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cancellation ideal

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Defines	cancellative

Let R be a commutative ring containing regular elements and $\mathfrak{S}$ be the multiplicative semigroup of the non-zero fractional ideals of R. A fractional ideal $\mathfrak{a}$ of R is called a *cancellation ideal* or simply *cancellative*, if it is a cancellative element of $\mathfrak{S}$, i.e. if

$$\mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{c} \Rightarrow \mathfrak{b} = \mathfrak{c} \quad \forall\ \mathfrak{b},\, \mathfrak{c} \in \mathfrak{S}.$$

- Each invertible ideal is cancellative.
- A finite product $\mathfrak{a}_1\mathfrak{a}_2...\mathfrak{a}_m$ of fractional ideals is cancellative iff every $\mathfrak{a}_i$ is such.
- The fractional ideal $\mathfrak{a}/r := \{ar^{-1} : \ a \in \mathfrak{a}\}$, where $\mathfrak{a}$ is an integral ideal of R and r a regular element of R, is cancellative if and only if $\mathfrak{a}$ is cancellative in the multiplicative semigroup of the non-zero integral ideals of R.
- If $r \in R$, then the principal ideal (r) of R is cancellative if and only if r is a regular element of the total ring of fractions of R.
- If $\mathfrak{a}_1{+}\mathfrak{a}_2{+}...{+}\mathfrak{a}_m$ is a cancellation ideal and n a positive integer, then

  $$(\mathfrak{a}_1{+}\mathfrak{a}_2{+}...{+}\mathfrak{a}_m)^n = \mathfrak{a}_1^n{+}\mathfrak{a}_2^n{+}...{+}\mathfrak{a}_m^n.$$

  In particular, if the ideal $(a_1,\, a_2,\, ...,\, a_m)$ of R is cancellative, then

  $$(a_1,\, a_2,\, ...,\, a_m)^n = (a_1^n,\, a_2^n,\, ...,\, a_m^n).$$

References

[1] R. Gilmer: *Multiplicative ideal theory.* Queens University Press. Kingston, Ontario (1968).

[2] M. Larsen & P. McCarthy: *Multiplicative theory of ideals.* Academic Press. New York (1971).