minimal model for an elliptic curve

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Let K be a local field, complete with respect to a discrete valuation ν (for example, K could be $\mathbb{Q}_p$, the field of http://planetmath.org/node/PAdicIntegers$p$-adic numbers, which is complete with respect to the http://planetmath.org/node/PAdicValuati adic valuation).

Let E/K be an elliptic curve defined over K given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where a_1, a_2, a_3, a_4, a_6 are constants in K. By a suitable change of variables, we may assume that $\nu(a_i) \geq 0$. As it is pointed out in http://planetmath.org/node/Weierstrass entry, any other Weierstrass equation for E is obtained by a change of variables of the form

$$x = u^2x' + r, \quad y = u^3y' + su^2x' + t$$

with $u, r, s, t \in K$ and $u \neq 0$. Moreover, by Proposition 2 in the same entry, the discriminants of both equations satisfy $\Delta = u^{12}\Delta'$, so they only differ by a 12th power of a non-zero number in K. Let us define a set:

$$S = \{\nu(\Delta) : \Delta \text{ is the discriminant of a Weierstrass eq. for } E \text{ and } \nu(\Delta) \geq 0\}$$

Since ν is a discrete valuation, the set S is a set of non-negative integers, therefore it has a minimum value $m \in S$. Moreover, by the remark above, m satisfies $0 \leq m < 12$ and m is the unique number $t \in S$ with $0 \leq t < 12$.

Definition. *Let E/K be an elliptic curve over a local field K, complete with respect to a discrete valuation ν. A Weierstrass equation for E with discriminant Δ is said to be a minimal model for E (at ν) if $\nu(\Delta) = m$, the minimum of the set S above.*

It follows from the discussion above that every elliptic curve over a local field K has a minimal model over K.

Definition. *Let F be a number field and let ν be an infinite or finite place (archimedean or non-archimedean prime) of F. Let E/F be an elliptic curve over F. A given Weierstrass model for E/F is said to be minimal at ν if the same model is minimal over F_ν, the completion of F at ν. A Weierstrass equation for E/F is said to be minimal if it is minimal at ν for all places ν of F.*

It can be shown that all elliptic curves over $\mathbb{Q}$ have a global minimal model. However, this is not true over general number fields. There exist elliptic curves over a number field F which do not have a global minimal model (i.e. any given model is not minimal at ν for every ν).