modules over algebars and homomorphisms between them

Canonical name	ModulesOverAlgebarsAndHomomorphismsBetweenThem
Date of creation	2013-03-22 19:16:32
Last modified on	2013-03-22 19:16:32
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	5
Author	joking (16130)
Entry type	Definition
Classification	msc 13B99
Classification	msc 20C99
Classification	msc 16S99

Let R be a ring and let A be an associative algebra (not necessarily unital).

Definition. A (left) A-module over R is a pair $(M, \circ)$ where M is a (left) R-module and

$$\circ : A \times M \to M$$

is a R-bilinear map such that the following conditions hold:

1. $(a \circ b) \circ x = a \circ (b \circ x)$
2. $r(a \circ x) = (ra) \circ x = a \circ (rx)$

for any $a, b \in A$, $x \in M$ and $r \in R$. We will simply use capital letters to denote modules.

Let M be an A-module over R. If $M' \subseteq M$ and $A' \subseteq A$ then by $A'M'$ we denote R-submodule of M generated by elements of the form am for $a \in A'$ and $m \in M'$. We will call M **unitary** if $AM = M$. Note, that if A has multiplicative identity 1, then M is unitary if and only if $1m = m$ for any $m \in M$.

The reason we use name ,,A-module over R" instead of ,,A-module" is that these to concepts may differ. The latter means that we treat A simply as a ring and take modules over it. But such module need not be equiped with a ,,good" R-module structure. On the other hand this is always the case, when M is unitary over unital algebra.

If M and N are two A-modules over R, then a function $f : M \to N$ is called an A-homomorphism iff f is an R-homomorphism and additionaly $f(am) = af(m)$ for any $a \in A$ and $m \in M$.

It can be easily checked that A-modules over R together with A-homomorphisms form a category which is abelian. Furthermore, if A is unital, then its full subcategory consisting unitary R-modules over A is equivalent to category of unitary A-modules.

In most cases it is important to assume that the base ring R is a field, even algebraically closed.