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properties of linear independence

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Let V be a vector space over a field k. Below are some basic properties of linear independence.

1. $S \subseteq V$ is never linearly independent if $0 \in S$.

   Proof. Since $1 \cdot 0 = 0$. $\square$

2. If S is linearly independent, so is any subset of S. As a result, if S and T are linearly independent, so is $S \cap T$. In addition, $\varnothing$ is linearly independent, its spanning set being the singleton consisting of the zero vector 0.

   Proof. If $r_1 v_1 + \cdots r_n v_n = 0$, where $v_i \in T$, then $v_i \in S$, so $r_i = 0$ for all $i = 1, \ldots, n$. $\square$

3. If $S_1 \subseteq S_2 \subseteq \cdots$ is a chain of linearly independent subsets of V, so is their union.

   Proof. Let S be the union. If $r_1 v_1 + \cdots r_n v_n = 0$, then $v_i \in S_{a(i)}$, for each i. Pick the largest $S_{a(i)}$ so that all v_i's are in it. Since this set is linearly independent, $r_i = 0$ for all i. $\square$

4. S is a basis for V iff S is a maximal linear independent subset of V. Here, maximal means that any proper superset of S is linearly dependent.

   Proof. If S is a basis for V, then it is linearly independent and spans V. If we take any vector $v \notin S$, then v can be expressed as a linear combination of elements in S, so that $S \cup \{v\}$ is no longer linearly independent, for the coefficient in front of v is non-zero. Therefore, S is maximal.

   Conversely, suppose S is a maximal linearly independent set in V. Let W be the span of S. If $W \neq V$, pick an element $v \in V - W$. Suppose $0 = r_1 v_1 + \cdots r_n v_n + rv$, where $v_i \in S$, then $-rv = r_1 v_1 + \cdots + r_n v_n$. If $r \neq 0$, then v would be in the span of S, contradicting the assumption. So $r = 0$, and as a result, $r_i = 0$, since S is linearly independent. This shows that $S \cup \{v\}$ is linearly independent, which is impossible since S is assumed to be maximal. Therefore, $W = V$. $\square$

Remark. All of the properties above can be generalized to modules over rings, except the last one, where the implication is only one-sided: basis implying maximal linear independence.