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p-adic valuation

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Entry type	Definition
Classification	msc 13A18
Synonym	p-adic valuation
Related topic	IndependenceOfPAdicValuations
Related topic	IntegralElement
Related topic	OrderValuation
Related topic	StrictDivisibility
Defines	p-integral rational number
Defines	normed p-adic valuation
Defines	normed archimedean valuation
Defines	dyadic valuation
Defines	triadic valuation
Defines	pentadic valuation
Defines	heptadic valuation

Let p be a positive prime number. For every non-zero rational number x there exists a unique integer n such that

$$x = p^n \cdot \frac{u}{v}$$

with some integers u and v indivisible by p. We define

$$|x|_p := \begin{cases} (\frac{1}{p})^n & \text{when } x \neq 0, \\ 0 & \text{when } x = 0, \end{cases}$$

obtaining a http://planetmath.org/TrivialValuationnon-trivial non-archimedean valuation, the so-called *p-adic valuation*

$$|\cdot|_p : \mathbb{Q} \to \mathbb{R}$$

of the field $\mathbb{Q}$.

The value group of the p-adic valuation consists of all integer-powers of the prime number p. The valuation ring of the valuation is called the ring of the *p-integral rational numbers*; their denominators, when http://planetmath.org/Fractionreduced to lowest terms, are not divisible by p.

The field of rationals has the *2-adic, 3-adic, 5-adic, 7-adic* and so on valuations (which may be called, according to Greek, *dyadic, triadic, pentadic, heptadic* and so on). They all are http://planetmath.org/EquivalentValuationsnon-equivalent with each other.

If one replaces the number $\frac{1}{p}$ by any positive ϱ less than 1, one obtains an http://planetmath.org/EquivalentValuationsequivalent p-adic valuation; among these the valuation with $\varrho = \frac{1}{p}$ is sometimes called the *normed p-adic valuation*. Analogously we can say that the absolute value is the normed archimedean valuation of $\mathbb{Q}$ which corresponds the infinite prime ∞ of $\mathbb{Z}$.

The product of all normed valuations of $\mathbb{Q}$ is the trivial valuation $|\cdot|_{\mathrm{tr}}$, i.e.

$$\prod_{p \,\mathrm{prime}} |x|_p = |x|_{\mathrm{tr}} \quad \forall x \in \mathbb{Q}.$$