tensor product (vector spaces)

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Definition. The classical conception of the tensor product operation involved finite dimensional vector spaces A, B, say over a field $\mathbb{K}$. To describe the tensor product $A \otimes B$ one was obliged to chose bases

$$\mathbf{a}_i \in A,\ i \in I, \quad \mathbf{b}_j \in B,\ j \in J$$

of A and B indexed by finite sets I and J, respectively, and represent elements of $\mathbf{a} \in A$ and $\mathbf{b} \in B$ by their coordinates relative to these bases, i.e. as mappings $a : I \to \mathbb{K}$ and $b : J \to \mathbb{K}$ such that

$$\mathbf{a} = \sum_{i \in I} a^i \mathbf{a}_i, \qquad \mathbf{b} = \sum_{j \in J} b^j \mathbf{b}_j.$$

One then represented $A \otimes B$ relative to this particular choice of bases as the vector space of mappings $c : I \times J \to \mathbb{K}$. These mappings were called "second-order contravariant tensors" and their values were customarily denoted by superscripts, a.k.a. contravariant indices:

$$c^{ij} \in \mathbb{K}, \quad i \in I,\ j \in J.$$

The canonical bilinear multiplication (also known as outer multiplication)

$$\otimes : A \times B \to A \otimes B$$

was defined by representing $\mathbf{a} \otimes \mathbf{b}$, relative to the chosen bases, as the tensor

$$c^{ij} = a^i b^j, \quad i \in I,\ j \in J.$$

In this system, the products

$$\mathbf{a}_i \otimes \mathbf{b}_j, \quad i \in I,\ j \in J$$

were represented by basic tensors, specified in terms of the Kronecker deltas as the mappings

$$(i', j') \mapsto \delta_i^{i'} \delta_j^{j'}, \quad i' \in I,\ j' \in J.$$

These gave a basis of $A \otimes B$.

The construction is independent of the choice of bases in the following sense. Let

$$\mathbf{a}'_i \in A,\ i \in I', \qquad \mathbf{b}'_j \in B,\ j \in J'$$

be different bases of A and B with indexing sets I' and J' respectively. Let

$$r : I \times I' \to \mathbb{K}, \qquad s : J \times J' \to \mathbb{K},$$

be the corresponding change of basis matrices determined by

$$\begin{aligned}
\mathbf{a}'_{i'} &= \sum_{i \in I} \left(r^i_{i'}\right) \mathbf{a}_i, \quad i' \in I' \\
\mathbf{b}'_{j'} &= \sum_{j \in I} \left(s^j_{j'}\right) \mathbf{b}_j, \quad j' \in J'.
\end{aligned}$$

One then stipulated that tensors $c : I \times J \to \mathbb{K}$ and $c' : I' \times J' \to \mathbb{K}$ represent the same element of $A \otimes B$ if

$$c^{ij} = \sum_{\substack{i' \in I' \\ j' \in J'}} \left(r^i_{i'}\right) \left(s^j_{j'}\right) (c')^{i'j'} \tag{1}$$

for all $i \in I, j \in J$. This relation corresponds to the fact that the products

$$\mathbf{a}'_i \otimes \mathbf{b}'_j, \quad i \in I',\ j \in J'$$

constitute an alternate basis of $A \otimes B$, and that the change of basis relations are

$$\mathbf{a}'_{i'} \otimes \mathbf{b}'_{j'} = \sum_{\substack{i \in I \\ j \in J}} \left(r^i_{i'}\right) \left(s^j_{j'}\right) \mathbf{a}_i \otimes \mathbf{b}_j, \quad i' \in I',\ j' \in J'. \tag{2}$$

Notes. Historically, the tensor product was called the *outer* product, and has its origins in the absolute differential calculus (the theory of manifolds). The old-time tensor calculus is difficult to understand because it is afflicted with a particularly lethal notation that makes coherent comprehension all but impossible. Instead of talking about an element $\mathbf{a}$ of a vector space, one was obliged to contemplate a symbol $\mathbf{a}^i$, which signified a list of real numbers indexed by $1, 2, \ldots, n$, and which was understood to represent $\mathbf{a}$ relative to some specified, but unnamed basis.

What makes this notation truly lethal is the fact a symbol $\mathbf{a}^j$ was taken to signify an alternate list of real numbers, also indexed by $1, \ldots, n$, and also representing $\mathbf{a}$, albeit relative to a different, but equally unspecified basis. Note that the choice of *dummy variables make all the difference.* Any sane system of notation would regard the expression

$$\mathbf{a}^i, \quad i = 1, \ldots, n$$

as representing a list of n symbols

$$\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^n.$$

However, in the classical system, one was strictly forbidden from using

$$\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^n$$

because where, after all, is the all important dummy variable to indicate choice of basis?

Thankfully, it is possible to shed some light onto this confusion (I have read that this is credited to Roger Penrose) by interpreting the symbol $\mathbf{a}^i$ as a mapping from some finite index set I to $\mathbb{R}$, whereas $\mathbf{a}^j$ is interpreted as a mapping from another finite index set J (of equal cardinality) to $\mathbb{R}$.

My own surmise is that the source of this notational difficulty stems from the reluctance of the ancients to deal with geometric objects directly. The prevalent superstition of the age held that in order to have meaning, a geometric entity had to be *measured* relative to some basis. Of course, it was understood that geometrically no one basis could be preferred to any other, and this leads directly to the definition of geometric entities as lists of measurements modulo the equivalence engendered by changing the basis.

It is also worth remarking on the contravariant nature of the relationship between the actual elements of $A \otimes B$ and the corresponding representation by tensors relative to a basis — compare equations (1) and (2). This relationship is the source of the terminology "contravariant tensor" and "contravariant index", and I surmise that it is this very medieval pit of darkness and confusion that spawned the present-day notion of "contravariant functor".

References.

1. Levi-Civita, "The Absolute Differential Calculus."