Hasse-Minkowski theorem

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The *Hasse-Minkowski theorem* is a classical example of the Hasse principle.

Let F be a global field, i.e. a number field or a rational function field over a finite field of characteristic not 2, X a finite dimensional vector space over F and ϕ a regular quadratic form over X.

A regular quadratic form ϕ over X is a quadratic form such that for every $x \neq 0$ in X there is a y in X with $b(x,y) \neq 0$. Here $b(x,y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$ is the associated bilinear form.

To every completion F_v of F with respect to a nontrivial valuation v we assign the vector space $X_v := F_v \otimes_F X$ and the induced quadratic form ϕ_v on X_v.

A quadratic form ϕ over X is an isotropic quadratic form if there is a nonzero vector $x \in X$ with $\phi(x) = 0$.

The Hasse-Minkowski theorem can now be stated as:

Theorem 1 *A regular quadratic form ϕ over a global field F is isotropic if and only if every completion ϕ_v is isotropic, where v runs through the nontrivial valuations of F.*

The case of $\mathbb{Q}$ was first proved by Minkowski. It can be proved using the Hilbert symbol and Dirichlet's theorem on primes in arithmetic progressions.

The general case was proved by Hasse. It can be proved using two local-global principles of class field theory, namely the *Hasse norm theorem*: For a cyclic field extension E/F of global fields an element $a \in F$ is a norm of E/F and only if it is a norm of E_v/F_v for every valuation v of E.

and the *Global square theorem*: An element a of a global field F is a square if and only if it is a square in every F_v.