matrix p-norm

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A class of matrix norms, denoted $\| \cdot \|_p$, is defined as

$$\| A \|_p = \sup_{x \neq 0} \frac{\| Ax \|_p}{\| x \|_p} \qquad x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}.$$

The matrix p-norms are defined in terms of the ***http://planetmath.org/VectorPNorm****vector p-norms*.

An alternate definition is

$$\| A \|_p = \max_{\| x \|_p = 1} \| Ax \|_p.$$

As with vector p-norms, the most important are the 1, 2, and ∞ norms. The 1 and ∞ norms are very easy to calculate for an arbitrary matrix:

$$\begin{aligned} \| A \|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \\ \| A \|_\infty &= \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|. \end{aligned}$$

It directly follows from this that $\| A \|_1 = \| A^T \|_\infty$.

The calculation of the 2-norm is more complicated. However, it can be shown that the 2-norm of A is the square root of the largest *eigenvalue* of $A^T A$. There are also various inequalities that allow one to make estimates on the value of $\| A \|_2$:

$$\frac{1}{\sqrt{n}} \| A \|_\infty \leq \| A \|_2 \leq \sqrt{m} \| A \|_\infty.$$

$$\frac{1}{\sqrt{m}} \| A \|_1 \leq \| A \|_2 \leq \sqrt{n} \| A \|_1.$$

$$\| A \|_2^2 \leq \| A \|_\infty \cdot \| A \|_1.$$

$$\| A \|_2 \leq \| A \|_F \leq \sqrt{n} \| A \|_2.$$

($\| A \|_F$ is the *Frobenius matrix norm*)