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Levy-Desplanques theorem

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Owner	Andrea Ambrosio (7332)
Last modified by	Andrea Ambrosio (7332)
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Author	Andrea Ambrosio (7332)
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A strictly diagonally dominant matrix is non-singular. In other words, let $A \in \mathbf{C}^{n,n}$ be a matrix satisfying the property

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \qquad \forall i;$$

then $\det(A) \neq 0$.

Proof: Let $\det(A) = 0$; then a non-zero vector $\mathbf{x}$ exists such that $A\mathbf{x} = \mathbf{0}$; let M be the index such that $|x_M| = \max(|x_1|, |x_2|, \cdots, |x_n|)$, so that $|x_j| \leq |x_M| \quad \forall j$; we have

$a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MM}x_M + \cdots + a_{Mn}x_n = 0$

which implies:

$|a_{MM}| |x_M| = |a_{MM}x_M| = \left|\sum_{j \neq M} a_{Mj}x_j\right| \leq \sum_{j \neq M} |a_{Mj}| |x_j| \leq |x_M| \sum_{j \neq M} |a_{Mj}|$

that is

$|a_{MM}| \leq \sum_{j \neq M} |a_{Mj}|,$

in contrast with strictly diagonally dominance definition.$\square$

Remark: the Levy-Desplanques theorem is equivalent to the well-known Gerschgorin circle theorem. In fact, let's assume Levy-Desplanques theorem is true, and let A a $n \times n$ complex-valued matrix, with an eigenvalue λ; let's apply Levy-Desplanques theorem to the matrix $B = A - \lambda I$, which is singular by definition of eigenvalue: an index i must exist for which $|a_{ii} - \lambda| = |b_{ii}| \leq \sum_{j \neq i}^{n} |b_{ij}| = \sum_{j \neq i}^{n} |a_{ij}|$, which is Gerschgorin circle theorem. On the other hand, let's assume Gerschgorin circle theorem is true, and let A be a strictly diagonally dominant $n \times n$ complex matrix. Then, since the absolute value of each disc center $|a_{ii}|$ is strictly greater than the same disc radius $\sum_{j \neq i}^{n} |a_{ij}|$, the point $\lambda = 0$ can't belong to any circle, so it doesn't belong to the spectrum of A, which therefore can't be singular.