positive cone

Canonical name	PositiveCone
Date of creation	2013-03-22 14:46:54
Last modified on	2013-03-22 14:46:54
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Definition
Classification	msc 13J25
Classification	msc 12D15
Related topic	PositivityInOrderedRing
Related topic	FormallyRealField
Defines	pre-positive cone

Let R be a commutative ring with 1. A subset P of R is called a *pre-positive cone* of R provided that

1. $P + P \subseteq P$ (P is additively closed)
2. $P \cdot P \subseteq P$ (P is multiplicatively closed)
3. $-1 \notin P$
4. $\operatorname{sqr}(R) := \{r^2 \mid r \in R\} \subseteq P$.

As it turns out, a field endowed with a pre-positive cone has an order structure. The field is called a `http://planetmath.org/FormallyRealField`formally real, orderable, or ordered field. Before defining what this "order" is, let's do some preliminary work. Let P_0 be a pre-positive cone of a field F. By Zorn's Lemma, the set of pre-positive cones extending P_0 has a maximal element P. It can be shown that P has two additional properties:

5. $P \cup (-P) = F$
6. $P \cap (-P) = (0)$.

Proof. First, suppose there is $a \in F - (P \cup (-P))$. Let $\overline{P} = P + Pa$. Then $a \in \overline{P}$ and so P is strictly contained in $\overline{P}$. Clearly, $\operatorname{sqr}(F) \subseteq \overline{P}$ and $\overline{P}$ is easily seen to be additively closed. Also, $\overline{P}$ is multiplicatively closed as the equation $(p_1 + q_1 a)(p_2 + q_2 a) = (p_1 p_2 + q_1 q_2 a^2) + (p_1 q_2 + q_1 p_2)a$ demonstrates. Since P is a maximal and $\overline{P}$ properly contains P, $\overline{P}$ is not a pre-positive cone, which means $-1 \in \overline{P}$. Write $-1 = p + qa$. Then $q(-a) = p + 1 \in P$. Since $q \in P$, $1/q = q(1/q)^2 \in P$, $-a = (1/q)(p+1) \in P$, contradicting the assumption that $a \notin -P$. Therefore, $P \cup (-P) = F$.

For the second part, suppose $a \in P \cap (-P)$. Since $a \in -P$, $-a \in P$. If $a \neq 0$, then $-1 = a(-a)(1/a)^2 \in P$, a contradiction. □

A subset P of a field F satisfying conditions 1, 2, 5 and 6 is called a *positive cone* of F. A positive cone is a pre-positive cone. If $a \in F$, then either $a \in P$ or $-a \in P$. In either case, $a^2 \in P$. Next, if $-1 \in P$, then $1 \in -P$. But $1 = 1^2 \in P$, we have $1 \in P \cap (-P)$, contradicting Condition 6 of P.

Now, define a binary relation $\leq$, on F by:

$$a \leq b \Longleftrightarrow b - a \in P$$

It is not hard to see that $\leq$ is a total order on F. In addition, with the additive and multiplicative structures on F, we also have the following two rules:

1. $a \leq b \Rightarrow a + c \leq b + c$
2. $0 \leq a$ and $0 \leq b \Rightarrow 0 \leq ab$.

Thus, F is a field ordered by $\leq$.

Remark. Positive cones may be defined for more general ordered algebraic structures, such as partially ordered groups, or partially ordered rings.

References

[1] A. Prestel, *Lectures on Formally Real Fields*, Springer, 1984