First Isomorphism Theorem for quivers

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Let $Q=(Q_0,Q_1,s,t)$ and $Q'=(Q'_0,Q'_1,s',t')$ be quivers. Assume, that $F:Q\to Q'$ is a morphism of quivers. Define an equivalence relation $\sim$ on Q as follows: for any $a,b\in Q_0$ and any $\alpha,\beta\in Q_1$ we have

$$a\sim_0 b \text{ if and only if } F_0(a)=F_0(b);$$

$$\alpha\sim_1 \beta \text{ if and only if } F_1(\alpha)=F_1(\beta).$$

It can be easily checked that $\sim=(\sim_0,\sim_1)$ is an equivalence relation on Q.

Using standard techniques we can prove the following:

First Isomorphism Theorem for quivers. The mapping

$$\overline{F}:(Q/\sim)\to \mathrm{Im}(F)$$

(where on the left side we have `http://planetmath.org/QuotientQuiver`the quotient quiver and on the right side `http://planetmath.org/SubquiverAndImageOfAQuiver`the image of a quiver) given by

$$\overline{F}_0([a])=F_0(a),\quad \overline{F}_1([\alpha])=F_1(\alpha)$$

is an isomorphism of quivers.

Proof. It easily follows from the definition of $\sim$ that $\overline{F}$ is a well-defined morphism of quivers. Thus it is enough to show, that $\overline{F}$ is both ,,onto" and ,,1-1" (in the sense that corresponding components of $\overline{F}$ are).

1. We will show, that $\overline{F}$ is onto, i.e. both $\overline{F}_0,\overline{F}_1$ are onto. Let $b\in\mathrm{Im}(F)_0$ and $\beta\in\mathrm{Im}(F)_1$. By definition

   $$F_0(a)=b,\quad F_1(\alpha)=\beta$$

   for some $a\in Q_0$, $\alpha\in Q_1$. It follows that

   $$\overline{F}_0([a])=b,\quad \overline{F}_1([\alpha])=\beta.$$

   which completes this part.

2. $\overline{F}$ is injective. Indeed, if

   $$\overline{F}_0([a])=\overline{F}_0([b])$$

   then $F_0(a)=F_0(b)$. But then $a\sim_0 b$ and thus $[a]=[b]$. Analogously we prove the statement for $\overline{F}_1$.

This completes the proof. $\square$