change of basis

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Defines	change of basis matrix

Let V be a vector space. Given a basis A for V, each vector $v \in V$ can be uniquely expressed in terms of the base elements $v_i \in A$ as follows:

$$v = \sum_{v_i \in A} r_i v_i$$

where the sum is taken over a finite number of elements in A. Suppose now that B is another basis for V. By a *change of basis* from A to B we mean re-expressing v in terms of base elements $w_i \in B$.

Formally, we can think of a change of basis as the identity function (viewed as a linear operator) on a vector space V, such that elements in the domain are expressed in terms of A and elements in the range are expressed in terms of B.

Note that, by the very design of a basis, a change of basis in a vector space is always possible.

Now, if V has dimension $n < \infty$. We can total order bases A and B. Then a change of basis (from A to B) has the matrix representation

$$[I]_B^A,$$

where $I : V \to V$ is the identity operator. $[I]_B^A$ is called a *change of basis matrix*. By applying $[I]_B^A$ to a vector v expressed in terms of A, we get v expressed in terms of B:

$$[v]_B = [I]_B^A [v]_A,$$

where $[v]_A$ and $[v]_B$ are v expressed in the two bases A and B respectively.

Since I is obviously invertible, $[I]_B^A$ is invertible also, whose inverse is $[I]_A^B$. Furthermore, $[I]_A = I_n$ for any basis A. Here, I_n is the identity matrix.

Examples.

1. Let $V = \mathbb{R}^3$ and the following two sets

$$A = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\} \text{ and } B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

be the two ordered bases for V, ordered in the way the elements are arranged in the set. For each $v_i \in A$, $I(v_i) = v_i = [v_i]_{E_3}$, we see that

$$[I]_B^A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

Notice that the columns of $[I]_B^A$ are exactly the elements of A. Indeed, each element of A is *already* written in terms of the standard basis elements (in B). For example, let v be the first basis element in A. Let us see what $[v]_A$ is, when expressed using base elements in B, the standard ordered basis:

$$[v]_B = [I]_B^A [v]_A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 0 \end{pmatrix} [v]_A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

exactly as we have expected.

2. Conversely, let w be the first basis element in B. What is w when expressed in terms of basis elements of A? In other words, we need to find

$$[w]_A = [I]_A^B [w]_B.$$

Now, $[w]_B$ is just $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, so $[w]_A$ is nothing more than the first column of $[I]_A^B$, which is just the inverse of the matrix $[I]_B^A$, so

$$[I]_A^B = ([I]_B^A)^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & -2/9 & 2/9 \\ -2/3 & 4/9 & 5/9 \\ 1/3 & 1/9 & -1/9 \end{pmatrix}.$$

Therefore, $[w]_A = \begin{pmatrix} 1/3 \\ -2/3 \\ 1/3 \end{pmatrix}$. A quick verification shows that this is indeed the case:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1/3) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + (-2/3) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (1/3) \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

3. Now let C be the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$. It is easy to check that C forms a basis for $\mathbb{R}^3$ (determinant is non-zero). Order C in the obvious manner. What is the change of basis matrix $[I]_A^C$? One way is to express

each element of C in terms of the elements of A. Another way is to use the formula $[I]_A^C = [I]_A^B[I]_B^C$. Applying the first example, we see that $[I]_B^C$ is just the matrix whose columns are elements of C. As a result:

$$[I]_A^C = [I]_A^B[I]_B^C = \begin{pmatrix} 1/3 & -2/9 & 2/9 \\ -2/3 & 4/9 & 5/9 \\ 1/3 & 1/9 & -1/9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 7/9 & 0 & 4/9 \\ 4/9 & 1 & -8/9 \\ 1/9 & 0 & 7/9 \end{pmatrix}.$$

Remarks. Let us summarize what we have learned from the examples above, as well as list some additional facts. Let V be a finite dimensional vector space of dimension n.

- If E is the standard basis (ordered), then for any ordered basis A, $[I]_E^A$ is the matrix whose columns are exactly the basis elements in A (assuming these elements have already been expressed in terms of E) such that the i-column corresponds to the i-th element in the ordered set A.

- This also means that every invertible matrix A corresponds to (in a one-to-one fashion) a change of basis from the basis S_A whose elements are columns of A to E, the standard basis: $A = [I]_E^{S_A}$.

- Continue to assume that E is the standard basis. Let A, B be any ordered bases for V. Using the above property, we can easily compute $[I]_B^A$, which is $[I]_B^E[I]_E^A = ([I]_E^B)^{-1}[I]_E^A$.

- Let A' be a re-ordering of the ordered basis A, where each $v_i' \in A'$ is just $v_{\pi(i)}$ for some permutation in S_n. Then $[I]_A^{A'}$ is the permutation matrix corresponding to the permutation π.

- Suppose T is a linear transformation from V to W (both finite dimensional). Under a bases $A \subset V$ and $B \subset W$, T has matrix representation $[T]_B^A$. Under changes of basis from A to A', and B to B', we have

$$[T]_{B'}^{A'} = [ITI]_{B'}^{A'} = [I]_{B'}^B[T]_B^A[I]_A^{A'}.$$

- If T is a linear operator on V, then setting $V = W$, $A = B$ and $A' = B'$ from above, we have that

$$[T]_{A'} = P^{-1}[T]_A P,$$

where P is the change of basis matrix $[I]_A^{A'}$. This shows that $[T]_A$ and $[T]_{A'}$ are similar matrices. In other words, under a change of basis, the linear transformation T is basically the same.

References

[1] Friedberg, Insell, Spence. *Linear Algebra*. Prentice-Hall Inc., 1997.