nilpotent matrix

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The square matrix A is said to be nilpotent if $A^n = \underbrace{AA \cdots A}_{\text{n times}} = \mathbf{0}$ for some positive integer n (here $\mathbf{0}$ denotes the matrix where every entry is 0).

Theorem (Characterization of nilpotent matrices)**.** *A matrix is nilpotent iff its eigenvalues are all 0.*

Proof. Let A be a nilpotent matrix. Assume $A^n = \mathbf{0}$. Let λ be an eigenvalue of A. Then $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$. By induction $\lambda^n\mathbf{x} = A^n\mathbf{x} = 0$, so $\lambda = 0$.

Conversely, suppose that all eigenvalues of A are zero. Then the chararacteristic polynomial of A: $\det(\lambda I - A) = \lambda^n$. It now follows from the Cayley-Hamilton theorem that $A^n = \mathbf{0}$. $\square$

Since the determinant is the product of the eigenvalues it follows that a nilpotent matrix has determinant 0. Similarly, since the trace of a square matrix is the sum of the eigenvalues, it follows that it has trace 0.

One class of nilpotent matrices are the `http://planetmath.org/node/4381`strictly triangular matrices (lower or upper), this follows from the fact that the eigenvalues of a triangular matrix are the diagonal elements, and thus are all zero in the case of *strictly* triangular matrices.

Note for 2×2 matrices A the theorem implies that A is nilpotent iff $A = \mathbf{0}$ or $A^2 = \mathbf{0}$.

Also it is worth noticing that any matrix that is similar to a nilpotent matrix is nilpotent.