there exist additive functions which are not linear

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Example 1. There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ which is additive but not linear.

Proof. Let V be the infinite dimensional vector space $\mathbb{R}$ over the field $\mathbb{Q}$. Since 1 and $\sqrt{2}$ are two independent vectors in V, we can extend the set $\{1, \sqrt{2}\}$ to a basis E of V (notice that here the axiom of choice is used).

Now we consider a linear function $f\colon V \to \mathbb{R}$ such that $f(1) = 1$ while $f(e) = 0$ for all $e \in E \setminus \{1\}$. This function is $\mathbb{Q}$-linear (i.e. it is additive on $\mathbb{R}$) but it is not $\mathbb{R}$-linear because $f(\sqrt{2}) = 0 \neq \sqrt{2} f(1)$. $\square$