Hadamard matrix

Canonical name	HadamardMatrix
Date of creation	2013-03-22 13:09:45
Last modified on	2013-03-22 13:09:45
Owner	Koro (127)
Last modified by	Koro (127)
Numerical id	13
Author	Koro (127)
Entry type	Definition
Classification	msc 15-00
Classification	msc 05B20
Synonym	Hadamard
Related topic	HadamardConjecture

An $n \times n$ matrix $H = (h_{ij})$ is a *Hadamard matrix* of order n if the entries of H are either $+1$ or -1 and such that $HH^T = nI$, where H^T is the transpose of H and I is the order n identity matrix.

In other words, an $n \times n$ matrix with only $+1$ and -1 as its elements is Hadamard if the inner product of two distinct rows is 0 and the inner product of a row with itself is n.

A few examples of Hadamard matrices are

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

These matrices were first considered as Hadamard determinants, because the determinant of a Hadamard matrix satisfies equality in Hadamard's determinant theorem, which states that if $X = (x_{ij})$ is a matrix of order n where $|x_{ij}| \leq 1$ for all i and j, then

$$det(X) \leq n^{n/2}$$

Property 1:

The order of a Hadamard matrix is $1, 2$ or $4n$, where n is an integer.

Property 2:

If the rows and columns of a Hadamard matrix are permuted, the matrix remains Hadamard.

Property 3:

If any row or column is multiplied by -1, the Hadamard property is retained.

Hence it is always possible to arrange to have the first row and first column of a Hadamard matrix contain only $+1$ entries. A Hadamard matrix in this form is said to be *normalized*.

Hadamard matrices are common in signal processing and coding applications.