tensor density

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0.1 Heuristic definition

A tensor density is a quantity whose transformation law under change of basis involves the determinant of the transformation matrix (as opposed to a tensor, whose transformation law does not involve the determinant).

0.2 Linear Theory

For any real number p, we may define a representation ρ_p of the group $GL(\mathbb{R}^k)$ on the vector space of tensor arrays of rank m, n as follows:

$$(\rho_p(M)T)^{i_1,\ldots,i_n}_{j_1,\ldots j_m} = (\det(M))^p M^{i_1}_{l_1} \cdots M^{i_n}_{l_n} (M^{-1})^{j_1}_{k_1} \cdots (M^{-1})^{j_m}_{k_m} T^{i_1,\ldots,i_n}_{j_1,\ldots j_m}$$

A *tensor density* T of rank m, n and weight p is an element of the vector space on which this representation acts.

Note that if the weight equals zero, the concept of tensor density reduces to that of a tensor.

0.3 Examples

The simplest example of such a quantity is a scalar density. Under a change of basis $y^i = M^i_j x^j$, a scalar density transforms as follows:

$$\rho_p(S) = (\det(M))^p S$$

An important example of a tensor density is the Levi-Civita permutation symbol. It is a density of weight 1 because, under a change of coordinates,

$$(\rho_1 \epsilon)_{j_1,\ldots j_m} = (\det(M))(M^{-1})^{j_1}_{k_1} \cdots (M^{-1})^{j_m}_{k_m} \epsilon^{i_1,\ldots,i_n}_{j_1,\ldots j_m} = \epsilon_{k_1,\ldots k_m}$$

0.4 Tensor Densities on Manifolds

As with tensors, it is possible to define tensor density fields on manifolds. On each coordinate neighborhood, the density field is given by a tensor array of functions. When two neighborhoods overlap, the tensor arrays are related by the change of variable formula

$$T^{i_1,\ldots,i_n}_{j_1,\ldots j_m}(x) = (\det(M))^p M^{i_1}_{l_1} \cdots M^{i_n}_{l_n} (M^{-1})^{j_1}_{k_1} \cdots (M^{-1})^{j_m}_{k_m} T^{i_1,\ldots,i_n}_{j_1,\ldots j_m}(y)$$

where M is the Jacobian matrix of the change of variables.