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ideals contained in a union of radical ideals

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Let R be a commutative ring and $I \subseteq R$ an ideal. Recall that *the radical of* I is defined as

$$r(I) = \{x \in R \mid \exists_{n \in \mathbb{N}}\ x^n \in I\}.$$

It can be shown, that $r(I)$ is again an ideal and $I \subseteq r(I)$. Let

$$V(I) = \{P \subseteq R \mid P \text{ is a prime ideal and } I \subseteq P\}.$$

Of course $V(I) \neq \emptyset$ (because I is contained in at least one maximal ideal) and it can be shown, that

$$r(I) = \bigcap_{P \in V(I)} P.$$

Finaly, recall that an ideal I is called *radical*, if $I = r(I)$.

Proposition. Let $I, R_1, \ldots, R_n$ be ideals in R, such that each R_i is radical. If

$$I \subseteq R_1 \cup \cdots \cup R_n,$$

then there exists $i \in \{1, \ldots, n\}$ such that $I \subseteq R_i$.

Proof. Assume that this not true, i.e. for every i we have $I \not\subseteq R_i$. Then for any $i \in \{1, \ldots, n\}$ there exists $P_i \in V(R_i)$ such that $I \not\subseteq P_i$ (this follows from the fact, that $R_i = r(R_i)$ and the characterization of radicals via prime ideals). But for any i we have $R_i \subseteq P_i$ and thus

$$I \subseteq P_1 \cup \cdots \cup P_n.$$

Contradiction, since each P_i is prime (see the parent object for details). □