criterion of Néron-Ogg-Shafarevich

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In this entry, we use the following notation. K is a local field, complete with respect to a discrete valuation ν, R is the ring of integers of K, $\mathcal{M}$ is the maximal ideal of R and $\mathbb{F}$ is the residue field of R.

Definition. *Let Ξ be a set on which $\mathrm{Gal}(\overline{K}/K)$ acts. We say that Ξ is unramified at ν if the action of the inertia group I_ν on Ξ is trivial, i.e. $\zeta^\sigma = \zeta$ for all $\sigma \in I_\nu$ and for all $\zeta \in \Xi$.*

Theorem (Criterion of Néron-Ogg-Shafarevich)**.** *Let E/K be an elliptic curve defined over K. The following are equivalent:*

1. *E has good reduction over K;*
2. *$E[m]$ is unramified at ν for all $m \geq 1$, $\gcd(m, \mathrm{char}(\mathbb{F})) = 1$;*
3. *The Tate module $T_l(E)$ is unramified at ν for some (all) l, $l \neq \mathrm{char}(\mathbb{F})$;*
4. *$E[m]$ is unramified at ν for infinitely many integers $m \geq 1$, $\gcd(m, \mathrm{char}(\mathbb{F})) = 1$.*

Corollary. *Let E/K be an elliptic curve. Then E has potential good reduction if and only if the inertia group I_ν acts on $T_l(E)$ through a finite quotient for some prime $l \neq \mathrm{char}(\mathbb{F})$.*

Proof of Corollary. ($\Rightarrow$) Assume that E has potential good reduction. By definition, there exists a finite extension of K, call it K', such that E/K' has good reduction. We can extend K' (if necessary) so K'/K is a Galois finite extension.

Let ν' and $I_{\nu'}$ be the corresponding valuation and inertia group for K'. Then the theorem above ((1)$\Rightarrow$(3)) implies that $T_l(E)$ is unramified at ν' for all l, $l \neq \mathrm{char}(\mathbb{F}) = \mathrm{char}(\mathbb{F}')$ (since $\mathbb{F}'$ is a finite extension of $\mathbb{F}$). So $I_{\nu'}$ acts trivially on $T_l(E)$ for all $l \neq \mathrm{char}(\mathbb{F}')$. Thus $I_\nu \hookrightarrow T_l(E)$ factors through the finite quotient $I_\nu/I_{\nu'}$.

($\Leftarrow$) Let $l \neq \mathrm{char}(\mathbb{F})$, and assume $I_\nu \hookrightarrow T_l(E)$ factors through a finite quotient, say I_ν/J. Let $\overline{K}^J$ be the fixed field of J, then $\overline{K}^J/\overline{K}^{I_\nu}$ is a finite extension, so we can find a finite extension K'/K so that $\overline{K}^J = K'\overline{K}^{I_\nu}$. So the inertia group of K' is equal to J, and J acts trivially on $T_l(E)$. Hence the criterion ((3)$\Rightarrow$(1)) implies that E has good reduction over K', and since K'/K is finite, E has potential good reduction. $\square$

Proposition. *Let E/K be an elliptic curve. Then E has potential good reduction if and only if its j-invariant is integral (i.e. $j(E) \in R$).*

Proof. ($\Leftarrow$) Assume $\mathrm{char}(\mathbb{F}) \neq 2$, it is easy to prove that we can extend K to a finite extension K' so that E has a Weierstrass equation:

$$E : y^2 = x(x-1)(x-\lambda) \quad \lambda \neq 0, 1 \tag{1}$$

Since we are assuming $j(E) \in R$, and:

$$(1 - \lambda(1-\lambda))^3 - j\lambda^2(1-\lambda)^2 = 0 \tag{2}$$

then $\lambda \in R$ and $\lambda \neq 0, 1 \mod \mathcal{M}'$ ($\Rightarrow \Delta' \in (R')^*$). Hence E/K' has good reduction, i.e. E has potential good reduction.

($\Rightarrow$) Assume that E has potential good reduction, so there exists K' so that E/K' has good reduction. Let Δ', c_4' the usual quantities associated to the Weierstrass equation over K'. Since E/K' has good reduction, $\Delta' \in (R')^*$, and so $j(E) = \frac{(c_4{}')^3}{\Delta'} \in R'$. But since E is defined over K, $j(E) \in K$, so $j(E) \in K \bigcap R' = R$. $\square$