motion in central-force field

Canonical name	MotionInCentralforceField
Date of creation	2013-03-22 18:52:41
Last modified on	2013-03-22 18:52:41
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	14
Author	pahio (2872)
Entry type	Derivation
Classification	msc 15A72
Classification	msc 51N20
Synonym	Kepler's first law
Related topic	CommonEquationOfConics
Related topic	PropertiesOfEllipse

Let us consider a body with m in a gravitational force http://planetmath.org/VectorFieldfie
exerted by the origin and directed always from the body towards the origin. Set the plane through the origin and the velocity vector $\vec{v}$ of the body. Apparently, the body is forced to move constantly in this plane, i.e. there is a question of a planar motion. We want to derive the trajectory of the body.

Equip the plane of the motion with a polar coordinate system r, φ and denote the position vector of the body by $\vec{r}$. Then the velocity vector is

$$\vec{v} \;=\; \frac{d\vec{r}}{dt} \;=\; \frac{d}{dt}(r\vec{r}^{\,0}) \;=\; \frac{dr}{dt}\vec{r}^{\,0} + r\frac{d\varphi}{dt}\vec{s}^{\,0}, \tag{1}$$

where $\vec{r}^{\,0}$ and $\vec{s}^{\,0}$ are the unit vectors in the direction of $\vec{r}$ and of $\vec{r}$ rotated 90 degrees anticlockwise ($\vec{r}^{\,0} = \vec{i}\cos\varphi + \vec{j}\sin\varphi$, whence $\frac{\vec{r}^{\,0}}{dt} = (-\vec{i}\sin\varphi + \vec{j}\cos\varphi)\frac{d\varphi}{dt} = \frac{d\varphi}{dt}\vec{s}^{\,0}$). Thus the kinetic energy of the body is

$$E_k \;=\; \frac{1}{2}m\left|\frac{d\vec{r}}{dt}\right|^2 \;=\; \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\varphi}{dt}\right)^2\right).$$

Because the gravitational force on the body is exerted along the position vector, its moment is 0 and therefore the angular momentum

$$\vec{L} \;=\; \vec{r}\times m\frac{d\vec{r}}{dt} \;=\; mr^2\frac{d\varphi}{dt}\vec{r}^{\,0}\times\vec{s}^{\,0}$$

of the body is constant; thus its magnitude is a constant,

$$mr^2\frac{d\varphi}{dt} \;=\; G,$$

whence

$$\frac{d\varphi}{dt} \;=\; \frac{G}{mr^2}. \tag{2}$$

The central force $\vec{F} := -\dfrac{k}{r^2}\vec{r}^{\,0}$ (where k is a constant) has the scalar potential $U(r) = -\frac{k}{r}$. Thus the total energy $E = E_k + U(r)$ of the body, which is constant, may be written

$$E \;=\; \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mr^2\left(\frac{G}{mr^2}\right)^2 - \frac{k}{r} \;=\; \frac{m}{2}\left(\frac{dr}{dt}\right)^2 + \frac{G^2}{2mr^2} - \frac{k}{r}.$$

This equation may be revised to

$$\left(\frac{dr}{dt}\right)^2+\frac{G^2}{m^2r^2}-\frac{2k}{mr}+\frac{k^2}{G^2} \;=\; \frac{2E}{m}+\frac{k^2}{G^2},$$

i.e.

$$\left(\frac{dr}{dt}\right)^2+\left(\frac{k}{G}-\frac{G}{mr}\right)^2 \;=\; q^2$$

where

$$q \;:=\; \sqrt{\frac{2}{m}\left(E+\frac{mk^2}{2G^2}\right)}$$

is a constant. We introduce still an auxiliary angle ψ such that

$$\frac{k}{G}-\frac{G}{mr} \;=\; q\cos\psi, \quad \frac{dr}{dt} \;=\; q\sin\psi. \tag{3}$$

Differentiation of the first of these equations implies

$$\frac{G}{mr^2}\cdot\frac{dr}{dt} \;=\; -q\sin\psi\frac{d\psi}{dt} \;=\; -\frac{dr}{dt}\cdot\frac{d\psi}{dt},$$

whence, by (2),

$$\frac{d\psi}{dt} \;=\; -\frac{G}{mr^2} \;=\; -\frac{d\varphi}{dt}.$$

This means that $\psi = C-\varphi$, where the constant C is determined by the initial conditions. We can then solve r from the first of the equations (3), obtaining

$$r \;=\; \frac{G^2}{km\left(1-\frac{Gq}{k}\cos(C-\varphi)\right)} \;=\; \frac{p}{1-\varepsilon\cos(\varphi-C)}, \tag{4}$$

where

$$p \;:=\; \frac{G^2}{km}, \quad \varepsilon \;:=\; \frac{Gq}{k}.$$

By the `http://planetmath.org/node/11724`parent entry, the result (4) shows that the trajectory of the body in the gravitational `http://planetmath.org/VectorField`fie of one point-like sink is always a conic section whose focus the sink causing the field.

As for the of the conic, the most interesting one is an ellipse. It occurs, by the http://planetmath.org/node/11724parent entry, when $\varepsilon < 1$. This condition is easily seen to be equivalent with a negative total energy E of the body.

One can say that any planet revolves around the Sun along an ellipse having the Sun in one of its foci — this is *Kepler's first law*.

References

[1] Я. Б. Зельдович & А. Д. Мышкис: *Элементы прикладной математики*. Издательство “Наука”. Москва (1976).