radical of an ideal

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Let R be a commutative ring. For any ideal I of R, the *radical* of I, written $\sqrt{I}$ or $\mathrm{Rad}(I)$, is the set

$$\{a \in R \mid a^n \in I \text{ for some integer } n > 0\}$$

The radical of an ideal I is always an ideal of R.

If $I = \sqrt{I}$, then I is called a *radical ideal*.

Every prime ideal is a radical ideal. If I is a radical ideal, the quotient ring R/I is a ring with no nonzero nilpotent elements.

More generally, the radical of an ideal in can be defined over an arbitrary ring. Let I be an ideal of a ring R, the radical of I is the set of $a \in R$ such that every m-system containing a has a non-empty intersection with I:

$$\sqrt{I} := \{a \in R \mid \text{if } S \text{ is an } m\text{-system, and } a \in S, \text{ then } S \cap I \neq \varnothing\}.$$

Under this definition, we see that $\sqrt{I}$ is again an ideal (two-sided) and it is a subset of $\{a \in R \mid a^n \in I \text{ for some integer } n > 0\}$. Furthermore, if R is commutative, the two sets coincide. In other words, this definition of a radical of an ideal is indeed a "generalization" of the radical of an ideal in a commutative ring.