basis

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A (Hamel) basis of a vector space is a linearly independent spanning set.

It can be proved that any two bases of the same vector space must have the same cardinality. This introduces the notion of dimension of a vector space, which is precisely the cardinality of the basis, and is denoted by $\dim(V)$, where V is the vector space.

The fact that `http://planetmath.org/EveryVectorSpaceHasABasis`every vector space has a Hamel basis is an important consequence of the axiom of choice (in fact, that proposition is equivalent to the axiom of choice.)

Examples.

- $\beta = \{e_i\}$, $1 \leq i \leq n$, is a basis for $\mathbb{R}^n$ (the n-dimensional vector space over the reals). For $n = 4$,

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- $\beta = \{1, x, x^2\}$ is a basis for the vector space of polynomials with degree at most 2, over a division ring.

- The set

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

  is a basis for the vector space of 2×2 matrices over a division ring, and assuming that the characteristic of the ring is not 2, then so is

$$\beta' = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \right\}.$$

- The empty set is a basis for the trivial vector space which consists of the unique element 0.

Remark. More generally, for any (left) right module M over a ring R, one may define a (left) right basis for M as a subset B of M such that B spans M and is linearly independent. However, unlike bases for a vector space, bases for a module may not have the same cardinality.