generators of inverse ideal

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Theorem. Let R be a commutative ring with non-zero unity and let T be the total ring of fractions of R. If $\mathfrak{a} = (a_1,\,\ldots,\,a_n)$ is an invertible http://planetmath.org/FractionalIdealOfCommutativeRing*fractional ideal* of R with $\mathfrak{ab} = R$, then also the inverse ideal $\mathfrak{b}$ can be generated by n elements of T.

Proof. The equation $\mathfrak{ab} = (1)$ implies the existence of the elements a'_i of $\mathfrak{a}$ and b'_i of $\mathfrak{b}$ $(i = 1, \ldots, m)$ such that $a'_1b'_1 + \cdots + a'_mb'_m = 1$. Because the a'_i's are in $\mathfrak{a}$, they may be expressed as

$$a'_i = \sum_{j=1}^{n} r_{ij}a_j \qquad (i = 1, \ldots, m),$$

where the r_{ij}'s are some elements of R. Now the unity acquires the form

$$1 = \sum_{i=1}^{m} a'_ib'_i = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}a_jb'_i = \sum_{j=1}^{n} a_j \sum_{i=1}^{m} r_{ij}b'_i = \sum_{j=1}^{n} a_jb_j,$$

in which

$$b_j = \sum_{i=1}^{m} r_{ij}b'_i \in R\mathfrak{b} = \mathfrak{b} \qquad (j = 1, \ldots, n).$$

Thus an arbitrary element b of the $\mathfrak{b}$ satisfies the condition

$$b = b\cdot 1 = \sum_{j=1}^{n} (a_jb)b_j \in Rb_1 + \cdots + Rb_n = (b_1,\,\ldots,\,b_n).$$

Consequently, $\mathfrak{b} \subseteq (b_1,\,\ldots,\,b_n)$. Since the inverse inclusion is apparent, we have the equality

$$\mathfrak{a}^{-1} = \mathfrak{b} = (b_1,\,\ldots,\,b_n).$$