Euclidean valuation

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Let D be an integral domain. A *Euclidean valuation* is a function from the nonzero elements of D to the nonnegative integers $\nu\colon D \setminus \{0_D\} \to \{x \in \mathbb{Z} : x \geq 0\}$ such that the following hold:

- For any $a, b \in D$ with $b \neq 0_D$, there exist $q, r \in D$ such that $a = bq + r$ with $\nu(r) < \nu(b)$ or $r = 0_D$.
- For any $a, b \in D \setminus \{0_D\}$, we have $\nu(a) \leq \nu(ab)$.

Euclidean valuations are important because they let us define greatest common divisors and use Euclid's algorithm. Some facts about Euclidean valuations include:

- The `http://planetmath.org/MinimalElement`minimal value of ν is $\nu(1_D)$. That is, $\nu(1_D) \leq \nu(a)$ for any $a \in D \setminus \{0_D\}$.
- $u \in D$ is a unit if and only if $\nu(u) = \nu(1_D)$.
- For any $a \in D \setminus \{0_D\}$ and any unit u of D, we have $\nu(a) = \nu(au)$.

These facts can be proven as follows:

- If $a \in D \setminus \{0_D\}$, then

  $$\nu(1_D) \leq \nu(1_D \cdot a) = \nu(a).$$

- If $u \in D$ is a unit, then let $v \in D$ be its `http://planetmath.org/MultiplicativeInverseinv`
  Thus,

  $$\nu(1_D) \leq \nu(u) \leq \nu(uv) = \nu(1_D).$$

  Conversely, if $\nu(u) = \nu(1_D)$, then there exist $q, r \in D$ with $\nu(r) < \nu(u) = \nu(1_D)$ or $r = 0_D$ such that

  $$1_D = qu + r.$$

  Since $\nu(r) < \nu(1_D)$ is impossible, we must have $r = 0_D$. Hence, q is the inverse of u.

- Let $v \in D$ be the inverse of u. Then

  $$\nu(a) \leq \nu(au) \leq \nu(auv) = \nu(a).$$

Note that an integral domain is a Euclidean domain if and only if it has a Euclidean valuation.

Below are some examples of Euclidean domains and their Euclidean valuations:

- Any field F is a Euclidean domain under the Euclidean valuation $\nu(a) = 0$ for all $a \in F \setminus \{0_F\}$.
- $\mathbb{Z}$ is a Euclidean domain with absolute value acting as its Euclidean valuation.
- If F is a field, then $F[x]$, the ring of polynomials over F, is a Euclidean domain with degree acting as its Euclidean valuation: If n is a nonnegative integer and $a_0, \dots, a_n \in F$ with $a_n \neq 0_F$, then

$$\nu\left(\sum_{j=0}^{n} a_j x^j\right) = n.$$

Due to the fact that the ring of polynomials over any field is always a Euclidean domain with degree acting as its Euclidean valuation, some refer to a Euclidean valuation as a *degree function*. This is done, for example, in Joseph J. Rotman's .