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Schur polynomial

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Defines	Schur function

A *Schur polynomial* is a special symmetric polynomial associated to a partition of an integer, or equivalently to a Young diagram. Schur polynomials also have a power series generalization, the *Schur functions*.

First we define some notation. Let λ be a partition of n, and let T be a filling of the Young diagram for λ. Then by x^T we mean the monomial

$$x^T = \prod_{i=1}^{\infty} x_i^{c_i(T)},$$

where $c_i(T)$ is the number of times the number i appears in the filling T. Since T only has finitely many boxes, the product is finite. For example, let $\lambda = (3, 3, 2, 2)$, and let T be the filling

2	4	1
5	2	3
1	4	
1	2	

Notice that 1 and 2 each appear three times in the filling, while 3, 4, and 5 each appear only once. Thus $x^T = x_1^3 x_2^3 x_3 x_4 x_5$.

For convenience let us use $\mathrm{sst}(\lambda, n)$ to denote the collection of fillings of semi-standard tableaux with shape λ by positive integers from 1 to n. Then we can define the *Schur polynomial* $s_\lambda(x_1, \ldots, x_n)$ to be the polynomial

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{sst}(\lambda, n)} x^T.$$

For example, take $n = 5$ and consider the partition $\lambda = (1, 1, 1)$ of 3. Then the Schur polynomial $s_\lambda(x_1, \ldots, x_5)$ is

$$s_{(1,1,1)}(x_1, \ldots, x_5) = \sum_{1 \leq i < j < k \leq 5} x_i x_j x_k.$$

Note that this is the elementary symmetric polynomial of degree 3 in the variables $x_1, \ldots, x_5$. In fact, $s_{(1^k)}(x_1, \ldots, x_n)$ is always the elementary symmetric polynomial of degree k in the variables $x_1, \ldots, x_n$. The polynomial $s_{(k)}(x_1, \ldots, x_n)$ is the complete symmetric polynomial of degree k in $x_1, \ldots, x_n$.

To define Schur functions, we consider the set $\mathrm{sst}(\lambda)$ of all fillings of semistandard tableaux with shape λ:

$$\mathrm{sst}(\lambda) := \bigcup_{n\geq 1} \mathrm{sst}(\lambda, n).$$

The *Schur function* associated to the partition λ is

$$s_\lambda(\mathbf{x}) = \sum_{x\in \mathrm{sst}(\lambda)} x^T$$

Thus the Schur functions are power series in infinitely many variables. For example,

$$s_{(1,1,1)}(\mathbf{x}) = \sum_{1\leq i<j<k} x_i x_j x_k = x_1 x_2 x_3 + \cdots + x_{14} x_{42} x_{132} + \ldots.$$

All Schur polynomials and Schur functions are symmetric functions. In fact, the Schur polynomials of degree n form a basis for the vector space of symmetric polynomials of degree n.

References

[1] William Fulton. *Young tableaux: with applications to representation theory and geometry*. Cambridge University Press, 1997.

[2] Bruce E. Sagan. *The symmetric group: representations, combinatorial algorithms, and symmetric functions*, 2nd ed. Springer, 2001.