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Pfaffian

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The *Pfaffian* is an analog of the determinant that is defined only for a $2n \times 2n$ antisymmetric matrix. It is a polynomial of the polynomial ring in elements of the matrix, such that its square is equal to the determinant of the matrix.

The Pfaffian is applied in the generalized Gauss-Bonnet theorem.

Examples

$Pf \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a,$

$Pf \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + dc.$

Standard definition

Let

$$A = \begin{bmatrix} 0 & a_{1,2} & \dots & a_{1,2n} \\ -a_{1,2} & 0 & \dots & a_{2,2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{2n,1} & -a_{2n,2} & \dots & 0 \end{bmatrix}.$$

Let Π be the set of all partition of $\{1, 2, \dots, 2n\}$ into pairs of elements $\alpha \in \Pi$, can be represented as

$$\alpha = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$$

with $i_k < j_k$ and $i_1 < i_2 < \dots < i_n$, let

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & 2n \\ i_1 & j_1 & i_2 & j_2 & \dots & j_n \end{bmatrix}$$

be a corresponding permutation and let us define $sgn(\alpha)$ to be the signature of a permutation π; clearly it depends only on the partition α and not on the particular choice of π. Given a partition α as above let us set $a_\alpha = a_{i_1,j_1} a_{i_2,j_2} \dots a_{i_n,j_n}$, then we can define the *Pfaffian* of A as

$$Pf(A) = \sum_{\alpha \in \Pi} sgn(\alpha) a_\alpha.$$

Alternative definition

One can associate to any antisymmetric $2n \times 2n$ matrix $A = \{a_{ij}\}$ a bivector :$\omega = \sum_{i<j} a_{ij} e_i \wedge e_j$ in a basis $\{e_1, e_2, \dots, e_{2n}\}$ of $\mathbb{R}^{2n}$, then

$$\omega^n = n! Pf(A) e_1 \wedge e_2 \wedge \cdots \wedge e_{2n},$$

where ω^n denotes exterior product of n copies of ω.

Identities

For any antisymmetric $2n \times 2n$ matrix A' and any $2n \times 2n$ matrix B

$$Pf(A)^2 = \det(A)$$
$$Pf(BAB^T) = \det(B) Pf(A)$$