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Morita equivalence

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Let R be a ring. Write $\mathcal{M}_R$ for the category of right modules over R. Two rings R and S are said to be *Morita equivalent* if $\mathcal{M}_R$ and $\mathcal{M}_S$ are `http://planetmath.org/EquivalenceOfCategories`equivalent as categories. What this means is: we have two functors

$$F : \mathcal{M}_R \to \mathcal{M}_S \qquad \text{and} \qquad G : \mathcal{M}_S \to \mathcal{M}_R$$

such that for any right R-module M and any right S-module N, we have

$$GF(M) \cong_R M \qquad \text{and} \qquad FG(N) \cong_S N,$$

where $A \cong_R B$ means that there is an R-module isomorphism between A and B.

Example. Any ring R with 1 is Morita equivalent to any matrix ring $M_n(R)$ over it.

Proof. Assume $n > 1$. For convenience, we will also say a module to mean a right module.

Let M be an R-module. Set $F(M) = \{(m_1, \dots, m_n) \mid m_i \in M\}$. Then $F(M)$ becomes a module over $M_n(R)$ if we adopt the standard matrix multiplication mA, where $m \in F(M)$ and $A \in M_n(R)$. If $f : M_1 \to M_2$ is an R-module homomorphism. Set $F(f) : F(M_1) \to F(M_2)$ by $F(f)(m_1, \dots, m_n) = (f(m_1), \dots, f(m_n)) \in F(M_2)$. Then F is a covariant functor by inspection.

Next, let N be an $M_n(R)$-module. Write $e(r)$ as the $n \times n$ matrix whose cell $(1, 1)$ is $r \in R$ and 0 everywhere else. For simplicity we write $e := e(1)$. Note that e is an idempotent in $M_n(R)$: $e = ee$, and e commutes with $e(r)$ for any $r \in R$: $ee(r) = e(r)e$.

Set $G(N) = \{se \mid s \in N\}$. For any $r \in R$, define $se \cdot r := see(r) = se(r)e$. Since $se(r) \in N$, this multiplication turns $G(N)$ into an R-module. If $g : N_1 \to N_2$ is an $M_n(R)$-module homomorphism, define $G(g) : G(N_1) \to G(N_2)$ by $G(g)(se) = g(s)e$. If $N_1 \xrightarrow{g} N_2 \xrightarrow{h} N_3$ are $M_n(R)$-module homomorphisms, then

$$G(h \circ g)(se) = (h \circ g)(s)e = h(g(s))e = G(h)[g(s)e] = G(h)[G(g)se] = G(h) \circ G(g)(se)$$

so that G is a covariant functor.

If M is any R-module, then $GF(M) = \{(m_1, \dots, m_n)e \mid m \in M\} = \{(m_1, 0, \dots, 0)^T \mid m \in M\} \cong M$, where m^T stands for the transpose of the row vector $m \in M$ into a column vector.

On the other hand, if N is any $M_n(R)$-module, then $FG(N) = \{(s_1e, \dots, s_ne) \mid s_i \in N\}$. Before proving that $FG(N) \cong N$, let's do some preliminary work.

Denote e_{ii} by the $n \times n$ matrix whose cell (i, i) is 1 and 0 everywhere else. Then each e_{ii} is idempotent, $e_{ii}e_{jj} = 0$ for $i \neq j$, and $e_{11} + \cdots + e_{nn} = 1$. From this, we see that $N = N_1 \oplus \cdots \oplus N_n$, where $N_i = Ne_{ii}$, and $N_i \cong N_j$ as $M_n(R)$-modules. Since $N_1 = Ne$ has an R-module structure as we had shown earlier, N_i are all R-modules. Let $\pi_i : N \to N_i$ be the projection map, $\psi_i : N_i \to N$ be the embedding of N_i into N, and $\phi_{ij} : N_i \to N_j$ be the isomorphism from N_i to N_j given by $\phi_{ij}(se_{ii}) = se_{jj}$. All these are $M_n(R)$-module homomorphisms since $e_{ii}A = Ae_{ii}$.

Now, take any $s \in N$, then $s \mapsto (\pi_1(s), \dots, \pi_n(s)) \mapsto (\phi_{11}\pi_1(s), \dots, \phi_{n1}\pi_n(s)) \in FG(N)$ is a homomorphism $\alpha : N \to FG(N)$. Conversely, $(s_1e, \dots, s_ne) \mapsto (\phi_{11}(s_1e), \dots, \phi_{1n}(s_ne)) \mapsto \psi_1(\phi_{11}(s_1e)) + \cdots + \psi_n(\phi_{1n}(s_ne)) \in N$ is also a homomorphism $\beta : FG(N) \to N$. By inspection, α and β are inverses of each other, and hence $FG(N) \cong N$. $\square$

Remark. A property P in the class of all rings is said to be *Morita invariant* if, whenever R has property P and S is Morita equivalent to R, then S has property P as well. By the example above, it is clear that commutativity is not a Morita invariant property.