determinant as a multilinear mapping

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Let $\mathbf{M} = (M_{ij})$ be an $n \times n$ matrix with entries in a field K. The matrix $\mathbf{M}$ is really the same thing as a list of n column vectors of size n. Consequently, the determinant operation may be regarded as a mapping

$$\det : \overbrace{K^n \times \ldots \times K^n}^{n \text{ times}} \to K$$

The determinant of a matrix $\mathbf{M}$ is then defined to be $\det(\mathbf{M}_1, \ldots, \mathbf{M}_n)$, where $\mathbf{M}_j \in K^n$ denotes the j^{th} column of $\mathbf{M}$.

Starting with the definition

$$\det(\mathbf{M}_1, \ldots, \mathbf{M}_n) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) M_{1\pi_1} M_{2\pi_2} \cdots M_{n\pi_n} \tag{1}$$

the following properties are easily established:

1. the determinant is multilinear;
2. the determinant is anti-symmetric;
3. the determinant of the identity matrix is 1.

These three properties uniquely characterize the determinant, and indeed can — some would say should — be used as the definition of the determinant operation.

Let us prove this. We proceed by representing elements of K^n as linear combinations of

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

the standard basis of K^n. Let $\mathbf{M}$ be an $n \times n$ matrix. The j^{th} column is represented as $\sum_i M_{ij} \mathbf{e}_i$; whence using multilinearity

$$\begin{aligned} \det(\mathbf{M}) &= \det\left(\sum_i M_{i1} \mathbf{e}_i \, , \sum_i M_{i2} \mathbf{e}_i \, , \; \ldots \; , \sum_i M_{in} \mathbf{e}_i \right) \\ &= \sum_{i_1, \ldots, i_n = 1}^{n} M_{i_1 1} M_{i_2 2} \cdots M_{i_n n} \det(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n}) \end{aligned}$$

The anti-symmetry assumption implies that the expressions $\det(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_n})$ vanish if any two of the indices $i_1, \dots, i_n$ coincide. If all n indices are distinct,

$$\det(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_n}) = \pm \det(\mathbf{e}_1, \dots, \mathbf{e}_n),$$

the sign in the above expression being determined by the number of transpositions required to rearrange the list $(i_1, \dots, i_n)$ into the list $(1, \dots, n)$. The sign is therefore the parity of the permutation $(i_1, \dots, i_n)$. Since we also assume that

$$\det(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1,$$

we now recover the original definition (**??**).