planetmath.org

Math for the people, by the people.

homogeneous polynomial

Canonical name	HomogeneousPolynomial1
Date of creation	2013-03-22 14:53:42
Last modified on	2013-03-22 14:53:42
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	17
Author	CWoo (3771)
Entry type	Definition
Classification	msc 16R99
Classification	msc 13B25
Classification	msc 16S36
Classification	msc 11E76
Synonym	polynomial form
Related topic	HomogeneousIdeal
Related topic	HomogeneousFunction
Related topic	HomogeneousEquation
Defines	homogeneous component
Defines	cubic form
Defines	linear form

Let R be an associative ring. A (multivariate) polynomial f over R is said to be *homogeneous of degree* r if it is expressible as an R-http://planetmath.org/LinearCombinat combination of monomials of degree r:

$$f(x_1,\dots,x_n)=\sum_{i=1}^{m} a_i {x_1}^{r_{i1}}\cdots {x_n}^{r_{in}},$$

where $r=r_{i1}+\cdots+r_{in}$ for all $i\in\{1,\dots,m\}$ and $a_i\in R$.

A general homogeneous polynomial is also known sometimes as a *polynomial form*. A homogeneous polynomial of degree 1 is called a *linear form*; a homogeneous polynomial of degree 2 is called a *quadratic form*; and a homogeneous polynomial of degree 3 is called a *cubic form*.

Remarks.

1. If f is a homogeneous polynomial over a ring R with $\deg(f)=r$, then $f(tx_1,\dots,tx_n)=t^r f(x_1,\dots,x_n)$. In fact, a homogeneous function that is also a polynomial is a homogeneous polynomial.

2. Every polynomial f over R can be expressed uniquely as a finite sum of homogeneous polynomials. The homogeneous polynomials that make up the polynomial f are called the *homogeneous components* of f.

3. If f and g are homogeneous polynomials of degree r and s over a domain R, then fg is homogeneous of degree $r+s$. From this, one sees that given a domain R, the ring $R[\boldsymbol{X}]$ is a graded ring, where $\boldsymbol{X}$ is a finite set of indeterminates. The condition that R does not have any zero divisors is essential here. As a counterexample, in $\mathbb{Z}_6[x,y]$, if $f(x,y)=2x+4y$ and $g(x,y)=3x+3y$, then $f(x,y)g(x,y)=0$.

Examples

- $f(x,y)=x^2+xy+yx+y^2$ is a homogeneous polynomial of degree 2. Notice the middle two monomials could be combined into the monomial 2xy if the variables are allowed to commute with one another.

- $f(x)=x^3+1$ is not a homogeneous polynomial.

- $f(x,y,z)=x^3+xyz+zyz+3xy^2+x^2-xy+y^2+zy+z^2+xz+y+2x+6$ is a polynomial that is the sum of four homogeneous polynomials: $x^3+xyz+zyz+3xy^2$ (with degree 3), $x^2-xy+y^2+zy+z^2+xz$ (degree $=2$), $y+2x$ (degree $=1$) and 6 ($\deg=0$).

- Every symmetric polynomial can be written as a sum of symmetric homogeneous polynomials.