Gram determinant

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Let V be an inner product space over a field k with $\langle\cdot,\cdot\rangle$ the inner product on V (note: since k is not restricted to be either $\mathbb{R}$ or $\mathbb{C}$, the inner product here shall mean a symmetric bilinear form on V). Let $x_1, x_2, \ldots, x_n$ be arbitrary vectors in V. Set $r_{ij} = \langle x_i, x_j\rangle$. The *Gram determinant* of $x_1, x_2, \ldots, x_n$ is defined to be the determinant of the symmetric matrix

$$\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix}$$

Let's denote this determinant by $\mathrm{Gram}[x_1, x_2, \ldots, x_n]$.

Properties.

1. $\mathrm{Gram}[x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n] = \mathrm{Gram}[x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n]$. More generally, $\mathrm{Gram}[x_1, \ldots, x_n] = \mathrm{Gram}[x_{\sigma(1)}, \ldots, x_{\sigma(n)}]$, where σ is a permutation on $\{1, \ldots, n\}$.

2. $\mathrm{Gram}[x_1, \ldots, ax_i+bx_j, \ldots, x_j, \ldots, x_n] = a^2\,\mathrm{Gram}[x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n]$, $a, b \in k$.

3. Setting $a = 0$ and $b = 1$ in Property 2, we get $\mathrm{Gram}[x_1, \ldots, x_j, \ldots, x_j, \ldots, x_n] = 0$.

4. Properties 2 and 3 can be generalized as follows: if x_i (in the ith term) is replaced by a linear combination $y = r_1x_1 + \cdots + r_nx_n$, then

$$\mathrm{Gram}[x_1, \ldots, y, \ldots, x_n] = r_i^2\,\mathrm{Gram}[x_1, \ldots, x_i, \ldots, x_n].$$

5. Suppose k is an ordered field. Then it can be shown that the Gram determinant is at least 0, and at most the product $\langle x_1, x_1\rangle \cdots \langle x_n, x_n\rangle$.

6. Suppose that in addition to k being ordered, that every positive element in k is a square, then the Gram determinant is equal to the square of the volume of the (hyper)parallelepiped generated by $x_1, \ldots, x_n$. (Recall that an n-dimensional parallelepiped is the set of vectors which are linear combinations of the form $r_1x_1 + \ldots + r_nx_n$ where $0 \le r_i \le 1$.)

7. It's now easy to see that in Property 5, the Gram determinant is 0 if the x_i's are linearly dependent, and attains its maximum if the x_i's are pairwise orthogonal (a quick proof: in the above matrix, $r_{ij} = 0$ if $i \neq j$), which corresponds exactly to the square of the volume of the hyperparallelepiped spanned by the x_i's.

8. If $e_1, \ldots, e_n$ are basis elements of a quadratic space V over an order field whose positive elements are squares, then V is , or , iff $\mathrm{Gram}[e_1, \ldots, e_n] = 0$.

References

[1] Georgi E. Shilov, "An Introduction to the Theory of Linear Spaces", translated from Russian by Richard A. Silverman, 2nd Printing, Prentice-Hall, 1963.