# matrix unit

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| Defines | full set of matrix units |

A *matrix unit* is a matrix (over some ring with 1) whose entries are all 0 except in one cell, where it is 1.

For example, among the $3 \times 2$ matrices,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are the matrix units.

Let $A$ and $B$ be $m \times n$ and $p \times q$ matrices over $R$, and $U_{ij}$ an $n \times p$ matrix unit (over $R$). Then

1. $AU_{ij}$ is the $m \times p$ matrix whose $j$th column is the $i$th column of $A$, and 0 everywhere else, and

2. $U_{ij}B$ is the $n \times q$ matrix whose $i$th row is the $j$th row of $B$ and 0 everywhere else.

**Remarks**. Let $M = M_{m \times n}(R)$ be the set of all $m$ by $n$ matrices with entries in a ring $R$ (with 1). Denote $U_{ij}$ the matrix unit in $M$ whose cell $(i, j)$ is 1.

- $M$ is a (left or right) $R$-module generated by the $m \times n$ matrix units.

- When $m = n$, $M$ has the structure of an algebra over $R$. The matrix units have the following properties:

  1. $U_{ij}U_{k\ell} = \delta_{jk}U_{i\ell}$, and
  2. $U_{11} + \cdots + U_{nn} = I_n$,

  where $\delta_{ij}$ is the Kronecker delta and $I_n$ is the identity matrix. Note that the $U_{ii}$ form a complete set of pairwise orthogonal idempotents, meaning $U_{ii}U_{ii} = U_{ii}$ and $U_{ii}U_{jj} = 0$ if $i \neq j$.

- In general, in a matrix ring $S$ (consisting of, say, all $n \times n$ matrices), any set of $n$ matrices satisfying the two properties above is called a *full set of matrix units* of $S$.

- For example, if $\{U_{ij} \mid 1 \leq i, j \leq 2\}$ is the set of $2 \times 2$ matrix units over $\mathbb{R}$, then for any invertible matrix $T$, $\{TU_{ij}T^{-1} \mid 1 \leq i, j \leq 2\}$ is a full set of matrix units.

- If we embed $R$ as a subring of $M_n(R)$, then $R$ is the centralizer of the matrix units of $M_n(R)$, meaning that the only elements in $M_n(R)$ that commute with the matrix units are the elements in $R$.

# References

[1] T. Y. Lam, *Lectures on Modules and Rings*, Springer, New York, 1998.