proof that a domain is Dedekind if its ideals are invertible

Canonical name	ProofThatADomainIsDedekindIfItsIdealsAreInvertible
Date of creation	2013-03-22 18:34:54
Last modified on	2013-03-22 18:34:54
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	5
Author	gel (22282)
Entry type	Proof
Classification	msc 13A15
Classification	msc 13F05
Related topic	DedekindDomain
Related topic	FractionalIdeal

Let R be an integral domain with field of fractions k. We show that the following are equivalent.

1. R is Dedekind. That is, it is http://planetmath.org/NoetherianNoetherian, integrally closed, and every prime ideal is http://planetmath.org/MaximalIdealmaximal.
2. Every nonzero (integral) ideal is invertible.
3. Every fractional ideal is invertible.

As every fractional ideal is the product of an element of k and an integral ideal, statements (**??**) and (**??**) are equivalent. We start by proving that (**??**) implies R is Dedekind.

Lemma. *If every fractional ideal is invertible, then R is Dedekind.*

Proof. First, every invertible ideal is finitely generated, so R is Noetherian.

Now, let $\mathfrak{p}$ be a prime ideal, and $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. As $\mathfrak{m}$ is invertible, there exists an ideal $\mathfrak{a}$ such that $\mathfrak{p} = \mathfrak{m}\mathfrak{a}$. That $\mathfrak{p}$ is a prime ideal implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{m} \subseteq \mathfrak{p}$. The first case gives $\mathfrak{p} \subseteq \mathfrak{m}\mathfrak{p}$ and, by cancelling the invertible ideal $\mathfrak{p}$ implies that $\mathfrak{m} = R$, a contradiction. So, the second case must be true and, by maximality of $\mathfrak{m}$, $\mathfrak{p} = \mathfrak{m}$, showing that all prime ideals are maximal.

Now let x be an element of the field of fractions k and be integral over R. Then, we can write $x^n = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$ for coefficients $c_k \in R$. Letting $\mathfrak{a}$ be the fractional ideal

$$\mathfrak{a} = (1, x, x^2, \ldots, x^{n-1})$$

gives $x^n \in \mathfrak{a}$, so $x\mathfrak{a} \subseteq \mathfrak{a}$. As $\mathfrak{a}$ is invertible, it can be cancelled to give $x \in R$, showing that R is integrally closed. □

It only remains to show the converse, that is if R is Dedekind then every nonzero ideal is invertible. We start with the following lemmas.

Lemma. *Every nonzero ideal $\mathfrak{a}$ contains a product of prime ideals. That is, $\mathfrak{p}_1 \cdots \mathfrak{p}_n \subseteq \mathfrak{a}$ for some nonzero prime ideals $\mathfrak{p}_k$.*

Proof. We use proof by contradiction, so suppose this is not the case. As R is Noetherian, the set of nonzero ideals which do not contain a product of nonzero primes has a maximal element (w.r.t. the partial order of set inclusion) say, $\mathfrak{a}$.

In particular $\mathfrak{a}$ cannot be prime itself, so there exist $x, y \in R$ such that $xy \in \mathfrak{a}$ and $x, y \notin \mathfrak{a}$. Therefore $\mathfrak{a}$ is strictly contained in $\mathfrak{a} + (x)$ and $\mathfrak{a} + (y)$ and, by the choice of $\mathfrak{a}$, these ideals must contain a product of primes. So,

$$(\mathfrak{a} + (x))(\mathfrak{a} + (y)) = \mathfrak{a}^2 + x\mathfrak{a} + y\mathfrak{a} + (xy) \subseteq \mathfrak{a}$$

contains a product of primes, which is the required contradiction. □

Lemma. *For any nonzero proper ideal $\mathfrak{a}$ there is an element $x \in k \setminus R$ such that $x\mathfrak{a} \subseteq R$.*

Proof. Let $\mathfrak{p}$ be a maximal ideal containing $\mathfrak{a}$ and a be a nonzero element of $\mathfrak{a}$. By the previous lemma there are prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ satisfying

$$\mathfrak{p}_1 \cdots \mathfrak{p}_n \subseteq (a) \subseteq \mathfrak{a} \subseteq \mathfrak{p}.$$

We choose n as small as possible. As $\mathfrak{p}$ is prime, this gives $\mathfrak{p}_k \subseteq \mathfrak{p}$ for some k and, as every prime ideal is maximal, this is an equality. Without loss of generality we may take $\mathfrak{p} = \mathfrak{p}_n$. As n was assumed to be as small as possible, $\mathfrak{p}_1 \cdots \mathfrak{p}_{n-1}$ is not a subset of (a), so there exists $b \in \mathfrak{p}_1 \cdots \mathfrak{p}_{n-1} \setminus (a)$. Then, $b \notin (a)$ gives $x \equiv a^{-1}b \notin R$ and

$$x\mathfrak{a} \subseteq a^{-1}b\mathfrak{p} \subseteq a^{-1}\mathfrak{p}_1 \cdots \mathfrak{p}_{n-1}\mathfrak{p} \subseteq a^{-1}(a) = R$$

as required. □

We finally show that every nonzero ideal $\mathfrak{a}$ is invertible. If its inverse exists then it should be the largest fractional ideal satisfying $\mathfrak{b}\mathfrak{a} \subseteq R$, so we set

$$\mathfrak{b} = \{x \in k : x\mathfrak{a} \subseteq R\}.$$

Choosing any nonzero $a \in \mathfrak{a}$ gives $a\mathfrak{b} \subseteq \mathfrak{b}\mathfrak{a} \subseteq R$ so $\mathfrak{b}$ is indeed a fractional ideal. It only remains to be shown that $\mathfrak{b}\mathfrak{a} = R$, for which we use proof by contradiction. If this were not the case then the previous lemma gives an $x \in k \setminus R$ such that $x\mathfrak{b}\mathfrak{a} \subseteq R$. By the definition of $\mathfrak{b}$, this gives $x\mathfrak{b} \subseteq \mathfrak{b}$ and therefore $\mathfrak{b}$ is an $R[x]$-module. Furthermore, as R is Noetherian, $\mathfrak{b}$ will be finitely generated as an R-module. This implies that x is integral over the integrally closed ring R, so $x \in R$, giving the required contradiction.