complexification of vector space

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Owner	stevecheng (10074)
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0.1 Complexification of vector space

If V is a real vector space, its *complexification* $V^{\mathbb{C}}$ is the complex vector space consisting of elements $x + iy$, where $x, y \in V$. Vector addition and scalar multiplication by complex numbers are defined in the obvious way:

$$\begin{aligned}
(x+iy) + (u+iv) &= (x+u) + i(y+v)\,, & x, y, u, v &\in V \\
(\alpha + i\beta)(x+iy) &= (\alpha x - \beta y) + i(\beta x + \alpha y), & x, y \in V, \alpha, \beta &\in \mathbb{R}\,.
\end{aligned}$$

If $v_1, \ldots, v_n$ is a basis for V, then $v_1 + i0, \ldots, v_n + i0$ is a basis for $V^{\mathbb{C}}$. Naturally, $x + i0 \in V^{\mathbb{C}}$ is often written just as x.

So, for example, the complexification of $\mathbb{R}^n$ is (isomorphic to) $\mathbb{C}^n$.

0.2 Complexification of linear transformation

If $T\colon V \to W$ is a linear transformation between two real vector spaces V and W, its complexification $T^{\mathbb{C}}\colon V^{\mathbb{C}} \to W^{\mathbb{C}}$ is defined by

$$T^{\mathbb{C}}(x+iy) = Tx + iTy\,.$$

It may be readily verified that $T^{\mathbb{C}}$ is complex-linear.

If $v_1, \ldots, v_n$ is a basis for V, $w_1, \ldots, w_m$ is a basis for W, and A is the matrix representation of T with respect to these bases, then A, regarded as a complex matrix, is also the representation of $T^{\mathbb{C}}$ with respect to the corresponding bases in $V^{\mathbb{C}}$ and $W^{\mathbb{C}}$.

So, the complexification process is a formal, coordinate-free way of saying: take the matrix A of T, with its real entries, but operate on it as a complex matrix. The advantage of making this abstracted definition is that we are not required to fix a choice of coordinates and use matrix representations when otherwise there is no need to. For example, we might want to make arguments about the complex eigenvalues and eigenvectors for a transformation $T\colon V \to V$, while, of course, non-real eigenvalues and eigenvectors, by definition, cannot exist for a transformation between real vector spaces. What we really mean are the eigenvalues and eigenvectors of $T^{\mathbb{C}}$.

Also, the complexification process generalizes without change for infinite-dimensional spaces.

0.3 Complexification of inner product

Finally, if V is also a real inner product space, its real inner product can be extended to a complex inner product for $V^{\mathbb{C}}$ by the obvious expansion:

$$\langle x + iy, u + iv \rangle = \langle x, u \rangle + \langle y, v \rangle + i(\langle y, u \rangle - \langle x, v \rangle)\,.$$

It follows that $\|x + iy\|^2 = \|x\|^2 + \|y\|^2$.

0.4 Complexification of norm

More generally, for a real normed vector space V, the equation

$$\|x + iy\|^2 = \|x\|^2 + \|y\|^2$$

can serve as a definition of the norm for $V^{\mathbb{C}}$.

References

[1] Vladimir I. Arnol'd. *Ordinary Differential Equations*. Springer-Verlag, 1992.