isomorphism swapping zero and unity

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Let $(R, +, \cdot)$ be a ring with unity 1. Define two new binary operations of R as follows:

$$a \oplus b \;=:\; a+b-1, \qquad a \odot b \;=:\; a+b-a\cdot b \tag{1}$$

Then we see that

$$a \oplus 1 \;=\; a \;=\; 1 \oplus a, \qquad a \odot 0 \;=\; a \;=\; 0 \odot a. \tag{2}$$

But moreover, the algebraic system $(R, \oplus, \odot)$ is a unitary ring, too, and isomorphic with the original ring.

In fact, we may define the bijective mapping

$$f\colon\; x \mapsto 1-x \tag{3}$$

from R to R and verify that it is homomorphic:

$$f(a) \oplus f(b) \;=\; (1-a) \oplus (1-b) \;=\; (1-a)+(1-b)-1 \;=\; 1-a-b \;=\; f(a+b),$$

$$f(a) \odot f(b) \;=\; (1-a) \odot (1-b) \;=\; (1-a)+(1-b)-(1-a)\cdot(1-b) \;=\; 1-a\cdot b \;=\; f(a\cdot b)$$

Thus $(R, \oplus, \odot)$ as a `http://planetmath.org/HomomorphicImageOfGroup`homomorphic image of the ring $(R, +, \cdot)$ is a ring, it's a question of two isomorphic rings.