place of field

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Defines	place of field

Let F be a field and ∞ an element not belonging to F. The mapping

$$\varphi:\ k \to F \cup \{\infty\},$$

where k is a field, is called a *place of the field* k, if it satisfies the following conditions.

- The preimage $\varphi^{-1}(F) = \mathfrak{o}$ is a subring of k.
- The restriction $\varphi|_{\mathfrak{o}}$ is a ring homomorphism from $\mathfrak{o}$ to F.
- If $\varphi(a) = \infty,$ then $\varphi(a^{-1}) = 0.$

It is easy to see that the subring $\mathfrak{o}$ of the field k is a valuation domain; so any place of a field determines a unique valuation domain in the field. Conversely, every valuation domain $\mathfrak{o}$ with field of fractions k determines a place of k:

Theorem. Let $\mathfrak{o}$ be a valuation domain with field of fractions k and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, consisting of the non-units of $\mathfrak{o}$. Then the mapping

$$\varphi:\ k \to \mathfrak{o}/\mathfrak{p} \cup \{\infty\}$$

defined by

$$\varphi(x) := \begin{cases} x+\mathfrak{p} & \text{when } x \in \mathfrak{o}, \\ \infty & \text{when } x \in k \smallsetminus \mathfrak{o}, \end{cases}$$

is a place of the field k.

Proof. Apparently, $\varphi^{-1}(\mathfrak{o}/\mathfrak{p}) = \mathfrak{o}$ and the restriction $\varphi|_{\mathfrak{o}}$ is the canonical homomorphism from the ring $\mathfrak{o}$ onto the residue-class ring $\mathfrak{o}/\mathfrak{p}$. Moreover, if $\varphi(x) = \infty,$ then x does not belong to the valuation domain $\mathfrak{o}$ and thus the inverse element x^{-1} must belong to it without being its unit. Hence x^{-1} belongs to the ideal $\mathfrak{p}$ which is the kernel of the homomorphism $\varphi|\mathfrak{o}$. So we see that $\varphi(x^{-1}) = 0$.

References

[1] Emil Artin: . Lecture notes. Mathematisches Institut, Göttingen (1959).