dimension formulae for vector spaces

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In this entry we look at various formulae involving the dimension of a vector space.

Throughout this entry, K will be a field, and V and W will be vector spaces over K. The dimension of a vector space U over K will be denoted by $\dim(U)$, or by $\dim_K(U)$ if the ground field needs to be emphasized.

All of these formulae potentially involve infinite cardinals, so the reader should have a basic knowledge of cardinal arithmetic in order to understand them in full generality.

Subspaces

If S and T are subspaces of V, then

$$\dim(S) + \dim(T) = \dim(S \cap T) + \dim(S + T).$$

Rank-nullity theorem

The rank-nullity theorem states that if $\phi\colon V \to W$ is a linear mapping, then the dimension of V is the sum of the dimensions of the image and kernel of ϕ:

$$\dim(V) = \dim(\operatorname{Im}\phi) + \dim(\operatorname{Ker}\phi).$$

In particular, if U is a subspace of V then

$$\dim(V) = \dim(V/U) + \dim(U).$$

The rank-nullity theorem can also be stated in terms of short exact sequences: if

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is a short exact sequence of vector spaces over K, then

$$\dim(V) = \dim(U) + \dim(W).$$

This can be generalized to infinite exact sequences: if

$$\cdots \longrightarrow V_{n+1} \longrightarrow V_n \longrightarrow V_{n-1} \longrightarrow \cdots$$

is an exact sequence of vector spaces over K, then

$$\sum_{n\ \text{even}} \dim(V_n) = \sum_{n\ \text{odd}} \dim(V_n).$$

(This is indeed a generalization, because any finite exact sequence of vectors spaces terminating with 0 at both ends can be considered as an infinite exact sequence in which all remaining terms are 0.)

Direct sums

If $(V_i)_{i\in I}$ is a family of vector spaces over K, then

$$\dim\left(\bigoplus_{i\in I} V_i\right) = \sum_{i\in I} \dim(V_i).$$

Cardinality of a vector space

The cardinality of a vector space is determined by its dimension and the cardinality of the ground field:

$$|V| = \begin{cases} |K|^{\dim(V)}, & \text{if } \dim(V) \text{ is finite;} \\ \max\{|K|, \dim(V)\}, & \text{if } \dim(V) \text{ is infinite.} \end{cases}$$

The effect of the above formula is somewhat different depending on whether K is `http://planetmath.org/FiniteField`finite or infinite. If K is finite, then it reduces to

$$|V| = \begin{cases} |K|^{\dim(V)}, & \text{if } \dim(V) \text{ is finite;} \\ \dim(V), & \text{if } \dim(V) \text{ is infinite.} \end{cases}$$

If K is infinite, then it can be expressed as

$$|V| = \begin{cases} 1, & \text{if } \dim(V) = 0; \\ |K|, & \text{if } 0 < \dim(V) \leq |K|; \\ \dim(V), & \text{if } \dim(V) \geq |K|. \end{cases}$$

Change of ground field

If F is a subfield of K, then V can be considered as a vector space over F. The dimensions of V over K and F are related by the formula

$$\dim_F(V) = [K : F] \cdot \dim_K(V).$$

In this formula, $[K : F]$ is the degree of the field extension K/F, that is, the dimension of K considered as a vector space over F.

Space of functions into a vector space

If S is any set, then the set K^S of all functions from S into K becomes a vector space over K if we define the operations pointwise, that is, $(f+g)(x) = f(x)+g(x)$ and $(\lambda f)(x) = \lambda f(x)$ for all $f, g \in K^S$, all $x \in S$, and all $\lambda \in K$. The dimension of this vector space is given by

$$\dim(K^S) = \begin{cases} |S|, & \text{if } S \text{ is finite;} \\ |K|^{|S|}, & \text{if } S \text{ is infinite.} \end{cases}$$

The case where S is infinite is not straightforward to prove. Proofs can be found in books by Baer[?] and Jacobson[?], among others.

More generally, we can consider the space V^S, which is really just the http://planetmath.org/DirectProductdirect product of copies of V indexed by S. We get

$$\dim(V^S) = \begin{cases} 0, & \text{if } \dim(V) = 0; \\ |S| \cdot \dim(V), & \text{if } S \text{ is finite;} \\ |V|^{|S|}, & \text{otherwise.} \end{cases}$$

Dual space

Given any basis B of V, the dual space V^* is isomorphic to K^B via the mapping $f \mapsto f|_B$. So the formula of the previous section can be applied to give a formula for the dimension of V^*:

$$\dim(V^*) = \begin{cases} \dim(V), & \text{if } \dim(V) \text{ is finite;} \\ |K|^{\dim(V)}, & \text{if } \dim(V) \text{ is infinite.} \end{cases}$$

In particular, this formula implies that V is isomorphic to V^* if and only if V is finite-dimensional. (Students who are familiar with the fact that an infinite-dimensional Banach space can be isomorphic to its dual are sometimes surprised to learn that an infinite-dimensional vector space cannot be isomorphic to its dual, for a Banach space is surely a vector space. But the term *dual* is used in different senses in these two statements, so there is no contradiction. In the theory of Banach spaces one is usually only interested in the *continuous* linear functionals, and the resulting 'continuous' dual is a subspace of the full dual used in the above formula.)

Space of linear mappings

The set $\mathrm{Hom}_K(V, W)$ of all linear mappings from V into W is itself a vector space over K, with the operations defined in the obvious way, namely $(f + g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda f(x)$ for all $f, g \in \mathrm{Hom}_K(V, W)$, all $x \in V$, and all $\lambda \in K$. The dual space $V^* = \mathrm{Hom}_K(V, K)$ considered in the previous section is a special case of this. For any basis B of V, the mapping $f \mapsto f|_B$ defines an isomorphism between $\mathrm{Hom}_K(V, W)$ and W^B, so that from an earlier section we get

$$\dim(\mathrm{Hom}_K(V, W)) = \begin{cases} 0, & \text{if } \dim(W) = 0; \\ \dim(V) \cdot \dim(W), & \text{if } \dim(V) \text{ is finite;} \\ |W|^{\dim(V)}, & \text{otherwise.} \end{cases}$$

In the special case $W = V$ this can be simplified to

$$\dim(\mathrm{End}_K(V)) = \begin{cases} \dim(V)^2, & \text{if } \dim(V) \text{ is finite;} \\ |K|^{\dim(V)}, & \text{otherwise.} \end{cases}$$

Tensor products

The dimension of the `http://planetmath.org/TensorProduct`tensor product of V and W is given by

$$\dim(V \otimes W) = \dim(V) \cdot \dim(W).$$

Banach spaces

The dimension of a Banach space, considered as a vector space, is sometimes called the *Hamel dimension*, in order to distinguish it from other concepts of dimension. For an infinite-dimensional Banach space B we have

$$\dim(B) = |B|.$$

The tricky part of establishing this formula is to show that the dimension is always at least the cardinality of the continuum. A short proof of this is given in a paper by Lacey[**?**].

The above formula suggests that Hamel dimension is not a very useful concept for infinite-dimensional Banach spaces, which is indeed the case. Nonetheless, it is interesting to see how Hamel dimension relates to the usual concept of dimension in Hilbert spaces. If H is a Hilbert space, and d is its dimension (meaning the cardinality of an orthonormal basis), then the Hamel dimension $\dim(H)$ is given by

$$\dim(H) = \begin{cases} d, & \text{if } d \text{ is finite;} \\ d^{\aleph_0}, & \text{if } d \text{ is infinite.} \end{cases}$$

References

[1] Reinhold Baer, *Linear Algebra and Projective Geometry*, Academic Press, 1952.

[2] Nathan Jacobson, *Lectures in Abstract Algebra*, Volume II: *Linear Algebra*, D. Van Nostrand Company Inc., 1953.

[3] H. Elton Lacey, *The Hamel Dimension of any Infinite Dimensional Separable Banach Space is c*, Amer. Math. Mon. 80 (1973), 298.