# unital path algebras

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Let $Q$ be a quiver and $k$ an arbitrary field.

**Proposition.** The path algebra $kQ$ is unitary if and only if $Q$ has a finite number of vertices.

*Proof.* „$\Rightarrow$" Assume, that $Q$ has an infinite number of vertices and let $1 \in kQ$ be an identity. Then we can express 1 as

$$1 = \sum_{i=1}^{n} \lambda_n \cdot w_n$$

where $\lambda_n \in k$ and $w_n$ are paths (they form a basis of $kQ$ as a vector space). Since $Q$ has an infinite number of vertices, then we can take a stationary path $e_x$ for some vertex $x$ such that there is no path among $w_1, \ldots, w_n$ ending in $x$. By definition of $kQ$ and by the fact that 1 is an identity we have:

$$e_x = 1 \cdot e_x = \left( \sum_{i=1}^{n} \lambda_n \cdot w_n \right) \cdot e_x = \sum_{i=1}^{n} \lambda_n \cdot (w_n \cdot e_x) = \sum_{i=1}^{n} \lambda_n \cdot 0 = 0.$$

Contradiction. □

„$\Leftarrow$" If the set $Q_0$ of vertices of $Q$ is finite, then put

$$1 = \sum_{q \in Q_0} e_q$$

where $e_q$ denotes the stationary path (note that 1 is well-defined, since the sum is finite). If $w$ is a path in $Q$ from $x$ to $y$, then $e_x \cdot w = w$ and $w \cdot e_y = w$. All other combinations of $w$ with $e_q$ yield 0 and thus we obtain that

$$1 \cdot w = w = w \cdot 1.$$

This completes the proof. □