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orthogonal matrices

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A real square $n \times n$ matrix Q is orthogonal if $Q^{\mathrm{T}}Q = I$, i.e., if $Q^{-1} = Q^{\mathrm{T}}$. The rows and columns of an orthogonal matrix form an orthonormal basis.

Orthogonal matrices play a very important role in linear algebra. Inner products are preserved under an orthogonal transform: $(Qx)^{\mathrm{T}}Qy = x^{\mathrm{T}}Q^{\mathrm{T}}Qy = x^{\mathrm{T}}y$, and also the Euclidean norm $||Qx||_2 = ||x||_2$. An example of where this is useful is solving the least squares problem $Ax \approx b$ by solving the equivalent problem $Q^{\mathrm{T}}Ax \approx Q^{\mathrm{T}}b$.

Orthogonal matrices can be thought of as the real case of unitary matrices. A unitary matrix $U \in \mathbb{C}^{n\times n}$ has the property $U^*U = I$, where $U^* = \overline{U^{\mathrm{T}}}$ (the conjugate transpose). Since $\overline{Q^{\mathrm{T}}} = Q^{\mathrm{T}}$ for real Q, orthogonal matrices are unitary.

An orthogonal matrix Q has $\det(Q) = \pm 1$.

Important orthogonal matrices are Givens rotations and Householder transformations. They help us maintain numerical stability because they do not amplify rounding errors.

Orthogonal 2×2 matrices are rotations or reflections if they have the form:

$$\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \text{or} \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}$$

respectively.

This entry is based on content from The Data Analysis Briefbook (`http://rkb.home.cern.ch/r`

References

[1] Friedberg, Insell, Spence. *Linear Algebra.* Prentice-Hall Inc., 1997.