determinant

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Overview

The *determinant* is an algebraic operation that transforms a square matrix M into a scalar. This operation has many useful and important properties. For example, the determinant is zero if and only the matrix M is singular (no inverse exists). The determinant also has an important geometric interpretation as the area of a parallelogram, and more generally as the volume of a higher-dimensional parallelepiped.

The notion of determinant predates matrices and linear transformations. Originally, the determinant was a number associated to a system of n linear equations in n variables. This number "determined" whether the system possessed a unique solution. In this sense, two-by-two determinants were considered by Cardano at the end of the 16th century and ones of arbitrary size (see the definition below) by Leibniz about 100 years later.

Definition

Let M be an $n \times n$ matrix with entries M_{ij} that are elements of a given field[1]. The determinant of M, or $\det M$ for short, is the scalar quantity

$$\det M = \begin{vmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{vmatrix} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) M_{1\pi_1} M_{2\pi_2} \cdots M_{n\pi_n}. \quad (1)$$

The index π in the above sum varies over all the permutations of $\{1, \dots, n\}$ (i.e., the elements of the symmetric group S_n.) Hence, there are $n!$ terms in the defining sum of the determinant. The symbol $\operatorname{sgn}(\pi)$ denotes the parity of the permutation; it is ± 1 according to whether π is an even or odd permutation. Using the Einstein summation convention one can also express the above definition as

$$\det M = \epsilon_{\pi_1 \pi_2 \dots \pi_n} M^{\pi_1}{}_1 M^{\pi_2}{}_2 \cdots M^{\pi_n}{}_n, \quad (2)$$

[1]Most scientific and geometric applications deal with matrices made up of real or complex numbers. However, the determinant of a matrix over any field is well defined sense and has all the properties of the more conventional determinant. Indeed, many properties of the determinant remain valid for matrices with entries in a commutative ring.

where we've raised the first index so that ${M^i}_j = M_{ij}$, and where

$$\epsilon_{\pi_1 \ldots \pi_n} = \operatorname{sgn}(\pi)$$

is known as the Levi-Civita permutation symbol.

By way of example, the determinant of a 2×2 matrix is given by

$$\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} = M_{11}M_{22} - M_{12}M_{21},$$

There are six permutations of the numbers $1, 2, 3$, namely

$$\overset{+}{123},\ \overset{+}{231},\ \overset{+}{312},\ \overset{-}{132},\ \overset{-}{321},\ \overset{-}{213};$$

the overset sign indicates the permutation's signature. Accordingly, the 3×3 deterimant is a sum of the following 6 terms:

$$\begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} = \begin{array}{r} M_{11}M_{22}M_{33} + M_{12}M_{23}M_{31} + M_{13}M_{21}M_{32} \\ -M_{11}M_{23}M_{32} - M_{13}M_{22}M_{31} - M_{12}M_{21}M_{33} \end{array}$$

Remarks and important properties

1. The determinant operation converts matrix multiplication into scalar multiplication;
$$\det(AB) = \det(A)\det(B),$$
where A, B are square matrices of the same size.

2. The determinant operation is multi-linear, and anti-symmetric with respect to the matrix's rows and columns. See the multi-linearity attachment for more details.

3. The determinant of a lower triangular, or an upper triangular matrix is the product of the diagonal entries, since all the other summands in (1) are zero.

4. `http://planetmath.org/SimilarMatrix`Similar matrices have the same determinant. To be more precise, let A and X be square matrices with X invertible. Then,
$$\det(XAX^{-1}) = \det(A).$$

In particular, if we let X be the matrix representing a change of basis, this shows that the determinant is independent of the basis. The same is true of the trace of a matrix. In fact, the whole characteristic polynomial of an endomorphism is definable without using a basis or a matrix, and it turns out that the determinant and trace are two of its coefficients.

5. The determinant of a matrix A is zero if and only if A is singular; that is, if there exists a non-trivial solution to the homogeneous equation

$$A\mathbf{x} = \mathbf{0}.$$

6. The transpose operation does not change the determinant:

$$\det A^{\mathrm{T}} = \det A.$$

7. The determinant of a diagonalizable transformation is equal to the product of its eigenvalues, counted with multiplicities.

8. The determinant is homogeneous of degree n. This means that

$$\det(kM) = k^n \det M, \quad k \text{ is a scalar.}$$