example of functor of points of a scheme

Canonical name	ExampleOfFunctorOfPointsOfAScheme
Date of creation	2013-03-22 14:11:07
Last modified on	2013-03-22 14:11:07
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Last modified by	archibal (4430)
Numerical id	4
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Entry type	Example
Classification	msc 14A15

Let X be an affine scheme of finite type over a field k. Then we must have

$$X = \operatorname{Spec} k[X_1, \ldots, X_n]/\left\langle f_1, \ldots, f_m \right\rangle,$$

with the structure morphism $X \to \operatorname{Spec} k$ induced from the natural embedding $k \to k[X_1, \ldots, X_n]$.

Let k' be some field extension of k. What are the k'-points of X? Recall that a k'-point of X is by definition a morphism $\operatorname{Spec} k' \to X$ (observe that since we have an embedding $k \to k'$ we have a morphism $\operatorname{Spec} k' \to \operatorname{Spec} k$, so $\operatorname{Spec} k'$ is natuarlly a k-scheme). Since X is affine, this must come from a ring homomorphism

$$k[X_1, \ldots, X_n]/\left\langle f_1, \ldots, f_m \right\rangle \to k'$$

which takes elements of k to themselves inside k'. Such a homomorphism is completely specified by specifying the images of $X_1, \ldots, X_n$; for it to be a homomorphism, these images must satisfy $f_1, \ldots, f_m$. In other words, a k'-point on X is identified with an element of $(k')^n$ satisfying all the polynomials f_i.

If k' is an algebraically closed field, a point on X corresponds uniquely to a point on an affine variety defined by the same equations as X. If k' is just any extension of k, then we have simply found which new points belong on X when we extend the base field. T

For an example of why schemes contain much more information than the list of points over their base field, take $X = \operatorname{Spec} \mathbb{R}[X]/\left\langle X^2 + 1 \right\rangle$. Then X has *no* points over $\mathbb{R}$, its natural base field. Over $\mathbb{C}$, it has two points, corresponding to i and $-i$.

This suggests that schemes may be the appropriate adaptation of varieties to deal with non-algebraically closed fields.

Observe that we never used the fact that k' (or in fact k) was a field. One often chooses k' as something other than a field in order to solve a problem. For example, one can take $k' = k[\epsilon]/\left\langle \epsilon^2 \right\rangle$. Then specifying a k'-point on X amounts to choosing an image $\kappa_i + \lambda_i \epsilon$ for each X_i. It is clear that the κ_i must satisfy the f_j. But upon reflection, we see that the λ_i must specify a tangent vector to X at the point specified by the κ_i. So the $k[\epsilon]/\left\langle \epsilon^2 \right\rangle$-points tell us about the tangent bundle to X. Observe that we made no assumption about the field k — we can extract these "tangent vectors" in positive characteristic or over a non-complete field.

The ring $k[\epsilon]/\langle\epsilon^2\rangle$ and rings like it (often any Artinian ring) can be used to define and study infinitesimal deformations of schemes, as a simple case of the study of families of schemes.