example of a projective module which is not free

Canonical name	ExampleOfAProjectiveModuleWhichIsNotFree
Date of creation	2013-03-22 18:49:55
Last modified on	2013-03-22 18:49:55
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	6
Author	joking (16130)
Entry type	Example
Classification	msc 16D40

Let R_1 and R_2 be two nontrivial, unital rings and let $R = R_1 \oplus R_2$. Furthermore let $\pi_i : R \to R_i$ be a projection for $i = 1, 2$. Note that in this case both R_1 and R_2 are (left) modules over R via

$$\cdot : R \times R_i \to R_i;$$

$$(r, s) \cdot x = \pi_i(r, s)x,$$

where on the right side we have the multiplication in a ring R_i.

Proposition. Both R_1 and R_2 are projective R-modules, but neither R_1 nor R_2 is free.

Proof. Obviously $R_1 \oplus R_2$ is isomorphic (as a R-modules) with R thus both R_1 and R_2 are projective as a direct summands of a free module.

Assume now that R_1 is free, i.e. there exists $\mathcal{B} = \{e_i\}_{i \in I} \subseteq R_1$ which is a basis. Take any $i_0 \in I$. Both R_1 and R_2 are nontrivial and thus $1 \neq 0$ in both R_1 and R_2. Therefore $(1, 0) \neq (1, 1)$ in R, but

$$(1, 1) \cdot e_{i_0} = \pi_1(1, 1)e_{i_0} = 1e_{i_0} = \pi_1(1, 0)e_{i_0} = (1, 0) \cdot e_{i_0}.$$

This situation is impossible in free modules (linear combination is uniquely determined by scalars). Contradiction. Analogously we prove that R_2 is not free. $\square$