variety

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Defines	complete
Defines	curve

Definition 1 *Let X be a scheme over a field k. Then X is said to be an* abstract variety *over k if it is integral, separated, and of finite type over k. Usually we simply say X is a* variety. *If X is proper over k, it is said to be* complete. *If the dimension of X is one, then X is said to be a* curve.

Some authors also require k to be algebraically closed, and some authors require curves to be nonsingular.

Calling X a variety would appear to conflict with the preexisting notion of an `http://planetmath.org/AffineVariety`affine or projective variety. However, it can be shown that if k is algebraically closed, then there is an equivalence of categories between affine abstract varieties over k and affine varieties over k, and another between projective abstract varieties over k and projective varieties over k.

This equivalence of categories identifies an abstract variety with the set of its k-points; this can be thought of as simply ignoring all the generic points. In the other direction, it identifies an affine variety with the prime spectrum of its coordinate ring: the variety in $\mathbb{A}^n$ defined by the ideal

$$\langle f_1, \ldots, f_m \rangle$$

is identified with

$$\operatorname{Spec} k[X_1, \ldots, X_n] / \langle f_1, \ldots, f_m \rangle .$$

A projective variety is identified as the gluing together of the affine varieties obtained by taking the complements of hyperplanes. To see this, suppose we have a projective variety in $\mathbb{P}^n$ given by the homogeneous ideal $\langle f_1, \ldots, f_m \rangle$. If we delete the hyperplane $X_i = 0$, then we obtain an affine variety: let $T_j = X_j / X_i$; then the affine variety is the set of common zeros of

$$\langle f_1(T_0, \ldots, T_n), \ldots, f_m(T_0, \ldots, T_n) \rangle .$$

In this way, we can get $n + 1$ overlapping affine varieties that cover our original projective variety. Using the theory of schemes, we can glue these affine varieties together to get a scheme; the result will be projective.

For more on this, see Hartshorne's book *Algebraic Geometry*; see the bibliography for algebraic geometry for more resources.