value group of completion

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Defines	value group of the completion

Let k be a field and $|\cdot|$ its non-archimedean valuation of http://planetmath.org/KrullValuat
one. Then its value group $|k\smallsetminus\{0\}|$ may be considered to be a subgroup of the multiplicative group of $\mathbb{R}$. In the completion K of the valued field k, the extension of the valuation is defined by

$$|x| \;=:\; \lim_{n\to\infty}|x_n|,$$

when the Cauchy sequence $x_1,\,x_2,\,\ldots,\,x_n,\,\ldots$ of elements of k determines the element x of K.

Theorem. The non-archimedean field k and its completion K have the same value group.

Proof. Of course, $|k| \subseteq |K|$. Let $x = \lim_{n\to\infty}x_n$ be any non-zero element of K, where x_j's form a Cauchy sequence in k. Then there exists a positive number n_0 such that

$$|x_n\!-\!x| \;<\; |x|$$

for all $n > n_0$. For all these values of n we have

$$|x_n| \;=\; |x\!+\!(x_n\!-\!x)| \;=\; |x|$$

according to the ultrametric triangle inequality. Thus we see that $|K| \subseteq |k|$.