reflexive module

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Defines	torsionless
Defines	reflexive

Let R be a ring, and M a right R-module. Then its dual, M^*, is given by $\hom(M, R)$, and has the structure of a left module over R. The dual of that, M^{**}, is in turn a right R-module. Fix any $m \in M$. Then for any $f \in M^*$, the mapping

$$f \mapsto f(m)$$

is a left R-module homomorphism from M^* to R. In other words, the mapping is an element of M^{**}. We call this mapping $\hat{m}$, since it only depends on m. For any $m \in M$, the mapping

$$m \mapsto \hat{m}$$

is a then a right R-module homomorphism from M to M^{**}. Let us call it θ.

Definition. Let R, M, and θ be given as above. If θ is injective, we say that M is *torsionless*. If θ is in addition an isomorphism, we say that M is *reflexive*. A torsionless module is sometimes referred to as being semi-reflexive.

An obvious example of a reflexive module is any vector space over a field (similarly, a right vector space over a division ring).

Some of the properties of torsionless and reflexive modules are

- any free module is torsionless.
- any direct sum of torsionless modules is torsionless; any submodule of a torsionless module is torsionless.
- based on the two properties above, any projective module is torsionless.
- R is reflexive.
- any finite direct sum of reflexive modules is reflexive; any direct summand of a reflexive module is reflexive.
- based on the two immediately preceding properties, any finitely generated projective module is reflexive.