proof of product of left and right ideal

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Theorem 1 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a ring R. Denote by $\mathfrak{ab}$ the subset of R formed by all finite sums of products ab with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Then if $\mathfrak{a}$ is a left and $\mathfrak{b}$ a right ideal, $\mathfrak{ab}$ is a two-sided ideal of R. If in addition both $\mathfrak{a}$ and $\mathfrak{b}$ are two-sided ideals, then $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b}$.*

Proof. We must show that the difference of any two elements of $\mathfrak{ab}$ is in $\mathfrak{ab}$, and that $\mathfrak{ab}$ is closed under multiplication by R. But both of these operations are linear in $\mathfrak{ab}$; that is, if they hold for elements of the form $ab, a \in \mathfrak{a}, b \in \mathfrak{b}$, then they hold for the general element of $\mathfrak{ab}$. So we restrict our analysis to elements ab.

Clearly if $a_1, a_2 \in \mathfrak{a}, b_1, b_2 \in \mathfrak{b}$, then $a_1b_1 - a_2b_2 \in \mathfrak{ab}$ by definition.

If $a \in \mathfrak{a}, b \in \mathfrak{b}, r \in R$, then

$$\begin{aligned} r \cdot ab &= (r \cdot a)b \in \mathfrak{ab} \text{ since } \mathfrak{a} \text{ is a left ideal} \\ ab \cdot r &= a(b \cdot r) \in \mathfrak{ab} \text{ since } \mathfrak{b} \text{ is a right ideal} \end{aligned}$$

and thus $\mathfrak{ab}$ is a two-sided ideal. This proves the first statement.

If $\mathfrak{a}, \mathfrak{b}$ are two-sided ideals, then $ab \in \mathfrak{a}$ since $b \in R$; similarly, $ab \in \mathfrak{b}$. This proves the second statement.