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elementary matrix operations as rank preserving operations

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Let M be a matrix over a division ring D. An *elementary operation* on M is any one of the eight operations below:

1. exchanging two rows
2. exchanging two columns
3. adding one row to another
4. adding one column to another
5. right multiplying a non-zero scalar to a row
6. left multiplying a non-zero scalar to a row
7. right multiplying a non-zero scalar to a column
8. left multiplying a non-zero scalar to a column

We want to determine the effects of these operations on the various ranks of M. To facilitate this discussion, let $M = (a_{ij})$ be an $n \times m$ matrix and $M' = (b_{ij})$ be the matrix after an application of one of the operations above to M. In addition, let $v_i = (a_{i1}, \cdots, v_{im})$ be the i-th row of M, and $w_i = (b_{i1}, \cdots, b_{im})$ be the i-th row of M'. In other words,

$$M = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \xrightarrow[\text{operation}]{\text{elementary}} \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = M'$$

Finally, let d be the left row rank of M.

Proposition 1. *Row and column exchanges preserve all ranks of M.*

Proof. Clearly, exchanging two rows of M do not change the subspace generated by the rows of M, and therefore d is preserved.

As exchanging rows do not affect d, let us assume that rows have been exchanged so that the first d rows of M are left linearly independent.

Now, let M' be obtained from M by exchanging columns i and j. So $w_1, \ldots, w_n$ are vectors obtained respectively from $v_1, \ldots, v_n$ by exchanging the i-th and j-th coordinates. Suppose $r_1 w_1 + \cdots + r_d w_d = 0$. Then we get an equation $r_1 b_{1k} + \cdots + r_d b_{dk} = 0$ for $1 \leq k \leq m$. Rearranging these equations, we see that $r_1 v_1 + \cdots + r_d v_d = 0$, which implies $r_1 = \cdots = r_d = 0$, showing that $w_1, \ldots, w_d$ are left linearly independent. This means that d is preserved by column exchanges.

Preservation of other ranks of M are similarly proved. □

Proposition 2. *Additions of rows and columns preserve all ranks of M.*

Proof. Let M' be the matrix obtained from M by replacing row i by vector $v_i + v_j$, and let V' be the left vector space spanned by the rows of M'. Since $v_i + v_j \in V$, we have $V' \subseteq V$. On other hand, $v_i = (v_i + v_j) - v_j \in V'$, so $V \subseteq V'$, and hence $V = V'$.

Next, let $w_1, \ldots, w_n$ be vectors obtained respectively from $v_1, \ldots, v_n$ such that the i-th coordinate of w_k is the sum of the i-th coordinate of v_k and the j-th coordinate of v_k, with all other coordinates remain the same. Again, by renumbering if necessary, let $v_1, \ldots, v_d$ be left linearly independent. Suppose $r_1 w_1 + \cdots + r_i w_i + \cdots + r_d w_d = 0$. A similar argument like in the previous proposition shows that $r_1 v_1 + \cdots + (r_i + r_j) v_j + r_d v_d = 0$, which implies $r_1 = \cdots = r_i + r_j = \cdots r_d = 0$. Since $r_i = 0$, $r_j = 0$ too. This shows that $w_1, \ldots, w_d$ are left linearly independent, which means that d is preserved by additions of columns.

Preservation of other ranks of M are proved similarly. □

Proposition 3. *Left (right) non-zero row scalar multiplication preserves left (right) row rank of M; left (right) non-zero column scalar multiplications preserves left (right) column rank of M.*

Proof. Let $w_1, \ldots, w_n$ be vectors obtained respectively from $v_1, \ldots, v_n$ such that the i-th vector $w_i = r v_i$, where $0 \neq r \in D$, and all other w_j's are the same as the v_j's. Assume that the first d rows of M are left linearly independent, and that $i \leq d$. Suppose $r_1 w_1 + \cdots + r_d w_d = 0$. Then $r_1 v_1 + \cdots + r_i (r v_i) + \cdots r_d v_d = 0$, which implies $r_1 = \cdots = r_i r = \cdots = r_d = 0$. Since $r \neq 0$, $r_i = 0$, and therefore $w_1, \ldots, w_d$ are left linearly independent.

The others are proved similarly. □

Proposition 4. *Left (right) non-zero row scalar multiplication preserves right (left) column rank of M; left (right) non-zero column scalar multiplication preserves right (left) row rank of M.*

Proof. Let us prove that right multiplying a column by a non-zero scalar r preserves the left row rank d of M. The others follow similarly.

Let $w_1, \ldots, w_n$ be vectors obtained respectively from $v_1, \ldots, v_n$ such that the i-th coordinate b_{ik} of w_k is $a_{ik} r$, where a_{ik} is the i-th coordinate of v_k. Suppose once again that the first d rows of M are left linearly independent, and suppose $r_1 w_1 + \cdots + r_d w_d = 0$. Then for each coordinate j we get an equation $r_1 b_{1j} + \cdots + r_d b_{dj} = 0$. In particular, for the i-th coordinate, we have $r_1 a_{1j} r + \cdots + r_d a_{dj} r = 0$. Since $r \neq 0$, right multiplying the equation by r^{-1} gives us $r_1 a_{1j} + \cdots + r_d a_{dj} = 0$. Re-collecting all the equations, we get $r_1 v_1 + \cdots + r_d w_d = 0$, which implies that $r_1 = \cdots = r_d = 0$, or that $w_1, \ldots, w_d$ are left linearly independent. $\square$