Kronecker product

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Synonym	tensor product (for matrices)
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Definition. Let $A = (a_{ij})$ be a $n \times n$ matrix and let B be a $m \times m$ matrix. Then the *Kronecker product* of A and B is the $mn \times mn$ block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

The Kronecker product is also known as the *direct product* or the *tensor product* [**?**].

Fundamental properties [**?**, **?**]

1. The product is bilinear. If k is a scalar, and A, B and C are square matrices, such that B and C are of the same order, then

$$\begin{aligned} A \otimes (B + C) &= A \otimes B + A \otimes C, \\ (B + C) \otimes A &= B \otimes A + C \otimes A, \\ k(A \otimes B) &= (kA) \otimes B = A \otimes (kB). \end{aligned}$$

2. If A, B, C, D are square matrices such that the products AC and BD exist, then $(A \otimes B)(C \otimes D)$ exists and

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

   If A and B are invertible matrices, then

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

3. If A and B are square matrices, then for the transpose (A^T) we have

$$(A \otimes B)^T = A^T \otimes B^T.$$

4. Let A and B be square matrices of orders n and m, respectively. If $\{\lambda_i | i = 1, \dots, n\}$ are the eigenvalues of A and $\{\mu_j | j = 1, \dots, m\}$ are the eigenvalues of B, then $\{\lambda_i \mu_j | i = 1, \dots, n,\ j = 1, \dots, m\}$ are the eigenvalues of $A \otimes B$. Also,

$$\begin{aligned} \det(A \otimes B) &= (\det A)^m (\det B)^n, \\ \operatorname{rank}(A \otimes B) &= \operatorname{rank} A \operatorname{rank} B, \\ \operatorname{trace}(A \otimes B) &= \operatorname{trace} A \operatorname{trace} B, \end{aligned}$$

References

[1] H. Eves, *Elementary Matrix Theory*, Dover publications, 1980.

[2] T. Kailath, A.H. Sayed, B. Hassibi, *Linear estimation*, Prentice Hall, 2000