vector space over an infinite field is not a finite union of proper subspaces

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Theorem 1. *A vector space V over an infinite field $\mathbb{F}$ cannot be a finite union of proper subspaces of itself.*

Proof. Let $V = V_1 \cup V_2 \cup \ldots \cup V_n$ where each V_i is a proper subspace of V and $n > 1$ is minimal. Because n is minimal, $V_n \not\subset V_1 \cup V_2 \cup \ldots \cup V_{n-1}$.

Let $u \notin V_n$ and let $v \in V_n \setminus (V_1 \cup V_2 \cup \ldots \cup V_{n-1})$.

Define $S = \{v + tu : t \in \mathbb{F}\}$. Since $u \notin V_n$ is not the zero vector and the field $\mathbb{F}$ is infinite, S must be infinite.

Since $S \subset V = V_1 \cup V_2 \cup \ldots \cup V_n$ one of the V_i must contain infinitely many vectors in S.

However, if V_n were to contain a vector, other than v, from S there would exist non-zero $t \in \mathbb{F}$ such that $v + tu \in V_n$. But then $tu = v + tu - v \in V_n$ and we would have $u \in V_n$ contrary to the choice of u. Thus V_n cannot contain infinitely many elements in S.

If some $V_i, 1 \leq i < n$ contained two distinct vectors in S, then there would exist distinct $t_1, t_2 \in \mathbb{F}$ such that $v + t_1 u, v + t_2 u \in V_i$. But then $(t_2 - t_1)\, v = t_2\,(v + t_1 u) - t_1\,(v + t_2 u) \in V_i$ and we would have $v \in V_i$ contrary to the choice of v. Thus for $1 \leq i < n, V_i$ cannot contain infinitely many elements in S either. $\square$