# diagonalization

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Let $V$ be a finite-dimensional linear space over a field $K$, and $T : V \to V$ a linear transformation. To *diagonalize* $T$ is to find a basis of $V$ that consists of eigenvectors. The transformation is called *diagonalizable* if such a basis exists. The choice of terminology reflects the fact that the matrix of a linear transformation relative to a given basis is diagonal if and only if that basis consists of eigenvectors.

Next, we give necessary and sufficient conditions for $T$ to be diagonalizable. For $\lambda \in K$ set

$$E_\lambda = \{u \in V : Tu = \lambda u\}.$$

It isn't hard to show that $E_\lambda$ is a subspace of $V$, and that this subspace is non-trivial if and only if $\lambda$ is an *eigenvalue* of $T$. In that case, $E_\lambda$ is called the eigenspace associated to $\lambda$.

**Proposition 1** *A transformation is diagonalizable if and only if*

$$\dim V = \sum_\lambda \dim E_\lambda,$$

*where the sum is taken over all eigenvalues of the transformation.*

**The Matrix Approach.** As was already mentioned, the term "diagonalize" comes from a matrix-based perspective. Let $M$ be a `http://planetmath.org/matrix`matrix representation of $T$ relative to some basis $B$. Let

$$P = [v_1, \ldots, v_n], \quad n = \dim V,$$

be a matrix whose column vectors are eigenvectors expressed relative to $B$. Thus,

$$Mv_i = \lambda_i v_i, \quad i = 1, \ldots, n$$

where $\lambda_i$ is the eigenvalue associated to $v_i$. The above $n$ equations are more succinctly as the matrix equation

$$MP = PD,$$

where $D$ is the diagonal matrix with $\lambda_i$ in the $i$-th position. Now the eigenvectors in question form a basis, if and only if $P$ is invertible. In that case, we may write

$$M = PDP^{-1}. \tag{1}$$

Thus in the matrix-based approach, to "diagonalize" a matrix $M$ is to find an invertible matrix $P$ and a diagonal matrix $D$ such that equation (**??**) is satisfied.

**Subtleties.** There are two fundamental reasons why a transformation $T$ can fail to be diagonalizable.

1. The characteristic polynomial of $T$ does not factor into linear factors over $K$.

2. There exists an eigenvalue $\lambda$, such that the kernel of $(T-\lambda I)^2$ is strictly greater than the kernel of $(T-\lambda I)$. Equivalently, there exists an invariant subspace where $T$ acts as a nilpotent transformation plus some multiple of the identity. Such subspaces manifest as non-trivial Jordan blocks in the Jordan canonical form of the transformation.