finding eigenvalues

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This example investigates eigenvalues and the similarity transformation used to diagonalize matrices. We seek the eigenvalues of the matrix A below. Afterward, we can transform this matrix into a diagonal matrix which has many useful applications.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Here, we need to solve the corresponding matrix equation;

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

$$AX = \lambda X$$

rearranging gives

$$AX - \lambda X = 0$$

or

$$(A - \lambda I)X = 0$$

We seek the values for λ and X. First, we need to solve the characteristic equation of A. We do this by finding $det(A - \lambda I)$. First, calculating $A - \lambda I$ gives;

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

Next, calculating $det(A - \lambda I)$ yields

$$det(A - \lambda I) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$$

Substituting $\lambda = 1$ into $(A - \lambda I)X$ gives...

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

so that $x_2 = -x_1$ and the corresponding eigenvector is

$$\begin{pmatrix} t \\ -t \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where $t \neq 0$.
Substituting $\lambda = 3$ gives...

$$\begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases}$$

so that $x_2 = x_1$ and the corresponding eigenvector is

$$\begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where $t \neq 0$.
Finally, to diagonalize A we let the eigenvectors be the columns of a new matrix

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and then since our eigenvectors are linearly independent we can also find;

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

then we create a diagonal matrix as follows...

$$D = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Computing powers of A is a very useful application of D. Solving for A lets us compute powers of A

$$A = PDP^{-1}$$

so that

$$A^n = PD^nP^{-1}$$

or

$$A^n = P \begin{pmatrix} 1^n & 0 \\ 0 & 3^n \end{pmatrix} P^{-1}$$