linear algebra

Canonical name	LinearAlgebra
Date of creation	2013-03-22 12:26:16
Last modified on	2013-03-22 12:26:16
Owner	rmilson (146)
Last modified by	rmilson (146)
Numerical id	10
Author	rmilson (146)
Entry type	Topic
Classification	msc 15-00
Classification	msc 15-01

Linear algebra is the branch of mathematics devoted to the theory of linear structure. The axiomatic treatment of linear structure is based on the notions of a *linear space* (more commonly known as a *vector space*), and a *linear mapping*. Broadly speaking, there are two fundamental questions considered by linear algebra:

- the solution of a linear equation, and
- diagonalization, a.k.a. the eigenvalue problem.

From the geometric point of view, "linear" is synonymous with "straight", and consequently linear algebra can be regarded as the branch of mathematics dealing with lines and planes, as well as with transformations of space that preserve "straightness", e.g. rotations and reflections. The two fundamental questions, in geometric terms, deal with

- the intersection of hyperplanes, and
- the principal axes of an ellipsoid.

Linearity is a very basic notion, and consequently linear algebra has applications in numerous areas of mathematics, science, and engineering. Diverse disciplines, such as differential equations, differential geometry, the theory of relativity, quantum mechanics, electrical circuits, computer graphics, and information theory benefit from the notions and techniques of linear algebra.

Euclidean geometry is related to a specialized branch of linear algebra that deals with linear measurement. Here the relevant notions are length and angle. A typical question is the determination of lines perpendicular to a given plane. A somewhat less specialized branch deals with affine structure, where the key notion is that of area and volume. Here determinants play an essential role.

Yet another branch of linear algebra is concerned with computation, algorithms, and numerical approximation. Important examples of such techniques include: Gaussian elimination, the method of least squares, LU factorization, QR decomposition, Gram-Schmidt orthogonalization, singular value decomposition, and a number of iterative algorithms for the calculation of eigenvalues and eigenvectors.

The following subject outline surveys key topics in linear algebra.

1. **Linear structure.**

(a) **Introduction:** systems of linear equations, Gaussian elimination, matrices, matrix operations.

(b) **Foundations:** fields and vector spaces, subspace, linear independence, basis, ordered basis, dimension, direct sum decomposition.

(c) **Linear mappings:** linearity axioms, kernels and images, injectivity, surjectivity, bijections, compositions, inverses, matrix representations, change of bases, conjugation, similarity.

2. **Affine structure.**

   (a) **Determinants:** characterizing properties, cofactor expansion, permutations, Cramer's rule, classical adjoint.

   (b) **Geometric aspects:** Euclidean volume, orientation, equiaffine transformations, determinants as geometric invariants of linear transformations.

3. **Diagonalization and Decomposition.**

   (a) **Basic notions:** eigenvector, eigenvalue, eigenspace, characteristic polynomial.

   (b) **Obstructions:** imaginary eigenvalues, nilpotent transformations, classification of 2-dimensional real transformations.

   (c) **Structure theory:** invariant subspaces, Cayley-Hamilton theorem, Jordan canonical form, rational canonical form.

4. **Multi-linearity.**

   (a) **Foundations:** vector space dual, bilinearity, bilinear transpose, Gram-Schmidt orthogonalization.

   (b) **Bilinearity:** bilinear forms, symmetric bilinear forms, quadratic forms, signature and Sylvester's theorem, orthogonal transformations, skew-symmetric bilinear forms, symplectic transformations.

   (c) **Tensor algebra:** tensor product, contraction, invariants of linear transformations, symmetry operations.

5. **Euclidean and Hermitian structure.**

(a) **Foundations:** inner product axioms, the adjoint operation, symmetric transformations, skew-symmetric transformations, self-adjoint transformations, normal transformations.

(b) **Spectral theorem:** diagonalization of self-adjoint transformations, diagonalization of quadratic forms.

6. **Computational and numerical methods.**

   (a) **Linear problems:** LU-factorization, QR decomposition, least squares, Householder transformations.

   (b) **Eigenvalue problems:** singular value decomposition, Gauss and Jacobi-Siedel iterative algorithms.