ind-variety

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Defines	ind-variety

Let $\mathbb{K}$ be a field. An *ind-variety* over $\mathbb{K}$ is a set X along with a filtration:

$$X_0 \subset X_1 \subset \cdots X_n \subset \cdots$$

such that

1. $X = \bigcup_{j \geq 0} X_j$
2. Each X_i is a finite dimensional algebraic variety over $\mathbb{K}$
3. The inclusions $i_j \colon X_j \to X_{j+1}$ are closed embeddings of algebraic varieties

The ring of regular functions on an ind-variety X is defined to be $\mathbb{K}[X] := \varprojlim \mathbb{K}[X_j]$ where the limit is taken with respect to the family of maps $\left\{ i_j^* \colon \mathbb{K}[X_{j+1}] \to \mathbb{K}[X_j] \right\}_{j \geq 0}$.

This ring is given the structure of a topological ring by letting each $\mathbb{K}[X_j]$ have the discrete topology and $\mathbb{K}[X]$ have the induced inverse limit topology, i.e. the topology induced from the canonical inclusion $\varprojlim \mathbb{K}[X_j] \subset \prod_j \mathbb{K}[X_j]$ and the product topology on $\prod_j \mathbb{K}[X_j]$.

An ind-variety is called *affine* (resp. *projective*) if each X_j is affine (resp. projective).

The notion of an ind-variety goes back to Igor Shafarevich in [?] and [?].

Examples

Let $\mathcal{K} := \mathbb{K}((t))$ be the ring of formal Laurant series over $\mathbb{K}$ and $\mathcal{O} := \mathbb{K}[[t]]$ be its ring of integers, the formal Taylor series. Let $V = \mathbb{K}^n$. Then the set X of $\mathcal{O}$-lattices ($\mathcal{O}$-submodules of maximal rank) in $V \otimes_{\mathbb{K}} \mathcal{K}$ is an example of a (non-finite dimensional) projective ind-variety using the filtration

$$X_i := \left\{ L \in X \mid t^i L_0 \subset L \subset t^{-i} L_0, \dim_{\mathbb{K}} L / t^i L_0 = in \right\}$$

where $L_0 := V \otimes_{\mathbb{K}} \mathcal{O}$.

(cf. [?] section 11, or [?] appendix C part 7)

References

[1] George Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Astérisque 101-102 (1983), pp. 208-229.

[2] Shrawan Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory.* Progress in Mathematics Vol. 204. Birkhauser, 2002.

[3] Igor Shafarevich, *On some infinite-dimensional groups. II* Math USSR Izvestija 18 (1982), pp. 185 - 194.

[4] Igor Shafarevich, *Letter to the editors: "On some infinite-dimensional groups. II"* Izv. Ross. Akad. Nauk. Ser. Mat. 59 (1995), pp. 224 - 224.