proof of cofactor expansion

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Let $M \in mat_N(K)$ be a $n \times n$-matrix with entries from a commutative field K. Let $e_1, \ldots, e_n$ denote the vectors of the canonical basis of K^n. For the proof we need the following

Lemma: Let M^*_{ij} be the matrix generated by replacing the i-th row of M by e_j. Then

$$\det M^*_{ij} = (-1)^{i+j} \det M_{ij}$$

where M_{ij} is the $(n-1) \times (n-1)$-matrix obtained from M by removing its i-th row and j-th column.

Proof. By adding appropriate of the i-th row of M^*_{ij} to its remaining rows we obtain a matrix with 1 at position (i, j) and 0 at positions (k, j) $(k \neq i)$. Now we apply the permutation

$$(12) \circ (23) \circ \cdots \circ ((i-1)i)$$

to rows and

$$(12) \circ (23) \circ \cdots \circ ((j-1)j)$$

to columns of the matrix. The matrix now looks like this:

- Row/column 1 is the vector e_1;
- under row 1 and right of column 1 is the matrix M_{ij}.

Since the determinant has changed its sign $i + j - 2$ times, we have

$$\det M^*_{ij} = (-1)^{i+j} \det M_{ij}.$$

Note also that only those permutations $\pi \in S_n$ are for the computation of the determinant of M^*_{ij} where $\pi(i) = j$. $\square$

Now we start out with

$$\det M = \sum_{\pi \in S_n} \operatorname{sgn}\pi \left(\prod_{j=1}^n m_{j\pi(j)} \right)$$

$$= \sum_{k=1}^n m_{ik} \left(\sum_{\pi \in S_n | \pi(i) = k} \operatorname{sgn}\pi \left(\prod_{1 \leq j \leq i} m_{j\pi(j)} \right) \cdot 1 \cdot \left(\prod_{i \leq j \leq n} m_{j-\pi(j)} \right) \right).$$

From the previous lemma, it follows that the   associated with M_{ik} is the determinant of M^*_{ij}. So we have

$$\det M = \sum_{k=1}^n M_{ik} \left((-1)^{i+k} \det M_{ik} \right).$$