ideals in matrix algebras

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Let R be a ring with 1. Consider the ring $M_{n\times n}(R)$ of $n \times n$-matrices with entries taken from R.

It will be shown that there exists a one-to-one correspondence between the (two-sided) ideals of R and the (two-sided) ideals of $M_{n\times n}(R)$.

For $1 \leq i, j \leq n$, let E_{ij} denote the $n\times n$-matrix having entry 1 at position (i, j) and 0 in all other places. It can be easily checked that

$$E_{ij} \cdot E_{kl} = \begin{cases} 0 & \text{iff} \qquad\qquad k \neq j \\ E_{il} & \text{otherwise.} \end{cases} \tag{1}$$

Let $\mathfrak{m}$ be an ideal in $M_{n\times n}(R)$.

Claim. *The set* $\mathfrak{i} \subseteq R$ *given by*

$$\mathfrak{i} = \{x \in R \mid x \quad \textit{is an entry of } A \in \mathfrak{m}\}$$

is an ideal in R*, and* $\mathfrak{m} = M_{n\times n}(\mathfrak{i})$*.*

Proof. $\mathfrak{i} \neq \emptyset$ since $0 \in \mathfrak{i}$. Now let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices in $\mathfrak{m}$, and $x, y \in R$ be entries of A and B respectively, say $x = a_{ij}$ and $y = b_{kl}$. Then the matrix $A \cdot E_{jl} + E_{ik} \cdot B \in \mathfrak{m}$ has $x + y$ at position (i, l), and it follows: If $x, y \in \mathfrak{i}$, then $x + y \in \mathfrak{i}$. Since $\mathfrak{i}$ is an ideal in $M_{n\times n}(R)$ it contains, in particular, the matrices $D_r \cdot A$ and $A \cdot D_r$, where

$$D_r := \sum_{i=1}^{n} r \cdot E_{ii}, r \in R.$$

thus, $rx, xr \in \mathfrak{i}$. This shows that $\mathfrak{i}$ is an ideal in R. Furthermore, $M_{n\times n}(\mathfrak{i}) \subseteq \mathfrak{m}$.

By construction, any matrix $A \in \mathfrak{m}$ has entries in $\mathfrak{i}$, so we have

$$A = \sum_{1\leq i,j\leq n} a_{ij}E_{ij}, a_{ij} \in \mathfrak{i}$$

so $A \in m_{n\times n}(\mathfrak{i})$. Therefore $\mathfrak{m} \subseteq M_{n\times n}(\mathfrak{i})$. □

A consequence of this is: If F is a field, then $M_{n\times n}(F)$ is simple.