example of non-diagonalizable matrices

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Some matrices with real entries which are not diagonalizable over $\mathbb{R}$ *are* diagonalizable over the complex numbers $\mathbb{C}$.

For instance,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has $\lambda^2 + 1$ as characteristic polynomial. This polynomial doesn't factor over the reals, but over $\mathbb{C}$ it does. Its roots are $\lambda = \pm i$.

Interpreting the matrix as a linear transformation $\mathbb{C}^2 \to \mathbb{C}^2$, it has eigenvalues i and $-i$ and linearly independent eigenvectors $(1, -i)$, $(-i, 1)$. So we can diagonalize A:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} .5 & .5i \\ .5i & .5 \end{pmatrix}$$

But there exist real matrices which aren't diagonalizable even if complex eigenvectors and eigenvalues are allowed.

For example,

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

cannot be written as UDU^{-1} with D diagonal.

In fact, the characteristic polynomial is λ^2 and it has only one double root $\lambda = 0$. However the eigenspace corresponding to the 0 (kernel) eigenvalue has dimension 1.

$B \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff v_2 = 0$ and thus the eigenspace is $ker(B) = span_{\mathbb{C}} \left\{ (1, 0)^T \right\}$, with only one dimension.

There isn't a change of basis where B is diagonal.