tensor product of chain complexes

Canonical name	TensorProductOfChainComplexes
Date of creation	2013-03-22 16:13:21
Last modified on	2013-03-22 16:13:21
Owner	Mazzu (14365)
Last modified by	Mazzu (14365)
Numerical id	13
Author	Mazzu (14365)
Entry type	Definition
Classification	msc 16E05
Classification	msc 18G35
Defines	tensor product of chain complexes

Let $C' = \{C'_n, \partial'_n\}$ and $C'' = \{C''_n, \partial''_n\}$ be two chain complexes of R-modules, where R is a commutative ring with unity. Their *tensor product* $C' \otimes_R C'' = \{(C' \otimes_R C'')_n, \partial_n\}$ is the chain complex defined by

$$(C' \otimes_R C'')_n = \bigoplus_{i+j=n} (C'_i \otimes_R C''_j),$$

$$\partial_n(t'_i \otimes_R s''_j) = \partial'_i(t'_i) \otimes_R s''_j + (-1)^i\, t'_i \otimes_R \partial''_j(s''_j), \quad \forall t'_i \in C'_i,\ s''_j \in C''_j,\ (i+j=n),$$

where $C'_i \otimes_R C''_j$ denotes the `http://planetmath.org/TensorProduct`tensor product of R-modules C'_i and C''_j.

Indeed, this defines a chain complex, because for each $t'_i \otimes_R s''_j \in C'_i \otimes_R C''_j \subseteq (C' \otimes_R C'')_{i+j}$ we have

$$\partial_{i+j-1}\partial_{i+j}(t'_i \otimes_R s''_j) = \partial_{i+j-1}\left(\partial'_i(t'_i) \otimes_R s''_j + (-1)^i\, t'_i \otimes_R \partial''_j(s''_j)\right) =$$

$$= (-1)^{i-1}\, \partial'_i(t'_i) \otimes_R \partial''_j(s''_j) + (-1)^i \partial'_i(t'_i) \otimes_R \partial''_j(s''_j) = 0,$$

thus $C' \otimes_R C''$ is a chain complex.