calculating the solid angle of disc

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We determine the solid angle formed by a disc when one is looking at it on the normal line of its plane set to the center of it.

Let us look the disc from the origin and let the disc with radius R situate such that its plane is parallel to the xy-plane and the center is on the z-axis at $(0,\,0,\,h)$ with $h>0$. Into the

$$\Omega = -\int_a d\vec{a}\cdot\nabla\frac{1}{r} = \int_a d\vec{a}\cdot\frac{\vec{r}}{|\vec{r}|^3} \tag{1}$$

of the http://planetmath.org/SolidAngleparent entry, we may substitute the position vector $\vec{r} = x\vec{i}+y\vec{j}+h\vec{k}$ of the directed surface element $d\vec{a} = \vec{k}\,da$, getting

$$\Omega = \int_a \frac{h\,da}{(x^2+y^2+h^2)^{3/2}}.$$

Now we can use a http://planetmath.org/Annulus2annulus-formed surface element $da = 2\pi\varrho\,d\varrho$ where $\varrho^2 = x^2+y^2$, whence the surface integral may be calculated as

$$\Omega = \pi h\int_0^R \frac{2\varrho\,d\varrho}{(\varrho^2+h^2)^{3/2}} = \frac{\pi h}{-2}\Big/_{\!\!\varrho=0}^{R}\frac{1}{\sqrt{\varrho^2+h^2}}.$$

Thus we have the result

$$\Omega = 2\pi h\left(\frac{1}{h}-\frac{1}{\sqrt{R^2+h^2}}\right).$$