proof of Cauchy-Schwarz inequality

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If a and b are linearly dependent, we write $\boldsymbol{b} = \lambda\boldsymbol{a}$. So we get:

$$\langle \boldsymbol{a}, \lambda\boldsymbol{a}\rangle^2 = \lambda^2\langle \boldsymbol{a}, \boldsymbol{a}\rangle^2 = \lambda^2||\boldsymbol{a}||^4 = ||\boldsymbol{a}||^2||\boldsymbol{b}||^2.$$

So we have equality if $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent. In the other case we look at the quadratic function

$$||x \cdot \boldsymbol{a} + \boldsymbol{b}||^2 = x^2||\boldsymbol{a}||^2 + 2x\langle \boldsymbol{a}, \boldsymbol{b}\rangle + ||\boldsymbol{b}||^2.$$

This function is positive for every real x, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent. Thus it has no real zeroes, which means that

$$\langle \boldsymbol{a}, \boldsymbol{b}\rangle^2 - ||\boldsymbol{a}||^2||\boldsymbol{b}||^2$$

is always negative. So we have:

$$\langle \boldsymbol{a}, \boldsymbol{b}\rangle^2 < ||\boldsymbol{a}||^2||\boldsymbol{b}||^2,$$

which is the Cauchy-Schwarz inequality if $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent.