proof of Cauchy-Schwarz inequality for real numbers

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Owner	stitch (17269)
Last modified by	stitch (17269)
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Author	stitch (17269)
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The version of the Cauchy-Schwartz inequality we want to prove is

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2,$$

where the a_k and b_k are real numbers, with equality holding only in the case of proportionality, $a_k = \lambda b_k$ for some real λ for all k.

The proof is by direct calculation:

$$\begin{aligned}
\sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 &= \sum_{k,l=1}^{n} a_k^2 b_l^2 - a_k b_k a_l b_l \\
&= \sum_{k,l=1}^{n} \frac{1}{2}(a_k^2 b_l^2 + a_l^2 b_k^2) - (a_k b_l)(a_l b_k) \\
&= \frac{1}{2} \sum_{k,l=1}^{n} (a_k b_l)^2 - 2(a_k b_l)(a_l b_k) + (a_l b_k)^2 \\
&= \frac{1}{2} \sum_{k,l=1}^{n} (a_k b_l - a_l b_k)^2 \\
&\geq 0.
\end{aligned}$$

The above identity implies that the Cauchy-Schwarz inequality holds. Moreover, it is an equality only when

$$a_k b_l - a_l b_k = 0 \quad \Longleftrightarrow \quad \frac{a_k}{b_k} = \frac{a_l}{b_l} \text{ or } \frac{b_k}{a_k} = \frac{b_l}{a_l} \text{ or } a_k = b_k = 0,$$

for all k and l. In other words, equality holds only when $a_k = \lambda b_k$ for all k for some real number λ.