quadratic space

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Defines	norm form
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Defines	generalized quaternion algebra
Defines	regular quadratic space

A *quadratic space* (over a field) is a vector space V equipped with a quadratic form Q on V. It is denoted by (V,Q). The dimension of the quadratic space is the dimension of the underlying vector space. Any vector space admitting a bilinear form has an induced quadratic form and thus is a quadratic space.

Two quadratic spaces (V_1,Q_1) and (V_2,Q_2) are said to be *isomorphic* if there exists an isomorphic linear transformation $T:V_1\to V_2$ such that for any $v\in V_1$, $Q_1(v)=Q_2(Tv)$. Since T is easily seen to be an isometry between V_1 and V_2 (over the symmetric bilinear forms induced by Q_1 and Q_2 respectively), we also say that (V_1,Q_1) and (V_2,Q_2) are isometric.

A quadratic space equipped with a regular quadratic form is called a *regular quadratic space*.

Example of a Qudratic Space. The *Generalized Quaternion Algebra*.

Let F be a field and $a,b\in \dot{F}:=F-\{0\}$. Let H be the algebra over F generated by i,j with the following defining relations:

1. $i^2=a$,
2. $j^2=b$, and
3. $ij=-ji$.

Then $\{1,i,j,k\}$, where $k:=ij$, forms a basis for the vector space H over F. For a direct proof, first note $(ij)^2=(ij)(ij)=i(ji)j=i(-ij)j=-ab\neq 0$, so that $k\in\dot{F}$. It's also not hard to show that k anti-commutes with both i,j: $ik=-ki$ and $jk=-kj$. Now, suppose $0=r+si+tj+uk$. Multiplying both sides of the equation on the right by i gives $0=ri+sa+tji+uki$. Multiplying both sides on the left by i gives $0=ri+sa+tij+uik$. Adding the two results and reduce, we have $0=ri+sa$. Multiplying this again by i gives us $0=ra+sai$, or $0=r+si$. Similarly, one shows that $0=r+tj$, so that $si=tj$. This leads to two equations, $sa=tij$ and $sa=tji$, if one multiplies it on the left and right by i. Adding the results then dividing by 2 gives $sa=0$. Since $a\neq 0$, $s=0$. Therefore, $0=r+si=r$. Same argument shows that $t=u=0$ as well.

Next, for any element $\alpha=r+si+tj+uk\in H$, define its conjugate $\overline{\alpha}$ by $r-si-tj-uk$. Note that $\alpha=\overline{\alpha}$ iff $\alpha\in F$. Also, it's not hard to see that

- $\overline{\overline{\alpha}}=\alpha$,
- $\overline{\alpha+\beta}=\overline{\alpha}+\overline{\beta}$,

- $\overline{\alpha\beta} = \overline{\beta}\overline{\alpha}$,

We next define the norm N on H by $N(\alpha) = \alpha\overline{\alpha}$. Since $\overline{N(\alpha)} = \overline{\alpha\overline{\alpha}} = \overline{\overline{\alpha}}\overline{\alpha} = \alpha\overline{\alpha} = N(\alpha)$, $N(\alpha) \in F$. It's easy to see that $N(r\alpha) = r^2 N(\alpha)$ for any $r \in F$.

Finally, if we define the trace T on H by $T(\alpha) = \alpha + \overline{\alpha}$, we have that $N(\alpha+\beta) - N(\alpha) - N(\beta) = T(\alpha\overline{\beta})$ is bilinear (linear each in α and β).

Therefore, N defines a quadratic form on H (N is commonly called a *norm form*), and H is thus a quadratic space over F. H is denoted by

$$\left(\frac{a,b}{F}\right).$$

It can be shown that H is a central simple algebra over F. Since H is four dimensional over F, it is a quaternion algebra. It is a direct generalization of the quaternions $\mathbb{H}$ over the reals

$$\left(\frac{-1,-1}{\mathbb{R}}\right).$$

In fact, every quaternion algebra (over a field F) is of the form $\left(\frac{a,b}{F}\right)$ for some $a, b \in F$.