morphisms of path algebras induced from morphisms of quivers

Canonical name	MorphismsOfPathAlgebrasInducedFromMorphismsOfQuivers
Date of creation	2013-03-22 19:17:03
Last modified on	2013-03-22 19:17:03
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	6
Author	joking (16130)
Entry type	Definition
Classification	msc 14L24

Let $Q=(Q_0,Q_1,s,t)$, $Q'=(Q'_0,Q'_1,s',t')$ be quivers and let $F:Q\to Q'$ be a morphism of quivers.

Proposition 1. If $w=(\alpha_1,\ldots,\alpha_n)$ is a path in Q, then

$$F(w)=\big(F_1(\alpha_1),\ldots,F_1(\alpha_n)\big)$$

is a path in Q'.

Proof. Indeed, for any $i=1,\ldots,n-1$ we calculate

$$t'\big(F_1(\alpha_i)\big)=F_0\big(t(\alpha_i)\big)=F_0\big(s(\alpha_{i+1})\big)=t'\big(F_1(\alpha_{i+1})\big),$$

which completes the proof. □

Proposition 2. Let w,u be paths in Q. If w is http://planetmath.org/PathAlgebraOfAQuive with u then $F(w)$ is http://planetmath.org/PathAlgebraOfAQuivercompatible with $F(u)$. The inverse implication holds if and only if F_0 is an injective function.

Proof. Assume that we have the following presentations:

$$w=(w_1,\ldots,w_n);$$

$$u=(u_1,\ldots,u_n).$$

If $t(w_n)=s(u_1)$, then

$$t'\big(F_1(w_n)\big)=F_0\big(t(w_n)\big)=F_0\big(s(u_1)\big)=s'\big(F_1(u_1)\big)$$

which shows the first part of the thesis.

For the second part note, that if F_0 is injective, then the above equalities can be reversed to obtain that $t(w_n)=s(u_1)$.

On the other hand assume that F_0 is not injective, i.e. $F_0(a)=F_0(b)$ for some distinct vertices $a,b\in Q_0$. Then for stationary paths e_a and e_b we have that

$$t'(F_1(e_a))=F_0(t(e_a))=F_0(a)=F_0(b)=F_0(s(e_b))=s'(F_1(e_b))$$

so paths $(F_1(e_a))$ and $(F_1(e_b))$ are http://planetmath.org/PathAlgebraOfAQuivercompatible, although (e_a), (e_b) are not. □

Definition. Let k be a field. The linear map

$$\overline{F}:kQ\to kQ'$$

defined on a basis of kQ by

$$\overline{F}(w) = F(w)$$

is said to be **induced from** F.

Proposition 3. The linear map $\overline{F} : kQ \to kQ'$ induced from $F : Q \to Q'$ is a homomorphism of algebras if and only if F_0 is injective.

Proof. Indeed, we will show that $\overline{F}$ preservers multiplication of http://planetmath.org/PathAl paths. If

$$w = (w_1, \ldots, w_n);$$

$$u = (u_1, \ldots, u_m)$$

are http://planetmath.org/PathAlgebraOfAQuivercompatible paths in Q, then

$$\overline{F}(w \cdot u) = \overline{F}\big((w_1, \ldots, w_n, u_1, \ldots, u_m)\big) = \big(F_1(w_1), \ldots, F_1(w_n), F_1(u_1), \ldots F_1(u_m)\big) = \overline{F}(w) \cdot \overline{F}(u),$$

which completes this part.

Now assume that w, u are paths, which are not http://planetmath.org/PathAlgebraOfAQuive
If F_0 is injective, then by proposition 2 $F(w)$ and $F(u)$ are also not http://planetmath.org/PathAl and thus

$$\overline{F}(w \cdot u) = \overline{F}(0) = 0 = \overline{F}(w) \cdot \overline{F}(u).$$

On the other hand, if F_0 is not injective, then there are paths w, u which are not http://planetmath.org/PathAlgebraOfAQuivercompatible, but $F(w)$, $F(u)$ are. Assume, that $\overline{F}$ is a homomorphism of algebras. Then

$$0 = \overline{F}(0) = \overline{F}(w \cdot u) = \overline{F}(w) \cdot \overline{F}(u) \neq 0$$

because of the http://planetmath.org/PathAlgebraOfAQuivercompatibility. The contradiction shows that $\overline{F}$ is not a homomorphism of algebras. This completes the proof. □