characterization of primary ideals

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Proposition. Let R be a commutative ring and $I \subseteq R$ an ideal. Then I is primary if and only if every zero divisor in R/I is nilpotent.

Proof. „$\Rightarrow$" Assume, that we have $x \in R$ such that $x+I$ is a zero divisor in R/I. In particular $x+I \neq 0+I$ and there is $y \in R$, $y+I \neq 0+I$ such that

$$0+I = (x+I)(y+I) = xy+I.$$

This is if and only if $xy \in I$. Thus either $y \in I$ or $x^n \in I$ for some $n \in \mathbb{N}$. Of course $y \notin I$, because $y+I \neq 0+I$ and thus $x^n \in I$. Therefore $x^n+I = 0+I$, which means that $x+I$ is nilpotent in R/I.

„$\Leftarrow$" Assume that for some $x, y \in R$ we have $xy \in I$ and $x, y \notin I$. Then

$$(x+I)(y+I) = xy+I = 0+I,$$

so both $x+I$ and $y+I$ are zero divisors in R/I. By our assumption both are nilpotent, and therefore there is $n, m \in \mathbb{N}$ such that $x^n+I = y^m+I = 0+I$. This shows, that $x^n \in I$ and $y^m \in I$, which completes the proof. $\square$