proof of finite separable extensions of Dedekind domains are Dedekind

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Let R be a Dedekind domain with field of fractions K and L/K be a http://planetmath.org/FiniteExtensionfinite separable extension of fields. We show that the integral closure A of R in L is also a Dedekind domain. That is, A is http://planetmath.org/NoetherianNoetherian, integrally closed and every nonzero prime ideal is http://planetmath.org/MaximalIdealmaximal.

First, as integral closures are themselves integrally closed, A is integrally closed. Second, as integral closures in separable extensions are finitely generated, A is finitely generated as an R-module. Then, any ideal $\mathfrak{a}$ of A is a submodule of A, so is finitely generated as an R-module and therefore as an A-module. So, A is Noetherian.

It only remains to show that a nonzero prime ideal $\mathfrak{p}$ of A is maximal. Choosing any $p \in \mathfrak{p} \setminus \{0\}$ there is a nonzero polynomial

$$f = \sum_{k=0}^{n} c_k X^k$$

for $c_k \in R$, $c_0 \neq 0$ and such that $f(p) = 0$. Then

$$c_0 = -p \sum_{k=1}^{n} c_k p^{k-1} \in \mathfrak{p} \cap R,$$

so $\mathfrak{p} \cap R$ is a nonzero prime ideal in R and is therefore a maximal ideal. So,

$$R/(\mathfrak{p} \cap R) \to A/\mathfrak{p}$$

gives an algebraic extension of the field $R/(\mathfrak{p} \cap R)$ to the integral domain $A/\mathfrak{p}$. Therefore, $A/\mathfrak{p}$ is a field (see a condition of algebraic extension) and $\mathfrak{p}$ is a maximal ideal.