linearization

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Linearization is the process of reducing a homogeneous polynomial into a multilinear map over a commutative ring. There are in general two ways of doing this:

- **Method 1.** Given any homogeneous polynomial f of degree n in m indeterminates over a commutative scalar ring R (scalar simply means that the elements of R commute with the indeterminates).

Step 1 If all indeterminates are linear in f, then we are done.

Step 2 Otherwise, pick an indeterminate x such that x is not linear in f. Without loss of generality, write $f = f(x, X)$, where X is the set of indeterminates in f excluding x. Define $g(x_1, x_2, X) := f(x_1 + x_2, X) - f(x_1, X) - f(x_2, X)$. Then g is a homogeneous polynomial of degree n in $m + 1$ indeterminates. However, the highest degree of x_1, x_2 is $n - 1$, one less that of x.

Step 3 Repeat the process, starting with Step 1, for the homogeneous polynomial g. Continue until the set X of indeterminates is enlarged to one X' such that each $x \in X'$ is linear.

- **Method 2.** This method applies only to homogeneous polynomials that are also homogeneous in each indeterminate, when the other indeterminates are held constant, i.e., $f(tx, X) = t^n f(x, X)$ for some $n \in \mathbb{N}$ and any $t \in R$. Note that if all of the indeterminates in f commute with each other, then f is essentially a monomial. So this method is particularly useful when indeterminates are non-commuting. If this is the case, then we use the following algorithm:

Step 1 If x is not linear in f and that $f(tx, X) = t^n f(x, X)$, replace x with a formal linear combination of n indeterminates over R:

$$r_1 x_1 + \cdots + r_n x_n, \text{ where } r_i \in R.$$

Step 2 Define a polynomial $g \in R\langle x_1, \ldots, x_n \rangle$, the non-commuting free algebra over R (generated by the non-commuting indeterminates x_i) by:

$$g(x_1, \ldots, x_n) := f(r_1 x_1 + \cdots + r_n x_n).$$

Step 3 Expand g and take the sum of the monomials in g whose coefficent is $r_1 \cdots r_n$. The result is a linearization of f for the indeterminate x.

Step 4 Take the next non-linear indeterminate and start over (with Step 1). Repeat the process until f is completely linearized.

Remarks.

1. The method of linearization is used often in the studies of Lie algebras, Jordan algebras, PI-algebras and quadratic forms.

2. If the characteristic of scalar ring R is 0 and f is a monomial in one indeterminate, we can recover f back from its linearization by setting all of its indeterminates to a single indeterminate x and dividing the resulting polynomial by $n!$:

$$f(x) = \frac{1}{n!}\,\text{linearization}(f)(x, \ldots, x).$$

Please see the first example below.

3. If f is a homogeneous polynomial of degree n, then the linearized f is a multilinear map in n indeterminates.

Examples.

- $f(x) = x^2$. Then $f(x_1 + x_2) - f(x_1) - f(x_2) = x_1x_2 + x_2x_1$ is a linearization of x^2. In general, if $f(x) = x^n$, then the linearization of f is

$$\sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{\sigma(i)},$$

where S_n is the symmetric group on $\{1, \ldots, n\}$. If in addition all the indeterminates commute with each other and $n! \neq 0$ in the ground ring, then the linearization becomes

$$n!x_1 \cdots x_n = \prod_{i=1}^{n} ix_i.$$

- $f(x, y) = x^3y^2 + xyxyx$. Since $f(tx, y) = t^3f(x, y)$ and $f(x, ty) = t^2f(x, y)$, f is homogeneous over x and y separately, and thus we can linearize f. First, collect all the monomials having coefficient abc in $(ax_1 + bx_2 + cx_3, y)$, we get

$$g(x_1, x_2, x_3, y) := \sum x_ix_jx_ky^2 + x_iyx_jyx_k,$$

where $i, j, k \in 1, 2, 3$ and $(i - j)(j - k)(k - i) \neq 0$. Repeat this for y and we have

$$h(x_1, x_2, x_3, y_1, y_2) := \sum x_i x_j x_k (y_1 y_2 + y_2 y_1) + (x_i y_1 x_j y_2 x_k + x_i y_2 x_j y_1 x_k).$$