dual space

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Synonym	algebraic dual
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Synonym	topological dual
Related topic	DualHomomorphism
Related topic	DoubleDualEmbedding
Related topic	BanachSpace
Related topic	Unimodular
Related topic	LinearFunctional
Related topic	BoundedLinearFunctionalsOnLpmu

Dual of a vector space; dual bases

Let V be a vector space over a field k. The *dual* of V, denoted by V^*, is the vector space of linear forms on V, i.e. linear mappings $V \to k$. The operations in V^* are defined pointwise:

$$(\varphi + \psi)(v) = \varphi(v) + \psi(v)$$

$$(\lambda\varphi)(v) = \lambda\varphi(v)$$

for $\lambda \in K$, $v \in V$ and $\varphi, \psi \in V^*$.

V is isomorphic to V^* if and only if the dimension of V is finite. If not, then V^* has a larger (infinite) dimension than V; in other words, the cardinal of any basis of V^* is strictly greater than the cardinal of any basis of V.

Even when V is finite-dimensional, there is no canonical or natural isomorphism $V \to V^*$. But on the other hand, a basis $\mathcal{B}$ of V does define a basis $\mathcal{B}^*$ of V^*, and moreover a bijection $\mathcal{B} \to \mathcal{B}^*$. For suppose $\mathcal{B} = \{b_1, \ldots, b_n\}$. For each i from 1 to n, define a mapping

$$\beta_i : V \to k$$

by

$$\beta_i(\sum_k x_k b_k) = x_i \ .$$

It is easy to see that the β_i are nonzero elements of V^* and are independent. Thus $\{\beta_1, \ldots, \beta_n\}$ is a basis of V^*, called the dual basis of $\mathcal{B}$.

The dual of V^* is called the *second dual* or *bidual* of V. There *is* a very simple canonical injection $V \to V^{**}$, and it is an isomorphism if the dimension of V is finite. To see it, let x be any element of V and define a mapping $x' : V^* \to k$ simply by

$$x'(\phi) = \phi(x) \ .$$

x' is linear by definition, and it is readily verified that the mapping $x \mapsto x'$ from V to V^{**} is linear and injective.

Dual of a topological vector space

If V is a topological vector space, the *continuous dual* V' of V is the subspace of V^* consisting of the *continuous* linear forms.

A *normed* vector space V is said to be *reflexive* if the natural embedding $V \to V''$ is an isomorphism. For example, any finite dimensional space is reflexive, and any Hilbert space is reflexive by the Riesz representation theorem.

Remarks

Linear forms are also known as linear functionals.

Another way in which a linear mapping $V \to V^*$ can arise is via a bilinear form

$$V \times V \to k \ .$$

The notions of duality extend, in part, from vector spaces to modules, especially free modules over commutative rings. A related notion is the duality in projective spaces.