vector spaces are isomorphic iff their bases are equipollent

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Theorem 1. *Vector spaces V and W are isomorphic iff their bases are equipollent (have the same cardinality).*

Proof. ($\Longrightarrow$) Let $\phi : V \to W$ be a linear isomorphism. Let A and B be bases for V and W respectively. The set

$$\phi(A) := \{\phi(a) \mid a \in A\}$$

is a basis for W. If

$$r_1\phi(a_1) + \cdots + r_n\phi(a_n) = 0,$$

with $a_i \in A$. Then

$$\phi(r_1a_1 + \cdots + r_na_n) = 0$$

since ϕ is linear. Furthermore, since ϕ is one-to-one, we have

$$r_1a_1 + \cdots + r_na_n = 0,$$

hence $r_i = 0$ for $i = 1, \ldots, n$, since A is linearly independent. This shows that $\phi(A)$ is linearly independent. Next, pick any $w \in W$, then there is $v \in V$ such that $\phi(v) = w$ since ϕ is onto. Since A spans V, we can write

$$v = r_1a_1 + \cdots + r_na_n,$$

so that

$$w = \phi(v) = r_1\phi(a_1) + \cdots + r_n\phi(a_n).$$

This shows that $\phi(A)$ spans W. As a result, $\phi(A)$ is a basis for W. A and $\phi(A)$ are equipollent because ϕ is one-to-one. But since B is also a basis for W, $\phi(A)$ and B are equipollent. Therefore

$$|A| = |\phi(A)| = |B|.$$

($\Longleftarrow$) Conversely, suppose A is a basis for V, B is a basis for W, and $|A| = |B|$. Let f be a bijection from A to B. We extend the domain of f to all of A, and call this extension ϕ, as follows: $\phi(a) = f(a)$ for any $a \in A$. For $v \in V$, write

$$v = r_1a_1 + \cdots + r_na_n$$

with $a_i \in A$, set

$$\phi(v) = r_1\phi(a_1) + \cdots + r_n\phi(a_n).$$

ϕ is a well-defined function since the expression of v as a linear combination of elements of A is unique. It is a routine verification to check that ϕ is indeed a linear transformation. To see that ϕ is one-to-one, let $\phi(v) = 0$. But this means that $v = 0$, again by the uniqueness of expression of 0 as a linear combination of elements of A. If $w \in W$, write it as a linear combination of elements of B:

$$w = s_1 b_1 + \cdots + s_m b_m.$$

Each $b_i \in B$ is the image of some $a \in A$ via f. For simplicity, let $f(a_i) = b_i$. Then

$$w = s_1 f(a_1) + \cdots + s_m f(a_m) = s_1 \phi(a_1) + \cdots + s_m \phi(a_m) = \phi(s_1 a_1 + \cdots + s_m a_m),$$

which shows that ϕ is onto. Hence ϕ is a linear isomorphism between V and W. $\square$