Dedekind domains with finitely many primes are PIDs

Canonical name	DedekindDomainsWithFinitelyManyPrimesArePIDs
Date of creation	2013-03-22 18:35:18
Last modified on	2013-03-22 18:35:18
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	6
Author	gel (22282)
Entry type	Theorem
Classification	msc 13F05
Classification	msc 11R04
Related topic	DivisorTheory

A commutative ring in which there are only finitely many maximal ideals is known as a semi-local ring. For such rings, the property of being a Dedekind domain and of being a principal ideal domain coincide.

Theorem. *A Dedekind domain in which there are only finitely many prime ideals is a principal ideal domain.*

This result is sometimes proven using the chinese remainder theorem or, alternatively, it follows directly from the fact that invertible ideals in semi-local rings are principal.

Suppose that R is a Dedekind domain such as the ring of algebraic integers in a number field. Although there are infinitely many prime ideals in such a ring, we can use the result that localizations of Dedekind domains are Dedekind and apply the above theorem to localizations of R.

In particular, if $\mathfrak{p}$ is a nonzero prime ideal, then $R_{\mathfrak{p}} \equiv (R \setminus \mathfrak{p})^{-1}R$ is a Dedekind domain with a unique nonzero prime ideal, so the theorem shows that it is a principal ideal domain.