dyad product

Canonical name	DyadProduct
Date of creation	2013-03-22 15:26:44
Last modified on	2013-03-22 15:26:44
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	15
Author	pahio (2872)
Entry type	Definition
Classification	msc 15A72
Related topic	Frame
Related topic	DotProduct
Related topic	CrossProduct
Related topic	PositionVector
Related topic	KalleVaisala
Defines	dyad
Defines	unit dyad
Defines	product of dyads

A third kind of "products" between two Euclidean vectors $\vec{a}$ and $\vec{b}$, besides the scalar product $\vec{a}\cdot\vec{b}$ and the vector product $\vec{a}\times\vec{b}$, is the *dyad product* $\vec{a}\,\vec{b}$, which is usually denoted without any multiplication symbol. The dyad products and the finite formal sums

$$\Phi := \sum_{\mu} \vec{a}_\mu \vec{b}_\mu \tag{1}$$

of them are called *dyads*.

A dyad is not a vector, but an operator. It on any vector $\vec{v}$ producing from it new vectors or new dyads according to the definitions

$$\Phi * \vec{v} := \sum_{\mu} \vec{a}_\mu (\vec{b}_\mu * \vec{v}), \quad \vec{v} * \Phi := \sum_{\mu} (\vec{v} * \vec{a}_\mu)\vec{b}_\mu. \tag{2}$$

Here the asterisks empty, in which case the vector $\vec{v}$ must be replaced by a scalar v; the products Φv and $v\Phi$ are dyads.

The dyad product obeys the distributive laws

$$\vec{a}(\vec{b}+\vec{c}) = \vec{a}\,\vec{b}+\vec{a}\,\vec{c}, \quad (\vec{b}+\vec{c})\vec{a} = \vec{b}\,\vec{a}+\vec{c}\,\vec{a},$$

which can be verified by multiplying an arbitrary vector $\vec{v}$ and both of these equations and then comparing the results. Likewise, the scalar factor transfer rule is valid. It follows that if we have $\vec{a} = a_1\vec{e_1} + a_2\vec{e_2} + a_3\vec{e_3}$ and $\vec{b} = b_1\vec{e_1} + b_2\vec{e_2} + b_3\vec{e_3}$ in the orthonormal basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ (for the brevity, we confine us to vectors of $\mathbb{R}^3$), their dyad product is the sum

$$\begin{aligned}\vec{a}\,\vec{b} = \; & a_1b_1\vec{e_1}\vec{e_1} + a_1b_2\vec{e_1}\vec{e_2} + a_1b_3\vec{e_1}\vec{e_3} + \\ & a_2b_1\vec{e_2}\vec{e_1} + a_2b_2\vec{e_2}\vec{e_2} + a_2b_3\vec{e_2}\vec{e_3} + \\ & a_3b_1\vec{e_3}\vec{e_1} + a_3b_2\vec{e_3}\vec{e_2} + a_3b_3\vec{e_3}\vec{e_3},\end{aligned}$$

which shows that the dyad product has been formed similarly as the matrix product of the vectors $(a_1, a_2, a_3)^{\mathrm{T}}$ and (b_1, b_2, b_3).

The *unit dyad*

$$\mathrm{I} := \vec{e_1}\vec{e_1}+\vec{e_2}\vec{e_2}+\vec{e_3}\vec{e_3} = \nabla\vec{r},$$

where $\vec{r}$ is the position vector,

$$\mathrm{I}\cdot\vec{v} = \vec{v}\cdot\mathrm{I} = \vec{v}$$

and

$$\mathrm{I}\times(\vec{u}\times\vec{v}) = \vec{v}\,\vec{u} - \vec{u}\,\vec{v}$$

for all vectors $\vec{u}$ and $\vec{v}$.

The *product* of two dyad products $\vec{a}\,\vec{b}$ and $\vec{c}\,\vec{d}$ is defined to be the dyad

$$(\vec{a}\,\vec{b})(\vec{c}\,\vec{d}) := (\vec{b}\cdot\vec{c})(\vec{a}\,\vec{d}) \tag{3}$$

and the product of such dyads as (1) to be the formal sum of individual products (3). The multiplication of dyads is associative and distributive over addition. The unit dyad acts as unity in the ring of dyads:

$$\mathrm{I}\Phi = \Phi\mathrm{I} = \Phi \quad \forall\Phi$$

References

[1] K. VÄISÄLÄ: *Vektorianalyysi*. Werner Söderström Osakeyhtiö, Helsinki (1961).