rotational invariance of cross product

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Theorem

Let $\mathbf{R}$ be a rotational 3×3 matrix, i.e., a real matrix with $\det \mathbf{R} = 1$ and $\mathbf{R}^{-1} = \mathbf{R}^T$. Then for all vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$,

$$\mathbf{R} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{R} \cdot \mathbf{u}) \times (\mathbf{R} \cdot \mathbf{v}).$$

Proof. Let us first fix some right hand oriented orthonormal basis in $\mathbb{R}^3$. Further, let $\{u^1, u^2, u^3\}$ and $\{v^1, v^2, v^3\}$ be the components of $\mathbf{u}$ and $\mathbf{v}$ in that basis. Also, in the chosen basis, we denote the entries of $\mathbf{R}$ by R_{ij}. Since $\mathbf{R}$ is rotational, we have $R_{ij}R_{kj} = \delta_{ik}$ where δ_{ik} is the Kronecker delta symbol. Here we use the Einstein summation convention. Thus, in the previous expression, on the left hand side, j should be summed over $1, 2, 3$. We shall use the Levi-Civita permutation symbol ε to write the cross product. Then the i:th coordinate of $\mathbf{u} \times \mathbf{v}$ equals $(\mathbf{u} \times \mathbf{v})^i = \varepsilon^{ijk} u^j v^k$. For the kth component of $(\mathbf{R} \cdot \mathbf{u}) \times (\mathbf{R} \cdot \mathbf{v})$ we then have

$$\begin{aligned} ((\mathbf{R} \cdot \mathbf{u}) \times (\mathbf{R} \cdot \mathbf{v}))^k &= \varepsilon^{imk} R_{ij} R_{mn} u^j v^n \\ &= \varepsilon^{iml} \delta_{kl} R_{ij} R_{mn} u^j v^n \\ &= \varepsilon^{iml} R_{kr} R_{lr} R_{ij} R_{mn} u^j v^n \\ &= \varepsilon^{jnr} \det \mathbf{R} \, R_{kr} u^j v^n. \end{aligned}$$

The last line follows since $\varepsilon^{ijk} R_{im} R_{jn} R_{kr} = \varepsilon^{mnr} \varepsilon^{ijk} R_{i1} R_{j2} R_{k3} = \varepsilon^{mnr} \det \mathbf{R}$. Since $\det \mathbf{R} = 1$, it follows that

$$\begin{aligned} ((\mathbf{R} \cdot \mathbf{u}) \times (\mathbf{R} \cdot \mathbf{v}))^k &= R_{kr} \varepsilon^{jnr} u^j v^n \\ &= R_{kr} (\mathbf{u} \times \mathbf{v})^r \\ &= (\mathbf{R} \cdot \mathbf{u} \times \mathbf{v})^k \end{aligned}$$

as claimed. $\square$