homogeneous ideal

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Let $R = \oplus_{g \in G} R_g$ be a graded ring. Then an element r of R is said to be *homogeneous* if it is an element of some R_g. An ideal I of R is said to be homogeneous if it can be generated by a set of homogeneous elements, or equivalently if it is the ideal generated by the set of elements $\bigcup_{g \in G} I \cap R_g$.

One observes that if I is a homogeneous ideal and $r = \sum_i r_{g_i}$ is the sum of homogeneous elements r_{g_i} for distinct g_i, then each r_{g_i} must be in I.

To see some examples, let k be a field, and take $R = k[X_1, X_2, X_3]$ with the usual grading by total degree. Then the ideal generated by $X_1^n + X_2^n - X_3^n$ is a homogeneous ideal. It is also a radical ideal. One reason homogeneous ideals in $k[X_1, \ldots, X_n]$ are of interest is because (if they are radical) they define projective varieties; in this case the projective variety is the `http://planetmath.org/FermatsLastTheorem`Fermat curve. For contrast, the ideal generated by $X_1 + X_2^2$ is not homogeneous.