Noetherian topological space

Canonical name	NoetherianTopologicalSpace
Date of creation	2013-03-22 13:03:33
Last modified on	2013-03-22 13:03:33
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	18
Author	mathcam (2727)
Entry type	Definition
Classification	msc 14A10
Related topic	Compact

A topological space X is called if it satisfies the descending chain condition for closed subsets: for any sequence

$$Y_1 \supseteq Y_2 \supseteq \cdots$$

of closed subsets Y_i of X, there is an integer m such that $Y_m = Y_{m+1} = \cdots$.

As a first example, note that all finite topological spaces are Noetherian.

There is a lot of interplay between the Noetherian condition and compactness:

- Every Noetherian topological space is quasi-compact.
- A Hausdorff topological space X is Noetherian if and only if every subspace of X is compact. (i.e. X is hereditarily compact)

Note that if R is a Noetherian ring, then $\operatorname{Spec}(R)$, the prime spectrum of R, is a Noetherian topological space.

Example of a Noetherian topological space:
The space $\mathbb{A}_k^n$ (affine n-space over a field k) under the Zariski topology is an example of a Noetherian topological space. By properties of the ideal of a subset of $\mathbb{A}_k^n$, we know that if $Y_1 \supseteq Y_2 \supseteq \cdots$ is a descending chain of Zariski-closed subsets, then $I(Y_1) \subseteq I(Y_2) \subseteq \cdots$ is an ascending chain of ideals of $k[x_1, \ldots, x_n]$.

Since $k[x_1, \ldots, x_n]$ is a Noetherian ring, there exists an integer m such that $I(Y_m) = I(Y_{m+1}) = \cdots$. But because we have a one-to-one correspondence between radical ideals of $k[x_1, \ldots, x_n]$ and Zariski-closed sets in $\mathbb{A}_k^n$, we have $V(I(Y_i)) = Y_i$ for all i. Hence $Y_m = Y_{m+1} = \cdots$ as required.