applying elementary symmetric polynomials

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Owner	pahio (2872)
Last modified by	pahio (2872)
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Author	pahio (2872)
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The method used in the proof of fundamental theorem of symmetric polynomials may be applied to concrete instances as follows.

We assume the given a symmetric polynomial $P(x_1,\,x_2,\,\ldots,\,x_n) = P$ of degree d be http://planetmath.org/HomogeneousPolynomialhomogeneous. Starting from the highest term of P we form all products

$$x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$$

where

$$\lambda_1 \;\geq\; \lambda_2 \;\geq\; \ldots \geq\; \lambda_n \;\geq\; 0 \quad \text{and} \quad \lambda_1+\lambda_2+\ldots+\lambda_n \;=\; d.$$

Then

$$P \;=\; Q(p_1,\,p_2,\,\ldots,\,p_n) \;=\; \sum_i m_i p_1^{\lambda_1-\lambda_2}p_2^{\lambda_2-\lambda_3}\cdots p_{n-1}^{\lambda_{n-1}-\lambda_n}p_n^{\lambda_n}, \qquad (1)$$

in which the coefficients m_i are determined by giving some suitable values to the indeterminates x_j.

Example 1. Express the polynomial $P = x_1^3x_2+x_1^3x_3+x_2^3x_1+x_2^3x_3+x_3^3x_1+x_3^3x_2$ in the elementary symmetric polynomials

$$p_1 \;=\; x_1+x_2+x_3, \quad p_2 \;=\; x_2x_3+x_3x_1+x_1x_2, \quad p_3 \;=\; x_1x_2x_3. \qquad (2)$$

We have four

$$4,\,0,\,0; \quad 3,\,1,\,0; \quad 2,\,2,\,0; \quad 2,\,1,\,1,$$

for which the corresponding p-products of the sum (1) are

$$p_1^4, \quad p_1^2p_2, \quad p_2^2, \quad p_1p_3,$$

respectively. Apparently, the first one is out of the question. Therefore, clearly

$$P \;=\; p_1^2p_2+ap_2^2+bp_1p_3.$$

Using $x_1 = x_2 = 1$ and $x_3 = 0$ makes $p_1 = 2$, $p_2 = 1$ and $p_3 = 0$, when

$$P \;=\; 2 \;=\; 4+a+0,$$

implying $a = -2$. Using similarly $x_1 = x_2 = x_3 = 1$ we get $p_1 = p_2 = 3$, $p_3 = 1$, which give

$$P \;=\; 6 \;=\; 27+9a+3b \;=\; 9+3b,$$

yielding $b = -1$. Hence we have the result

$$P \;=\; p_1^2 p_2 - 2p_2^2 - p_1 p_3,$$

i.e.

$$x_1^3x_2{+}x_1^3x_3{+}x_2^3x_1{+}x_2^3x_3{+}x_3^3x_1{+}x_3^3x_2 \;=\; (x_1{+}x_2{+}x_3)^2(x_2x_3{+}x_3x_1{+}x_1x_2){-}2(x_2x_3{+}x_3x_1{+}x_1x_2)^2{-}(x_1{+}x_2{+}x_3)$$

Example 2. Let $P = x_1^4{+}x_2^4{+}\ldots{+}x_n^4$. If we suppose that $n \geqq 4$, the possible highest terms are

$$x_1^4, \quad x_1^3x_2, \quad x_1^2x_2^2, \quad x_1^2x_2x_3, \quad x_1x_2x_3x_4$$

whence we may write

$$P \;=\; p_1^4{+}ap_1^2p_2{+}bp_2^2{+}cp_1p_3{+}dp_4. \tag{3}$$

For determining the coefficients, evidently we can put $x_5 = x_6 = \ldots = x_n = 0$ and in as follows.
$1°$. $x_1 = 1,\ x_2 = -1,\ x_3 = x_4 = 0$. Then we have $P = 2,\ p_1 = 0,\ p_2 = -1,\ p_3 = p_4 = 0$. Thus (3) gives $b = 2$.
$2°$. $x_1 = x_2 = 1,\ x_3 = x_4 = -1$. Now $P = 4,\ p_1 = 0,\ p_2 = -2,\ p_3 = 0,\ p_4 = 1$, whence (3) reads $4 = 4b{+}d = 8{+}d$, giving $d = -4$.
$3°$. $x_1 = x_2 = 1,\ x_3 = x_4 = 0$. We get $P = 2,\ p_1 = 2,\ p_2 = 1,\ p_3 = p_4 = 0$. These yield $2 = 16{+}4a{+}b = 18{+}4a$, i.e. $a = -4$.
$4°$. $x_1 = x_2 = 2,\ x_3 = -1,\ x_4 = 0$. In this case, $P = 33,\ p_1 = 3,\ p_2 = 0,\ p_3 = -4,\ p_4 = 0$, whence $33 = 81 - 12c$, or $c = 4$. Consequently, we obtain from (3) the result

$$P \;=\; p_1^4{-}4p_1^2p_2{+}2p_2^2{+}4p_1p_3{-}4p_4. \tag{4}$$

Although it has been derived by supposing $n \geqq 4$ (= the degree of P), it holds without this supposition. One has only to see that e.g. in the case $n = 2$, one must substitute to (4) the values $p_3 = p_4 = 0$, which changes the to the form $P \;=\; p_1^4{-}4p_1^2p_2{+}2p_2^2$.