matrix operations

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Entry type	Definition
Classification	msc 15-00
Defines	matrix addition
Defines	matrix multiplication
Defines	matrix product
Defines	standard matrix multiplication

A *matrix* is an array, or a rectangular grid, of numbers. An $m \times n$ matrix is one which has m rows and n columns. Examples of matrices include:

- The 2×3 matrix
$$A = \begin{pmatrix} 1 & -2 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$
- The 3×3 matrix
$$B = \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 5 \\ 4 & 2 & 0 \end{pmatrix}$$
- The 3×1 matrix
$$C = \begin{pmatrix} 7 \\ 7 \\ 2 \end{pmatrix}$$
- The 1×2 matrix
$$D = \begin{pmatrix} \frac{1}{2} & 2 \end{pmatrix}$$

All of our example matrices (except the last one) have entries which are integers. In general, matrices are allowed to have their entries taken from any ring R. The set of all $m \times n$ matrices with entries in a ring R is denoted $M_{m \times n}(R)$. If a matrix has exactly as many rows as it has columns, we say it is a square matrix.

Addition of two matrices is allowed provided that both matrices have the same number of rows and the same number of columns. The sum of two such matrices is obtained by adding their respective entries. For example,

$$\begin{pmatrix} 7 \\ 7 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 4.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 + (-1) \\ 7 + 4.5 \\ 2 + 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 11.5 \\ 2 \end{pmatrix}$$

Multiplication of two matrices is allowed provided that the number of columns of the first matrix equals the number of rows of the second matrix. (For example, multiplication of a 2×3 matrix with a 3×3 is allowed, but multiplication of a 3×3 matrix with a 2×3 matrix is not allowed, since the first matrix has 3 columns, and the second matrix has 2 rows, and 3 doesn't equal 2.) In this case the matrix multiplication is defined by

$$(AB)_{ij} = \sum_k (A)_{ik}(B)_{kj} \ .$$

We will describe how matrix multiplication works is with an example. Let

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 5 \\ 4 & 2 & 0 \end{pmatrix}$$

be the two matrices that we used above as our very first two examples of matrices. Since A is a 2×3 matrix, and B is a 3×3 matrix, it is legal to multiply A and B, but it is not legal to multiply B and A. The method for computing the product AB is to place A below and to the left of B, as follows:

$$\begin{array}{cc} & \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 5 \\ 4 & 2 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -2 & 2 \\ 1 & 0 & 2 \end{pmatrix} & \begin{pmatrix} X & Y & \\ & & \end{pmatrix} \end{array}$$

A is always in the bottom left corner, B is in the top right corner, and the product, AB, is always in the bottom right corner. We see from the picture that AB will be a 2×3 matrix. (In general, AB has as many rows as A, and as many columns as B.)

Let us compute the top left entry of AB, denoted by X in the above picture. The way to calculate this entry of AB (or any other entry) is to take the dot product of the stuff above it [which is $(1, 3, 4)$] and the stuff to the left of it [which is $(1, -2, 2)$]. In this case, we have

$$X = 1 \cdot 1 + 3 \cdot (-2) + 4 \cdot 2 = 3.$$

Similarly, the top middle entry of AB (where the Y is in the above picture) is gotten by taking the dot product of the stuff above it: $(0, -1, 2)$, and the stuff to the left of it: $(1, -2, 2)$, which gives

$$Y = 0 \cdot 1 + (-1) \cdot (-2) + 2 \cdot 2 = 6$$

Continuing in this way, we can compute every entry of AB one by one to get

$$\begin{array}{cc} & \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 5 \\ 4 & 2 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -2 & 2 \\ 1 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 3 & 6 & -9 \\ 9 & 4 & 1 \end{pmatrix} \end{array}$$

and so

$$AB = \begin{pmatrix} 3 & 6 & -9 \\ 9 & 4 & 1 \end{pmatrix}.$$

If one tries to compute the **illegal** product BA using this procedure, one winds up with

$$\begin{array}{cc} & \begin{pmatrix} 1 & -2 & 2 \\ 1 & 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 5 \\ 4 & 2 & 0 \end{pmatrix} & \begin{pmatrix} ? & & \\ & & \\ & & \end{pmatrix} \end{array}$$

The top left entry of this illegal product (marked with a ? above) would have to be the dot product of the stuff above it: $(1, 1)$, and the stuff to the left of it: $(1, 0, 1)$, but these vectors do not have the same length, so it is impossible to take their dot product, and consequently it is impossible to take the product of the matrices BA.

Under the correspondence of matrices and linear transformations, one can show that matrix multiplication is equivalent to composition of linear transformations, which explains why matrix multiplication is defined in a manner which is so odd at first sight, and why this strange manner of multiplication is so useful in mathematics.