rotation matrix

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Related topic	DerivationOfRotationMatrixUsingPolarCoordinates
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Related topic	TransitionToSkewAngledCoordinates

Definition 1. *A* rotation matrix *is a (real) orthogonal matrix whose determinant is* $+1$*. All* $n \times n$ *rotation matrices form a group called the* special orthogonal group *and it is denoted by* $\mathrm{SO}(n)$*.*

Examples

1. The identity matrix in $\mathbb{R}^n$ is a rotation matrix.

2. The most general rotation matrix in $\mathbb{R}^2$ can be written as

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

   where $\theta \in \mathbb{R}$. Multiplication (from the left) with this matrix rotates a vector (in $\mathbb{R}^2$) θ radians in the anti-clockwise direction.

Properties

1. Suppose $v \in \mathbb{R}^n$ is a unit vector. Then there exists a rotation matrix R such that $R \cdot v = (1, 0, \ldots, 0)$.

2. In fact, for $v \in \mathbb{R}^n$, $n \geq 3$, there are many rotation matrices $\mathbf{R} \in \mathrm{SO}(n)$ such that $R \cdot v = (1, 0, \ldots, 0)^T$. To see this, let f be the mapping $f\colon \mathrm{SO}(n-1) \to \mathrm{SO}(n)$, defined as

$$f(Q) = \begin{pmatrix} 1 & 0_{1\times n-1} \\ 0_{n-1\times 1} & Q_{n-1\times n-1} \end{pmatrix}.$$

   Then for each $Q \in \mathrm{SO}(n-1)$, $f(Q)$ maps $(1, 0, \ldots, 0)^T$ onto itself. Thus, if $R_0 \in \mathrm{SO}(n)$ satisfies $R \cdot v = (1, 0, \ldots, 0)^T$, then $f(Q) \cdot R$ satisfies the same property for all $Q \in \mathrm{SO}(n-1)$.