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adjugate

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The *adjugate*, $\mathrm{adj}(A)$, of an $n \times n$ matrix A, is the $n \times n$ matrix

$$\mathrm{adj}(A)_{ij} = (-1)^{i+j}\, M_{ji}(A) \tag{1}$$

where $M_{ji}(A)$ is the indicated minor of A (the determinant obtained by deleting row j and column i from A). The adjugate is also known as the *classical adjoint*, to distinguish it from the `http://planetmath.org/AdjointEndomorphism`usual usage of "adjoint" which denotes the conjugate transpose operation.

An equivalent characterization of the adjugate is the following:

$$\mathrm{adj}(A)A = \det(A)I. \tag{2}$$

The equivalence of (**??**) and (**??**) follows easily from the `http://planetmath.org/DeterminantAsA` linearity properties of the determinant. Thus, the adjugate operation is closely related to the matrix inverse. Indeed, if A is invertible, the adjugate can be defined as

$$\mathrm{adj}(A) = \det(A)A^{-1}$$

Yet another definition of the adjugate is the following:

$$\begin{aligned} \mathrm{adj}(A) = p_{n-1}(A)I - p_{n-2}(A)A + p_{n-3}(A)A^2 - \cdots \\ + (-1)^{n-2}p_1(A)A^{n-2} + (-1)^{n-1}A^{n-1}, \end{aligned} \tag{3}$$

where $p_1(A) = \mathrm{tr}(A), p_2(A), \ldots, p_n(A) = \det(A)$ are the elementary invariant polynomials of A. The latter arise as coefficients in the characteristic polynomial $p(t)$ of A, namely

$$p(t) = \det(tI - A) = t^n - p_1(A)t^{n-1} + \cdots + (-1)^n p_n(A).$$

The equivalence of (**??**) and (**??**) follows from the Cayley-Hamilton theorem. The latter states that $p(A) = 0$, which in turn implies that

$$A(A^{n-1} - p_1(A)A^{n-2} + \cdots + (-1)^{n-1}p_{n-1}(A)) = (-1)^{n-1}\det(A)I$$

The adjugate operation enjoys a number of notable properties:

$$\mathrm{adj}(AB) = \mathrm{adj}(B)\,\mathrm{adj}(A), \tag{4}$$

$$\mathrm{adj}(A^t) = \mathrm{adj}(A)^t, \tag{5}$$

$$\det(\mathrm{adj}(A)) = \det(A)^{n-1}. \tag{6}$$