proof of rank-nullity theorem

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Let $T : V \to W$ be a linear mapping, with V finite-dimensional. We wish to show that

$$\dim V = \dim \operatorname{Ker} T + \dim \operatorname{Img} T$$

The images of a basis of V will span $\operatorname{Img} T$, and hence $\operatorname{Img} T$ is finite-dimensional. Choose then a basis $w_1, \ldots, w_n$ of $\operatorname{Img} T$ and choose preimages $v_1, \ldots, v_n \in U$ such that

$$w_i = T(v_i), \quad i = 1 \ldots n$$

Choose a basis $u_1, \ldots, u_k$ of $\operatorname{Ker} T$. The result will follow once we show that $u_1, \ldots, u_k, v_1, \ldots, v_n$ is a basis of V.

Let $v \in V$ be given. Since $T(v) \in \operatorname{Img} T$, by definition, we can choose scalars $b_1, \ldots, b_n$ such that

$$T(v) = b_1 w_1 + \ldots b_n w_n.$$

Linearity of T now implies that $T(b_1 v_1 + \ldots + b_n v_n - v) = 0$, and hence we can choose scalars $a_1, \ldots, a_k$ such that

$$b_1 v_1 + \ldots + b_n v_n - v = a_1 u_1 + \ldots a_k u_k.$$

Therefore $u_1, \ldots, u_k, v_1, \ldots, v_n$ span V.

Next, let $a_1, \ldots, a_k, b_1, \ldots, b_n$ be scalars such that

$$a_1 u_1 + \ldots + a_k u_k + b_1 v_1 + \ldots + b_n v_n = 0.$$

By applying T to both sides of this equation it follows that

$$b_1 w_1 + \ldots + b_n w_n = 0,$$

and since $w_1, \ldots, w_n$ are linearly independent that

$$b_1 = b_2 = \ldots = b_n = 0.$$

Consequently

$$a_1 u_1 + \ldots + a_k u_k = 0$$

as well, and since $u_1, \ldots, u_k$ are also assumed to be linearly independent we conclude that

$$a_1 = a_2 = \ldots = a_k = 0$$

also. Therefore $u_1, \ldots, u_k, v_1, \ldots, v_n$ are linearly independent, and are therefore a basis. Q.E.D.