Shioda-Tate formula

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The main references for this part are the works of Shioda and Tate [?], [?], [?].

Let k be a field and let $\overline{k}$ be a fixed algebraic closure of k. Let $\mathcal{E}$ be an elliptic surface over a curve C/k and let $K = k(C)$ be the function field of C. Let $\overline{\mathcal{E}} = \mathcal{E}(\overline{k})$ (or more precisely $\overline{\mathcal{E}} = \mathcal{E} \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$). The Néron-Severi group of $\overline{\mathcal{E}}$, denoted by $\mathrm{NS}(\overline{\mathcal{E}})$, is by definition the group of divisors on $\overline{\mathcal{E}}$ modulo algebraic equivalence. Under the previous assumptions, $\mathrm{NS}(\overline{\mathcal{E}})$ is a finitely generated abelian group (this is a consequence of the so-called 'theorem of the base' which can be found in [?]). The Néron-Severi group of $\mathcal{E}$, denoted by $\mathrm{NS}(\mathcal{E})$, is simply the image of the group of divisors on $\mathcal{E}$ in $\mathrm{NS}(\overline{\mathcal{E}})$. Let $T \subset \mathrm{NS}(\mathcal{E})$ be the subgroup generated by the image of the zero-section σ_0 and all the irreducible components of the fibers of π. T is sometimes called the "trivial part" of $\mathrm{NS}(\mathcal{E})$.

Theorem (Shioda-Tate formula). *For each $t \in C$ let n_t be the number of irreducible components on the fiber at t, i.e. $\pi^{-1}(t)$. Then:*

$$\begin{aligned} \operatorname{rank}_{\mathbb{Z}}(\mathcal{E}/K) &= \operatorname{rank}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{E})) - \operatorname{rank}_{\mathbb{Z}}(T) \\ &= \operatorname{rank}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{E})) - 2 - \sum_{t \in C}(n_t - 1). \end{aligned}$$

References

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[3] T. Shioda, *An Explicit Algorithm for Computing the Picard Number of Certain Algebraic Surfaces*, Amer. J. Math. 108 (1986), 415-432.

[4] T. Shioda, *On the Mordell-Weil Lattices*, Commentarii Mathematici Universitatis Sancti Pauli, Vol 39, No. 2, 1990, pp. 211-239.

[5] J. Tate, *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog*, Séminaire Bourbaki, 9, Soc. Math. France, Paris, 1966, Exp. No. 306, 415-440, 1995.