dimension theorem for symplectic complement (proof)

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We denote by $V^\star$ the dual space of V, i.e., linear mappings from V to $\mathbb{R}$. Moreover, we assume known that $\dim V = \dim V^*$ for any vector space V.

We begin by showing that the mapping $S : V \to V^*$, $a \mapsto \omega(a, \cdot)$ is an linear isomorphism. First, linearity is clear, and since ω is non-degenerate, $\ker S = \{0\}$, so S is injective. To show that S is surjective, we apply the `http://planetmath.org/node/2238`rank-nullity theorem to S, which yields $\dim V = \dim \operatorname{img} S$. We now have $\operatorname{img} S \subset V^*$ and $\dim \operatorname{img} S = \dim V^*$. (The first assertion follows directly from the definition of S.) Hence $\operatorname{img} S = V^*$ (see `http://planetmath.org/VectorSubspace`this page), and S is a surjection. We have shown that S is a linear isomorphism.

Let us next define the mapping $T : V \to W^*$, $a \mapsto \omega(a, \cdot)$. Applying the `http://planetmath.org/node/2238`rank-nullity theorem to T yields

$$\dim V = \dim \ker T + \dim \operatorname{img} T. \tag{1}$$

Now $\ker T = W^\omega$ and $\operatorname{img} T = W^*$. To see the latter assertion, first note that from the definition of T, we have $\operatorname{img} T \subset W^*$. Since S is a linear isomorphism, we also have $\operatorname{img} T \supset W^*$. Then, since $\dim W = \dim W^*$, the result follows from equation **??**. $\Box$