valuation domain is local

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Theorem. Every valuation domain is a local ring.

Proof. Let R be a valuation domain and K its field of fractions. We shall show that the set of all non-units of R is the only maximal ideal of R.

Let a and b first be such elements of R that $a-b$ is a unit of R; we may suppose that $ab \neq 0$ since otherwise one of a and b is instantly stated to be a unit. Because R is a valuation domain in K, therefore e.g. $\frac{a}{b} \in R$. Because now $\frac{a-b}{b} = 1 - \frac{a}{b}$ and $(a-b)^{-1}$ belong to R, so does also the product $\frac{a-b}{b} \cdot (a-b)^{-1} = \frac{1}{b}$, i.e. b is a unit of R. We can conclude that the difference $a-b$ must be a non-unit whenever a and b are non-units.

Let a and b then be such elements of R that ab is its unit, i.e. $a^{-1}b^{-1} \in R$. Now we see that

$$a^{-1} = b \cdot a^{-1}b^{-1} \in R, \;\; b^{-1} = a \cdot a^{-1}b^{-1} \in R,$$

and consequently a and b both are units. So we conclude that the product ab must be a non-unit whenever a is an element of R and b is a non-unit.

Thus the non-units form an ideal $\mathfrak{m}$. Suppose now that there is another ideal $\mathfrak{n}$ of R such that $\mathfrak{m} \subset \mathfrak{n} \subseteq R$. Since $\mathfrak{m}$ contains all non-units, we can take a unit ε in $\mathfrak{n}$. Thus also the product $\varepsilon^{-1}\varepsilon$, i.e. 1, belongs to $\mathfrak{n}$, or $R \subseteq \mathfrak{n}$. So we see that $\mathfrak{m}$ is a maximal ideal. On the other hand, any maximal ideal of R contains no units and hence is contained in $\mathfrak{m}$; therefore $\mathfrak{m}$ is the only maximal ideal.