isomorphism of rings of real and complex matrices

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Note that http://planetmath.org/Submatrixsubmatrix notation will be used within this entry. Also, for any positive integer n, $M_{n\times n}(R)$ will be used to denote the ring of $n \times n$ matrices with entries from the ring R, and R_n will be used to denote the following subring of $M_{2n\times 2n}(\mathbb{R})$:

$$R_n = \left\{ P \in M_{2n\times 2n}(\mathbb{R}) : P = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \text{ for some } A, B \in M_{n\times n}(\mathbb{R}) \right\}$$

Theorem. *For any positive integer n, $R_n \cong M_{n\times n}(\mathbb{C})$.*

Proof. Define $\varphi\colon R_n \to M_{n\times n}(\mathbb{C})$ by $\varphi\left(\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\right) = A + iB$ for $A, B \in M_{n\times n}(\mathbb{R})$.

Let $A, B, C, D \in M_{n\times n}(\mathbb{R})$ such that $\varphi\left(\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} C & D \\ -D & C \end{pmatrix}\right)$. Then $A+iB = C+iD$. Therefore, $A = C$ and $B = D$. Hence, $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} C & D \\ -D & C \end{pmatrix}$. It follows that φ is injective.

Let $Z \in M_{n\times n}(\mathbb{C})$. Then there exist $X, Y \in M_{n\times n}(\mathbb{R})$ such that $X + iY = Z$. Since $\varphi\left(\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}\right) = X + iY = Z$, it follows that φ is surjective.

Let $A, B, C, D \in M_{n\times n}(\mathbb{R})$. Then

$$\begin{aligned}
\varphi\left(\begin{pmatrix} A & B \\ -B & A \end{pmatrix} + \begin{pmatrix} C & D \\ -D & C \end{pmatrix}\right) &= \varphi\left(\begin{pmatrix} A+C & B+D \\ -B-D & A+C \end{pmatrix}\right) \\
&= A + C + i(B+D) \\
&= A + iB + C + iD \\
&= \varphi\left(\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\right) + \varphi\left(\begin{pmatrix} C & D \\ -D & C \end{pmatrix}\right)
\end{aligned}$$

and

$$
\begin{aligned}
\varphi\left(\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\begin{pmatrix} C & D \\ -D & C \end{pmatrix}\right) &= \varphi\left(\begin{pmatrix} AC-BD & AD+BC \\ -AD-BC & AC-BD \end{pmatrix}\right) \\
&= AC-BD+i(AD+BC) \\
&= (A+iB)(C+iD) \\
&= \varphi\left(\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\right)\varphi\left(\begin{pmatrix} C & D \\ -D & C \end{pmatrix}\right).
\end{aligned}
$$

It follows that φ is an `http://planetmath.org/RingIsomorphism`isomorphism.

$\square$