Householder transformation

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This entry describes the *Householder transformation* $u = Hv$, the most frequently used algorithm for performing QR decomposition. The key object here is the *Householder matrix* H, a symmetric and orthogonal matrix of the form

$$H = I - 2xx^T,$$

where I is the identity matrix and we have used any normalized vector x with $||x||_2^2 = x^T x = 1$.

The Householder transformation zeroes the last $m-1$ elements of a column vector below the first element:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \to \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{with } c = \pm ||v||_2 = \pm \left(\sum_{i=1}^{m} v_i^2 \right)^{1/2}$$

One can verify that

$$x = f \begin{bmatrix} v_1 - c \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \text{with } f = \frac{1}{\sqrt{2c(c - v_1)}}$$

fulfils $x^T x = 1$ and that with $H = I - 2xx^T$ one obtains the vector $\begin{bmatrix} c & 0 & \cdots & 0 \end{bmatrix}^T$.

To perform the decomposition of the $m \times n$ matrix $A = QR$ (with $m \geq n$) we construct an $m \times m$ matrix $H^{(1)}$ to change the $m-1$ elements of the first column to zero. Similarly, an $m-1 \times m-1$ matrix $G^{(2)}$ will change the $m-2$ elements of the second column to zero. With $G^{(2)}$ we produce the $m \times m$ matrix

$$H^{(2)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & G^{(2)} & \\ 0 & & & \end{bmatrix}.$$

After n such orthogonal transformations ($n-1$ times in the case that $m = n$), we let

$$R = H^{(n)} \cdots H^{(2)} H^{(1)} A.$$

R is upper triangular and the orthogonal matrix Q becomes

$$Q = H^{(1)} H^{(2)} \cdots H^{(n)}.$$

In practice the $H^{(i)}$ are never explicitly computed.

References

- Originally from The Data Analysis Briefbook (`http://rkb.home.cern.ch/rkb/titleA.html`