Teichmüller character

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Before we define the Teichmüller character, we begin with a corollary of Hensel's lemma.

Corollary. *Let p be a prime number. The ring of* `http://planetmath.org/PAdicIntegers`*p-adic integers $\mathbb{Z}_p$ contains exactly $p-1$ distinct $(p-1)$th roots of unity. Furthermore, every $(p-1)$th root of unity is distinct modulo p.*

Proof. Notice that $\mathbb{Q}_p$, the p-adic rationals, is a field. Therefore $f(x) = x^{p-1}-1$ has at most $p-1$ roots in $\mathbb{Q}_p$ (see `http://planetmath.org/APolynomialOfDegreeNOverAFi` entry). Moreover, if we let $a \in \mathbb{Z}$ with $1 \leq a \leq p-1$ then $f(a) = a^{p-1}-1 \equiv 0 \mod p$ by Fermat's little theorem. Since $f'(a) = (p-1) \cdot a^{p-2}$ is non-zero modulo p, the trivial case of Hensel's lemma implies that there exist a root of $x^{p-1}-1$ in $\mathbb{Z}_p$ which is congruent to a modulo p. Hence, there are at least $p-1$ roots in $\mathbb{Z}_p$, and we can conclude that there are exactly $p-1$ roots. □

Definition. *The Teichmüller character is a homomorphism of multiplicative groups:*

$$\omega \colon \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$$

such that $\omega(a)$ is the unique $(p-1)$th root of unity in $\mathbb{Z}_p$ which is congruent to a modulo p (which exists by the corollary above). The map ω is sometimes called the Teichmüller lift of $\mathbb{F}_p$ to $\mathbb{Z}_p$ ($0 \mod p$ would lift to $0 \in \mathbb{Z}_p$).

Remark. Some authors define the Teichmüller character to be the homomorphism:

$$\hat{\omega} \colon \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$$

defined by

$$\hat{\omega}(z) = \lim_{n\to\infty} z^{p^n}.$$

Notice that for any $z \in \mathbb{Z}_p^\times$, $\hat{\omega}(z)$ is a $(p-1)$th root of unity:

$$(\hat{\omega}(z))^p = \left(\lim_{n\to\infty} z^{p^n}\right)^p = \lim_{n\to\infty} z^{p^{n+1}} = \hat{\omega}(z).$$

Thus, the value $\hat{\omega}(z)$ is the same than $\omega(z \mod p)$.