sheaf cohomology

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Let X be a topological space. The category of sheaves of abelian groups on X has enough injectives. So we can define the sheaf cohomology $H^i(X,\mathcal{F})$ of a sheaf $\mathcal{F}$ to be the right derived functors of the global sections functor $\mathcal{F} \to \Gamma(X,\mathcal{F})$.

Usually we are interested in the case where X is a scheme, and $\mathcal{F}$ is a coherent sheaf. In this case, it does not matter if we take the derived functors in the category of sheaves of abelian groups or coherent sheaves.

Sheaf cohomology can be explicitly calculated using `http://planetmath.org/CechCohomology` cohomology. Choose an open cover $\{U_i\}$ of X. We define

$$C^i(\mathcal{F}) = \prod \mathcal{F}(U_{j_0 \cdots j_i})$$

where the product is over $i+1$ element subsets of $\{1,\ldots,n\}$ and $U_{j_0 \cdots j_i} = U_{j_0} \cap \cdots \cap U_{j_i}$. If $s \in \mathcal{F}(U_{j_0 \cdots j_i})$ is thought of as an element of $C^i(\mathcal{F})$, then the differential

$$\partial(s) = \prod_{\ell} \left(\prod_{k=j_\ell+1}^{j_{\ell+1}-1} (-1)^{\ell} s|_{U_{j_0 \cdots j_\ell k j_{\ell+1} \cdots j_i}} \right)$$

makes $C^*(\mathcal{F})$ into a chain complex. The cohomology of this complex is denoted $\check{H}^i(X,\mathcal{F})$ and called the *Čech cohomology* of $\mathcal{F}$ with respect to the cover $\{U_i\}$. There is a natural map $H^i(X,\mathcal{F}) \to \check{H}^i(X,\mathcal{F})$ which is an isomorphism for sufficiently fine covers. (A cover is *sufficiently fine* if $H^i(U_j,\mathcal{F}) = 0$ for all $i > 0$, for every j and for every sheaf $\mathcal{F}$). In the category of schemes, for example, any cover by open affine schemes has this property. What this means is that if one can find a finite fine enough cover of X, sheaf cohomology becomes computable by a finite process. In fact in [**?**], this is how the cohomology of projective space is explicitly calculated.

References

[1] Grothendieck, A. *Sur quelques points d'algèbre homologique*, Tôhoku Math. J., Second Series, 9 (1957), 119–221.

[2] Hartshorne, R. *Algebraic Geometry*, Springer-Verlag Graduate Texts in Mathematics 52, 1977