examples of ring of integers of a number field

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Definition 1. *Let K be a number field. The ring of integers of K, usually denoted by $\mathcal{O}_K$, is the set of all elements $\alpha \in K$ which are roots of some monic polynomial with coefficients in $\mathbb{Z}$, i.e. those $\alpha \in K$ which are integral over $\mathbb{Z}$. In other words, $\mathcal{O}_K$ is the integral closure of $\mathbb{Z}$ in K.*

Example 1. Notice that the only rational numbers which are roots of monic polynomials with integer coefficients are the integers themselves. Thus, the ring of integers of $\mathbb{Q}$ is $\mathbb{Z}$.

Example 2. Let $\mathcal{O}_K$ denote the ring of integers of $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. Then:

$$\mathcal{O}_K \cong \begin{cases} \mathbb{Z} \oplus \frac{1+\sqrt{d}}{2}\mathbb{Z}, & \text{if } d \equiv 1 \mod 4, \\ \mathbb{Z} \oplus \sqrt{d}\,\mathbb{Z}, & \text{if } d \equiv 2, 3 \bmod 4. \end{cases}$$

In other words, if we let

$$\alpha = \begin{cases} \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \mod 4, \\ \sqrt{d}, & \text{if } d \equiv 2, 3 \bmod 4. \end{cases}$$

then

$$\mathcal{O}_K = \{n + m\alpha : n, m \in \mathbb{Z}\}.$$

Example 3. Let $K = \mathbb{Q}(\zeta_n)$ be a cyclotomic extension of $\mathbb{Q}$, where ζ_n is a primitive nth root of unity. Then the ring of integers of K is $\mathcal{O}_K = \mathbb{Z}[\zeta_n]$, i.e.

$$\mathcal{O}_K = \{a_0 + a_1\zeta_n + a_2\zeta_n^2 + \ldots + a_{n-1}\zeta_n^{n-1} : a_i \in \mathbb{Z}\}.$$

Example 4. Let α be an algebraic integer and let $K = \mathbb{Q}(\alpha)$. It is *not true in general* that $\mathcal{O}_K = \mathbb{Z}[\alpha]$ (as we saw in Example 2, for $d \equiv 1 \mod 4$).

Example 5. Let p be a prime number and let $F = \mathbb{Q}(\zeta_p)$ be a cyclotomic extension of $\mathbb{Q}$, where ζ_p is a primitive pth root of unity. Let F^+ be the maximal real subfield of F. It can be shown that:

$$F^+ = \mathbb{Q}(\zeta_p + \zeta_p^{-1}).$$

Moreover, it can also be shown that the ring of integers of F^+ is $\mathcal{O}_{F^+} = \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$.