idempotent classifications

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Defines	division idempotent
Defines	local idempotent

An a unital ring R, an idempotent $e \in R$ is called a *division idempotent* if $eRe = \{ere : r \in R\}$, with the product of R, forms a division ring. If instead eRe is a local ring – here this means a ring with a unique maximal ideal $\mathfrak{m}$ where $eRe/\mathfrak{m}$ a division ring – then e is called a *local idempotent*.

Lemma 1. *Any integral domain R has only the trivial idempotents 0 and 1. In particular, every division ring has only trivial idempotents.*

Proof. Suppose $e \in R$ with $e \neq 0$ and $e^2 = e = 1e$. Then by cancellation $e = 1$. □

The integers are an integral domain which is not a division ring and they serve as a counter-example to many conjectures about idempotents of general rings as we will explore below. However, the first important result is to show the hierarchy of idempotents.

Theorem 2. *Every local ring R has only trivial idempotents 0 and 1.*

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of R. Then $\mathfrak{m}$ is the Jacobson radical of R. Now suppose $e \in \mathfrak{m}$ is an idempotent. Then $1 - e$ must be left invertible (following the `http://planetmath.org/JacobsonRadical`element characterization of Jacobson radicals). So there exists some $u \in R$ such that $1 = u(1 - e)$. However, this produces

$$e = u(1 - e)e = u(e - e^2) = u(e - e) = 0.$$

Thus every non-trivial idempotent $e \in R$ lies outside $\mathfrak{m}$. As $R/\mathfrak{m}$ is a division ring, the only idempotents are 0 and 1. Thus if $e \in R$, $e \neq 0$ is an idempotent then it projects to an idempotent of $R/\mathfrak{m}$ and as $e \notin \mathfrak{m}$ it follows e projects onto 1 so that $e = 1 + z$ for some $z \in \mathfrak{m}$. As $e^2 = e$ we find $0 = z + z^2$ (often called an anti-idempotent). Once again as $z \in \mathfrak{m}$ we know there exists a $u \in R$ such that $1 = u(1 + z)$ and $z = u(1 + z)z = u(z + z^2) = 0$ so indeed $e = 1$. □

Corollary 3. *Every division idempotent is a local idempotent, and every local idempotent is a primitive idempotent.*

Example 4. *Let R be a unital ring. Then in $M_n(R)$ the standard idempotents* *are the matrices*

$$E_{ii} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}, \qquad 1 \leq i \leq n.$$

(i) If R has only trivial idempotents (i.e.: 0 and 1) then each E_{ii} is a primitive idempotent of $M_n(R)$.

(ii) If R is a local ring then each E_{ii} is a local idempotent.

(iii) If R is a division ring then each E_{ii} is a division idempotent.

When $R = \mathbb{R} \oplus \mathbb{R}$ then (i) is not satisfied and consequently neither are (ii) and (iii). When $R = \mathbb{Z}$ then (i) is satisfied but not (ii) nor (iii). When $R = \mathbb{R}[[x]]$ – the formal power series ring over $\mathbb{R}$ – then (i) and (ii) are satisfied but not (iii). Finally when $R = \mathbb{R}$ then all three are satisfied.

A consequence of the Wedderburn-Artin theorems classifies all Artinian simple rings as matrix rings over a division ring. Thus the primitive idempotents of an Artinian ring are all local idempotents. Without the Artinian assumption this may fail as we have already seen with $\mathbb{Z}$.