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linear extension

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Related topic	basis
Related topic	Basis
Defines	bilinear extension
Defines	multilinear extension
Defines	n-linear extension

Let R be a commutative ring, M a free R-module, B a basis of M, and N a further R-module. Each element $m \in M$ then has a unique representation

$$m = \sum_{b \in B} m_b b,$$

where $m_b \in R$ for all $b \in B$, and only finitely many m_b are non-zero. Given a set map $f_1 \colon B \to N$ we may therefore define the R-module homomorphism $\varphi_1 \colon M \to N$, called the *linear extension* of f_1, such that

$$m \mapsto \sum_{b \in B} m_b f_1(b).$$

The map φ_1 is the unique homomorphism from M to N whose restriction to B is f_1.

The above observation has a convenient reformulation in terms of category theory. Let RMod denote the category of R-modules, and Set the category of sets. Consider the adjoint functors $U \colon \mathsf{RMod} \to \mathsf{Set}$, the forgetful functor that maps an R-module to its underlying set, and $F \colon \mathsf{Set} \to \mathsf{RMod}$, the free module functor that maps a set to the free R-module generated by that set. To say that U is right-adjoint to F is the same as saying that every set map from B to $U(N)$, the set underlying N, corresponds naturally and bijectively to an R-module homomorphism from $M = F(B)$ to N.

Similarly, given a map $f_2 \colon B^2 \to N$, we may define the *bilinear extension*

$$\varphi_2 \colon M^2 \to N \qquad\qquad (m, n) \mapsto \sum_{b \in B} \sum_{c \in B} m_b n_c f_2(b, c),$$

which is the unique bilinear map from M^2 to N whose restriction to B^2 is f_2.

Generally, for any positive integer n and a map $f_n \colon B^n \to N$, we may define the *n-linear extension*

$$\varphi_n \colon M^n \to N \qquad\qquad m \mapsto \sum_{b \in B^n} m_b f_n(b)$$

quite compactly using multi-index notation: $m_b = \prod_{k=1}^{n} m_{k,b_k}$.

Usage

The notion of linear extension is typically used as a *manner-of-speaking*. Thus, when a multilinear map is defined explicitly in a mathematical text, the images of the basis elements are given accompanied by the phrase “by multilinear extension” or similar.