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properties of non-archimedean valuations

Canonical name	PropertiesOfNonarchimedeanValuations
Date of creation	2013-03-22 18:01:11
Last modified on	2013-03-22 18:01:11
Owner	rm50 (10146)
Last modified by	rm50 (10146)
Numerical id	6
Author	rm50 (10146)
Entry type	Theorem
Classification	msc 13F30
Classification	msc 13A18
Classification	msc 12J20
Classification	msc 11R99
Related topic	CompleteUltrametricField

If K is a field, and $|\cdot|$ a nontrivial non-archimedean valuation (or absolute value) on K, then $|\cdot|$ has some properties that are counterintuitive (and that are false for archimedean valuations).

Theorem 1. *Let K be a field with a non-archimedean absolute value $|\cdot|$. For $r > 0$ a real number, $x \in K$, define*

$$B(x, r) = \{y \in K \mid |x - y| < r\}, \text{ the open ball of radius } r \text{ at } x$$
$$\bar{B}(x, r) = \{y \in K \mid |x - y| \leq r\}, \text{ the closed ball of radius } r \text{ at } x$$

Then

1. *$B(x, r)$ is both open and closed;*
2. *$\bar{B}(x, r)$ is both open and closed;*
3. *If $y \in B(x, r)$ (resp. $\bar{B}(x, r)$) then $B(x, r) = B(y, r)$ (resp. $\bar{B}(x, r) = \bar{B}(y, r)$);*
4. *$B(x, r)$ and $B(y, r)$ (resp. $\bar{B}(x, r)$ and $\bar{B}(y, r)$) are either identical or disjoint;*
5. *If $B_1 = B(x, r)$ and $B_2 = B(y, s)$ are not disjoint, then either $B_1 \subset B_2$ or $B_2 \subset B_1$;*
6. *If (x_n) is a sequence of elements of K with $\lim_{n\to\infty} x_n = 0$, then $\sum_{n=1}^{\infty} x_n$ is Cauchy (and thus if K is complete, a sufficient condition for convergence of a series is that the terms tend to zero)*

Proof. We start by proving (3). Suppose $y \in B(x, r)$. If $z \in B(x, r)$, then since the absolute value is non-archimedean, we have

$$|z - y| = |(z - x) + (x - y)| \leq \max(|z - x|, |x - y|) < r$$

so that $z \in B(y, r)$. Clearly $x \in B(y, r)$, so reversing the roles of x and y, we see that $B(x, r) = B(y, r)$. Finally, replacing B by $\bar{B}$ and $<$ by $\leq$, we get equality of closed balls as well.

(4) is now trivial: If $B(x, r) \cap B(y, r) \neq \emptyset$, choose $z \in B(x, r) \cap B(y, r)$; then by (3), $B(x, r) = B(z, r) = B(y, r)$. An identical argument proves the result for closed balls.

To prove (5), choose $z \in B_1 \cap B_2$. Assume first that $r \leq s$; then $B(z, r) = B_1$, and $B(z, r) \subset B(z, s) = B_2$, so that $B_1 \subset B_2$. If $s \leq r$, then we have identically that $B_2 \subset B_1$. (Note that (4) is a special case when $r = s$).

(1) and (2) now follow: for (1), note that $B(x, r)$ is obviously open; its complement consists of a union of open balls of radius r disjoint with $B(x, r)$ and its complement is therefore open. Thus $B(x, r)$ is closed. For (2), $\bar{B}(x, r)$ is obviously closed; to see that it is open, take any $y \in \bar{B}(x, r)$; then $\bar{B}(x, r) = \bar{B}(y, r)$ and thus $B(y, s) \subset \bar{B}(y, r)$ for $s < r$ is an open neighborhood of y contained in $\bar{B}(x, r)$, which is therefore open.

Finally, to prove (6), we must show that given ϵ, we can find $N > 0$ sufficiently large such that $|\sum_{i=m}^{n} x_i| < \epsilon$ whenever $m, n > N$. Simply choose N such that $|x_i| < \epsilon$ for $i > N$; then

$$\left| \sum_{i=m}^{n} x_i \right| \leq \max(|x_m|, \ldots, |x_n|) < \epsilon$$