nilpotent transformation

Canonical name	NilpotentTransformation
Date of creation	2013-03-22 12:19:52
Last modified on	2013-03-22 12:19:52
Owner	rmilson (146)
Last modified by	rmilson (146)
Numerical id	7
Author	rmilson (146)
Entry type	Definition
Classification	msc 15-00
Synonym	nilpotent
Related topic	LinearTransformation

A linear transformation $N : U \to U$ is called nilpotent if there exists a $k \in \mathbb{N}$ such that

$$N^k = 0.$$

A nilpotent transformation naturally determines a flag of subspaces

$$\{0\} \subset \ker N^1 \subset \ker N^2 \subset \ldots \subset \ker N^{k-1} \subset \ker N^k = U$$

and a signature

$$0 = n_0 < n_1 < n_2 < \ldots < n_{k-1} < n_k = \dim U, \qquad n_i = \dim \ker N^i.$$

The signature is governed by the following constraint, and characterizes N up to linear isomorphism.

Proposition 1 *A sequence of increasing natural numbers is the signature of a nil-potent transformation if and only if*

$$n_{j+1} - n_j \leq n_j - n_{j-1}$$

for all $j = 1, \ldots, k-1$. Equivalently, there exists a basis of U such that the matrix of N relative to this basis is block diagonal

$$\begin{pmatrix} N_1 & 0 & 0 & \ldots & 0 \\ 0 & N_2 & 0 & \ldots & 0 \\ 0 & 0 & N_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N_k \end{pmatrix},$$

with each of the blocks having the form

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Letting d_i denote the number of blocks of size i, the signature of N is given by

$$n_i = n_{i-1} + d_i + d_{i+1} + \ldots + d_k, \quad i = 1, \ldots, k$$