primary decomposition

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Entry type	Definition
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Synonym	shortest primary decomposition
Defines	decomposable ideal
Defines	minimal primary decomposition

Let R be a commutative ring and A be an ideal in R. A *decomposition* of A is a way of writing A as a finite intersection of primary ideals:

$$A = \bigcap_{i=1}^{n} Q_i,$$

where the Q_i are primary in R.

Not every ideal admits a primary decomposition, so we define a *decomposable ideal* to be one that does.

Example. Let $R = \mathbb{Z}$ and take $A = (180)$. Then A is decomposable, and a primary decomposition of A is given by

$$A = (4) \cap (9) \cap (5),$$

since (4), (9), and (5) are all primary ideals in $\mathbb{Z}$.

Given a primary decomposition $A = \cap Q_i$, we say that the decomposition is a *minimal primary decomposition* if for all i, the prime ideals $P_i = \mathrm{rad}(Q_i)$ (where rad denotes the radical of an ideal) are distinct, and for all $1 \leq i \leq n$, we have

$$Q_i \not\subset \bigcap_{j \neq i} Q_j$$

In the example above, the decomposition $(4) \cap (9) \cap (5)$ of A is minimal, where as $A = (2) \cap (4) \cap (3) \cap (9) \cap (5)$ is not.

Every primary decomposition can be refined to admit a minimal primary decomposition.