integral closures in separable extensions are finitely generated

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The theorem below generalizes to arbitrary integral ring extensions (under certain conditions) the fact that the ring of integers of a number field is finitely generated over $\mathbb{Z}$. The proof parallels the proof of the number field result.

Theorem 1. *Let B be an integrally closed Noetherian domain with field of fractions K. Let L be a finite separable extension of K, and let A be the integral closure of B in L. Then A is a finitely generated B-module.*

Proof. We first show that the `http://planetmath.org/Trace2`trace Tr_K^L maps A to B. Choose $u \in A$ and let $f = Irr(u, K) \in K[x]$ be the minimal polynomial for u over K; assume f is of degree d. Let the conjugates of u in some splitting field be $u = a_1, \ldots, a_d$. Then the a_i are all integral over B since they satisfy u's monic polynomial in $B[x]$. Since the coefficients of F are polynomials in the a_i, they too are integral over B. But the coefficients are in K, and B is integrally closed (in K), so the coefficients are in B. But $Tr_K^L(u)$ is just the coefficient of x^{d-1} in f, and thus $Tr_K^L(u) \in B$. This proves the claim.

Now, choose a basis $\omega_1, \ldots, \omega_d$ of L/K. We may assume $\omega_i \in A$ by multiplying each by an appropriate element of B. (To see this, let $Irr(\omega_i, K) \in K[x] = x^d + k_1 x^{d-1} + \ldots + k_d$. Choose $b \in B$ such that $bk_i \in B\ \forall i$. Then $(b\omega)^d + bk_1(b\omega)^{d-1} + \ldots + b^d k_d = 0$ and thus $b\omega \in A$). Define a linear map $\varphi : L \to K^d : a \mapsto (Tr_K^L(a\omega_1), \ldots, Tr_K^L(a\omega_d))$.

φ is 1-1, since if $u \in \ker\varphi, u \neq 0$, then $Tr(uL) = 0$. But $uL = L$, so Tr_K^L is identically zero, which cannot be since L is separable over K (it is a standard result that separability is equivalent to nonvanishing of the trace map; see for example [**?**], Chapter 8).

But $Tr_K^L : A \to B$ by the above, so $\varphi : A \hookrightarrow B^d$. Since B is Noetherian, any submodule of a finitely generated module is also finitely generated, so A is finitely generated as a B-module. $\square$

References

[1] P. Morandi, *Field and Galois Theory*, Springer, 2006.