linear transformation is continuous if its domain is finite dimensional

Canonical name	LinearTransformationIsContinuousIfItsDomainIsFiniteDimensional
Date of creation	2013-03-22 15:17:59
Last modified on	2013-03-22 15:17:59
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Last modified by	matte (1858)
Numerical id	7
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Entry type	Theorem
Classification	msc 15A04

Theorem 1. *A linear transformation is continuous if the domain is finite dimensional.*

Proof. Suppose $L\colon X \to Y$ is the transformation, $\dim X = n$, and $\|\cdot\|_X$, $\|\cdot\|_Y$ are the norms on X, Y, respectively. By `http://planetmath.org/ContinuityIsPreservedWhenCo` result and `http://planetmath.org/SubspaceTopologyInAMetricSpace`this result, it suffices to prove that $L\colon X \to L(X)$ is continuous when $L(X)$ is equipped with the topology given by $\|\cdot\|_Y$ restricted onto $L(X)$. Also, since continuity and boundedness are equivalent, it suffices to prove that L is bounded. Let $e_1, \dots, e_n$ be a basis for X such that L is invertible on $\operatorname{span}\{e_1, \dots, e_k\}$ and $\ker L = \operatorname{span}\{e_{k+1}, \dots, e_n\}$ for $k = 1, \dots, n$. (The zero map is always continuous.) Let $f_i = L(e_i)$ for $i = 1, \dots, k$, so that $\operatorname{span}\{f_1, \dots, f_k\} = L(X)$. Let us define new norms on X and $L(X)$,

$$\begin{aligned}
\|x\|'_X &= \sqrt{\sum_{i=1}^{n} \alpha_i^2}, \\
\|y\|'_X &= \sqrt{\sum_{i=1}^{k} \beta_i^2},
\end{aligned}$$

for $x = \sum_{i=1}^n \alpha_i e_i \in X$ and $y = \sum_{i=1}^k \beta_i f_i \in Y$. Since norms on finite dimensional vector spaces are equivalent, it follows that

$$\begin{aligned}
1/C\|x\|'_X \leq \|x\|_X \leq C\|x\|'_X, &\quad x \in X \\
1/D\|y\|'_Y \leq \|y\|_Y \leq D\|y\|'_Y, &\quad y \in L(X)
\end{aligned}$$

for some constants $C, D > 0$. For $x = \sum_{i=1}^n \alpha_i e_i \in X$,

$$\begin{aligned}
\|L(x)\|_Y &\leq D\|\sum_{i=1}^{k} \alpha_i f_i\|'_Y \\
&= D\sqrt{\sum_{i=1}^{k} \alpha_i^2} \\
&\leq D\sqrt{\sum_{i=1}^{n} \alpha_i^2} \\
&= D\|x\|'_X \\
&= CD\|x\|_X.
\end{aligned}$$

Thus $L \colon X \to L(X)$ is bounded. $\square$