criterion for maximal ideal

Canonical name	CriterionForMaximalIdeal
Date of creation	2013-03-22 19:10:40
Last modified on	2013-03-22 19:10:40
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	6
Author	pahio (2872)
Entry type	Theorem
Classification	msc 16D25
Classification	msc 13A15
Related topic	MaximalIdealIsPrime

Theorem. In a commutative ring R with non-zero unity, an ideal $\mathfrak{m}$ is maximal if and only if

$$\forall a \in R \smallsetminus \mathfrak{m} \;\; \exists r \in R \quad \text{such that} \;\; 1+ar \;\in\; \mathfrak{m}. \tag{1}$$

Proof. 1°. Let first $\mathfrak{m}$ be a maximal ideal of R and $a \in R \smallsetminus \mathfrak{m}$. Because $\mathfrak{m}+(a) = R$, there exist some elements $m \in \mathfrak{m}$ and $-r \in R$ such that $m-ar = 1$. Consequently, $1+ar = m \in \mathfrak{m}$.
2°. Assume secondly that the ideal $\mathfrak{m}$ satisfies the condition (1). Now there must be a maximal ideal $\mathfrak{m}'$ of R such that

$$\mathfrak{m} \;\subseteq\; \mathfrak{m}' \;\subset\; R.$$

Let us make the antithesis that $\mathfrak{m}' \smallsetminus \mathfrak{m}$ is non-empty. Choose an element

$$a \;\in\; \mathfrak{m}' \smallsetminus \mathfrak{m} \;\subset\; R \smallsetminus \mathfrak{m}.$$

By our assumption, we can choose another element r of R such that

$$s \;=\; 1+ar \;\in\; \mathfrak{m} \;\subset\; \mathfrak{m}'.$$

Then we have

$$1 \;=\; s-ar \;\in\; \mathfrak{m}'+\mathfrak{m}' \;=\; \mathfrak{m}'$$

which is impossible since with 1 the ideal $\mathfrak{m}'$ would contain the whole R. Thus the antithesis is wrong and $\mathfrak{m} = \mathfrak{m}'$ is maximal.