Schur decomposition, proof of

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The columns of the unitary matrix Q in Schur's decomposition theorem form an orthonormal basis of $\mathbb{C}^n$. The matrix A takes the upper-triangular form $D+N$ on this basis. Conversely, if $v_1, \ldots, v_n$ is an orthonormal basis for which A is of this form then the matrix Q with v_i as its i-th column satisfies the theorem.

To find such a basis we proceed by induction on n. For $n = 1$ we can simply take $Q = 1$. If $n > 1$ then let $v \in \mathbb{C}^n$ be an eigenvector of A of unit length and let $V = v^{\perp}$ be its orthogonal complement. If π denotes the orthogonal projection onto the line spanned by v then $(1-\pi)A$ maps V into V.

By induction there is an orthonormal basis $v_2, \ldots, v_n$ of V for which $(1-\pi)A$ takes the desired form on V. Now $A = \pi A + (1-\pi)A$ so $Av_i \equiv (1-\pi)Av_i \pmod{v}$ for $i \in \{2, \ldots, n\}$. Then $v, v_2, \ldots, v_n$ can be used as a basis for the Schur decomposition on $\mathbb{C}^n$.