multiplicatively closed

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Let R be a ring. A subset S of R is said to be *multiplicatively closed* if $S \neq \varnothing$, and whenever $a, b \in S$, then $ab \in S$. In other words, S is a multiplicative set where the multiplication defined on S is the multiplication inherited from R.

For example, let $a \in R$, the set $S := \{a^i, a^{i+1}, \cdots, a^n, \cdots\}$ is multiplicatively closed for any positive integer i. Another simple example is the set $\{1\}$, if R is unital.

Remarks. Let R be a commutative ring.

- If P is a prime ideal in R, then $R - P$ is multiplicatively closed.
- Furthermore, an ideal maximal with respect to the being disjoint from a multiplicative set not containing 0 is a prime ideal.
- In particular, assuming $1 \in R$, any ideal maximal with respect to being disjoint from $\{1\}$ is a maximal ideal.

A multiplicatively closed set S in a ring R is said to be *saturated* if for any $a \in S$, every divisor of a is also in S.

In the example above, if $i = 1$ and a has no divisors, then S is saturated.

Remarks.

- In a unital ring, a saturated multiplicatively closed set always contains $U(R)$, the group of units of R (since it contains 1, and therefore, all divisors of 1). In particular, $U(R)$ itself is saturated multiplicatively closed.
- Assume R is commutative. $S \subseteq R$ is saturated multiplicatively closed and $0 \notin S$ iff $R - S$ is a union of prime ideals in R.

  Proof. This can be shown as follows: if let T be a union of prime ideals in R and $a, b \in R - T$. if $ab \notin R - T$, then $ab \in P \subseteq T$ for some prime ideal P. Therefore, either a or $b \in P \subseteq T$. This contradicts the assumption that $a, b \notin T$. So $R - T$ is multiplicatively closed. If $ab \in R - T$ with $a \notin R - T$, then $a \in P \subseteq T$ for some prime ideal P, which implies $ab \in P \subseteq T$ also. This contradicts the assumption that $ab \notin T$. This shows that $R - T$ is saturated. Of course, $0 \notin R - T$, since 0 lies in any ideal of R.

  Conversely, assume S is saturated multiplicatively closed and $0 \notin S$. For any $r \notin S$, we want to find a prime ideal P containing r such that

$P \cap S = \varnothing$. Once we show this, then take the union T of these prime ideals and that $S = R - T$ is immediate. Let $\langle r \rangle$ be the principal ideal generated by r. Since S is saturated, $\langle r \rangle \cap S = \varnothing$. Let M be the set of all ideals containing $\langle r \rangle$ and disjoint from S. M is non-empty by construction, and we can order M by inclusion. So M is a poset and Zorn's lemma applies. Take any chain C in M containing $\langle r \rangle$ and let P be the maximal element in C. Then any ideal larger than P must not be disjoint from S, so P is prime by the second remark in the first set of remarks. $\square$

- The notion of multiplicative closure can be generalized to be defined over any non-empty set with a binary operation (multiplication) defined on it.

References

[1] I. Kaplansky, *Commutative Rings*. University of Chicago Press, 1974.