proof that a Euclidean domain is a PID

Canonical name	ProofThatAEuclideanDomainIsAPID
Date of creation	2013-03-22 12:43:11
Last modified on	2013-03-22 12:43:11
Owner	rm50 (10146)
Last modified by	rm50 (10146)
Numerical id	7
Author	rm50 (10146)
Entry type	Result
Classification	msc 13F07
Related topic	PID
Related topic	UFD
Related topic	IntegralDomain
Related topic	EuclideanValuation

Let D be a Euclidean domain, and let $\mathfrak{a} \subseteq D$ be a nonzero ideal. We show that $\mathfrak{a}$ is principal. Let

$$A = \{\nu(x) : x \in \mathfrak{a}, x \neq 0\}$$

be the set of Euclidean valuations of the non-zero elements of $\mathfrak{a}$. Since A is a non-empty set of non-negative integers, it has a minimum m. Choose $d \in \mathfrak{a}$ such that $\nu(d) = m$. Claim that $\mathfrak{a} = (d)$. Clearly $(d) \subseteq \mathfrak{a}$. To see the reverse inclusion, choose $x \in \mathfrak{a}$. Since D is a Euclidean domain, there exist elements $y, r \in D$ such that

$$x = yd + r$$

with $\nu(r) < \nu(d)$ or $r = 0$. Since $r \in \mathfrak{a}$ and $\nu(d)$ is minimal in A, we must have $r = 0$. Thus $d|x$ and $x \in (d)$.