irreducible of a UFD is prime

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Any irreducible element of a factorial ring D is a prime element of D.

Proof. Let p be an arbitrary irreducible element of D. Thus p is a non-unit. If $ab \in (p) \smallsetminus \{0\}$, then $ab = cp$ with $c \in D$. We write a, b, c as products of irreducibles:

$$a \;=\; p_1 \cdots p_l, \quad b \;=\; q_1 \cdots q_m, \quad c \;=\; r_1 \cdots r_n$$

Here, one of those first two products may me empty, i.e. it may be a unit. We have

$$p_1 \cdots p_l \, q_1 \cdots q_m \;=\; r_1 \cdots r_n \, p. \tag{1}$$

Due to the uniqueness of prime factorization, every factor r_k is an associate of certain of the $l+m$ irreducibles on the left hand side of (1). Accordingly, p has to be an associate of one of the p_i's or q_j's. It means that either $a \in (p)$ or $b \in (p)$. Thus, (p) is a prime ideal of D, and its generator must be a prime element.