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similar matrix

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Definition A square matrix A is *similar* (or *conjugate*) to a square matrix B if there exists a nonsingular square matrix S such that

$$A = S^{-1}BS. \tag{1}$$

Note that, given S as above, we can define $R = S^{-1}$ and have $A = RBR^{-1}$. Thus, whether the inverse comes first or last does not matter.

Transformations of the form $S^{-1}BS$ (or SBS^{-1}) are called *similarity transformations*.

Discussion Similarity is useful for turning recalcitrant matrices into pliant ones. The canonical example is that a diagonalizable matrix A is similar to the diagonal matrix of its eigenvalues Λ, with the matrix of its eigenvectors acting as the similarity transformation. That is,

$$\begin{aligned} A &= T\Lambda T^{-1} && (2)\\ &= \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^{-1}. && (3) \end{aligned}$$

This follows directly from the equation defining eigenvalues and eigenvectors,

$$AT = T\Lambda. \tag{4}$$

If A is http://planetmath.org/SymmetricMatrixsymmetric for example, then through this transformation, we have turned A into the product of two orthogonal matrices and a diagonal matrix. This can be very useful. As an application, see the solution for the normalizing constant of a multidimensional Gaussian integral.

Properties of similar matrices

1. Similarity is http://planetmath.org/Reflexivereflexive: All square matrices A are similar to themselves via the similarity transformation $A = I^{-1}AI$, where I is the identity matrix with the same dimensions as A.

2. Similarity is http://planetmath.org/Symmetricsymmetric: If A is similar to B, then B is similar to A, as we can define a matrix $R = S^{-1}$ and have

$$B = R^{-1}AR \tag{5}$$

3. Similarity is http://planetmath.org/Transitive3transitive: If A is similar to B, which is similar to C, we have

$$A = S^{-1}BS = S^{-1}(R^{-1}CR)S = (S^{-1}R^{-1})C(RS) = (RS)^{-1}C(RS). \tag{6}$$

4. Because of 1, 2 and 3, similarity defines an equivalence relation () on square matrices, http://planetmath.org/Partitionpartitioning the space of such matrices into a disjoint set of equivalence classes.

5. If A is similar to B, then their determinants are equal; http://planetmath.org/Iei.e., $\det A = \det B$. This is easily verified:

$$\det A = \det(S^{-1}BS) = \det(S^{-1})\det B \det S = (\det S)^{-1}\det B \det S = \det B. \tag{7}$$

   In fact, similar matrices have the same characteristic polynomial, which implies this result directly, the determinant being the constant term of the characteristic polynomial (up to sign).

6. Similar matrices represent the same linear transformation after a change of basis.

7. It can be shown that a matrix A and its transpose A^T are always similar.