theorem for normal triangular matrices

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Theorem 1 *([?], pp. 82) A square matrix is diagonal if and only if it is normal and triangular.*

Proof. If A is a diagonal matrix, then the complex conjugate A^* is also a diagonal matrix. Since arbitrary diagonal matrices commute, it follows that $A^*A = AA^*$. Thus any diagonal matrix is a normal triangular matrix.

Next, suppose $A = (a_{ij})$ is a normal upper triangular matrix. Thus $a_{ij} = 0$ for $i > j$, so for the diagonal elements in A^*A and AA^*, we obtain

$$\begin{aligned} (A^*A)_{ii} &= \sum_{k=1}^{i} |a_{ki}|^2, \\ (AA^*)_{ii} &= \sum_{k=i}^{n} |a_{ik}|^2. \end{aligned}$$

For $i = 1$, we have

$$|a_{11}|^2 = |a_{11}|^2 + |a_{12}|^2 + \cdots + |a_{1n}|^2.$$

It follows that the only non-zero entry on the first row of A is a_{11}. Similarly, for $i = 2$, we obtain

$$|a_{12}|^2 + |a_{22}|^2 = |a_{22}|^2 + \cdots + |a_{2n}|^2.$$

Since $a_{12} = 0$, it follows that the only non-zero element on the second row is a_{22}. Repeating this for all rows, we see that A is a diagonal matrix. Thus any normal upper triangular matrix is a diagonal matrix.

Suppose then that A is a normal lower triangular matrix. Then it is not difficult to see that A^* is a normal upper triangular matrix. Thus, by the above, A^* is a diagonal matrix, whence also A is a diagonal matrix. $\square$

References

[1] V.V. Prasolov, *Problems and Theorems in Linear Algebra*, American Mathematical Society, 1994.