Hurwitz genus formula

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The following formula, due to Hurwitz, is extremely useful when trying to compute the genus of an algebraic curve. In this entry K is a perfect field (i.e. every algebraic extension of K is separable). Recall that a non-constant map of curves $\psi : C_1 \to C_2$ over K is separable if the extension of function fields $K(C_1)/\psi^* K(C_2)$ is a separable extension of fields.

Theorem (Hurwitz Genus Formula)**.** *Let C_1 and C_2 be two smooth curves defined over K of genus g_1 and g_2, respectively. Let $\psi : C_1 \to C_2$ be a non-constant and separable map. Then*

$$2g_1 - 2 \geq (\deg \psi)(2g_2 - 2) + \sum_{P \in C_1} (e_\psi(P) - 1)$$

where $e_\psi(P)$ is the ramification index of ψ at P. Moreover, there is equality if and only if either $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) = p > 0$ *and p does not divide $e_\psi(P)$ for all $P \in C_1$.*

Example. As an application of the Hurwitz genus formula, we show that an elliptic curve $E : y^2 = x(x-\alpha)(x-\beta)$ defined over a field K of characteristic 0 has genus 1. Notice that the fact that E is an elliptic curve over K implies that $0, \alpha$ and β are distinct elements of K, otherwise E would be a singular curve. We define a map:

$$\psi : E \to \mathbb{P}^1, \quad [x, y, z] \mapsto [x, z]$$

and notice that $[0, 1, 0]$, the "point at infinity" of E, maps to $[1, 0]$, the point at infinity of $\mathbb{P}^1$. The degree of this map is 2: generically every point in $\mathbb{P}^1$ has two preimages, namely $[x, y, z]$ and $[x, -y, z]$. Moreover, the genus of $\mathbb{P}^1$ is 0 and the map ψ is ramified exactly at 4 points, namely $P_1 = [0, 0, 1], P_2 = [\alpha, 0, 1], P_3 = [\beta, 0, 1]$ and the point at infinity. It is easily checked that the ramification index at each point is $e_\psi(P_i) = 2$. Hence, the Hurwitz formula reads:

$$2g_1 - 2 = 2(2 \cdot 0 - 2) + \sum_{i=1}^{4} (e_\psi(P_i) - 1) = -4 + 4 = 0.$$

We conclude that $g_1 = 1$, as claimed.