Noetherian and Artinian properties are inherited in short exact sequences

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Theorem 1. *Let M, M', M'' be A-modules and $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ a short exact sequence. Then*

1. *M is Noetherian if and only if M' and M'' are Noetherian;*

2. *M is Artinian if and only if M' and M'' are Artinian.*

For $\Leftarrow$, we will need a lemma that essentially says that a submodule of M is uniquely determined by its image in M'' and its intersection with M':

Lemma 1. *In the situation of the theorem, if $N_1, N_2 \subset M$ are submodules with $N_1 \subset N_2$, $\pi(N_1) = \pi(N_2)$, and $N_1 \cap \iota(M') = N_2 \cap \iota(M')$, then $N_1 = N_2$.*

Proof. The proof is essentially a diagram chase. Choose $x \in N_2$. Then $\pi(x) = \pi(x')$ for some $x' \in N_1$, and thus $\pi(x - x') = 0$, so that $x - x' \in \operatorname{im} \iota$, and $x - x' \in N_2$ since $N_1 \subset N_2$. Hence $x - x' \in N_2 \cap \iota(M') = N_1 \cap \iota(M') \subset N_1$. Since $x' \in N_1$, it follows that $x \in N_1$ so that $N_1 = N_2$. □

Proof. ($\Rightarrow$): If M is Noetherian (Artinian), then any ascending (descending) chain of submodules of M' (or of M'') gives rise to a similar sequence in M, which must therefore terminate. So the original chain terminates as well.
($\Leftarrow$): Assume first that M', M'' are Noetherian, and choose any ascending chain $M_1 \subset M_2 \subset \dots$ of submodules of M. Then the ascending chain $\pi(M_1) \subset \pi(M_2) \subset \dots$ and the ascending chain $M_1 \cap \iota(M') \subset M_2 \cap \iota(M') \subset \dots$ both stabilize since M' and M'' are Noetherian. We can choose n large enough so that both chains stabilize at n. Then for $N \geq n$, we have (by the lemma) that $M_N = M_n$ since $\pi(M_N) = \pi(M_n)$ and $M_N \cap \iota(M') = M_n \cap \iota(M')$. Thus M is Noetherian. For the case where M is Artinian, an identical proof applies, replacing ascending chains by descending chains. □

References

[1] M.F. Atiyah, I.G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley 1969.