fractional ideal

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1 Basics

Let A be an integral domain with field of fractions K. Then K is an A–module, and we define a *fractional ideal* of A to be a submodule of K which is finitely generated as an A–module.

The product of two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A is defined to be the submodule of K generated by all the products $x \cdot y \in K$, for $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. This product is denoted $\mathfrak{a} \cdot \mathfrak{b}$, and it is always a fractional ideal of A as well. Note that, if A itself is considered as a fractional ideal of A, then $\mathfrak{a} \cdot A = \mathfrak{a}$. Accordingly, the set of fractional ideals is always a monoid under this product operation, with identity element A.

We say that a fractional ideal $\mathfrak{a}$ is *invertible* if there exists a fractional ideal $\mathfrak{a}'$ such that $\mathfrak{a} \cdot \mathfrak{a}' = A$. It can be shown that if $\mathfrak{a}$ is invertible, then its inverse must be $\mathfrak{a}' = (A : \mathfrak{a})$, the annihilator[1] of $\mathfrak{a}$ in A.

2 Fractional ideals in Dedekind domains

We now suppose that A is a Dedekind domain. In this case, every nonzero fractional ideal is invertible, and consequently the nonzero fractional ideals in A form a group under ideal multiplication, called the *ideal group* of A.

The *unique factorization of ideals* theorem states that every fractional ideal in A factors uniquely into a finite product of prime ideals of A and their (fractional ideal) inverses. It follows that the ideal group of A is freely generated as an abelian group by the nonzero prime ideals of A.

A fractional ideal of A is said to be *principal* if it is generated as an A–module by a single element. The set of nonzero principal fractional ideals is a subgroup of the ideal group of A, and the quotient group of the ideal group of A by the subgroup of principal fractional ideals is nothing other than the ideal class group of A.

[1] In general, for any fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$, the annihilator of $\mathfrak{b}$ in $\mathfrak{a}$ is the fractional ideal $(\mathfrak{a} : \mathfrak{b})$ consisting of all $x \in K$ such that $x \cdot \mathfrak{b} \subset \mathfrak{a}$.