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norm-Euclidean number field

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Definition. An algebraic number field K is a *norm-Euclidean number field*, if for every pair $(\alpha,\,\beta)$ of the `http://planetmath.org/AlgebraicInteger`integers of K, where $\beta\neq 0$, there exist $\varkappa$ and ϱ of the field such that

$$\alpha\;=\;\varkappa\beta+\varrho,\quad |\mathrm{N}(\varrho)|<|\mathrm{N}(\beta)|.$$

Here N means the norm function in K.

Theorem 1. A field K is norm-Euclidean if and only if each number γ of K is in the form

$$\gamma\;=\;\varkappa+\delta \tag{1}$$

where $\varkappa$ is an of the field and $|\mathrm{N}(\delta)|<1$.

Proof. First assume the condition (1). Let α and β be integers of K, $\beta\neq 0$. Then there are the numbers $\varkappa,\,\delta\in K$ such that $\varkappa$ is integer and

$$\frac{\alpha}{\beta}\;=\;\varkappa+\delta,\quad |\mathrm{N}(\delta)|\;<\;1.$$

Thus we have

$$\alpha\;=\;\varkappa\beta+\beta\delta\;=\;\varkappa\beta+\varrho.$$

Here $\varrho=\beta\delta$ is integer, since α and $\varkappa\beta$ are integers. We also have

$$|\mathrm{N}(\varrho)|\;=\;|\mathrm{N}(\beta)|\cdot|\mathrm{N}(\delta)|\;<\;|\mathrm{N}(\beta)|\cdot 1\;=\;|\mathrm{N}(\beta)|.$$

Accordingly, K is a norm-Euclidean number field. Secondly assume that K is norm-Euclidean. Let γ be an arbitrary element of the field. We `http://planetmath.org/MultiplesOfAnAlgebraicNumber`can determine a rational integer $m\,(\neq 0)$ such that $m\gamma$ is an algebraic integer of K. The assumption guarantees the integers $\varkappa,\,\varrho$ of K such that

$$m\gamma\;=\;\varkappa m+\varrho,\quad \mathrm{N}(\varrho)\;<\;\mathrm{N}(m).$$

Thus

$$\gamma=\frac{m\gamma}{m}=\varkappa+\frac{\varrho}{m},\quad \left|\mathrm{N}\left(\frac{\varrho}{m}\right)\right|=\frac{|\mathrm{N}(\varrho)|}{|\mathrm{N}(m)|}<1,$$

Q.E.D.

Theorem 2. In a norm-Euclidean number field, any two non-zero have a greatest common divisor.

Proof. We recall that the *greatest common divisor* of two elements of a commutative ring means such a common divisor of the elements that it is divisible by each common divisor of the elements. Let now ϱ_0 and ϱ_1 be two algebraic integers of a norm-Euclidean number field K. According the definition there are the integers $\varkappa_i$ and ϱ_i of K such that

$$\begin{cases} \varrho_0 = \varkappa_2\varrho_1 + \varrho_2, & |\mathrm{N}(\varrho_2)| < |\mathrm{N}(\varrho_1)| \\ \varrho_1 = \varkappa_3\varrho_2 + \varrho_3, & |\mathrm{N}(\varrho_3)| < |\mathrm{N}(\varrho_2)| \\ \varrho_2 = \varkappa_4\varrho_3 + \varrho_4, & |\mathrm{N}(\varrho_4)| < |\mathrm{N}(\varrho_3)| \\ \qquad \cdots\cdots \\ \varrho_{n-2} = \varkappa_n\varrho_{n-1} + \varrho_n, & |\mathrm{N}(\varrho_n)| < |\mathrm{N}(\varrho_{n-1})| \\ \varrho_{n-1} = \varkappa_{n+1}\varrho_n + 0, \end{cases}$$

The ends to the remainder 0, because the numbers $|\mathrm{N}(\varrho_i)|$ form a descending sequence of non-negative rational integers — see the entry norm and trace of algebraic number. As in the Euclid's algorithm in $\mathbb{Z}$, one sees that the last divisor ϱ_n is one greatest common divisor of ϱ_0 and ϱ_1. N.B. that ϱ_0 and ϱ_1 may have an infinite amount of their greatest common divisors, depending the amount of the units in K.

Remark. The ring of integers of any norm-Euclidean number field is a unique factorization domain and thus all ideals of the ring are principal ideals. But not all algebraic number fields with ring of integers a `http://planetmath.org/UFD`UFD are norm-Euclidean, e.g. $\mathbb{Q}(\sqrt{14})$.

Theorem 3. The only norm-Euclidean quadratic fields $\mathbb{Q}(\sqrt{d})$ are those with

$d \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$