Serre duality

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The most general version of Serre duality states that on certain schemes X of dimension n, including all projective varieties over any algebraically closed field k, there is a natural perfect http://planetmath.org/BilinearMappairing

$$\mathrm{Ext}^i(\mathcal{F},\omega) \times H^{n-i}(X,\mathcal{F}) \to k,$$

where $\mathcal{F}$ is any coherent sheaf on X and ω is a sheaf, called the *dualizing sheaf*. Here "perfect" means that the natural map above induces an isomorphism

$$\mathrm{Ext}^i(\mathcal{F},\omega) \cong \mathrm{Hom}(H^{n-i}(X,\mathcal{F}),k).$$

In special cases, this reduces to more approachable forms. If X is nonsingular (or more generally, Cohen-Macaulay), then ω is simply $\bigwedge^n \Omega$, where Ω is the sheaf of differentials on X.

If $\mathcal{F}$ is locally free, then

$$\mathrm{Ext}^i(\mathcal{F},\omega) \cong \mathrm{Ext}^i(\mathcal{O}_X,\mathcal{F}^* \otimes \omega) \cong H^i(X,\mathcal{F}^* \otimes \omega),$$

so that we obtain the somewhat more familiar looking fact that there is a perfect pairing $H^i(X,\mathcal{F}^* \otimes \omega) \times H^{n-i}(X,\mathcal{F}) \to k$.

While Serre duality is not in a strict sense a generalization of Poincaré duality, they are philosophically similar, and both fit into a larger pattern on duality results.