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I-adic topology

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Let R be a ring and I an ideal in R such that

$$\bigcap_{k=1}^{\infty} I^k = \{0\}.$$

Though not usually explicitly done, we can define a metric on R by defining $ord_I(r)$ for a $r \in R$ by $ord_I(r) = k$ where k is the largest integer such that $r \in I^k$ (well-defined by the intersection assumption, and I^0 is taken to be the entire ring) and by $ord_I(0) = \infty$, and then defining for any $r_1, r_2 \in R$,

$$d_I(r_1, r_2) = 2^{-ord_I(r_1 - r_s)}.$$

The topology induced by this metric is called the I-adic topology. Note that the number 2 was chosen rather arbitrarily. Any other real number greater than 1 will induce an equivalent topology.

Except in the case of the similarly-defined p-adic topology, it is rare that reference is made to the actual I-adic metric. Instead, we usually refer to the I-adic topology.

In particular, a sequence of elements in $\{r_i\} \in R$ is Cauchy with respect to this topology if for any k there exists an N such that for all $m, n \geq N$ we have $(a_m - a_n) \in I^k$. (Note the parallel with the metric version of Cauchy, where k plays the part analogous to an arbitrary ϵ). The ring R is complete with respect to the I-adic topology if every such Cauchy sequence converges to an element of R.