sum of ideals

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Synonym	greatest common divisor of ideals
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Related topic	TwoGeneratorProperty
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Related topic	AlgebraicLattice
Related topic	LatticeOfIdeals
Related topic	MaximalIdealIsPrime
Related topic	AnyDivisorIsGcdOfTwoPrincipalDivisors
Related topic	GcdDomain
Defines	sum ideal
Defines	sum of the ideals
Defines	addition of ideals
Defines	factor of ideal
Defines	greatest common divisor of ideals
Defines	least common multiple of ideals

Definition. Let's consider some set of ideals (left, right or two-sided) of a ring. The *sum of the ideals* is the smallest ideal of the ring containing all those ideals. The sum of ideals is denoted by using "+" and "$\sum$" as usually.

It is not difficult to be persuaded of the following:

- The sum of a finite amount of ideals is

$$\mathfrak{a}_1+\mathfrak{a}_2+\cdots+\mathfrak{a}_k \;=\; \{a_1+a_2+\cdots+a_k\vdots\quad a_i\in\mathfrak{a}_i\ \forall i\}.$$

- The sum of any set of ideals consists of all finite sums $\displaystyle\sum_j a_j$ where every a_j belongs to one $\mathfrak{a}_j$ of those ideals.

Thus, one can say that the sum ideal is *generated by* the set of all elements of the individual ideals; in fact it suffices to use all generators of these ideals.

Let $\mathfrak{a}+\mathfrak{b}=\mathfrak{d}$ in a ring R. Because $\mathfrak{a}\subseteq\mathfrak{d}$ and $\mathfrak{b}\subseteq\mathfrak{d}$, we can say that $\mathfrak{d}$ is a of both $\mathfrak{a}$ and $\mathfrak{b}$.[1] Moreover, $\mathfrak{d}$ is contained in every common factor $\mathfrak{c}$ of $\mathfrak{a}$ and $\mathfrak{b}$ by virtue of its minimality. Hence, $\mathfrak{d}$ may be called the *greatest common divisor* of the ideals $\mathfrak{a}$ and $\mathfrak{b}$. The notations

$$\mathfrak{a}+\mathfrak{b} \;=\; \gcd(\mathfrak{a},\,\mathfrak{b}) \;=\; (\mathfrak{a},\,\mathfrak{b})$$

are used, too.

In an analogous way, the intersection of ideals may be designated as the *least common* of the ideals.

The by "$\subseteq$" partially ordered set of all ideals of a ring forms a lattice, where the least upper bound of $\mathfrak{a}$ and $\mathfrak{b}$ is $\mathfrak{a}+\mathfrak{b}$ and the greatest lower bound is $\mathfrak{a}\cap\mathfrak{b}$. See also the example 3 in algebraic lattice.

[1]This may be motivated by the situation in $\mathbb{Z}$: $(n)\subseteq(m)$ iff m is a factor of n.