Frobenius method

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Let us consider the linear homogeneous differential equation

$$\sum_{\nu=0}^{n} k_\nu(x)y^{(n-\nu)}(x) \;=\; 0$$

of http://planetmath.org/DifferentialEquationorder n. If the coefficient functions $k_\nu(x)$ are continuous and the coefficient $k_0(x)$ of the highest http://planetmath.org/HigherOrderDerivativesorder derivative does not vanish on a certain interval (resp. a http://planetmath.org/Domain2domain in $\mathbb{C}$), then all solutions $y(x)$ are continuous on this interval (resp.). If all coefficients have the continuous derivatives up to a certain , the same concerns the solutions.

If, instead, $k_0(x)$ vanishes in a point x_0, this point is in general a singular point. After dividing the differential equation by $k_0(x)$ and then getting the form

$$y^{(n)}(x)+\sum_{\nu=1}^{n} c_\nu(x)y^{(n-\nu)}(x) \;=\; 0,$$

some new coefficients $c_\nu(x)$ are discontinuous in the singular point. However, if the discontinuity is so, that the products

$$(x-x_0)c_1(x), \quad (x-x_0)^2c_2(x), \quad \ldots, \quad (x-x_0)^nc_n(x)$$

are continuous, and analytic in x_0, the point x_0 is a regular singular point of the differential equation.

We introduce the so-called *Frobenius method* for finding solution functions in a neighbourhood of the regular singular point x_0, confining us to the case of a http://planetmath.org/DifferentialEquationsecond order differential equation. When we use the http://planetmath.org/Divisionquotient forms

$$(x-x_0)c_1(x) \;:=\; \frac{p(x)}{r(x)}, \quad (x-x_0)^2c_2(x) \;:=\; \frac{q(x)}{r(x)},$$

where $r(x)$, $p(x)$ and $q(x)$ are analytic in a neighbourhood of x_0 and $r(x) \neq 0$, our differential equation reads

$$(x-x_0)^2r(x)y''(x)+(x-x_0)p(x)y'(x)+q(x)y(x) \;=\; 0. \tag{1}$$

Since a change $x - x_0 \mapsto x$ of variable brings to the case that the singular point is the origin, we may suppose such a starting situation. Thus we can study the equation

$$x^2 r(x) y''(x) + x p(x) y'(x) + q(x) y(x) \;=\; 0, \tag{2}$$

where the coefficients have the converging power series expansions

$$r(x) \;=\; \sum_{n=0}^{\infty} r_n x^n, \quad p(x) \;=\; \sum_{n=0}^{\infty} p_n x^n, \quad q(x) \;=\; \sum_{n=0}^{\infty} q_n x^n \tag{3}$$

and

$$r_0 \;\neq\; 0.$$

In the Frobenius method one examines whether the equation (2) allows a series solution of the form

$$y(x) \;=\; x^s \sum_{n=0}^{\infty} a_n x^n \;=\; a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + \ldots, \tag{4}$$

where s is a constant and $a_0 \neq 0$.

Substituting (3) and (4) to the differential equation (2) converts the left hand to

$$\begin{aligned}
&[r_0 s(s-1) + p_0 s + q_0] a_0 x^s + \\
&[[r_0(s+1)s + p_0(s+1) + q_0] a_1 + [r_1 s(s-1) + p_1 s + q_1] a_0] x^{s+1} + \\
&[[r_0(s+2)(s+1) + p_0(s+2) + q_0] a_2 + [r_1(s+1)s + p_1(s+1) + q_1] a_1 + [r_2 s(s-1) + p_2 s + q_2] a_0] x^{s+2} + \ldots
\end{aligned}$$

Our equation seems clearer when using the notations $f_\nu(s) := r_\nu s(s-1) + p_\nu s + q_n u$:

$$f_0(s) a_0 x^s + [f_0(s+1) a_1 + f_1(s) a_0] x^{s+1} + [f_0(s+2) a_2 + f_1(s+1) a_1 + f_2(s) a_0] x^{s+2} + \ldots \;=\; 0 \tag{5}$$

Thus the condition of satisfying the differential equation by (4) is the infinite system of equations

$$\begin{cases}
f_0(s) a_0 \;=\; 0 \\
f_0(s+1) a_1 + f_1(s) a_0 \;=\; 0 \\
f_0(s+2) a_2 + f_1(s+1) a_1 + f_2(s) a_0 \;=\; 0 \\
\quad \ldots \quad \ldots \quad \ldots
\end{cases} \tag{6}$$

In the first , since $a_0 \neq 0$, the *indicial equation*

$$f_0(s) \equiv r_0 s^2 + (p_0 - r_0)s + q_0 \;=\; 0 \tag{7}$$

must be satisfied. Because $r_0 \neq 0$, this quadratic equation determines for s two values, which in special case may coincide.

The first of the equations (6) leaves $a_0\,(\neq 0)$ arbitrary. The next linear equations in a_n allow to solve successively the constants a_1, a_2, ... provided that the first coefficients $f_0(s{+}1)$, $f_0(s{+}2), \ldots$ do not vanish; this is evidently the case when the `http://planetmath.org/Equation`roots of the indicial equation don't differ by an integer (e.g. when the are complex conjugates or when s is the having greater real part). In any case, one obtains at least for one of the of the indicial equation the definite values of the coefficients a_n in the series (4). It is not hard to show that then this series converges in a neighbourhood of the origin.

For obtaining the solution of the differential equation (2) it suffices to have only one solution $y_1(x)$ of the form (4), because another solution $y_2(x)$, linearly independent on $y_1(x)$, is gotten via mere integrations; then it is possible in the cases $s_1{-}s_2 \in \mathbb{Z}$ that $y_2(x)$ has no expansion of the form (4).

References

[1] Pentti Laasonen: *Matemaattisia erikoisfunktioita*. Handout No. 261. Teknillisen Korkeakoulun Ylioppilaskunta; Otaniemi, Finland (1969).