Einstein summation convention

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The *Einstein summation convention* implies that when an index occurs more than once in the same expression, the expression is implicitly summed over all possible values for that index. Therefore, in order to use the summation convention, it must be clear from the context over what range indices should be summed.

The Einstein summation convention is illustrated in the below examples.

1. Let $\{e_i\}_{i=1}^n$ be a orthogonal basis in $\mathbb{R}^n$. Then the inner product of the vectors $u = u^i e_i = (\sum_{i=1}^n) u^i e_i$ and $v = v^i e_i = (\sum_{i=1}^n) v^i e_i$, is

$$\begin{aligned} u \cdot v &= u^i v^j e_i \cdot e_j \\ &= \delta_{ij} u^i v^j. \end{aligned}$$

2. Let V be a vector space with basis $\{e_i\}_{i=1}^n$ and a dual basis $\{e^i\}_{i=1}^n$. Then, for a vector $v = v^i e_i$ and dual vectors $\alpha = \alpha_i e^i$ and $\beta = \beta_i e^i$, we have

$$\begin{aligned} (\alpha + \beta)(v) &= \alpha_i v^i + \beta_j v^j \\ &= (\alpha_i + \beta_i) v^i. \end{aligned}$$

   This example shows that the summation convention is "distributive" in a natural way.

3. Chain rule. Let $F : \mathbb{R}^m \to \mathbb{R}^n$, $x \mapsto (F^1, \cdots, F^n)$, and $G : \mathbb{R}^n \to \mathbb{R}^p$, $y \mapsto (G^1(y), \cdots, G^p(y))$ be smooth functions. Then

$$\frac{\partial (G \circ F)^i}{\partial x^j}(x) = \frac{\partial G^i}{\partial y^k}\big(F(x)\big)\frac{\partial F^k}{\partial x^j}(x),$$

   where the right hand side is summed over $k = 1, \ldots, n$.

An index which is summed is called a *dummy index* or *dummy variable*. For instance, i is a dummy index in $v^i e_i$. An expression does not depend on a dummy index, i.e., $v^i e_i = v^j e_j$. It is common that one must change the name of dummy indices. For instance, above, in Example 2 when we calculated $u \cdot v$, it was necessary to change the index i in $v = v^i e_i$ to j so that it would not clash with $u = u^i e_i$.

When using the Einstein summation convention, objects are usually indexed so that when summing, one index will always be an "upper index" and the other a "lower index". Then summing should only take place over upper and lower indices. In the above examples, we have followed this rule. Therefore we did not write $\delta_{ij}u^iv^j = u^iv^i$ in the first example since u^iv^i has two upper indices. This is consistent; it is not possible to take the inner product of two vectors without a metric, which is here δ_{ij}. The last example illustrates that when we consider k as a "lower index" in $\frac{\partial G^i}{\partial y^k}$, then the chain rule obeys this upper-lower rule for the indices.