proof of Bauer-Fike theorem

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We can assume $\tilde{\lambda} \notin \sigma(A)$ (otherwise, we can choose $\lambda = \tilde{\lambda}$ and theorem is proven, since $\kappa_p(X) > 1$). Then $(A - \tilde{\lambda}I)^{-1}$ exists, so we can write:

$$\tilde{u} = (A - \tilde{\lambda}I)^{-1} r = X(D - \tilde{\lambda}I)^{-1} X^{-1} r$$

since A is diagonalizable; taking the `http://planetmath.org/VectorPnormp`-norm of both sides, we obtain:

$$\begin{aligned} 1 &= \|\tilde{u}\|_p \\ &= \|X(D - \tilde{\lambda}I)^{-1} X^{-1} r\|_p \leq \|X\|_p \|(D - \tilde{\lambda}I)^{-1}\|_p \|X^{-1}\|_p \|r\|_p \\ &= \kappa_p(X) \|(D - \tilde{\lambda}I)^{-1}\|_p \|r\|_p. \end{aligned}$$

But, since $(D - \tilde{\lambda}I)^{-1}$ is a diagonal matrix, the p-norm is easily computed, and yields:

$$\|(D - \tilde{\lambda}I)^{-1}\|_p = \max_{\|x\|_p \neq 0} \frac{\|(D - \tilde{\lambda}I)^{-1} x\|_p}{\|x\|_p} = \max_{\lambda \in \sigma(A)} \frac{1}{|\lambda - \tilde{\lambda}|} = \frac{1}{\min\limits_{\lambda \in \sigma(A)} |\lambda - \tilde{\lambda}|}$$

whence:

$$\min_{\lambda \in \sigma(A)} |\lambda - \tilde{\lambda}| \leq \kappa_p(X) ||r||_p.$$