divisor as factor of principal divisor

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Let an integral domain $\mathcal{O}$ have a divisor theory $\mathcal{O}^* \to \mathfrak{D}$. The http://planetmath.org/Diviso of divisor theory implies that for any divisor $\mathfrak{a}$, there exists an element ω of $\mathcal{O}$ such that $\mathfrak{a}$ divides the principal divisor (ω), i.e. that $\mathfrak{ac} = (\omega)$ with $\mathfrak{c}$ a divisor. The following theorem states that $\mathfrak{c}$ may always be chosen such that it is coprime with any beforehand given divisor.

Theorem. For any two divisors $\mathfrak{a}$ and $\mathfrak{b}$, there is a principal divisor (ω) such that

$$\mathfrak{ac} \;=\; (\omega)$$

and

$$\gcd(\mathfrak{b},\, \mathfrak{c}) \;=\; (1).$$

Proof. Let $\mathfrak{p}_1,\, \ldots,\, \mathfrak{p}_s$ all distinct prime divisors, which divide the product $\mathfrak{ab}$, and let the divisor $\mathfrak{a}$ be http://planetmath.org/ExactlyDividesexactly divisible by the powers $\mathfrak{p}_1^{a_1},\, \ldots,\, \mathfrak{p}_s^{a_s}$ (the cases $a_i = 0$ are not excluded). For each $i = 1,\, \ldots,\, s,$ we choose a nonzero element α_i of $\mathcal{O}$ being exactly divisible by the power $\mathfrak{p}_i^{a_i}$; the choosing is possible, since any nonzero element of the ideal determined by the divisor $\mathfrak{p}_i^{a_i}$, not belonging to the sub-ideal determined by the divisor $\mathfrak{p}_i^{a_i+1}$, will do. According to the http://planetmath.org/ChineseRemainderTheore remainder theorem, there exists a nonzero element ω of the ring $\mathcal{O}$ such that

$$\omega \;\equiv\; \alpha_i \quad \mathrm{mod}\ \mathfrak{p}_i^{a_i+1} \quad (i \;=\; 1,\, \ldots,\, s). \tag{1}$$

Because α_i is divisible by $\mathfrak{p}_i^{a_i}$, the element ω is divisible by $\mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_s^{a_s} = \mathfrak{a},$ i.e. $(\omega) = \mathfrak{ac}$. If one of the divisors $\mathfrak{p}_i$ would divide $\mathfrak{c}$, then (ω) would be divisible by $\mathfrak{p}_i^{a_i+1}$ and thus by (1), also α_i were divisible by $\mathfrak{p}_i^{a_i+1}$. Therefore, no one of the prime divisors $\mathfrak{p}_1,\, \ldots,\, \mathfrak{p}_s$ divides $\mathfrak{c}$. On the other hand, every prime divisor dividing the divisor $\mathfrak{b}$ divides $\mathfrak{ab}$ and thus is one of $\mathfrak{p}_1,\, \ldots,\, \mathfrak{p}_s$. Accordingly, the divisors $\mathfrak{b}$ and $\mathfrak{c}$ have no common prime divisor, i.e. $\gcd(\mathfrak{b},\, \mathfrak{c}) = (1)$.

References

[1] М. М. Постников: *Введение в теорию алгебраических чисел.* Издательство “Наука”. Москва (1982).