Smith normal form

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This topic gives a version of the Gauss elimination algorithm for a commutative principal ideal domain which is usually described only for a field.

Let $A \neq 0$ be a $m \times n$-matrix with entries from a commutative principal ideal domain R. For $a \in R \setminus \{0\}$ $\delta(a)$ denotes the number of prime factors of a. Start with $t = 1$ and choose j_t to be the smallest column index of A with a non-zero entry.

(I) If $a_{t,j_t} = 0$ and $a_{k,j_t} \neq 0$, exchange rows 1 and k.

(II) If there is an entry at position (k, j_t) such that $a_{t,j_t} \nmid a_{k,j_t}$, then set $\beta = \gcd(a_{t,j_t}, a_{k,j_t})$ and choose $\sigma, \tau \in R$ such that

$$\sigma \cdot a_{t,j_t} - \tau \cdot a_{k,j_t} = \beta.$$

By left-multiplication with an appropriate matrix it can be achieved that row 1 of the matrix product is the sum of row 1 multiplied by σ and row k multiplied by $(-\tau)$. Then we get β at position (t, j_t), where $\delta(\beta) < \delta(a_{t,j_t})$. Repeating these steps one obtains a matrix having an entry at position (t, j_t) that divides all entries in column j_t.

(III) Finally, adding appropriate multiples of row t, it can be achieved that all entries in column j_t except for that at position (t, j_t) are zero. This can be achieved by left-multiplication with an appropriate matrix.

Applying the steps described above to the remaining non-zero columns of the resulting matrix (if any), we get an $m \times n$-matrix with column indices $j_1, \ldots, j_r$ where $r \leq \min(m, n)$, each of which satisfies the following:

1. the entry at position (l, j_l) is non-zero;
2. all entries below and above position (l, j_l) as well as entries left of (l, j_l) are zero.

Furthermore, all rows below the r-th row are zero.

Now we can re-order the columns of this matrix so that elements on positions (i, i) for $1 \leq i \leq r$ are nonzero and $\delta(a_{i,i}) \leq \delta(a_{i+1,i+1})$ for $1 \leq i < r$; and all columns right of the r-th column (if present) are zero. For short set α_i for the element at position (i, i). δ has non-negative integer values; so $\delta(\alpha_1) = 0$ is equivalent to α_1 being a unit of R. $\delta(\alpha_i) = \delta(\alpha_{i+1})$ can either happen if α_i and α_{i+1} differ by a unit factor, or if they are relatively prime. In the latter case one can add column $i+1$ to column i (which doesn't change

α_i) and then apply appropriate row manipulations to get $\alpha_i = 1$. And for $\delta(\alpha_i) < \delta(\alpha_{i+1})$ and $\alpha_i \nmid \alpha_{i+1}$ one can apply step (II) after adding column $i+1$ to column i. This diminishes the minimum δ-values for non-zero entries of the matrix, and by reordering columns etc. we end up with a matrix whose diagonal elements α_i satisfy $\alpha_i \mid \alpha_{i+1}\ \forall\ 1 \leq i < r$.

Since all row and column manipulations involved in the process are , this shows that there exist invertible $m \times m$ and $n \times n$-matrices S, T so that SAT is

$$\begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_r \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

This is the **Smith normal form** of the matrix. The elements α_i are unique up to associates and are called **elementary divisors**.