isotropic quadratic space

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Defines	isotropic vector
Defines	isotropic quadratic form
Defines	anisotropic vector
Defines	anisotropic quadratic form
Defines	anisotropic quadratic space
Defines	totally isotropic quadratic space
Defines	totally isotropic quadratic form

A vector v (an element of V) in a quadratic space (V, Q) is *isotropic* if

1. $v \neq 0$ and

2. $Q(v) = 0$.

Otherwise, it is called *anisotropic*. A quadratic space (V, Q) is isotropic if it contains an isotropic vector. Otherwise, it is anisotropic. A quadratic space (V, Q) is *totally isotropic* if every one of its non-zero vector is isotropic, or that $Q(V) = 0$.

Similarly, an *isotropic quadratic form* is one which has a non-trivial kernel, or that there exists a vector v such that $Q(v) = 0$. The definitions for that of an *anisotropic quadratic form* and that of a *totally isotropic quadratic form* should now be clear from the above discussion (anisotropic: $\ker(Q) = 0$; totally isotropic: $\ker(Q) = V$).

Examples.

- Consider the quadratic form $Q(x, y) = x^2 + y^2$ in the vector space $\mathbb{R}^2$ over the reals. It is clearly anisotropic since there are no real numbers a, b not both 0, such that $a^2 + b^2 = 0$.

- However, the same form is isotropic in $\mathbb{C}^2$ over $\mathbb{C}$, since $1^2 + i^2 = 0$; the complex numbers are algebraically closed.

- Again, using the same form x^2+y^2, but in $\mathbb{R}^3$ over the reals , we see that it is isotropic since the z term is missing, so that $Q(0, 0, 1) = 0^2+0^2 = 0$.

- If we restrict Q to the subspace consisting of the z-axis ($x = y = 0$) and call it Q_z, then Q_z is totally isotropic, and the z-axis is a totally isotropic subspace.

- The quadratic form $Q(x, y) = x^2 - y^2$ is clearly isotropic in any vector space over any field. In general, this is true if the coefficients of a diagonal quadratic form Q consist of $1, -1, 0$ (0 is optional) and nothing else.