tensor product of subspaces of vector spaces

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Proposition. Let V, W be vector spaces over a field k. Moreover let $A \subseteq V$, $B \subseteq W$ be subspaces. Then $V \otimes B \cap A \otimes W = A \otimes B$.

Proof. The inclusion „$\supseteq$" is obvious. We will show the inclusion „$\subseteq$".

Let $\{e_i\}_{i\in I}$ and $\{e'_j\}_{j\in P}$ be bases of A and B respectively. Moreover let $\{e_i\}_{i\in I'}$ be a completion of given basis of A to the basis of V, i.e. $\{e_i\}_{i\in I\cup I'}$ is a basis of V. Analogously let $\{e'_j\}_{j\in P'}$ be a completion of a basis of B to the basis of W. Then each element $q \in V \otimes W$ can be uniquely written in a form

$$q = \sum_{i\in I, j\in P} \alpha_{i,j}\, e_i \otimes e'_j \;+ \sum_{i\in I', j\in P} \beta_{i,j}\, e_i \otimes e'_j \;+$$
$$+ \sum_{i\in I, j\in P'} \delta_{i,j}\, e_i \otimes e'_j \;+ \sum_{i\in I', j\in P'} \gamma_{i,j}\, e_i \otimes e'_j.$$

Assume that $q \in V \otimes B \cap A \otimes W$. Let $i \in I'$ and $j \in P'$. Consider the following linear map: $f_i : V \to k$ such that $f_i(e_t) = 1$ if $i = t$ and $f_i(e_t) = 0$ if $i \neq t$. Analogously we define $g_j : W \to k$. Then we combine these two mappings into one, i.e.

$$f_i \otimes g_j : V \otimes W \to k;$$
$$(f_i \otimes g_j)(v \otimes w) = f_i(v) g_j(w).$$

Furthermore we have

$$(f_i \otimes g_j)(q) = \gamma_{i,j}.$$

Note that since $q \in V \otimes B$, then $(f_i \otimes g_j)(q) = 0$ and thus

$$\gamma_{i,j} = 0.$$

Similarly we obtain that all $\beta_{i,j}$ and $\delta_{i,j}$ are equal to 0. Thus

$$q = \sum_{i\in I, j\in P} \alpha_{i,j}\, e_i \otimes e'_j \in A \otimes B,$$

which completes the proof. $\square$