proof of necessary and sufficient condition for diagonalizability

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First, suppose that T is diagonalizable. Then V has a basis whose elements $\{v_1, \dots, v_n\}$ are eigenvectors of T associated with the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ respectively. For each $i = 1, ..., n$, as v_i is an eigenvector, its annihilator polynomial is $m_{v_i} = X - \lambda_i$. As these vectors form a basis of V, we have that the `http://planetmath.org/MinimalPolynomialEndomorphism`minimal polynomial of T is $m_T = \mathrm{lcm}(X - \lambda_1, \dots, X - \lambda_n)$ which is trivially a product of linear factors.

Now, suppose that $m_T = (X - \lambda_1) \dots (X - \lambda_p)$ for some p. Let $v \in V$. Consider the T - cyclic subspace generated by v, $Z(v, T) = \langle v, Tv, \dots, T^r v \rangle$. Let T_v be the restriction of T to $Z(v, T)$. Of course, v is a cyclic vector of $Z(v, T_v)$, and then $m_v = m_{T_v} = \chi_T$. This is really easy to see: the dimension of $Z(v, T)$ is $r + 1$, and it's also the degree of m_v. But as m_v divides m_{T_v} (because $m_{T_v} v = 0$), and m_T divides χ_{T_v} (Cayley-Hamilton theorem), we have that m_v divides χ_{T_v}. As these are two monic polynomials of degree $r + 1$ and one divides the other, they are equal. And then by the same reasoning $m_v = m_{T_v} = \chi_T$. But as m_v divides m_T, then as $m_v = m_{T_v}$, we have that m_{T_v} divides m_T, and then m_{T_v} has no multiple roots and they all lie in k. But then so does χ_{T_v}. Suppose that these roots are $\lambda_1, \dots, \lambda_{r+1}$. Then $Z(v, T) = \bigoplus_{\lambda_i} E_{\lambda_i}$, where E_{λ_i} is the eigenspace associated to λ_i. Then v is a sum of eigenvectors. QED.