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exponent valuation

Canonical name	ExponentValuation
Date of creation	2013-03-22 17:59:31
Last modified on	2013-03-22 17:59:31
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	12
Author	pahio (2872)
Entry type	Definition
Classification	msc 13F30
Classification	msc 13A18
Classification	msc 12J20
Classification	msc 11R99
Synonym	exponent of field
Related topic	DiscreteValuation
Related topic	OrderValuation
Related topic	UltrametricTriangleInequality
Related topic	DivisorTheoryAndExponentValuations
Related topic	DivisorTheory
Defines	exponent of a field
Defines	exponent of the field

Definition. A function ν defined in a field K is called an *exponent valuation* or shortly an *exponent* of the field, if it satisfies the following conditions:

1. $\nu(0) = \infty$ and $\nu(\alpha)$ runs all rational integers when α runs the nonzero elements of K.
2. $\nu(\alpha\beta) = \nu(\alpha)+\nu(\beta)$.
3. $\nu(\alpha+\beta) \geqq \min\{\nu(\alpha),\, \nu(\beta)\}$.

Note that because of the discrete value set $\mathbb{Z}$, an exponent valuation belongs to the discrete valuations, and because of notational causes, to the order valuations.

Properties.
$\nu(1) = 0$
$\nu(-\alpha) = \nu(\alpha)$
$\nu\left(\frac{\alpha}{\beta}\right) = \nu(\alpha)-\nu(\beta)$
$\nu(\alpha^n) = n\,\nu(\alpha)$
$\nu(\alpha_1 + \ldots + \alpha_n) \geqq \min\{\nu(\alpha),\, \ldots,\, \nu(\alpha_n)\}$
$\nu(\alpha+\beta) = \min\{\nu(\alpha),\, \nu(\beta)\} \quad \text{if} \quad \nu(\alpha) \neq \nu(\beta)$

Example. If an integral domain $\mathcal{O}$ has a divisor theory $\mathcal{O}^* \to \mathfrak{D}$, then for each prime divisor $\mathfrak{p}$ there is an exponent valuation $\nu_\mathfrak{p}$ of the quotient field K of $\mathcal{O}$. It is given by

$$\nu_\mathfrak{p}(\alpha) \;=:\; \begin{cases} \infty \quad \text{when } \alpha = 0, \\ \max\,\{k \in \mathbb{Z} \vdots\; \mathfrak{p}^k \mid (\alpha)\} \quad \text{when } \alpha \neq 0; \end{cases}$$

$$\nu_\mathfrak{p}(\xi) \;=:\; \nu_\mathfrak{p}(\alpha) - \nu_\mathfrak{p}(\beta) \;\text{ when }\; \xi = \frac{\alpha}{\beta} \text{ with } \alpha,\, \beta \in \mathcal{O}^*.$$

Hence, $\mathfrak{p}^{\nu_\mathfrak{p}(\alpha)}$ exactly divides α. Apparently, $\nu_\mathfrak{p}(\xi)$ does not depend on the quotient form $\frac{\alpha}{\beta}$ for ξ. It is not hard to show that $\nu_\mathfrak{p}$ defined above is an exponent of the field K.

Different prime divisors $\mathfrak{p}$ and $\mathfrak{q}$ determine different exponents $\nu_\mathfrak{p}$ and $\nu_\mathfrak{q}$, since the condition 3 of the `http://planetmath.org/DivisorTheory`definition of divisor theory guarantees such an element γ of $\mathcal{O}$ which in divisible by $\mathfrak{p}$

but not by $\mathfrak{q}$; then $\nu_{\mathfrak{p}}(\gamma) \geqq 1,\ \nu_{\mathfrak{q}}(\gamma) = 0$.

Theorem. Let $\nu_1, \ldots, \nu_r$ be different exponents of a field K. Then for arbitrary set $n_1, \ldots, n_r$ of integers, there exists in K an element ξ such that

$$\nu_1(\xi) = n_1, \ \ldots, \ \nu_r(\xi) = n_r.$$

The proof of this theorem is found in [1].

References

[1] S. Borewicz & I. Safarevic: *Zahlentheorie*. Birkhäuser Verlag. Basel und Stuttgart (1966).