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projective space

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Projective space and homogeneous coordinates. Let $\mathbb{K}$ be a field. *Projective space* of dimension n over $\mathbb{K}$, typically denoted by $\mathbb{K}\mathrm{P}^n$, is the set of lines passing through the origin in $\mathbb{K}^{n+1}$. More formally, consider the equivalence relation $\sim$ on the set of non-zero points $\mathbb{K}^{n+1}\backslash\{0\}$ defined by

$$\mathbf{x} \sim \lambda\mathbf{x}, \quad \mathbf{x} \in \mathbb{K}^{n+1}\backslash\{0\}, \quad \lambda \in \mathbb{K}\backslash\{0\}.$$

Projective space is defined to be the set of the corresponding equivalence classes.

Every $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{K}^{n+1}\backslash\{0\}$ determines an element of projective space, namely the line passing through $\mathbf{x}$. Formally, this line is the equivalence class $[\mathbf{x}]$, or $[x_0 : x_1 : \ldots : x_n]$, as it is commonly denoted. The numbers $x_0, \ldots, x_n$ are referred to as *homogeneous coordinates* of the line. Homogeneous coordinates differ from ordinary coordinate systems in that a given element of projective space is labeled by multiple homogeneous "coordinates".

Affine coordinates. Projective space also admits a more conventional type of coordinate system, called affine coordinates. Let $A_0 \subset \mathbb{K}\mathrm{P}^n$ be the subset of all elements $p = [x_0 : x_1 : \ldots : x_n] \in \mathbb{K}\mathrm{P}^n$ such that $x_0 \neq 0$. We then define the functions

$$X_i : A_0 \to \mathbb{K}^n, \quad i = 1, \ldots, n,$$

according to

$$X_i(p) = \frac{x_i}{x_0},$$

where $(x_0, x_1, \ldots, x_n)$ is *any* element of the equivalence class representing p. This definition makes sense because other elements of the same equivalence class have the form

$$(y_0, y_1, \ldots, y_n) = (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$$

for some non-zero $\lambda \in \mathbb{K}$, and hence

$$\frac{y_i}{y_0} = \frac{x_i}{x_0}.$$

The functions $X_1, \ldots, X_n$ are called *affine coordinates* relative to the hyperplane

$$H_0 = \{x_0 = 1\} \subset \mathbb{K}^{n+1}.$$

Geometrically, affine coordinates can be described by saying that the elements of A_0 are lines in $\mathbb{K}^{n+1}$ that are not parallel to H_0, and that every such line intersects H_0 in one and exactly one point. Conversely points of H_0 are represented by tuples $(1, x_1, \ldots, x_n)$ with $(x_1, \ldots, x_n) \in \mathbb{K}^n$, and each such point uniquely labels a line $[1 : x_1 : \ldots : x_n]$ in A_0.

It must be noted that a single system of affine coordinates does not cover *all* of projective space. However, it is possible to define a system of affine coordinates relative to every hyperplane in $\mathbb{K}^{n+1}$ that does not contain the origin. In particular, we get $n+1$ different systems of affine coordinates corresponding to the hyperplanes $\{x_i = 1\},\ i = 0, 1, \ldots, n$. Every element of projective space is covered by at least one of these $n+1$ systems of coordinates.

Projective automorphisms. A projective automorphism, also known as a projectivity, is a bijective transformation of projective space that preserves all incidence relations. For $n \geq 2$, every automorphism of $\mathbb{K}\mathrm{P}^n$ is engendered by a semilinear invertible transformation of $\mathbb{K}^{n+1}$. Let $A : \mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ be an invertible semilinear transformation. The corresponding projectivity $[A] : \mathbb{K}\mathrm{P}^n \to \mathbb{K}\mathrm{P}^n$ is the transformation

$$[\mathbf{x}] \mapsto [A\mathbf{x}], \quad \mathbf{x} \in \mathbb{K}^{n+1}.$$

For every non-zero $\lambda \in \mathbb{K}$ the transformation λA gives the same projective automorphism as A. For this reason, it is convenient we identify the group of projective automorphisms with the quotient

$$\mathrm{P\Gamma L}_{n+1}(\mathbb{K}) = \Gamma\mathrm{L}_{n+1}(\mathbb{K})/\mathbb{K}.$$

Here $\Gamma\mathrm{L}$ refers to the group of invertible semi-linear transformations, while the quotienting $\mathbb{K}$ refers to the subgroup of scalar multiplications.

A collineation is a special kind of projective automorphism, one that is engendered by a strictly linear transformation. The group of projective collineations is therefore denoted by $\mathrm{PGL}_{n+1}(\mathbb{K})$ Note that for fields such as $\mathbb{R}$ and $\mathbb{C}$, the group of projective collineations is also described by the projectivizations $\mathrm{PSL}_{n+1}(\mathbb{R}), \mathrm{PSL}_{n+1}(\mathbb{C})$, of the corresponding unimodular group.

Also note that if a field, such as $\mathbb{R}$, lacks non-trivial automorphisms, then all semi-linear transformations are linear. For such fields all projective automorphisms are collineations. Thus,

$$\mathrm{P\Gamma L}_{n+1}(\mathbb{R}) = \mathrm{PSL}_{n+1}(\mathbb{R}) = \mathrm{SL}_{n+1}(\mathbb{R})/\{\pm\, I_{n+1}\}.$$

By contrast, since $\mathbb{C}$ possesses non-trivial automorphisms, complex conjugation for example, the group of automorphisms of complex projective space is larger than $\mathrm{PSL}_{n+1}(\mathbb{C})$, where the latter denotes the quotient of $\mathrm{SL}_{n+1}(\mathbb{C})$ by the subgroup of scalings by the $(n+1)$st roots of unity.