equivalent formulation of Nakayama's lemma

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The following is equivalent to Nakayama's lemma.

Let A be a ring, M be a finitely-generated A-module, N a submodule of M, and $\mathfrak{a}$ an ideal of A contained in its Jacobson radical. Then $M = \mathfrak{a}M + N \Rightarrow M = N$.

Clearly this statement implies Nakayama's Lemma, by setting N to 0. To see that it follows from Nakayama's Lemma, note first that by the second isomorphism theorem for modules,

$$\frac{\mathfrak{a}M + N}{N} = \frac{\mathfrak{a}M}{\mathfrak{a}M \cap N}$$

and the obvious map

$$\mathfrak{a}M \to \mathfrak{a}\frac{M}{N} : am \mapsto a(m + N)$$

is surjective; the kernel is clearly $\mathfrak{a}M \cap N$. Thus

$$\frac{\mathfrak{a}M + N}{N} \cong \mathfrak{a}\frac{M}{N}$$

So from $M = \mathfrak{a}M + N$ we get $M/N = \mathfrak{a}(M/N)$. Since $\mathfrak{a}$ is contained in the Jacobson radical of M, it is contained in the Jacobson radical of M/N, so by Nakayama, $M/N = 0$, i.e. $M = N$.