n-system

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Let R be a ring. A subset S of R is said to be an n-system if

- $S \neq \varnothing$, and
- for every $x \in S$, there is an $r \in R$, such that $xrx \in S$.

n-systems are a generalization of http://planetmath.org/MSystem$m$-systems in a ring. Every m-system is an n-system, but not conversely. For example, for any distinct $x, y \in R$, inductively define the elements

$$a_0 = x, \quad \text{and} \quad a_{i+1} = a_i y^i a_i \quad \text{for } i = 0, 1, 2, \ldots.$$

Form the set $A = \{a_n \mid n \text{ is a non-negative integer}\}$. In addition, inductively define

$$b_0 = y, \quad \text{and} \quad b_{j+1} = b_j x^j b_j \quad \text{for } j = 0, 1, 2 \ldots,$$

and form $B = \{b_m \mid m \text{ is a non-negative integer}\}$. Then both A and B are m-systems (as well as n-systems). Furthermore, $S = A \cup B$ is an n-system which is not an m-system.

The example above suggests that, given an n-system S and any $x \in S$, we can "construct" an m-system $T \subseteq S$ such that $x \in T$. Start with $a_0 = x$, inductively define $a_{i+1} = a_i y_i a_i$, where the existence of $y_i \in R$ such that $a_{i+1} \in S$ is guaranteed by the fact that S is an n-system. Then the collection $T := \{a_i \mid i \text{ is a non-negative integer}\}$ is a subset of S that is an m-system. For if we pick any a_i and a_j, if $i \leq j$, then a_i is both the left and right sections of a_j, meaning that there are $r, s \in R$ such that $a_j = ra_i = a_i s$ (this can be easily proved inductively). As a result, $a_i(sy_j)a_j = a_j y_j a_j \in S$, and $a_j(y_j r)a_i = a_j y_j a_j \in S$.

Remark n-systems provide another characterization of a semiprime ideal: an ideal $I \subseteq R$ is semiprime iff $R - I$ is an n-system.

Proof. Suppose I is semiprime. Let $x \in R - I$. Then $xRx \nsubseteq I$, which means there is an element $y \in R$ such that $xyx \notin I$. So $R - I$ is an n-system. Now suppose that $R - I$ is an n-system. Let $x \in R$ with the condition that $xRx \subseteq I$. This means $xyx \in I$ for all $y \in R$. If $x \in R - I$, then there is some $y \in R$ with $xyx \in R - I$, contradicting condition on x. Therefore, $x \in I$, and I is semiprime. $\square$