separability is required for integral closures to be finitely generated

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The parent theorem assumed that L was a separable extension of K. Here is an example that shows that separability is in fact a necessary condition for the result to hold.

Let $k = (\mathbb{Z}/p\mathbb{Z})(b_1, b_2, \ldots)$ where the b_i are indeterminates.

Let B be the subring of $k[[t]]$ (power series in t over k) consisting of

$$\left\{ \sum_{i=0}^{\infty} c_i t^i \right\}$$

such that $\{c_0, c_1, c_2, \ldots\}$ generates a finite extension of $(\mathbb{Z}/p\mathbb{Z})(b_1^p, b_2^p, \ldots)$. Then every element of B is a power of t times a unit (first factor out the largest power of t. What's left is $a(1 + c_1 t + c_2 t^2 + \cdots)$; its inverse is $a^{-1}(1 - c_1 t - \cdots))$. Hence the ideals of B are powers of t, so B is a PID (in fact, it is a DVR).

Now, let $u = b_0 + b_1 t + b_2 t^2 + \cdots$. Now, $u \notin B$ because it uses all of the b_i and thus the coefficients do not define a finite extension of $(\mathbb{Z}/p\mathbb{Z})(b_1^p, b_2^p, \ldots)$. However, u is integral over B: $b_i^p \in B \Rightarrow u^p \in B$ which implies that the degree of u over the field of fractions is p. Hence a basis for $K(u)/K$ is $\{1, u, u^2, \ldots, u^{p-1}\}$. There are other elements integral over B:

$$\begin{aligned} b_1 + b_2 t + b_3 t^2 + \cdots &= \frac{u - b_0}{t} = \frac{-b_0}{t} + \frac{1}{t} u \\ b_2 + b_3 t + b_4 t^2 + \cdots &= \frac{u - b_0 - b_1 t}{t^2} = \frac{-b_0 - b_1 t}{t^2} + \frac{1}{t^2} u \\ &\vdots \end{aligned}$$

Clearly the denominators are getting bigger, so the integral closure of B cannot be finitely generated as a B-module.