Peetre's inequality

Canonical name	PeetresInequality
Date of creation	2013-03-22 13:55:26
Last modified on	2013-03-22 13:55:26
Owner	Koro (127)
Last modified by	Koro (127)
Numerical id	10
Author	Koro (127)
Entry type	Theorem
Classification	msc 15-00
Classification	msc 15A39

Theorem **[Peetre's inequality]** [**?**, **?**] If t is a real number and x, y are vectors in $\mathbb{R}^n$, then

$$\Big(\frac{1+|x|^2}{1+|y|^2}\Big)^t \le 2^{|t|}(1+|x-y|^2)^{|t|}.$$

Proof. (Following [**?**].) Suppose b and c are vectors in $\mathbb{R}^n$. Then, from $(|b|-|c|)^2 \ge 0$, we obtain

$$2|b|\cdot|c| \le |b|^2 + |c|^2.$$

Using this inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
1+|b-c|^2 &= 1+|b|^2 - 2b\cdot c + |c|^2 \\
&\le 1+|b|^2 + 2|b||c| + |c|^2 \\
&\le 1+2|b|^2 + 2|c|^2 \\
&\le 2\big(1+|b|^2+|c|^2+|b|^2|c|^2\big) \\
&= 2(1+|b|^2)(1+|c|^2)
\end{aligned}$$

Let us define $a = b - c$. Then for any vectors a and b, we have

$$\frac{1+|a|^2}{1+|b|^2} \le 2(1+|a-b|^2). \tag{1}$$

Let us now return to the given inequality. If $t = 0$, the claim is trivially true for all x, y in $\mathbb{R}^n$. If $t > 0$, then raising both sides in inequality **??** to the power of t, using $t = |t|$, and setting $a = x$, $b = y$ yields the result. On the other hand, if $t < 0$, then raising both sides in inequality **??** to the power to $-t$, using $-t = |t|$, and setting $a = y$, $b = x$ yields the result. □

References

[1] J. Barros-Neta, *An introduction to the theory of distributions*, Marcel Dekker, Inc., 1973.

[2] F. Treves, *Introduction To Pseudodifferential and Fourier Integral Operators*, Vol. I, Plenum Press, 1980.