nilpotency is not a radical property

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Nilpotency is not a radical property, because a ring does not, in general, contain a largest nilpotent ideal.

Let k be a field, and let $S = k[X_1, X_2, \dots]$ be the ring of polynomials over k in infinitely many variables $X_1, X_2, \dots$. Let I be the ideal of S generated by $\{X_n^{n+1} \mid n \in (N)\}$. Let $R = S/I$. Note that R is commutative.

For each n, let $A_n = \sum_{k=1}^{n} RX_n$. Let $A = \bigcup A_n = \sum_{k=1}^{\infty} RX_n$.

Then each A_n is nilpotent, since it is the sum of finitely many nilpotent ideals (see proof `http://planetmath.org/node/5650`here). But A is nil, but not nilpotent. Indeed, for any n, there is an element $x \in A$ such that $x^n \neq 0$, namely $x = X_n$, and so we cannot have $A^n = 0$.

So R cannot have a largest nilpotent ideal, for this ideal would have to contain all the ideals A_n and therefore A.