basis-free definition of determinant

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The definition of determinant as a multilinear mapping on rows can be modified to provide a basis-free definition of determinant. In order to make it clear that we are not using bases. we shall speak in terms of an endomorphism of a vector space over k rather than speaking of a matrix whose entries belong to k. We start by recalling some preliminary facts.

Suppose V is a finite-dimensional vector space of dimension n over a field k. Recall that a multilinear map $f\colon V^n \to k$ is alternating if $f(x) = 0$ whenever there exist distinct indices $i, j \in [n] = \{1, \dots, n\}$ such that $x_i = x_j$. Every alternating map $f\colon V^n \to k$ is skew-symmetric, that is, for each permutation $\pi \in \mathfrak{S}_n$, we have that $f(x) = \operatorname{sgn}(\pi) f(x^\pi)$, where x^π denotes $(x_{\pi(i)})_{i\in[n]}$, the result of π permuting the entries of x.

Since the trivial map $0\colon V^n \to k$ is alternating and any linear combination of alternating maps is alternating, it follows that alternating maps form a subspace of the space of multilinear maps. In the following proposition we show that this subspace is one-dimensional.

Theorem. *Suppose V is a finite-dimensional vector space of dimension n over a field k. Then the space of alternating maps from V^n to k is one-dimensional.*

Proof. We use a basis here, but we will throw it away later. We need the basis here because each map we will consider has exactly as many elements as a basis of V. So let $B = \{b_i \colon i \in [n]\}$ be a basis of V.

Suppose f and g are nontrivial alternating maps from V^n to k. We claim that f and g are linearly dependent. Let $x \in V^n$. We may assume that the entries of x are basis vectors, that is, that $X = \{x_i \colon i \in [n]\} \subset \{b_i \colon i \in [n]\}$. If $X \subsetneq B$, then there exist distinct indices $i, j \in [n]$ such that $x_i = x_j$. Since f and g are alternating, it follows that $f(x) = g(x) = 0$, which implies that $f(b)g(x) = g(b)f(x)$. On the other hand, if $X = B$, then there is a permutation $\pi \in \mathfrak{S}_n$ such that $x = b^\pi$. Since f and g are skew-symmetric, it follows that

$$f(b)g(x) = \operatorname{sgn}(\pi) f(b)g(b) = g(b)f(x).$$

In either case we find that $f(b)g(x) = g(b)f(x)$. Since $f(b)$ and $g(b)$ are fixed scalars, it follows that f and g are linearly dependent.

So far we have shown only that the dimension of the space of alternating maps is less than or equal to one. In order to show that the space is one-dimensional we simply need to find a nontrivial alternating form. To do this, let $\{b_i^* \colon i \in [n]\}$ be the natural basis of V^*, so that $b_i^*(b_j)$ is the Kronecker delta of i and j for any $i, j \in [n]$. Define a map $f\colon V^n \to k$ by

$$f(x) = \sum_{\pi \in \mathfrak{S}_n} \operatorname{sgn}(\pi) \prod_{i \in [n]} b_i^*(x_{\pi(i)}).$$

One can check that f is multilinear and alternating. Moreover, $f(b) = 1$, so it is nontrivial. Hence the space of alternating maps is one-dimensional. $\square$

For an alternate view of the above results, we could look instead at linear maps from the exterior product $\bigwedge^n V$ into k. The proposition above can be viewed as saying that the dimension of $\bigwedge^n V$ is $\binom{n}{n} = 1$.

We define the determinant of an endomorphism in terms of the action of the endomorphism on alternating maps. Recall that if $M\colon V \to V$ is an endomorphism, its pullback M^* is the unique operator such that

$$(M^* f)(x_i)_{i \in [n]} = f(M(x_i))_{i \in [n]}.$$

Since the space of alternating maps is one-dimensional and endomorphisms of a one-dimensional space reduce to scalar multiplication, it follows that $M^* f$ is a scalar multiple of f. We call this scalar the determinant. It is well-defined because the scalar depends on M but not on f.

Definition. *Suppose V is a finite-dimensional vector space of dimension n over a field k, and let $M\colon V \to V$ be an endomorphism. Then the* determinant *of M is the unique scalar* $\det(M)$ *such that*

$$M^* f = \det(M) f$$

for all alternating maps $f\colon V^n \to k$.