skew-Hermitian matrix

Canonical name	SkewHermitianMatrix
Date of creation	2013-03-22 13:36:14
Last modified on	2013-03-22 13:36:14
Owner	matte (1858)
Last modified by	matte (1858)
Numerical id	21
Author	matte (1858)
Entry type	Definition
Classification	msc 15A57
Synonym	anti-Hermitian matrix
Related topic	HermitianMatrix
Related topic	SymmetricMatrix
Related topic	SkewSymmetricMatrix

Definition. A square matrix A with complex entries is *skew-Hermitian*, if

$$A^* = -A.$$

Here $A^* = \overline{A^{\mathrm{T}}}$, A^{T} is the transpose of A, and $\overline{A}$ is is the complex conjugate of the matrix A.

Properties.

1. The trace of a skew-Hermitian matrix is `http://planetmath.org/node/2017`imaginary.
2. The eigenvalues of a skew-Hermitian matrix are `http://planetmath.org/node/2017`imagina

Proof. Property (1) follows directly from property (2) since the trace is the sum of the eigenvalues. But one can also give a simple proof as follows. Let x_{ij} and y_{ij} be the real respectively imaginary parts of the elements in A. Then the diagonal elements of A are of the form $x_{kk} + iy_{kk}$, and the diagonal elements in A^* are of the form $-x_{kk} + iy_{kk}$. Hence x_{kk}, i.e., the real part for the diagonal elements in A must vanish, and property (1) follows. For property (2), suppose A is a skew-Hermitian matrix, and x an eigenvector corresponding to the eigenvalue λ, i.e.,

$$Ax = \lambda x. \quad (1)$$

Here, x is a complex column vector. Since x is an eigenvector, x is not the zero vector, and $x^*x > 0$. Without loss of generality we can assume $x^*x = 1$. Thus

$$\begin{aligned}
\overline{\lambda} &= x^*\overline{\lambda}x \\
&= (x^*\lambda x)^* \\
&= (x^*Ax)^* \\
&= x^*A^*x \\
&= x^*(-A)x \\
&= -x^*\lambda x \\
&= -\lambda.
\end{aligned}$$

Hence the eigenvalue λ corresponding to x is `http://planetmath.org/node/2017`imaginary. □