proof that a domain is Dedekind if its ideals are products of primes

Canonical name	ProofThatADomainIsDedekindIfItsIdealsAreProductsOfPrimes
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We show that for an integral domain R, the following are equivalent.

1. R is a Dedekind domain.
2. every nonzero proper ideal is a product of maximal ideals.
3. every nonzero proper ideal is a product of prime ideals.

For the equivalence of **??** and **??** see proof that a domain is Dedekind if its ideals are products of maximals. Also, as every maximal ideal is prime, it is immediate that **??** implies **??**. So, we just need to consider the case where **??** is satisfied and show that **??** follows, for which it enough to show that every nonzero prime ideal is maximal.

We first suppose that $\mathfrak{p}$ is an `http://planetmath.org/FractionalIdeal`invertible prime ideal and show that it is maximal. To do this it is enough to show that any $a \in R \setminus \mathfrak{p}$ gives $\mathfrak{p} + (a) = R$. First, we have the following inclusions,

$$\mathfrak{p} \subsetneq \mathfrak{p} + (a^2) \subseteq \mathfrak{p} + (a).$$

Then, consider the prime factorizations

$$\mathfrak{p} + (a) = \mathfrak{p}_1 \cdots \mathfrak{p}_m, \tag{1}$$
$$\mathfrak{p} + (a^2) = \mathfrak{q}_1 \cdots \mathfrak{q}_n. \tag{2}$$

We write $\bar{a}$ for the image of a under the `http://planetmath.org/NaturalHomomorphism`natural homorphism $R \to R/\mathfrak{p}$ and $\bar{\mathfrak{a}}$ for the image of any ideal $\mathfrak{a}$. Equations (**??**) and (**??**) give

$$\bar{\mathfrak{q}}_1 \cdots \bar{\mathfrak{q}}_n = (\bar{a})^2 = (\bar{\mathfrak{p}}_1 \cdots \bar{\mathfrak{p}}_m)^2. \tag{3}$$

As $\mathfrak{p}$ is strictly contained in $\mathfrak{p} + (a)$ and $\mathfrak{p} + (a^2)$, it must also be strictly contained in $\mathfrak{p}_k$ and $\mathfrak{q}_k$. So, $\bar{\mathfrak{p}}_k, \bar{\mathfrak{q}}_k$ are nonzero prime ideals, and by uniqueness of prime factorization (see, prime ideal factorization is unique) Equation (**??**) gives $n = 2m$ and $\bar{\mathfrak{p}}_k = \bar{\mathfrak{q}}_k = \bar{\mathfrak{q}}_{k+m}$, after reordering of the factors. So $\mathfrak{p}_k = \mathfrak{q}_k = \mathfrak{q}_{k+m}$ and,

$$\mathfrak{p} + (a^2) = \mathfrak{q}_1 \cdots \mathfrak{q}_n = (\mathfrak{p}_1 \cdots \mathfrak{p}_m)^2 = (\mathfrak{p} + (a))^2 = \mathfrak{p}^2 + a\mathfrak{p} + (a^2). \tag{4}$$

Then, $a \notin \mathfrak{p}$ gives $\mathfrak{p} \cap (a^2) \subseteq a\mathfrak{p}$ and taking the intersection of both sides of (**??**) with $\mathfrak{p}$,

$$\mathfrak{p} = \mathfrak{p}^2 + a\mathfrak{p} + \mathfrak{p} \cap (a^2) \subseteq \mathfrak{p}^2 + a\mathfrak{p} = (\mathfrak{p} + (a))\,\mathfrak{p}.$$

But $\mathfrak{p}$ was assumed to be invertible, and can be cancelled giving $R \subseteq \mathfrak{p} + (a)$, showing that $\mathfrak{p}$ is maximal.

Now let $\mathfrak{p}$ be any prime ideal and $a \in \mathfrak{p} \setminus \{0\}$. Factoring into a product of primes

$$\mathfrak{p}_1 \cdots \mathfrak{p}_n = (a) \subseteq \mathfrak{p}, \tag{5}$$

each of the $\mathfrak{p}_k$ is invertible and, by the above argument, must be maximal. Finally, as $\mathfrak{p}$ is prime, (**??**) gives $\mathfrak{p}_k \subseteq \mathfrak{p}$ for some k, so $\mathfrak{p} = \mathfrak{p}_k$ is maximal.