free objects in the category of commutative algebras

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Let R be a commutative ring and let $\mathcal{ALG}_c(R)$ be the category of all commutative algebras over R and algebra homomorphisms. This category together with the forgetful functor is a construct (i.e. it is a concrete category over the category of sets $\mathcal{SET}$). Therefore we can talk about free objects in $\mathcal{ALG}_c(R)$ (see `http://planetmath.org/FreeObjectsInConcreteCategories2`this entry for definitions).

Theorem. For any set $\mathbb{X}$ the polynomial algebra $R[\mathbb{X}]$ (see parent object) is a free object in $\mathcal{ALG}_c(R)$ with $\mathbb{X}$ being a basis. This means that for any commutative algebra A and any function

$$f : \mathbb{X} \to A$$

there exists a unique algebra homomorphism $F : R[\mathbb{X}] \to A$ such that

$$F(x) = f(x)$$

for any $x \in \mathbb{X}$.

Proof. Assume that $f : \mathbb{X} \to A$ is a function. If $W \in R[\mathbb{X}]$, then there are finite subsets $A_1, \ldots, A_n \subseteq \mathbb{X}$ (not necessarily disjoint) and natural numbers $n(x,i)$, $i = 1, \ldots, n$ such that W can be uniquely expressed as

$$W = \sum_{i=1}^{n} \left(\lambda_i \cdot \prod_{x \in A_i} x^{n(x,i)} \right)$$

with $\lambda_i \in R$. Define $F(W)$ by putting

$$F(W) = \sum_{i=1}^{n} \left(\lambda_i \cdot \prod_{x \in A_i} f(x)^{n(x,i)} \right).$$

Of course F is well defined and obviously $F(x) = f(x)$. We leave as a simple exercise that F is an algebra homomorphism. The uniqueness of F again follows from the explicit form of W. It is easily seen that $F(W)$ depends only on $F(x)$ for $x \in \mathbb{X}$. This completes the proof. $\square$

Corollary 1. If $\mathbb{X}$ is a set and $\mathbb{Y} \subseteq \mathbb{X}$, then the inclusion $i : \mathbb{Y} \to \mathbb{X}$ induces an algebra monomorphism

$$I : R[\mathbb{Y}] \to R[\mathbb{X}].$$

In particular we can treat $R[\mathbb{Y}]$ as a subalgebra of $R[\mathbb{X}]$.

Proof. We have a well-defined function $i : \mathbb{Y} \to R[\mathbb{X}]$, $i(y) = y$. By the theorem we have an extension

$$I : R[\mathbb{Y}] \to R[\mathbb{X}]$$

such that $I(y) = y$. It remains to show, that I is ,,1-1". Indeed, assume that $I(W) = 0$ for some polynomial $W \in R[\mathbb{Y}]$. But if we recall the expression of W as in proof of the theorem and remember that I is an algebra homomorphism, then it is easy to see that $I(y) = y$ implies that

$$I(W) = W.$$

In particular $W = 0$, which completes the proof. □

Corollary 2. If A is an R-algebra, then there exists a set $\mathbb{X}$ such that

$$A \simeq R[\mathbb{X}]/I$$

for some ideal I.

Proof. Let $\mathbb{X} = A$ as a set. Define

$$f : \mathbb{X} \to A$$

by $f(x) = x$. By the theorem we have an algebra homomorphism

$$F : R[\mathbb{X}] \to A$$

such that $F(x) = x$ for $x \in \mathbb{X}$. In particular F is ,,onto" and thus by the First Isomorphism Theorem for algebras we have

$$A \simeq R[\mathbb{X}]/\mathrm{Ker}F$$

which completes the proof. □