proof of Cayley-Hamilton theorem in a commutative ring

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Let R be a commutative ring with identity and let A be an order n matrix with elements from $R[x]$. For example, if A is $\begin{pmatrix} x^2+2x & 7x^2 \\ x+1 & 5 \end{pmatrix}$

then we can also associate with A the following polynomial having matrix coefficents:

$$A^\sigma = \begin{bmatrix} 0 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 7 \\ 0 & 0 \end{bmatrix} x^2.$$

In this way we have a mapping $A \longrightarrow A^\sigma$ which is an isomorphism of the rings $M_n(R[x])$ and $M_n(R)[x]$.

Now let $A \in M_n(R)$ and consider the characteristic polynomial of A: $p_A(x) = \det(xI - A)$, which is a monic polynomial of degree n with coefficients in R. Using a property of the adjugate matrix we have

$$(xI - A)\operatorname{adj}(xI - A) = p_A(x)I.$$

Now view this as an equation in $M_n(R)[x]$. It says that $xI - A$ is a left factor of $p_A(x)$. So by the factor theorem, the left hand value of $p_A(x)$ at $x = A$ is 0. The coefficients of $p_A(x)$ have the form cI, for $c \in R$, so they commute with A. Therefore right and left hand values are the same.

References

[1] Malcom F. Smiley. Algebra of Matrices. Allyn and Bacon, Inc., 1965. Boston, Mass.