ordinary generating function for linear recursive relations

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Let (a_n) be a linear recursive sequence with values in a (commutative) ring R, i.e. there exist constants $\beta_1, \dots, \beta_k \in R$ such that for any $n > k$ we have

$$a_n = \beta_1 \cdot a_{n-1} + \cdots + \beta_i \cdot a_{n-i} + \cdots + \beta_k \cdot a_{n-k}.$$

Now consider the ordinary generating function for this sequence

$$f(t) = \sum_{n \geqslant 0} a_n \cdot t^n$$

which is a formal power series in the ring of formal power series $R[[t]]$. We will try to find the closed form of $f(t)$.

First write down first k elements of this power series:

$$f(t) = \sum_{n=0}^{k} a_n \cdot t^n + \sum_{n>k} a_n \cdot t^n$$

and note that for $n > k$ we can use our recursive relation:

$$f(t) = \sum_{n=0}^{k} a_n \cdot t^n + \sum_{n>k} (\beta_1 \cdot a_{n-1} + \cdots + \beta_k \cdot a_{n-k}) \cdot t^n$$

which gives us

$$f(t) = \sum_{n=0}^{k} a_n \cdot t^n + \left(\beta_1 \cdot t \cdot \sum_{n>k} a_{n-1} \cdot t^{n-1} \right) + \cdots + \left(\beta_k \cdot t^k \cdot \sum_{n>k} a_{n-k} \cdot t^{n-k} \right). \tag{1}$$

Now focus on those sums on the right. For any j we have

$$\sum_{n>k} a_{n-j} \cdot t^{n-j} = \left(\sum_{n \geqslant 0} a_n \cdot t^n \right) - \left(\sum_{n=0}^{k-j} a_n \cdot t^n \right) = f(t) - \left(\sum_{n=0}^{k-j} a_n \cdot t^n \right).$$

Inserting this into (1) gives us

$$f(t) = \sum_{n=0}^{k} a_n \cdot t^n + \left(\beta_1 \cdot t \cdot (f(t) - \sum_{n=0}^{k-1} a_n \cdot t^n) \right) + \cdots + \left(\beta_k \cdot t^k \cdot (f(t) - \sum_{n=0}^{0} a_n \cdot t^n) \right)$$

which can be simplified as follows:

$$f(t) = \sum_{n=0}^{k} a_n \cdot t^n + \sum_{j=1}^{k} \left(\beta_j \cdot t^j \cdot (f(t) - \sum_{n=0}^{k-j} a_n \cdot t^n) \right).$$

Taking the components with $f(t)$ to the left side gives us

$$f(t) - \sum_{j=1}^{k} \beta_j \cdot t^j \cdot f(t) = \sum_{n=0}^{k} a_n \cdot t^n - \sum_{j=1}^{k} \left(\beta_j \cdot t^j \cdot \sum_{n=0}^{k-j} a_n \cdot t^n \right),$$

which finally gives us the closed formula for $f(t)$ (note that all sums are finite):

$$f(t) = \frac{\sum_{n=0}^{k} a_n \cdot t^n - \sum_{j=1}^{k} \left(\beta_j \cdot t^j \cdot \sum_{n=0}^{k-j} a_n \cdot t^n \right)}{1 - \sum_{j=1}^{k} \beta_j \cdot t^j}.$$

Note that in the denominator we have a power series with the constant term 1, thus (by general properties of formal power series) it is invertible in $R[[t]]$. Thus we proved the following theorem:

Theorem. If (a_n) is a recursive sequnce given by

$$a_n = \beta_1 \cdot a_{n-1} + \cdots + \beta_k \cdot a_{n-k}$$

for all $n > k$ then the ordinary generating function

$$f(t) = \sum_{n \geqslant 0} a_n \cdot t^n$$

has its closed form given by

$$f(t) = \frac{\sum_{n=0}^{k} a_n \cdot t^n - \sum_{j=1}^{k} \left(\beta_j \cdot t^j \cdot \sum_{n=0}^{k-j} a_n \cdot t^n \right)}{1 - \sum_{j=1}^{k} \beta_j \cdot t^j}.$$

Remark. Note that if we replace R with (for example) the reals $\mathbb{R}$ then the theorem is still valid if we treat those power series as functions. Of course such equalites hold only there where those functions are defined.