in a vectorspace, $\lambda v = 0$ if and only if $\lambda = 0$ or v is the zero vector

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Theorem Let V be a vector space over the field F. Further, let $\lambda \in F$ and $v \in V$. Then $\lambda v = 0$ if and only if λ is zero, or if v is the zero vector, or if both λ and v are zero.

Proof. Let us denote by 0_F and by 1_F the zero and unit elements in F respectively. Similarly, we denote by 0_V the zero vector in V. Suppose $\lambda = 0_F$. Then, by `http://planetmath.org/VectorSpace`axiom 8, we have that

$$1_F v + 0_F v = 1_F v,$$

for all $v \in V$. By `http://planetmath.org/VectorSpace`axiom 6, there is an element in V that cancels $1_F v$. Adding this element to both yields $0_F v = 0_V$. Next, suppose that $v = 0_V$. We claim that $\lambda 0_V = 0_V$ for all $\lambda \in F$. This follows from the previous claim if $\lambda = 0$, so let us assume that $\lambda \neq 0_F$. Then λ^{-1} exists, and `http://planetmath.org/VectorSpace`axiom 7 implies that

$$\lambda\lambda^{-1} v + \lambda 0_V = \lambda(\lambda^{-1} v + 0_V)$$

holds for all $v \in V$. Then using `http://planetmath.org/VectorSpace`axiom 3, we have that

$$v + \lambda 0_V = v$$

for all $v \in V$. Thus $\lambda 0_V$ satisfies the axiom for the zero vector, and $\lambda 0_V = 0_V$ for all $\lambda \in F$.

For the other direction, suppose $\lambda v = 0_V$ and $\lambda \neq 0_F$. Then, using `http://planetmath.org/VectorSpace`axiom 3, we have that

$$v = 1_F v = \lambda^{-1}\lambda v = \lambda^{-1} 0_V = 0_V.$$

On the other hand, suppose $\lambda v = 0_V$ and $v \neq 0_V$. If $\lambda \neq 0$, then the above calculation for v is again valid whence

$$0_V \neq v = 0_V,$$

which is a contradiction, so $\lambda = 0$. □

This result with proof can be found in [**?**], page 6.

References

[1] W. Greub, *Linear Algebra*, Springer-Verlag, Fourth edition, 1975.