Cauchy-Binet formula

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Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Then the determinant of their product $C = AB$ can be written as a sum of products of minors of A and B:

$$|C| = \sum_{1 \le k_1 < k_2 < \cdots < k_m \le n} A \begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} B \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 2 & \cdots & m \end{pmatrix}.$$

Basically, the sum is over the maximal (m-th order) minors of A and B. See the entry on `http://planetmath.org/MinorOfAMatrix`minors for notation.

If $m > n$, then neither A nor B have minors of rank m, so $|C| = 0$. If $m = n$, this formula reduces to the usual multiplicativity of determinants $|C| = |AB| = |A||B|$.

Proof. Since $C = AB$, we can write its elements as $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. Then its determinant is

$$\begin{aligned} |C| &= \begin{vmatrix} \sum_{k_1=1}^{n} a_{1k_1} b_{k_1 1} & \cdots & \sum_{k_m=1}^{n} a_{1k_m} b_{k_m m} \\ \vdots & \ddots & \vdots \\ \sum_{k_1=1}^{n} a_{mk_1} b_{k_1 1} & \cdots & \sum_{k_m=1}^{n} a_{mk_m} b_{k_m m} \end{vmatrix} \\ &= \sum_{k_1,\ldots,k_m=1}^{n} \begin{vmatrix} a_{1k_1} b_{k_1 1} & \cdots & a_{1k_m} b_{k_m m} \\ \vdots & \ddots & \vdots \\ a_{mk_1} b_{k_1 1} & \cdots & a_{mk_m} b_{k_m m} \end{vmatrix} \\ &= \sum_{k_1,\ldots,k_m=1}^{n} A \begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} b_{k_1 1} b_{k_2 2} \cdots b_{k_m m}. \end{aligned}$$

In both steps above, we have used the property that the determinant is multilinear in the colums of a matrix.

Note that the terms in the last sum with any two k's the same will make the minor of A vanish. And, for $\{k_1, \cdots, k_m\}$'s that differ only by a permutation, the minor of A will simply change sign according to the parity of the permutation. Hence the determinant of C can be rewritten as

$$|C| = \sum_{1 \le k_1 < \cdots < k_m \le n} A \begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\, b_{k_{\sigma(1)} 1} b_{k_{\sigma(2)} 2} \cdots b_{k_{\sigma(m)} m},$$

where S_m is the permutation group on m elements. But the last sum is none other than the determinant $B \left(\begin{smallmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 2 & \cdots & m \end{smallmatrix}\right)$. Hence we write

$$|C| = \sum_{1 \le k_1 < \cdots < k_m \le n} A \begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} B \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 2 & \cdots & m \end{pmatrix},$$

which is the Cauchy-Binet formula. $\square$