additive inverse of a sum in a ring

Canonical name	AdditiveInverseOfASumInARing
Date of creation	2013-03-22 15:45:02
Last modified on	2013-03-22 15:45:02
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	11
Author	rspuzio (6075)
Entry type	Theorem
Classification	msc 16B70

Let R be a ring with elements $a, b \in R$. Suppose we want to find the inverse of the element $(a+b) \in R$. (Note that we call the element $(a+b)$ the sum of a and b.) So we want the unique element $c \in R$ so that $(a+b)+c = 0$. Actually, let's put $c = (-a) + (-b)$ where $(-a) \in R$ is the additive inverse of a and $(-b) \in R$ is the additive inverse of b. Because addition in the ring is both associative and commutative we see that

$$\begin{aligned} (a+b) + ((-a) + (-b)) &= (a + (-a)) + (b + (-b)) \\ &= 0 + 0 = 0 \end{aligned}$$

since $(-a) \in R$ is the additive inverse of a and $(-b) \in R$ is the additive inverse of b. Since additive inverses are unique this means that the additive inverse of $(a+b)$ must be $(-a) + (-b)$. We write this as

$$-(a+b) = (-a) + (-b).$$

It is important to note that we *cannot* just distribute the minus sign across the sum because this would imply that $-1 \in R$ which is not the case if our ring is not with unity.