fractional ideal of commutative ring

Canonical name	FractionalIdealOfCommutativeRing
Date of creation	2015-05-06 14:40:32
Last modified on	2015-05-06 14:40:32
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	16
Author	pahio (2872)
Entry type	Definition
Classification	msc 13B30
Related topic	FractionalIdeal
Related topic	GeneratorsOfInverseIdeal
Related topic	IdealClassesFormAnAbelianGroup
Defines	fractional ideal
Defines	integral ideal
Defines	invertible ideal
Defines	invertible
Defines	inverse ideal
Defines	class group of a ring
Defines	unit ideal

Definition. Let R be a commutative ring having a regular element and let T be the total ring of fractions of R. An http://planetmath.org/Submodule$R$-submodule $\mathfrak{a}$ of T is called *fractional ideal* of R, provided that there exists a regular element d of R such that $\mathfrak{a}d \subseteq R$. If a fractional ideal is contained in R, it is a usual ideal of R, and we can call it an *integral ideal* of R.

Note that a fractional ideal of R is not necessarily a subring of T. The set of all fractional ideals of R form under the multiplication an commutative semigroup with identity element $R' = R+\mathbb{Z}e$, where e is the unity of T.

An ideal $\mathfrak{a}$ (or fractional) of R is called *invertible*, if there exists another ideal $\mathfrak{a}^{-1}$ of R such that $\mathfrak{a}\mathfrak{a}^{-1} = R'$. It is not hard to show that any invertible ideal $\mathfrak{a}$ is finitely generated and http://planetmath.org/RegularIdealregular, moreover that the *inverse ideal* $\mathfrak{a}^{-1}$ is uniquely determined (see the entry "http://planetmath.org/InvertibleIdealIsFinitelyGeneratedinvertible ideal is finitely generated") and may be generated by the http://planetmath.org/GeneratorsOfI amount of generators as $\mathfrak{a}$.

The set of all invertible fractional ideals of R forms an Abelian group under the multiplication. This group has a normal subgroup consisting of all regular principal fractional ideals; the corresponding factor group is called the of the ring R.

Note. In the special case that the ring R has a unity 1, R itself is the principal ideal (1), being the identity element of the semigroup of fractional ideals and the group of invertible fractional ideals. It is called the *unit ideal.* The unit ideal is the only integral ideal containing units of the ring.