linear involution

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Definition. Let V be a vector space. A *linear involution* is a linear operator $L : V \to V$ such that L^2 is the identity operator on V. An equivalent definition is that a linear involution is a linear operator that equals its own inverse.

Theorem 1. Let V be a vector space and let $A : V \to V$ be a linear involution. Then the eigenvalues of A are ± 1. Further, if V is $\mathbb{C}^n$, and A is a $n \times n$ complex matrix, then we have that:

1. $\det A = \pm 1$.
2. The characteristic polynomial of A, $p(\lambda) = \det(A - \lambda I)$, is a reciprocal polynomial, i.e.,
$$p(\lambda) = \pm \lambda^n p(1/\lambda).$$

(`http://planetmath.org/EigenvaluesOfAnInvolution`proof.)

The next theorem gives a correspondence between involution operators and projection operators.

Theorem 2. Let L and P be linear operators on a vector space V over a field of characteristic not 2, and let I be the identity operator on V. If L is an involution then the operators $\frac{1}{2}\big(I \pm L\big)$ are projection operators. Conversely, if P is a projection operator, then the operators $\pm(2P - I)$ are involutions.

Involutions have important application in expressing hermitian-orthogonal operators, that is, $H^t = \overline{H} = H^{-1}$. In fact, it may be represented as

$$H = Le^{iS},$$

being L a real symmetric involution operator and S a real skew-symmetric operator permutable with it, i.e.

$$L = \overline{L} = L^t, \qquad L^2 = I, \qquad S = \overline{S} = -S^t, \qquad LS = SL.$$