properties of spanning sets

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Let V be a vector space over a field k. Let S be a subset of V. We denote $\mathrm{Sp}(S)$ the span of the set S. Below are some basic properties of spanning sets.

1. If $S \subseteq T$, then $\mathrm{Sp}(S) \subseteq \mathrm{Sp}(T)$. In particular, if $\mathrm{Sp}(S) = V$, every superset of S spans (generates) V.

   Proof. If $v \in \mathrm{Sp}(S)$, then $v = r_1v_1 + \cdots + r_nv_n$ for $v_i \in S$. But $v_i \in T$ by assumption. So $v \in \mathrm{Sp}(T)$ as well. If $\mathrm{Sp}(S) = V$, and $S \subseteq T$, then $V = \mathrm{Sp}(S) \subseteq \mathrm{Sp}(T) \subseteq V$. □

2. If S contains 0, then $\mathrm{Sp}(S - \{0\}) = \mathrm{Sp}(S)$.

   Proof. Let $T = S - \{0\}$. So $\mathrm{Sp}(T) \subseteq \mathrm{Sp}(S)$ by 1 above. If $v \in \mathrm{Sp}(S)$, then $v = r_1v_1 + \cdots + r_nv_n$. If one of the v_i's, say v_i, is 0, then $v = r_2v_2 + \cdots + r_nv_n \in \mathrm{Sp}(T)$. □

3. It is not true that if $S_1 \supseteq S_2 \supseteq \cdots$ is a chain of subsets, each spanning the same subspace W of V, so does their intersection.

   Proof. Take $V = \mathbb{R}^n$, the Euclidean space in n dimensions. For each $i = 1, 2, \ldots$, let S_i be the closed ball centered at the origin, with radius $1/i$. Then $\mathrm{Sp}(S_i) = V$. But the intersection of these S_i's is just the origin, whose span is itself, not V. □

4. S is a basis for V iff S is a minimal spanning set of V. Here, minimal means that any deletion of an element of S is no longer a spanning set of V.

   Proof. If S is a basis for V, then S spans V and S is linearly independent. Let T be the set obtained from S with $v \in S$ deleted. If T spans V, then v can be written as a linear combination of elements in T. But then $S = T \cup \{v\}$ would no longer be linearly independent, contradiction the assumption. Therefore, S is minimal.

   Conversely, suppose S is a minimal spanning set for V. Furthermore, suppose that S is linearly dependent. Let $0 = r_1v_1 + \cdots r_nv_n$, with $r_1 \neq 0$. Then

   $$v_1 = s_2v_2 + \cdots + s_nv_n, \tag{1}$$

where $s_i = -r_i/r_1$. So any linear combination of elements in S involving v_1 can be replaced by a linear combination not involving v_1 through equation (1). Therefore $\mathrm{Sp}(S) = \mathrm{Sp}(S - \{v\})$. But this means that S is not minimal, contrary to our assumption. Therefore, S must be linearly independent. $\square$

Remark. All of the properties above can be generalized to modules over rings, except the last one, where the implication is only one-sided: basis implying minimal spanning set.