example of a right noetherian ring that is not left noetherian

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This example, due to Lance Small, is briefly described in *Noncommutative Rings*, by I. N. Herstein, published by the Mathematical Association of America, 1968.

Let R be the ring of all 2×2 matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ such that a is an integer and b, c are rational. The claim is that R is right noetherian but not left noetherian.

It is relatively straightforward to show that R is not left noetherian. For each natural number n, let

$$I_n = \{\begin{pmatrix} 0 & \frac{m}{2^n} \\ 0 & 0 \end{pmatrix} \mid m \in \mathbb{Z}\}.$$

Verify that each I_n is a left ideal in R and that $I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$.

It is a bit harder to show that R is right noetherian. The approach given here uses the fact that a ring is right noetherian if all of its right ideals are finitely generated.

Let I be a right ideal in R. We show that I is finitely generated by checking all possible cases. In the first case, we assume that every matrix in I has a zero in its upper left entry. In the second case, we assume that there is some matrix in I that has a nonzero upper left entry. The second case splits into two subcases: either every matrix in I has a zero in its lower right entry or some matrix in I has a nonzero lower right entry.

CASE 1: Suppose that for all matrices in I, the upper left entry is zero. Then every element of I has the form

$$\begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix} \text{ for some } y, z \in \mathbb{Q}.$$

Note that for any $c \in \mathbb{Q}$ and any $\begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix} \in I$, we have $\begin{pmatrix} 0 & cy \\ 0 & cz \end{pmatrix} \in I$ since

$$\begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & cy \\ 0 & cz \end{pmatrix}$$

and I is a right ideal in R. So I looks like a rational vector space.

Indeed, note that $V = \{(y, z) \in \mathbb{Q}^2 \mid \begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix} \in I\}$ is a subspace of the two dimensional vector space $\mathbb{Q}^2$. So in V there exist two (not necessarily linearly independent) vectors (y_1, z_1) and (y_2, z_2) which span V.

Now, an arbitrary element $\begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix}$ in I corresponds to the vector (y, z) in V and $(y, z) = (c_1y_1 + c_2y_2, c_1z_1 + c_2z_2)$ for some $c_1, c_2 \in \mathbb{Q}$. Thus

$$\begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & c_1y_1 + c_2y_2 \\ 0 & c_1z_1 + c_2z_2 \end{pmatrix} = \begin{pmatrix} 0 & y_1 \\ 0 & z_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c_1 \end{pmatrix} + \begin{pmatrix} 0 & y_2 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$$

and it follows that I is finitely generated by the set $\{\begin{pmatrix} 0 & y_1 \\ 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & y_2 \\ 0 & z_2 \end{pmatrix}\}$ as a right ideal in R.

CASE 2: Suppose that some matrix in I has a nonzero upper left entry. Then there is a least positive integer n occurring as the upper left entry of a matrix in I. It follows that every element of I can be put into the form

$$\begin{pmatrix} kn & y \\ 0 & z \end{pmatrix} \text{ for some } k \in \mathbb{Z};\ y, z \in \mathbb{Q}.$$

By definition of n, there is a matrix of the form $\begin{pmatrix} n & b \\ 0 & c \end{pmatrix}$ in I. Since I is a right ideal in R and since $\begin{pmatrix} n & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$, it follows that $\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$ is in I. Now break off into two subcases.

case 2.1: Suppose that every matrix in I has a zero in its lower right entry. Then an arbitrary element of I has the form

$$\begin{pmatrix} kn & y \\ 0 & 0 \end{pmatrix} \text{ for some } k \in \mathbb{Z}, y \in \mathbb{Q}.$$

Note that $\begin{pmatrix} kn & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k & \frac{y}{n} \\ 0 & 0 \end{pmatrix}$. Hence, $\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$ generates I as a right ideal in R.

case 2.2: Suppose that some matrix in I has a nonzero lower right entry. That is, in I we have a matrix

$$\begin{pmatrix} mn & y_1 \\ 0 & z_1 \end{pmatrix} \text{ for some } m \in \mathbb{Z};\ y_1, z_1 \in \mathbb{Q};\ z_1 \neq 0.$$

Since $\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} \in I$, it follows that $\begin{pmatrix} n & y_1 \\ 0 & z_1 \end{pmatrix} \in I$. Let $\begin{pmatrix} kn & y \\ 0 & z \end{pmatrix}$ be an arbitrary element of I. Since $\begin{pmatrix} kn & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} n & y_1 \\ 0 & z_1 \end{pmatrix} \begin{pmatrix} k & \frac{1}{n}(y - \frac{y_1 z}{z_1}) \\ 0 & \frac{z}{z_1} \end{pmatrix}$, it follows that $\begin{pmatrix} n & y_1 \\ 0 & z_1 \end{pmatrix}$ generates I as a right ideal in R.

In all cases, I is a finitely generated.