prime ideals by Artin are prime ideals

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Theorem. Due to Artin, a prime ideal of a commutative ring R is the maximal element among the ideals not intersecting a multiplicative subset S of R. This is http://planetmath.org/Equivalent3equivalent to the usual criterion

$$ab \in \mathfrak{p} \quad \Rightarrow \quad a \in \mathfrak{p} \;\lor\; b \in \mathfrak{p} \tag{1}$$

of prime ideal (see the entry http://planetmath.org/PrimeIdealprime ideal).

Proof. 1º. Let $\mathfrak{p}$ be a prime ideal by Artin, corresponding the semigroup S, and let the ring product ab belong to $\mathfrak{p}$. Assume, contrary to the assertion, that neither of a and b lies in $\mathfrak{p}$. When $(\mathfrak{p},\, x)$ generally means the least ideal containing $\mathfrak{p}$ and an element x, the antithesis implies that

$$\mathfrak{p} \subset (\mathfrak{p},\, a) \quad \land \quad \mathfrak{p} \subset (\mathfrak{p},\, a),$$

whence by the maximality of $\mathfrak{p}$ we have

$$(\mathfrak{p},\, a) \cap S \neq \varnothing \quad \land \quad (\mathfrak{p},\, b) \cap S \neq \varnothing.$$

Therefore we can chose such elements $s_i = p_i + r_i a + n_i a$ of S (N.B. the multiples) that

$$p_i \in \mathfrak{p},\;\; r_i \in R,\;\; n_i \in \mathbb{Z} \quad (i = 1,\, 2).$$

But then

$$s_1 s_2 \;=\; (p_2{+}r_2 b{+}n_2 b)p_1{+}(r_1 a{+}n_1 a)p_2{+}(r_1 r_2{+}n_2 r_1{+}n_1 r_2)ab{+}(n_1 n_2)ab \in \mathfrak{p}.$$

This is however impossible, since the product $s_1 s_2$ belongs to the semigroup S and $\mathfrak{p} \cap S = \varnothing$. Because the antithesis thus is wrong, we must have $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

2º. Let us then suppose that an ideal $\mathfrak{p}$ satisfies the condition (1) for all $a,\, b \in R$. It means that the set $S = R \smallsetminus \mathfrak{p}$ is a multiplicative semigroup. Accordingly, the $\mathfrak{p}$ is the greatest ideal not intersecting the semigroup S, Q.E.D.

Remark. It follows easily from the theorem, that if $\mathfrak{p}$ is a prime ideal of the commutative ring $\mathfrak{O}$ and $\mathfrak{o}$ is a subring of $\mathfrak{O}$, then $\mathfrak{p} \cap \mathfrak{o}$ is a prime ideal of $\mathfrak{o}$.