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generalized eigenspace

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Let V be a vector space (over a field k), and T a linear operator on V, and λ an eigenvalue of T. The set E_λ of all generalized eigenvectors of T corresponding to λ, together with the zero vector 0, is called the *generalized eigenspace* of T corresponding to λ. In short, the generalized eigenspace of T corresponding to λ is the set

$$E_\lambda := \{v \in V \mid (T - \lambda I)^i(v) = 0 \text{ for some positive integer } i\}.$$

Here are some properties of E_λ:

1. $W_\lambda \subseteq E_\lambda$, where W_λ is the eigenspace of T corresponding to λ.
2. E_λ is a subspace of V and E_λ is T-invariant.
3. If V is finite dimensional, then $\dim(E_\lambda)$ is the algebraic multiplicity of λ.
4. $E_{\lambda_1} \cap E_{\lambda_2} = 0$ iff $\lambda_1 \neq \lambda_2$. More generally, $E_A \cap E_B = 0$ iff A and B are disjoint sets of eigenvalues of T, and E_A (or E_B) is defined as the sum of all E_λ, where $\lambda \in A$ (or B).
5. If V is finite dimensional and T is a linear operator on V such that its characteristic polynomial p_T splits (over k), then
$$V = \bigoplus_{\lambda \in S} E_\lambda,$$
where S is the set of all eigenvalues of T.
6. Assume that T and V have the same properties as in (5). By the Jordan canonical form theorem, there exists an ordered basis β of V such that $[T]_\beta$ is a Jordan canonical form. Furthermore, if we set $\beta_i = \beta \cap E_{\lambda_i}$, then $[T|_{E_{\lambda_i}}]_{\beta_i}$, the matrix representation of $T|_{E_\lambda}$, the restriction of T to E_{λ_i}, is a Jordan canonical form. In other words,
$$[T]_\beta = \begin{pmatrix} J_1 & O & \cdots & O \\ O & J_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & J_n \end{pmatrix}$$
where each $J_i = [T|_{E_{\lambda_i}}]_{\beta_i}$ is a Jordan canonical form, and O is a zero matrix.

7. Conversely, for each E_{λ_i}, there exists an ordered basis β_i for E_{λ_i} such that $J_i := [T|_{E_{\lambda_i}}]_{\beta_i}$ is a Jordan canonical form. As a result, $\beta := \bigcup_{i=1}^n \beta_i$ with linear order extending each β_i, such that $v_i < v_j$ for $v_i \in \beta_i$ and $v_j \in \beta_j$ for $i < j$, is an ordered basis for V such that $[T]_\beta$ is a Jordan canonical form, being the direct sum of matrices J_i.

8. Each J_i above can be further decomposed into Jordan blocks, and it turns out that the number of Jordan blocks in each J_i is the dimension of W_{λ_i}, the eigenspace of T corresponding to λ_i.

More to come...

References

[1] Friedberg, Insell, Spence. *Linear Algebra*. Prentice-Hall Inc., 1997.