proof of Jordan canonical form theorem

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This theorem can be proved combining the cyclic decomposition theorem and the primary decomposition theorem. By hypothesis, the characteristic polynomial of T factorizes completely over F, and then so does the minimal polynomial of T (or its annihilator polynomial). This is because the minimal polynomial of T has exactly the same factors on $F[X]$ as the characteristic polynomial of T. Let's suppose then that the minimal polynomial of T factorizes as $m_T = (X - \lambda_1)^{\alpha_1} \dots (X - \lambda_r)^{\alpha_r}$. We know, by the primary decomposition theorem, that

$$V = \bigoplus_{i=1}^{r} \ker((T - \lambda_i I)^{\alpha_i}).$$

Let T_i be the restriction of T to $\ker((T - \lambda_i I)^{\alpha_i})$. We apply now the cyclic decomposition theorem to every linear operator

$$(T_i - \lambda_i I) \colon \ker(T - \lambda_i I)^{\alpha_i} \to \ker(T - \lambda_i I)^{\alpha_i}.$$

We know then that $\ker(T - \lambda_i I)^{\alpha_i}$ has a basis B_i of the form $B_i = B_{1,i} \bigcup B_{2,i} \bigcup \dots \bigcup B_{d_i,i}$ such that each $B_{s,i}$ is of the form

$$B_{s,i} = \{v_{s,i}, (T - \lambda_i)v_{s,i}, (T - \lambda_i)^2 v_{s,i}, \dots, (T - \lambda_i)^{k_{s,i}} v_{s,i}\}.$$

Let's see that T in each of this "cyclic sub-basis" $B_{s,i}$ is a Jordan block: Simply notice the following fact about this polynomials:

$$\begin{aligned} X(X - \lambda_i)^j &= (X - \lambda_i)^{j+1} + X(X - \lambda_i)^j - (X - \lambda_i)^{j+1} \\ &= (X - \lambda_i)^{j+1} + (X - X + \lambda_i)(X - \lambda_i)^j \\ &= (X - \lambda_i)^{j+1} + \lambda_i (X - \lambda_i)^j \end{aligned}$$

and then

$$T(T - \lambda_i I)^j (v_{s,i}) = (T - \lambda_i)^{j+1}(v_{s,i}) + \lambda_i (T - \lambda_i I)^j (v_{s,i}).$$

So, if we also notice that $(T - \lambda_i I)^{k_{s,i}+1}(v_{s,i}) = 0$, we have that T in this sub-basis is the Jordan block

$$\begin{pmatrix} \lambda_i & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_i & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda_i \end{pmatrix}$$

So, taking the basis $B = B_1 \bigcup B_2 \bigcup \ldots \bigcup B_r$, we have that T in this basis has a Jordan form.

This form is unique (except for the order of the blocks) due to the uniqueness of the cyclic decomposition.