an example for Schur decomposition

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Let

$$A = \begin{pmatrix} 5 & 7 \\ -2 & -4 \end{pmatrix}.$$

We will find an orthogonal matrix P and an upper triangular matrix T such that $P^t AP = T$ applying the proof of Schur's decomposition. We 're following the steps below

- We find the eigenvalues of A
  The eigenvalues of a matrix are precisely the solutions to the equation

  $$\det(\lambda I - A) = 0 \leftrightarrow \lambda^2 - \lambda - 6 = 0$$

  Hence the roots of the http://planetmath.org/QuadraticFormulaquadratic equation are the eigenvalues $\lambda_1 = -2, \lambda_2 = 3$

- We find the eigenvectors
  For each eigenvalue λ_i, solving the system

  $$(A - \lambda_i I)X_i = 0$$

  So we have that for $\lambda_1 = -2$

  $$(A + 2I) = 0 \leftrightarrow \begin{pmatrix} 7 & 7 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to X_1 = (1, -1)$$

  Analogously for $\lambda_2 = 3$ the eigenvector $X_2 = (7, -2)$

- We get an orthonormal set of eigenvectors using Gram-Schmidt orthogonalization
  Consider the above two eigenvectors which are linearly independent but are not orthogonal

  $$X_1 = (1, -1)$$
  $$X_2 = (7, -2)$$

  First we take $w_1 = X_1 = (1, -1)$. Therefore

  $$w_2 = X_2 - \frac{w_1 \cdot X_2}{\|w_1\|^2} w_1$$

that is,

$$w_2 = (\frac{5}{2}, \frac{5}{2})$$

and finally the orthonormal set is $\{w_1/\|w_1\|, w_2/\|w_2\|\} = \{(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$
So

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$T = P^t A P = \begin{pmatrix} -2 & 9 \\ 0 & 3 \end{pmatrix}.$$