# eigenvalue

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| Related topic | EigenvalueProblem |
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| Defines | eigenvalue |
| Defines | spectral value |

Let $V$ be a vector space over a field $k$, and let $A$ be an endomorphism of $V$ (meaning a linear mapping of $V$ into itself). A scalar $\lambda \in k$ is said to be an *eigenvalue* of $A$ if there is a nonzero $x \in V$ for which

$$Ax = \lambda x \ . \tag{1}$$

Geometrically, one thinks of a vector whose direction is unchanged by the action of $A$, but whose magnitude is multiplied by $\lambda$.

If $V$ is finite dimensional, elementary linear algebra shows that there are several equivalent definitions of an eigenvalue:
(2) The linear mapping

$$B = \lambda I - A$$

i.e. $B : x \mapsto \lambda x - Ax$, has no inverse.
(3) $B$ is not injective.
(4) $B$ is not surjective.
(5) $\det(B) = 0$, i.e. $\det(\lambda I - A) = 0$.

But if $V$ is of infinite dimension, (5) has no meaning and the conditions (2) and (4) are not equivalent to (1). A scalar $\lambda$ satisfying (2) (called a *spectral value* of $A$) need not be an eigenvalue. Consider for example the complex vector space $V$ of all sequences $(x_n)_{n=1}^{\infty}$ of complex numbers with the obvious operations, and the map $A : V \to V$ given by

$$A(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots) \ .$$

Zero is a spectral value of $A$, but clearly not an eigenvalue.

Now suppose again that $V$ is of finite dimension, say $n$. The function

$$\chi(\lambda) = \det(B)$$

is a polynomial of degree $n$ over $k$ in the variable $\lambda$, called the *characteristic polynomial* of the endomorphism $A$. (Note that some writers define the characteristic polynomial as $\det(A - \lambda I)$ rather than $\det(\lambda I - A)$, but the two have the same zeros.)

If $k$ is $\mathbb{C}$ or any other algebraically closed field, or if $k = \mathbb{R}$ and $n$ is odd, then $\chi$ has at least one zero, meaning that $A$ has at least one eigenvalue. In no case does $A$ have more than $n$ eigenvalues.

Although we didn't need to do so here, one can compute the coefficients of $\chi$ by introducing a basis of $V$ and the corresponding matrix for $B$. Unfortunately, computing $n \times n$ determinants and finding roots of polynomials of degree $n$ are computationally messy procedures for even moderately large $n$, so for most practical purposes variations on this naive scheme are needed. See the eigenvalue problem for more information.

If $k = \mathbb{C}$ but the coefficients of $\chi$ are real (and in particular if $V$ has a basis for which the matrix of $A$ has only real entries), then the non-real eigenvalues of $A$ appear in conjugate pairs. For example, if $n = 2$ and, for some basis, $A$ has the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then $\chi(\lambda) = \lambda^2 + 1$, with the two zeros $\pm i$.

Eigenvalues are of relatively little importance in connection with an infinite-dimensional vector space, unless that space is endowed with some additional structure, typically that of a Banach space or Hilbert space. But in those cases the notion is of great value in physics, engineering, and mathematics proper. Look for "spectral theory" for more on that subject.