Gerstenhaber - Serezhkin theorem

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Let $\mathbb{F}$ be an arbitrary field. Consider $\mathcal{M}_n(\mathbb{F})$, the vector space of all $n \times n$ matrices over $\mathbb{F}$. Define

- $\mathcal{N} = \{A \in \mathcal{M}_n(\mathbb{F}) : A \text{ is nilpotent}\}$,
- $\mathcal{GL}_n(\mathbb{F}) = \{A \in \mathcal{M}_n(\mathbb{F}) : \det(A) \neq 0\}$,
- $\mathcal{T} = \{A \in \mathcal{M}_n(\mathbb{F}) : A \text{ is strictly upper triangular}\}$.

Notice that $\mathcal{T}$ is a linear subspace of $\mathcal{M}_n(\mathbb{F})$. Moreover, $\mathcal{T} \subseteq \mathcal{N}$ and $\dim \mathcal{T} = n(n-1)/2$.

The Gerstenhaber – Serezhkin theorem on linear subspaces contained in the nilpotent cone [**?**, **?**] reads as follows.

Theorem 1 *Let $\mathcal{L}$ be a linear subspace of $\mathcal{M}_n(\mathbb{F})$. Assume that $\mathcal{L} \subseteq \mathcal{N}$. Then*

(i) $\dim \mathcal{L} \leq n(n-1)/2$,

(ii) $\dim \mathcal{L} = n(n-1)/2$ *if and only if there exists* $U \in \mathcal{GL}_n(\mathbb{F})$ *such that* $\{UAU^{-1} : A \in \mathcal{L}\} = \mathcal{T}$

An alternative simple proof of inequality *(i)* can be found in [**?**].

References

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