example of bounded operator with no eigenvalues

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Owner	asteroid (17536)
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Author	asteroid (17536)
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In this entry we show that there are operators with no eigenvalues. Moreover, we exhibit an operator T in a Hilbert space which is bounded, self-adjoint, has a non-empty spectrum but no eigenvalues.

Consider the Hilbert space http://planetmath.org/L2SpacesAreHilbertSpaces$L^2([0,1])$ and let $f:[0,1]\longrightarrow\mathbb{C}$ be the function $f(t)=t$.

Let $T:L^2([0,1])\longrightarrow L^2([0,1])$ be the http://planetmath.org/MultiplicationOperatorOnMa of multiplication by f

$$T(\varphi)=f\varphi\,,\qquad\qquad \varphi\in L^2([0,1])$$

Thus, T is a bounded operator, since it is a multiplication operator (see http://planetmath.org/OperatorNormOfMultiplicationOperatorOnL2this entry). Also, it is easily seen that T is self-adjoint.

We now prove that T has no eigenvalues: suppose $\lambda\in\mathbb{C}$ is an eigenvalue of T and φ is an eigenvector. Then,

$$T\varphi=\lambda\varphi$$

This means that $(f-\lambda)\varphi=0$, but this is impossible for $\varphi\neq 0$ since $f-\lambda$ has at most one zero. Hence, T has no eigenvalues.

Of course, since the Hilbert space is complex, the spectrum of T is non-empty (see http://planetmath.org/SpectrumIsANonEmptyCompactSetthis entry). Moreover, the spectrum of T can be easily computed and seen to be the whole interval $[0,1]$, as we explain now:

It is known that an operator of multiplication by a continuous function g is invertible if and only if g is invertible. Thus, for every $\lambda\in\mathbb{C}$, $T-\lambda I$ is easily seen to be the operator of multiplication by $(f-\lambda)$. Hence, $T-\lambda I$ is not invertible if and only if $\lambda\in[0,1]$, i.e. $\sigma(T)=[0,1]$.