continuation of exponent

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Theorem. Let K/k be a finite field extension and ν an exponent valuation of the extension field K. Then there exists one and only one positive integer e such that the function

$$(1) \qquad \nu_0(x) := \begin{cases} \infty & \text{when } x = 0, \\ \frac{\nu(x)}{e} & \text{when } x \neq 0, \end{cases}$$

defined in the base field k, is an `http://planetmath.org/ExponentValuation`exponent of k.

Proof. The exponent ν of K attains in the set $k\smallsetminus\{0\}$ also non-zero values; otherwise k would be included in $\mathcal{O}_\nu$, the ring of the exponent ν. Since any element ξ of K are integral over k, it would then be also integral over $\mathcal{O}_\nu$, which is integrally closed in its quotient field K (see theorem 1 in ring of exponent); the situation would mean that $\xi \in \mathcal{O}_\nu$ and thus the whole K would be contained in $\mathcal{O}_\nu$. This is impossible, because an exponent of K attains also negative values. So we infer that ν does not vanish in the whole $k\smallsetminus\{0\}$. Furthermore, ν attains in $k\smallsetminus\{0\}$ both negative and positive values, since $\nu(a)+\nu(a^{-1}) = \nu(aa^{-1}) = \nu(1) = 0$.

Let p be such an element of k on which ν attains as its value the least possible positive integer e in the field k and let a be an arbitrary non-zero element of k. If

$$\nu(a) = m = qe + r \quad (q,\, r \in \mathbb{Z},\;\; 0 \leqq r < e),$$

then $\nu(ap^{-q}) = m - qe = r$, and thus $r = 0$ on grounds of the choice of p. This means that $\nu(a)$ is always divisible by e, i.e. that the values of the function ν_0 in $k\smallsetminus\{0\}$ are integers. Because $\nu_0(p) = 1$ and $\nu_0(p^l) = l$, the function attains in k every integer value. Also the conditions

$$\nu_0(ab) = \nu_0(a) + \nu_0(b), \quad \nu_0(a+b) \geqq \min\{\nu_0(a),\, \nu_0(b)\}$$

are in , whence ν_0 is an exponent of the field k.

Definition. Let K/k be a finite field extension. If the exponent ν_0 of k is tied with the exponent ν of K via the condition (1), one says that ν *induces* ν_0 to k and that ν is the *continuation* of ν_0 to K. The positive integer e, uniquely determined by (1), is the *ramification index* of ν with respect to ν_0 (or with respect to the subfield k).

References

[1] S. Borewicz & I. Safarevic: *Zahlentheorie*. Birkhäuser Verlag. Basel und Stuttgart (1966).