inflection points and canonical forms of non-singular cubic curves

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In this entry, we shall investigate the inflection points of non-singular complex cubic curves and show that the equations of such curves can always be put in various canonical forms. The reason for lumping these two subjects in a single entry is that, being related, it is more efficient to develop them both at the same time than it would be to treat either one in isolation — on the one hand, putting the equation of the curve in a canonical form simplifies calculations relating to inflection points whilst, on the other hand, information about inflection points allows one to construct canonical forms.

We shall consider our curves as projective curves and describe them with homogenous equations; i.e. a curve is a locus in $\mathbb{CP}^2$ where $F(x, y, z) = 0$ for some third- order homogenous polynomial F. For the purpose at hand, the term "inflection point" may be taken to mean a point on the curve where the tangent intersects the curve with multiplicity 3 — a point on the curve will have this property if and only if it is a zero of the Hessian.

We begin by presenting a crude canonical form. With further refinement, this will become the Weierstrass canonical form.

Theorem 1. *Suppose C is a non-singular complex cubic curve and P a point on that curve. Then we may choose homogenous coordiantes in which the coordiantes of P are $(0, 0, 1)$ and equation of the curve looks as follows:*

$$yz^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z = 0$$

Proof. The most general equation of a cubic is as follows:

$$AX^3+BX^2Y+CX^2Z+DXY^2+EXYZ+FXZ^2+GY^3+HY^2Z+KYZ^2+LZ^3 = 0$$

Requiring that the curve pass through $(0, 0, 1)$ means that $L = 0$. Since our curve is not singular, it has a tangent at all points, in particular, a tangent through P. We may ask that the equation of the tangent through P be $Y = 0$, which requires that $F = 0$. Since our curve is not singular at $(0, 0, 1)$, both F and K cannot be zero so, without loss of generality, we may take $K = 1$.

We may then make the following linear transform:

$$\begin{aligned} X &= x \\ Y &= y \\ Z &= z - Ex/2 - Hy/2 \end{aligned}$$

Upon doing so, our equation assumes the form

$$yz^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z = 0$$

where the coefficients a, b, c, d, e are defined as follows:

$$\begin{aligned} a &= A - CE/2 \\ b &= B - CH/2 - E^2/4 \\ c &= D - EH/2 \\ d &= G - H^2/4 \\ e &= C \end{aligned}$$

□

Next, we turn our attention to the Hessian to begin our investigation of inflection points.

Theorem 2. *If C is a non-singular complex cubic curve with homogenous equation $F(x, y, z) = 0$, then the Hessian of F does not equal a multiple of F.*

Proof. By the foregoing theorem, we know that there exists a system of coordinates in which

$$F(x, y, z) = yz^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z.$$

Let us examine the Hessian at the point $(0, 0, 1)$:

$$H(0, 0, 1) = \begin{vmatrix} 2e & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = 8e$$

If $e \neq 0$, then the Hessian cannot me a multiple of F because $F(0, 0, 1) = 0$. Assume then that $e = 0$. In that case, we must have $a \neq 0$ because, if $a = 0$, then F would factor as a product of y and a quadratic polynomial, which is impossible because the curve C is assumed not to be singular.[1] Evaluating the Hessian on the line $y = 0$,

$$H(x, 0, z) = \begin{vmatrix} 6ax & 2bx & 0 \\ 2bx & 2cx & 2z \\ 0 & 2z & 0 \end{vmatrix} = 24axz^2$$

[1]Were $F(x, y, z) = yQ(x, y, z)$ for some quadratic polynomial Q, then the gradient of F would vanish at the intersection of the line $y = 0$ and the conic $Q(x, y, z) = 0$, so the variety described by $F(x, y, z) = 0$ would be singular at the point(s) of intersection.

Since a cannnot be zero, this does not vanish identically and is clearly not a non-zero multiple of $F(x, 0, z)$. $\square$

This theorem tells us that $H(x, y, z) = 0$ is the equation of a cubic curve distinct from C. Hence, we know that there must be at least one and at most nine point of intersection of these two curves, i.e. at least one and at most nine inflection points. Knowing that inflection points exist, we can improve our canonical form by choosing P as an inflection point. With a little tidying up, this will give us Weierstrass' canonical form.

Theorem 3. *Suppose C is a non-singular complex cubic curve and P a point of inflection of on that curve. Then we may choose homogenous coordinates in which the coordiantes of P are $(0, 0, 1)$ and equation of the curve looks as follows:*

$$yz^2 + 4x^3 + g_2xy^2 + g_3y^3 = 0$$

Proof. By our earlier result, we know that there exist coordinates in which the equation of the curve is

$$YZ^2 + aX^3 + bX^2Y + cXY^2 + dY^3 + eX^2Z = 0$$

and the coordinates of P are $(0, 0, 1)$. By the calculation made in the proof of the previous theorem, we know that $H(0, 0, 1) = 8e$. Since P is a point of inflection, this implies that $e = 0$.

As noted in the previous proof, we cannot have $a = 0$ because that would make our curve singular. Hence, we can divide by $a^{1/3}$ to make the following coordinate transformation:

$$\begin{aligned} X &= x - by/3 \\ Y &= ay/4 \\ Z &= z \end{aligned}$$

After canceling a factor of a, our equation becomes

$$yz^2 + 4x^3 + g_2xy^2 + g_3y^3 = 0,$$

where

$$\begin{aligned} g_2 &= \frac{1}{4}ac - \frac{7}{12}b^2 \\ g_3 &= \frac{11}{432}b^3 + \frac{1}{48}abc + \frac{1}{16}a^2d \end{aligned}$$

$\square$

Using this canonical form, it is easy to show that we have the maximum number of inflection points possible.

Theorem 4. *A non-singular complex cubic curve has nine inflection points.*

Proof. Given an inflection point, the foregoing theorem tells us that we can choose a coordinate system in which that point has coordinates $(0,0,1)$ and the equation of the curve assumes the form $F(x,y,z)=0$ where

$$F(x,y,z) = yz^2 + 4x^3 + g_2xy^2 + g_3y^3.$$

Computing the Hessian, we have

$$H(x,y,z) = \begin{vmatrix} 24x & 2g_2y & 0 \\ 2g_2y & 2g_2x + 6g_3y & 2z \\ 0 & 2z & 2y \end{vmatrix} = 96g_2x^2y + 288g_3xy^2 - 8g_2^2y^3 - 96xz^2$$

Computing its gradient at $(0,0,1)$, we find that

$$\left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z}\right)(0,0,1) = (-96,0,0).$$

For comparison, we have

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)(0,0,1) = (0,1,0).$$

Thus, we see that the curves $H(x,y,z)=0$ and $F(x,y,z)=0$ are both smooth at $(0,0,1)$ and intersect transversely, hence they have intersection multiplicity 1 there. Because we could choose our coordinates to place any inflection point at $(0,0,1)$, this means that every intersection of our curve with its Hessian has intersection multiplicity 1. Since both F and H are of third order, Bezout's theorem implies that there are nine distinct intersection points, i.e. nine distinct points of inflection. $\square$

Knowing that there exist multiple inflection points, we will now cast the equation in a form which places two inflection points at priveleged locations.

Theorem 5. *Suppose C is a non-singular complex cubic curve and P and Q are points of inflection of on that curve. Then we may choose homogeneous coordinates in which the coordinates of P are $(1,0,0)$, the coordinates of Q are $(0,1,0)$ and the equation of the curve looks as follows:*

$$x^2y + y^2x + z^3 + axyz = 0$$

Proof. Since P is a point of its inflection, the tangent to C at P intersects C with multiplicity 3 at P. Since C is a cubic, this means that this tangent cannot intersect C at any other points; in particular, it cannot pass through Q. Likewise, the tangent through Q cannot pass through P. Hence, the tangent at P and the tangent at Q must intersect at some third point R which is not collinear with P and Q.

We may choose our coordinates so as to place P at $(1,0,0)$, Q at $(0,1,0)$ and R at $(0,0,1)$. The equation of our curve is $F(x,y,z) = 0$, where

$$F(x,y,z) = axyz + bz^3 + cyz^2 + dy^2z + ey^3 + fxz^2 + gx^2z + hxy^2 + jx^2y + kx^3.$$

Since $F(1,0,0) = k$ and $F(0,1,0) = e$, we must have $k = 0$ and $e = 0$ in order for P and Q to lie on C. The equation of the line PR is $y = 0$. Restricting to this line, we have

$$F(x,0,z) = bz^3 + fz^2 + gx^2z.$$

Since P is an inflection point, the curve must have third order contact with this tangent line, the restriction of F to this line should have a triple root at P, i.e. should be a multiple of z^3. Hence, $f = 0$ and $g = 0$.

Likewise, the equation of the line QR is $x = 0$. Restricting to this line, we have

$$F(0,y,z) = bz^3 + cyz^2 + dy^2z.$$

Again, since Q is an inflection point, we must have a triple root, so this quantity should be a multiple of z^3, so we must have $c = 0$ and $d = 0$.

Summarizing our progress so far, we have found that

$$F(x,y,z) = axyz + bz^3 + hxy^2 + jx^2y.$$

Next, we note that h cannot be zero because that would imply that C had a singularity at Q. Likewise, we must not have j be zero because that would mean a singularity at P. Also, were $b = 0$, we would have $F(x,y,z) = xy(z + hy + jx)$ which would have a singularity at R. Thus, by rescaling x, y, and z if necessarry, we can take $h = j = b = 1$, so our equation assumes the form

$$F(x,y,z) = axyz + z^3 + xy^2 + x^2y.$$

□

Upon examining the form of the equation just derived more closely, we learn an important geometric fact:

Theorem 6. *Suppose that C is a non-singular complex cubic curve and that P and Q are inflection points of C. Then the line PQ intersects C in a third point R, which is also an inflection point.*

Proof. By the foregoing result, we know that we can write the equation for C as $F(x, y, z) = 0$ with

$$F(x, y, z) = xy^2 + x^2y + z^3 + axyz$$

with P located at $(1, 0, 0)$ and Q located at $(0, 1, 0)$. The line PQ then has equation $z = 0$. Restricting F to this line, $F(x, y, 0) = xy^2 + x^2y = x(x+y)y$, so R, the third point of intersection of C with PQ, has coordinates $(1, -1, 0)$.

Next, we compute the Hessian determinant at R:

$$H(1, -1, 0) = \begin{vmatrix} -1 & 0 & -a \\ 0 & 1 & a \\ -a & a & 0 \end{vmatrix} = 0$$

Since it equals zero, R is an inflection point. □

Since our curve is of the third order, a line cannot intersect more than three points. Hence, we see that a line passing through two inflection points of a non-singular complex cubic plane curve must pass through exactly three inflection points. As it turns out, this observation, together with the fact that there are exactly nine inflection points suffices to determine the locations of the inflection points up to collineation. However, rather than pursuing this line of reasoning here, we will instead cast the equation of the curve in a form which makes it easy to locate all the inflection points and makes the symmetry of the curve apparent.

Theorem 7. *Suppose C is a non-singular complex cubic curve. Then we may choose homogeneous coordinates the equation of the curve assumes the form*

$$x^3 + y^3 + z^3 + 6mxyz = 0.$$

Proof. We know that there exists a straight line L which intersects C in exactly three points, all of which are inflection points of C. Choose a homogenous coordinate system in which the equation of L is $Z = 0$. Since the three points of intersection of C with L are distinct, we may place them at any three locations; we shall choose $(1, -1, 0)$, $(1, -\rho, 0)$, and $(1, -\rho^2, 0)$, where $\rho = (-1 + \sqrt{-3})/2$ is a primitive cube root of unity.

With this choice, the equation of C becomes

$$X^3 + Y^3 + DX^2Z + EXYZ + FY^2Z + BXZ^2 + CYZ^2 + AZ^3 = 0.$$

By making the change of variables

$$x = X - FZ/3$$

$$y' = Y - DZ/3$$

we simplify the equation of the curve to

$$x^3 + y^3 + ExyZ + bxZ^2 + cyZ^2 + aZ^3 = 0.$$

where

$$a = A - \frac{BD}{3} + \frac{2D^3}{27} - \frac{CF}{3} + \frac{DEF}{9} + \frac{2F^3}{27}$$

$$b = B - \frac{D^2}{3} - \frac{EF}{3}$$

$$c = C - \frac{DE}{3} - \frac{F^2}{3}$$

Note that this transformation leaves the coordinates of the three points of intersection of L and C unchanged.

Next, we impose the requirement that the three points lying on L be inflecton points of C. Setting Z to zero, the Hessian becomes

$$\begin{vmatrix} 6x & 0 & Ey \\ 0 & 6y & Ex \\ Ey & Ex & 2bx + 2cy \end{vmatrix} = 72bx^2y + 72cxy^2 - 6E^2(x^3 + y^3)$$

In order for the three points intersection of L and C to be inflection points, we must have this be a multiple of $x^3 + y^3$. This requires that b and c both be zero.

Were a zero as well, our cubic would have a singularity at $(0, 0, 1)$. Since C is assumed to not be singular, that means that $a \neq 0$, so we can make the further transform

$$z = a^{1/3}Z$$

$$m = a^{-1/3}E/6$$

to put the equation for C in the form $x^3 + y^3 + z^3 + 6mxyz = 0$. $\square$

While we have shown that we can transform the equation of any non-singular curve into our canonical form, there remains the possibility that a curve whose equation is expressible in our canonical form may be singular or degenerate. We will now examine this possibility.

Theorem 8. *A equation $x^3+y^3+z^3+6mxyz=0$ describes a singular curve if and only if $8m^3+1=0$.*

Proof. The curve described by the equation $f(x,y,z)=0$ is singular if and only if there exists a point on the curve at which all three partial derivatives of f go zero. Taking derivatives and doing a little bit of algebraic manipulation, this means that, for the curve to be singular, there must be a non-trivial solution to the system of equations

$$x^2 = -2myz$$

$$y^2 = -2mxz$$

$$z^2 = -2mxy.$$

Multiplying the last three equations together, we obtain $x^2y^2z^2 = -8m^3x^2y^2z^2 = 0$, or $(8m^3+1)x^2y^2z^2=0$. If $8m^3+1 \neq 0$, the only way for this to be satisfied is to have either $x=0$ or $y=0$ or $z=0$. However, suppose that $x=0$. Then, by the last two equations, we would also have $y=0$ and $z=0$, so the solution would be trivial. Likewise, if $y=0$, then it follows that $x=0$ and $z=0$; and, if $z=0$, it follows that $x=0$ and $y=0$. Hence, when $8m^3+1 \neq 0$, we only have the trivial solution, so the curve is non-singular.

We will finish by explicitly showing that the curve degenerates when $8m^3+1=0$, i.e. when $m=-1/2$ or $m=-\rho/2$ or $m=-\rho^2/2$. (As before, ρ is a primitive cube root of unity.) In each of these cases, we can factor our equation into a product of three linear terms:

$$x^3+y^3+z^3-3xyz = (x+y+z)(x+\rho y+\rho^2 z)(x+\rho^2 y+\rho z)$$

$$x^3+y^3+z^3-3\rho xyz = \rho^2(x+y+\rho z)(x+\rho y+z)(\rho x+y+z)$$

$$x^3+y^3+z^3-3\rho^2 xyz = \rho(x+y+\rho^2 z)(x+\rho^2 y+z)(\rho^2 x+y+z)$$

Thus, when $8m^3+1=0$, our curve degenerates into a triangle (whose vertices are the singularities). □

Having obtained this canonical form, it is quite easy to exhibit all nine inflection points.

Theorem 9. *Given a non-singular complex cubic plane curve, there exists a coordinate system in which the inflection points of that curve have the following coordinates:*

$$\begin{array}{lll} (1,-\rho,0) & (1,-1,0) & (1,-\rho^2,0) \\ (-\rho,0,1) & (-1,0,1) & (-\rho^2,0,1). \\ (0,1,-\rho) & (0,1,-1) & (0,1,-\rho^2) \end{array}$$

As previously, ρ denotes a primitive third root of unity.

Proof. For convenience, set $f = x^3 + y^3 + z^3 + 6mxyz$ and let h be $1/216$ times the Hessian of f. Computing, we have

$$h = \frac{1}{216}\begin{vmatrix} 6x & 6mz & 6my \\ 6mz & 6y & 6mx \\ 6my & 6mx & 6z \end{vmatrix} = (2m^3+1)xyz - m^2(x^3+y^3+z^3)$$

Thus the Hessian is of the same form, but with m replaced by $-(2m^3+1)/6m^2$. Next, we form two combinations of f ande h:

$$m^2 f + h = (8m^3+1)xyz$$

$$(2m^3+1)f - 6mh = (8m^3+1)(x^3+y^3+z^3)$$

As we have just seen, the condition for the curve not to be singular is exactly that $8m^3+1 \neq 0$, so we may cancel to conclude that

$$x^3+y^3+z^3 = 0$$

and

$$xyz = 0.$$

The latter equation will be satisfied if either $x=0$ or $y=0$ or $z=0$. When $x=0$, the former equation reduces to $y^3+z^3=0$, which gives us the solutions

$$(0,1,-\rho), (0,1,-1), (0,1,-\rho^2)$$

Likewise, when $y=0$, it reduces to $x^3+z^3=0$ with the solutions

$$(-\rho,0,1), (-1,0,1), (-\rho^2,0,1)$$

and, when $z=0$, it reduces to $x^3+y^3=0$, with solutions

$$(1,-\rho,0), (1,-1,0), (1,-\rho^2,0).$$

$\square$