uniqueness of division algorithm in Euclidean domain

Canonical name	UniquenessOfDivisionAlgorithmInEuclideanDomain
Date of creation	2013-03-22 17:53:00
Last modified on	2013-03-22 17:53:00
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	7
Author	pahio (2872)
Entry type	Theorem
Classification	msc 13F07
Related topic	KrullValuation
Related topic	Quotient
Defines	incomplete quotient

Theorem. Let a, b be non-zero elements of a Euclidean domain D with the Euclidean valuation ν. The *incomplete quotient* q and the remainder r of the division algorithm

$$a = qb + r \quad \text{where} \quad r = 0 \ \text{ or } \ \nu(r) < \nu(b)$$

are unique if and only if

$$\nu(a+b) \leqq \max\{\nu(a),\, \nu(b)\}. \tag{1}$$

Proof. Assume first (1) for the elements a, b of D. If we had

$$\begin{cases} a = qb + r \quad \text{with} \quad r = 0 \quad \vee \quad \nu(r) < \nu(b), \\ a = q'b + r' \quad \text{with} \quad r' = 0 \quad \vee \quad \nu(r') < \nu(b) \end{cases}$$

and $r' \neq r$, $q' \neq q$, then the http://planetmath.org/EuclideanValuationproperties of the Euclidean valuation and the assumption yield the of inequalities

$$\nu(b) \leqq \nu((q'-q)b) = \nu(r'-r) \leqq \max\{\nu(r'),\, \nu(-r)\} < \nu(b)$$

which is impossible. We must infer that $r'-r = 0$ or $q'-q = 0$. But these two conditions are http://planetmath.org/Equivalent3equivalent. Thus the division algorithm is unique.

Conversely, assume that (1) is not true for non-zero elements a, b of D, i.e.

$$\nu(a+b) > \max\{\nu(a),\, \nu(b)\}.$$

Then we obtain two repsesentations

$$b = 0(a+b) + b = 1(a+b) - a$$

where $\nu(b) < \nu(a+b)$ and $\nu(-a) = \nu(a) < \nu(a+b)$. Thus the incomplete quotient and the remainder are not unique.