 3y^2 - 2 \\ 3 & -2y & 3y^2 - 2 & 0 \\ 0 & 3 & -2y & 3y^2 - 2 \end{vmatrix}$$

This determinant evaluates to $144y^4 - 96y^2$. Hence, in order for the system of equations to have a solution, y must satisfy the equation

$$144y^4 - 96y^2 = 0$$

We can factor the polynomial as

$$144y^4 - 96y^2 = 144(y + \frac{\sqrt{2}}{3})y^2(y - \frac{\sqrt{2}}{3})$$

Hence, the solutions are

$$y = -\frac{\sqrt{2}}{3}$$

$$y = 0$$

$$y = +\frac{\sqrt{2}}{3}$$

Note that the solution $y = 0$ occurs with multiplicity 2. We shall see what that means shortly.

Having found the possible values of y, let us now find the corresponding values for x. Substituting the possible value $y = -\frac{\sqrt{2}}{3}$ into the equation $f(x, y) = 0$, we obtain

$$3x^2 - \frac{2\sqrt{2}}{3}x = 0$$

Hence, either $x = 0$ or $x = +\frac{2\sqrt{2}}{3}$. If we substitute $x = 0$ and $y = -\frac{\sqrt{2}}{3}$ into $g(x, y)$, we obtain zero so

$$x = 0 \quad y = -\frac{\sqrt{2}}{3}$$

is a solution of our system. However, if we substitute $x = +\frac{2\sqrt{2}}{3}$ and $y = -\frac{\sqrt{2}}{3}$ into $g(x, y)$, we obtain $\frac{16}{9}$, so this root does not lead to a solution of the original system of equations.

Substituting the possible value $y = +\frac{\sqrt{2}}{3}$ into the equation $f(x, y) = 0$, we obtain

$$3x^2 + \frac{2\sqrt{2}}{3}x = 0$$

Hence, either $x = 0$ or $x = -\frac{2\sqrt{2}}{3}$. If we substitute $x = 0$ and $y = +\frac{\sqrt{2}}{3}$ into $g(x, y)$, we obtain zero so

$$x = 0 \quad y = +\frac{\sqrt{2}}{3}$$

is a solution of our system. However, if we substitute $x = -\frac{2\sqrt{2}}{3}$ and $y = +\frac{\sqrt{2}}{3}$ into $g(x, y)$, we obtain $-\frac{16}{9}$, so this root does not lead to a solution of the original system of equations.

Finally, let us consider the value $y = 0$. Substituting this value into $f(x, y)$, we obtain the equation

$$3x^2 - 2 = 0$$

This equation has the solutions $x = -\sqrt{2}/3$ and $x = +\sqrt{2}/3$. Substituting $y = 0$ and $x = -\sqrt{2}/3$ into $g(x, y)$, we obtain 0, so

$$y = 0 \qquad x = -\sqrt{2}/3$$

is a solution of the system of equations. Likewise, substituting $y = 0$ and $x = +\sqrt{2}/3$ into $g(x, y)$, we obtain 0, so

$$y = 0 \qquad x = +\sqrt{2}/3$$

is a solution of the system of equations. In this case, we obtained two solutions to the system of equations.

At this point, recall the remark that $y = 0$ was a double root of the resultant. This fact explains why both values of x gave rise to solutions of the system when $y = 0$. In general, the number of solutions (counted with multiplicity) of the system of equations for a particular value of y equals the mutiplicity of that value of y as a root of the resultant.