proof of Hilbert's Nullstellensatz

Canonical name	ProofOfHilbertsNullstellensatz
Date of creation	2013-03-22 15:27:46
Last modified on	2013-03-22 15:27:46
Owner	pbruin (1001)
Last modified by	pbruin (1001)
Numerical id	4
Author	pbruin (1001)
Entry type	Proof
Classification	msc 13A10

Let K be an algebraically closed field, let $n \geq 0$, and let I be an ideal of the polynomial ring $K[x_1, \ldots, x_n]$. Let $f \in K[x_1, \ldots, x_n]$ be a polynomial with the property that

$$f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in V(I).$$

Suppose that $f^r \notin I$ for all $r > 0$; in particular, I is strictly smaller than $K[x_1, \ldots, x_n]$ and $f \neq 0$. Consider the ring

$$R = K[x_1, \ldots, x_n, 1/f] \subset K(x_1, \ldots, x_n).$$

The R-ideal RI is strictly smaller than R, since

$$RI = \bigcup_{r=0}^{\infty} f^{-r} I$$

does not contain the unit element. Let y be an indeterminate over $K[x_1, \ldots, x_n]$, and let J be the inverse image of RI under the homomorphism

$$\phi \colon K[x_1, \ldots, x_n, y] \to R$$

acting as the identity on $K[x_1, \ldots, x_n]$ and sending y to $1/f$. Then J is strictly smaller than $K[x_1, \ldots, x_n, y]$, so the weak Nullstellensatz gives us an element $(a_1, \ldots, a_n, b) \in K^{n+1}$ such that $g(a_1, \ldots, a_n, b) = 0$ for all $g \in J$. In particular, we see that $g(a_1, \ldots, a_n) = 0$ for all $g \in I$. Our assumption on f therefore implies $f(a_1, \ldots, a_n) = 0$. However, J also contains the element $1 - yf$ since ϕ sends this element to zero. This leads to the following contradiction:

$$0 = (1 - yf)(a_1, \ldots, a_n, b) = 1 - bf(a_1, \ldots, a_n) = 1.$$

The assumption that $f^r \notin I$ for all $r > 0$ is therefore false, i.e. there is an $r > 0$ with $f^r \in I$.