well-definedness of product of finitely generated ideals

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Ler R be of a commutative ring with nonzero unity. If

$$\mathfrak{a} \;=\; (a_1,\, \ldots,\, a_m) \;=\; (\alpha_1,\, \ldots,\, \alpha_\mu) \tag{1}$$

and

$$\mathfrak{b} \;=\; (b_1,\, \ldots,\, b_n) \;=\; (\beta_1,\, \ldots,\, \beta_\nu) \tag{2}$$

are two finitely generated ideals of R, both with two , then the ideals

$$\mathfrak{c} \;:=\; (a_1b_1,\, \ldots,\, a_ib_j,\, \ldots,\, a_mb_n)$$

and

$$\mathfrak{d} \;:=\; (\alpha_1\beta_1,\, \ldots,\, \alpha_i\beta_j,\, \ldots,\, \alpha_\mu\beta_\nu)$$

are equal.

Proof. By (1) and (2), for every i, j, there are elements r_{ik}, s_{jl} of R such that

$$a_i \;=\; r_{i1}\alpha_1 + \ldots + r_{i\mu}\alpha_\mu, \quad b_j \;=\; s_{j1}\beta_1 + \ldots + s_{j\nu}\beta_\nu. \tag{3}$$

Multiplying the equations (3) we see that

$$a_ib_j \;=\; (r_{i1}s_{j1})(\alpha_1\beta_1) + (r_{i2}s_{j1})(\alpha_2\beta_1) + \ldots + (r_{i\mu}s_{j\nu})(\alpha_\mu\beta_\nu),$$

whence the generators a_ib_j of $\mathfrak{c}$ belong to $\mathfrak{d}$ and consecuently $\mathfrak{c} \subseteq \mathfrak{d}$. The reverse containment is seen similarly.