central idempotent

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Let R be a ring. An element $e \in R$ is called a *central idempotent* if it is an idempotent and is in the center $Z(R)$ of R.

It is well-known that if $e \in R$ is an idempotent, then eRe has the structure of a ring with unity, with e being the unity. Thus, if e is central, $eRe = eR = Re$ is a ring with unity e.

It is easy to see that the operation of ring multiplication preserves central idempotency: if e, f are central idempotents, so is ef. In addition, if R has a multiplicative identity 1, then $f := 1 - e$ is also a central idempotent. Furthermore, we may characterize central idempotency in a ring with 1 as follows:

Proposition 1. *An idempotent e in a ring R with 1 is central iff $eRf = fRe = 0$, where $f = 1 - e$.*

Proof. If e is central, then clearly $eRf = fRe = 0$. Conversely, for any $r \in R$, we have $er = er - erf = er(1-f) = ere = (1-f)re = re - fre = re$. $\square$

Another interesting fact about central idempotents in a ring with unity is the following:

Proposition 2. *The set C of all central idempotents of a ring R with 1 has the structure of a Boolean ring.*

Proof. First, note that $0, 1 \in C$. Next, for $e, f \in C$, we define addition $\oplus$ and multiplication $\odot$ on C as follows:

$$e \oplus f := e + f - ef \qquad \text{and} \qquad e \odot f := ef.$$

As discussed above, $\oplus$ and $\odot$ are well-defined (as C is closed under these operations). In addition, for any $e, f, g \in C$, we have

1. $(C, 1, \odot)$ is a commutative monoid, in which every element is an idempotent (with respect to $\odot$). This fact is clear.

2. $\oplus$ is commutative, since $C \subseteq Z(R)$.

3. $\oplus$ is associative:

$$\begin{aligned} e \oplus (f \oplus g) &= e + (f + g - fg) - e(f + g - fg) \\ &= e + f + g - ef - fg - eg + efg \\ &= (e + f - ef) + g - (e + f - ef)g \\ &= (e \oplus f) \oplus g. \end{aligned}$$

4. $\odot$ distributes over $\oplus$: we only need to show left distributivity (since $\odot$ is commutative by 1 above):

$$\begin{aligned} e \odot (f \oplus g) &= e(f + g - fg) = ef + eg - efg \\ &= ef + eg - eefg = ef + eg - efeg \\ &= ef \oplus eg = (e \odot f) \oplus (e \odot g). \end{aligned}$$

This shows that $(C, 0, 1, \oplus, \odot)$ is a Boolean ring. $\square$