minimal Gershgorin set

Canonical name	MinimalGershgorinSet
Date of creation	2013-03-22 15:35:57
Last modified on	2013-03-22 15:35:57
Owner	Andrea Ambrosio (7332)
Last modified by	Andrea Ambrosio (7332)
Numerical id	11
Author	Andrea Ambrosio (7332)
Entry type	Definition
Classification	msc 15A42

related to Gershgorin's theorem is the so called "minimal Gershgorin set". For every $A \in \mathbf{C}^{n,n}$, $\mathbf{x} > 0$ meaning $x_i > 0 \quad \forall i$, let's define its *minimal Gershgorin set* $G(A)$ as:

$$G(A) = \bigcap_{\mathbf{x}>0} G_{\mathbf{x}}(A),$$

where

$$G_{\mathbf{x}}(A) = \bigcup_{i=1}^{n} \left\{ z \in \mathbf{C} : |z - a_{ii}| \leq \frac{1}{x_i} \sum_{j \neq i} |a_{ij}| \, x_j \right\}.$$

Theorem: Let $A \in \mathbf{C}^{n,n}$, let $\sigma(A)$ be the spectrum of A and let $G(A)$ be its minimal Gershgorin set defined as above. Then

$$\sigma(A) \subseteq G(A).$$

Proof. Given $\mathbf{x} > 0$, let $X = diag\,\{x_1, x_2, \ldots, x_n\}$and let $B_X = X^{-1}AX$. Then A and B_X share the same spectrum, being similar. Due to definition, and keeping in mind that $X^{-1} = diag\,\{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$, we have $b_{ij}^{(X)} = a_{ij}\frac{x_j}{x_i}$ and, applying Gershgorin theorem to B_X, we get:

$\sigma(A) = \sigma(B_X) \subseteq \bigcup_{i=1}^{n} \left\{ z \in \mathbf{C} : |z - a_{ii}| \leq \frac{1}{x_i} \sum_{j \neq i} |a_{ij}| \, x_j \right\}$

and, since this is true for any $\mathbf{x} > 0$, we finally get the thesis. $\square$