Farkas lemma, proof of

Canonical name	FarkasLemmaProofOf
Date of creation	2013-03-22 14:12:51
Last modified on	2013-03-22 14:12:51
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	13
Author	CWoo (3771)
Entry type	Proof
Classification	msc 15A39

We begin by showing that at least one of the systems has a solution.

Suppose that system 2 has no solution. Let S be the cone in $\mathbb{R}^n$ generated by nonnegative linear combinations of the rows $a_1, \ldots, a_m$ of A. The set S is closed and convex. Since system 2 is unsolvable, the vector c is not in S; therefore, there exist a scalar α and n-column vector x such that the hyperplane $x^T v = \alpha$ separates c from S in $\mathbb{R}^n$. This hyperplane can be selected so that for any point $v \in S$,

$$x^T c^T > \alpha > x^T v^T.$$

Since $0 \in S$, this implies that $\alpha > 0$. Hence for any $w \geq 0$,

$$\alpha > x^T (wA)^T = wAx = \sum_{i=0}^{m} w_i (Ax)_i.$$

Each $(Ax)_i$ is nonpositive. Otherwise, by selecting w with w_i sufficiently large and all other $w_j = 0$, we would get a contradiction. We have now shown that x satisfies $Ax \leq 0$ and $cx = (cx)^T = x^T c^T > \alpha > 0$, which means that x is a solution of system 1. Thus at least one of the systems is solvable.

We claim that systems 1 and 2 are not simultaneously solvable. Suppose that x is a solution of system 1 and w is a solution of system 2. Then for each i, the inequality $w_i (Ax)_i \leq 0$ holds, and so $w(Ax) \leq 0$. However,

$$(wA)x = cx > 0,$$

a contradiction. This completes the proof.