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proof of Wielandt-Hoffman theorem

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Since both A and B are normal, they can be diagonalized by unitary transformations:

$$A = V^\dagger C V \quad \text{and} \quad B = W^\dagger D W,$$

where C and D are diagonal, V and W are unitary, and $(\ )^\dagger$ denotes the conjugate transpose. The Frobenius matrix norm is defined by the quadratic form $\|A\|_F^2 = \mathrm{tr}[A^\dagger A]$ and is invariant under unitary transformations, hence

$$\|A - B\|_F^2 = \|V^\dagger C V - W^\dagger D W\|_F^2 = \|C\|_F^2 + \|D\|_F^2 - 2\,\mathrm{Re}\,\mathrm{tr}[C^\dagger U^\dagger D U],$$

where $U = WV^\dagger$. The matrix U is also unitary, let its matrix elements be given by $(U)_{ij} = u_{ij}$. Unitarity implies that the matrix with elements $|u_{ij}|^2$ has its row and column sums equal to 1, in other words, it is doubly stochastic.

The diagonal elements $C_{ii} = a_i$ are eigenvalues of A and $D_{ii} = b_i$ are those of B. Writing out the Frobenius norm explicitly, we get

$$\|A-B\|_F^2 = \sum_i (|a_i|^2+|b_i|^2) - 2\,\mathrm{Re}\sum_{ij} \overline{a}_i |u_{ij}|^2 b_j \geq \sum_i (|a_i|^2+|b_i|^2) - 2\min_S \mathrm{Re}\sum_{ij} \overline{a}_i s_{ij} b_j,$$

where the minimum is taken over all doubly stochastic matrices S, whose elements are $(S)_{ij} = s_{ij}$. By the Birkoff-von Neumann theorem, doubly stochastic matrices form a closed `http://planetmath.org/ConvexSet`convex polyhedron with permutation matrices at the vertices. The expression $\sum_{ij} \overline{a}_i s_{ij} b_j$ is a linear functional on this polyhedron, hence its minimum is achieved at one of the vertices, that is when S is a permutation matrix.

If S represents the permutation σ, its action can be written as $\sum_j s_{ij} b_j = b_{\sigma(i)}$. Finally, we can write the last inequality as

$$\|A - B\|_F^2 \geq \sum_i (|a_i|^2 + |b_{\sigma(i)}|^2) - 2\min_\sigma \mathrm{Re}\sum_{ij} \overline{a}_i b_{\sigma(i)} = \min_\sigma |a_i - b_{\sigma(i)}|^2,$$

which is exactly the statement of the Wielandt-Hoffman theorem.