parameterization of equitable matrices

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A $n \times n$ matrix is *equitable* if and only if it can be expressed in the form

$$m_{ij} = \exp(\lambda_i - \lambda_j)$$

for real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_1 = 0$.

Assume that m_{ij} are the entries of an equitable matrix.

Since all the elements of an equitable matrix are positive by definition, we can write

$$m_{ij} = \exp \mu_{ij}$$

with the quantities μ_{ij} being real numbers (which may be positive, negative or zero).

In terms of this representation, the defining identity for an equitable matrix becomes

$$\mu_{ik} = \mu_{ij} + \mu_{jk}$$

Since this comprises a system of linear equations for the quantities μ_{ij}, we could solve it using the usual methods of matrix theory. However, for this particular system of linear equations, there is a much simpler approach.

Consider the special case of the identity when $i = j = k$:

$$\mu_{ii} = \mu_{ii} + \mu_{ii}.$$

This simplifies to

$$\mu_{ii} = 0.$$

In other words, all the diagonal entries are zero.

Consider the case when $i = k$ (but does not equal j).

$$\mu_{ij} + \mu_{ji} = \mu_{ii}$$

By wat we have just shown, the right hand side of this equation equals zero. Hence, we have

$$\mu_{ij} = -\mu_{ji}.$$

In other words, the matrix of μ's is antisymmetric.

We may express any entry in terms of the n entries μ_{i1}:

$$\mu_{ij} = \mu_{i1} + \mu_{1j} = \mu_{i1} - \mu_{j1}$$

We will conclude by noting that if, given any n numbers λ_i with $\lambda_1 = 0$, but the remaining λ's arbitrary, we define

$$\mu_{ij} = \lambda_i - \lambda_j,$$

then

$$\mu_{ij} + \mu_{jk} = \lambda_i - \lambda_j + \lambda_j - \lambda_k = \lambda_i - \lambda_k = \mu_{ik}$$

Hence, we obtain a solution of the equations

$$\mu_{ik} = \mu_{ij} + \mu_{jk}.$$

Moreover, by what we what we have seen, if we set $\lambda_i = \mu_{i1}$, all solutions of these equations can be so described.