generalized Kronecker delta symbol

Canonical name	GeneralizedKroneckerDeltaSymbol
Date of creation	2013-03-22 13:31:38
Last modified on	2013-03-22 13:31:38
Owner	matte (1858)
Last modified by	matte (1858)
Numerical id	5
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Entry type	Definition
Classification	msc 15A99
Related topic	LeviCivitaPermutationSymbol3

Let l and n be natural numbers such that $1 \leq l \leq n$. Further, let i_k and j_k be natural numbers in $\{1, \cdots, n\}$ for all k in $\{1, \cdots, l\}$. Then the *generalized Kronecker delta symbol*, denoted by $\delta^{i_1 \cdots i_l}_{j_1 \cdots j_l}$, is zero if $i_r = i_s$ or $j_r = j_s$ for some $r \neq s$, or if $\{i_1, \cdots, i_l\} \neq \{j_1, \cdots, j_l\}$ as sets. If none of the above conditions are met, then $\delta^{i_1 \cdots i_l}_{j_1 \cdots j_l}$ is defined as the sign of the permutation that maps $i_1 \cdots i_l$ to $j_1 \cdots j_l$.

From the definition, it follows that when $l = 1$, the generalized Kronecker delta symbol reduces to the traditional delta symbol δ^i_j. Also, for $l = n$, we obtain

$$\begin{aligned} \delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} &= \varepsilon^{i_1 \cdots i_n} \varepsilon_{j_1 \cdots j_n}, \\ \delta^{1 \cdots n}_{j_1 \cdots j_n} &= \varepsilon_{j_1 \cdots j_n}, \end{aligned}$$

where $\varepsilon_{j_1 \cdots j_n}$ is the Levi-Civita permutation symbol.

For any l we can write the generalized delta function as a determinant of traditional delta symbols. Indeed, if $S(l)$ is the permutation group of l elements, then

$$\begin{aligned} \delta^{i_1 \cdots i_l}_{j_1 \cdots j_l} &= \sum_{\tau \in S(l)} \operatorname{sign} \tau \, \delta^{i_{\tau(1)}}_{j_1} \cdots \delta^{i_{\tau(l)}}_{j_l} \\ &= \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_l}_{j_1} \\ \vdots & \ddots & \vdots \\ \delta^{i_1}_{j_l} & \cdots & \delta^{i_l}_{j_l} \end{pmatrix}. \end{aligned}$$

The first equality follows since the sum one the first line has only one non-zero term; the term for which $i_{\tau(k)} = j_k$. The second equality follows from the definition of the determinant.