free modules over a ring which is not a PID

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Let R be a unital ring. In the following modules will be left modules.

We will say that R has *the free submodule property* if for any free module F over R and any submodule $F' \subseteq F$ we have that F' is also free. It is well known, that if R is a PID, then R has the free submodule property. One can ask whether the converse is also true? We will try to answer this question.

Proposition. If R is a commutative ring, which is not a PID, then R does not have the free submodule property.

Proof. Assume that R is not a PID. Then there are two possibilities: either R is not a domain or there is an ideal $I \subseteq R$ which is not principal. Assume that R is not a domain and let $a, b \in R$ be two zero divisors, i.e. $a \neq 0$, $b \neq 0$ and $a \cdot b = 0$. Let $(b) \subseteq R$ be an ideal generated by b. Then obviously (b) is a submodule of R (regarded as a R-module). Assume that (b) is free. In particular there exists $m \in (b)$, $m \neq 0$ such that $r \cdot m = 0$ if and only if $r = 0$. But m is of the form $\lambda \cdot b$ and because R is commutative we have

$$a \cdot m = a \cdot (\lambda \cdot b) = \lambda \cdot (a \cdot b) = 0.$$

Contradiction, because $a \neq 0$. Thus (b) is not free although (b) is a submodule of a free module R.

Assume now that there is an ideal $I \subseteq R$ which is not principal and assume that I is free as a R-module. Since I is not principal, then there exist $a, b \in I$ such that $\{a, b\}$ is linearly independent. On the other hand $a, b \in R$ and 1 is a free generator of R. Thus $\{1, a\}$ is linearly dependent, so

$$\lambda \cdot 1 + \alpha \cdot a = 0$$

for some nonzero $\lambda, \alpha \in R$ (note that in this case both λ, α are nonzero, more precisely $\lambda = a$ and $\alpha = -1$). Multiply the equation by b. Thus we have

$$\lambda \cdot b + (\alpha \cdot b) \cdot a = 0.$$

Note that here we used commutativity of R. Since $\{a, b\}$ is linearly independend (in I), then the last equation implies that $\lambda = 0$. Contradiction. $\square$

Corollary. Commutative ring is a PID if and only if it has the free submodule property.