uniqueness of a sparse solution

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Let $\mathbb{F}$ be a field, and let $m, n \in \mathbb{N} \setminus \{0\}$. We denote by $\|x\|_0$ the Hamming weight of a column vector $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{F}^n$, i.e., $\|x\|_0 = \#\{j \in \{1, \ldots, n\} : x_j \neq 0\}$. Consider a non-negative integer k, and an $m \times n$ matrix A whose entries belong to $\mathbb{F}$.

Theorem 1 (Donoho and Elad) *The following conditions are equivalent:*

(1) *for each column vector $y \in \mathbb{F}^m$ there exists at most one $x \in \mathbb{F}^n$ such that $Ax = y$ and $\|x\|_0$*

(2) $k < \frac{1}{2}\mathrm{spark}(A)$.

The standard proof of the theorem (see, for instance, [**?**]) goes as follows.
Proof. First, suppose that condition *(1)* is not satisfied. Then there exist column vectors $v, w \in \mathbb{F}^n$ such that $v \neq w$, $Av = Aw$, and $\max\{\|v\|_0, \|w\|_0\} \leq k$. Consequently, $v - w \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ and $A(v - w) = \mathbf{0}$. Moreover, by the definition of the Hamming weight, $\|v - w\|_0 \leq 2k$. Thus, $\mathrm{spark}(A) = \inf\{\|x\|_0 : x \in \mathbb{F}^n \setminus \{\mathbf{0}\}, Ax = \mathbf{0}\} \leq 2k$, which means that condition *(2)* is not satisfied.
Next, suppose that *(2)* is not satisfied. Then there exists a column vector $u \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ such that $Au = \mathbf{0}$ and $\|u\|_0 \leq 2k$. It is easy to see that $u = p - q$ for some $p, q \in \mathbb{F}^n$ with $\max\{\|p\|_0, \|q\|_0\} \leq k$. (If $h := \|u\|_0 > k$, $u = [u_1, \ldots, u_n]^{\mathrm{T}}$, and $\{j \in \{1, \ldots, n\} : u_j \neq 0\} = \{j_1, \ldots, j_h\}$, define $p = [p_1, \ldots, p_n]^{\mathrm{T}}$ by

$$p_j = \begin{cases} u_j, & \text{if } j \in \{j_1, \ldots, j_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

If $\|u\|_0 \leq k$, define $p = u$.) Since $p \neq q$ and $Ap = Aq$, condition *(1)* is not satisfied. $\square$

References

[1] D. L. Donoho and M. Elad, Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ^1 minimization, *Proc. Natl. Acad. Sci. USA* **100**, No. 5: 2197–2202 (2003).

[2] *Compressed Sensing: Theory and Applications*, edited by Y. C. Eldar and G. Kutyniok, Cambridge University Press, 2012.