ideals with maximal radicals are primary

Canonical name	IdealsWithMaximalRadicalsArePrimary
Date of creation	2013-03-22 19:04:31
Last modified on	2013-03-22 19:04:31
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	4
Author	joking (16130)
Entry type	Theorem
Classification	msc 13C99

Proposition. Assume that R is a commutative ring and $I \subseteq R$ is an ideal, such that the radical $r(I)$ of I is a maximal ideal. Then I is a primary ideal.

Proof. We will show, that every zero divisor in R/I is nilpotent (please, see parent object for details).

First of all, recall that $r(I)$ is an intersection of all prime ideals containing I (please, see `http://planetmath.org/ACharacterizationOfTheRadicalOfAnIdeal`this entry for more details). Since $r(I)$ is maximal, it follows that there is exactly one prime ideal $P = r(I)$ such that $I \subseteq P$. In particular the ring R/I has only one prime ideal (because there is one-to-one correspondence between prime ideals in R/I and prime ideals in R containing I). Thus, in R/I an ideal $r(0)$ is prime.

Now assume that $\alpha \in R/I$ is a zero divisor. In particular $\alpha \neq 0 + I$ and for some $\beta \neq 0 + I \in R/I$ we have

$$\alpha\beta = 0 + I.$$

But $0 + I \in r(0)$ and $r(0)$ is prime. This shows, that either $\alpha \in r(0)$ or $\beta \in r(0)$.

Obviously $\alpha \in r(0)$ (and $\beta \in r(0)$), because $r(0)$ is the only maximal ideal in R/I (the ring R/I is local). Therefore elements not belonging to $r(0)$ are invertible, but α cannot be invertible, because it is a zero divisor.

On the other hand $r(0) = \{x + I \in R/I \mid (x + I)^n = 0 \text{ for some } n \in \mathbb{N}\}$. Therefore α is nilpotent and this completes the proof. $\square$

Corollary. Let $p \in \mathbb{N}$ be a prime number and $n \in \mathbb{N}$. Then the ideal $(p^n) \subseteq \mathbb{Z}$ is primary.

Proof. Of course the ideal (p) is maximal and we have

$$r\big((p^n)\big) = r\big((p)^n\big) = (p),$$

since for any prime ideal P (in arbitrary ring R) we have $r(P^n) = P$. The result follows from the proposition. $\square$