proof of Hölder inequality

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First we prove the more general form (in measure spaces).

Let (X,μ) be a measure space and let $f \in L^p(X)$, $g \in L^q(X)$ where $p,q \in [1,+\infty]$ and $\frac{1}{p}+\frac{1}{q}=1$.

The case $p=1$ and $q=\infty$ is obvious since

$$|f(x)g(x)| \leq \|g\|_{L^\infty}|f(x)|.$$

Also if $f=0$ or $g=0$ the result is obvious. Otherwise notice that (applying http://planetmath.org/node/YoungInequalityYoung inequality) we have

$$\frac{\|fg\|_1}{\|f\|_p\cdot\|g\|_q} = \int_X \frac{|f|}{\|f\|_p}\cdot\frac{|g|}{\|g\|_q}\,d\mu \leq \frac{1}{p}\int_X\left(\frac{|f|}{\|f\|_p}\right)^p d\mu+\frac{1}{q}\int_X\left(\frac{|g|}{\|g\|_q}\right)^q d\mu = \frac{1}{p}+\frac{1}{q}=1$$

hence the desired inequality holds

$$\int_X |fg| = \|fg\|_1 \leq \|f\|_p\cdot\|g\|_q = \left(\int_X |f|^p\right)^{\frac{1}{p}}\left(\int_X |g|^q\right)^{\frac{1}{q}}.$$

If x and y are vectors in $\mathbb{R}^n$ or vectors in ℓ^p and ℓ^q-spaces we can specialize the previous result by choosing μ to be the counting measure on $\mathbb{N}$.

In this case the proof can also be rewritten, without using measure theory, as follows. If we define

$$\|x\|_p = \left(\sum_k |x_k|^p\right)^{\frac{1}{p}}$$

we have

$$\frac{|\sum_k x_k y_k|}{\|x\|_p\cdot\|y\|_q} \leq \frac{\sum_k |x_k||y_k|}{\|x\|_p\cdot\|y\|_q} = \sum_k \frac{|x_k|}{\|x\|_p}\frac{|y_k|}{\|y\|_q} \leq \frac{1}{p}\sum_k\frac{|x_k|^p}{\|x\|_p^p}+\frac{1}{q}\sum_k\frac{|y_k|^q}{\|y\|_q^q} = \frac{1}{p}+\frac{1}{q}=1.$$