proof of theorem on equivalent valuations

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It is easy to see that $|\cdot|$ and $|\cdot|^c$ are equivalent valuations for any constant $c > 0$ — it follows from the fact that $0 \leq x^c < 1$ if and only if $0 < x \leq 1$.

Assume that the valuations $|\cdot|_1$ and $|\cdot|_2$ are equivalent. Let b be an element of K such that $0 < |b|_1 < 1$. Because the valuations are assumed to be equivalent, it is also the case that $0 < |b|_2 < 1$. Hence, there must exist positive constants c_1 and c_2 such that $|b|_1^{c_1} = \frac{1}{2}$ and $|b|_2^{c_2} = \frac{1}{2}$.

We will show that show that $|x|_1^{c_1} = |x|_2^{c_2}$ for all $a \in K$ by contradiction.

Let a be any element of k such that $0 < |a|_1 < 1$. Assume that $|a|_1^{c_1} \neq |a|_2^{c_2}$. Then either $|a|_1^{c_1} < |a|_2^{c_2}$ or $|a|_1^{c_1} > |a|_2^{c_2}$. We may assume that $|a|_1^{c_1} < |a|_2^{c_2}$ without loss of generality.

Since $|a|_2^{c_2}/|a|_1^{c_1} > 1$, there exists an integer $m > 0$ such that $(|a|_2^{c_2}/|a|_1^{c_1})^m > 2$. Let n be the least integer such that $2^n|a|_2^{mc_2} > 1$. Then we have

$$2^n|a|_1^{mc_1} < 2^{n-1}|a|_2^{mc_2} < 1 < 2^n|a|_2^{mc_2}.$$

Since $2 = |b^{-1}|_1^{c_1} = |b^{-1}|_2^{c_2}$, this implies that

$$\left|\frac{a^m}{b^n}\right|_1^{c_1} < 1 < \left|\frac{a^m}{b^n}\right|_2^{c_2},$$

but then

$$\left|\frac{a^m}{b^n}\right|_1 < 1$$

and

$$\left|\frac{a^m}{b^n}\right|_2 > 1,$$

which is impossible because the two valuations are assumed to be equivalent.

Q.E.D