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Frobenius theorem on linear determinant preservers

Canonical name	FrobeniusTheoremOnLinearDeterminantPreservers
Date of creation	2013-03-22 19:19:52
Last modified on	2013-03-22 19:19:52
Owner	kammerer (26336)
Last modified by	kammerer (26336)
Numerical id	7
Author	kammerer (26336)
Entry type	Theorem
Classification	msc 15A04
Classification	msc 15A15
Related topic	DieudonneTheoremOnLinearPreserversOfTheSingularMatrices

Let $\mathbb{F}$ be an arbitrary field. Consider $\mathcal{M}_n(\mathbb{F})$, the vector space of all $n \times n$ matrices over $\mathbb{F}$. Let $\mathcal{GL}_n(\mathbb{F})$ be the set of all nonsingular matrices $P \in \mathcal{M}_n(\mathbb{F})$.

Definition 1. *A linear endomorphism $\varphi : \mathcal{M}_n(\mathbb{F}) \longrightarrow \mathcal{M}_n(\mathbb{F})$ is said to be in standard form, if either $\exists\, P, Q \in \mathcal{GL}_n(\mathbb{F})\, \forall\, A \in \mathcal{M}_n(\mathbb{F}) :\ \varphi(A) = PAQ$ or $\exists\, P, Q \in \mathcal{GL}_n(\mathbb{F})\, \forall\, A \in \mathcal{M}_n(\mathbb{F}) :\ \varphi(A) = PA^\top Q$.*

The classical on linear preservers of the determinant function [**?**] reads as follows.

Theorem 2. *If $\varphi : \mathcal{M}_n(\mathbb{C}) \longrightarrow \mathcal{M}_n(\mathbb{C})$ is a linear automorphism such that $\det(\varphi(A)) = \det(A)$ for all $A \in \mathcal{M}_n(\mathbb{C})$, then φ is in standard form with $\det(PQ) = 1$.*

It is well known that the can be strengthened.

Theorem 3. *Let $\mathbb{F}$ be an arbitrary field and let $\varphi : \mathcal{M}_n(\mathbb{F}) \longrightarrow \mathcal{M}_n(\mathbb{F})$ be a linear endomorphism. Then the following conditions are equivalent:*

(i) $\det(\varphi(A)) = \det(A)$ *for all* $A \in \mathcal{M}_n(\mathbb{F})$,

(ii) φ *is in standard form with* $\det(PQ) = 1$.

The above strengthened version of the can be derived from the Dieudonné theorem on linear preservers of the singular matrices.

References

[GF] G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen*, Sitzungsber., Preuss. Akad. Wiss., Berlin, 1897 (994–1015).