planetmath.org

Math for the people, by the people.

spanning sets of dual space

Canonical name	SpanningSetsOfDualSpace
Date of creation	2013-03-22 17:17:28
Last modified on	2013-03-22 17:17:28
Owner	stevecheng (10074)
Last modified by	stevecheng (10074)
Numerical id	6
Author	stevecheng (10074)
Entry type	Theorem
Classification	msc 15A99

Theorem. *Let X be a vector space and $\phi_1, \dots, \phi_n \in X^*$ be functionals belonging to the dual space. A linear functional $f \in X^*$ belongs to the linear span of $\phi_1, \dots, \phi_n$ if and only if* $\ker f \supseteq \bigcap_{i=1}^n \ker \phi_i$.

ker refers to the kernel. Note that the domain X need not be finite-dimensional.

Proof. The "only if" part is easy: if $f = \sum_{i=1}^n \lambda_i \phi_i$ for some scalars λ_i, and $x \in X$ is such that $\phi_i(x) = 0$ for all i, then clearly $f(x) = 0$ too.

The "if" part will be proved by induction on n.

Suppose $\ker f \supseteq \ker \phi_1$. If $f = 0$, then the result is trivial. Otherwise, there exists $y \in X$ such that $f(y) \neq 0$. By hypothesis, we also have $\phi_1(y) \neq 0$. Every $z \in X$ can be decomposed into $z = x + ty$ where $x \in \ker \phi_1 \subseteq \ker f$, and t is a scalar. Indeed, just set $t = \phi_1(z)/\phi_1(y)$, and $x = z - ty$. Then we propose that

$$f(z) = \frac{f(y)}{\phi_1(y)} \phi_1(z)\,, \text{ for all } z \in X.$$

To check this equation, simply evaluate both sides using the decomposition $z = x + ty$.

Now suppose we have $\ker f \supseteq \bigcap_{i=1}^n \ker \phi_i$ for $n > 1$. Restrict each of the functionals to the subspace $W = \ker \phi_n$, so that $\ker f|_W \supseteq \bigcap_{i=1}^{n-1} \ker \phi_i|_W$. By the induction hypothesis, there exist scalars $\lambda_1, \dots, \lambda_{n-1}$ such that $f|_W = \sum_{i=1}^{n-1} \lambda_i \phi_i|_W$. Then $\ker(f - \sum_{i=1}^{n-1} \lambda_i \phi_i) \supseteq W = \ker \phi_n$, and the argument for the case $n = 1$ can be applied anew, to obtain the final λ_n. $\square$