determinant inequalities

Canonical name	DeterminantInequalities
Date of creation	2013-03-22 15:34:46
Last modified on	2013-03-22 15:34:46
Owner	Andrea Ambrosio (7332)
Last modified by	Andrea Ambrosio (7332)
Numerical id	12
Author	Andrea Ambrosio (7332)
Entry type	Result
Classification	msc 15A15

There are a number of interesting inequalities bounding the determinant of a $n \times n$ complex matrix A, where ρ is its spectral radius:

1) $|\det(A)| \leq \rho^n(A)$

2) $|\det(A)| \leq \prod_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|\right) = \prod_{i=1}^n \|a_i\|_1$

3) $|\det(A)| \leq \prod_{j=1}^n \left(\sum_{i=1}^n |a_{ij}|\right) = \prod_{j=1}^n \|a_j\|_1$

4) $|\det(A)| \leq \prod_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}} = \prod_{i=1}^n \|a_i\|_2$

5) $|\det(A)| \leq \prod_{j=1}^n \left(\sum_{i=1}^n |a_{ij}|^2\right)^{\frac{1}{2}} = \prod_{j=1}^n \|a_j\|_2$

6) if A is Hermitian positive semidefinite, $\det(A) \leq \prod_{i=1}^n a_{ii}$, with equality if and only if A is diagonal.

Inequalities 4)-6) are known as "Hadamard's inequalities".

(Note that inequalities 2)-5) may suggest the idea that such inequalities could hold: $|\det(A)| \leq \prod_{i=1}^n \|a_i\|_p$ or $|\det(A)| \leq \prod_{j=1}^n \|a_j\|_p$ for any $p \in \mathbf{N}$; however, this is not true, as one can easily see with $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $p = 3$. Actually, inequalities 2)-5) give the best possible estimate of this kind.)

Proofs:

1) $|\det(A)| = |\prod_{i=1}^n \lambda_i| = \prod_{i=1}^n |\lambda_i| \leq \prod_{i=1}^n \rho(A) = \rho^n(A)$.

2) If A is singular, the thesis is trivial. Let then $\det(A) \neq 0$. Let's define $B = DA$, $D = diag(d_{11}, d_{22}, \cdots, d_{nn})$,$d_{ii} = \left(\sum_{j=1}^n |a_{ij}|\right)^{-1}$. (Note that d_{ii} exist for any i, because $\det(A) \neq 0$ implies no all-zero row exists.) So $\|B\|_\infty = \max_i \left(\sum_{j=1}^n |b_{ij}|\right) = 1$ and, since $\rho(B) \leq \|B\|_\infty$, we have:

$|\det(B)| = |\det(D)|\,|\det(A)| = \left(\prod_{i=1}^n \sum_{j=1}^n |a_{ij}|\right)^{-1} |\det(A)| \leq \rho^n(B) \leq \|B\|_\infty^n = 1$,

from which:

$|\det(A)| \leq \prod_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|\right)$.

3) Same as 2), but applied to A^T.

4)-6) See related proofs attached to "Hadamard's inequalities".