properties of the Jacobson radical

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Theorem:

Let R, T be rings and $\varphi : R \to T$ be a surjective homomorphism. Then $\varphi(J(R)) \subseteq J(T)$.

Proof:

We shall use the characterization of the Jacobson radical as the set of all $a \in R$ such that for all $r \in R$, $1 - ra$ is left invertible.

Let $a \in J(R), t \in T$. We claim that $1 - t\varphi(a)$ is left invertible:

Since φ is surjective, $t = \varphi(r)$ for some $r \in R$. Since $a \in J(R)$, we know $1 - ra$ is left invertible, so there exists $u \in R$ such that $u(1 - ra) = 1$. Then we have

$$\varphi(u)\left(\varphi(1) - \varphi(r)\varphi(a)\right) = \varphi(u)\varphi(1 - ra) = \varphi(1) = 1$$

So $\varphi(a) \in J(T)$ as required.

Theorem:

Let R, T be rings. Then $J(R \times T) \subseteq J(R) \times J(T)$.

Proof:

Let $\pi_1 : R \times T \to R$ be a (surjective) projection. By the previous theorem, $\pi_1(J(R \times T)) \subseteq J(R)$.

Similarly let $\pi_2 : R \times T \to T$ be a (surjective) projection. We see that $\pi_2(J(R \times T)) \subseteq J(T)$.

Now take $(a, b) \in J(R \times T)$. Note that $a = \pi_1(a, b) \in J(R)$ and $b = \pi_2(a, b) \in J(T)$. Hence $(a, b) \in J(R) \times J(T)$ as required.