Botta - Pierce - Watkins theorem

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Let $\mathbb{F}$ be an arbitrary field, and let n be a positive integer. Consider $\mathcal{M}_n(\mathbb{F})$, the vector space of all $n \times n$ matrices over $\mathbb{F}$. Define

- $\mathfrak{sl}_n(\mathbb{F}) = \{A \in \mathcal{M}_n(\mathbb{F}) : \operatorname{tr}(A) = 0\}$,
- $\mathcal{N} = \{A \in \mathcal{M}_n(\mathbb{F}) : A \text{ is nilpotent}\}$,
- $\mathcal{GL}_n(\mathbb{F}) = \{A \in \mathcal{M}_n(\mathbb{F}) : \det(A) \neq 0\}$.

Notice that $\mathfrak{sl}_n(\mathbb{F})$ is a linear subspace of $\mathcal{M}_n(\mathbb{F})$ and $\mathcal{N} \subseteq \mathfrak{sl}_n(\mathbb{F})$.

The Botta – Pierce – Watkins theorem on linear preservers of the nilpotent matrices [**?**] can be formulated as follows.

Theorem 1 *Let* $\varphi : \mathfrak{sl}_n(\mathbb{F}) \longrightarrow \mathfrak{sl}_n(\mathbb{F})$ *be a linear automorphism. Assume that* $\varphi(\mathcal{N}) \subseteq \mathcal{N}$. *Then either* $\exists\, P \in \mathcal{GL}_n(\mathbb{F})\, \exists\, c \in \mathbb{F} \setminus \{0\}\, \forall\, A \in \mathfrak{sl}_n(\mathbb{F}) : \varphi(A) = cPAP^{-1}$, *or* $\exists\, P \in \mathcal{GL}_n(\mathbb{F})\, \exists\, c \in \mathbb{F} \setminus \{0\}\, \forall\, A \in \mathfrak{sl}_n(\mathbb{F}) : \varphi(A) = cPA^{\mathrm{T}}P^{-1}$.

The original proof is based on the Gerstenhaber - Serezhkin theorem, some elementary algebraic geometry, and the fundamental theorem of projective geometry.

References

[BPW] P. Botta, S. Pierce, W. Watkins, Linear transformations that preserve the nilpotent matrices, *Pacific J. Math.* **104** (No. 1): 39–46 (1983).