# alternating form

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A bilinear form $A$ on a vector space $V$ (over a field $k$) is called an *alternating form* if for all $v \in V$, $A(v,v) = 0$.

Since for any $u, v \in V$,

$$0 = A(u+v, u+v) = A(u,u) + A(u,v) + A(v,u) + A(v,v) = A(u,v) + A(v,u),$$

we see that $A(u,v) = -A(v,u)$. So an alternating form is automatically a anti-symmetric, or skew symmetric form. The converse is true if the characteristic of $k$ is not 2.

Let $V$ be a two dimensional vector space over $k$ with an alternating form $A$. Let $\{e_1, e_2\}$ be a basis for $V$. The matrix associated with $A$ looks like

$$\begin{pmatrix} A(e_1,e_1) & A(e_1,e_2) \\ A(e_2,e_1) & A(e_2,e_2) \end{pmatrix} = r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = rS,$$

where $r = A(e_1, e_2)$. The skew symmetric matrix $S$ has the property that its diagonal entries are all 0. $S$ is called the $2 \times 2$ *alternating* or *symplectic matrix*.

$A$ is called *non-singular* or *non-degenerate* if there exist a vectors $u, v \in V$ such that $A(u,v) \neq 0$. $u, v$ are necessarily non-zero. Note that the associated matrix $rS$ is non-singular iff $r \neq 0$ iff $A$ is non-singular.

In the two dimensional vector space case above, if $A$ is non-singular, we can re-scale the basis elements so that $r = 1$. This means that the matrix associated with $A$ is the alternating matrix. A two-dimensional vector space which carries a non-singular alternating form is sometimes called an *alternating* or *symplectic hyperbolic plane*. Some authors also call it simply a hyperbolic plane. But here on PlanetMath, we will reserve the shorter name for its cousin in the category of quadratic spaces. Let's denote an alternating hyperbolic plane by $\mathcal{A}$.

**Remark.** In general, it can be shown that if $V$ is an $n$-dimensional vector space equipped with a non-singular alternating form $A$, then $V$ can be written as an orthogonal direct sum of the alternating hyperbolic planes $\mathcal{A}$. In other words, the associated matrix for $A$ has the block form

$$\begin{pmatrix} S & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & S & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & S \end{pmatrix}, \text{ where } \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, $n$ is even. $V$ is called a symplectic vector space.