idempotent

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Defines	orthogonal idempotents
Defines	complete set of orthogonal idempotents

An element x of a ring is called an *idempotent element*, or simply an *idempotent* if $x^2 = x$.

The set of idempotents of a ring can be partially ordered by putting $e \leq f$ iff $e = ef = fe$.

The element 0 is a minimum element in this partial order. If the ring has an identity element, 1, then 1 is a maximum element in this partial order.

Since the above definitions refer only to the multiplicative structure of the ring, they also hold for semigroups (with the proviso, of course, that a semigroup may have neither a zero element nor an identity element). In the special case of a semilattice, this partial order is the same as the one described in the entry for semilattice.

If a ring has an identity, then $1 - e$ is always an idempotent whenever e is an idempotent, and $e(1-e) = (1-e)e = 0$.

In a ring with an identity, two idempotents e and f are called a *pair of orthogonal idempotents* if $e + f = 1$, and $ef = fe = 0$. Obviously, this is just a fancy way of saying that $f = 1 - e$.

More generally, a set $\{e_1, e_2, \ldots, e_n\}$ of idempotents is called a *complete set of orthogonal idempotents* if $e_i e_j = e_j e_i = 0$ whenever $i \neq j$ and if $1 = e_1 + e_2 + \cdots + e_n$.

If $\{e_1, e_2, \ldots, e_n\}$ is a complete set of orthogonal idempotents, and in addition each e_i is in the centre of R, then each Re_i is a subring, and

$$R \cong Re_1 \times Re_2 \times \cdots \times Re_n.$$

Conversely, whenever $R_1 \times R_2 \times \cdots \times R_n$ is a direct product of rings with identities, write e_i for the element of the product corresponding to the identity element of R_i. Then $\{e_1, e_2, \ldots, e_n\}$ is a complete set of central orthogonal idempotents of the product ring.

When a complete set of orthogonal idempotents is not central, there is a more complicated : see the entry on the Peirce decomposition for the details.