rank of a matrix

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Defines	left row rank
Defines	left column rank
Defines	right row rank
Defines	right column rank

Let D be a division ring, and M an $m \times n$ matrix over D. There are four numbers we can associate with M:

1. the dimension of the subspace spanned by the columns of M viewed as elements of the n-dimensional right vector space over D.

2. the dimension of the subspace spanned by the columns of M viewed as elements of the n-dimensional left vector space over D.

3. the dimension of the subspace spanned by the rows of M viewed as elements of the m-dimensional right vector space over D.

4. the dimension of the subspace spanned by the rows of M viewed as elements of the m-dimensional left vector space over D.

The numbers are respectively called the *right column rank*, *left column rank*, *right row rank*, and *left row rank* of M, and they are respectively denoted by $\operatorname{rc.rnk}(M)$, $\operatorname{lc.rnk}(M)$, $\operatorname{rr.rnk}(M)$, and $\operatorname{lr.rnk}(M)$.

Since the columns of M are the rows of its transpose M^T, we have

$$\operatorname{lc.rnk}(M) = \operatorname{lr.rnk}(M^T), \qquad \text{and} \qquad \operatorname{rc.rnk}(M) = \operatorname{rr.rnk}(M^T).$$

In addition, it can be shown that for a given matrix M,

$$\operatorname{lc.rnk}(M) = \operatorname{rr.rnk}(M), \qquad \text{and} \qquad \operatorname{rc.rnk}(M) = \operatorname{lr.rnk}(M).$$

For any $0 \neq r \in D$, it is also easy to see that the left column and row ranks of rM are the same as those of M. Similarly, the right column and row ranks of Mr are the same as those of M.

If D is a field, $\operatorname{lc.rnk}(M) = \operatorname{rc.rnk}(M)$, so that all four numbers are the same, and we simply call this number the *rank* of M, denoted by $\operatorname{rank}(M)$.

Rank can also be defined for matrices M (over a fixed D) that satisfy the identity $M = rM^T$, where r is in the center of D. Matrices satisfying the identity include symmetric and anti-symmetric matrices.

However, the left column rank is not necessarily the same as the right row rank of a matrix, if the underlying division ring is not commutative, as can be shown in the following example: let $u = (1, j)$ and $v = (i, k)$ be vectors over the Hamiltonian quaternions $\mathbb{H}$. They are columns in the 2×2 matrix

$$M := \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}$$

Since $iu = (i, ij) = (i, k) = v$, they are left linearly dependent, and therefore the left column rank of M is 1. Now, suppose $ur + vs = (0, 0)$, with $r, s \in \mathbb{H}$. Since $ui = (i, ji) = (i, -k)$, then $ui(-ir) + vs = 0$, which boils down to two equations $ir = s$ and $-ir = s$, and which imply that $s = r = 0$, showing that u, v are right linearly independent. Thus the right column rank of M is 2.