proof that a domain is Dedekind if its ideals are products of maximals

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Let R be an integral domain. We show that it is a Dedekind domain if and only if every nonzero proper ideal can be expressed as a product of maximal ideals. To do this, we make use of the characterization of Dedekind domains as integral domains in which every nonzero integral ideal is invertible (proof that a domain is Dedekind if its ideals are invertible).

First, let us suppose that every nonzero proper ideal in R is a product of maximal ideals. Let $\mathfrak{m}$ be a maximal ideal and choose a nonzero $x \in \mathfrak{m}$. Then, by assumption,

$$(x) = \mathfrak{m}_1 \cdots \mathfrak{m}_\mathfrak{n}$$

for some $n \geq 0$ and maximal ideals $\mathfrak{m}_k$. As (x) is a principal ideal, each of the factors $\mathfrak{m}_k$ is invertible. Also,

$$\mathfrak{m}_1 \cdots \mathfrak{m}_\mathfrak{n} \subseteq \mathfrak{m}.$$

As $\mathfrak{m}$ is prime, this gives $\mathfrak{m}_k \subseteq \mathfrak{m}$ for some k. However, $\mathfrak{m}_k$ is maximal so must equal $\mathfrak{m}$, showing that $\mathfrak{m}$ is indeed invertible. Then, every nonzero proper ideal is a product of maximal, and hence invertible, ideals and so is invertible, and it follows that R is Dedekind.

We now show the reverse direction, so suppose that R is Dedekind. Proof by contradiction will be used to show that every nonzero ideal is a product of maximals, so suppose that this is not the case. Then, as R is defined to be `http://planetmath.org/Noetherian`Noetherian, there is an ideal $\mathfrak{a}$ `http://planetmath.org/MaximalElement`maximal (w.r.t. the partial order of set inclusion) among those proper ideals which are not a product of maximal ideals. Then $\mathfrak{a}$ cannot be a maximal ideal itself, so is strictly contained in a maximal ideal $\mathfrak{m}$ and, as $\mathfrak{m}$ is invertible, we can write $\mathfrak{a} = \mathfrak{m}\mathfrak{b}$ for an ideal $\mathfrak{b}$.

Therefore $\mathfrak{a} \subseteq \mathfrak{b}$ and we cannot have equality, otherwise cancelling $\mathfrak{a}$ from $\mathfrak{a} = \mathfrak{m}\mathfrak{a}$ would give $\mathfrak{m} = R$. So, $\mathfrak{b}$ is strictly larger than $\mathfrak{a}$ and, by the choice of $\mathfrak{a}$, is therefore a product of maximal ideals. Finally, $\mathfrak{a} = \mathfrak{m}\mathfrak{b}$ is then also a product of maximal ideals.