Clifford algebra

Canonical name	CliffordAlgebra
Date of creation	2013-03-22 13:18:05
Last modified on	2013-03-22 13:18:05
Owner	rmilson (146)
Last modified by	rmilson (146)
Numerical id	9
Author	rmilson (146)
Entry type	Definition
Classification	msc 15A66
Classification	msc 11E88

Let V be a vector space over a field k, and $Q : V \times V \to k$ a symmetric bilinear form. Then the Clifford algebra $\mathrm{Cliff}(Q, V)$ is the quotient of the tensor algebra $\mathcal{T}(V)$ by the relations

$$v \otimes w + w \otimes v = -2Q(v, w) \qquad \forall v, w \in V.$$

Since the above relationship is not homogeneous in the usual $\mathbb{Z}$-grading on $\mathcal{T}(V)$, $\mathrm{Cliff}(Q, V)$ does not inherit a $\mathbb{Z}$-grading. However, by reducing mod 2, we also have a $\mathbb{Z}_2$-grading on $\mathcal{T}(V)$, and the relations above are homogeneous with respect to this, so $\mathrm{Cliff}(Q, V)$ has a natural $\mathbb{Z}_2$-grading, which makes it into a superalgebra.

In addition, we do have a filtration on $\mathrm{Cliff}(Q, V)$ (making it a filtered algebra), and the associated graded algebra $\mathrm{Gr}\,\mathrm{Cliff}(Q, V)$ is simply $\Lambda^* V$, the exterior algebra of V. In particular,

$$\dim \mathrm{Cliff}(Q, V) = \dim \Lambda^* V = 2^{\dim V}.$$

The most commonly used Clifford algebra is the case $V = \mathbb{R}^n$, and Q is the standard inner product with orthonormal basis $e_1, \ldots, e_n$. In this case, the algebra is generated by $e_1, \ldots, e_n$ and the identity of the algebra 1, with the relations

$$\begin{aligned} e_i^2 &= -1 \\ e_i e_j &= -e_j e_i \quad (i \neq j) \end{aligned}$$

Trivially, $\mathrm{Cliff}(\mathbb{R}^0) = \mathbb{R}$, and it can be seen from the relations above that $\mathrm{Cliff}(\mathbb{R}) \cong \mathbb{C}$, the complex numbers, and $\mathrm{Cliff}(\mathbb{R}^2) \cong \mathbb{H}$, the quaternions.

On the other ha nd, for $V = \mathbb{C}^n$ we get the particularly answer of

$$\mathrm{Cliff}(\mathbb{C}^{2k}) \cong \mathrm{M}_{2^k}(\mathbb{C}) \qquad \mathrm{Cliff}(\mathbb{C}^{2k+1}) = \mathrm{M}_{2^k}(\mathbb{C}) \oplus \mathbf{M}_{2^k}(\mathbb{C}).$$