proof of Hensel's lemma

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Lemma: Using the setup and terminology of the statement of Hensel's Lemma, for $i \geq 0$,

$$\begin{array}{ll} \text{i)} & |f'(\alpha_i)| = |f'(\alpha_0)| \\ \text{ii)} & \left|\dfrac{f(\alpha_i)}{f'(\alpha_i)^2}\right| \leq D^{2^i} \\ \text{iii)} & |\alpha_i - \alpha_0| \leq D \\ \text{iv)} & \alpha_i \in \mathcal{O}_K \end{array}$$

where $D = \left|\frac{f(\alpha_0)}{f'(\alpha_0)^2}\right|$.

Proof: All four statements clearly hold when $i = 0$. Suppose they are true for i. The proof for $i + 1$ essentially uses Taylor's formula. Let $\delta = \left|\frac{-f(\alpha_i)}{f'(\alpha_i)}\right|$. Then

$$f'(\alpha_{i+1}) = f'(\alpha_i + \delta) = f'(\alpha_i) + \delta u$$

$$f(\alpha_{i+1}) = f(\alpha_i + \delta) = f(\alpha_i) + f'(\alpha_i)\delta + \delta^2 v$$

for $u, v \in \mathcal{O}_K$. $|\delta| \leq D^{2^i}|f'(\alpha_i)|$ by induction, and since $D < 1$, it follows that $|\delta| < |f'(\alpha_i)|$. Since the norm is non-Archimedean, we see that

$$f'(\alpha_{i+1}) = f'(\alpha_i)$$

proving i).

$f(\alpha_i) + f'(\alpha_i)\delta = 0$ by definition of δ, so $f(\alpha_{i+1}) = \delta^2 v$ and hence $|f(\alpha_{i+1})| \leq |\delta^2|$. Hence

$$\left|\frac{f(\alpha_{i+1})}{f'(\alpha_{i+1})^2}\right| \leq \frac{|\delta|^2}{|f'(\alpha_{i+1})|^2} = \frac{|\delta|^2}{|f'(\alpha_i)|^2} = \left(\frac{|\delta|}{|f'(\alpha_i)|}\right)^2 = \left(\frac{|f(\alpha_i)|}{|f'(\alpha_i)|^2}\right)^2 \leq D^{2^{i+1}}$$

where the last equality follows by induction. This proves ii).

To prove iii), note that $|\alpha_{i+1} - \alpha_i| = |\delta|$ by the definitions of δ and α_{i+1}, so $|\alpha_{i+1} - \alpha_i| \leq D^{2^i}|f'(\alpha_i)| = D^{2^i}|f'(\alpha_0) < D$ when $i > 0$ since $D^2 < D = \left|\frac{f(\alpha_0)}{f'(\alpha_0)^2}\right|$. So by induction, $|\alpha_{i+1} - \alpha_0| \leq D$.

Finally, to prove iv) and the proof of the lemma, $\delta \in \mathcal{O}_K$ since $|\delta| < \left|\frac{f(\alpha_0)}{f'(\alpha_0)}\right| \leq 1$ and hence is in the valuation ring of K. So by induction, $\alpha_{i+1} = \alpha_i + \delta \in \mathcal{O}_K$.

Proof of Hensel's Lemma:

To prove Hensel's lemma from the above lemma, note that $\delta = \delta_i \to 0$ since $|\delta| \leq D^{2^i}|f'(\alpha_0)|$, so $\{\alpha_i\}$ converges to $\alpha \in \mathcal{O}_K$ since K is complete. Thus $f(\alpha_i) \to f(\alpha)$ by continuity. But $|f(\alpha_i)| \leq |\delta^2| = D^{2^{i+1}}|f'(\alpha_0)|$, so $|f(\alpha_i)| \to 0$, so $f(\alpha) = 0$ and the proof is complete.