multiplicative sets in rings and prime ideals

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Proposition. Let R be a commutative ring, $S \subseteq R$ a mutliplicative subset of R such that $0 \notin S$. Then there exists prime ideal $P \subseteq R$ such that $P \cap S = \emptyset$.

Proof. Consider the family $\mathcal{A} = \{I \subseteq R \mid I \text{ is an ideal and } I \cap S = \emptyset\}$. Of course $\mathcal{A} \neq \emptyset$, because the zero ideal $0 \in \mathcal{A}$. We will show, that $\mathcal{A}$ is inductive (i.e. satisfies Zorn's Lemma's assumptions) with respect to inclusion.

Let $\{I_k\}_{k \in K}$ be a chain in $\mathcal{A}$ (i.e. for any $a, b \in K$ either $I_a \subseteq I_b$ or $I_b \subseteq I_a$). Consider $I = \bigcup_{k \in K} I_k$. Obviously I is an ideal. Furthermore, if $x \in I \cap S$, then there is $k \in K$ such that $x \in I_k \cap S = \emptyset$. Thus $I \cap S = \emptyset$, so $I \in \mathcal{A}$. Lastely each $I_k \subseteq I$, which completes this part of proof.

By Zorn's Lemma there is a maximal element $P \in \mathcal{A}$. We will show that this ideal is prime. Let $x, y \in R$ be such that $xy \in P$. Assume that neither $x \notin P$ nor $y \notin P$. Then $P \subset P + (x)$ and $P \subset P + (y)$ and these inclusions are proper. Therefore both $P + (x)$ and $P + (y)$ do not belong to $\mathcal{A}$ (because P is maximal). This implies that there exist $a \in (P + (x)) \cap S$ and $b \in (P + (y)) \cap S$. Thus

$$a = m_1 + r_1 x \in S; \qquad b = m_2 + r_2 y \in S;$$

where $m_1, m_2 \in P$ and $r_1, r_2 \in R$. Note that $ab \in S$. We calculate

$$ab = (m_1 + r_1 x)(m_2 + r_2 y) = m_1 m_2 + m_2 r_1 x + m_1 r_2 y + x y r_1 r_2.$$

Of course $m_1 m_2, m_2 r_1 x, m_1 r_2 y \in P$, because $m_1, m_2 \in P$ and $x y r_1 r_2 \in P$ by our assumption that $xy \in P$. This shows, that $ab \in P$. But $ab \in S$ and $P \in \mathcal{A}$. Contradiction. $\square$

Corollary. Let R be a commutative ring, I an ideal in R and $S \subseteq R$ a multiplicative subset such that $I \cap S = \emptyset$. Then there exists prime ideal P in R such that $I \subseteq P$ and $P \cap S = \emptyset$.

Proof. Let $\pi : R \to R/I$ be the projection. Then $\pi(S) \subseteq R/I$ is a multiplicative subset in R/I such that $0 + I \notin \pi(S)$ (because $I \cap S = \emptyset$). Thus, by proposition, there exists a prime ideal P in R/I such that $P \cap \pi(S) = \emptyset$. Of course the preimage of a prime ideal is again a prime ideal. Furthermore $I \subseteq \pi^{-1}(P)$. Finaly $\pi^{-1}(P) \cap S = \emptyset$, because $P \cap \pi(S) = \emptyset$. This completes the proof.