determinant in terms of traces of powers

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It is possible to express the determinant of a matrix in of traces of powers of a matrix.

The easiest way to derive these expressions is to specialize to the case of diagonal matrices. For instance, suppose we have a 2×2 matrix $M = \mathrm{diag}(u, v)$. Then

$$\begin{aligned} \det M &= uv \\ \operatorname{tr} M &= u + v \\ \operatorname{tr} M^2 &= u^2 + v^2 \end{aligned}$$

From the algebraic identity $(u + v)^2 = u^2 + v^2 + 2uv$, it can be concluded that $\det M = \frac{1}{2}(\operatorname{tr} M)^2 - \frac{1}{2}\operatorname{tr}(M^2)$.

Likewise, one can derive expressions for the determinants of larger matrices from the identities for elementary symmetric polynomials in of power sums. For instance, from the identity

$$xyz = \frac{1}{6}(x + y + z)^3 - \frac{1}{2}(x^2 + y^2 + z^2)(x + y + z) + \frac{1}{3}(x^3 + y^3 + z^3),$$

it can be concluded that

$$\det M = \frac{1}{6}(\operatorname{tr} M)^3 - \frac{1}{2}(\operatorname{tr} M^2)(\operatorname{tr} M) + \frac{1}{3}\operatorname{tr} M^3$$

for a 3×3 matrix M.