quasi-regularity

Canonical name	Quasiregularity
Date of creation	2013-03-22 13:12:59
Last modified on	2013-03-22 13:12:59
Owner	mclase (549)
Last modified by	mclase (549)
Numerical id	8
Author	mclase (549)
Entry type	Definition
Classification	msc 16N20
Synonym	quasi regular
Synonym	quasi regularity
Related topic	JacobsonRadical
Related topic	RegularIdeal
Related topic	HomotopesAndIsotopesOfAlgebras
Defines	quasi-regular
Defines	right quasi-regular
Defines	left quasi-regular
Defines	quasi-inverse
Defines	quasi-regular ideal
Defines	quasi-regular ring

An element x of a ring is called *right quasi-regular* [resp. *left quasi-regular*] if there is an element y in the ring such that $x + y + xy = 0$ [resp. $x + y + yx = 0$].

For calculations with quasi-regularity, it is useful to introduce the operation $*$ defined:

$$x * y = x + y + xy.$$

Thus x is right quasi-regular if there is an element y such that $x * y = 0$. The operation $*$ is easily demonstrated to be associative, and $x * 0 = 0 * x = x$ for all x.

An element x is called *quasi-regular* if it is both left and right quasi-regular. In this case, there are elements y and z such that $x + y + xy = 0 = x + z + zx$ (equivalently, $x * y = z * x = 0$). A calculation shows that

$$y = 0 * y = (z * x) * y = z * (x * y) = z.$$

So $y = z$ is a unique element, depending on x, called the *quasi-inverse* of x.

An ideal (one- or two-sided) of a ring is called *quasi-regular* if each of its elements is quasi-regular. Similarly, a ring is called *quasi-regular* if each of its elements is quasi-regular (such rings cannot have an identity element).

Lemma. *Let A be an ideal (one- or two-sided) in a ring R. If each element of A is right quasi-regular, then A is a quasi-regular ideal.*

This lemma means that there is no extra generality gained in defining terms such as right quasi-regular left ideal, etc.

Quasi-regularity is important because it provides elementary characterizations of the Jacobson radical for rings without an identity element:

- The Jacobson radical of a ring is the sum of all quasi-regular left (or right) ideals.
- The Jacobson radical of a ring is the largest quasi-regular ideal of the ring.

For rings with an identity element, note that x is [right, left] quasi-regular if and only if $1 + x$ is [right, left] invertible in the ring.