some facts about injective and surjective linear maps

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Let k be a field and V, W be vector spaces over k.

Proposition. Let $f : V \to W$ be an injective linear map. Then there exists a (surjective) linear map $g : W \to V$ such that $g \circ f = \mathrm{id}_V$.

Proof. Of course $\mathrm{Im}(f)$ is a subspace of W so $f : V \to \mathrm{Im}(f)$ is a linear isomorphism. Let $(e_i)_{i \in I}$ be a basis of $\mathrm{Im}(f)$ and $(e_j)_{j \in J}$ be its completion to the basis of W, i.e. $(e_i)_{i \in I \cup J}$ is a basis of W. Define $g : W \to V$ on the basis as follows:

$$g(e_i) = f^{-1}(e_i), \text{ if } i \in I;$$

$$g(e_j) = 0, \text{ if } j \in J.$$

We will show that $g \circ f = \mathrm{id}_V$.

Let $v \in V$. Then

$$f(v) = \sum_{i \in I} \alpha_i e_i,$$

where $\alpha_i \in k$ (note that the indexing set is I). Thus we have

$$(g \circ f)(v) = g(\sum_{i \in I} \alpha_i e_i) = \sum_{i \in I} \alpha_i g(e_i) = \sum_{i \in I} \alpha_i f^{-1}(e_i) =$$

$$= f^{-1}(\sum_{i \in I} \alpha_i e_i) = f^{-1}(f(v)) = v.$$

It is clear that the equality $g \circ f = \mathrm{id}_V$ implies that g is surjective. $\square$

Proposition. Let $g : W \to V$ be a surjective linear map. Then there exists a (injective) linear map $f : V \to W$ such that $g \circ f = \mathrm{id}_V$.

Proof. Let $(e_i)_{i \in I}$ be a basis of V. Since g is onto, then for any $i \in I$ there exist $w_i \in W$ such that $g(w_i) = e_i$. Now define $f : V \to W$ by the formula

$$f(e_i) = w_i.$$

It is clear that $g \circ f = \mathrm{id}_V$, which implies that f is injective. $\square$

If we combine these two propositions, we have the following corollary:

Corollary. There exists an injective linear map $f : V \to W$ if and only if there exists a surjective linear map $g : W \to V$.