algebra (module)

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Defines	Jacobi identity

Given a commutative ring R, an algebra over R is a module M over R, endowed with a law of composition

$$f : M \times M \to M$$

which is R-bilinear.

Most of the important algebras in mathematics belong to one or the other of two classes: the unital associative algebras, and the Lie algebras.

1 Unital associative algebras

In these cases, the "product" (as it is called) of two elements v and w of the module, is denoted simply by vw or $v \centerdot w$ or the like.

Any unital associative algebra is an algebra in the sense of djao (a sense which is also used by Lang in his book *Algebra* (Springer-Verlag)).

Examples of unital associative algebras:

– tensor algebras and quotients of them

– Cayley algebras, such as the ring of quaternions

– polynomial rings

– the ring of endomorphisms of a vector space, in which the bilinear product of two mappings is simply the composite mapping.

2 Lie algebras

In these cases the bilinear product is denoted by $[v, w]$, and satisfies

$$[v, v] = 0 \text{ for all } v \in M$$

$$[v, [w, x]] + [w, [x, v]] + [x, [v, w]] = 0 \text{ for all } v, w, x \in M$$

The second of these formulas is called the Jacobi identity. One proves easily

$$[v, w] + [w, v] = 0 \text{ for all } v, w \in M$$

for any Lie algebra M.

Lie algebras arise naturally from Lie groups, q.v.