factoring all-one polynomials using the grouping method

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The method of grouping terms can be used to factor all-one polynomials, i.e. polynomials of the form

$$\sum_{m=0}^{n-1} x^m$$

when n is composite. (When n is prime, these polynomials are irreducible, so there is nothing to do in that case.)

Let us consider a few examples:

$n = 4$:

$$\begin{aligned} 1 + x + x^2 + x^3 &= \\ (1 + x) + (x^2 + x^3) &= \\ (1 + x) + x^2(1 + x) &= \\ &(1 + x)(1 + x^2) \end{aligned}$$

$n = 6$:

$$\begin{aligned} 1 + x + x^2 + x^3 + x^4 + x^5 &= \\ (1 + x + x^2) + (x^3 + x^4 + x^5) &= \\ (1 + x + x^2) + x^3(1 + x + x^2) &= \\ &(1 + x^3)(1 + x + x^2) \end{aligned}$$

$n = 8$:

$$\begin{aligned} 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 &= \\ (1 + x + x^2 + x^3) + (x^4 + (x^5 + x^6 + x^7) &= \\ (1 + x + x^2 + x^3) + x^4(1 + x + x^2 + x^3) &= \\ &(1 + x^4)(1 + x + x^2 + x^3) \end{aligned}$$

Combining this result with the factorization we have for the case $n = 4$, we obtain the following:

$$\begin{aligned} 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 &= \\ &(1 + x)(1 + x^2)(1 + x^4) \end{aligned}$$

$n = 9$:

$$\begin{aligned} 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 &= \\ (1 + x + x^2) + (x^3 + x^4 + x^5) + (x^6 + x^7 + x^8) &= \\ (1 + x + x^2) + x^3(1 + x + x^2) + x^6(1 + x + x^2) &= \\ &(1 + x + x^2)(1 + x^3 + x^6) \end{aligned}$$

$n = 12$:

$$\begin{aligned}
1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} &= \\
(1 + x + x^2) + (x^3 + x^4 + x^5) + (x^6 + x^7 + x^8) + (x^9 + x^{10} + x^{11}) &= \\
(1 + x + x^2) + x^3(1 + x + x^2) + x^6(1 + x + x^2) + x^9(1 + x + x^2) &= \\
(1 + x + x^2)(1 + x^3 + x^6 + x^9) &= \\
(1 + x + x^2)((1 + x^3) + (x^6 + x^9)) &= \\
(1 + x + x^2)((1 + x^3) + x^6(1 + x^3)) &= \\
(1 + x + x^2)(1 + x^3)(1 + x^6)
\end{aligned}$$

It might be worth pointing out that the polynomials produced by this factorization are not all irreducible. For instance,

$$1 + x^3 = (1 + x)(1 - x + x^2).$$

However, to obtain this factorization, one needs to use some techique other than the grouping method. Likewise. the polynomial $1 + x^6$ is also reducible.