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monoid bialgebra is a Hopf algebra if and only if monoid is a group

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Assume that H is a Hopf algebra with comultiplication Δ, counit ε and antipode S. It is well known, that if $c \in H$ and $\Delta(c) = \sum_{i=1}^{n} a_i \otimes b_i$, then $\sum_{i=1}^{n} S(a_i)b_i = \varepsilon(c)1 = \sum_{i=1}^{n} a_i S(b_i)$ (actualy, this condition defines the antipode), where on the left and right side we have multiplication in H.

Now let G be a monoid and k a field. It is well known that kG is a bialgebra (please, see parent object for details), but one may ask, when kG is a Hopf algebra? We will try to answer this question.

Proposition. A monoid bialgebra kG is a Hopf algebra if and only if G is a group.

Proof. ,,$\Leftarrow$" If G is a group, then define $S : kG \to kG$ by $S(g) = g^{-1}$. It is easy to check, that S is the antipode, thus kG is a Hopf algebra.

,,$\Rightarrow$" Assume that kG is a Hopf algebra, i.e. we have the antipode $S : kG \to kG$. Then, for any $g \in G$ we have $S(g)g = gS(g) = 1$ (because $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$). Here 1 is the identity in both G and kG. Of course $S(g) \in kG$, so

$$S(g) = \sum_{h \in G} \lambda_h h.$$

Thus we have

$$1 = \big(\sum_{h \in G} \lambda_h h\big) g = \sum_{h \in G} \lambda_h h g.$$

Of course G is a basis, so this decomposition is unique. Therefore, there exists $g' \in G$ such that $\lambda_{g'} = 1$ and $\lambda_{h'} = 0$ for $h' \neq g'$. We obtain, that $1 = g'g$, thus g is left invertible. Since g was arbitrary it implies that g is invertible. Thus, we've shown that G is a group. $\square$