proof of Riemann-Roch theorem

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For a divisor D, let $\mathfrak{L}(D)$ be the associated line bundle. By Serre duality, $H^0(\mathfrak{L}(K-D)) \cong H^1(\mathfrak{L}(D))$, so $\ell(D) - \ell(K-D) = \chi(D)$, the Euler characteristic of $\mathfrak{L}(D)$. Now, let p be a point of C, and consider the divisors D and $D+p$. There is a natural injection $\mathfrak{L}(D) \to \mathfrak{L}(D+p)$. This is an isomorphism anywhere away from p, so the quotient $\mathcal{E}$ is a skyscraper sheaf supported at p. Since skyscraper sheaves are flasque, they have trivial higher cohomology, and so $\chi(\mathcal{E}) = 1$. Since Euler characteristics add along exact sequences (because of the long exact sequence in cohomology) $\chi(D+p) = \chi(D) + 1$. Since $\deg(D+p) = \deg(D) + 1$, we see that if Riemann-Roch holds for D, it holds for $D+p$, and vice-versa. Now, we need only confirm that the theorem holds for a single line bundle. $\mathcal{O}_X$ is a line bundle of degree 0. $\ell(0) = 1$ and $\ell(K) = g$. Thus, Riemann-Roch holds here, and thus for all line bundles.