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commuting matrices

Canonical name	CommutingMatrices
Date of creation	2013-03-22 15:54:12
Last modified on	2013-03-22 15:54:12
Owner	Algeboy (12884)
Last modified by	Algeboy (12884)
Numerical id	20
Author	Algeboy (12884)
Entry type	Definition
Classification	msc 15A04
Related topic	SimultaneousTriangularisationOfCommutingMatricesOverAnyField2
Related topic	CommonEigenvectorOfADiagonalElementCrossSection

We consider the properties of commuting matrices and linear transformations over a vector space V . Two linear transformations $\varphi_i : V \rightarrow V$, $i = 1, 2$ are said to commute if for every $v \in V$,

$$\varphi_1(\varphi_2(v)) = \varphi_2(\varphi_1(v)).$$

If V has finite dimension n and we fix a basis of V then we may represent the linear transformations as $n \times n$ matrices A_i and here the condition of commuting linear transformations is equivalent to testing if their corresponding matrices commute:

$$A_1 A_2 = A_2 A_1.$$

Simultaneous triangularisation of commuting matrices over any field can be achieved but may require an extension of the field. The reason begins to be apparent from the study of eigenvalues.

Remark 1. *Because the implication of commuting matrices is best expressed through eigenvectors, we prefer the treatment of linear transformations for the .*

Recall a linear transformation $f : V \rightarrow V$ is said to leave a subspace $E \leq V$ invariant if $f(E) \leq E$.

Proposition 2. *If $\{\varphi_i\}_{i \in I}$ are commuting linear transformations and E is an eigenspace of φ_{i_0} for some $i_0 \in I$, then for all $i \in I$, $\varphi_i(E) \leq E$.*

Proof. Let λ be the eigenvalue of φ_{i_0} on E . Take any $i \in I$ and $v \in E$. Then

$$\varphi_{i_0}(\varphi_i(v)) = \varphi_i(\varphi_{i_0}(v)) = \varphi_i(\lambda v) = \lambda \varphi_i(v).$$

Therefore $\varphi_i(v) \in E$ as E is the λ eigenspace of φ_{i_0} . In particular, $\varphi_i(E) \leq E$. \square

We have just shown that commuting linear transformations preserve each other's eigenspaces. This property does not depend on a finite dimension for V or a finite set of commuting transformations. However, to characterize commuting linear transformations further will require that V have finite dimension.

Proposition 3. *Let V be a finite dimensional vector space and let $\{\varphi_i\}_{i \in I}$ be a family of commuting diagonalizable linear transformations from V to V . Then φ_i can be simultaneously diagonalized.*

Proof. If a finite dimensional linear transformation is diagonalizable over its field then it has all its eigenvalues in the field (under some basis the matrix is diagonal and the eigenvalues are simply those elements on the diagonal.)

If all the eigenvalues of a linear transformation are the same then the associated diagonal matrix is scalar. If all φ_i are scalar then they are simultaneously diagonalized.

Now presume that each φ_i is not a scalar transformation. Hence there are at least two distinct eigenspaces. It follows each eigenspace of φ_i has dimension less than that of V .

Now we set up an induction on the dimension of V . When the dimension of V is 1, all linear transformations are scalar. Now suppose that for all vector spaces of dimension n , any commuting diagonalizable linear transformations can be simultaneously diagonalized. Then in the case where $\dim V = n + 1$, either all the linear transformations are scalar and so simultaneously diagonalized, or at least one is not scalar in which case its eigenspaces are proper subspaces. Since the maps commute they respect each others eigenspaces. So we restrict the maps to any eigenspace and by induction simultaneously diagonalize on this subspace. As the linear transformations are diagonalizable, the sum of the eigenspaces of any φ_i is V so this process simultaneously diagonalizes each of the φ_i . \square

Of course it is possible to have commuting matrices which are not diagonalizable. At the other extreme are unipotent matrices, that is, matrices with all eigenvalues 1. Aside from the identity matrix, unipotent matrices are never diagonal. Yet they often commute. But here the generalized eigenspaces substitute for the usual eigenspaces.

It is generally not true that two unipotent matrices commute, even if they share the same eigenspace. For example, the set of unitriangular matrices forms a nilpotent group which is abelian only for 2×2 -matrices.

However, if we consider unipotent matrices of the form

$$\begin{bmatrix} I_k & * \\ 0 & I_j \end{bmatrix}$$

we find these to correspond to $k \times j$ matrices under addition. Thus this large family of unipotent matrices do commute.