proof of properties of trace of a matrix

Canonical name	ProofOfPropertiesOfTraceOfAMatrix
Date of creation	2013-03-22 13:42:54
Last modified on	2013-03-22 13:42:54
Owner	Daume (40)
Last modified by	Daume (40)
Numerical id	4
Author	Daume (40)
Entry type	Proof
Classification	msc 15A99

Proof of Properties:

1. Let us check linearity. For sums we have

$$\begin{aligned}
\operatorname{trace}(A+B) &= \sum_{i=1}^{n}(a_{i,i}+b_{i,i}) \qquad \text{(property of matrix addition)}\\
&= \sum_{i=1}^{n} a_{i,i} + \sum_{i=1}^{n} b_{i,i} \quad \text{(property of sums)}\\
&= \operatorname{trace}(A) + \operatorname{trace}(B).
\end{aligned}$$

   Similarly,

$$\begin{aligned}
\operatorname{trace}(cA) &= \sum_{i=1}^{n} c \cdot a_{i,i} \quad \text{(property of matrix scalar multiplication)}\\
&= c \cdot \sum_{i=1}^{n} a_{i,i} \quad \text{(property of sums)}\\
&= c \cdot \operatorname{trace}(A).
\end{aligned}$$

2. The second property follows since the transpose does not alter the entries on the main diagonal.

3. The proof of the third property follows by exchanging the summation order. Suppose A is a $n \times m$ matrix and B is a $m \times n$ matrix. Then

$$\begin{aligned}
\operatorname{trace} AB &= \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}B_{j,i}\\
&= \sum_{j=1}^{m}\sum_{i=1}^{n} B_{j,i}A_{i,j} \quad \text{(changing summation order)}\\
&= \operatorname{trace} BA.
\end{aligned}$$

4. The last property is a consequence of Property 3 and the fact that matrix multiplication is associative;

$$\begin{aligned}
\operatorname{trace}(B^{-1}AB) &= \operatorname{trace}\big((B^{-1}A)B\big)\\
&= \operatorname{trace}\big(B(B^{-1}A)\big)\\
&= \operatorname{trace}\big((BB^{-1})A\big)\\
&= \operatorname{trace}(A).
\end{aligned}$$