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quotient of ideals

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Let R be a commutative ring having regular elements and let T be its total ring of fractions. If $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals of R, then one can define two different or *residuals* of $\mathfrak{a}$ by $\mathfrak{b}$:

- $\mathfrak{a}\!:\!\mathfrak{b} \;:=\; \{r \in R|\quad r\mathfrak{b} \subseteq \mathfrak{a}\}$
- $[\mathfrak{a}\!:\!\mathfrak{b}] \;:=\; \{t \in T|\quad t\mathfrak{b} \subseteq \mathfrak{a}\}$

They both are fractional ideals of R, and the former in fact an integral ideal of R. It is clear that

$$\mathfrak{a}\!:\!\mathfrak{b} \;=\; [\mathfrak{a}\!:\!\mathfrak{b}] \cap R.$$

In the special case that R has non-zero unity and $\mathfrak{b}$ has the inverse ideal $\mathfrak{b}^{-1}$, we have

$$[\mathfrak{a}\!:\!\mathfrak{b}] \;=\; \mathfrak{a}\mathfrak{b}^{-1},$$

in particular

$$[R\!:\!\mathfrak{b}] \;=\; \mathfrak{b}^{-1}.$$

Some rules concerning the former of quotient (the corresponding rules are valid also for the latter):

1. $\mathfrak{a} \subseteq \mathfrak{b} \quad\Rightarrow\quad \mathfrak{a} : \mathfrak{c} \subseteq \mathfrak{b} : \mathfrak{c} \;\wedge\; \mathfrak{c} : \mathfrak{a} \supseteq \mathfrak{c} : \mathfrak{b}$
2. $\mathfrak{a} : (\mathfrak{b}\mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}$
3. $\mathfrak{a} : (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}) \cap (\mathfrak{a} : \mathfrak{c})$
4. $(\mathfrak{a} \cap \mathfrak{b}) : \mathfrak{c} = (\mathfrak{a} : \mathfrak{c}) \cap (\mathfrak{b} : \mathfrak{c})$

Remark. In a Prüfer ring R the `http://planetmath.org/SumOfIdeals`addition and intersection of ideals are dual operations of each other in the sense that there we have the duals

$\mathfrak{a} : (\mathfrak{b} \cap \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}) + (\mathfrak{a} : \mathfrak{c})$

$(\mathfrak{a} + \mathfrak{b}) : \mathfrak{c} = (\mathfrak{a} : \mathfrak{c}) + (\mathfrak{b} : \mathfrak{c})$

of the two last rules if the are finitely generated.