elementary matrix

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Defines	elementary column operation
Defines	elementary row operation
Defines	basic diagonal matrix
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Elementary Operations on Matrices

Let $\mathbb{M}$ be the set of all $m \times n$ matrices (over some commutative ring R). An operation on $\mathbb{M}$ is called an *elementary row operation* if it takes a matrix $M \in \mathbb{M}$, and does one of the following:

1. interchanges of two rows of M,
2. multiply a row of M by a non-zero element of R,
3. add a (constant) multiple of a row of M to another row of M.

An *elementary column operation* is defined similarly. An operation on $\mathbb{M}$ is an *elementary operation* if it is either an elementary row operation or elementary column operation.

For example, if $M = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$, then the following operations correspond respectively to the three types of elementary row operations described above

1. $\begin{pmatrix} a & b \\ e & f \\ c & d \end{pmatrix}$ is obtained by interchanging rows 2 and 3 of M,
2. $\begin{pmatrix} a & b \\ rc & rd \\ e & f \end{pmatrix}$ is obtained by multiplying $r \neq 0$ to the second row of M,
3. $\begin{pmatrix} a & b \\ c & d \\ sa+e & sb+f \end{pmatrix}$ is obtained by adding to row 1 multiplied by s to row 3 of M.

Some immediate observation: elementary operations of type 1 and 3 are always invertible. The inverse of type 1 elementary operation is itself, as interchanging of rows twice gets you back the original matrix. The inverse of type 3 elementary operation is to add the negative of the multiple of the first row to the second row, thus returning the second row back to its original form. Type 2 is invertible provided that the multiplier has an inverse.

Some notation: for each type k (where $k = 1, 2, 3$) of elementary operations, let $E_c^k(A)$ be the set of all matrices obtained from A via an elementary column operation of type k, and $E_r^k(A)$ the set of all matrices obtained from A via an elementary row operation of type k.

Elementary Matrices

Now, assume R has 1. An $n \times n$ *elementary matrix* is a (square) matrix obtained from the identity matrix I_n by performing an elementary operation. As a result, we have three types of elementary matrices, each corresponding to a type of elementary operations:

1. *transposition matrix* T_{ij}: an matrix obtained from I_n with rows i and j switched,

2. *basic diagonal matrix* $D_i(r)$: a diagonal matrix whose entries are 1 except in cell (i, i), whose entry is a non-zero element r of R

3. *row replacement matrix* $E_{ij}(s)$: $I_n + sU_{ij}$, where $s \in R$ and U_{ij} is a matrix unit with $i \neq j$.

For example, among the 3×3 matrices, we have

$$T_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_3(r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{pmatrix}, \quad \text{and} \quad E_{32}(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 1 \end{pmatrix}$$

For each positive integer n, let $\mathbb{E}^k(n)$ be the collection of all $n \times n$ elementary matrices of type k, where $k = 1, 2, 3$.

Below are some basic properties of elementary matrices:

- $T_{ij} = T_{ji}$, and $T_{ij}^2 = I_n$.
- $D_i(r)D_i(r^{-1}) = I_n$, provided that r^{-1} exists.
- $E_{ij}(s)E_{ij}(-s) = I_n$.
- $\det(T_{ij}) = -1$, $\det(D_i(r)) = r$, and $\det(E_{ij}(s)) = 1$.
- If A is an $m \times n$ matrix, then

  $$E_c^k(A) = \{AE \mid E \in \mathbb{E}^k(n)\} \qquad \text{and} \qquad E_r^k(A) = \{EA \mid E \in \mathbb{E}^k(m)\}.$$

- Every non-singular matrix can be written as a product of elementary matrices. This is the same as saying: given a non-singular matrix, one can perform a finite number of elementary row (column) operations on it to obtain the identity matrix.

Remarks.

- One can also define elementary matrix operations on matrices over general rings. However, care must be taken to consider left scalar multiplications and right scalar multiplications as separate operations.
- The discussion above pertains to elementary linear algebra. In algebraic K-theory, an elementary matrix is defined only as a row replacement matrix (type 3) above.