orthogonal direct sum

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Let (V_1, B_1) and (V_2, B_2) be two vector spaces, each equipped with a symmetric bilinear form. Form the direct sum of the two vector spaces $V := V_1 \oplus V_2$. Next define a symmetric bilinear form B on V by

$$B((u_1, u_2), (v_1, v_2)) := B_1(u_1, v_1) + B_2(u_2, v_2),$$

where $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$. Since $B((u_1, 0), (u_2, 0)) = B_1(u_1, u_2)$, we see that $B = B_1$ when the domain of B is restricted to V_1. Therefore, V_1 can be viewed as a subspace of V with respect to B. The same holds for V_2.

Now suppose $(u, 0) \in V_1$ and $(0, v) \in V_2$ are two arbitrary vectors. Then $B((u, 0), (0, v)) = B_1(u, 0) + B_2(0, v) = 0 + 0 = 0$. In other words, V_1 and V_2 are "orthogonal" to one another with respect to B.

From the above discussion, we say that (V, B) is the *orthogonal direct sum* of (V_1, B_1) and (V_2, B_2). Clearly the above construction is unique (up to linear isomorphisms respecting the bilinear forms). As vectors from V_1 and V_2 can be seen as being "perpendicular" to each other, we appropriately write V as

$$V_1 \perp V_2.$$

Orthogonal Direct Sums of Quadratic Spaces. Since a symmetric biliner form induces a quadratic form (on the same space), we can speak of orthogonal direct sums of quadratic spaces. If (V_1, Q_1) and (V_2, Q_2) are two quadratic spaces, then the orthogonal direct sum of V_1 and V_2 is the direct sum of V_1 and V_2 with the corresponding quadratic form defined by

$$Q((u, v)) := Q_1(u) + Q_2(v).$$

It may be shown that any n-dimensional quadratic space (V, Q) is an orthogonal direct sum of n one-dimensional quadratic subspaces. The quadratic form associated with a one-dimensional quadratic space is nothing more than ax^2 (the form is uniquely determined by the single coefficient a), and the space associated with this form is generally written as $\langle a \rangle$. A finite dimensional quadratic space V is commonly written as

$$\langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle, \text{ or simply } \langle a_1, \ldots, a_n \rangle,$$

where n is the dimension of V.

Remark. The orthogonal direct sum may also be defined for vector spaces associated with bilinear forms that are `http://planetmath.org/AlternatingForm`alternati skew symmetric or Hermitian. The construction is similar to the one discussed above.