finite morphism

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Affine schemes

Let X and Y be affine schemes, so that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Let $f\colon X \to Y$ be a morphism, so that it induces a homomorphism of rings $g\colon B \to A$.

The homomorphism g makes A into a B-algebra. If A is finitely-generated as a B-algebra, then f is said to be a morphism of *finite type*.

If A is in fact finitely generated as a B-module, then f is said to be a *finite* morphism.

For example, if k is a field, the scheme $\mathbb{A}^n(k)$ has a natural morphism to $\operatorname{Spec} k$ induced by the ring homomorphism $k \to k[X_1, \ldots, X_n]$. This is a morphism of finite type, but if $n > 0$ then it is not a finite morphism.

On the other hand, if we take the affine scheme $\operatorname{Spec} k[X,Y]/\left\langle Y^2 - X^3 - X\right\rangle$, it has a natural morphism to $\mathbb{A}^1$ given by the ring homomorphism $k[X] \to k[X,Y]/\left\langle Y^2 - X^3 - X\right\rangle$. Then this morphism is a finite morphism. As a morphism of schemes, we see that every fiber is finite.

General schemes

Now, let X and Y be arbitrary schemes, and let $f\colon X \to Y$ be a morphism. We say that f is of *finite type* if there exist an open cover of Y by affine schemes $\{U_i\}$ and a finite open cover of each U_i by affine schemes $\{V_{ij}\}$ such that $f|_{V_{ij}}$ is a morphism of finite type for every i and j. We say that f is *finite* if there exists an open cover of Y by affine schemes $\{U_i\}$ such that each inverse image, $V_i = f^{-1}(U_i)$ is itself affine, and such that $f|_{V_i}$ is a finite morphism of affine schemes.

Example. Let $X = \mathbb{P}^1(k)$ and $Y = \operatorname{Spec} k$. We cover X by two copies of $\mathbb{A}^1$ and consider the natural morphisms from each of these copies to $\operatorname{Spec} k$. Both of these affine morphisms are of finite type, but are not finite. The covering morphisms patch together to give a morphism from $\mathbb{P}^1$ to $\operatorname{Spec} k$. The overall morphism is of finite type, but again is not finite.

References.

D. Eisenbud and J. Harris, *The Geometry of Schemes*, Springer.