semilinear transformation

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Let K be a field and k its prime subfield. For example, if K is $\mathbb{C}$ then k is $\mathbb{Q}$, and if K is the finite field of order $q = p^i$, then k is $\mathbb{Z}_p$.

Definition 1. *Given a field automorphism θ of K, a function $f : V \to W$ between two K vector spaces V and W is θ-semilinear, or simply semilinear, if for all $x, y \in V$ and $l \in K$ it follows: (shown here first in left hand notation and then in the preferred right hand notation.)*

1. $f(x + y) = f(x) + f(y)$, *(in right hand notation: $(x + y)f = xf + yf$.)*
2. $f(lx) = l^\theta f(x)$, *(in right hand notation: $(lx)f = l^\theta xf$.)*

where l^θ denotes the image of l under θ.

Remark 2. *θ must be a field automorphism for f to remain additive, for example, θ must fix the prime subfield as*

$$n^\theta xf = (nx)f = (x + \cdots + x)f = n(xf).$$

Also

$$(l_1 + l_2)^\theta xf = ((l_1 + l_2)x)f = (l_1x)f + (l_2x)f = (l_1^\theta + l_2^\theta)xf$$

so $(l_1 + l_2)^\theta = l_1^\theta + l_2^\theta$. Finally,

$$(l_1l_2)^\theta xf = ((l_1l_2x)f = l_1^\theta(l_2x)f = l_1^\theta l_2^\theta xf.$$

Every linear transformation is semilinear, but the converse is generally not true. If we treat V and W as vector spaces over k, (by considering K as vector space over k first) then every θ-semilinear map is a k-linear map, where k is the prime subfield of K.

Example

- Let $K = \mathbb{C}$, $V = \mathbb{C}^n$ with standard basis $e_1, \ldots, e_n$. Define the map $f : V \to V$ by
$$f\left(\sum_{i=1} z_ie_i\right) = \sum_{i=1}^{n} \bar{z}_ie_i.$$
f is semilinear (with respect to the complex conjugation field automorphism) but not linear.

- Let $K = GF(q)$ – the Galois field of order $q = p^i$, p the characteristic. Let $l^\theta = l^p$, for $l \in K$. By the Freshman's dream it is known that this is a field automorphism. To every linear map $f : V \to W$ between vector spaces V and W over K we can establish a θ-semilinear map

$$\left(\sum_{i=1} l_i e_i\right) \tilde{f} = \sum_{i=1}^{n} l_i^\theta e_i f.$$

□

Indeed every linear map can be converted into a semilinear map in such a way. This is part of a general observation collected into the following result.

Definition 3. *Given a vector space V, the set of all invertible semilinear maps (over all field automorphisms) is the group $\Gamma L(V)$.*

Proposition 4. *Given a vector space V over K, and k the prime subfield of K, then $\Gamma L(V)$ decomposes as the semidirect product*

$$\Gamma L(V) = GL(V) \rtimes Gal(K/k)$$

where $Gal(K/k)$ is the Galois group of K/k.

Remark 5. *We identify $Gal(K/k)$ with a subgroup of $\Gamma L(V)$ by fixing a basis B for V and defining the semilinear maps:*

$$\sum_{b\in B} l_b b \mapsto \sum_{b\in B} l_b^\sigma b$$

for any $\sigma \in Gal(K/k)$. We shall denoted this subgroup by $Gal(K/k)_B$. We also see these complements to $GL(V)$ in $\Gamma L(V)$ are acted on regularly by $GL(V)$ as they correspond to a change of basis.

Proof. Every linear map is semilinear thus $GL(V) \leq \Gamma L(V)$. Fix a basis B of V. Now given any semilinear map f with respect to a field automorphism $\sigma \in Gal(K/k)$, then define $g : V \to V$ by

$$\left(\sum_{b\in B} l_b b\right) g = \sum_{b\in B} (l_b^{\sigma^{-1}} b) f = \sum_{b\in B} l_b (b) f.$$

As $(B)f$ is also a basis of V, it follows g is simply a basis exchange of V and so linear and invertible: $g \in GL(V)$.

Set $h := g^{-1}f$. For every $v = \sum_{b\in B} l_b \neq 0$ in V,

$$vh = vg^{-1}f = \sum_{b\in B} l_b^{\sigma} b$$

thus h is in the $Gal(K/k)$ subgroup relative to the fixed basis B. This factorization is unique to the fixed basis B. Furthermore, $GL(V)$ is normalized by the action of $Gal(K/k)_B$, so $\Gamma L(V) = GL(V) \rtimes Gal(K/k)$. $\square$

The $\Gamma L(V)$ groups extend the typical classical groups in $GL(V)$. The importance in considering such maps follows from the consideration of projective geometry.

The projective geometry of a vector space V, denoted $PG(V)$, is the lattice of all subspaces of V. Although the typical semilinear map is not a linear map, it does follow that every semilinear map $f : V \to W$ induces an order-preserving map $f : PG(V) \to PG(W)$. That is, every semilinear map induces a projectivity. The converse of this observation is the Fundamental Theorem of Projective Geometry. Thus semilinear maps are useful because they define the automorphism group of the projective geometry of a vector space.

References

[1] Gruenberg, K. W. and Weir, A.J. *Linear Geometry 2nd Ed.* (English) [B] Graduate Texts in Mathematics. 49. New York - Heidelberg - Berlin: Springer-Verlag. X, 198 p. DM 29.10; $ 12.80 (1977).