additive function

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Definition 1. *Let $f\colon V \to \mathbb{R}$ be a function on a real vector space V (more generally we can consider a vector space V over a field F). We say that f is* additive *if*

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in V$.

If f is additive, we find that

1. $f(0) = 0$. In fact $f(0) = f(0+0) = f(0) + f(0) = 2f(0)$.
2. $f(nx) = nf(x)$ for $n \in \mathbb{N}$. In fact $f(nx) = f(x) + \cdots + f(x) = nf(x)$.
3. $f(nx) = nf(x)$ for $n \in \mathbb{Z}$. In fact $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$ so that $f(-x) = -f(x)$ and hence $f(-nx) = -f(nx) = -nf(x)$.
4. $f(qx) = qf(x)$ for $q \in \mathbb{Q}$. In fact $qf(px/q) = f(q(px/q)) = f(px) = pf(x)$ so that $f(px/q) = pf(x)/q$.

This means that f is $\mathbb{Q}$ linear. Quite surprisingly it is possible to show that there exist additive functions which are not linear (for example when V is a vector space over the field $\mathbb{R}$).