infinitude of inverses

Canonical name	InfinitudeOfInverses
Date of creation	2013-03-22 18:17:30
Last modified on	2013-03-22 18:17:30
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	4
Author	CWoo (3771)
Entry type	Theorem
Classification	msc 16U99

Proposition 1. *Let R be a ring with 1.*

1. *If $a \in R$ has a right inverse but no left inverses, then a has infinitely many right inverses.*

2. *If $a \in R$ has more than one right inverse, then a has infinitely many right inverses.*

Proof.

1. Let $ab = 1$. Define $b_0 = b, b_1 = 1 - b_0 a + b_0, \ldots, b_{i+1} = 1 - b_i a + b_i, \ldots$ Then, by induction, we see that $ab_i = a - ab_{i-1}a + ab_{i-1} = a - a + 1 = 1$. Next we want to show that $b_i \neq b_j$ if $i \neq j$. Suppose $i > j$ and $b_i = b_j$. Again by induction, we have

$$b_j = b_i = 1 + (1-a) + \cdots + (1-a)^{i-j-1} + b_j(1-a)^{i-j} \tag{1}$$

If we let $c = 1 + (1-a) + \cdots + (1-a)^{i-j-1}$ then $(1-a)c = c(1-a) = (1-a) + (1-a)^2 + \cdots + (1-a)^{i-j} = c - 1 + (1-a)^{i-j}$. So Equation 3 can be rewritten as $c = b_j - b_j(1-a)^{i-j} = b_j(1-(1-a)^{i-j}) = b_j ca$. Then $cb_j = b_j cab_j = b_j c$. Now, note that for $m \leq n$, $(1-a)^n b_j^m = (1-a)^{n-m}(b_j - 1)^m$. This implies that

$$\begin{aligned} c{b_j}^{i-j-1} &= {b_j}^{i-j-1} + (b_j - 1){b_j}^{i-j-2} + \cdots + (b_j - 1)^{i-j-1} \\ &= g(b_j) + (b_j - 1)^{i-j-1}. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} c{b_j}^{i-j-1} &= b_j c {b_j}^{i-j-2} \\ &= b_j({b_j}^{i-j-2} + (b_j - 1)^{i-j-3} + \cdots + (1-a)(b_j - 1)^{i-j-2}) \\ &= g(b_j) + b_j(1-a)(b_j - 1)^{i-j-2}. \end{aligned}$$

So combining the above two equations, we get $(b_j - 1)^{i-j-1} = b_j(1 - a)(b_j - 1)^{i-j-2}$. Let $d = (b_j - 1)^{i-j-2}$, then $(b_j - 1)d = b_j(1-a)d = b_j d - b_j a d$. Simplify, we have $d = b_j a d$. Expanding d, then

$$\begin{aligned} {b_j}^{i-j-2} + \cdots + (-1)^{i-j-2} &= (b_j a)({b_j}^{i-j-2} + \cdots + (-1)^{i-j-2}) \\ &= b_j a {b_j}^{i-j-2} + \cdots + b_j a(-1)^{i-j-2} \\ &= {b_j}^{i-j-2} + \cdots + (-1)^{i-j-2} b_j a. \end{aligned}$$

Then $1 = b_j a$ and we have reached a contradiction.

2. For the next part, notice that if b and c are two distinct right inverses of a, then neither one of them can be a left inverse of a, for if, say, $ba = 1$, then $c = (ba)c = b(ca) = b$. So we can apply the same technique used in the previous portion of the problem. Note that if $b_j a = 1$, then

$$1 = b_j a = (1 - b_{j-1}a + b_{j-1})a = a - b_{j-1}a^2 + b_{j-1}a.$$

Multiply b_{j-1} from the right, we have

$$b_{j-1} = ab_{j-1} - b_{j-1}a^2 b_{j-1} + b_{j-1}ab_{j-1} = 1 - b_{j-1}a + b_{j-1}$$

Thus $b_{j-1}a = 1$. Keep going until we reach $ba = 1$, again a contradiction.

□

Remark. The first part of the above proposition implies that a finite ring is Dedekind-finite.