centralizer of matrix units

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Theorem - Let R be a ring with identity 1 and $M_n(R)$ the ring of $n \times n$ matrices with entries in R. The centralizer of the matrix units is the set $R \cdot Id$, consisting of all multiples of the identity matrix.

: It is clear that the multiples of the identity matrix commute with all matrix units, and therefore belong to their centralizer. We will now prove the converse.

We will regard the elements of $M_n(R)$ as endomorphisms of the module $\oplus_{i=1}^n R$. We will denote by $\{e_i\}$ the canonical basis of $\oplus_{i=1}^n R$ and by E_{ij} the matrix unit whose entry (i, j) is 1.

Let $S = [s_{ij}] \in M_n(R)$ be an element of the centralizer of the matrix units. For all i, j, k we must have

$$SE_{ij}\, e_k = E_{ij}S\, e_k \tag{1}$$

But a straightforward computation shows that $SE_{ij}\, e_j = S\, e_i$ and $E_{ij}S\, e_j = s_{jj}\, e_i$. Since j is arbitrary we see, by equality (1), that all s_{jj} are equal, say $s_{jj} = s \in R$.

Hence, $S\, e_i = s\, e_i$, wich means that $S = s\, Id$. We conclude that S must be a multiple of the identity matrix. $\square$