derivative of homogeneous function

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Theorem 1. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable positively homogeneous function of degree r. Then $\frac{\partial f}{\partial x^i}$ is a positively homogeneous function of degree $r-1$.*

Proof. By considering component functions if necessary, we can assume that $m = 1$. For $\lambda \in \mathbb{R}$, let M_λ be the multiplication map,

$$\begin{array}{rcl} M_\lambda\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ v & \mapsto & \lambda v. \end{array}$$

For $\lambda > 0$ and $v \in \mathbb{R}^n$, we have

$$\begin{aligned} \frac{\partial f}{\partial x^i}(\lambda v) &= \frac{\partial (f \circ M_\lambda \circ M_{1/\lambda})}{\partial x^i}(\lambda v) \\ &= \sum_{l=1}^{n} \frac{\partial (f \circ M_\lambda)}{\partial x^l}(v)\, \frac{\partial (x \mapsto x/\lambda)^l}{\partial x^i}(\lambda v) \\ &= \frac{\partial (f \circ M_\lambda)}{\partial x^i}(v)\, \frac{1}{\lambda} \\ &= \lambda^{r-1} \frac{\partial f}{\partial x^i}(v) \end{aligned}$$

as claimed. □