maximal ideals of ring of formal power series

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Suppose that R is a commutative ring with non-zero unity.

If $\mathfrak{m}$ is a maximal ideal of R, then $\mathfrak{M} := \mathfrak{m}+(X)$ is a maximal ideal of the ring $R[[X]]$ of formal power series.

Also the converse is true, i.e. if $\mathfrak{M}$ is a maximal ideal of $R[[X]]$, then there is a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{M} = \mathfrak{m}+(X)$.

Note. In the special case that R is a field, the only maximal ideal of which is the zero ideal (0), this corresponds to the only maximal ideal (X) of $R[[X]]$ (see `http://planetmath.org/node/12087`formal power series over field).

We here prove the first assertion. So, $\mathfrak{m}$ is assumed to be maximal. Let

$$f(x) := a_0+a_1X+a_2X^2+\ldots$$

be any formal power series in $R[[X]]\smallsetminus\mathfrak{M}$. Hence, the constant term a_0 cannot lie in $\mathfrak{m}$. According to the criterion for maximal ideal, there is an element r of R such that $1+ra_0 \in \mathfrak{m}$. Therefore

$$1+rf(X) = (1+ra_0)+r(a_1+a_2X+a_3X^2+\ldots)X \in \mathfrak{m}+(X) = \mathfrak{M},$$

whence the same criterion says that $\mathfrak{M}$ is a maximal ideal of $R[[X]]$.