rules of calculus for derivative of polynomial

Canonical name	RulesOfCalculusForDerivativeOfPolynomial
Date of creation	2013-03-22 18:20:05
Last modified on	2013-03-22 18:20:05
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Last modified by	rspuzio (6075)
Numerical id	10
Author	rspuzio (6075)
Entry type	Derivation
Classification	msc 13P05
Classification	msc 11C08
Classification	msc 12E05
Related topic	ProofOfPropertiesOfDerivativesByPureAlgebra

In this entry, we will derive the properties of derivatives of polynomials in a rigorous fashion. We begin by showing that the derivative exists.

Theorem 1. *If A is a commutative ring and p is a polynomial in $A[x]$, then there exist unique polynomials q and A such that $p(x+y) = p(x) + y\,q(x) + y^2\,r(x,y)$.*

Proof. We will first show existence, then uniqueness. Define $f(y) = p(x+y) - p(x)$. Since f is a polynomial in y with coefficients in the ring $A[x]$ and $f(0) = 0$, we must have y be a factor of $f(y)$, so $f(y) = y\,g(x,y)$ for some g in $A[x,y]$. By definition of f, this means that $p(x+y) - p(x) = y\,g(x,y)$. [1] Define $q(x) = g(x,0)$ and $h(x,y) = g(x,y) - g(x,0)$. Regarding h as a polynomial in y with coefficients in $A[x]$, we may, similiarly to what we did earlier, note that, since $h(0) = 0$ by construction, y must be a factor of $h(y)$. Hence there exists a polynomial r with coefficients in $A[x,y]$ such that $h(y) = y\,r(x,y)$. Combining our definitions, we conclude that $p(x+y) = p(x) + y\,q(x) + y^2\,r(x,y)$.

We will now show uniqueness. Assume that there exists polymonomials q, r, Q, R such that $p(x+y) = p(x)+y\,q(x)+y^2\,r(x,y)$ and $p(x+y) = p(x)+y\,Q(x) + y^2\,R(x,y)$. Subtracting and rearranging terms, $y(q(x) - Q(x)) = y^2(R(x,y) - r(x,y))$. Cancelling y^2, we have $q(x) - Q(x) = y(R(x,y) - r(x,y))$. Substituting 0 for y, we have $q(x) - Q(x) = 0$. Replacing this in our equation, $y(R(x,y) - r(x,y)) = 0$. Cancelling another y, $R(x,y) - r(x,y) = 0$. Hence, we conclude that $Q = q$ and $R = r$, so our is unique. $\square$

Hence, the following is well-defined:

Definition 1. *Let A be a commutative ring and let p be polynomial in $A[x]$. Then p' is the unique element of $A[x]$ such that $p(x+y) = p(x) + y\,p'(x) + y^2\,r[x,y]$ for some $r \in A[x,y]$*

We will now derive some of the rules for manipulating derivatives familiar form calculus for polynomials using purely algebraic operations with no limits involved.

[1] We are here making use of the identification of $A[x][y]$ with $A[x,y]$ to write the polynomial g either as a polynomial in y with coefficients in $A[x]$ or as a polynomial in x and y with coefficients in A.

[2] Note that, in general, the cancellation law need not hold. However, even if A has divisors of zero, it still will be the case that the polynomial y cannot divide zero, so we may cancel it.

Theorem 2. *If A is a commutative ring and $p, q \in A[x]$, then $(p+q)' = p'+q'$.*

Proof. Let us write $p(x+y) = p(x) + y\,p'(x) + y^2\,r(x,y)$ and $q(x+y) = q(x) + y\,q'(y) + y^2\,s(x,y)$. Adding, we have

$$p(x,y) + q(x,y) = p(x) + q(x) + y(p'(x) + q'(x)) + y^2(r(x,y) + s(x,y)).$$

By definition of derivative, this means that $(p+q)' = p' + q'$. $\square$

Theorem 3. *If A is a commutative ring and $p, q \in A[x]$, then $(p \cdot q)' = p' \cdot q + p \cdot q'$.*

Proof. Let us write $p(x+y) = p(x) + y\,p'(x) + y^2\,r(x,y)$ and $q(x+y) = q(x) + y\,q'(y) + y^2\,s(x,y)$. Multiplying, grouping terms, and pulling out some common factors, we have

$$\begin{aligned} p(x+y)q(x+y) = p(x)q(y) &+ y(p'(x)q(x) + p(x)q'(x)) \\ &+ y^2(p(x)s(x,y) + q(x)r(x,y) + p'(x)q'(y) \\ &\quad + y\,p'(x)s(x,y) + y\,q'(x)r(x,y) + y^2\,r(x,y)s(x,y)). \end{aligned}$$

By definition of derivative, this means that $(p \cdot q)' = p' \cdot q + p \cdot q'$. $\square$

Theorem 4. *If A is a commutative ring and $p, q \in A[x]$, then $(p \circ q)' = (p' \circ q) \cdot q'$.*

Proof. Let us write $p(x+y) = p(x) + y\,p'(x) + y^2\,r(x,y)$ and $q(x+y) = q(x) + y\,q'(y) + y^2\,s(x,y)$. Composing, grouping terms, and pulling out some common factors, we have

$$\begin{aligned} p(q(x+y)) &= p\left(q(x) + y\,q'(y) + y^2\,s(x,y)\right) \\ &= p(q(x)) + \left(y\,q'(y) + y^2\,s(x,y)\right) p'(q(x)) \\ &\quad + \left(y\,q'(y) + y^2\,s(x,y)\right)^2 r\left(q(x), y\,q'(y) + y^2\,s(x,y)\right) \\ &= p(q(x)) + y\,p'(q(x))q'(y) \\ &\quad + y^2\left(s(x,y)p'(q(x)) + (q'(y) + y\,s(x,y))^2\,r\left(q(x), y\,q'(y) + y^2\,s(x,y)\right)\right) \end{aligned}$$

By definition of derivative, this means that $(p \circ q)' = (p' \circ q) \cdot q'$. $\square$