weak approximation theorem

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The weak approximation theorem allows selection, in a Dedekind ring, of an element having specific valuations at a specific finite set of primes, and nonnegative valuations at all other primes. It is essentially a generalization of the Chinese Remainder theorem, as is evident from its proof.

Theorem 1 (Weak). *Let A be a Dedekind domain with fraction field K. Then for any finite set $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ of primes of A and integers $a_1, \ldots, a_k$, there is $x \in K^\star$ such that $\nu_{\mathfrak{p}_i}((x)) = a_i$ and for all other prime ideals $\mathfrak{p}$, $\nu_{\mathfrak{p}}((x)) \geq 0$. Here $\nu_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic valuation associated with a prime ideal $\mathfrak{p}$.*

Proof. Assume first that all $a_i \geq 0$. By the Chinese Remainder Theorem,

$$A/\mathfrak{p}_1^{a_1+1} \times \cdots A/\mathfrak{p}_k^{a_k+1} \cong A/\mathfrak{p}_1^{a_1+1} \cdots \mathfrak{p}_k^{a_k+1}$$

Thus the map

$$A \to A/\mathfrak{p}_1^{a_1+1} \times \cdots A/\mathfrak{p}_k^{a_k+1}$$

is surjective. Now choose $x_i \in p_i^{a_i}, x_i \notin p_i^{a_i+1}$; this is possible since these two ideals are unequal by unique factorization. Choose $x \in A$ with image $(x_1, \ldots, x_k)$. Clearly $\nu_{\mathfrak{p}_i}((x)) = a_i$. But $x \in A$, so all other valuations are nonnegative.

In the general case, assume wlog that we are given a set $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of primes of A and integers $a_1, \ldots, a_r \geq 0$, and a set $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ of primes with integers $b_1, \ldots, b_t < 0$. First choose $y \in K^\star$ (using the case already proved above) so that

$$\begin{cases} \nu_{\mathfrak{p}}((y)) = 0 & \mathfrak{p} = \mathfrak{p}_i \\ \nu_{\mathfrak{p}}((y)) = -b_i & \mathfrak{p} = \mathfrak{q}_j \\ \nu_{\mathfrak{p}}((y)) \geq 0 & \text{otherwise} \end{cases}$$

Now, there are only a finite number of primes $\mathfrak{p}'_k$ such that $\mathfrak{p}'_k$ is not the same as any of the $\mathfrak{q}_j$ and $\nu_{\mathfrak{p}'_k}((y)) > 0$. Let $\nu_{\mathfrak{p}'_k}((y)) = c_k > 0$. Again using the case proved above, choose $x \in K^\star$ such that

$$\begin{cases} \nu_{\mathfrak{p}}((x)) = a_i & \mathfrak{p} = \mathfrak{p}_i \\ \nu_{\mathfrak{p}}((x)) = 0 & \mathfrak{p} = \mathfrak{q}_j \\ \nu_{\mathfrak{p}}((x)) = c_k & \mathfrak{p} = \mathfrak{p}'_k \\ \nu_{\mathfrak{p}}((x)) \geq 0 & \text{otherwise} \end{cases}$$

Then x/y is the required element. □