proof of Cramer's rule

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Since $\det(A) \neq 0$, by properties of the determinant we know that A is invertible.

We claim that this implies that the equation $Ax = b$ has a unique solution. Note that $A^{-1}b$ is a solution since $A(A^{-1}b) = (AA^{-1})b = b$, so we know that a solution exists.

Let s be an arbitrary solution to the equation, so $As = b$. But then $s = (A^{-1}A)s = A^{-1}(As) = A^{-1}b$, so we see that $A^{-1}b$ is the only solution.

For each integer i, $1 \leq i \leq n$, let a_i denote the ith column of A, let e_i denote the ith column of the identity matrix I_n, and let X_i denote the matrix obtained from I_n by replacing column i with the column vector x.

We know that for any matrices A, B that the kth column of the product AB is simply the product of A and the kth column of B. Also observe that $Ae_k = a_k$ for $k = 1, \ldots, n$. Thus, by multiplication, we have:

$$\begin{aligned} AX_i &= A(e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n) \\ &= (Ae_1, \ldots, Ae_{i-1}, Ax, Ae_{i+1}, \ldots, Ae_n) \\ &= (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \\ &= M_i \end{aligned}$$

Since X_i is I_n with column i replaced with x, computing the determinant of X_i with cofactor expansion gives:

$$\det(X_i) = (-1)^{(i+i)} x_i \det(I_{n-1}) = 1 \cdot x_i \cdot 1 = x_i$$

Thus by the multiplicative property of the determinant,

$$\det(M_i) = \det(AX_i) = \det(A)\det(X_i) = \det(A)x_i$$

and so $x_i = \frac{\det(M_i)}{\det(A)}$ as required.