finite-dimensional linear problem

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Defines	system of linear equations

Let $L : U \to V$ be a linear mapping, and let $v \in V$ be given. When both the domain U and codomain V are finite-dimensional, a linear equation

$$L(u) = v,$$

where $u \in U$ is the unknown, can be solved by means of row reduction. To do so, we need to choose a basis $a_1, \ldots, a_m$ of the domain U, and a basis $b_1, \ldots, b_n$ of the codomain V. Let M be the $n \times m$ transformation matrix of L relative to these bases, and let $y \in \mathbb{R}^n$ be the coordinate vector of v relative to the basis of V. Expressing this in terms of matrix notation, we have

$$\begin{bmatrix} L(a_1), \ldots, L(a_m) \end{bmatrix} = \begin{bmatrix} b_1, \ldots, b_n \end{bmatrix} \begin{bmatrix} M_{11} & \ldots & M_{1m} \\ \vdots & \ddots & \vdots \\ M_{n1} & \ldots & M_{nm} \end{bmatrix},$$

$$v = \begin{bmatrix} b_1, \ldots, b_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

We can now restate the abstract linear equation as the matrix-vector equation

$$Mx = y,$$

with $x \in \mathbb{R}^m$ unknown, or equivalently, as the following system of n linear equations

$$\begin{array}{ccccc} M_{11}x_1 + & \cdots & +M_{1m}\,x_m & = & y_1 \\ \vdots & \ddots & \vdots & & \vdots \\ M_{n1}x_1 + & \cdots & +M_{nm}\,x_m & = & y_n \end{array}$$

with $x_1, \ldots, x_m$ unknown. Solutions $u \in U$ of the abstract linear equation $L(u) = v$ are in one-to-one correspondence with solutions of the matrix-vector equation $Mx = y$. The correspondence is given by

$$u = \begin{bmatrix} a_1, \ldots, a_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

Note that the dimension of the domain is the number of variables, while the dimension of the codomain is the number of equations. The equation is called under-determined or over-determined depending on whether the former is greater than the latter, or vice versa. In general, over-determined systems are inconsistent, while under-determined ones have multiple solutions. However, this is a "rule of thumb" only, and exceptions are not hard to find. A full understanding of consistency, and multiple solutions relies on the notions of kernel, image, rank, and is described by the rank-nullity theorem.

Remark. Elementary applications exclusively on the coefficient matrix and the right-hand vector, and neglect to mention the underlying linear mapping. This is unfortunate, because the concept of a linear equation is much more general than the traditional notion of "variables and equations", and relies in an essential way on the idea of a linear mapping. See the `http://planetmath.org/UnderDeterminedPolynomialInterpolation`example on polynomial as a case in point. Polynomial interpolation is a linear problem, but one that is specified abstractly, rather than in terms of variables and equations.