UFD's are integrally closed

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Theorem: Every UFD is integrally closed.

Proof: Let R be a UFD, K its field of fractions, $u \in K, u$ integral over R. Then for some $c_0, \ldots, c_{n-1} \in R$,

$$u^n + c_{n-1}u^{n-1} + \ldots + c_0 = 0$$

Write $u = \frac{a}{b}, a, b \in R$, where a, b have no non-unit common divisor (which we can assume since R is a UFD). Multiply the above equation by b^n to get

$$a^n + c_{n-1}ba^{n-1} + \ldots + c_0 b^n = 0$$

Let d be an irreducible divisor of b. Then d is prime since R is a UFD. Now, $d | a^n$ since it divides all the other terms and thus (since d is prime) $d | a$. But a, b have no non-unit common divisors, so d is a unit. Thus b is a unit and hence $u \in R$.