# corollary of Schur decomposition



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**theorem:** $A \in \mathbb{C}^{n\times n}$ is a normal matrix if and only if there exists a unitary matrix $Q \in \mathbb{C}^{n\times n}$ such that $Q^H AQ = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ *(the diagonal matrix)* where $^H$ is the conjugate transpose. [?]

**proof:** Firstly we show that if there exists a unitary matrix $Q \in \mathbb{C}^{n\times n}$ such that $Q^H AQ = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ then $A \in \mathbb{C}^{n\times n}$ is a normal matrix. Let $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ then $A$ may be written as $A = QDQ^H$. Verifying that A is normal follows by the following observation $AA^H = QDQ^H QD^H Q^H = QDD^H Q^H$ and $A^H A = QD^H Q^H QDQ^H = QD^H DQ^H$. Therefore $A$ is normal matrix because $DD^H = \mathrm{diag}(\lambda_1 \bar{\lambda_1}, \ldots, \lambda_n \bar{\lambda_n}) = D^H D$.
Secondly we show that if $A \in \mathbb{C}^{n\times n}$ is a normal matrix then there exists a unitary matrix $Q \in \mathbb{C}^{n\times n}$ such that $Q^H AQ = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. By Schur decompostion we know that there exists a $Q \in \mathbb{C}^{n\times n}$ such that $Q^H AQ = T$ *($T$ is an upper triangular matrix)*. Since $A$ is a normal matrix then $T$ is also a normal matrix. The result that $T$ is a diagonal matrix comes from showing that a normal upper triangular matrix is diagonal (see theorem for normal triangular matrices).
QED

# References


[GVL] Golub, H. Gene, Van Loan F. Charles: Matrix Computations *(Third Edition)*. The Johns Hopkins University Press, London, 1996.