properties of nil and nilpotent ideals

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Lemma 1. *Let $A \subset B$ be ideals of a ring R. If A is nil and B/A is nil, then B is nil. If A is nilpotent and B/A is nilpotent, then B is nilpotent.*

Proof. Suppose that A and B/A are nil. Let $x \in B$. Then $x^n \in A$ for some n, since B/A is nil. But A is nil, so there is an m such that $x^{nm} = (x^n)^m = 0$. Thus B is nil.

Suppose that A and B/A are nilpotent. Then there are natural numbers n and m such that $A^m = 0$ and $B^n \subseteq A$. Therefore, $B^{nm} = 0$. $\square$

Lemma 2. *The sum of an arbitrary family of nil ideals is nil.*

Proof. Let R be a ring, and let $\mathcal{F}$ be a family of nil ideals of R. Let $S = \sum_{I \in \mathcal{F}} I$. We must show that there is an n with $x^n = 0$ for every $x \in S$. Now, any such x is actually in a sum of only finitely many of the ideals in $\mathcal{F}$. So it suffices to prove the lemma in the case that $\mathcal{F}$ is finite. By induction, it is enough to show that the sum of two nil ideals is nil.

Let A and B be nil ideals of a ring R. Then $A \subset A + B$, and $A + B/A \cong B/(A \cap B)$, which is nil. So by the first lemma, $A + B$ is nil. $\square$

Lemma 3. *The sum of a finite family of nilpotent left or right ideals is nilpotent.*

Proof. We prove this for right ideals. Again, by induction, it suffices to prove it for the case of two right ideals.

Let A and B be nilpotent right ideals of a ring R. Then there are natural numbers n and m such that $A^n = 0$ and $b^m = 0$.

Let $k = n + m - 1$. Let $z_1, z_2, \ldots, z_k$ be elements of $A + B$. We may write $z_i = a_i + b_i$ for each i, with $a_i \in A$ and $b_i \in B$. If we expand the product $z_1 z_2 \cdots z_k$ we get a sum of terms of the form $x_1 x_2 \ldots x_k$ where each $x_i \in \{a_i, b_i\}$.

Consider one of these terms $x_1 x_2 \cdots x_k$. Then by our choice of k, it must contain at least n of the a_i's or at least m of the b_i's. Without loss of generality, assume the former. So there are indices $i_1 < i_2 < \cdots < i_n$ with $x_{i_j} \in A$ for each j. For $1 \leq j \leq n-1$, define $y_j = x_{i_j} x_{i_j+1} \cdots x_{i_{j+1}-1}$, and define $y_n = x_{i_n} x_{i_n+1} \cdots x_k$. Since A is a right ideal, $y_j \in A$.

Then $x_1 x_2 \cdots x_k = x_1 x_2 \cdots x_{i_1-1} y_1 y_2 \cdots y_n \in x_1 x_2 \cdots x_{i_1-1} A^n = 0$.

This is true for all choices of the x_i, and so $z_1 z_2 \cdots z_k = 0$. But this says that $(A + B)^k = 0$. $\square$