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every vector space has a basis

Canonical name	EveryVectorSpaceHasABasis
Date of creation	2013-03-22 13:04:48
Last modified on	2013-03-22 13:04:48
Owner	GrafZahl (9234)
Last modified by	GrafZahl (9234)
Numerical id	14
Author	GrafZahl (9234)
Entry type	Theorem
Classification	msc 15A03
Synonym	every vector space has a Hamel basis
Related topic	ZornsLemma
Related topic	AxiomOfChoice
Related topic	ZermelosWellOrderingTheorem
Related topic	HaudorffsMaximumPrinciple
Related topic	KuratowskisLemma

This result, trivial in the finite case, is in fact rather surprising when one thinks of infinite dimensionial vector spaces, and the definition of a basis: just try to imagine a basis of the vector space of all continuous mappings $f\colon \mathbb{R} \to \mathbb{R}$. The theorem is equivalent to the axiom of choice family of axioms and theorems. Here we will only prove that Zorn's lemma implies that every vector space has a basis.

Theorem. *Let X be any vector space over any field F and assume Zorn's lemma. Then if L is a linearly independent subset of X, there exists a basis of X containing L. In particular, X does have a basis at all.*

Proof. Let $\mathcal{A}$ be the set of linearly independent subsets of X containing L (in particular, $\mathcal{A}$ is not empty), then $\mathcal{A}$ is partially ordered by inclusion. For each chain $C \subseteq \mathcal{A}$, define $\widehat{C} = \bigcup C$. Clearly, $\widehat{C}$ is an upper bound of C. Next we show that $\widehat{C} \in \mathcal{A}$. Let $V := \{v_1, \dots, v_n\} \subseteq \widehat{C}$ be a finite collection of vectors. Then there exist sets $C_1, \dots, C_n \in C$ such that $v_i \in C_i$ for all $1 \leq i \leq n$. Since C is a chain, there is a number k with $1 \leq k \leq n$ such that $C_k = \bigcup\limits_{i=1}^{n} C_i$ and thus $V \subseteq C_k$, that is V is linearly independent. Therefore, $\widehat{C}$ is an element of $\mathcal{A}$.

According to Zorn's lemma $\mathcal{A}$ has a maximal element, M, which is linearly independent. We show now that M is a basis. Let $\langle M \rangle$ be the span of M. Assume there exists an $x \in X \setminus \langle M \rangle$. Let $\{x_1, \dots, x_n\} \subseteq M$ be a finite collection of vectors and $a_1, \dots, a_{n+1} \in F$ elements such that

$$a_1 x_1 + \cdots + a_n x_n - a_{n+1} x = 0.$$

If a_{n+1} was necessarily zero, so would be the other a_i, $1 \leq i \leq n$, making $\{x\} \cup M$ linearly independent in contradiction to the maximality of M. If $a_{n+1} \neq 0$, we would have

$$x = \frac{a_1}{a_{n+1}} x_1 + \cdots + \frac{a_n}{a_{n+1}} x_n,$$

contradicting $x \notin \langle M \rangle$. Thus such an x does not exist and $X = \langle M \rangle$, so M is a generating set and hence a basis.

Taking $L = \emptyset$, we see that X does have a basis at all. $\square$