characterization of prime ideals

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This entry gives a number of equivalent `http://planetmath.org/node/5865`characterizations of prime ideals in rings of different generality.

We start with a general ring R.

Theorem 1. *Let R be a ring and $P \subsetneq R$ a two-sided ideal. Then the following statements are equivalent:*

1. *Given (left, right or two-sided) ideals I, J of P such that the product of ideals $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.*
2. *If $x, y \in R$ such that $xRy \subseteq P$, then $x \in P$ or $y \in P$.*

Proof.

- "**??**⇒**??**":

  Let $x, y \in R$ such that $xRy \subseteq P$. Let (x) and (y) be the (left, right or two-sided) ideals generated by x and y, respectively. Then each element of the product of ideals $(x)R(y)$ can be expanded to a finite sum of products each of which contains or is a factor of the form $\pm xry$ for a suitable $r \in R$. Since P is an ideal and $xRy \subseteq P$, it follows that $(x)R(y) \subseteq P$. Assuming statement **??**, we have $(x) \subseteq P$, $R \subseteq P$ or $(y) \subseteq P$. But $P \subsetneq R$, so we have $(x) \subseteq P$ or $(y) \subseteq P$ and hence $x \in P$ or $y \in P$.

- "**??**⇒**??**":

  Let I, J be (left, right or two-sided) ideals, such that the product of ideals $IJ \subseteq P$. Now $RJ \subseteq J$ or $IR \subseteq I$ (depending on what type of ideal we consider), so $IRJ \subseteq IJ \subseteq P$. If $I \subseteq P$, nothing remains to be shown. Otherwise, let $i \in I \setminus P$, then $iRj \subseteq P$ for all $j \in J$. Since $i \notin P$ we have by statement **??** that $j \in P$ for all $j \in J$, hence $J \subseteq P$. □

There are some additional properties if our ring is commutative.

Theorem 2. *Let R a commutative ring and $P \subsetneq R$ an ideal. Then the following statements are equivalent:*

1. *Given ideals I, J of P such that the product of ideals $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.*
2. *The quotient ring R/P is a cancellation ring.*
3. *The set $R \setminus P$ is a subsemigroup of the multiplicative semigroup of R.*

4. *Given $x, y \in R$ such that $xy \in P$, then $x \in P$ or $y \in P$.*
5. *The ideal P is maximal in the set of such ideals of R which do not intersect a subsemigroup S of the multiplicative semigroup of R.*

Proof.

- "**??**$\Rightarrow$**??**":

  Let $\bar{x}, \bar{y} \in R/P$ be arbitrary nonzero elements. Let x and y be representatives of $\bar{x}$ and $\bar{y}$, respectively, then $x \notin P$ and $y \notin P$. Since R is commutative, each element of the product of ideals $(x)(y)$ can be written as a product involving the factor xy. Since P is an ideal, we would have $(x)(y) \subseteq P$ if $xy \in P$ which by statement **??** would imply $(x) \subseteq P$ or $(y) \subseteq P$ in contradiction with $x \notin P$ and $y \notin P$. Hence, $xy \notin P$ and thus $\bar{x}\bar{y} \neq 0$.

- "**??**$\Rightarrow$**??**":

  Let $x, y \in R \setminus P$. Let $\pi\colon R \to R/P$ be the canonical projection. Then $\pi(x)$ and $\pi(y)$ are nonzero elements of R/P. Since π is a homomorphism and due to statement **??**, $\pi(x)\pi(y) = \pi(xy) \neq 0$. Therefore $xy \notin P$, that is $R \setminus P$ is closed under multiplication. The associative property is inherited from R.

- "**??**$\Rightarrow$**??**":

  Let $x, y \in R$ such that $xy \in P$. If both x, y were not elements of P, then by statement **??** xy would not be an element of P. Therefore at least one of x, y is an element of P.

- "**??**$\Rightarrow$**??**":

  Let I, J be ideals of R such that $IJ \subseteq P$. If $I \subseteq P$, nothing remains to be shown. Otherwise, let $i \in I \setminus P$. Then for all $j \in J$ the product $ij \in IJ$, hence $ij \in P$. It follows by statement **??** that $j \in P$, and therefore $J \subseteq P$.

- "**??**$\Rightarrow$**??**":

  The condition **??** that the set $S = R \setminus P$ is a multiplicative semigroup. Now P is trivially the greatest ideal which does not intersect S.

- "**??**$\Rightarrow$**??**":

We presume that P is maximal of the ideals of R which do not intersect a semigroup S and that $xy \in P$. Assume the contrary of the assertion, i.e. that $x \notin P$ and $y \notin P$. Therefore, P is a proper subset of both (P, x) and (P, y). Thus the maximality of P implies that

$$(P, x) \cap S \neq \{\}, \quad (P, y) \cap S \neq \{\}.$$

So we can choose the elements s_1 and s_2 of S such that

$$s_1 = p_1 + r_1 x + n_1 x, \quad s_2 = p_2 + r_2 y + n_2 y,$$

where $p_1, p_2 \in P$, $r_1, r_2 \in R$ and $n_1, n_2 \in \mathbb{Z}$. Then we see that the product

$$s_1 s_2 = (p_1 + r_2 y + n_2 y) p_1 + (r_1 x + n_1 x) p_2 + (r_1 r_2 + n_2 r_1 + n_1 r_2) xy + (n_1 n_2) xy$$

would belong to the ideal P. But this is impossible because $s_1 s_2$ is an element of the multiplicative semigroup S and P does not intersect S. Thus we can conclude that either x or y belongs to the ideal P.

□

If R has an identity element 1, statements **??** and **??** of the preceding theorem become stronger:

Theorem 3. *Let R be a commutative ring with identity element 1. Then an ideal P of R is a prime ideal if and only if R/P is an integral domain. Furthermore, P is prime if and only if $R \backslash P$ is a monoid with identity element 1 with respect to the multiplication in R.*

Proof. Let P be prime, then $1 \notin P$ since otherwise P would be equal to R. Now by theorem **??** R/P is a cancellation ring. The canonical projection $\pi \colon R \to R/P$ is a homomorphism, so $\pi(1)$ is the identity element of R/P. This in turn implies that the semigroup $R \setminus P$ is a monoid with identity element 1. □