product of finitely generated ideals

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Related topic	UniqueFactorizationAndIdealsInRingOfIntegers
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Related topic	WellDefinednessOfProductOfFinitelyGeneratedIdeals
Defines	Dedekind–Mertens lemma

Let R be a commutative ring having at least one regular element and T its total ring of fractions. Let $\mathfrak{a} := (a_0,\, a_1,\, \ldots,\, a_{m-1})$ and $\mathfrak{b} := (b_0,\, b_1,\, \ldots,\, b_{n-1})$ be two fractional ideals of R (see the entry "fractional ideal of commutative ring"). Then the product submodule $\mathfrak{ab}$ of T is also a of R and is generated by all the elements a_ib_j, thus having a generating set of mn elements.

Such a generating set may be condensed in the case of any Dedekind domain, especially for the of any algebraic number field one has the multiplication formula

$$\mathfrak{ab} = (a_0b_0,\, a_0b_1+a_1b_0,\, a_0b_2+a_1b_1+a_2b_0,\, \ldots,\, a_{m-1}b_{n-1}). \tag{1}$$

Here, the number of generators is only $m+n-1$ (in principle, every ideal of a Dedekind domain has a http://planetmath.org/TwoGeneratorPropertygenerating system of two elements). The formula is http://planetmath.org/Characterizationcharacteristi still for a wider class of rings R which may contain zero divisors, viz. for the Prüfer rings (see [1]), but then at least one of $\mathfrak{a}$ and $\mathfrak{b}$ must be a regular ideal.

Note that the generators in (1) are formed similarly as the coefficients in the product of the polynomials $f(X) := f_0+f_1X+\cdots+f_{m-1}X^{m-1}$ and $g(X) := g_0+g_1X+\cdots+g_{n-1}X^{n-1}$. Thus we may call the fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R the *coefficient modules* $\mathfrak{m}_f$ and $\mathfrak{m}_g$ of the polynomials f and g (they are R-modules). Hence the formula (1) may be rewritten as

$$\mathfrak{m}_f\mathfrak{m}_g = \mathfrak{m}_{fg}. \tag{2}$$

This formula says the same as Gauss's lemma I for a unique factorization domain R.

Arnold and Gilmer [2] have presented and proved the following generalisation of (2) which is valid under much less stringent assumptions than the ones requiring R to be a Prüfer ring (initially: a Prüfer domain); the proof is somewhat simplified in [1].

Theorem (Dedekind–Mertens lemma). Let R be a subring of a commutative ring T. If f and g are two arbitrary polynomials in the polynomial ring $T[X]$, then there exists a non-negative integer n such that the R-submodules of T generated by the coefficients of the polynomials f, g and fg satisfy the equality

$$\mathfrak{m}_f^{n+1}\,\mathfrak{m}_g = \mathfrak{m}_f^{n}\,\mathfrak{m}_{fg}. \tag{3}$$

References

[1] J. Pahikkala: "Some formulae for multiplying and inverting ideals". – *Ann. Univ. Turkuensis* **183** (A) (1982).

[2] J. Arnold & R. Gilmer: "On the contents of polynomials". – *Proc. Amer. Math. Soc.* **24** (1970).