proof of calculus theorem used in the Lagrange method

Canonical name	ProofOfCalculusTheoremUsedInTheLagrangeMethod
Date of creation	2013-03-22 13:29:51
Last modified on	2013-03-22 13:29:51
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Last modified by	mathcam (2727)
Numerical id	6
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Entry type	Proof
Classification	msc 15A18
Classification	msc 15A42

Let $f(\mathbf{x})$ and $g_i(\mathbf{x}), i = 0, \ldots, m$ be differentiable scalar functions; $\mathbf{x} \in R^n$.

We will find local extremes of the function $f(\mathbf{x})$ where $\nabla f = 0$. This can be proved by contradiction:

$$\nabla f \neq 0$$

$$\Leftrightarrow \exists \epsilon_0 > 0, \forall \epsilon; 0 < \epsilon < \epsilon_0 : f(\mathbf{x} - \epsilon \nabla f) < f(\mathbf{x}) < f(\mathbf{x} + \epsilon \nabla \mathbf{f})$$

but then $f(\mathbf{x})$ is not a local extreme.

Now we put up some conditions, such that we should find the $\mathbf{x} \in S \subset R^n$ that gives a local extreme of f. Let $S = \bigcap_{i=1}^m S_i$, and let S_i be defined so that $g_i(\mathbf{x}) = 0 \forall \mathbf{x} \in S_i$.

Any vector $\mathbf{x} \in R^n$ can have one component perpendicular to the subset S_i (for visualization, think $n = 3$ and let S_i be a flat surface). ∇g_i will be perpendicular to S_i, because:

$$\exists \epsilon_0 > 0, \forall \epsilon; 0 < \epsilon < \epsilon_0 : g_i(\mathbf{x} - \epsilon \nabla g_i) < g_i(\mathbf{x}) < g_i(\mathbf{x} + \epsilon \nabla g_i)$$

But $g_i(\mathbf{x}) = 0$, so any vector $\mathbf{x} + \epsilon \nabla g_i$ must be outside S_i, and also outside S. (todo: I have proved that there might exist a component perpendicular to each subset S_i, but not that there exists only one; this should be done)

By the argument above, ∇f must be zero - but now we can ignore all components of ∇f perpendicular to S. (todo: this should be expressed more formally and proved)

So we will have a local extreme within S_i if there exists a λ_i such that

$$\nabla f = \lambda_i \nabla g_i$$

We will have local extreme(s) within S where there exists a set $\lambda_i, i = 1, \ldots, m$ such that

$$\nabla f = \sum \lambda_i \nabla g_i$$