example of under-determined polynomial interpolation

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Consider the following interpolation problem:

> *Given* $x_1, y_1, x_2, y_2 \in \mathbb{R}$ *with* $x_1 \neq x_2$ *to determine all cubic polynomials*
>
> $$p(x) = ax^3 + bx^2 + cx + d, \quad x, a, b, c, d \in \mathbb{R}$$
>
> *such that*
>
> $$p(x_1) = y_1, \quad p(x_2) = y_2.$$

This is a linear problem. Let $\mathcal{P}_3$ denote the vector space of cubic polynomials. The underlying linear mapping is the multi-evaluation mapping

$$E : \mathcal{P}_3 \to \mathbb{R}^2,$$

given by

$$p \mapsto \begin{pmatrix} p(x_1) \\ p(x_2) \end{pmatrix}, \quad p \in \mathcal{P}_3$$

The interpolation problem in question is represented by the equation

$$E(p) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $p \in \mathcal{P}_3$ is the unknown. One can recast the problem into the traditional form by taking standard bases of $\mathcal{P}_3$ and $\mathbb{R}^2$ and then seeking all possible $a, b, c, d \in \mathbb{R}$ such that

$$\begin{pmatrix} (x_1)^3 & (x_1)^2 & x_1 & 1 \\ (x_2)^3 & (x_2)^2 & x_2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

However, it is best to treat this problem at an abstract level, rather than mucking about with row reduction. The Lagrange interpolation formula gives us a particular solution, namely the linear polynomial

$$p_0(x) = \frac{x - x_1}{x_2 - x_1} y_1 + \frac{x - x_2}{x_1 - x_2} y_2, \quad x \in \mathbb{R}$$

The general solution of our interpolation problem is therefore given as $p_0 + q$, where $q \in \mathcal{P}_3$ is a solution of the homogeneous problem

$$E(q) = 0.$$

A basis of solutions for the latter is, evidently,

$$q_1(x) = (x - x_1)(x - x_2), \quad q_2(x) = xq_1(x), \qquad x \in \mathbb{R}$$

The general solution to our interpolation problem is therefore

$$p(x) = \frac{x - x_1}{x_2 - x_1} y_1 + \frac{x - x_2}{x_1 - x_2} y_2 + (ax + b)(x - x_1)(x - x_2), \quad x \in \mathbb{R},$$

with $a, b \in \mathbb{R}$ arbitrary. The general under-determined interpolation problem is treated in an entirely analogous manner.