modules over bound quiver algebra and bound quiver representations

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Let (Q, I) be a bound quiver over a fixed field k. Denote by Mod$A$ (resp. mod$A$) the category of all (resp. all finite-dimensional) (right) modules over algebra A and by $\mathrm{REP}_{Q,I}$ (resp. $\mathrm{rep}_{Q,I}$) the category of all (resp. all finite-dimensional, (see `http://planetmath.org/QuiverRepresentationsAndRepresentationMorphis` entry for details) bound representations.

We will also allow $I = 0$ (which is an admissible ideal only if lengths of paths in Q are bounded, in particular when Q is finite and acyclic). In this case bound representations are simply representations.

Theorem. If Q is a connected and finite quiver, I and admissible ideal in kQ and $A = kQ/I$, then there exists a k-equivalence of categories

$$F : \mathrm{Mod}A \to \mathrm{REP}_{Q,I}$$

which restricts to the equivalence of categories

$$F' : \mathrm{mod}A \to \mathrm{rep}_{Q,I}.$$

Sketch of the proof. We will only define functor F and its quasi-inverse G. For proof that F is actually an equivalence please see [**?**, Theorem 1.6] (this not difficult, but rather technical proof).

Let e_a be a stationary path in $a \in Q_0$ and put $\epsilon_a = e_a + I \in A$. Now if M is a module in Mod$A$, then define a representation

$$F(M) = (M_a, M_\alpha)$$

by putting $M_a = M\epsilon_a$ (M is a right module over A). Now for an arrow $\alpha \in Q_1$ define $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ by putting $M_\alpha(x) = x\overline{\alpha}$, where $\overline{\alpha} = \alpha + I \in A$. It can be shown (see [**?**]) that $F(M)$ is a bound representation.

On module morphisms F acts as follows. If $f : M \to M'$ is a module morphism, then define

$$F(f) = (f_a)_{a \in Q_0}$$

where $f_a : M_a \to M'_a$ is a restriction, i.e. $f_a(x) = f(x)$. It can be shown that f_a is well-defined (i.e. $f_a(x) \in M'_a$) and in this manner F is a functor.

The inverse functor is defined on objects as follows: for a representation (M_a, M_α) put

$$G(M) = \bigoplus_{a \in Q_0} M_a.$$

Now we will define right kQ-module structure on $G(M)$. For a stationary path e_a in $a \in Q_0$ and for $x = (x_a) \in G(M)$ put

$$x \cdot e_a = x_a.$$

Now for a path $w = (a_1, \dots, a_n)$ from a to b in kQ we consider the evaluation map (see `http://planetmath.org/RepresentationsOfABoundQuiver`this entry for details) $f_w : M_a \to M_b$ and we put

$$(x \cdot w)_c = \delta_{bc} f_w(x_a),$$

where δ_{bc} denotes the Kronecker delta. It can be shown that $G(M)$ is a kQ-module with the property that $G(M)I = 0$. In particular $G(M)$ is a kQ/I-module.

Now, if $f = (f_a) : M \to M'$ is a morphism of representations then we define

$$G(f) = \bigoplus_{a \in Q_0} f_a : G(M) \to G(M).$$

It can be shown that $G(f)$ is indeed an A-homomorphism and that G is a functor.

Also, it follows easily from definitions that both F and G take finite-dimensional objects to finite-dimensional.

It remains to show that these two functors are quasi-inverse. For the proof please see [**?**, Theorem 1.6]. □

Corollary. If Q is a finite, connected and acyclic quiver, then there exists an equivalence of categories $\mathrm{Mod}kQ \simeq \mathrm{REP}_Q$ which restricts to the equivalence of categories $\mathrm{mod}kQ \simeq \mathrm{rep}_Q$.

Proof. Since Q is finite and acyclic, then the zero ideal $I = 0$ is admissible (because lengths of paths are bounded, so $R_Q^m = 0$ for some $m \geqslant 1$, where R_Q denotes the arrow ideal). The thesis follows from the theorem. □

References

[1] I. Assem, D. Simson, A. Skowroski, *Elements of the Representation Theory of Associative Algebras, vol 1.*, Cambridge University Press 2006, 2007