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algebraic sets and polynomial ideals

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Defines	zero set
Defines	algebraic set
Defines	ideal of an algebraic set
Defines	affine algebraic set

Suppose k is a field. Let $\mathbb{A}_k^n$ denote affine n-space over k.
For $S \subseteq k[x_1, \ldots, x_n]$, define $V(S)$, the *zero set of* S, by

$$V(S) = \{(a_1, \ldots, a_n) \in k^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in S\}$$

We say that $Y \subseteq \mathbb{A}_k^n$ is an (affine) *algebraic set* if there exists $T \subseteq k[x_1, \ldots, x_n]$ such that $Y = V(T)$. Taking these subsets of $\mathbb{A}_k^n$ as a definition of the closed sets of a topology induces the Zariski topology over $\mathbb{A}_k^n$.
For $Y \subseteq \mathbb{A}_k^n$, define the *deal of* Y *in* $k[x_1, \ldots, x_n]$ by

$$I(Y) = \{f \in k[x_1, \ldots, x_n] \mid f(P) = 0 \text{ for all } P \in Y\}.$$

It is easily shown that $I(Y)$ is an ideal of $k[x_1, \ldots, x_n]$.
Thus we have defined a function V mapping from subsets of $k[x_1, \ldots, x_n]$ to algebraic sets in $\mathbb{A}_k^n$, and a function I mapping from subsets of $\mathbb{A}^n$ to ideals of $k[x_1, \ldots, x_n]$.
We remark that the theory of algebraic sets presented herein is most cleanly stated over an algebraically closed field. For example, over such a field, the above have the following properties:

1. $S_1 \subseteq S_2 \subseteq k[x_1, \ldots, x_n]$ implies $V(S_1) \supseteq V(S_2)$.
2. $Y_1 \subseteq Y_2 \subseteq \mathbb{A}_k^n$ implies $I(Y_1) \supseteq I(Y_2)$.
3. For any ideal $\mathfrak{a} \subset k[x_1, \ldots, x_n]$, $I(V(\mathfrak{a})) = \mathrm{Rad}(\mathfrak{a})$.
4. For any $Y \subset \mathbb{A}_k^n$, $V(I(Y)) = \overline{Y}$, the closure of Y in the Zariski topology.

From the above, we see that there is a 1-1 correspondence between algebraic sets in $\mathbb{A}_k^n$ and radical ideals of $k[x_1, \ldots, x_n]$. Furthermore, an algebraic set $Y \subseteq \mathbb{A}_k^n$ is an affine variety if and only if $I(Y)$ is a prime ideal. As an example of how things can go wrong, the radical ideals (1) and $(x^2 + 1)$ in $\mathbb{R}[x]$ define the same zero locus (the empty set) inside of $\mathbb{R}$, but are not the same ideal, and hence there is no such 1-1 correspondence.