characterization of isomorphisms of quivers

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Let $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q'_0, Q'_1, s', t')$ be quivers. Recall, that a morphism $F : Q \to Q'$ is an isomorphism if and only if there is a morphism $G : Q' \to Q$ such that $FG = \mathrm{Id}(Q')$ and $GF = \mathrm{Id}(Q)$, where

$$\mathrm{Id}(Q) : Q \to Q$$

is given by $\mathrm{Id}(Q) = (\mathrm{Id}(Q)_0, \mathrm{Id}(Q)_1)$, where both $\mathrm{Id}(Q)_0$ and $\mathrm{Id}(Q)_1$ are the identities on Q_0, Q_1 respectively.

Proposition. A morphism of quivers $F : Q \to Q'$ is an isomorphism if and only if both F_0 and F_1 are bijctions.

Proof. „$\Rightarrow$" It follows from the definition of isomorphism that $F_0G_0 = \mathrm{Id}(Q')_0$ and $G_0F_0 = \mathrm{Id}(Q)_0$ for some $G_0 : Q'_0 \to Q_0$. Thus F_0 is a bijection. The same argument is valid for F_1.

„$\Leftarrow$" Assume that both F_0 and F_1 are bijections and define $G : Q'_0 \to Q_0$ and $H : Q'_1 \to Q_1$ by

$$G = F_0^{-1}, \quad H = F_1^{-1}.$$

Obviously (G, H) is „the inverse" of F in the sense, that the equalites for compositions hold. What is remain to prove is that (G, H) is a morphism of quivers. Let $\alpha \in Q'_1$. Then there exists an arrow $\beta \in Q_1$ such that

$$F_1(\beta) = \alpha.$$

Thus

$$H(\alpha) = \beta.$$

Since F is a morphism of quivers, then

$$s'(\alpha) = s'(F_1(\beta)) = F_0(s(\beta)),$$

which implies that

$$G(s'(\alpha)) = s(\beta) = s(H(\alpha)).$$

The same arguments hold for the target function t, which completes the proof. $\square$