# dual homomorphism of the derivative

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Let $\mathcal{P}_n$ denote the vector space of real polynomials of degree $n$ or less, and let $\mathrm{D}_n : \mathcal{P}_n \to \mathcal{P}_{n-1}$ denote the ordinary derivative. Linear forms on $\mathcal{P}_n$ can be given in terms of evaluations, and so we introduce the following notation. For every scalar $k \in \mathbb{R}$, let $\mathrm{Ev}_k^{(n)} \in (\mathcal{P}_n)^*$ denote the evaluation functional

$$\mathrm{Ev}_k^{(n)} : p \mapsto p(k), \quad p \in \mathcal{P}_n.$$

Note: the degree superscript matters! For example:

$$\mathrm{Ev}_2^{(1)} = 2\,\mathrm{Ev}_1^{(1)} - \mathrm{Ev}_0^{(1)},$$

whereas $\mathrm{Ev}_0^{(2)}, \mathrm{Ev}_1^{(2)}, \mathrm{Ev}_2^{(2)}$ are linearly independent. Let us consider the dual homomorphism $\mathrm{D}_2^*$, i.e. the adjoint of $\mathrm{D}_2$. We have the following relations:

$$\begin{aligned} \mathrm{D}_2^*\left(\mathrm{Ev}_0^{(1)}\right) &= -\tfrac{3}{2}\,\mathrm{Ev}_0^{(2)} + 2\,\mathrm{Ev}_1^{(2)} - \tfrac{1}{2}\,\mathrm{Ev}_2^{(2)}, \\ \mathrm{D}_2^*\left(\mathrm{Ev}_1^{(1)}\right) &= -\tfrac{1}{2}\,\mathrm{Ev}_0^{(2)} + \tfrac{1}{2}\,\mathrm{Ev}_2^{(2)}. \end{aligned}$$

In other words, taking $\mathrm{Ev}_0^{(1)}, \mathrm{Ev}_1^{(1)}$ as the basis of $(\mathcal{P}_1)^*$ and $\mathrm{Ev}_0^{(2)}, \mathrm{Ev}_1^{(2)}, \mathrm{Ev}_2^{(2)}$ as the basis of $(\mathcal{P}_2)^*$, the matrix that represents $\mathrm{D}_2^*$ is just

$$\begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 2 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Note the contravariant relationship between $\mathrm{D}_2$ and $\mathrm{D}_2^*$. The former turns second degree polynomials into first degree polynomials, where as the latter turns first degree evaluations into second degree evaluations. The matrix of $\mathrm{D}_2^*$ has 2 columns and 3 rows precisely because $\mathrm{D}_2^*$ is a homomorphism from a 2-dimensional vector space to a 3-dimensional vector space.

By contrast, $\mathrm{D}_2$ will be represented by a $2 \times 3$ matrix. The dual basis of $\mathcal{P}_1$ is

$$-x + 1, \quad x$$

and the dual basis of $\mathcal{P}_2$ is

$$\frac{1}{2}(x-1)(x-2), \quad x(2-x), \quad \frac{1}{2}x(x-1).$$

Relative to these bases, $\mathrm{D}_2$ is represented by the transpose of the matrix for $\mathrm{D}_2^*$, namely

$$\begin{pmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

This corresponds to the following three relations:

$$
\begin{array}{lcrr}
\mathrm{D}_2\left[\frac{1}{2}(x-1)(x-2)\right] & = & -\frac{3}{2}\,(-x+1) & -\frac{1}{2}\,x \\
\mathrm{D}_2\left[x(2-x)\right] & = & 2\,(-x+1) & +0\,x \\
\mathrm{D}_2\left[\frac{1}{2}x(x-1)\right] & = & -\frac{1}{2}\,(-x+1) & +\frac{1}{2}\,x
\end{array}
$$