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integral closure is ring

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Theorem. Let A be a subring of a commutative ring B having nonzero unity. Then the integral closure of A in B is a subring of B containing A.

Proof. Let x be an arbitrary element of the integral closure A' of A in B. Then there are the elements $a_0,\,a_1,\,\ldots,\,a_{n-1}$ of A such that

$$a_0{+}a_1x{+}\ldots{+}a_{n-1}x^{n-1}{+}x^n \;=\; 0$$

where $n > 0$. If $f(X) = c_0{+}c_1X{+}\ldots{+}c_mX^m$ is a polynomial in $A[X]$ with degree $m > n$, we have

$$\begin{aligned} f(x) \;&=\; c_0{+}c_1x{+}\ldots{+}c_{m-1}x^{m-1}{+}c_mx^{m-n}(-a_0{-}a_1x{-}\ldots{-}a_{n-1}x^{n-1}) \\ &=\; c_0'{+}c_1'x{+}\ldots{+}c_{m-1}'x^{m-1} \end{aligned}$$

where the elements c_i' belong to A. This procedure may be repeated until we see that $f(x)$ is an element of the A-module generated by $1,\,x,\,\ldots,\,x^n$. Accordingly,

$$A[x] \;=\; A + Ax + \ldots + Ax^n$$

is a finitely generated A-module.

Now we have evidently $A \subseteq A'$. Let y be another element of A'. Then

$$A[x,\,y] \;=\; A[x][y]$$

is a finitely generated $A[x]$-module, whence $A[x,\,y]$ is a finitely generated A-module. Because the elements $x{-}y$ and xy belong to $A[x,\,y]$, they are integral over A and thus belong to A'. Consequently, A' is a subring of B (see the http://planetmath.org/node/2738subring condition).

References

[1] M. Larsen & P. McCarthy: *Multiplicative theory of ideals.* Academic Press, New York (1971).