formal power series over field

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Theorem. If K is a field, then the ring $K[[X]]$ of formal power series is a discrete valuation ring with (X) its unique maximal ideal.

Proof. We show first that an arbitrary ideal I of $K[[X]]$ is a principal ideal. If $I = (0)$, the thing is ready. Therefore, let $I \neq (0)$. Take an element

$$f(X) \;:=\; \sum_{i=0}^{\infty} a_i X^i$$

of I such that it has the least possible amount of successive zero coefficients in its beginning; let its first non-zero coefficient be a_k. Then

$$f(X) \;=\; X^k(a_k + a_{k+1}X + \ldots).$$

Here we have in the parentheses an invertible formal power series $g(X)$, whence get the equation

$$X^k \;=\; f(X)[g(X)]^{-1}$$

implying $X^k \in I$ and consequently $(X^k) \subseteq I$.
For obtaining the reverse inclusion, suppose that

$$h(X) \;:=\; b_n X^n + b_{n+1} X^{n+1} + \ldots$$

is an arbitrary nonzero element of I where $b_n \neq 0$. Because $n \geq k$, we may write

$$h(X) \;=\; X^k(b_n X^{n-k} + b_{n+1} X^{n-k+1} + \ldots).$$

This equation says that $h(X) \in (X^k)$, whence $I \subseteq (X^k)$.
Thus we have seen that I is the principal ideal (X^k), so that $K[[X]]$ is a principal ideal domain.
Now, all ideals of the ring $K[[X]]$ form apparently the strictly descending chain

$$(X) \;\supset\; (X^2) \;\supset\; (X^3) \;\supset\; \ldots \;\supset\; (0),$$

whence the ring has the unique maximal ideal (X). A principal ideal domain with only one maximal ideal is a discrete valuation ring.