proof of Dedekind−Mertens lemma

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Let R be subring of the commutative ring T and

$$f(X) = f_0 + f_1X + \ldots + f_mX^m \quad \text{and} \quad g(X) = g_0 + g_1X + \ldots + g_nX^n$$

be arbitrary polynomials in $T[X]$. We will prove by induction on n that the R-submodules of T generated by the coefficients of the polynomials f, g, and fg satisfy

$$M_f^{n+1}M_g \;=\; M_f^nM_{fg} \tag{1}$$

where the product modules are generated by the products of their generators.

The generators of the right hand side of (1) belong obviously to the left hand side, whence only the containment

$$M_f^{n+1}M_g \;\subseteq\; M_f^nM_{fg} \tag{2}$$

has to be proved.

Firstly, (2) is trivial in the case $n = 0$. Let now $n > 0$. Define

$$f_j \;:=\; 0 \quad \text{for} \quad j < 0 \quad \text{or} \quad j > m$$

and let G_n be the R-submodule of T generated by $g_0, g_1, \ldots, g_{n-1}$. We have

$$\sum_{i<n} f_{k-i}g_i \;=\; h_k - f_{k-n}g_n \;\in\; M_{fg} + g_nM_f$$

where h_k is the coefficient of X^k of the polynomial fg, and thus by induction we can write

$$M_f^nG_n \;\subseteq\; M_f^{n-1}(M_{fg} + g_nM_f) \;\subseteq\; M_f^{n-1}M_{fg} + M_f^ng_n.$$

This implies the containment

$$f_iM_f^nG_n \;\subseteq\; M_f^nM_{fg} + M_f^nf_ig_n$$

for every i. In addition, we have

$$f_ig_n \;\in\; M_{fg} + f_{i+1}G_n + M_{i+2}G_n + \ldots + f_nG_n,$$

whence

$$f_iM_f^nG_n \;\subseteq\; M_f^nM_{fg} + f_{i+1}M_f^nG_n + \ldots + f_nM_f^nG_n.$$

From this we infer that

$$f_iM_f^nG_n \;\subseteq\; M_f^nM_{fg}$$

is true for each $i = m$, $m{-}1$, $\ldots$, 0. Thus also (2) is true.

References

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[2] J. Arnold & R. Gilmer: "On the contents of polynomials". – *Proc. Amer. Math. Soc.* **24** (1970).

[3] T. Coquand: "*A direct proof of Dedekind–Mertens lemma*". University of Gothenburg 2006. (Available `http://www.cse.chalmers.se/ coquand/mertens.pdf`here.)