partial fractions in Euclidean domains

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This entry states and proves the existence of partial fraction decompositions on an Euclidean domain.

In the following, we use ν to denote the Euclidean valuation function of an Euclidean domain E, with the convention that $\nu(0) = -\infty$.

For a gentle introduction:

1. See `http://planetmath.org/PartialFractions`partial fractions of fractional numbers for the case when E consists of the integers and $\nu(k) = |k|$ for $k \neq 0$.

2. See partial fractions of expressions for the case when E consists of polynomials over the complex field, with $\nu(p)$ being the degree of the polynomial p.

3. See partial fractions for polynomials for the case when E is the ring of polynomials over any field, and ν is the degree of polynomials.

Theorem 1. *Let p, $q_1 \neq 0$ and $q_2 \neq 0$ be elements of an Euclidean domain E, with q_1 and q_2 be relatively prime. Then there exist α_1 and α_2 in E such that*

$$\frac{p}{q_1\, q_2} = \frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2}\,.$$

Proof. By the Euclidean algorithm, we can obtain elements s_1 and s_2 in E such that

$$1 = s_1\, q_1 + s_2\, q_2\,.$$

Then

$$\frac{p}{q_1\, q_2} = \frac{p\, s_2}{q_1} + \frac{p\, s_1}{q_2}\,,$$

so we can take $\alpha_1 = p\, s_2$ and $\alpha_2 = p\, s_1$. □

Theorem 2. *Let p and $q \neq 0$ be elements of an Euclidean domain E, and n be any positive integer. Then there exist elements $\alpha_1, \ldots, \alpha_n, \beta$ in E such that*

$$\frac{p}{q^n} = \beta + \frac{\alpha_1}{q} + \frac{\alpha_2}{q^2} + \cdots + \frac{\alpha_n}{q^n}\,, \quad \nu(\alpha_j) < \nu(q)\,.$$

Proof. Let $r_0 = p$. Iterating through $k = 1, \ldots, n$ in order, using the division algorithm, we can find elements r_k and s_k such that

$$r_{k-1} = r_k\, q + s_k\,, \quad \nu(s_k) < \nu(q)\,.$$

Then

$$
\begin{aligned}
p = r_0 &= r_1\, q + s_1 \\
&= (r_2 q + s_2)\, q + s_1 \\
&= \dots \\
&= r_n\, q^n + s_n\, q^{n-1} + s_{n-1}\, q^{n-2} + \cdots + s_2\, q + s_1 \\
\frac{p}{q^n} &= r_n + \frac{s_n}{q} + \frac{s_{n-1}}{q^2} + \cdots + \frac{s_2}{q^{n-1}} + \frac{s_1}{q^n}\,.
\end{aligned}
$$

So set $\beta = r_n$ and $\alpha_j = s_{n-j+1}$. $\square$

Theorem 3. *Let p and $q \neq 0$ be elements of an Euclidean domain E. Let $q = \phi_1^{n_1}\, \phi_2^{n_2}\, \cdots\, \phi_k^{n_k}$ be a factorization of q to prime factors ϕ_i. Then there exist elements α_{ij}, β in E such that*

$$
\frac{p}{q} = \beta + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\alpha_{ij}}{\phi_i^j}\,, \quad \nu(\alpha_{ij}) < \nu(\phi_i)\,.
$$

Proof. Apply Theorem **??** inductively to obtain elements s_i in E such that

$$
\frac{p}{q} = \sum_{i=1}^{k} \frac{s_i}{\phi_i^{n_i}}
$$

(the factors ϕ_i are relatively prime). Then apply Theorem **??** to obtain elements α_{ij} and β_i in E such that

$$
\frac{s_i}{\phi_i^{n_i}} = \beta_i + \sum_{j=1}^{n_i} \frac{\alpha_{ij}}{\phi_i^j}
$$

with $\nu(\alpha_{ij}) < \nu(\phi_i)$. Take $\beta = \beta_1 + \cdots + \beta_k$. $\square$