duality with respect to a non-degenerate bilinear form

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Definition 1. *Let V and W be finite dimensional vector spaces over a field F and let $B : V \times W \to F$ be a non-degenerate bilinear form. Then we say that V and W are dual with respect to B.*

Example 1. Let V be a finite dimensional vector space and let $W = V^*$ be the dual space of V, i.e. W is the vector space formed by all linear transformations $V \to F$. Let $B : V \times V^* \to F$ be defined by $B(v, f) = f(v)$ for all $v \in V$ and all $f : V \to F$ in V^*. Then B is a non-degenerate bilinear form and V and V^* are dual with respect to B.

Definition 2. *Let $f : V \to V$ and $g : W \to W$ be linear transformations. We say that f and g are transposes of each other with respect to B if*

$$B(f(v), w) = B(v, g(w))$$

for all $v \in V$ and $w \in W$.

The reasons why the terms "dual" and "transpose" are used are explained in the following theorems (here V^* denotes the dual vector space of V). Notice that for a fixed element $w \in W$ one can define a linear form $V \to F$ which sends v to $B(v, w)$.

Theorem 1. *Let V, W be finite dimensional vector spaces over F which are dual with respect to a non-degenerate bilinear form $B : V \times W \to F$. Then there exist canonical isomorphisms $V \cong W^*$ and $W \cong V^*$ given by*

$$W \to V^*,\ w \mapsto (v \mapsto B(v, w)); \quad V \to W^*,\ v \mapsto (w \mapsto B(v, w)).$$

Theorem 2. *Let V, W be finite dimensional vector spaces over F which are dual with respect to a non-degenerate bilinear form $B : V \times W \to F$. Moreover, suppose $f : V \to V$ and $g : W \to W$ are transposes of each other with respect to B. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V and let $\mathcal{C} = \{w_1, \ldots, w_n\}$ be the basis of W which maps to the dual basis of $\mathcal{B}$ via the isomporphism $W \cong V^*$ defined in the previous theorem. If A is the matrix of f in the basis $\mathcal{B}$ then the matrix of g in the basis $\mathcal{C}$ is A^T, the transpose matrix of A.*

Proof of Theorem 2. Let V and W be dual with respect to a non-degenerate bilinear form B and let f and g be transposes of each other, also with respect to B so that:

$$B(f(v), w) = B(v, g(w))$$

for all $v \in V$ and $w \in W$. By Theorem 1, we have $W \cong V^*$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V and let $\mathcal{C} = \{w_1, \ldots, w_n\}$ be a basis for W which corresponds to the dual basis of V^* via the isomorphism $W \cong V^*$. Then $B(v_i, w_j) = 1$ for $i = j$ and equal to 0 otherwise. Let $A = (\alpha_{ij})$ be the matrix of f with respect to $\mathcal{B}$. Then

$$f(v_j) = \sum_{i=1}^{n} \alpha_{ij} v_i.$$

Let $A' = (\beta_{ij})$ be the matrix of g with respect to $\mathcal{C}$ so that $g(w_j) = \sum_i \beta_{ij} w_i$. We will show that $A' = A^T$, the transpose of A. Indeed:

$$B(f(v_j), w_k) = B(\sum_i \alpha_{ij} v_i, w_k) = \alpha_{kj}$$

and also

$$B(f(v_j), w_k) = B(v_j, g(w_k)) = B(v_j, \sum_i \beta_{ik} w_i) = \beta_{jk}.$$

Therefore $\beta_{jk} = \alpha_{kj}$ for all k and j, as desired. $\square$