m-system

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Let R be a ring. A subset S of R is called an *m-system* if

- $S \neq \varnothing$, and
- for every two elements x, $y \in S$, there is an element $r \in R$ such that $xry \in S$.

m-Systems are a generalization of multiplicatively closet subsets in a ring. Indeed, every multiplicatively closed subset of R is an m-system: any $x, y \in S$, then $xy \in S$, hence $xyy \in S$. However, the converse is not true. For example, the set

$$\{r^n \mid r \in R \text{ and } n \text{ is an odd positive integer}\}$$

is an m-system, but not multiplicatively closed in general (unless, for example, if $r = 1$).

Remarks. m-Systems and prime ideals of a ring are intimately related. Two basic relationships between the two notions are

1. An ideal P in a ring R is a prime ideal iff $R - P$ is an m-system.

   Proof. P is prime iff $xRy \subseteq P$ implies x or $y \in P$, iff $x, y \in R - P$ implies that there is $r \in R$ with $xry \notin P$ iff $R - P$ is an m-system. $\square$

2. Given an m-system S of R and an ideal I with $I \cap S = \varnothing$. Then there exists a prime ideal $P \subseteq R$ with the property that P contains I and $P \cap S = \varnothing$, and P is the largest among all ideals with this property.

   Proof. Let $\mathcal{C}$ be the collection of all ideals containing I and disjoint from S. First, $I \in \mathcal{C}$. Second, any chain K of ideals in $\mathcal{C}$, its union $\bigcup K$ is also in $\mathcal{C}$. So Zorn's lemma applies. Let P be a maximal element in $\mathcal{C}$. We want to show that P is prime. Suppose otherwise. In other words, $aRb \subseteq P$ with $a, b \notin P$. Then $\langle P, a\rangle$ and $\langle P, b\rangle$ both have non-empty intersections with S. Let

   $$c = p + fag \in \langle P, a\rangle \cap S \quad \text{and} \quad d = q + hbk \in \langle P, b\rangle \cap S,$$

   where $p, q \in P$ and $f, g, h, k \in R$. Then there is $r \in R$ such that $crd \in S$. But this implies that

   $$crd = (p + fag)r(q + hbk) = p(rq + rhbk) + (fagr)q + f\big(a(grh)b\big)k \in P$$

   as well, contradicting $P \cap S = \varnothing$. Therefore, P is prime. $\square$

m-Systems are also used to define the non-commutative version of the radical of an ideal of a ring.