Ostrowski theorem

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Let A be a complex $n \times n$ matrix, $R_i = \sum_{j \neq i} |a_{ij}|, C_j = \sum_{i \neq j} |a_{ij}| \quad 1 \leq i \leq n, 1 \leq j \leq n$. Let's consider, for any $\alpha \in (0,1)$, the circles of this kind: $O_i = \left\{ z \in \mathbf{C} : |z - a_{ii}| \leq R_i^{\alpha} C_i^{1-\alpha} \right\} \quad 1 \leq i \leq n$.

Theorem (A. Ostrowski): For any $\alpha \in (0,1)$, all the eigenvalues of A lie in the union of these n circles:$\sigma(A) \subseteq \bigcup_i O_i$.

Proof. If $R_i = 0$, the theorem says a_{ii} is an eigenvalue, which is obviously true. Let's then concentrate on the $R_i \neq 0$. By eigenvalue definition, we have:

$$(\lambda - a_{ii})x_i = \sum_{j \neq i} a_{ij} x_j$$

so that, recalling Hölder's inequality with $p = 1/\alpha$ and $q = 1/(1-\alpha)$ (to have $p, q > 1$, we must have $\alpha \in (0,1)$)

$$\begin{aligned}
|\lambda - a_{ii}| \, |x_i| &\leq \sum_{j \neq i} |a_{ij}| \, |x_j| \\
&= \sum_{j \neq i} |a_{ij}|^{\alpha} \, |a_{ij}|^{1-\alpha} \, |x_j| \\
&\leq \left(\sum_{j \neq i} (|a_{ij}|^{\alpha})^{1/\alpha} \right)^{\alpha} \left(\sum_{j \neq i} (|a_{ij}|^{1-\alpha} \, |x_j|)^{1/(1-\alpha)} \right)^{1-\alpha} \\
&= \left(\sum_{j \neq i} |a_{ij}| \right)^{\alpha} \left(\sum_{j \neq i} |a_{ij}| \, |x_j|^{1/(1-\alpha)} \right)^{1-\alpha} \\
&= R_i^{\alpha} \left(\sum_{j \neq i} |a_{ij}| \, |x_j|^{1/(1-\alpha)} \right)^{1-\alpha}
\end{aligned}$$

which means

$$\frac{|\lambda - a_{ii}|^{1/(1-\alpha)}}{R_i^{\alpha/(1-\alpha)}} |x_i|^{1/(1-\alpha)} \leq \sum_{j \neq i} |a_{ij}| \, |x_j|^{1/(1-\alpha)}$$

Summing over all i, one obtains

$$\sum_{i=1}^{n} \frac{|\lambda - a_{ii}|^{1/(1-\alpha)}}{R_i^{\alpha/(1-\alpha)}} |x_i|^{1/(1-\alpha)} \leq \sum_{i=1}^{n} \sum_{j \neq i} |a_{ij}| \, |x_j|^{1/(1-\alpha)} = \sum_{j=1}^{n} C_j \, |x_j|^{1/(1-\alpha)}$$

If, for each i, the coefficient of $|x_i|^{1/(1-\alpha)}$ in the first sum would be greater than the coefficient of the same term in the right-hand side, inequality couldn't hold. So we can conclude that at least one index p exists such as

$$\frac{|\lambda - a_{pp}|^{1/(1-\alpha)}}{R_p^{\alpha/(1-\alpha)}} \leq C_p$$

that is

$$|\lambda - a_{pp}| \leq R_p^{\alpha} C_p^{1-\alpha}$$

which is the thesis. □

Remarks:

The Gershgorin theorem is obtained as a limit for $\alpha \to 0$ or for $\alpha \to 1$; in other words, Ostrowski's theorem represents a kind of "continuous deformation" between the two Gershgorin rows and columns sets.

References

[1] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985