tensor algebra

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Let R be a commutative ring, and M an R-module. The *tensor algebra*

$$\mathcal{T}(M) = \bigoplus_{n=0}^{\infty} \mathcal{T}_n(M)$$

is the graded R-algebra with n^{th} graded component simply the n^{th} tensor power:

$$\mathcal{T}_n(M) = M^{\otimes n} = \overbrace{M \otimes \cdots \otimes M}^{n \text{ times}}, \quad n = 1, 2, \ldots,$$

and $\mathcal{T}_0(M) = R$. The multiplication $m : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ is given by the usual tensor product:

$$m(a, b) = a \otimes b, \quad a \in M^{\otimes n},\ b \in M^{\otimes m}.$$

Remark 1. One can generalize the above definition to cover the case where the ground ring R is non-commutative by requiring that the module M is a bimodule with R acting on both the left and the right.

Remark 2. From the point of view of category theory, one can describe the tensor algebra construction as a functor $\mathcal{T}$ from the category of R-module to the category of R-algebras that is left-adjoint to the forgetful functor $\mathcal{F}$ from algebras to modules. Thus, for M an R-module and S an R-algebra, every module homomorphism $M \to \mathcal{F}(S)$ extends to a unique algebra homomorphism $\mathcal{T}(M) \to S$.