polynomial function

Canonical name	PolynomialFunction
Date of creation	2013-03-22 15:40:34
Last modified on	2013-03-22 15:40:34
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	13
Author	pahio (2872)
Entry type	Definition
Classification	msc 13A99
Synonym	ring of polynomial functions
Related topic	NotationInSetTheory
Related topic	ProductAndQuotientOfFunctionsSum
Related topic	ZeroOfPolynomial
Related topic	PolynomialFunctionIsAProperMap
Related topic	DerivativeOfPolynomial
Defines	polynomial function

Definition. Let R be a commutative ring. A function $f : R \to R$ is called a *polynomial function of* R, if there are some elements $a_0, a_1, \ldots, a_m$ of R such that

$$f(x) \;=\; a_0{+}a_1x{+}\cdots{+}a_mx^m \quad \forall x \in R.$$

Remark. The coefficients a_i in a polynomial function need not be unique; e.g. if $R = \{0, 1\}$ is the ring (and field) of two elements, then the polynomials X and X^2 both may be used for the same polynomial function. However, if we stipulate that R is an infinite integral domain, the coefficients are guaranteed to be unique.

The set of all polynomial functions of R, being a subset of the set R^R of all functions from R to R, is here denoted by $R/^R$.

Theorem. If R is a commutative ring, then the set $R/^R$ of all polynomial functions of R, equipped with the operations

$$(f{+}g)(x) \;:=\; f(x){+}g(x), \quad (f{\cdot}g)(x) \;:=\; f(x)g(x) \quad \forall x \in R, \tag{1}$$

is a commutative ring.

Proof. It's straightforward to show that the function set R^R forms a commutative ring when equipped with the operations "+" and "·" defined as (1). We show now that $R/^R$ forms a subring of R^R. Let f and g be any two polynomial functions given by

$$f(x) \;=\; a_0{+}a_1x{+}\cdots{+}a_mx^m, \;\; g(x) \;=\; b_0{+}b_1x{+}\cdots{+}b_nx^n.$$

Then we can give $f{+}g$ by

$$(f{+}g)(x) \;=\; \sum_{i=0}^{k}(a_i{+}b_i)x^i$$

where $k = \max\{m, n\}$ and $a_i = 0$ (resp. $b_i = 0$) for $i > m$ (resp. $i > n$). This means that $f{+}g \in R/^R$. Secondly, the equation

$$(f{\cdot}g)(x) \;=\; a_0b_0 + (a_0b_1{+}a_1b_0)x + (a_0b_2{+}a_1b_1{+}a_2b_0)x^2{+}\cdots{+}a_mb_nx^{m+n}$$

signifies that $f{\cdot}g \in R/^R$. Because also the function $-f$ given by

$$(-f)(x) \;=\; -a_0{-}a_1x{-}\cdots{-}a_mx^m$$

and satisfying $-f{+}f = 0 : x \mapsto 0$ belongs to $R/^R$, the subset $R/^R$ is a subring of R^R.