invertible matrix

Canonical name	InvertibleMatrix
Date of creation	2013-03-22 19:23:09
Last modified on	2013-03-22 19:23:09
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	11
Author	CWoo (3771)
Entry type	Definition
Classification	msc 15-01
Classification	msc 15A09
Classification	msc 15A33

Let R be a ring and M an $m \times n$ matrix over R. M is said to be *left invertible* if there is an $n \times m$ matrix such that $NM = I_n$, where I_n is the $n \times n$ identity matrix. We call N a *left inverse* of M. Similarly, M is *right invertible* if there is an $n \times m$ matrix P, called a *right inverse* of M, such that $MP = I_m$, where I_m is the $m \times m$ identity matrix. If M is both left invertible and right invertible, we say that M is *invertible*. If R is an associative ring, and M is invertible, then it has a unique left and a unique right inverse, and they are in fact equal, we call this matrix the *inverse* of M.

If R is a division ring, then it can be shown that for any matrix M over R, M is left invertible iff it is invertible iff it is right invertible. In addition, when M is invertible, it is a square matrix. Furthermore, R is a field iff for any square matrix M (over R), M is invertible implies that M^T, its transpose, is invertible as well. Invertibility of matrices over a division ring can also be determined by quantities known as ranks and determinants. It can be shown that a matrix over a division ring is invertible iff its left row rank (or right column rank) is full iff its determinant is non-zero. For example, the 2×2 matrix

$$\begin{pmatrix} 1 & j \\ i & k \end{pmatrix}$$

over the Hamiltonian quaternions is not invertible, as its determinant $k - ji = 0$. It is interesting to note that, however, its transpose

$$\begin{pmatrix} 1 & i \\ j & k \end{pmatrix}$$

is invertible, whose determinant is $2k \neq 0$. The relationship between determinants and matrix invertibility can also be used to prove the following: preservation of matrix invertibility upon matrix transposition implies commutativity of division ring D. This can be done as follows: given any $a, b \in D$, the 2×2 matrix

$$\begin{pmatrix} ab & b \\ a & 1 \end{pmatrix}$$

is not invertible because its determinant is 0. Therefore, its transpose

$$\begin{pmatrix} ab & a \\ b & 1 \end{pmatrix}$$

is also not invertible, and its determinant is $0 = ab - ba$, whence D is a field.