minimal polynomial (endomorphism)

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Let T be an endomorphism of an n-dimensional vector space V.

Definitions. We define the , $M_T(X)$, to be the unique monic polynomial of such that $M_T(T) = 0$. We say that $P(X)$ is a *zero* for T if $P(T)$ is the zero endomorphism.

Note that the minimal polynomial exists by virtue of the Cayley-Hamilton theorem, which provides a zero polynomial for T.

. Firstly, $\mathrm{End}(V)$ is a vector space of dimension n^2. Therefore the $n^2 + 1$ vectors, $i_v, T, T^2, \ldots T^{n^2}$, are linearly dependant. So there are coefficients, a_i not all zero such that $\sum_{i=0}^{n^2} a_i T^i = 0$. We conclude that a non-trivial zero polynomial for T exists. We take $M_T(X)$ to be a zero polynomial for T of minimal degree with leading coefficient one.

: If $P(X)$ is a zero polynomial for T then $M_T(X) \mid P(X)$.

Proof. By the division algorithm for polynomials, $P(X) = Q(X)M_T(X) + R(X)$ with $degR < degM_T$. We note that $R(X)$ is also a zero polynomial for T and by minimality of $M_T(X)$, must be just 0. Thus we have shown $M_T(X) \mid P(X)$. $\square$

The minimal polynomial has a number of interesting properties:

1. The roots are exactly the eigenvalues of the endomorphism

2. If the minimal polynomial of T splits into linear factors then T is upper-triangular with respect to some basis

3. The minimal polynomial of T splits into *distinct* linear factors (i.e. no repeated roots) if and only if T is diagonal with respect to some basis.

It is then a corollary of the fundamental theorem of algebra that every endomorphism of a finite dimensional vector space over $\mathbb{C}$ may be upper-triangularized.

The minimal polynomial is intimately related to the characteristic polynomial for T. For let $\chi_T(X)$ be the characteristc polynomial. Since $\chi_T(T) = 0$, we have by the above lemma that $M_T(X) \mid \chi_T(X)$. It is also a fact that the eigenvalues of T are exactly the roots of χ_T. So when split into linear factors the only difference between $M_T(X)$ and $\chi_T(X)$ is the algebraic multiplicity of the roots.

In general they may not be the same - for example any diagonal matrix with repeated eigenvalues.