Zorn's lemma and bases for vector spaces

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In this entry, we illustrate how Zorn's lemma can be applied in proving the existence of a basis for a vector space. Let V be a vector space over a field k.

Proposition 1. *Every linearly independent subset of V can be extended to a basis for V.*

This has already been proved in `http://planetmath.org/EveryVectorSpaceHasABasis`this entry. We reprove it here for completion.

Proof. Let A be a linearly independent subset of V. Let $\mathcal{S}$ be the collection of all linearly independent supersets of A. First, $\mathcal{S}$ is non-empty since $A \in \mathcal{S}$. In addition, if $A_1 \subseteq A_2 \subseteq \cdots$ is a chain of linearly independent supersets of A, then their union is again a linearly independent superset of A (for a proof of this, see `http://planetmath.org/PropertiesOfLinearIndependence`here). So by Zorn's Lemma, $\mathcal{S}$ has a maximal element B. Let $W = \operatorname{span}(B)$. If $W \neq V$, pick $b \in V - W$. If $0 = rb + r_1b_1 + \cdots + r_nb_n$, where $b_i \in B$, then $-rb = r_1b_1 + \cdots + r_nb_n$, so that $-rb \in \operatorname{span}(B) = W$. But $b \notin W$, so $b \neq 0$, which implies $r = 0$. Consequently $r_1 = \cdots = r_n = 0$ since B is linearly independent. As a result, $B \cup \{b\}$ is a linearly independent superset of B in $\mathcal{S}$, contradicting the maximality of B in $\mathcal{S}$. $\square$

Proposition 2. *Every spanning set of V has a subset that is a basis for V.*

Proof. Let A be a spanning set of V. Let $\mathcal{S}$ be the collection of all linearly independent subsets of A. $\mathcal{S}$ is non-empty as $\varnothing \in \mathcal{S}$. Let $A_1 \subseteq A_2 \subseteq \cdots$ be a chain of linearly independent subsets of A. Then the union of these sets is again a linearly independent subset of A. Therefore, by Zorn's lemma, $\mathcal{S}$ has a maximal element B. In other words, B is a linearly independent subset A. Let $W = \operatorname{span}(B)$. Suppose $W \neq V$. Since A spans V, there is an element $b \in A$ not in W (for otherwise the span of A must lie in W, which would imply $W = V$). Then, using the same argument as in the previous proposition, $B \cup \{b\}$ is linearly independent, which contradicts the maximality of B in $\mathcal{S}$. Therefore, B spans V and thus a basis for V. $\square$

Corollary 1. *Every vector space has a basis.*

Proof. Either take $\varnothing$ to be the linearly independent subset of V and apply proposition 1, or take V to be the spanning subset of V and apply proposition 2. $\square$

Remark. The two propositions above can be combined into one: If $A \subseteq C$ are two subsets of a vector space V such that A is linearly independent and C spans V, then there exists a basis B for V, with $A \subseteq B \subseteq C$. The proof again relies on Zorn's Lemma and is left to the reader to try.