example of injective module

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In the category of unitary $\mathbb{Z}$-modules (which is the category of Abelian groups), every divisible Group is injective, i.e. every Group G such that for any $g \in G$ and $n \in \mathbb{N}$, there is a $h \in G$ such that $nh = g$. For example, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible, and therefore injective.

Proof. We have to show that, if G is a divisible Group, $\varphi : U \to G$ is any homomorphism, and U is a subgroup of a Group H, there is a homomorphism $\psi : H \to G$ such that the restriction $\psi|_U = \varphi$. In other words, we want to extend φ to a homomorphism $H \to G$.

Let $\mathcal{D}$ be the set of pairs (K, ψ) such that K is a subgroup of G containing U and $\psi : K \to G$ is a homomorphism with $\psi|_U = \varphi$. Then $\mathcal{D}$ ist non-empty since it contains (U, φ), and it is partially ordered by

$$(K, \psi) \leq (K', \psi') :\Longleftrightarrow K \subseteq K' \text{ and } \psi'|_K = \psi.$$

For any ascending chain

$$(K_1, \psi_1) \leq (K_2, \psi_2) \leq \dots,$$

in $\mathcal{D}$, the pair $(\bigcup_{i \in \mathbb{N}} K_i, \bigcup_{i \in \mathbb{N}} \psi_i)$ is in $\mathcal{D}$, and it is an upper bound for this chain. Therefore, by Zorn's Lemma, $\mathcal{D}$ contains a maximal element (M, χ).

It remains to show that $M = H$. Suppose the opposite, and let $h \in H \backslash M$. Let $\langle h \rangle$ denote the subgroup of H generated by h. If $\langle h \rangle \cap M = \{0\}$, the sum $M + \langle h \rangle$ is in fact a direct sum, and we can extend χ to $M + \langle h \rangle$ by choosing an arbitrary image of h in G and extending linearly. This contradicts the maximality of (M, χ).

Let us therefore suppose $\langle h \rangle \cap M$ contains an element nh, with $n \in \mathbb{N}$ minimal. Since $nh \in M$, and χ is defined on M, $\chi(nh)$ exists, and furthermore, since G is divisible, there is a $g \in G$ such that $ng = \chi(nh)$. It is now easy to see that we can extend χ to $M + \langle h \rangle$ by defining $\chi(h) := g$, in contradiction to the maximality of (M, χ).

Therefore, $M = H$. This proves the statement. □