invertibility of regularly generated ideal

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Lemma. Let R be a commutative ring containing regular elements. If $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are three ideals of R such that $\mathfrak{b}+\mathfrak{c}$, $\mathfrak{c}+\mathfrak{a}$ and $\mathfrak{a}+\mathfrak{b}$ are `http://planetmath.org/FractionalIdealOfCommutativeRing`invertible, then also their sum ideal $\mathfrak{a}+\mathfrak{b}+\mathfrak{c}$ is .

Proof. We may assume that R has a unity, therefore the product of an ideal and its `http://planetmath.org/FractionalIdealOfCommutativeRing`inverse is always R. Now, the ideals $\mathfrak{b}+\mathfrak{c}$, $\mathfrak{c}+\mathfrak{a}$ and $\mathfrak{a}+\mathfrak{b}$ have the $\mathfrak{f}_1$, $\mathfrak{f}_2$ and $\mathfrak{f}_3$, respectively, so that

$$(\mathfrak{b}+\mathfrak{c})\mathfrak{f}_1 \;=\; (\mathfrak{c}+\mathfrak{a})\mathfrak{f}_2 \;=\; (\mathfrak{a}+\mathfrak{b})\mathfrak{f}_3 \;=\; R.$$

Because $\mathfrak{a}\mathfrak{f}_2 \subseteq R$ and $\mathfrak{c}\mathfrak{f}_1 \subseteq R$, we obtain

$$\begin{aligned}(\mathfrak{a}+\mathfrak{b}+\mathfrak{c})(\mathfrak{a}\mathfrak{f}_2\mathfrak{f}_3+\mathfrak{c}\mathfrak{f}_1\mathfrak{f}_2) \;&=\; (\mathfrak{a}+\mathfrak{b})\mathfrak{a}\mathfrak{f}_2\mathfrak{f}_3+\mathfrak{c}(\mathfrak{a}\mathfrak{f}_2)\mathfrak{f}_3+\mathfrak{a}(\mathfrak{c}\mathfrak{f}_1)\mathfrak{f}_2+(\mathfrak{b}+\mathfrak{c})\mathfrak{c}\mathfrak{f}_1\mathfrak{f}_2\\ &=\; \mathfrak{a}\mathfrak{f}_2+\mathfrak{c}\mathfrak{f}_2 \;=\; (\mathfrak{c}+\mathfrak{a})\mathfrak{f}_2\\ &=\; R.\end{aligned}$$

Theorem. Let R be a commutative ring containing regular elements. If every ideal of R generated by two regular elements is , then in R also every ideal generated by a finite set of regular elements is .

Proof. We use induction on n, the number of the regular elements of the generating set. We thus assume that every ideal of R generated by n regular elements $(n \geqq 2)$ is . Let $\{r_1,\, r_2,\, \ldots,\, r_{n+1}\}$ be any set of regular elements of R. Denote

$$\mathfrak{a} \;=:\; (r_1), \quad \mathfrak{b} \;=:\; (r_2,\, \ldots,\, r_n), \quad \mathfrak{c} \;=:\; (r_{n+1}).$$

The sums $\mathfrak{b}+\mathfrak{c}$, $\mathfrak{c}+\mathfrak{a}$ and $\mathfrak{a}+\mathfrak{b}$ are, by the assumptions, . Then the ideal

$$(r_1,\, r_2,\, \ldots,\, r_n,\, r_{n+1}) \;=\; \mathfrak{a}+\mathfrak{b}+\mathfrak{c}$$

is, by the lemma, , and the induction proof is complete.

References

[1] R. Gilmer: *Multiplicative ideal theory*. Queens University Press. Kingston, Ontario (1968).