localization of a module

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Let R be a commutative ring and M an R-module. Let $S \subset R$ be a non-empty multiplicative set. Form the Cartesian product $M \times S$, and define a binary relation $\sim$ on $M \times S$ as follows:

> $(m_1, s_1) \sim (m_2, s_2)$ if and only if there is some $t \in S$ such that $t(s_2 m_1 - s_1 m_2) = 0$

Proposition 1. *$\sim$ on $M \times S$ is an equivalence relation.*

Proof. Clearly $(m, s) \sim (m, s)$ as $t(sm - sm) = 0$ for any $t \in S$, where $S \neq \varnothing$. Also, $(m_1, s_1) \sim (m_2, s_2)$ implies that $(m_2, s_2) \sim (m_1, s_1)$, since $t(s_2 m_1 - s_1 m_2) = 0$ implies that $t(s_1 m_2 - s_2 m_1) = 0$. Finally, given $(m_1, s_1) \sim (m_2, s_2)$ and $(m_2, s_2) \sim (m_3, s_3)$, we are led to two equations $t(s_2 m_1 - s_1 m_2) = 0$ and $u(s_3 m_2 - s_2 m_3) = 0$ for some $t, u \in S$. Expanding and rearranging these, then multiplying the first equation by us_3 and the second by ts_1, we get $tus_2(s_3 m_1 - s_1 m_3) = 0$. Since $tus_2 \in S$, $(m_1, s_1) \sim (m_3, s_3)$ as required. $\square$

Let M_S be the set of equivalence classes in $M \times S$ under $\sim$. For each $(m, s) \in M \times S$, write

$$[(m, s)] \text{ or more commonly } \frac{m}{s}$$

the equivalence class in M_S containing (m, s). Next,

- define a binary operation $+$ on M_S as follows:

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} := \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}.$$

- define a function $\cdot : R_S \times M_S \to M_S$ as follows:

$$\frac{r}{s} \cdot \frac{m}{t} := \frac{rm}{st}$$

  where R_S is the localization of R over S.

Proposition 2. *M_S together with $+$ and $\cdot$ defined above is a unital module over R_S.*

Proof. That $+$ and $\cdot$ are well-defined is based on the following: if $(m_1, s_1) \sim (m_2, s_2)$, then

$$\frac{m}{s} + \frac{m_1}{s_1} = \frac{m}{s} + \frac{m_2}{s_2}, \qquad \frac{m_1}{s_1} + \frac{m}{s} = \frac{m_2}{s_2} + \frac{m}{s}, \quad \text{and} \quad \frac{r}{s} \cdot \frac{m_1}{s_1} = \frac{r}{s} \cdot \frac{m_2}{s_2},$$

which are clear by Proposition 1. Furthermore $+$ is commutative and associative and that $\cdot$ distributes over $+$ on both sides, which are all properties inherited from M. Next, $\dfrac{0}{s}$ is the additive identity in M_S and $\dfrac{-m}{s} \in M_S$ is the additive inverse of $\dfrac{m}{s}$. So M_S is a module over R_S. Finally, since $(mt, st) \sim (m, s)$ for any $t \in S$, $\dfrac{t}{t} \cdot \dfrac{m}{s} = \dfrac{m}{s}$ so that M_S is unital. $\square$

Definition. M_S, as an R_S-module, is called the *localization* of M at S. M_S is also written $S^{-1}M$.

Remarks.

- The notion of the localization of a module generalizes that of a ring in the sense that R_S is the localization of R at S as an R_S-module.
- If $S = R - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in R, then M_S is usually written $M_{\mathfrak{p}}$.