invertible formal power series

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Theorem. Let R be a commutative ring with non-zero unity. A formal power series

$$f(X) \ := \ \sum_{i=0}^{\infty} a_i X^i \tag{1}$$

is invertible in the ring $R[[X]]$ iff a_0 is invertible in the ring R.

Proof. 1°. Let $f(X)$ have the multiplicative inverse $g(X) := \sum_{i=0}^{\infty} b_i X^i$. Since

$$f(X)g(X) \ = \ \sum_{i=0}^{\infty} \sum_{j=0}^{i} a_j b_{i-j} X^i \ = \ 1,$$

we see that $a_0 b_0 = 1$, i.e. a_0 is an invertible element (unit) of R.

2°. Assume conversely that a_0 is invertible in R. For making from a formal power series

$$g(X) := \sum_{i=0}^{\infty} b_i X^i \tag{2}$$

the inverse of $f(X) = \sum_{i=0}^{\infty} a_i X^i$, we first choose $b_0 := a_0^{-1}$. For all already defined coefficients $b_0,\, b_1,\, \ldots,\, b_{i-1}$ let the next coefficient be defined as

$$b_i \ := \ -a_0^{-1}(a_1 b_{i-1} + a_2 b_{i-2} + \ldots + a_i b_0).$$

This equation means that

$$\sum_{j=0}^{i} a_j b_{i-j} \ = \ a_0 b_i + a_1 b_{i-1} + a_2 b_{i-2} + \ldots + a_i b_0$$

vanishes for all $i = 1,\, 2,\, \ldots;$ since $a_0 b_0 = 1,$ the product of the formal power series (1) and (2) becomes simply equal to 1. Accordingly, $f(x)$ is invertible.