number field that is not norm-Euclidean

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Proposition. The real quadratic field $\mathbb{Q}(\sqrt{14})$ is not norm-Euclidean.

Proof. We take the number $\gamma = \frac{1}{2}+\frac{1}{2}\sqrt{14}$ which is not integer of the field ($14 \equiv 2 \pmod 4$). Antithesis: $\gamma = \varkappa+\delta$ where $\varkappa = a+b\sqrt{14}$ is an integer of the field ($a,\, b \in \mathbb{Z}$) and

$$|\mathrm{N}(\delta)| = \left|\left(\frac{1}{2}-a\right)^2 - 14\left(\frac{1}{2}-b\right)^2\right| < 1.$$

Thus we would have

$$|\underbrace{(2a-1)^2-14(2b-1)^2}_{E}| < 4.$$

And since $(2a-1)^2 = 4(a-1)a+1 \equiv 1 \pmod 8$, it follows $E \equiv 1-14\cdot 1 \equiv 3 \pmod 8$, i.e. $E = 3$. So we must have

$$(2a-1)^2 \equiv (2a-1)^2-14(2b-1)^2 \equiv 3 \pmod 7. \tag{1}$$

But $\{0,\, \pm 1,\, \pm 2,\, \pm 3\}$ is a complete residue system modulo 7, giving the set $\{1,\, 2,\, 4\}$ of possible quadratic residues modulo 7. Therefore (1) is impossible. The antithesis is wrong, whence the `http://planetmath.org/EuclideanNumberField`theorem 1 of the parent entry says that the number field is not norm-Euclidean.

Note. The function N used in the proof is the usual

$$\mathrm{N}:\ r+s\sqrt{14} \mapsto r^2-14s^2 \quad (r,\, s \in \mathbb{Q})$$

defined in the field $\mathbb{Q}(\sqrt{14})$. The notion of norm-Euclidean number field is based on the `http://planetmath.org/NormAndTraceOfAlgebraicNumber`norm. There exists a fainter function, the so-called Euclidean valuation, which can be defined in the maximal orders of some `http://planetmath.org/NumberField`algebraic number fields; such a maximal order, i.e. the ring of integers of the number field, is then a Euclidean domain. The existence of a Euclidean valuation guarantees that the maximal order is a UFD and thus a PID. Recently it has been shown the existence of the Euclidean domain $\mathbb{Z}[\frac{1+\sqrt{69}}{2}]$ in the field $\mathbb{Q}(\sqrt{69})$ but the field is not norm-Euclidean.

The maximal order $\mathbb{Z}[\sqrt{14}]$ of $\mathbb{Q}(\sqrt{14})$ has also been proven to be a Euclidean domain (Malcolm Harper 2004 in *Canadian Journal of Mathematics*).