proof of determinant lower bound of a strict diagonally dominant matrix

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Owner	Andrea Ambrosio (7332)
Last modified by	Andrea Ambrosio (7332)
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Author	Andrea Ambrosio (7332)
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Let's define, for any $i = 1, 2, ..., n$

$$h_i = |a_{ii}| - \sum_{j=1, j \neq i} |a_{ij}|$$

Then, by strict diagonally dominance, one has $h_i > 0 \quad \forall i$. Let $D = diag\{(h_1)^{-1}, (h_2)^{-1}, ..., (h_n)^{-1}\}$ and $B = DA$, so that the i-th row of B matrix is equal to the corresponding row of A matrix multiplied by $(h_i)^{-1}$. In this way , one has

$$\begin{aligned} d_i &= |b_{ii}| - \sum_{j=1, j \neq i} |b_{ij}| \\ &= \frac{|a_{ii}|}{h_i} - \sum_{j=1, j \neq i} \frac{|a_{ij}|}{h_i} \\ &= 1 \end{aligned}$$

Now, let λ be an eigenvalue of B, and $v = [v_1, v_2, ..., v_n]$ the corresponding eigenvector; let moreover p be the index of the maximal component of v, i.e.

$$|v_p| \geq |v_i| \quad \forall i$$

Of course, by definition of eigenvector, $|v_p| > 0$. Writing the p-th characteristic equation, we have:

$$\begin{aligned} \lambda v_p &= \sum_{j=1}^{n} b_{pj} v_j \\ &= b_{pp} v_p + \sum_{j=1, j \neq p}^{n} b_{pj} v_j \end{aligned}$$

so that, being $\left|\frac{v_j}{v_p}\right| \leq 1$,

$$\begin{aligned}
\lambda &= b_{pp} + \sum_{j=1, j\neq p}^{n} b_{pj} \frac{v_j}{v_p} \\
|\lambda| &= \left| b_{pp} + \sum_{j=1, j\neq p}^{n} b_{pj} \frac{v_j}{v_p} \right| \\
&\geq \left| |b_{pp}| - \left| \sum_{j=1, j\neq p}^{n} b_{pj} \frac{v_j}{v_p} \right| \right| \\
&\geq \left| |b_{pp}| - \sum_{j=1, j\neq p}^{n} |b_{pj}| \left| \frac{v_j}{v_p} \right| \right| \qquad (*) \\
&\geq \left| |b_{pp}| - \sum_{j=1, j\neq p}^{n} |b_{pj}| \right| \qquad (**) \\
&= |b_{pp}| - \sum_{j=1, j\neq p}^{n} |b_{pj}| \\
&= d_p = 1
\end{aligned}$$

In this way, we found that each eigenvalue of B is greater than one in absolute value; for this reason,

$$|\det(B)| = \left| \prod_{i=1}^{n} \lambda_i \right| \geq 1$$

Finally,

$$\det(D) = \prod_{i=1}^{n} (h_i)^{-1} = \left(\prod_{i=1}^{n} h_i \right)^{-1}$$

so that

$$\begin{aligned}
1 &\leq |\det(B)| \\
&= |\det(D)|\, |\det(A)| \\
&= \left(\prod_{i=1}^{n} h_i \right)^{-1} |\det(A)|
\end{aligned}$$

whence the thesis.

Remark: Perhaps it could be not immediately evident where the hypothesis of strict diagonally dominance is employed in this proof; in fact, inequality (*) and (**) would be, in a general case, *not valid*; they can be stated only because we can assure, by virtue of strict diagonally dominance, that the final argument of the absolute value $(|b_{pp}| - \sum_{j=1,j\neq p}^{n} |b_{pj}|)$ does remain positive.