valuation determined by valuation domain

Canonical name	ValuationDeterminedByValuationDomain
Date of creation	2013-03-22 14:54:58
Last modified on	2013-03-22 14:54:58
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	10
Author	pahio (2872)
Entry type	Theorem
Classification	msc 13F30
Classification	msc 13A18
Classification	msc 12J20
Classification	msc 11R99
Related topic	ValuationDomainIsLocal
Related topic	KrullValuationDomain
Related topic	PlaceOfField

Theorem. Every valuation domain determines a Krull valuation of the field of fractions.

Proof. Let R be a valuation domain, K its field of fractions and E the group of units of R. Then E is a normal subgroup of the multiplicative group $K^* = K \smallsetminus \{0\}$. So we can form the factor group K^*/E, consisting of all cosets aE where $a \in K^*$, and attach to it the additional "coset" $0E$ getting thus a multiplicative group K/E equipped with zero. If $\mathfrak{m} = R \smallsetminus E$ is the maximal ideal of R (any valuation domain has a unique maximal ideal — cf. valuation domain is local), then we denote $\mathfrak{m}^* = \mathfrak{m} \smallsetminus \{0\}$ and $S = \mathfrak{m}^*/E = \{aE : \ a \in \mathfrak{m}^*\}$. Then the subsemigroup S of K/E makes K/E an ordered group equipped with zero. It is not hard to check that the mapping

$$x \mapsto |x| := xE$$

from K to K/E is a Krull valuation of the field K.