example of an Artinian module which is not Noetherian

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It is well known, that left (right) Artinian ring is left (right) Noetherian (Akizuki-Hopkins-Levitzki theorem). We will show that this no longer holds for modules.

Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Q}$ the field of rationals. Let $p \in \mathbb{Z}$ be a prime number and consider

$$G = \{\frac{a}{p^n} \in \mathbb{Q} \mid a \in \mathbb{Z};\ n \geq 0\}.$$

Of course G is a $\mathbb{Z}$-module via standard multiplication and addition. For $n \geq 0$ consider

$$G_n = \{\frac{a}{p^n} \in \mathbb{Q} \mid a \in \mathbb{Z}\}.$$

Of course each $G_n \subseteq G$ is a submodule and it is easy to see, that

$$\mathbb{Z} = G_0 \subset G_1 \subset G_2 \subset G_3 \subset \cdots ,$$

where each inclusion is proper. We will show that $G/\mathbb{Z}$ is Artinian, but it is not Noetherian.

Let $\pi : G \to G/\mathbb{Z}$ be the canonical projection. Then $G'_n = \pi(G_n)$ is a submodule of $G/\mathbb{Z}$ and

$$0 = G'_0 \subset G'_1 \subset G'_2 \subset G'_3 \subset G'_4 \subset \cdots .$$

The inclusions are proper, because for any $n > 0$ we have

$$G'_{n+1}/G'_n \simeq \big(G_{n+1}/\mathbb{Z}\big)/\big(G_n/\mathbb{Z}\big) \simeq G_{n+1}/G_n \neq 0,$$

due to Third Isomorphism Theorem for modules. This shows, that $G/\mathbb{Z}$ is not Noetherian.

In order to show that $G/\mathbb{Z}$ is Artinian, we will show, that each proper submodule of $G/\mathbb{Z}$ is of the form G'_n. Let $N \subseteq G/\mathbb{Z}$ be a proper submodule. Assume that for some $a \in \mathbb{Z}$ and $n \geq 0$ we have

$$\frac{a}{p^n} + \mathbb{Z} \in N.$$

We may assume that $\gcd(a, p^n) = 1$. Therefore there are $\alpha, \beta \in \mathbb{Z}$ such that

$$1 = \alpha a + \beta p^n.$$

Now, since N is a $\mathbb{Z}$-module we have

$$\frac{\alpha a}{p^n} + \mathbb{Z} \in N$$

and since $0 + \mathbb{Z} = \beta + \mathbb{Z} = \frac{\beta p^n}{p^n} + \mathbb{Z} \in N$ we have that

$$\frac{1}{p^n} + \mathbb{Z} = \frac{\alpha a + \beta p^n}{p^n} + \mathbb{Z} \in N.$$

Now, let $m > 0$ be the smallest number, such that $\frac{1}{p^m} + \mathbb{Z} \notin N$. What we showed is that

$$N = G'_{m-1} = \pi(G_{m-1}),$$

because for every $0 \leq n \leq m-1$ (and only for such n) we have $\frac{1}{p^n} + \mathbb{Z} \in N$ and thus N is a image of a submodule of G, which is generated by $\frac{1}{p^n}$ and this is precisely G_{m-1}. Now let

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$$

be a chain of submodules in $G/\mathbb{Z}$. Then there are natural numbers $n_1, n_2, \ldots$ such that $N_i = G'_{n_i}$. Note that $G'_k \supseteq G'_s$ if and only if $k \geq s$. In particular we obtain a sequence of natural numbers

$$n_1 \geq n_2 \geq n_3 \geq \cdots$$

This chain has to stabilize, which completes the proof. $\square$