ultrametric triangle inequality

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Theorem 1. Let K be a field and G an ordered group equipped with zero. Suppose that the function $|\cdot|\colon\ K \to G$ satisfies the postulates 1 and 2 of Krull valuation. Then the *non-archimedean* or *ultrametric triangle inequality*

3. $\qquad |x+y| \ \leqq\ \max\{|x|,\, |y|\}$

in the field is with the condition

(*) $\qquad |x| \leqq 1 \quad\Rightarrow\quad |x+1| \leqq 1.$

Proof. The value $y = 1$ in the ultrametric triangle inequality gives the (*) as result. Secondly, let's assume the condition (*). Let x and y be non-zero elements of the field K (if $xy = 0$ then 3 is at once verified), and let e.g. $|x| \leqq |y|$. Then we get $|\frac{x}{y}| = |x| \cdot |y|^{-1} \leqq 1,$ and thus according to (*),

$$|x+y| \cdot |y|^{-1} \ =\ \left|\frac{x+y}{y}\right| \ =\ \left|\frac{x}{y}+1\right| \leqq 1.$$

So we see that $|x+y| \leqq |y| = \max\{|x|,\, |y|\}$.

Theorem 2. The Krull valuation (and any http://planetmath.org/Valuationnon-archimedean valuation) $|\cdot|$ of the field K satisfies the sharpening

$$|x+y| \ =\ \max\{|x|,\, |y|\} \quad \text{for } |x| \neq |y|$$

of the ultrametric triangle inequality.

Proof. Let e.g. $|x| > |y|$. Surely $|x+y| \leqq |x|$, but also $|x| = |(x+y)-y| \leqq \max\{|x+y|,\, |y|\}$; this maximum is $|x+y|$ since otherwise one would have $|x| \leqq |y|$. Thus the result is: $|x+y| = |x|$.

Note. The metric defined by a non-archimedean valuation of the field K is the *ultrametric* of K. Theorem 2 implies, that every triangle of K with vertices $A,\ B,\ C\ (\in\ K)$ is isosceles: if $|B-C| \neq |C-A|$, then $|A-B| = \max\{|B-C|,\, |C-A|\}$.

Theorem 3. The http://planetmath.org/Valuationvaluation $|\cdot| : K \to \mathbb{R}$ of the field K is archimedean if and only if the set

$$\{|1|,\, |1+1|,\, |1+1+1|,\, \ldots\}$$

of the "values" of the multiples of the unity is not bounded.

Proof. If $|\cdot|$ is non-archimedean, then $|n\cdot 1| = |1+\ldots+1| \leqq \max\{|1|\} = 1$, and the multiples are bounded. Conversely, let $|n \cdot 1| < M \ \forall n \in \mathbb{Z}_+$. Now one obtains, when $|x| \leqq 1$:

$$|x+1|^n \ \leqq \ \sum_{j=0}^{n} \left| \binom{n}{j} \right| \cdot |x|^j \ < \ (n+1)M,$$

or $|x+1| < \sqrt[n]{(n+1)M}$ for all n. As n tends to infinity, this n^{th} root has the limit 1. Therefore one gets the limit inequality $|x+1| \leqq 1$, i.e. the valuation is non-archimedean.

References

[1] Emil Artin: *Theory of Algebraic Numbers.* Lecture notes. Mathematisches Institut, Göttingen (1959).