a finite integral domain is a field

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Proof:

Let R be a finite integral domain. Let a be nonzero element of R.

Define a function $\varphi\colon R \to R$ by $\varphi(r) = ar$.

Suppose $\varphi(r) = \varphi(s)$ for some $r, s \in R$. Then $ar = as$, which implies $a(r-s) = 0$. Since $a \neq 0$ and R is a cancellation ring, we have $r - s = 0$. So $r = s$, and hence φ is injective.

Since R is finite and φ is injective, by the pigeonhole principle we see that φ is also surjective. Thus there exists some $b \in R$ such that $\varphi(b) = ab = 1_R$, and thus a is a unit.

Thus R is a finite division ring. Since it is commutative, it is also a field.

Note:

A more general result is that an Artinian integral domain is a field.