tensor product and dual spaces

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Let k be a field and V be a vector space over k. Recall that

$$V^* = \{f : V \to k \mid f \text{ is linear}\}$$

denotes the dual space of V (which is also a vector space over k).

Proposition. Let V and W be vector spaces. Consider the map $\phi : V^* \otimes W^* \to (V \otimes W)^*$ such that

$$\phi(f \otimes g)(v \otimes w) = f(v)g(w).$$

Then ϕ is a monomorphism. Moreover if one of the spaces V, W is finite dimensional, then ϕ is an isomorphism.

Proof. One can easily check that ϕ is a well defined linear map, thus it is sufficient to show that $\mathrm{Ker}(\phi) = 0$. So assume that $F \in V^* \otimes W^*$ is such that $\phi(F) = 0$. It is clear that F can be (uniquely) expressed in the form

$$F = \sum_{i,j} \alpha_{i,j} f_i \otimes g_j,$$

where (f_i) is a basis of V^*, (g_j) is a basis of W^* and $\alpha_{i,j} \in k$. Then for any $v \in V$ and $w \in W$ we have:

$$0 = \phi(F)(v \otimes w) = \phi(\sum_{i,j} \alpha_{i,j} f_i \otimes g_j)(v \otimes w) =$$

$$= \sum_{i,j} \alpha_{i,j} \phi(f_i \otimes g_j)(v \otimes w) = \sum_{i,j} \alpha_{i,j} f_i(v) g_j(w).$$

Since $w \in W$ is arbitrary then we can write this equality in the form:

$$0 = \sum_{i,j} \alpha_{i,j} f_i(v) g_j = \sum_j (\sum_i \alpha_{i,j} f_i(v)) g_j$$

and since (g_j) are linearly independent we obtain that $\sum_i \alpha_{i,j} f_i(v) = 0$ for all j. Again since $v \in V$ was arbitrary we obtain that $\sum_i \alpha_{i,j} f_i = 0$ for all j. Now since (f_i) are linearly independent we obtain that $\alpha_{i,j} = 0$ for all i, j. Thus $F = 0$.

Now assume that $\dim_k V = q < +\infty$. Let $(v_i)_{i=1}^q$ be a basis of V and let $(v_i^*)_{i=1}^q$ be an induced basis of V^*. Moreover let $(w_p)_{p \in P}$ be a basis of W.

We wish to show that ϕ is onto, so let $f : V \otimes W \to k$ be an element of $(V \otimes W)^*$. Define $F \in V^* \otimes W^*$ by the formula:

$$F = \sum_{i=1}^{q} v_i^* \otimes g_i,$$

where $g_i : W \to k$ is such that $g_i(w_p) = f(v_i \otimes w_p)$. Then for any v_j from $(v_i)_{i=1}^q$ and for any w_p from $(w_p)_{p\in P}$ we have:

$$\phi(F)(v_j \otimes w_p) = \phi(\sum_{i=1}^{q} v_i^* \otimes g_i)(v_j \otimes w_p) = \sum_{i=1}^{q} \phi(v_i^* \otimes g_i)(v_j \otimes w_p) =$$

$$= \sum_{i=1}^{q} v_i^*(v_j) g_i(w_p) = g_j(w_p) = f(v_j \otimes w_p)$$

and thus $\phi(F) = f$. □

Remark. The map ϕ from the previous proposition is very important in studying algebras and coalgebras (more precisely it is an essence in defining dual (co)algebras). Unfortunetly ϕ does not have to be an isomorphism in general. Nevertheless, the spaces $(V \otimes W)^*$ and $V^* \otimes W^*$ are always isomorphic (see `http://planetmath.org/TensorProductOfDualSpacesIsADualSpaceOfTensorProduct` entry for more details).