coboundary definition of exterior derivative

Canonical name	CoboundaryDefinitionOfExteriorDerivative
Date of creation	2013-03-22 15:38:06
Last modified on	2013-03-22 15:38:06
Owner	rmilson (146)
Last modified by	rmilson (146)
Numerical id	15
Author	rmilson (146)
Entry type	Definition
Classification	msc 15A69
Classification	msc 58A10
Related topic	LieAlgebraCohomology

Let M be a smooth manifold, and

- let $C^\infty(M)$ denote the algebra of smooth functions on M;
- let $V(M)$ denote the Lie-algebra of smooth vector fields;
- and let $\Omega^k(M)$ denote the vector space of smooth, differential k-forms.

Recall that a differential form $\alpha \in \Omega^k(M)$ is a multilinear, alternating mapping

$$\alpha : V(M) \times \cdots \times V(M)(\text{k times}) \to C^\infty(M)$$

such that, in local coordinates, α looks like a multilinear combination of its vector field arguments. Thus, employing the Einstein summation convention and local coordinates , we have

$$\alpha(u, v, \ldots, w) = \alpha_{ij\ldots k}\, u^i v^j \cdots w^k,$$

where $u, v, \ldots, w$ is a list of k vector fields. Recall also that $C^\infty(M)$ is a $V(M)$ module. The action is given by a directional derivative, and takes the form

$$v(f) = v^i \partial_i f, \quad v \in V(M),\ f \in C^\infty(M).$$

With these preliminaries out of the way, we have the following description of the exterior derivative operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$. For $\omega \in \Omega^k(M)$, we have

$$\begin{aligned}(d\omega)(v_0, v_1, \ldots, v_k) = \sum_{0 \le i \le k} (-1)^k v_i \omega(\ldots, \widehat{v}_i, \ldots) + \\ + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([v_i, v_j], \ldots, \widehat{v}_i, \ldots \widehat{v}_j, \ldots),\end{aligned} \tag{1}$$

where $\widehat{v}_i$ indicates the omission of the argument v_i.

The above expression (**??**) of $d\omega$ can be taken as the definition of the exterior derivative. Letting the v_i arguments be coordinate vector fields, it is not hard to show that the above definition is equivalent to the usual definition of d as a derivation of the exterior algebra of differential forms, or the local coordinate definition of d. The nice feature of (**??**) is that it is equivalent to the definition of the coboundary operator for Lie algebra cohomology. Thus, we see that de Rham cohomology, which is the cohomology of the cochain complex $d : \Omega^k(M) \to \Omega^{k+1}(M)$, is just zeroth-order Lie algebra cohomology of $V(M)$ with coefficients in $C^\infty(M)$. The bit about "zeroth order" means that we are considering cochains that are zeroth order differential operators of their arguments — in other words, differential forms.