generalized Bézout theorem on matrices

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Owner	perucho (2192)
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Author	perucho (2192)
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Generalized Bézout theorem 1. *Let $M[x]$ be an arbitrary matrix polynomial of order n and A a square matrix of the same order. Then, when the matrix polynomial is divided on the right (left) by the characteristic polynomial $xI - A$, the remainder is $M(A)$ $(\widehat{M}(A))$.*

Proof. Consider $M[x]$ given by

$$M[x] = M_0 x^m + M_1 x^{m-1} + \cdots + M_m, \qquad (M_0 \neq 0). \tag{1}$$

The polynomial can also be written as

$$M[x] = x^m M_0 + x^{m-1} M_1 + \cdots + M_m. \tag{2}$$

We are now substituting the scalar argument (real or complex) x by the matrix A and therefore (1) and (2) will, in general, be distinct, as the powers of A need not be permutable with the polynomial matrix coefficients. So that,

$$M(A) = M_0 A^m + M_1 A^{m-1} + \cdots + M_m$$

and

$$\widehat{M}(A) = A^m M_0 + A^{m-1} M_1 + \cdots + M_m,$$

calling $M(A)$ $(\widehat{M}(A))$ the right (left) value of $M[x]$ on substitution of A for x.

If we divide $M[x]$ by the binomial $xI - A$ (I is the correspondent identity matrix), we shall prove that the right (left) remainder R $(\widehat{R})$ does not depend on x. In fact,

$$\begin{aligned}
M[x] =& M_0 x^m + M_1 x^{m-1} + \cdots + M_m \\
=& M_0 x^{m-1}(xI - A) + (M_0 A + M_1) x^{m-1} + M_2 x^{m-2} + \cdots + M_m \\
=& [M_0 x^{m-1} + (M_0 A + M_1) x^{m-2}](xI - A) + (M_0 A^2 + M_1 A + M_2) x^{m-2} + M_3 x^{m-3} + \cdots + M_m \\
=& [M_0 x^{m-1} + (M_0 A + M_1) x^{m-2} + (M_0 A^2 + M_1 A + M_2) x^{m-3}](xI - A) \\
&+ (M_0 A^3 + M_1 A^2 + M_2 A + M_3) x^{m-3} + M_4 x^{m-4} + \cdots + M_m \\
=& [M_0 x^{m-1} + (M_0 A + M_1) x^{m-2} + (M_0 A^2 + M_1 A + M_2) x^{m-3} + \cdots \\
&+ (M_0 A^{m-1} + M_1 A^{m-2} + \cdots + M_{m-1})](xI - A) + M_0 A^m + M_1 A^{m-1} + \cdots + M_m,
\end{aligned}$$

whence we have found that

$$R = M_0 A^m + M_1 A^{m-1} + \cdots + M_m \equiv M(A),$$

and analogously that

$$\widehat{R} = A^m M_0 + A^{m-1} M_1 + \cdots + M_m \equiv \widehat{M}(A),$$

which proves the theorem. $\square$

From this theorem we have the following

Corollary 1. *A polynomial $M[x]$ is divisible by the characteristic polynomial $xI - A$ on the right (left) without remainder iff $M(A) = 0$ ($\widehat{M}(A) = 0$).*