multiplication rule gives inverse ideal

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Theorem. Let R be a commutative ring with non-zero unity. If an ideal $(a,\,b)$ of R, with a or b http://planetmath.org/RegularElementregular, obeys the multiplication rule

$$(a,\,b)(c,\,d) = (ac,\,ad{+}bc,\,bd) \tag{1}$$

with all ideals $(c,\,d)$ of R, then $(a,\,b)$ is an invertible ideal.

Proof. The rule gives

$$(a,\,b)^2 = (a,\,-b)(a,\,b) = (a^2,\,ab{-}ba,\,b^2) = (a^2,\,b^2).$$

Thus the product ab may be written in the form

$$ab = ua^2{+}vb^2,$$

where u and v are elements of R. Let's assume that e.g. a is regular. Then a has the multiplicative inverse a^{-1} in the total ring of fractions R. Again applying the rule yields

$$(a,\,b)(va,\,a{-}vb)(a^{-2}) = (va^2,\,a^2{-}vab{+}vab,\,ab{-}vb^2)(a^{-2}) = (va^2,\,a^2,\,ua^2)(a^{-2}) = (v,\,1,\,u) = R.$$

Consequently the ideal $(a,\,b)$ has an inverse ideal (which may be a http://planetmath.org/Fract ideal); this settles the proof.

Remark. The rule (1) in the theorem may be replaced with the rule

$$(a,\,b)(c,\,d) = (ac,\,(a{+}b)(c{+}d),\,bd) \tag{2}$$

as is seen from the identical equation $(a{+}b)(c{+}d){-}ac{-}bd = ad + bc$.