# eigenspace

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Let $V$ be a vector space over a field $k$. Fix a linear transformation $T$ on $V$. Suppose $\lambda$ is an eigenvalue of $T$. The set $\{v \in V \mid Tv = \lambda v\}$ is called the *eigenspace* (of $T$) corresponding to $\lambda$. Let us write this set $W_\lambda$.

Below are some basic properties of eigenspaces.

1. $W_\lambda$ can be viewed as the kernel of the linear transformation $T - \lambda I$. As a result, $W_\lambda$ is a subspace of $V$.
2. The dimension of $W_\lambda$ is called the geometric multiplicity of $\lambda$. Let us denote this by $g_\lambda$. It is easy to see that $1 \leq g_\lambda$, since the existence of an eigenvalue means the existence of a non-zero eigenvector corresponding to the eigenvalue.
3. $W_\lambda$ is an invariant subspace under $T$ ($T$-invariant).
4. $W_{\lambda_1} \cap W_{\lambda_2} = 0$ iff $\lambda_1 \neq \lambda_2$.
5. In fact, if $W'_\lambda$ is the sum of eigenspaces corresponding to eigenvalues of $T$ other than $\lambda$, then $W_\lambda \cap W'_\lambda = 0$.

From now on, we assume $V$ finite-dimensional.

Let $S_T$ be the set of all eigenvalues of $T$ and let $W = \oplus_{\lambda \in S} W_\lambda$. We have the following properties:

1. If $m_\lambda$ is the algebraic multiplicity of $\lambda$, then $g_\lambda \leq m_\lambda$.
2. Suppose the characteristic polynomial $p_T(x)$ of $T$ can be factored into linear terms, then $T$ is diagonalizable iff $m_\lambda = g_\lambda$ for every $\lambda \in S_T$.
3. In other words, if $p_T(x)$ splits over $k$, then $T$ is diagonalizable iff $V = W$.

For example, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x, x + y)$. Using the standard basis, $T$ is represented by the matrix

$$M_T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

From this matrix, it is easy to see that $p_T(x) = (x - 1)^2$ is the characteristic polynomial of $T$ and 1 is the only eigenvalue of $T$ with $m_1 = 2$. Also, it is not hard to see that $T(x, y) = (x, y)$ only when $y = 0$. So $W_1$ is a one-dimensional subspace of $\mathbb{R}^2$ generated by $(1, 0)$. As a result, $T$ is not diagonalizable.