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antisymmetric mapping

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Let U and V be a vector spaces over a field K. A bilinear mapping $B : U \times U \to V$ is said to be *antisymmetric* if

$$B(u, u) = 0 \tag{1}$$

for all $u \in U$.

If B is antisymmetric, then the polarization of the anti-symmetry relation gives the condition:

$$B(u, v) + B(v, u) = 0 \tag{2}$$

for all $u, v \in U$. If the characteristic of K is not 2, then the two conditions are equivalent.

A multlinear mapping $M : U^k \to V$ is said to be *totally antisymmetric*, or simply antisymmetric, if for every $u_1, \ldots, u_k \in U$ such that

$$u_{i+1} = u_i$$

for some $i = 1, \ldots, k-1$ we have

$$M(u_1, \ldots, u_k) = 0.$$

Proposition 1 *Let $M : U^k \to V$ be a totally antisymmetric, multlinear mapping, and let π be a permutation of $\{1, \ldots, k\}$. Then, for every $u_1, \ldots, u_k \in U$ we have*

$$M(u_{\pi_1}, \ldots, u_{\pi_k}) = \operatorname{sgn}(\pi) M(u_1, \ldots, u_k),$$

where $\operatorname{sgn}(\pi) = \pm 1$ according to the parity of π.

Proof. Let $u_1, \ldots, u_k \in U$ be given. multlinearity and anti-symmetry imply that

$$\begin{aligned} 0 &= M(u_1 + u_2, u_1 + u_2, u_3, \ldots, u_k) \\ &= M(u_1, u_2, u_3, \ldots, u_k) + M(u_2, u_1, u_3, \ldots, u_k) \end{aligned}$$

Hence, the proposition is valid for $\pi = (12)$ (see cycle notation). Similarly, one can show that the proposition holds for all transpositions

$$\pi = (i, i+1), \quad i = 1, \ldots, k-1.$$

However, such transpositions generate the group of permutations, and hence the proposition holds in full generality.

Note. The determinant is an excellent example of a totally antisymmetric, multlinear mapping.