Chow's theorem

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For the purposes of this entry, let us define as any complex analytic variety of $\mathbb{P}^n$, the n dimensional complex projective space. Let $\sigma\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the natural projection. That is, the map that takes $(z_1, \ldots, z_{n+1})$ to $[z_1 : \ldots : z_{n+1}]$ in homogeneous coordinates. We define *algebraic projective variety* of $\mathbb{P}^n$ as a set $\sigma(V)$ where $V \subset \mathbb{C}^{n+1}$ is the common zero set of a finite family of homogeneous holomorphic polynomials. It is not hard to show that $\sigma(V)$ is a in the above sense. Usually an algebraic projective variety is just called a *projective variety* partly because of the following theorem.

Theorem (Chow)**.** *Every complex analytic projective variety is algebraic.*

We follow the proof by Cartan, Remmert and Stein. Note that the application of the Remmert-Stein theorem is the key point in this proof.

Proof. Suppose that we have a complex analytic variety $X \in \mathbb{P}^n$. It is not hard to show that that $\sigma^{-1}(X)$ is a complex analytic subvariety of $\mathbb{C}^{n+1} \setminus \{0\}$. By the theorem of Remmert-Stein the set $V = \sigma^{-1}(X) \cup \{0\}$ is a subvariety of $\mathbb{C}^{n+1}$. Furthermore V is a complex cone, that is if $z = (z_1, \ldots, z_{n+1}) \in V$, then $tz \in V$ for all $t \in \mathbb{C}$.

Final step is to show that if a complex analytic subvariety $V \subset \mathbb{C}^{n+1}$ is a complex cone, then it is given by the vanishing of finitely many homogeneous polynomials. Take a finite set of defining functions of V near the origin. I.e. take $f_1, \ldots, f_k$ defined in some open ball $B = B(0, \epsilon)$, such that in $B \cap V = \{z \in B \mid f_1(z) = \cdots = f_k(z) = 0\}$. We can suppose that ϵ is small enough that the power series for f_j converges in B for all j. Expand f_j in a power series near the origin and group together homogeneous terms as $f_j = \sum_{m=0}^{\infty} f_{jm}$, where f_{jm} is a homogeneous polynomial of degree m. For $t \in \mathbb{C}$ we write

$$f_j(tz) = \sum_{m=0}^{\infty} f_{jm}(tz) = \sum_{m=0}^{\infty} t^m f_{jm}(z)$$

For a fixed $z \in V$ we know that $f_j(tz) = 0$ for all $|t| < 1$, hence we have a power series in one variable that is identically zero, and so all coefficients are zero. Thus f_{jm} vanishes on $V \cap B$ and hence on V. It follows that V is defined by a family of homogeneous polynomials. Since the ring of polynomials is Noetherian we need only finitely many, and we are done. $\square$

References

[1] Hassler Whitney. . Addison-Wesley, Philippines, 1972.