proof of Hadamard's inequality

Canonical name	ProofOfHadamardsInequality
Date of creation	2013-03-22 15:37:02
Last modified on	2013-03-22 15:37:02
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Numerical id	17
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Entry type	Proof
Classification	msc 15A45

Let's first prove the second inequality. If A is singular, the thesis is trivially verified, since for a Hermitian positive semidefinite matrix the right-hand side is always nonnegative, all the diagonal entries being nonnegative. Let's thus assume $\det(A) \neq 0$, which means, A being Hermitian positive semidefinite, $\det(A) > 0$. Then no diagonal entry of A can be 0 (otherwise, since for a Hermitian positive semidefinite matrix, $0 \leq \lambda_{min} \leq a_{ii} \leq \lambda_{max}$, λ_{min} and λ_{max} being respectively the minimal and the maximal eigenvalue, this would imply $\lambda_{min} = 0$, that is A is singular); for this reason we can define $D = diag(d_{11}, d_{22}, \ldots, d_{nn})$, with $d_{ii} = a_{ii}^{-\frac{1}{2}} \in \mathbb{R}$, since all $a_{ii} \in \mathbb{R}^+$. Let's furthermore define $B = DAD$. It's easy to check that B too is Hermitian positive semidefinite, so its eigenvalues λ_B are all non-negative (actually, since $A = A^H$ and since D is real and diagonal, $B^H = (DAD)^H = D^H(DA)^H = D^H A^H D^H = DAD = B$; on the other hand, for any $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^H B\mathbf{x} = \mathbf{x}^H DAD\mathbf{x} = (\mathbf{x}^H D)A(D\mathbf{x}) = (D^H\mathbf{x})^H A(D\mathbf{x}) = (D\mathbf{x})^H A(D\mathbf{x}) = \mathbf{y}^H A\mathbf{y} \geq 0$). Moreover, we have obviously $b_{ii} = d_{ii}a_{ii}d_{ii} = 1$ so that $tr(B) = n$ and, recalling the `http://planetmath.org/ArithmeticGeometricMeansInequality`geometric-arithmetic mean inequality, which holds in this case because the eigenvalues of B are all non-negative,

$\det(B) = \prod_{i=1}^n \lambda_B^{(i)} \leq \left(\frac{1}{n}\sum_{i=1}^n \lambda_B^{(i)}\right)^n = \left(\frac{1}{n}tr(B)\right)^n = 1$,

and since $\det(B) = \det(D)^2 \det(A) = \left(\prod_{i=1}^n a_{ii}\right)^{-1} \det(A)$, we have the thesis. Since $\det(A) = \prod_{i=1}^n a_{ii}$ if and only if $\det(B) = 1$ and since in the geometric-arithmetic inequality equality holds if and only if all terms are equal, we must have $\lambda_B^{(i)} = \lambda_B$, so that $\prod_{i=1}^n \lambda_B^{(i)} = \lambda_B^n = 1$, whence $\lambda_B = 1$ (λ_B having to be non-negative), and since B is Hermitian and hence is diagonalizable, we obtain $B = I$, and so $A = D^{-1}BD^{-1} = D^{-2} = diag(a_{11}, a_{22}, \ldots, a_{nn})$. So we can conclude that equality holds if and only if A is diagonal.

Let's now derive the more general first inequality. Let A be a complex-valued $n \times n$ matrix. If A is singular, the thesis is trivially verified. Let's thus assume $\det(A) \neq 0$; then $B = AA^H$ is a Hermitian positive semidefinite matrix (actually, $B^H = (AA^H)^H = (A^H)^H A^H = AA^H = B$ and, for any $\mathbf{x} \neq \mathbf{0}$,$\mathbf{x}^H B\mathbf{x} = \mathbf{x}^H AA^H\mathbf{x} = (\mathbf{x}^H A)(A^H\mathbf{x}) = (A^H\mathbf{x})^H(A^H\mathbf{x}) = \mathbf{y}^H\mathbf{y} = \|\mathbf{y}\|_2^2 \geq 0$). Therefore, the second inequality can be applied to B, yielding:

$|\det(A)|^2 = \det(A)\det{}^*(A) = \det(A)\det(A^H) = \det(AA^H) =$
$= \det(B) \leq \prod_{i=1}^n b_{ii} = \prod_{i=1}^n \sum_{j=1}^n a_{ij}(a^H)_{ji} = \prod_{i=1}^n \sum_{j=1}^n a_{ij}a_{ij}^* = \prod_{i=1}^n \sum_{j=1}^n |a_{ij}|^2$.

As we proved above, for $\det(B)$ to be equal to $\prod_{i=1}^n b_{ii}$, B must be diagonal, which means that $\sum_{k=1}^n a_{ik}a_{jk}^* = |a_{ii}|^2\delta_{ij}$. So we can conclude that equality holds if and only if the rows of A are orthogonal. $\square$

References

[1] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985