path algebra of a disconnected quiver

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Let Q be a disconnected quiver, i.e. Q can be written as a disjoint union of two quivers Q' and Q'' (which means that there is no path starting in Q' and ending in Q'' and vice versa) and let k be an arbitrary field.

Proposition. The path algebra kQ is isomorphic to the product of path algebras $kQ' \times kQ''$.

Proof. If w is a path in Q, then w belongs either to Q' or Q''. Define linear map

$$T : kQ \to kQ' \times kQ''$$

by $T(w) = (w, 0)$ if $w \in Q'$ or $T(w) = (0, w)$ if $w \in Q''$ and extend it linearly to entire kQ. We will show that T is an isomorphism of algebras.

If w, w' are paths in Q, then since Q' and Q'' are disjoint, then each of them entirely lies in Q' or Q''. Now since Q' and Q'' don't have common vertices it follows that $w \cdot w' = w' \cdot w = 0$. Without loss of generality we may assume, that w is in Q' and w' is in Q''. Then we have

$$T(w \cdot w') = T(0) = (0, 0) = (w, 0) \cdot (0, w') = T(w) \cdot T(w').$$

If both lie in the same component, for example in Q', then

$$T(w \cdot w') = (w \cdot w', 0) = (w, 0) \cdot (w', 0) = T(w) \cdot T(w').$$

Since T preservers multiplication on paths, then T preserves multiplication and thus T is an algebra homomorphism.

Obviously by definition T is 1-1.

It remains to show, that T is onto. Assume that $(a, b) \in kQ' \oplus kQ''$. Then we can write

$$(a, b) = \sum_{i,j} \lambda_{i,j}(v_i, w_j) = \sum_{i,j} \lambda_{i,j}(v_i, 0) + \sum_{i,j} \lambda_{i,j}(0, w_j),$$

where v_i are paths in Q' and w_j are paths in Q''. It can be easily checked, that

$$T\left(\sum_{i,j} \lambda_{i,j}(v_i + w_j)\right) = (a, b).$$

Here we consider all v_i and w_j as paths in Q.

Thus T is an isomorphism, which completes the proof. $\square$