cardinalities of bases for modules

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Let R be a ring and M a left module over R.

Proposition 1. *If M has a finite basis, then all bases for M are finite.*

Proof. Suppose $A = \{a_1, \ldots, a_n\}$ is a finite basis for M, and B is another basis for M. Each element in A can be expressed as a finite linear combination of elements in B. Since A is finite, only a finite number of elements in B are needed to express elements of A. Let $C = \{b_1, \ldots, b_m\}$ be this finite subset (of B). C is linearly independent because B is. If $C \neq B$, pick $b \in B - C$. Then b is expressible as a linear combination of elements of A, and subsequently a linear combination of elements of C. This means that $b = r_1 b_1 + \cdots + r_m b_m$, or $0 = -b + r_1 b_1 + \cdots r_m b_m$, contradicting the linear independence of C. □

Proposition 2. *If M has an infinite basis, then all bases for M have the same cardinality.*

Proof. Suppose A be a basis for M with $|A| \geq \aleph_0$, the smallest infinite cardinal, and B is another basis for M. We want to show that $|B| = |A|$. First, notice that $|B| \geq \aleph_0$ by the previous proposition. Each element $a \in A$ can be expressed as a *finite* linear combination of elements of B, so let B_a be the collection of these elements. Now, B_a is uniquely determined by a, as B is a basis. Also, B_a is finite. Let

$$B' = \bigcup_{a \in A} B_a.$$

Since A spans M, so does B'. If $B' \neq B$, pick $b \in B - B'$, so that b is a linear combination of elements of B'. Moving b to the other side of the expression and we have expressed 0 as a non-trivial linear combination of elements of B, contradicting the linear independence of B. Therefore $B' = B$. This means

$$|B| = \left| \bigcup_{a \in A} B_a \right| \leq \aleph_0 |A| = |A|.$$

Similarly, every element in B is expressible as a finite linear combination of elements in A, and using the same argument as above,

$$|A| \leq \aleph_0 |B| \leq |B|.$$

By Schroeder-Bernstein theorem, the two inequalities can be combined to form the equality $|A| = |B|$. □