proof that commuting matrices are simultaneously triangularizable

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Proof by induction on n, order of matrix.
For $n = 1$ we can simply take $Q = 1$. We assume that there exists a common unitary matrix S that triangularizes simultaneously commuting matrices ,$(n-1) \times (n-1)$.
So we have to show that the statement is valid for commuting matrices, $n \times n$.
From hypothesis A and B are commuting matrices $n \times n$ so these matrices have a common eigenvector.
Let $Ax = \lambda x$, $Bx = \mu x$ where x be the common eigenvector of unit length and λ, μ are the eigenvalues of A and B respectively. Consider the matrix, $R = \begin{pmatrix} x & X \end{pmatrix}$ where X be orthogonal complement of x and $R^H R = I$, then we have that

$$R^H A R = \begin{pmatrix} \lambda & x^H A X \\ 0 & X^H A X \end{pmatrix}$$

$$R^H B R = \begin{pmatrix} \mu & x^H B X \\ 0 & X^H B X \end{pmatrix}$$

It is obvious that the above matrices and also $X^H B X$, $X^H A X$,$(n-1) \times (n-1)$ matrices are commuting matrices. Let $B_1 = X^H B X$ and $A_1 = X^H A X$ then there exists unitary matrix S such that $S^H B_1 S = \bar{T}_2$, $S^H A_1 S = \bar{T}_1$. Now $Q = R \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$ is a unitary matrix, $Q^H Q = I$ and we have

$$Q^H A Q = \begin{pmatrix} 1 & 0 \\ 0 & S^H \end{pmatrix} R^H A R \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} = \begin{pmatrix} \lambda & x^H A X S \\ 0 & \bar{T}_1 \end{pmatrix} = T_1.$$

Analogously we have that

$$Q^H B Q = T_2.$$