Krull valuation domain

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Theorem. Any Krull valuation $|\cdot|$ of a field K determines a unique valuation domain $R = \{a \in K : \ |x| \leqq 1\}$, whose field of fraction is K.

Proof. We first see that $1 \in R$ since $|1| = 1$. Let then a, b be any two elements of R. The non-archimedean triangle inequality shows that $|a-b| \leqq \max\{|a|, |b|\} \leqq 1$, i.e. that the difference $a-b$ belongs to R. Using the `http://planetmath.org/OrderedGroup`multiplication rule 4 of inequalities we obtain

$$|ab| = |a| \cdot |b| \leqq 1 \cdot 1 = 1$$

which shows that also the product ab is element of R. Thus, R is a subring of the field K, and so an integral domain. Let now c be an arbitrary element of K not belonging to R. This implies that $1 < |c|$, whence $|c^{-1}| = |c|^{-1} < 1$ (see the `http://planetmath.org/OrderedGroup`inverse rule 5). Consequently, the inverse c^{-1} belongs to R, and we conclude that R is a valuation domain. The $a = \frac{a}{1}$ and $c = \frac{1}{c^{-1}}$ make evident that K is the field of fractions of R.