square root of positive definite matrix

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Suppose M is a positive definite Hermitian matrix. Then M has a diagonalization

$$M = P^* \operatorname{diag}(\lambda_1, \ldots, \lambda_n) P$$

where P is a unitary matrix and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M, which are all positive.

We can now define the *square root* of M as the matrix

$$M^{1/2} = P^* \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) P.$$

The following properties are clear

1. $M^{1/2} M^{1/2} = M$,
2. $M^{1/2}$ is Hermitian and positive definite.
3. $M^{1/2}$ and M commute
4. $(M^{1/2})^T = (M^T)^{1/2}$.
5. $(M^{1/2})^{-1} = (M^{-1})^{1/2}$, so one can write $M^{-1/2}$
6. If the eigenvalues of M are $(\lambda_1, \ldots, \lambda_n)$, then the eigenvalues of $M^{1/2}$ are $(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$.