AB and BA are almost isospectral

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0.1 General case

Let A and B be endomorphisms of a vector space V. Let $\sigma(AB)$ and $\sigma(BA)$ denote, respectively, the `http://planetmath.org/spectrum`spectra of AB and BA.

The next result shows that AB and BA are "almost" isospectral, in the sense that their spectra is the same except possibly the value 0.

Theorem - Let A and B be as above. We have

1. $\sigma(AB)\cup\{0\}=\sigma(BA)\cup\{0\}$, and moreover

2. AB and BA have the same eigenvalues, except possibly the zero eigenvalue.

Proof : Let $\lambda\neq 0$.

1. If $\lambda\in\sigma(AB)$ then $\lambda^{-1}AB-I$ is not invertible. By the result in the parent entry, this implies that $\lambda^{-1}BA-I$ is not invertible either, hence $\lambda\in\sigma(BA)$.

   A similar argument proves that every non-zero element of $\sigma(BA)$ also belongs to $\sigma(AB)$. Hence $\sigma(AB)\cup\{0\}=\sigma(BA)\cup\{0\}$.

2. If λ is an eigenvalue of AB, then $I-\lambda^{-1}AB$ is not injective. By the result in the parent entry, this implies that $I-\lambda^{-1}BA$ is also not injective, hence λ is an eigenvalue of BA.

   A similar argument proves that non-zero eigenvalues of BA are also eigenvalues of AB. $\square$

Remark : Note that for infinite dimensional vector spaces the spectrum of a linear mapping does not consist solely of its eigenvalues. Hence, 1 and 2 above are two different statements.

0.2 Finite dimensional case

When the vector space V is finite dimensional we can strengthen the above result.

Theroem - AB and BA are isospectral, i.e. they have the same spectrum. Since V is finite dimensional, this means that AB and BA have the same eigenvalues.

Proof : By the above result we only need to prove that: AB is invertible if and only if BA is invertible.

Suppose AB is not invertible. Hence, A is not invertible or B is not invertible.

For finite dimensional vector spaces invertibility, injectivity and surjectivity are the same thing. Thus, the above statement can be rewritten as: A is not injective or B is not surjective.

Either way BA is not invertible.

A similar argument shows that if BA is not invertible, then AB is also not invertible, which concludes the proof. $\square$

0.3 Comments

The first theorem can be proven in a more general context : If A and B are elements of an arbitrary unital algebra, then

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}.$$

This humble result plays an important role in the spectral theory of operator algebras.