Burnside ring

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Let G be a finite group. Recall that by *G-set* we understand a pair $(X, \circ)$, where X is a set and $\circ : G \times X \to X$ is a group action of G on X. For short notation the pair notation will be omitted and G-sets will be simply denoted by capital letters.

Recall that for each subgroup $H \subseteq G$ we have canonical G-set $G/H = \{gH;\ g \in G\}$ where group action is defined as follows: for any $g, k \in G$ we have $(g, kH) \longmapsto gkH$.

Let X and Y be G-sets. Recall that by *G-map* from X to Y we understand any function $F : X \to Y$ such that for any $g \in G$ and $x \in X$ we have $F(gx) = gF(x)$.

It is easy to see that family of all G-sets and G-maps forms a category (with standard comoposition). We shall denote this category by $G - \mathbb{S}$. Moreover, by $G - \mathbb{S}_0$ we shall denote full subcategory of $G - \mathbb{S}$ whose objects are all finite G-sets.

From G-sets X and Y one can construct another G-set in two interesting (from our point of view) ways, i.e. by taking disjoint union $X \sqcup Y$ with obvious group action and by taking product $X \times Y$ with group action as follows: $(g, (x, y)) \longmapsto (gx, gy)$. Moreover it is clear that when X and Y are finite, so are $X \sqcup Y$ and $X \times Y$.

Consider a finite G-set X. Then there exist a natural number $n \in \mathbb{N}$, finite family $\{H_i\}_{i=1}^n$ of subgroups of G and an isomorphism (in $G - \mathbb{S}_0$ category)

$$X \simeq \coprod_{i=1}^{n} G/H_i.$$

Therefore (since G is finite) family of isomorphism classes of $G - \mathbb{S}_0$ forms a countable set.

Denote by $\Omega^+(G) = \{[X];\ X \in G - \mathbb{S}_0\}$ the set of isomorphism classes of category $G - \mathbb{S}_0$. Then one can turn $\Omega^+(G)$ into a semiring as follows: for any finite G-sets X and Y define

$$[X] + [Y] = [X \sqcup Y];$$

$$[X][Y] = [X \times Y].$$

Note that here we treat the empty set as a G-set (with one and unique group action), therefore $\Omega^+(G)$ has zero element $[\emptyset]$ (the other way is to formally add the zero to $\Omega^+(G)$ - this is just technical thing).

Define by $\Omega(G) = K(\Omega^+(G))$ the Grothendieck group of $(\Omega^+(G), +)$. If A is an abelian semigroup and $f : A \times A \to A$ is a bilinear map, then it can be uniquely extended to a bilinear map $K(f) : K(A) \times K(A) \to K(A)$, therefore $\Omega(G)$ can be uniquely turned into a ring from $\Omega^+(G)$. This ring is called *the Burnside ring* of G.

Some properties:

(0) each element of $\Omega(G)$ can be expressed as a formal diffrence $[X] - [Y]$;

(1) $\Omega(G)$ is a commutative, unital ring, where $[G/G]$ is the unity of $\Omega(G)$;

(2) Ω can be turned into a contravariant functor from the category of finite groups to the category of commutative, unital rings;

(3) $(\Omega^+(G), +)$ is a cancellative semigroup, therefore it embedds into $\Omega(G)$;

(4) for the trivial group E there is a ring isomorphism $\Omega(E) \simeq \mathbb{Z}$;

(5) for any group G there is a ring monomorphism $\varphi : \Omega(G) \to \bigoplus_{i=1}^{n} \mathbb{Z}$ for some natural number $n \in \mathbb{N}$; this is called the characteristic embedding;

(6) for any two groups G, H we have: if $\Omega(G)$ and $\Omega(H)$ are isomorphic (as a rings), then $|G| = |H|$; generally G need not be isomorphic to H.