evaluation homomorphism

Canonical name	EvaluationHomomorphism
Date of creation	2013-03-22 14:13:51
Last modified on	2013-03-22 14:13:51
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	6
Author	mathcam (2727)
Entry type	Theorem
Classification	msc 13P05
Classification	msc 11C08
Classification	msc 12E05
Synonym	substitution homomorphism
Related topic	LectureNotesOnPolynomialInterpolation
Defines	evaluation homomorphism

Let R be a commutative ring and let $R[X]$ be the ring of polynomials with coefficients in R.

Theorem 1. *Let S be a commutative ring, and let $\psi\colon R \to S$ be a homomorphism. Further, let $s \in S$. Then there is a unique homomorphism $\phi\colon R[X] \to S$ taking X to s and taking every $r \in R$ to $\psi(r)$.*

This amounts to saying that polynomial rings are free objects in the category of R-algebras; the theorem then states that they are projective. This is true in much greater generality; in fact, the property of being projective is intended to extract the essential property of being free.

Proof. We first prove existence. Let $f \in R[X]$. Then by definition there is some finite list of a_i such that $f = \sum_i a_i X^i$. Then define $\phi(f)$ to be $\sum_i \psi(a_i) s^i$. It is clear from the definition of addition and multiplication on polynomials that ϕ is a homomorphism; the definition makes it clear that $\phi(X) = s$ and $\phi(r) = \psi(r)$.

Now, to show uniqueness, suppose γ is any homomorphism satisfying the conditions of the theorem, and let $f \in R[X]$. Write $f = \sum_i a_i X^i$ as before. Then $\gamma(a_i) = \psi(a_i)$ and $\gamma(s)$ by assumption. But then since γ is a homomorphism, $\gamma(a_i X^i) = \psi(a_i) s^i$ and $\gamma(f) = \sum_i \psi(a_i) s^i = \phi(f)$. $\square$