cofactor expansion

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Let M be an $n \times n$ matrix with entries M_{ij} that are elements of a commutative ring. Let m_{ij} denote the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j of M, and let

$$C_{ij} = (-1)^{i+j} m_{ij}.$$

The *subdeterminants* m_{ij} are called the *minors* of M, and the C_{ij} are called the *cofactors*.

We have the following useful formulas for the cofactors of a matrix. First, if we regard $\det M$ as a polynomial in the entries M_{ij}, then we may write

$$C_{ij} = \frac{\partial M}{\partial M_{ij}} \tag{1}$$

Second, we may regard the determinant of $M = (M_1, \ldots, M_n)$ as a multi-linear, skew-symmetric function of its columns:

$$\det M = \det(M_1, \ldots, M_n).$$

This point of view leads to the following formula:

$$C_{ij} = \det(M_1, \ldots, \hat{M}_j, \mathbf{e}_i, \ldots, M_n), \tag{2}$$

where the notation indicates that column j has been replaced by the ith standard vector.

As a consequence, we obtain the following representation of the determinant in terms of cofactors:

$$\begin{aligned} \det(M) &= \det(M_1, \ldots, M_{1j}\mathbf{e}_1 + \cdots + M_{nj}\mathbf{e}_n, \ldots, M_n) \\ &= \sum_{i=1}^{n} M_{ij} C_{ij}, \quad j = 1, \ldots, n. \end{aligned}$$

The above identity is often called the cofactor expansion of the determinant along column j. If we regard the determinant as a multi-linear, skew-symmetric function of n row-vectors, then we obtain the analogous cofactor expansion along a row:

$$\det(M) = \sum_{i=1}^{n} M_{ji} C_{ji}.$$

Example. Consider a general 3×3 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3.$$

The above can equally well be expressed as a cofactor expansion along the first row:

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1); \end{aligned}$$

or along the second column:

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ &= -a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1); \end{aligned}$$

or indeed as four other such expansion corresponding to rows 2 and 3, and columns 1 and 3.