rank of an elliptic curve

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Entry type	Definition
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Synonym	rank
Related topic	EllipticCurve
Related topic	HeightFunction
Related topic	MordellWeilTheorem
Related topic	SelmerGroup
Related topic	MazursTheoremOnTorsionOfEllipticCurves
Related topic	NagellLutzTheorem
Related topic	ArithmeticOfEllipticCurves
Defines	weak Mordell-Weil theorem
Defines	rank of an elliptic curve

Let K be a number field and let E be an elliptic curve over K. By $E(K)$ we denote the set of points in E with coordinates in K.

Theorem 1 (Mordell-Weil). *$E(K)$ is a finitely generated abelian group.*

Proof. The proof of this theorem is fairly involved. The main two ingredients are the so called "weak Mordell-Weil theorem" (see below), the concept of height function for abelian groups and the "descent" theorem.
See [**?**], Chapter VIII, page 189. □

Theorem 2 (Weak Mordell-Weil). *$E(K)/mE(K)$ is finite for all $m \geq 2$.*

The Mordell-Weil theorem implies that for any elliptic curve E/K the group of points has the following structure:

$$E(K) \simeq E_{\text{torsion}}(K) \bigoplus \mathbb{Z}^R$$

where $E_{\text{torsion}}(K)$ denotes the set of points of finite order (or torsion group), and R is a non-negative integer which is called the *rank* of the elliptic curve. It is not known how big this number R can get for elliptic curves over $\mathbb{Q}$. The largest rank known for an elliptic curve over $\mathbb{Q}$ is 28 `http://www.math.hr/ duje/tors/tors.htmlE` (2006).

Note: see Mazur's theorem for an account of the possible torsion subgroups over $\mathbb{Q}$.

Examples:

1. The elliptic curve $E_1/\mathbb{Q}\colon y^2 = x^3 + 6$ has rank 0 and $E_1(\mathbb{Q}) \simeq 0$.

2. Let $E_2/\mathbb{Q}\colon y^2 = x^3 + 1$, then $E_2(\mathbb{Q}) \simeq \mathbb{Z}/6\mathbb{Z}$. The torsion group is generated by the point $(2, 3)$.

3. Let $E_3/\mathbb{Q}\colon y^2 = x^3 + 109858299531561$, then $E_3(\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z} \bigoplus \mathbb{Z}^5$. See `http://math.bu.edu/people/alozano/Torsion.html`generators here.

4. Let $E_4/\mathbb{Q}\colon y^2 + 1951/164xy - 3222367/40344y = x^3 + 3537/164x^2 - 40302641/121032x$, then $E_4(\mathbb{Q}) \simeq \mathbb{Z}^{10}$. See `http://math.bu.edu/people/alozano/Example` here.

References

[1] James Milne, *Elliptic Curves*, online course notes. http://www.jmilne.org/math/CourseNotes/math679.htmlhttp://www.jmilne.org/math/Co

[2] Joseph H. Silverman, *The Arithmetic of Elliptic Curves*. Springer-Verlag, New York, 1986.

[3] Joseph H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*. Springer-Verlag, New York, 1994.

[4] Goro Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*. Princeton University Press, Princeton, New Jersey, 1971.