proof of finite inseparable extensions of Dedekind domains are Dedekind

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Let R be a Dedekind domain with field of fractions K and L/K be a field extension. We suppose that K has `http://planetmath.org/characteristic`characteristic $p>0$ and that there is a $q=p^r$ such that $x^q\in K$ for all $x\in L$. In particular, this is satisfied if it is a purely inseparable and finite extension.

We show that the integral closure A of R in L is a Dedekind domain.

We cannot apply the same method of proof as for the proof of finite separable extensions of Dedekind domains are Dedekind, because here A does not have to be finitely generated as an R-module.

We use the characterization of Dedekind domains as integral domains in which all nonzero ideals are invertible (see proof that a domain is Dedekind if its ideals are invertible). Note that for any $x\in A$, x^q is in K and is integral over R so, by integral closure, $x^q\in R$.

So, let $\mathfrak{a}$ be a nonzero ideal in A, and let $\mathfrak{b}$ be the ideal of R generated by terms of the form a^q for $a\in\mathfrak{a}$,

$$\mathfrak{b}=(a^q:a\in\mathfrak{a})_R\,.$$

Then, as R is a Dedekind domain, there is a fractional ideal $\mathfrak{b}^{-1}$ of R such that $\mathfrak{b}\mathfrak{b}^{-1}=R$, and write $\mathfrak{b}_A^{-1}$ for the fractional ideal of A generated by $\mathfrak{b}^{-1}$. Then,

$$1\in R=\mathfrak{b}\mathfrak{b}^{-1}\subseteq\mathfrak{a}^q\mathfrak{b}_A^{-1}. \tag{1}$$

On the other hand, if $a_1,\ldots,a_q\in\mathfrak{a}$ and $b\in\mathfrak{b}^{-1}$ then

$$(a_1\cdots a_qb)^q=(a_1^qb)\cdots(a_q^qb)\in R,$$

so $a_1\cdots a_qb$ is integral over R and is in A. Therefore, $\mathfrak{a}^q\mathfrak{b}_A^{-1}\subseteq A$. Combining with (**??**) gives $\mathfrak{a}^q\mathfrak{b}_A^{-1}=A$, so $\mathfrak{a}$ is invertible with inverse $\mathfrak{a}^{q-1}\mathfrak{b}_A^{-1}$.