commutativity theorems on rings

Canonical name	CommutativityTheoremsOnRings
Date of creation	2013-03-22 17:54:55
Last modified on	2013-03-22 17:54:55
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	12
Author	CWoo (3771)
Entry type	Theorem
Classification	msc 16B99

Since Wedderburn proved his celebrated theorem that any finite division ring is commutative, the interest in studying properties on a ring that would render the ring commutative dramatically increased. Below is a list of some of the so-called "commutativity theorems" on a ring, showing how much one can generalize the result that Wedderburn first obtained. In the list below, R is assumed to be unital ring.

Theorem 1. *In each of the cases below, R is commutative:*

1. *(Wedderburn's theorem) R is a finite division ring.*

2. *(Jacobson) If for every element $a \in R$, there is a positive integer $n > 1$ (depending on a), such that $a^n = a$.*

3. *(Jacobson-Herstein) For every $a, b \in R$, if there is a positive integer $n > 1$ (depending on a, b) such that*

$$(ab - ba)^n = ab - ba.$$

4. *(Herstein) If there is an integer $n > 1$ such that for every element $a \in R$ such that $a^n - a \in Z(R)$, the center of R.*

5. *(Herstein) If for every $a \in R$, there is a polynomial $p \in \mathbb{Z}[X]$ (p depending on a) such that $a^2 p(a) - a \in Z(R)$.*

6. *(Herstein) If for every $a, b \in R$, such that there is an integer $n > 1$ (depending on a, b) with*

$$(a^n - a)b = b(a^n - a).$$

Some of the commutativity problems can be derived fairly easily, such as the following examples:

Theorem 2. *If R is a ring with 1 such that $(ab)^2 = a^2 b^2$ for all $a, b \in R$, then R is commutative.*

Proof. Let $a, b \in R$. From the assumption, we have $((a + 1)b)^2 = (a + 1)^2 b^2$. Expanding the LHS, we get $(ab)^2 + (ab)b + b(ab) + b^2$. Expanding the RHS, we get $a^2 b^2 + 2ab^2 + b^2$. Equating both sides and eliminating common terms, we have

$$bab = ab^2 \tag{1}$$

Similarly, from $(a(b+1))^2 = a^2(b+1)^2$, we expand the equations and get

$$(ab)^2 + (ab)a + a(ab) + a^2 = a^2b^2 + 2a^2b + a^2.$$

Hence

$$aba = a^2b \tag{2}$$

Finally, expanding out $((a+1)(b+1))^2 = (a+1)^2(b+1)^2$ and eliminating common terms, keeping in mind Equations (1) and (2) from above, we get $ab = ba$. □

Corollary 3. *If each element of a ring R is idempotent, then R is commutative.*

Proof. If R contains 1, then we can apply Theorem 2: for $(st)^2 = st = s^2t^2$ for any $s, t \in R$. Otherwise, we do the following trick: first $2s = (2s)^2 = 4s^2 = 4s$, so that $2s = 0$ for all $s \in R$. Next, $s+t = (s+t)^2 = s^2+st+ts+t^2 = s+st+ts+t$, so $0 = st+ts$, which implies $st = st+(st+ts) = 2st+ts = ts$, and the result follows.

The corollary also follows directly from part 2 of Theorem 1. □

References

[1] I. N. Herstein, *Noncommutative Rings*, The Mathematical Association of America (1968).