proof of inverse of matrix with small-rank adjustment

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We will first prove the formula when $A = I$.

Suppose that $R^{-1} + Y^T X$ is invertible. Thus

$$(R^{-1} + Y^T X)(R^{-1} + Y^T X)^{-1} = I.$$

and

$$R^{-1}(R^{-1} + Y^T X)^{-1} + Y^T X(R^{-1} + Y^T X)^{-1} = I.$$

Multiply by XR from the left, and multiply by Y^T from the right, we get

$$X(R^{-1} + Y^T X)^{-1} Y^T + XRY^T X(R^{-1} + Y^T X)^{-1} Y^T = XRY^T.$$

The right hand side is equal to $B - I$, while the left hand side can be factorized as

$$(I + XRY^T)X(R^{-1} + Y^T X)^{-1} Y^T.$$

So,

$$B \cdot (X(R^{-1} + Y^T X)^{-1} Y^T) = B - I.$$

After rearranging, we obtain

$$I = B(I - X(R^{-1} + Y^T X)^{-1} Y^T).$$

Therefore

$$(I + XRY^T)^{-1} = I - X(R^{-1} + Y^T X)^{-1} Y^T \qquad (*)$$

For the general case $B = A + XRY^T$, consider

$$BA^{-1} = I + XRY^T A^{-1}.$$

We can apply (*) with Y^T replaced by $Y^T A^{-1}$.