example of quasi-affine variety that is not affine

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Let k be an algebraically closed field. Then the affine plane $\mathbb{A}^2$ is certainly affine. If we remove the point $(0, 0)$, then we obtain a quasi-affine variety A.

The ring of regular functions of A is the same as the ring of regular functions of $\mathbb{A}^2$. To see this, first observe that the two varieties are clearly birational, so they have the same function field. Clearly also any function regular on $\mathbb{A}^2$ is regular on A. So let f be regular on A. Then it is a rational function on $\mathbb{A}^2$, and its poles (if any) have codimension one, which means they will have support on A. Thus it must have no poles, and therefore it is regular on $\mathbb{A}^2$.

We know that the morphisms $A \to \mathbb{A}^2$ are in natural bijection with the morphisms from the coordinate ring of $\mathbb{A}^2$ to the coordinate ring of A; so isomorphisms would have to correspond to automorphisms of $k[X, Y]$, but this is just the set of invertible linear transformations of X and Y; none of these yield an isomorphism $A \to \mathbb{A}^2$.

Alternatively, one can use Čech cohomology to show that $H^1(A, \mathcal{O}_A)$ is nonzero (in fact, it is infinite-dimensional), while every affine variety has zero higher cohomology groups.

For further information on this sort of subject, see Chapter I of Hartshorne's (which lists this as exercise I.3.6). See the bibliography for algebraic geometry for this and other books.