any divisor is gcd of two principal divisors

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Using the exponent valuations, one can easily prove the

Theorem. In any divisor theory, each divisor is the greatest common divisor of two principal divisors.

Proof. Let $\mathcal{O}^* \to \mathfrak{D}$ be a divisor theory and $\mathfrak{d}$ an arbitrary divisor in $\mathfrak{D}$. We may suppose that $\mathfrak{d}$ is not a principal divisor (if $\mathfrak{D}$ contains exclusively principal divisors, then $\mathfrak{d} = \gcd(\mathfrak{d}, \mathfrak{d})$ and the proof is ready). Let

$$\mathfrak{d} = \prod_{i=1}^{r} \mathfrak{p}_i^{k_i}$$

where the $\mathfrak{p}_i$'s are pairwise distinct prime divisors and every $k_i > 0$. Then third condition in the theorem concerning divisors and exponents allows to choose an element α of the ring $\mathcal{O}$ such that

$$\nu_{\mathfrak{p}_1}(\alpha) = k_1, \quad \ldots, \quad \nu_{\mathfrak{p}_r}(\alpha) = k_r.$$

Let the principal divisor corresponding to α be

$$(\alpha) = \prod_{i=1}^{r} \mathfrak{p}_i^{k_i} \prod_{j=1}^{s} \mathfrak{q}_j^{l_j} = \mathfrak{d}\mathfrak{d}',$$

where the prime divisors $\mathfrak{q}_j$ are pairwise different among themselves and with the divisors $\mathfrak{p}_i$. We can then choose another element β of $\mathcal{O}$ such that

$$\nu_{\mathfrak{p}_1}(\beta) = k_1, \quad \ldots, \quad \nu_{\mathfrak{p}_r}(\beta) = k_r, \quad \nu_{\mathfrak{q}_1}(\beta) = \ldots = \nu_{\mathfrak{q}_s}(\beta) = 0.$$

Then we have $(\beta) = \mathfrak{d}\mathfrak{d}''$, where $\mathfrak{d}'' \in \mathfrak{D}$ and

$$\gcd(\mathfrak{d}', \mathfrak{d}'') = \mathfrak{q}^0 \cdots \mathfrak{q}^0 = \mathfrak{e} = (1).$$

The gcd of the principal divisors (α) and (β) is apparently $\mathfrak{d}$, whence the proof is settled.