henselian field

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Defines	valuation ring
Defines	residue field
Defines	residue class field
Defines	Hensel property
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Let $|\cdot|$ be a non-archimedean valuation on a field K. Let $V = \{x : |x| \leq 1\}$. Since $|\cdot|$ is ultrametric, V is closed under addition and in fact an additive group. The other valuation axioms ensure that V is a ring. We call V the *valuation ring* of K with respect to the valuation $|\cdot|$. Note that the field of fractions of V is K.

The set $\mu = \{x : |x| < 1\}$ is a maximal ideal of V. The factor $R := V/\mu$ is called the *residue field* or the *residue class field*.

The map $\mathrm{res} : V \to V/\mu$ given by $x \mapsto x + \mu$ is called the *residue map*. We extend the definition of the residue map to sequences of elements from V, and hence to $V[X]$ so that if $f(X) \in V[X]$ is given by $\sum_{i \leq n} a_i X^i$ then $\mathrm{res}(f) \in R[X]$ is given by $\sum_{i \leq n} \mathrm{res}(ai) X^i$.

Hensel property: Let $f(x) \in V[x]$. Suppose $\mathrm{res}(f)(x)$ has a simple root $e \in k$. Then $f(x)$ has a root $e' \in V$ and $\mathrm{res}(e') = e$.

Any valued field satisfying the Hensel property is called *henselian*. The completion of a non-archimedean valued field K with respect to the valuation (cf. constructing the reals from the rationals as the completion with respect to the standard metric) is a henselian field.

Every non-archimedean valued field K has a unique (up to isomorphism) smallest henselian field K^h containing it. We call K^h the *henselisation* of K.