reduced ring

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A ring R is said to be a *reduced ring* if R contains no non-zero nilpotent elements. In other words, $r^2 = 0$ implies $r = 0$ for any $r \in R$.

Below are some examples of reduced rings.

- A reduced ring is semiprime.
- A ring is a `http://planetmath.org/CancellationRing`domain iff it is `http://planetmath.org/PrimeRing`prime and reduced.
- A commutative semiprime ring is reduced. In particular, all integral domains and Boolean rings are reduced.
- Assume that R is commutative, and let A be the set of all nilpotent elements. Then A is an ideal of R, and that R/A is reduced (for if $(r+A)^2 = 0$, then $r^2 \in A$, so r^2, and consequently r, is nilpotent, or $r \in A$).

An example of a reduced ring with zero-divisors is $\mathbb{Z}^n$, with multiplication defined componentwise: $(a_1, \ldots, a_n)(b_1, \ldots, b_n) := (a_1b_1, \ldots, a_nb_n)$. A ring of functions taking values in a reduced ring is also reduced.

Some prototypical examples of rings that are not reduced are $\mathbb{Z}_8$, since $4^2 = 0$, as well as any matrix ring over any ring; as illustrated by the instance below

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$