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Moore-Penrose generalized inverse

Canonical name	MoorePenroseGeneralizedInverse
Date of creation	2013-03-22 14:31:31
Last modified on	2013-03-22 14:31:31
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	8
Author	CWoo (3771)
Entry type	Definition
Classification	msc 15A09
Classification	msc 60J10
Synonym	Moore-Penrose pseudoinverse
Related topic	DrazinInverse
Related topic	Pseudoinverse

Let A be an $m \times n$ matrix with entries in $\mathbb{C}$. The *Moore-Penrose generalized inverse*, denoted by $A^\dagger$, is an $n \times m$ matrix with entries in $\mathbb{C}$, such that

1. $AA^\dagger A = A$
2. $A^\dagger AA^\dagger = A^\dagger$
3. $AA^\dagger$ and $A^\dagger A$ are both Hermitian

Remarks

- The Moore-Penrose generalized inverse of a given matrix is unique.
- If $A^\dagger$ is the Moore-Penrose generalized inverse of A, then $(A^\dagger)^{\mathrm{T}}$ is the Moore-Penrose generalized inverse of A^{T}.
- If $A = BC$ such that
  1. $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times r}$, and $C \in \mathbb{C}^{r\times n}$,
  2. $r = \operatorname{rank}(A) = \operatorname{rank}(B) = \operatorname{rank}(C)$, then

$$A^\dagger = C^*(CC^*)^{-1}(B^*B)^{-1}B^*.$$

For example, let

$$A = \begin{pmatrix} 1 & 1 & i \\ 0 & 1 & 0 \end{pmatrix}.$$

Transform A to its row echelon form to get a decomposition of $A = BC$, where

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \end{pmatrix}.$$

It is readily verified that $2 = \operatorname{rank}(A) = \operatorname{rank}(B) = \operatorname{rank}(C)$. So

$$A^\dagger = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 2 \\ -i & i \end{pmatrix}.$$

We check that

$$AA^\dagger = I \text{ and } A^\dagger A = \frac{1}{2}\begin{pmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{pmatrix}$$

are both Hermitian. Furthermore, $AA^\dagger A = A$ and $A^\dagger AA^\dagger = A^\dagger$. So, $A^\dagger$ is the Moore-Penrose generalized inverse of A.