Prüfer ring

Canonical name	PruferRing
Date of creation	2015-05-05 15:21:07
Last modified on	2015-05-05 15:21:07
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	89
Author	pahio (2872)
Entry type	Theorem
Classification	msc 13C13
Classification	msc 13F05
Related topic	LeastCommonMultiple
Related topic	GeneratorsOfInverseIdeal
Related topic	ProductOfIdeals
Related topic	MultiplicationRing
Related topic	PruferDomain
Related topic	InvertibilityOfRegularlyGeneratedIdeal
Related topic	MultiplicationRuleGivesInverseIdeal
Related topic	ContentOfPolynomial
Related topic	ProductOfFinitelyGeneratedIdeals
Defines	Prüfer ring
Defines	coefficient module
Defines	Gaussian ring

Definition. A commutative ring R with non-zero unity is a *Prüfer ring* (cf. Prüfer domain) if every finitely generated regular ideal of R is invertible. (It can be proved that if every ideal of R generated by two elements is invertible, then all finitely generated ideals are invertible; cf. invertibility of regularly generated ideal.)

Denote generally by $\mathfrak{m}_p$ the R-module generated by the coefficients of a polynomial p in $T[x]$, where T is the total ring of fractions of R. Such *coefficient modules* are, of course, fractional ideals of R.

Theorem 1 (Pahikkala 1982). Let R be a commutative ring with non-zero unity and let T be the total ring of fractions of R. Then, R is a Prüfer ring iff the equation

$$\mathfrak{m}_f\mathfrak{m}_g = \mathfrak{m}_{fg} \tag{1}$$

holds whenever f and g belong to the polynomial ring $T[x]$ and at least one of the fractional ideals $\mathfrak{m}_f$ and $\mathfrak{m}_g$ is . (See also product of finitely generated ideals.)

Theorem 2 (Pahikkala 1982). The commutative ring R with non-zero unity is Prüfer ring iff the multiplication rule

$$(a,\, b)(c,\, d) = (ac,\, ad+bc,\, bd)$$

for the integral ideals of R holds whenever at least one of the generators a, b, c and d is not zero divisor.

The proofs are found in the paper

J. Pahikkala 1982: "Some formulae for multiplying and inverting ideals". – *Annales universitatis turkuensis* 183. Turun yliopisto (University of Turku).

Cf. the entries "`http://planetmath.org/MultiplicationRuleGivesInverseIdeal`multiplication rule gives inverse ideal" and "`http://planetmath.org/TwoGeneratorProperty`two-generator property".

An additional characterization of Prüfer ring is found here in the entry "`http://planetmath.org/LeastCommonMultiple`least common multiple", several other characterizations in [1] (p. 238–239).

Note. A commutative ring R satisfying the equation (1) for all polynomials f, g is called a *Gaussian ring.* Thus any `http://planetmath.org/PruferDomain`Prüfer domain is always a Gaussian ring, and `http://planetmath.org/Converse`conversely, an integral domain, which is a Gaussian ring, is a Prüfer domain. Cf. [2].

References

[1] M. Larsen & P. McCarthy: *Multiplicative theory of ideals.* Academic Press. New York (1971).

[2] Sarah Glaz: "The weak dimensions of Gaussian rings". – *Proc. Amer. Math. Soc.* (2005).