singular value decomposition

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Any real $m \times n$ matrix A can be decomposed into

$$A = USV^T$$

where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, and S is a unique $m \times n$ diagonal matrix with real, non-negative elements σ_i, $i = 1, \dots, \min(m, n)$, in descending order:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0$$

The σ_i are the *singular values* of A and the first $\min(m, n)$ columns of U and V are the left and right (respectively) *singular vectors* of A. S has the form:

$$\begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \text{ if } m \geq n \text{ and } \begin{bmatrix} \Sigma & 0 \end{bmatrix} \text{ if } m < n,$$

where Σ is a diagonal matrix with the diagonal elements $\sigma_1, \sigma_2, \dots, \sigma_{\min(m,n)}$. We assume now $m \geq n$. If $r = \text{rank}(A) < n$, then

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0.$$

If $\sigma_r \neq 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$, then r is the rank of A. In this case, S becomes an $r \times r$ matrix, and U and V shrink accordingly. SVD can thus be used for rank determination.

The SVD provides a numerically robust solution to the least-squares problem. The matrix-algebraic phrasing of the least-squares solution x is

$$x = (A^T A)^{-1} A^T b$$

Then utilizing the SVD by making the replacement $A = USV^T$ we have

$$x = V \begin{bmatrix} \Sigma^{-1} & 0 \end{bmatrix} U^T b.$$

References

- Originally from The Data Analysis Briefbook (`http://rkb.home.cern.ch/rkb/titleA.html`