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Dedekind-Hasse valuation

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Defines	Dedekind-Hasse valuation

If D is an integral domain then it is a PID iff it has a *Dedekind-Hasse valuation*, that is, a function $\nu : D - \{0\} \to \mathbb{Z}^+$ such that for any $a, b \in D - \{0\}$ either

- $a \in (b)$

  or

- $\exists \alpha \in (a) \exists \beta \in (b) \, [0 < \nu(\alpha + \beta) < \nu(b)]$

Proof: First, let ν be a Dedekind-Hasse valuation and let I be an ideal of an integral domain D. Take some $b \in I$ with $\nu(b)$ minimal (this exists because the integers are well-ordered) and some $a \in I$ such that $a \neq 0$. I must contain both (a) and (b), and since it is closed under addition, $\alpha + \beta \in I$ for any $\alpha \in (a), \beta \in (b)$.

Since $\nu(b)$ is minimal, the second possibility above is ruled out, so it follows that $a \in (b)$. But this holds for any $a \in I$, so $I = (b)$, and therefore every ideal is princple.

For the converse, let D be a PID. Then define $\nu(u) = 1$ for any unit. Any non-zero, non-unit can be factored into a finite product of irreducibles (since `http://planetmath.org/node/PIDsareUFDs`every PID is a UFD), and every such factorization of a is of the same length, r. So for $a \in D$, a non-zero non-unit, let $\nu(a) = r + 1$. Obviously $r \in \mathbb{Z}^+$.

Then take any $a, b \in D - \{0\}$ and suppose $a \notin (b)$. Then take the ideal of elements of the form $\{\alpha + \beta | \alpha \in (a), \beta \in (b)\}$. Since this is a PID, it is a principal ideal (c) for some $r \in D - \{0\}$, and since $0 + b = b \in (c)$, there is some non-unit $x \in D$ such that $xc = b$. Then $N(b) = N(xr)$. But since x is not a unit, the factorization of b must be longer than the factorization of c, so $\nu(b) > \nu(c)$, so ν is a Dedekind-Hasse valuation.