module-finite extensions are integral

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Theorem Suppose $B \subset A$ is module-finite. Then A is integral over B.

Proof. Choose $u \in A$.

For clarity, assume A is spanned by two elements ω_1, ω_2. The proof given clearly generalizes to the case where a spanning set for A has more than two elements.

Write

$$\begin{aligned} u\omega_1 &= b_{11}\omega_1 + b_{12}\omega_2 \\ u\omega_2 &= b_{21}\omega_1 + b_{22}\omega_2 \end{aligned}$$

Consider

$$C = \begin{pmatrix} u - b_{11} & -b_{12} \\ -b_{21} & u - b_{22} \end{pmatrix}$$

and let C^{adj} be the adjugate of C. Then $C \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = 0$, so $C^{\text{adj}} C \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = 0$.

Now, $C^{\text{adj}} C$ is a diagonal matrix with $\det C$ on the diagonal, so

$$\begin{pmatrix} f(u) & 0 \\ 0 & f(u) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $f \in B[x]$ is monic.

But neither ω_1 nor ω_2 is zero, so $f(u)$ must be.

Note that, as with the field case, the converse is not true. For example, the algebraic integers are integral but not finite over $\mathbb{Z}$.