global dimension of a subring

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Let S be a ring with identity and $R \subset S$ a subring, such that R is contained in the center of S. In this case S is a (left) R-module via multiplication. Throughout by modules we will understand left modules and by global dimension we will understand left global dimension (we will denote it by $\mathrm{gl}\,\dim(S)$).

Proposition. Assume that $\mathrm{gl}\,\dim(S) = n < \infty$. If S is free as a R-module, then $\mathrm{gl}\,\dim(R) \leq n+1$.

Proof. Let M be a R-module. Then, there exists exact sequence

$$0 \to K \to P_n \to \cdots \to P_0 \to M \to 0,$$

of R-modules, where each P_i is projective (module K is just a kernel of a map $P_n \to P_{n-1}$). We will show, that K is also projective (and since M is arbitrary, it will show that $\mathrm{gl}\,\dim(R) \leq n+1$).

Since S is free as a R-module, then the extension of scalars $(- \otimes_R S)$ is an exact functor from the cateogry of R-modules to the category of S-modules. Furthermore for any projective R-module M, the S-module $M \otimes_R S$ is projective (in the category of S-modules). Thus we have following exact sequence of S-modules

$$0 \to K \otimes_R S \to P_n \otimes_R S \to \cdots \to P_0 \otimes_R S \to M \otimes_R S \to 0,$$

where each $P_i \otimes_R S$ is a projective S-module. But projective dimension of $M \otimes_R S$ is at most n (since $\mathrm{gl}\,\dim(S) = n$). Thus $K \otimes_R S$ is a projective S-module (please, see `http://planetmath.org/ExactSequencesForModulesWithFiniteProjective` entry for more details).

Note that the restriction of scalars functor also maps projective S-modules into projective R-modules. Thus $K \otimes_R S$ is a projective R-module. But S is free R-module, so

$$S \simeq \bigoplus_{i \in I} R,$$

for some index set I. Finally we have

$$K \otimes_R S \simeq K \otimes_R \big(\bigoplus_{i \in I} R\big) \simeq \bigoplus_{i \in I} \big(K \otimes_R R\big) \simeq \bigoplus_{i \in I} K.$$

This shows, that K is a direct summand of a projective R-module $K \otimes_R R$ and therefore K is projective, which completes the proof. □