criteria for cyclic rings to be isomorphic

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Owner	Wkbj79 (1863)
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Author	Wkbj79 (1863)
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Theorem. *Two cyclic rings are isomorphic if and only if they have the same order and the same behavior.*

Proof. Let R be a cyclic ring with behavior k and r be a `http://planetmath.org/Generator`genera of the additive group of R with $r^2 = kr$. Also, let S be a cyclic ring.

If R and S have the same order and the same behavior, then let s be a generator of the additive group of S with $s^2 = ks$. Define $\varphi \colon R \to S$ by $\varphi(cr) = cs$ for every $c \in \mathbb{Z}$. This map is clearly well defined and surjective. Since R and S have the same order, φ is injective. Since, for every $a, b \in \mathbb{Z}$, $\varphi(ar) + \varphi(br) = as + bs = (a + b)s = \varphi((a + b)r) = \varphi(ar + br)$ and

$$\begin{aligned}
\varphi(ar)\varphi(br) &= (as)(bs) \\
&= (ab)s^2 \\
&= (ab)(ks) \\
&= (abk)s \\
&= \varphi((abk)r) \\
&= \varphi((ab)(kr)) \\
&= \varphi((ab)r^2) \\
&= \varphi((ar)(br)),
\end{aligned}$$

it follows that φ is an isomorphism.

Conversely, let $\psi \colon R \to S$ be an isomorphism. Then R and S must have the same order. If R is infinite, then S is infinite, and k is a nonnegative integer. If R is finite, then k `http://planetmath.org/Divisibility`divides $|R|$, which equals $|S|$. In either case, k is a candidate for the behavior of S. Since r is a generator of the additive group of R and ψ is an isomorphism, $\psi(r)$ is a generator of the additive group of S. Since $(\psi(r))^2 = \psi(r^2) = \psi(kr) = k\psi(r)$, it follows that S has behavior k. $\square$