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outer multiplication

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Note: the present entry employs the terminology and notation defined and described in the entry on tensor arrays. To keep things reasonably self contained we mention that the symbol $\mathrm{T}^{p,q}$ refers to the vector space of type (p,q) tensor arrays, i.e. maps

$$I^p \times I^q \to \mathbb{K},$$

where I is some finite list of index labels, and where $\mathbb{K}$ is a field.

Let p_1, p_2, q_1, q_2 be natural numbers. Outer multiplication is a bilinear operation

$$\mathrm{T}^{p_1,q_1} \times \mathrm{T}^{p_2,q_2} \to \mathrm{T}^{p_1+p_2,q_1+q_2}$$

that combines a type (p_1, q_1) tensor array X and a type (p_2, q_2) tensor array Y to produce a type (p_1+p_2, q_1+q_2) tensor array XY (also written as $X \otimes Y$), defined by

$$(XY)^{i_1 \ldots i_{p_1} i_{p_1+1} \ldots i_{p_1+p_2}}_{j_1 \ldots j_{q_1} j_{q_1+1} \ldots j_{q_1+q_2}} = X^{i_1 \ldots i_{p_1}}_{j_1 \ldots j_{q_1}} Y^{i_{p_1+1} \ldots i_{p_1+p_2}}_{j_{q_1+1} \ldots j_{q_1+q_2}}$$

Speaking informally, what is going on above is that we multiply every value of the X array by every possible value of the Y array, to create a new array, XY. Quite obviously then, the size of XY is the size of X times the size of Y, and the index slots of the product XY are just the union of the index slots of X and of Y.

Outer multiplication is a non-commutative, associative operation. The type $(0,0)$ arrays are the scalars, i.e. elements of $\mathbb{K}$; they commute with everything. Thus, we can embed $\mathbb{K}$ into the direct sum

$$\bigoplus_{p,q \in \mathbb{N}} \mathrm{T}^{p,q},$$

and thereby endow the latter with the structure of an $\mathbb{K}$-algebra[1].

By way of illustration we mention that the outer product of a column vector, i.e. a type $(1,0)$ array, and a row vector, i.e. a type $(0,1)$ array, gives a matrix, i.e. a type $(1,1)$ tensor array. For instance:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{pmatrix}, \quad a,b,c,x,y,z \in \mathbb{K}$$

[1]We will not pursue this line of thought here, because the topic of algebra structure is best dealt with in the a more abstract context. The same comment applies to the use of the tensor product sign $\otimes$ in denoting outer multiplication. These topics are dealt with in the entry pertaining to abstract tensor algebra.