dense ring of linear transformations

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Let D be a division ring and V a vector space over D. Let R be a subring of the ring of endomorphisms (linear transformations) $\mathrm{End}_D(V)$ of V. Then R is said to be a *dense ring of linear transformations* (over D) if we are given

1. any positive integer n,
2. any set $\{v_1, \ldots, v_n\}$ of linearly independent vectors in V, and
3. any set $\{w_i, \ldots, w_n\}$ of arbitrary vectors in V,

then there exists an element $f \in R$ such that

$$f(v_i) = w_i \quad \text{for } i = 1, \ldots, n.$$

Note that the linear independence of the v_i's is essential in insuring the existence of a linear transformation f. Otherwise, suppose $0 = \sum d_i v_i$ where $d_1 \neq 0$. Pick w_i's so that they are linearly independent. Then $0 = f(\sum d_i v_i) = \sum d_i f(v_i) = \sum d_i w_i$, contradicting the linear independence of the w_i's.

The notion of "dense" comes from topology: if V is given the discrete topology and $\mathrm{End}_D(V)$ the compact-open topology, then R is dense in $\mathrm{End}_D(V)$ iff R is a dense ring of linear transformations of V.

Proof. First, assume that R is a dense ring of linear transformations of V. Recall that the compact-open topology on $\mathrm{End}_D(V)$ has subbasis of the form $B(K, U) := \{f \mid f(K) \subseteq U\}$, where U is open and K is compact in V. Since V is discrete, K is finite. Now, pick a point $g \in \mathrm{End}_D(V)$ and let

$$B = \bigcup_{\alpha \in I} \bigcap_{i=1}^{n(\alpha)} B(K_{i\alpha}, U_{i\alpha})$$

be a neighborhood of g, I some index set. Then for some $\alpha \in I$, $g \in \bigcap B(K_{i\alpha}, U_{i\alpha})$. This means that $g(K_{i\alpha}) \subseteq U_{i\alpha}$ for all $i = 1, \ldots, n(\alpha)$. Since each $K_{i\alpha}$ is finite, so is $K := \bigcup K_{i\alpha}$. After some re-indexing, let $\{v_1, \ldots, v_n\}$ be a maximal linearly independent subset of K. Set $w_j = g(v_j)$, $j = 1, \ldots, n$. By assumption, there is an $f \in R$ such that $f(v_j) = w_j$, for all j. For any $v \in K$, v is a linear combination of the v_j's: $v = \sum d_j v_j$, $d_j \in D$. Then $f(v) = \sum d_j f(v_j) = \sum d_j g(_j) = g(v) \in U_{i\alpha}$ for some i. This shows that $f(K_{i\alpha}) \subseteq U_{i\alpha}$ and we have $f \in \bigcap B(K_{i\alpha}, U_{i\alpha}) \subseteq B$.

Conversely, assume that the ring R is a dense subset of the space $\text{End}_D(V)$. Let $v_1, \dots, v_n$ be linearly independent, and $w_1, \dots, w_n$ be arbitrary vectors in V. Let W be the subspace spanned by the v_i's. Because the v_i's are linearly independent, there exists a linear transformation g such that $g(v_i) = w_i$ and $g(v) = 0$ for $v \notin W$. Let $K_i = \{v_i\}$ and $U_i = \{w_i\}$. Then the K_i's are compact and the U_i's are open in the discrete space V. Clearly $g \in \{h \mid h(v_i) = w_i\} = B(K_i, U_i)$ for each $i = 1, \dots, n$. So g lies in the neighborhood $B = \cap B(K_i, U_i) \subseteq \text{End}_D(V)$. Since R is dense in $\text{End}_D(V)$, there is an $f \in R \cap B$. This implies that $f(v_i) = w_i$ for all i. $\square$

Remarks.

- If V is finite dimensional over D, then any dense ring of linear transformations $R = \text{End}_D(V)$. This can be easily observed by using the second half of the proof above. Take a basis $v_1, \dots, v_n$ of V and any set of n vectors $w_1, \dots, w_n$ in V. Let g be the linear transformation that maps v_i to w_i. The above proof shows that there is an $f \in R$ such that f agrees with g on the basis elements. But then they must agree on all of V as a result, which is precisely the statement that $g = f \in R$.

- It can be shown that a ring R is a primitive ring iff it is isomorphic to a dense ring of linear transformations of a vector space over a division ring. This is known as the . It is a generalization of the special case of the Wedderburn-Artin Theorem when the ring in question is a simple Artinian ring. In the general case, the finite chain condition is dropped.