another proof of rank-nullity theorem

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Let $\phi : V \to W$ be a linear transformation from vector spaces V to W. Recall that the rank of ϕ is the dimension of the image of ϕ and the nullity of ϕ is the dimension of the kernel of ϕ.

Proposition 1. $\dim(V) = \operatorname{rank}(\phi) + \operatorname{nullity}(\phi)$.

Proof. Let $K = \ker(\phi)$. K is a subspace of V so it has a unique algebraic complement L such that $V = K \oplus L$. It is evident that

$$\dim(V) = \dim(K) + \dim(L)$$

since K and L have disjoint bases and the union of their bases is a basis for V.

Define $\phi' : L \to \phi(V)$ by restriction of ϕ to the subspace L. ϕ' is obviously a linear transformation. If $\phi'(v) = 0$, then $\phi(v) = \phi'(v) = 0$ so that $v \in K$. Since $v \in L$ as well, we have $v \in K \cap L = \{0\}$, or $v = 0$. This means that ϕ' is one-to-one. Next, pick any $w \in \phi(V)$. So there is some $v \in V$ with $\phi(v) = w$. Write $v = x + y$ with $x \in K$ and $y \in L$. So $\phi'(y) = \phi(y) = 0 + \phi(y) = \phi(x) + \phi(y) = \phi(v) = w$, and therefore ϕ' is onto. This means that L is isomorphic to $\phi(V)$, which is equivalent to saying that $\dim(L) = \dim(\phi(V)) = \operatorname{rank}(\phi)$. Finally, we have

$$\dim(V) = \dim(K) + \dim(L) = \operatorname{nullity}(\phi) + \operatorname{rank}(\phi).$$

□

Remark. The dimension of V is not assumed to be finite in this proof. For another approach (where finite dimensionality of V is assumed), please see `http://planetmath.org/ProofOfRankNullityTheorem`this entry.