derivative of polynomial

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Defines	derivative of the polynomial

Let R be an arbitrary commutative ring. If

$$f(X) := \sum_{i=1}^{n} a_i X^i$$

is a polynomial in the ring $R[X]$, one can form in a polynomial ring $R[X,\,Y]$ the polynomial

$$f(X+Y) = \sum_{i=1}^{n} a_i (X+Y)^i.$$

Expanding this by the http://planetmath.org/GeneralAssociativitypowers of Y yields uniquely the form

$$f(X+Y) := f(X) + f_1(X)\,Y + f_2(X,\,Y)\,Y^2, \tag{1}$$

where $f_1(X) \in R[X]$ and $f_2(X,\,Y) \in R[X,\,Y]$.

We define the polynomial $f_1(X)$ in (1) the *derivative of the polynomial* $f(X)$ and denote it by $f'(X)$ or $\dfrac{df}{dX}$.

It is apparent that this algebraic definition of derivative of polynomial is in harmony with the definition of http://planetmath.org/Derivative2derivative of analysis when R is $\mathbb{R}$ or $\mathbb{C}$; then we identify substitution homomorphism and polynomial function.

It is easily shown the linearity of the derivative of polynomial and the product rule

$$(fg)' = f'g + g'f$$

with its generalisations. Especially:

$$(X^n)' = nX^{n-1} \quad \text{for } n = 1,\,2,\,3,\,\ldots$$

Remark. The polynomial ring $R[X]$ may be thought to be a subring of $R[[X]]$, the ring of formal power series in X. The http://planetmath.org/FormalPowerSeriesder defined in $R[[X]]$ extend the concept of derivative of polynomial and obey laws.

If we have a polynomial $f \in R[X_1, X_2, \ldots, X_m]$, we can analogically define the *partial derivatives* of f, denoting them by $\frac{\partial f}{\partial X_i}$. Then, e.g. the "http://planetmath.org/EulersTheoremOnHomogeneousFunctionsEuler's theorem on homogeneous functions"

$$X_1\frac{\partial f}{\partial X_1} + X_2\frac{\partial f}{\partial X_2} + \ldots + X_m\frac{\partial f}{\partial X_m} \;=\; nf$$

is true for a homogeneous polynomial f of degree n.