proof of the determinant condition for a sequence of vectors

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Owner	GeraW (6138)
Last modified by	GeraW (6138)
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Author	GeraW (6138)
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Theorem. *Let $x_1, x_2, ...$ be a sequence of d dimensional vectors. Assume that there is $C : \mathbb{N}^d \to \mathbb{R} \setminus \{0\}$ such that*

$$\sum_{\substack{n_1+\cdots+n_d=n \\ 0<n_1<\cdots<n_d}} C(n_1, ..., n_d) \det[x_{n_1}, x_{n_2}, ..., x_{n_d}] = 0 \tag{1}$$

for every $n \in \mathbb{N}$. Then $\det[x_{n_1}, x_{n_2}, ..., x_{n_d}]=0$ for all $(n_1, ..., n_d) \in \mathbb{N}^d$.

Proof. Introduce a linear order over the set of ordered tuples: $(n_1, n_2, ..., n_d) \prec (\hat{n}_1, \hat{n}_2, ..., \hat{n}_d)$ if $\left(\sum_{i=1}^d n_i, \hat{n}_d, \hat{n}_{d-1}, ..., \hat{n}_1\right)$ precedes $\left(\sum_{i=1}^d \hat{n}_i, n_d, n_{d-1}, ..., n_1\right)$ lexicographically. Let $(n_1, n_2, ..., n_d)$ be the minimal (according to the above order) ordered tuple for which

$$\det[x_{n_1}, x_{n_2}, ..., x_{n_d}] \neq 0. \tag{2}$$

Take another ordered tuple, $(\hat{n}_1, \hat{n}_2, ..., \hat{n}_d)$, such that $\sum_{i=1}^d n_i = \sum_{i=1}^d \hat{n}_i$. By minimality, if $(n_d, n_{d-1}, ..., n_1)$ precedes $(\hat{n}_d, \hat{n}_{d-1}, ..., \hat{n}_1)$ lexicographically then $\det[x_{\hat{n}_1}, x_{\hat{n}_2}, ..., x_{\hat{n}_d}] = 0$. Otherwise, let $i \in \{0, 1, ..., d-1\}$ be the first index such that $n_{d-i} \neq \hat{n}_{d-i}$ (more specifically, $n_{d-i} > \hat{n}_{d-i}$). Then, $\hat{n}_{d-j} = n_{d-j}$ for $j = 0, ..., i-1$ and $\hat{n}_{d-j} < n_{d-i}$ for $j = i, ..., d-1$. Therefore,

$$\det[x_{n_1}, ..., x_{n_{d-i-1}}, x_{\hat{n}_m}, x_{n_{d-i+1}}, ..., x_{n_d}] = 0$$

for all $m = 1, 2, ..., d$ (some because of repeated columns and the others because $\sum_{j=1}^d n_j - n_{d-i} + \hat{n}_m < \sum_{j=1}^d n_j$). Since the vectors $x_{n_1}, x_{n_2}, ..., x_{n_d}$ are linearly independent, we get that

$$\{x_{\hat{n}_1}, x_{\hat{n}_2}, ..., x_{\hat{n}_d}\} \subset \operatorname{span}\left(\{x_{n_1}, x_{n_2}, ..., x_{n_d}\} \setminus \{x_{n_{d-i}}\}\right).$$

In particular $\det[x_{\hat{n}_1}, x_{\hat{n}_2}, ..., x_{\hat{n}_d}] = 0$. Therefore, (**??**) reduces to $\det[x_{n_1}, x_{n_2}, ..., x_{n_d}] = 0$ which contradicts (**??**).

$\square$