proof of Hilbert basis theorem

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Let R be a noetherian ring and let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in R[x]$ with $a_n \neq 0$. Then call a_n the *initial coefficient* of f.

Let I be an ideal in $R[x]$. We will show I is finitely generated, so that $R[x]$ is noetherian. Now let f_0 be a polynomial of least degree in I, and if $f_0, f_1, \ldots, f_k$ have been chosen then choose f_{k+1} from $I \setminus (f_0, f_1, \ldots, f_k)$ of minimal degree. Continuing inductively gives a sequence (f_k) of elements of I.

Let a_k be the initial coefficient of f_k, and consider the ideal $J = (a_1, a_2, a_3, \ldots)$ of initial coefficients. Since R is noetherian, $J = (a_0, \ldots, a_N)$ for some N.

Then $I = (f_0, f_1, \ldots, f_N)$. For if not then $f_{N+1} \in I \setminus (f_0, f_1, \ldots, f_N)$, and $a_{N+1} = \sum_{k=0}^{N} u_k a_k$ for some $u_1, u_2, \ldots, u_N \in R$. Let $g(x) = \sum_{k=0}^{N} u_k f_k x^{\nu_k}$ where $\nu_k = \deg(f_{N+1}) - \deg(f_k)$.

Then $\deg(f_{N+1} - g) < \deg(f_{N+1})$, and $f_{N+1} - g \in I$ and $f_{N+1} - g \notin (f_0, f_1, \ldots, f_N)$. But this contradicts minimality of $\deg(f_{N+1})$.

Hence, $R[x]$ is noetherian.$\square$