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direct sum of Hermitian and skew-Hermitian matrices

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In this example, we show that any square matrix with complex entries can uniquely be decomposed into the sum of one Hermitian matrix and one skew-Hermitian matrix. A fancy way to say this is that complex square matrices is the direct sum of Hermitian and skew-Hermitian matrices.

Let us denote the vector space (over $\mathbb{C}$) of complex square $n \times n$ matrices by M. Further, we denote by M_+ respectively M_- the vector subspaces of Hermitian and skew-Hermitian matrices. We claim that

$$M = M_+ \oplus M_-. \tag{1}$$

Since M_+ and M_- are vector subspaces of M, it is clear that $M_+ + M_-$ is a vector subspace of M. Conversely, suppose $A \in M$. We can then define

$$\begin{aligned} A_+ &= \frac{1}{2}\big(A + A^*\big), \\ A_- &= \frac{1}{2}\big(A - A^*\big). \end{aligned}$$

Here $A^* = \overline{A}^{\mathrm{T}}$, and $\overline{A}$ is the complex conjugate of A, and A^{T} is the transpose of A. It follows that A_+ is Hermitian and A_- is anti-Hermitian. Since $A = A_+ + A_-$, any element in M can be written as the sum of one element in M_+ and one element in M_-. Let us check that this decomposition is unique. If $A \in M_+ \cap M_-$, then $A = A^* = -A$, so $A = 0$. We have established equation **??**.

Special cases

- In the special case of 1×1 matrices, we obtain the decomposition of a complex number into its real and imaginary components.
- In the special case of real matrices, we obtain the decomposition of a $n \times n$ matrix into a symmetric matrix and anti-symmetric matrix.