invertible matrices are dense in set of nxn matrices

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If A is any $n \times n$ matrix with real or complex entries, Then there are invertible matrices arbitrarily close to A, under any norm for the $n \times n$ matrices.

This is easily proven as follows. Take any invertible matrix B (e.g. $B = I$), and consider the function (for $t \in \mathbb{R}$ or $\mathbb{C}$)

$$p(t) = \det\bigl((1-t)A + tB\bigr) .$$

Clearly, p is a polynomial function. It is not identically zero, for $p(1) = \det B \neq 0$. But a non-zero polynomial has only finitely many zeroes, So given any single point t_0, if t is close enough but unequal to t_0, $p(t)$ must be non-zero. In particular, applying this for $t_0 = 0$, we see that the matrix $(1-t)A + tB$ is invertible for small $t \neq 0$. And the distance of this matrix from A is $|t|\,\|B - A\|$, which becomes small as t gets small.