formal power series as inverse limits

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1 Motivation and Overview

The ring of formal power series can be described as an inverse limit.[1] The fundamental idea behind this approach is that of truncation — given a formal power series $a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ and an integer $n \geq 0$, we can truncate the series to order n to obtain $a_0 + a_1 t + \cdots + a_n t^n + O(t^{n+1})$.[2] (Indeed, we must do this in practical computation since it is only possible to write down a finite number of terms at a time; thus the approach taken here has the advantage of being close to actual practice.) Furthermore, this procedure of truncation commutes with ring operations — given two formal power series, the truncation of their sum is the sum of their truncations and the truncation of their product is the product of their truncations. Thus, for every integer n the set of power series truncated to order n forms a ring and truncation is a morphism from the ring of formal power series to this ring.

To obtain our definition, we will proceed in the opposite direction. We will begin by defining rings of truncated power series and exhibiting truncation morphisms between different truncations. Then we will show that these rings and morphisms form an inverse system which has a limit which we will take as the definition of the ring of formal power series. Finally, we will complete the circle by demonstrating that the object so constructed is isomorphic with the usual definition for ring of formal power series.

2 Formal Development

In this section, we will carry out the develpment outlined above in rigorous detail. We begin by formalizing this notion of truncation.

Theorem 1. *Let A be a commutative ring and let n be a positive integer. Then $A[[x]]/\langle x^n \rangle$ is isomorphic to $A[x]/\langle x^n \rangle$.*

Proof. We may identify $A[x]$ with the subring of $A[[x]]$ consisting of series which have all but a finite number of coeficients equal to zero. Consider an element $f = \sum_{k=0}^{\infty} c_n x^n$ of $A[[x]]$. We may write $f = \sum_{k=0}^{n-1} c_n x^k + x^n \sum_{k=0}^{\infty} c_{k+n} x^k$. Thus, every element of $A[[x]]$ is equivalent to an element of

[1] It is worth pointing out that, since we are dealing with formal series, the concept of limit used here has nothing to do with convergence but is purely algebraic.

[2] Here, the symbol "$O(t^{n+1})$" is not used in the sense of Landau notation but merely as an indicator that the power series has been truncated to order n.

the subring $A[x]$ modulo x^k. Hence, if follows rather immediately from the definition of quotient ring that $A[[x]]\langle x^n\rangle$ is isomorphic to $A[x]/\langle x^n\rangle$. $\square$

Let us call the isomorphism between $A[[x]]/\langle x^n\rangle$ and $A[x]/\langle x^n\rangle$ which is described above I_n. We now define a few more morphisms.

Definition 1. *Suppose m, n are integers satifying the inequalities $m > n \geq 0$. Then define the morphisms t_{mn}, T_{mn}, p_n, P_n as follows:*

- *Define $t_{nm}\colon A[x]/\langle x^m\rangle \to A[x]/\langle x^n\rangle$ as the map which sends each equivalence class a modulo x^n to the unique equivalence class b modulo x^m such that $a \subset b$.*
- *Define $T_{nm}\colon A[[x]]/\langle x^m\rangle \to A[[x]]/\langle x^{n+1}\rangle$ as the map which sends each equivalence class a modulo x^n to the unique equivalence class b modulo x^m such that $a \subset b$.*
- *For every integer $n > 0$, let Q_n be the quotient map from $A[[x]]$ to $A[[x]]/\langle x^n\rangle$.*
- *For every integer $n > 0$, let q_n be the quotient map from $A[x]$ to $A[x]/\langle x^n\rangle$.*

These morphisms commute with each other in ways which are described by the next theorem:

Theorem 2. *Suppose m, n, k are integers satifying the inequalities $m > n > k \geq 0$. Then we have the following relations:*

1. $t_{nk} \circ t_{mn} = t_{mk}$
2. $I_n \circ T_{mn} = t_{mn} \circ I_m$
3. $T_{nk} \circ T_{mn} = T_{mk}$
4. $t_{mn} \circ q_m = q_n$
5. $T_{mn} \circ Q_m = Q_n$

[More to come]