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order valuation

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Related topic	PAdicValuation
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Related topic	ExponentValuation2
Related topic	DivisibilityOfPrimePowerBinomialCoefficients
Defines	exponent of field
Defines	zero
Defines	zero of an element
Defines	pole
Defines	pole of an element

Given a Krull valuation $|.|$ of a field K as a mapping of K to an ordered group G (with operation "$\cdot$") equipped with 0, one may use for the an alternative notation "ord":

The "$<$" of G is reversed and the operation of G is denoted by "$+$". The element 0 of G is denoted as ∞, thus ∞ is greater than any other element of G. When we still call the valuation the *order* of K and instead of $|x|$ write $\operatorname{ord} x$, the valuation postulates read as follows.

1. $\operatorname{ord} x \;=\; \infty \;\text{ iff }\; x = 0;$
2. $\operatorname{ord} xy \;=\; \operatorname{ord} x + \operatorname{ord} y;$
3. $\operatorname{ord}(x+y) \;\geqq\; \min\{\operatorname{ord} x,\, \operatorname{ord} y\}.$

We must emphasize that the order valuation is nothing else than a Krull valuation. The name *order* comes from complex analysis, where the "places" http://planetmath.org/ZeroOfAFunctionzero and http://planetmath.org/Polepole of a meromorphic function with their orders have a fully analogous meaning as have the corresponding concepts http://planetmath.org/PlaceOfFieldplace and order valuation in the valuation theory. Thus also a place φ of a field is called a *zero* of an element a of the field, if $\varphi(a) = 0$, and a *pole* of an element b of the field, if $\varphi(b) = \infty$; then e.g. the equation $\varphi(a) = 0$ implies always that $\operatorname{ord} a > 0$.

Example. Let p be a given positive prime number. Any non-zero rational number x can be uniquely expressed in the form

$$x = p^n u,$$

in which n is an integer and the rational number u is by p indivisible, i.e. when reduced to lowest terms, p divides neither its numerator nor its denominator. If we define

$$\operatorname{ord}_p x \;=\; \begin{cases} \infty \;\text{ for }\; x = 0, \\ n \;\text{ for }\; x = p^n u \neq 0, \end{cases}$$

then the function ord_p, defined in $\mathbb{Q}$, clearly satisfies the above postulates of the order valuation.

In [2], an order valuation having only integer values is called the *exponent of the field* (*der Exponent des Körpers*); this name apparently motivated by the exponent n of p. Such an order valuation is a special case of the discrete valuation. Note, that an arbitrary order valuation need not be a discrete valuation, since the values need not be real numbers.

References

[1] E. Artin: *Theory of Algebraic Numbers*. Lecture notes. Mathematisches Institut, Göttingen (1959).

[2] S. Borewicz & I. Safarevic: *Zahlentheorie*. Birkhäuser Verlag. Basel und Stuttgart (1966).