regular ideal

Canonical name	RegularIdeal
Date of creation	2013-03-22 15:43:05
Last modified on	2013-03-22 15:43:05
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	12
Author	pahio (2872)
Entry type	Definition
Classification	msc 14K99
Classification	msc 16D25
Classification	msc 11N80
Classification	msc 13A15
Related topic	QuasiRegularIdeal
Related topic	QuasiRegularity

An ideal $\mathfrak{a}$ of a ring R is called a , iff $\mathfrak{a}$ a regular element of R.

Proposition. If m is a positive integer, then the only regular ideal in the residue class ring $\mathbb{Z}_m$ is the unit ideal (1).

Proof. The ring $\mathbb{Z}_m$ is a principal ideal ring. Let (n) be any regular ideal of the ring $\mathbb{Z}_m$. Then n can not be zero divisor, since otherwise there would be a non-zero element r of $\mathbb{Z}_m$ such that $nr = 0$ and thus every element sn of the principal ideal would satisfy $(sn)r = s(nr) = s0 = 0$. So, n is a regular element of $\mathbb{Z}_m$ and therefore we have $\gcd(m,\, n) = 1$. Then, according to `http://planetmath.org/BezoutsLemma`Bézout's lemma, there are such integers x and y that $1 = xm+yn$. This equation gives the congruence $1 \equiv yn \pmod{m}$, i.e. $1 = yn$ in the ring $\mathbb{Z}_m$. With 1 the principal ideal (n) contains all elements of $\mathbb{Z}_m$, which means that $(n) = \mathbb{Z}_m = (1)$.

Note. The above notion of "regular ideal" is used in most books concerning ideals of commutative rings, e.g. [1]. There is also a different notion of "regular ideal" mentioned in [2] (p. 179): Let I be an ideal of the commutative ring R with non-zero unity. This ideal is called *regular*, if the quotient ring R/I is a regular ring, in other words, if for each $a \in R$ there exists an element $b \in R$ such that $a^2b - a \in I$.

References

[1] M. Larsen and P. McCarthy: "*Multiplicative theory of ideals*". Academic Press. New York (1971).

[2] D. M. Burton: "*A first course in rings and ideals*". Addison-Wesley. Reading, Massachusetts (1970).