prime factors of $x^n - 1$

Canonical name	PrimeFactorsOfXn1
Date of creation	2013-03-22 16:29:51
Last modified on	2013-03-22 16:29:51
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	12
Author	pahio (2872)
Entry type	Result
Classification	msc 13G05
Related topic	GausssLemmaII
Related topic	IrreducibilityOfBinomialsWithUnityCoefficients
Related topic	FactorsOfNAndXn1
Related topic	ExamplesOfCyclotomicPolynomials

We list prime factor of the binomials x^n–1 in $\mathbb{Q}$, i.e. in the polynomial ring $\mathbb{Q}[x]$. The prime factors can always be chosen to be with integer coefficients and the number of the prime factors equals to `http://planetmath.org/TauFunction`$\tau(n)$; see `http://planetmath.org/FactorsOfNAndXn1`the proof.

$x-1$

$x^2-1=(x+1)(x-1)$

$x^3-1=(x^2+x+1)(x-1)$

$x^4-1=(x^2+1)(x+1)(x-1)$

$x^5-1=(x^4+x^3+x^2+x+1)(x-1)$

$x^6-1=(x^2+x+1)(x^2-x+1)(x+1)(x-1)$

$x^7-1=(x^6+x^5+x^4+x^3+x^2+x+1)(x-1)$

$x^8-1=(x^4+1)(x^2+1)(x+1)(x-1)$

$x^9-1=(x^6+x^3+1)(x^2+x+1)(x-1)$

$x^{10}-1=(x^4+x^3+x^2+1)(x^4-x^3+x^2-x+1)(x+1)(x-1)$

$x^{11}-1=(x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1)(x-1)$

$x^{12}-1=(x^4-x^2+1)(x^2+x+1)(x^2-x+1)(x^2+1)(x+1)(x-1)$

$x^{13}-1=(x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1)(x-1)$

$x^{14}-1=(x^6+x^5+x^4+x^3+x^2+x+1)(x^6-x^5+x^4-x^3+x^2-x+1)(x+1)(x-1)$

$x^{15}-1=(x^8-x^7+x^5-x^4+x^3-x+1)(x^4+x^3+x^2+x+1)(x^2+x+1)(x-1)$

$x^{16}-1=(x^8+1)(x^4+1)(x^2+1)(x+1)(x-1)$

$x^{17}-1=(x^{16}+x^{15}+x^{14}+\ldots+x^2+x+1)(x-1)$

$x^{18}-1=(x^6+x^3+1)(x^6-x^3+1)(x^2+x+1)(x^2-x+1)(x+1)(x-1)$

$x^{19}-1=(x^{18}+x^{17}+x^{16}+\ldots+x^2+x+1)(x-1)$

$x^{20}-1=(x^8-x^6+x^4-x^2+1)(x^4+x^3+x^2+x+1)(x^4-x^3+x^2-x+1)(x^2+1)(x+1)(x-1)$

$x^{21}-1=(x^{12}-x^{11}+x^9-x^8+x^6-x^4+x^3-x+1)(x^6+x^5+x^4+x^3+x^2+x+1)(x^2+x+1)(x-1)$

$x^{22}-1=(x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1)(x^{10}-x^9+x^8-x^7+x^6-x^5+x^4-x^3+x^2-x+1)(x+1)(x-1)$

$x^{23}-1=(x^{22}+x^{21}+x^{20}+\ldots+x^2+x+1)(x-1)$

$x^{24}-1=(x^8-x^4+1)(x^4-x^2+1)(x^4+1)(x^2+x+1)(x^2-x+1)(x^2+1)(x+1)(x-1)$

Note 1. All factors shown above are irreducible polynomials (in the field $\mathbb{Q}$ of their own coefficients), but of course they (except $x\pm1$) may be split into factors of positive degree in certain extension fields; so e.g.

$$x^4+1=(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1)\quad\text{in the field }\ \mathbb{Q}(\sqrt{2}).$$

Note 2. The 24 examples of factorizations are true also in the fields of characteristic $\neq 0$, but then many of the factors can be simplified or factored onwards (e.g. $x^2+1 \equiv (x+1)^2$ if the `http://planetmath.org/Characteristic`characteristic is 2).