characteristic polynomial

Canonical name	CharacteristicPolynomial
Date of creation	2013-03-22 12:17:47
Last modified on	2013-03-22 12:17:47
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Definition
Classification	msc 15A18
Related topic	Equation
Defines	characteristic equation

Characteristic Polynomial of a Matrix

Let A be a $n \times n$ matrix over some field k. The *characteristic polynomial* $p_A(x)$ of A in an indeterminate x is defined by the determinant:

$$p_A(x) := \det(A - xI) = \begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x \end{vmatrix}$$

Remarks

- The polynomial $p_A(x)$ is an nth-degree polynomial over k.
- If A and B are similar matrices, then $p_A(x) = p_B(x)$, because

$$\begin{aligned} p_A(x) &= \det(A - xI) = \det(P^{-1}BP - xI) \\ &= \det(P^{-1}BP - P^{-1}xIP) = \det(P^{-1})\det(B - xI)\det(P) \\ &= \det(P)^{-1}\det(B - xI)\det(P) = \det(B - xI) = p_B(x) \end{aligned}$$

  for some invertible matrix P.
- The *characteristic equation* of A is the equation $p_A(x) = 0$, and the solutions to which are the eigenvalues of A.

Characteristic Polynomial of a Linear Operator

Now, let T be a linear operator on a vector space V of dimension $n < \infty$. Let α and β be any two ordered bases for V. Then we may form the matrices $[T]_\alpha$ and $[T]_\beta$. The two matrix representations of T are similar matrices, related by a change of bases matrix. Therefore, by the second remark above, we define the *characteristic polynomial* of T, denoted by $p_T(x)$, in the indeterminate x, by

$$p_T(x) := p_{[T]_\alpha}(x).$$

The characteristic equation of T is defined accordingly.