proof that a gcd domain is integrally closed

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Proposition 1. *Every gcd domain is integrally closed.*

Proof. Let D be a gcd domain. For any $a, b \in D$, let $\mathrm{GCD}(a,b)$ be the collection of all gcd's of a and b. For this proof, we need two facts:

1. $\mathrm{GCD}(ma, mb) = m\,\mathrm{GCD}(a,b)$.
2. If $\mathrm{GCD}(a,b) = [1]$ and $\mathrm{GCD}(a,c) = [1]$, then $\mathrm{GCD}(a,bc) = [1]$.

The proof of the two properties above can be found `http://planetmath.org/PropertiesOfAGc` For convenience, we let $\gcd(a,b)$ be any one of the representatives in $\mathrm{GCD}(a,b)$.

Let K be the field of fraction of D, and $a/b \in K$ ($a, b \in D$ and $b \neq 0$) is a root of a monic polynomial $p(x) \in D[x]$. We may, from property (1) above, assume that $\gcd(a,b) = 1$.

Write

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0.$$

So we have

$$0 = (a/b)^n + c_{n-1}(a/b)^{n-1} + \cdots + c_0.$$

Multiply the equation by b^n then rearrange, and we get

$$-a^n = c_{n-1}ba^{n-1} + \cdots + c_0b^n = b(c_{n-1}a^{n-1} + \cdots + c_0b^{n-1}).$$

Therefore, $b \mid a^n$. Since $\gcd(a,b) = 1$, $1 = \gcd(a^n, b) = b$, by repeated applications of property (2), and one application of property (1) above. Therefore b is an associate of 1, hence a unit and we have $a/b \in D$.

$\square$

Together with the additional property (call it property 3)

if $\mathrm{GCD}(a,b) = [1]$ and $a \mid bc$, then $a \mid c$ (proof found `http://planetmath.org/PropertiesOf`

we have the following

Proposition 2. *Every gcd domain is a Schreier domain.*

Proof. That a gcd domain is integrally closed is clear from the previous paragraph. We need to show that D is pre-Schreier, that is, every non-zero element is primal. Suppose c is non-zero in D, and $c \mid ab$ with $a, b \in D$. Let $r = \gcd(a,c)$ and $rt = a$, $rs = c$. Then $1 = \gcd(s,t)$ by property (1) above. Next, since $c \mid ab$, write $cd = ab$ so that $rsd = rtb$. This implies that $sd = tb$. So $s \mid tb$ together with $\gcd(s,t) = 1$ show that $s \mid b$ by property (3). So we have just shown the existence of $r, s \in D$ with $c = rs$, $r \mid a$ and $s \mid b$. Therefore, c is primal and D is a Schreier domain.

$\square$