Q is the prime subfield of any field of characteristic 0, proof that

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The following two propositions show that $\mathbb{Q}$ can be embedded in any field of characteristic 0, while $\mathbb{F}_p$ can be embedded in any field of characteristic p.

Proposition. $\mathbb{Q}$ is the prime subfield of any field of characteristic 0.

Proof. Let F be a field of characteristic 0. We want to find a one-to-one field homomorphism $\phi : \mathbb{Q} \to F$. For $\frac{m}{n} \in \mathbb{Q}$ with m, n coprime, define the mapping ϕ that takes $\frac{m}{n}$ into $\frac{m1_F}{n1_F} \in F$. It is easy to check that ϕ is a well-defined function. Furthermore, it is elementary to show

1. additive: for $p, q \in \mathbb{Q}$, $\phi(p+q) = \phi(p) + \phi(q)$;
2. multiplicative: for $p, q \in \mathbb{Q}$, $\phi(pq) = \phi(p)\phi(q)$;
3. $\phi(1) = 1_F$, and
4. $\phi(0) = 0_F$.

This shows that ϕ is a field homomorphism. Finally, if $\phi(p) = 0$ and $p \neq 0$, then $1 = \phi(1) = \phi(pp^{-1}) = \phi(p)\phi(p^{-1}) = 0 \cdot \phi(p^{-1}) = 0$, a contradiction. □

Proposition. $\mathbb{F}_p$ ($\cong \mathbb{Z}/p\mathbb{Z}$) is the prime subfield of any field of characteristic p.

Proof. Let F be a field of characteristic p. The idea again is to find an injective field homomorphism, this time, from $\mathbb{F}_p$ into F. Take ϕ to be the function that maps $m \in \mathbb{F}_p$ to $m \cdot 1_F$. It is well-defined, for if $m = n$ in $\mathbb{F}_p$, then $p \mid (m-n)$, meaning $(m-n)1_F = 0$, or that $m \cdot 1_F = n \cdot 1_F$, (showing that one element in $\mathbb{F}_p$ does not get "mapped" to more than one element in F). Since the above argument is reversible, we see that ϕ is one-to-one.

To complete the proof, we next show that ϕ is a field homomorphism. That $\phi(1) = 1_F$ and $\phi(0) = 0_F$ are clear from the definition of ϕ. Additivity and multiplicativity of ϕ are readily verified, as follows:

- $\phi(m+n) = (m+n) \cdot 1_F = m \cdot 1_F + n \cdot 1_F = \phi(m) + \phi(n)$;
- $\phi(mn) = mn \cdot 1_F = mn \cdot 1_F \cdot 1_F = (m \cdot 1_F)(n \cdot 1_F) = \phi(m)\phi(n)$.

This shows that ϕ is a field homomorphism. □