topological proof of the Cayley-Hamilton theorem

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We begin by showing that the theorem is true if the characteristic polynomial does not have repeated roots, and then prove the general case.

Suppose then that the discriminant of the characteristic polynomial is non-zero, and hence that $T : V \to V$ has $n = \dim V$ distinct eigenvalues once we extend[1] to the algebraic closure of the ground field. We can therefore choose a basis of eigenvectors, call them $\mathbf{v}_1, \ldots, \mathbf{v}_n$, with $\lambda_1, \ldots, \lambda_n$ the corresponding eigenvalues. From the definition of characteristic polynomial we have that

$$c_T(x) = \prod_{i=1}^{n} (x - \lambda_i).$$

The factors on the right commute, and hence

$$c_T(T)\mathbf{v}_i = 0$$

for all $i = 1, \ldots, n$. Since $c_T(T)$ annihilates a basis, it must, in fact, be zero.

To prove the general case, let $\delta(p)$ denote the discriminant of a polynomial p, and let us remark that the discriminant mapping

$$T \mapsto \delta(c_T), \quad T \in \operatorname{End}(V)$$

is polynomial on $\operatorname{End}(V)$. Hence the set of T with distinct eigenvalues is a dense open subset of $\operatorname{End}(V)$ relative to the Zariski topology. Now the characteristic polynomial map

$$T \mapsto c_T(T), \quad T \in \operatorname{End}(V)$$

is a polynomial map on the vector space $\operatorname{End}(V)$. Since it vanishes on a dense open subset, it must vanish identically. Q.E.D.

[1] Technically, this means that we must work with the vector space $\bar{V} = V \otimes \bar{k}$, where $\bar{k}$ is the algebraic closure of the original field of scalars, and with $\bar{T} : \bar{V} \to \bar{V}$ the extended automorphism with action

$$\bar{T}(v \otimes a) \to T(V) \otimes a, \quad v \in V,\ a \in \bar{k}.$$