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distance of non-parallel lines

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As an application of the http://planetmath.org/CrossProductvector product we derive the expression of the d between two non-parallel lines in $\mathbb{R}^3$.

Suppose that the position vectors of the points of the two non-parallel lines are expressed in parametric forms

$$\vec{r} = \vec{a}+s\vec{u}$$

and

$$\vec{r} = \vec{b}+t\vec{v},$$

where s and t are parameters. A common vector of the lines is the cross product $\vec{u} \times \vec{v}$ of the direction vectors of the lines, and it may be normed to a unit vector

$$\vec{n} := \frac{\vec{u}\times\vec{v}}{|\vec{u}\times\vec{v}|}$$

by dividing it by its , which is distinct from 0 because of the non-parallelity. The vectors $\vec{a}$ and $\vec{b}$ are the position vectors of certain points A and B on the lines, and thus their difference $\vec{a}-\vec{b}$ is the vector from B to A. If we project $\vec{a}-\vec{b}$ on the unit normal $\vec{n}$, the obtained vector

$$\vec{d} := [(\vec{a}-\vec{b})\cdot\vec{n}]\,\vec{n}$$

has the sought $d = |(\vec{a}-\vec{b})\cdot\vec{n}|$, i.e.

$$d = \frac{|(\vec{a}-\vec{b})\cdot(\vec{u}\times\vec{v})|}{|\vec{u}\times\vec{v}|}.$$

For illustrating that d is the minimal distance between points of the two lines we underline, that the construction of d guarantees that it connects two points on the lines and is perpendicular to both lines — thus any displacement of its end point makes it longer.

Notes. The numerator is the absolute value of a triple scalar product. If the lines intersect each other, then the connecting vector $\vec{a}-\vec{b}$ is at right angles to the common normal vector $\vec{n}$ of their plane and thus the dot product of these vectors vanishes, i.e. also $d = 0$. If the lines do not intersect, they are called *agonic lines* or *skew lines*; then $d > 0$.