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divisor theory and exponent valuations

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A divisor theory $\mathcal{O}^* \to \mathfrak{D}$ of an integral domain $\mathcal{O}$ determines via its prime divisors a certain set N of exponent valuations on the quotient field of $\mathcal{O}$. Assume to be known this set of http://planetmath.org/ExponentValuation2exponents $\nu_\mathfrak{p}$ corresponding the prime divisors $\mathfrak{p}$. There is a bijective correspondence between the elements of N and of the set of all prime divisors. The set of the prime divisors determines completely the of the free monoid $\mathfrak{D}$ of all divisors in question. The homomorphism $\mathcal{O}^* \to \mathfrak{D}$ is then defined by the condition

$$\alpha \;\mapsto\; \prod_i \mathfrak{p_i}^{\nu_{\mathfrak{p}_i}(\alpha)} = (\alpha), \tag{1}$$

since for any element α of $\mathcal{O}^*$ there exists only a finite number of exponents $\nu_{\mathfrak{p}_i}$ which do not vanish on α (corresponding the different prime divisor http://planetmath.org/DivisibilityInRingsfactors of the principal divisor (α)).

One can take the concept of exponent as foundation for divisor theory:

Theorem. Let $\mathcal{O}$ be an integral domain with quotient field K and N a given set of http://planetmath.org/ExponentValuation2exponents of K. The exponents in N determine, as in (1), a divisor theory of $\mathcal{O}$ iff the following three conditions are in :

- For every $\alpha \in \mathcal{O}$ there is at most a finite number of exponents $\nu \in N$ such that $\nu(\alpha) \neq 0$.
- An element $\alpha \in K$ belongs to $\mathcal{O}$ if and only if $\nu(\alpha) \geqq 0$ for each $\nu \in N$.
- For any finite set $\nu_1, \ldots, \nu_n$ of distinct exponents in N and for the arbitrary set $k_1, \ldots, k_n$ of non-negative integers, there exists an element α of $\mathcal{O}$ such that

$$\nu_1(\alpha) = k_1, \;\; \ldots, \;\; \nu_n(\alpha) = k_n.$$

For the proof of the theorem, we mention only how to construct the divisors when we have the exponent set N fulfilling the three conditions of the theorem. We choose a commutative monoid $\mathfrak{D}$ that allows unique prime factorisation and that may be mapped bijectively onto N. The exponent in

N which corresponds to arbitrary prime element $\mathfrak{p}$ is denoted by $\nu_{\mathfrak{p}}$. Then we obtain the homomorphism

$$\alpha \mapsto \prod_{\nu} \mathfrak{p}^{\nu_{\mathfrak{p}}(\alpha)} := (\alpha)$$

which can be seen to satisfy all required properties for a divisor theory $\mathcal{O}^* \to \mathfrak{D}$.

References

[1] S. Borewicz & I. Safarevic: *Zahlentheorie.* Birkhäuser Verlag. Basel und Stuttgart (1966).