height function

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Definition 1 *Let A be an abelian group. A height function on A is a function $h\colon A \to \mathbb{R}$ with the properties:*

1. *For all $Q \in A$ there exists a constant C_1, depending on A and Q, such that for all $P \in A$:*

$$h(P+Q) \leq 2h(P) + C_1$$

2. *There exists an integer $m \geq 2$ and a constant C_2, depending on A, such that for all $P \in A$:*

$$h(mP) \geq m^2 h(P) - C_2$$

3. *For all $C_3 \in \mathbb{R}$, the following set is finite:*

$$\{P \in A : h(P) \leq C_3\}$$

Examples:

1. For $t = p/q \in \mathbb{Q}$, a fraction in lower terms, define $H(t) = \max\{\mid p \mid, \mid q \mid\}$. Even though this is not a height function as defined above, this is the prototype of what a height function should look like.

2. Let E be an elliptic curve over $\mathbb{Q}$. The function on $E(\mathbb{Q})$, the points in E with coordinates in $\mathbb{Q}$, $h_x\colon E(\mathbb{Q}) \to \mathbb{R}$:

$$h_x(P) = \begin{Bmatrix} \log H(x(P)), & if\ P \neq 0 \\ 0, & if\ P = 0 \end{Bmatrix}$$

is a height function (H is defined as above). Notice that this depends on the chosen Weierstrass model of the curve.

3. The *canonical height* of $E/\mathbb{Q}$ (due to Neron and Tate) is defined by:

$$h_C(P) = 1/2 \lim_{N\to\infty} 4^{(-N)} h_x([2^N]P)$$

where h_x is defined as in (2).

Finally we mention the fundamental theorem of "descent", which highlights the importance of the height functions:

Theorem 1 (Descent) *Let A be an abelian group and let $h\colon A \to \mathbb{R}$ be a height function. Suppose that for the integer m, as in property (2) of height, the quotient group A/mA is finite. Then A is finitely generated.*

References

[1] Joseph H. Silverman, *The Arithmetic of Elliptic Curves*. Springer-Verlag, New York, 1986.