example of free module

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from the definition, $\mathbb{Z}^n$ is `http://planetmath.org/node/FreeModule`free as a $\mathbb{Z}$-module for any positive integer n.

A more interesting example is the following:

Theorem 1. *The set of rational numbers $\mathbb{Q}$ do* not *form a `http://planetmath.org/node/FreeMo`*
$\mathbb{Z}$-module.

Proof. First note that any two elements in $\mathbb{Q}$ are $\mathbb{Z}$-linearly dependent. If $x = \frac{p_1}{q_1}$ and $y = \frac{p_2}{q_2}$, then $q_1 p_2 x - q_2 p_1 y = 0$. Since `http://planetmath.org/Basis`basis elements must be linearly independent, this shows that any basis must consist of only one element, say $\frac{p}{q}$, with p and q relatively prime, and without loss of generality, $q > 0$. The $\mathbb{Z}$-span of $\{\frac{p}{q}\}$ is the set of rational numbers of the form $\frac{np}{q}$. I claim that $\frac{1}{q+1}$ is not in the set. If it were, then we would have $\frac{np}{q} = \frac{1}{q+1}$ for some n, but this implies that $np = \frac{q}{q+1}$ which has no solutions for $n, p \in \mathbb{Z}$,$q \in \mathbb{Z}^+$, giving us a contradiction. $\square$