Givens rotation

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Let A be an $m \times n$ matrix with $m \geq n$ and full rank (viz. rank n). An orthogonal matrix triangularization (QR Decomposition) consists of determining an $m \times m$ orthogonal matrix Q such that

$$Q^T A = \begin{bmatrix} R \\ 0 \end{bmatrix}$$

with the $n \times n$ upper triangular matrix R. One only has then to solve the triangular system $Rx = Py$, where P consists of the first n rows of Q.

Householder transformations clear whole columns except for the first element of a vector. If one wants to clear parts of a matrix one element at a time, one can use Givens rotation, which is particularly practical for parallel implementation .

A matrix

$$G = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

with properly chosen $c = \cos(\varphi)$ and $s = \sin(\varphi)$ for some rotation angle φ can be used to zero the element a_{ki}. The elements can be zeroed column by column from the bottom up in the following order:

$$(m, 1), (m, -1, 1), \ldots, (2, 1), (m, 2), \ldots, (3, 2), \ldots, (m, n), \ldots, (n + 1, n).$$

Q is then the product of $g = \frac{(2m-n-1)n}{2}$ Givens matrices $Q = G_1 G_2 \cdots G_g$.

To annihilate the bottom element of a 2×1 vector:

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix}^T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$$

the conditions $sa + cb = 0$ and $c^2 + s^2 = 1$ give:

$$c = \frac{a}{\sqrt{a^2 + b^2}}, s = \frac{b}{\sqrt{a^2 + b^2}}$$

For “Fast Givens”, see [Golub89].

References

- Originally from The Data Analysis Briefbook (`http://rkb.home.cern.ch/rkb/titleA.html`

Golub89 Gene H. Golub and Charles F. van Loan: Matrix Computations, 2nd edn., The John Hopkins University Press, 1989.