ideals of a discrete valuation ring are powers of its maximal ideal

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Theorem 1. *Let R be a discrete valuation ring. Then all nonzero ideals of R are powers of its maximal ideal $\mathfrak{m}$.*

Proof. Let $\mathfrak{m} = (\pi)$ (that is, π is a uniformizer for R). Assume that R is not a field (in which case the result is trivial), so that $\pi \neq 0$. Let $I = (\alpha) \subset R$ be any ideal; claim $(\alpha) = \mathfrak{m}^k$ for some k. By the Krull intersection theorem, we have

$$\bigcap_{n \geq 0} \mathfrak{m}^n = (0)$$

so that we may choose $k \geq 0$ with $\alpha \in \mathfrak{m}^k - \mathfrak{m}^{k+1}$. Since $\alpha \in \mathfrak{m}^k$, we have $\alpha = u\pi^k$ for $u \in R$. $u \notin \mathfrak{m}$, since otherwise $\alpha \in \mathfrak{m}^{k+1}$, so that α is a unit (in a DVR, the maximal ideal consists precisely of the nonunits). Thus $(\alpha) = (\pi)^k$.

Corollary 1. *Let R be a Noetherian local ring with a principal maximal ideal. Then all nonzero ideals are powers of the maximal ideal $\mathfrak{m}$.*

Proof. Let $I = (\alpha_1, \ldots, \alpha_n)$ be an ideal of R. Then by the above argument, for each i, $\alpha_i = u_i\pi^{k_i}$ for u_i a unit, and thus $I = (\pi^{k_1}, \ldots, \pi^{k_n}) = (\pi^k)$ for $k = \min(k_1, \ldots, k_n)$.