prime spectrum

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1 Spec as a set

Let R be any commutative ring with identity. The *prime spectrum* $\operatorname{Spec}(R)$ of R is defined to be the set

$$\{P \subsetneq R \mid P \text{ is a prime ideal of } R\}.$$

For any subset A of R, we define the *variety* of A to be the set

$$V(A) := \{P \in \operatorname{Spec}(R) \mid A \subset P\} \subset \operatorname{Spec}(R)$$

It is enough to restrict attention to subsets of R which are ideals, since, for any subset A of R, we have $V(A) = V(I)$ where I is the ideal generated by A. In fact, even more is true: $V(I) = V(\sqrt{I})$ where $\sqrt{I}$ denotes the radical of the ideal I.

2 Spec as a topological space

We impose a topology on $\operatorname{Spec}(R)$ by defining the sets $V(A)$ to be the collection of closed subsets of $\operatorname{Spec}(R)$ (that is, a subset of $\operatorname{Spec}(R)$ is open if and only if it equals the complement of $V(A)$ for some subset A). The equations

$$\begin{aligned} \bigcap_{\alpha} V(I_\alpha) &= V\left(\bigcup_{\alpha} I_\alpha\right) \\ \bigcup_{i=1}^{n} V(I_i) &= V\left(\bigcap_{i=1}^{n} I_i\right), \end{aligned}$$

for any ideals I_α, I_i of R, establish that this collection does constitute a topology on $\operatorname{Spec}(R)$. This topology is called the *Zariski topology* in light of its relationship to the Zariski topology on an algebraic variety (see Section **??** below). Note that a point $P \in \operatorname{Spec}(R)$ is closed if and only if $P \subset R$ is a maximal ideal.

A *distinguished open set* of $\operatorname{Spec}(R)$ is defined to be an open set of the form

$$\operatorname{Spec}(R)_f := \{P \in \operatorname{Spec}(R) \mid f \notin P\} = \operatorname{Spec}(R) \setminus V(\{f\}),$$

for any element $f \in R$. The collection of distinguished open sets forms a topological basis for the open sets of $\operatorname{Spec}(R)$. In fact, we have

$$\operatorname{Spec}(R) \setminus V(A) = \bigcup_{f \in A} \operatorname{Spec}(R)_f.$$

The topological space $\operatorname{Spec}(R)$ has the following additional properties:

- $\operatorname{Spec}(R)$ is compact (but almost never Hausdorff).
- A subset of $\operatorname{Spec}(R)$ is an irreducible closed set if and only if it equals $V(P)$ for some prime ideal P of R.
- For $f \in R$, let R_f denote the localization of R at f. Then the topological spaces $\operatorname{Spec}(R)_f$ and $\operatorname{Spec}(R_f)$ are naturally homeomorphic, via the correspondence sending a prime ideal of R not containing f to the induced prime ideal in R_f.
- For $P \in \operatorname{Spec}(R)$, let R_P denote the localization of R at the prime ideal P. Then the topological spaces $V(P) \subset \operatorname{Spec}(R)$ and $\operatorname{Spec}(R_P)$ are naturally homeomorphic, via the correspondence sending a prime ideal of R contained in P to the induced prime ideal in R_P.

3 Spec as a sheaf

For convenience, we adopt the usual convention of writing X for $\operatorname{Spec}(R)$. For any $f \in R$ and $P \in X_f$, let $\iota_{f,P} : R_f \longrightarrow R_P$ be the natural inclusion map. Define a presheaf of rings $\mathcal{O}_X$ on X by setting

$$\mathcal{O}_X(U) := \left\{ (s_P) \in \prod_{P \in U} R_P \;\middle|\; \begin{array}{l} U \text{ has an open cover } \{X_{f_\alpha}\} \text{ with elements } s_\alpha \in R_{f_\alpha} \\ \text{such that } s_P = \iota_{f_\alpha,P}(s_\alpha) \text{ whenever } P \in X_{f_\alpha} \end{array} \right\},$$

for each open set $U \subset X$. The restriction map $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ is the map induced by the projection map

$$\prod_{P \in U} R_P \longrightarrow \prod_{P \in V} R_P,$$

for each open subset $V \subset U$. The presheaf $\mathcal{O}_X$ satisfies the following properties:

1. $\mathcal{O}_X$ is a sheaf.

2. $\mathcal{O}_X(X_f) = R_f$ for every $f \in R$.

3. The stalk $(\mathcal{O}_X)_P$ is equal to R_P for every $P \in X$. (In particular, X is a locally ringed space.)

4. The restriction sheaf of $\mathcal{O}_X$ to X_f is isomorphic as a sheaf to $\mathcal{O}_{\mathrm{Spec}(R_f)}$.

4 Relationship to algebraic varieties

$\mathrm{Spec}(R)$ is sometimes called an *affine scheme* because of the close relationship between affine varieties in $\mathbb{A}_k^n$ and the Spec of their corresponding coordinate rings. In fact, the correspondence between the two is an equivalence of categories, although a complete statement of this equivalence requires the notion of *morphisms of schemes* and will not be given here. Nevertheless, we explain what we can of this correspondence below.

Let k be a field and write as usual $\mathbb{A}_k^n$ for the vector space k^n. Recall that an affine variety V in $\mathbb{A}_k^n$ is the set of common zeros of some prime ideal $I \subset k[X_1, \dots, X_n]$. The coordinate ring of V is defined to be the ring $R := k[X_1, \dots, X_n]/I$, and there is an embedding $i : V \hookrightarrow \mathrm{Spec}(R)$ given by

$$i(a_1, \dots, a_n) := (X_1 - a_1, \dots, X_n - a_n) \in \mathrm{Spec}(R).$$

The function i is not a homeomorphism, because it is not a bijection (its image is contained inside the set of maximal ideals of R). However, the map i does define an order preserving bijection between the open sets of V and the open sets of $\mathrm{Spec}(R)$ in the Zariski topology. This isomorphism between these two lattices of open sets can be used to equate the sheaf $\mathrm{Spec}(R)$ with the structure sheaf of the variety V, showing that the two objects are identical in every respect except for the minor detail of $\mathrm{Spec}(R)$ having more points than V.

The additional points of $\mathrm{Spec}(R)$ are valuable in many situations and a systematic study of them leads to the general notion of schemes. As just one example, the classical Bezout's theorem is only valid for algebraically closed fields, but admits a scheme–theoretic generalization which holds over non–algebraically closed fields as well. We will not attempt to explain the theory of schemes in detail, instead referring the interested reader to the references below.

Remark. The spectrum Spec(R) of a ring R may be generalized to the case when R is not commutative, as long as R contains the multiplicative identity. For a ring R with 1, the Spec(R), like above, is the set of all proper prime ideals of R. This definition is used to develop the noncommutative version of Hilbert's Nullstellensatz.

References

[1] Robin Hartshorne, *Algebraic Geometry*, Springer–Verlag New York, Inc., 1977 (GTM **52**).

[2] David Mumford, *The Red Book of Varieties and Schemes, Second Expanded Edition*, Springer–Verlag, 1999 (LNM **1358**).

[3] Louis H. Rowen, *Ring Theory, Vol. 1*, Academic Press, New York, 1988.