direct sum of matrices

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Direct sum of matrices

Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. By the direct sum of A and B, written $A \oplus B$, we mean the $(m+p) \times (n+q)$ matrix of the form

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

where the O's represent zero matrices. The O on the top right is an $m \times q$ matrix, while the O on the bottom left is $n \times p$.

For example, if $A = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 4 & 0 \\ -7 & 8 \end{pmatrix}$, then

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & -7 & 8 \end{pmatrix}$$

Remark. It is not hard to see that the $\oplus$ operation on matrices is associative:

$$(A \oplus B) \oplus C = A \oplus (B \oplus C),$$

because both sides lead to

$$\begin{pmatrix} A & O & O \\ O & B & O \\ O & O & C \end{pmatrix}$$

In fact, we can inductively define the direct sum of n matrices unambiguously.

Direct sums of linear transformations

The direct sum of matrices is closely related to the direct sum of vector spaces and linear transformations. Let A and B be as above, over some field k. We may view A and B as linear transformations $T_A : k^n \to k^m$ and $T_B : k^q \to k^p$ using the standard ordered bases. Then $A \oplus B$ may be viewed as the linear transformation

$$T_{A \oplus B} : k^{n+q} \to k^{m+p}$$

using the standard ordered basis, such that

- the restriction of $T_{A\oplus B}$ to the subspace k^n (embedded in k^{n+q}) is T_A, and

- the restriction of $T_{A\oplus B}$ to k^q is T_B.

The above suggests that we can define direct sums on linear transformations. Let $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$ be linear transformations, where V_i and W_j are finite dimensional vector spaces over some field k such that $V_1 \cap V_2 = 0$. Then define $T_1 \oplus T_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$ such that for any $v \in V_1 \oplus V_2$,

$$(T_1 \oplus T_2)(v_1, v_2) := (T_1(v_1), T_2(v_2))$$

where $v_i \in V_i$. Based on this definition, it is not hard to see that

$$T_{A\oplus B} = T_A \oplus T_B$$

for any matrices A and B.

More generally, if β_i is an ordered basis for V_i, then $\beta := \beta_1 \cup \beta_2$ extending the linear orders on β_i, such that if $v_i \in \beta_1$ and $v_j \in \beta_2$, then $v_i < v_j$ is an ordered basis for $V_1 \oplus V_2$, and

$$[T_1 \oplus T_2]_\beta = [T_1]_{\beta_1} \oplus [T_2]_{\beta_2}.$$