irreducible ideal

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Let R be a ring. An ideal I in R is said to be if, whenever I is an intersection of two ideals: $I = J \cap K$, then either $I = J$ or $I = K$.

Irreducible ideals are closely related to the notions of irreducible elements in a ring. In fact, the following holds:

Proposition 1. *If D is a gcd domain, and x is an irreducible element, then $I = (x)$ is an irreducible ideal.*

Proof. If x is a unit, then $I = D$ and we are done. So we assume that x is not a unit for the remainder of the proof.

Let $I = J \cap K$ and suppose $a \in J - I$ and $b \in K - I$. Then $ab = x^n$ for some $n \in \mathbb{N}$. Let c be a gcd of a and x. So

$$cd = x$$

for some $d \in D$. Since x is irreducible, either c is a unit or d is. The proof now breaks down into two cases:

- c is a unit. Let t be a lcm of a and x. Then tc is an associate of ax. But c is a unit, t and ax are associates, so that ax is a lcm of a and x. As $ab = x^n$, both $a \mid ab$ and $x \mid ab$ hold, which imply that $ax \mid ab$. Write $axr = ab$, where $r \in D$. Then $b = xr \in I$, which is impossible by assumption.

- d is a unit. So c is an associate of x. Because c divides a, we get that $x \mid a$ as well, or $a \in I$, which is again impossible by assumption.

Therefore, the assumption that $J - I \neq \varnothing$ and $K - I \neq \varnothing$ is false, which is the same as saying $J \subseteq I$ or $K \subseteq I$. But $I \subseteq J$ and $I \subseteq K$, either $I = J$ or $I = K$, or I is irreducible. $\square$

Remark. In a commutative Noetherian ring, the notion of an irreducible ideal can be used to prove the Lasker-Noether theorem: every ideal (in a Noetherian ring) has a primary decomposition.

References

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[3] M. Reid, *Undergraduate Commutative Algebra*, Cambridge University Press, 1996.