proof of theorem for normal matrices

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1) ($A^H = g(A) \to A$ is normal)

Keeping in mind that every matrix commutes with its own powers, let's compute

$$AA^H = Ag(A) = A\sum_{i=0}^{n-1} a_i A^i = \sum_{i=0}^{n-1} a_i AA^i = \sum_{i=0}^{n-1} a_i A^i A = \left(\sum_{i=0}^{n-1} a_i A^i\right) A = g(A)A = A^H A$$

which shows A to be normal.

2) (A is normal $\to A^H = g(A)$)

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$, $1 \leq r \leq n$ be the distinct eigenvalues of A, and let $\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$; then it's possible to find a $(r-1)$-degree polynomial $g(t)$ such that $g(\lambda_i) = \lambda_i^*$ $1 \leq i \leq r$, solving the $r \times r$ linear Vandermonde system:

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{r-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_{r-1} & \lambda_{r-1}^2 & \cdots & \lambda_{r-1}^{r-1} \\ 1 & \lambda_r & \lambda_r^2 & \cdots & \lambda_r^{r-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{r-1} \end{bmatrix} = \begin{bmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_3^* \\ \vdots \\ \lambda_r^* \end{bmatrix}$$

Since these r eigenvalues are distinct, the Vandermonde matrix is full rank, and the linear system admits a unique solution; so a $(r-1)$-degree polynomial $g(t)$ can be found such that $g(\lambda_i) = \lambda_i^*$ $1 \leq i \leq r$ and therefore $g(\lambda_i) = \lambda_i^*$ $1 \leq i \leq n$. Writing these equations in matrix form, we have

$$g(\Lambda) = \Lambda^*$$

By Schur's decomposition theorem, a unitary matrix U and an upper triangular matrix T exist such that

$$A = UTU^H$$

and since A is normal we have $T = \Lambda$.

Let's evaluate $g(A)$.

$$g(A) = g(U\Lambda U^H) = \sum_{i=0}^{r-1} a_i (U\Lambda U^H)^i$$

But, keeping in mind that $U^H U = I$,

$$(U\Lambda U^H)^i = \overbrace{U\Lambda U^H U\Lambda U^H U\Lambda U^H \cdots U\Lambda U^H}^{i\ times} = U\Lambda^i U^H$$

and so

$$
\begin{aligned}
g(A) &= \sum_{i=0}^{r-1} a_i (U\Lambda^i U^H) \\
&= U\left(\sum_{i=0}^{r-1} a_i \Lambda^i\right) U^H \\
&= U g(\Lambda) U^H \\
&= U\Lambda^* U^H \\
&= U\Lambda^H U^H \\
&= (U\Lambda U^H)^H = A^H
\end{aligned}
$$

which is the thesis.

Remark: note that this is a constructive proof, giving explicitly a way to find $g(t)$ polynomial by solving Vandermonde system in the eigenvalues.

Example:

Let $A = \frac{1}{2}\begin{bmatrix} 1+j & -1-j \\ 1+j & 1+j \end{bmatrix}$ (which is easily checked to be normal),

with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -j \\ j & -1 \end{bmatrix}$. Then $\sigma(A) = \{1, j\}$ and the Vandermonde system is

$$
\begin{bmatrix} 1 & 1 \\ 1 & j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -j \end{bmatrix}
$$

from which we find

$$
g(t) = (1-j) + jt
$$

A simple calculation yields

$$
g(A) = (1-j)I + jA = \frac{1}{2}\begin{bmatrix} 1-j & 1-j \\ -1+j & 1-j \end{bmatrix} = A^H
$$