proof of invertible ideals are projective

Canonical name	ProofOfInvertibleIdealsAreProjective
Date of creation	2013-03-22 18:35:51
Last modified on	2013-03-22 18:35:51
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	5
Author	gel (22282)
Entry type	Proof
Classification	msc 16D40
Classification	msc 13A15

We show that a nonzero fractional ideal $\mathfrak{a}$ of an integral domain R is invertible if and only if it is http://planetmath.org/ProjectiveModuleprojective as an R-module.

Let $\mathfrak{a}$ be an invertible fractional ideal and $f\colon M \to \mathfrak{a}$ be an epimorphism of R-modules. We need to show that f has a right inverse. Letting $\mathfrak{a}^{-1}$ be the inverse ideal of $\mathfrak{a}$, there exists $a_1, \ldots, a_n \in \mathfrak{a}$ and $b_1, \ldots, b_n \in \mathfrak{a}^{-1}$ such that

$$a_1 b_1 + \cdots + a_n b_n = 1$$

and, as f is onto, there exist $e_k \in M$ such that $f(e_k) = a_k$. For any $x \in \mathfrak{a}$, $x b_k \in \mathfrak{a}\mathfrak{a}^{-1} = R$, so we can define $g\colon \mathfrak{a} \to M$ by

$$g(x) \equiv (x b_1) e_1 + \cdots + (x b_n) e_n.$$

Then

$$f \circ g(x) = (x b_1) f(e_1) + \cdots + (x b_n) f(e_n) = x(b_1 a_1 + \cdots b_n a_n) = x,$$

so g is indeed a right inverse of f, and $\mathfrak{a}$ is projective.

Conversely, suppose that $\mathfrak{a}$ is projective and let $(a_i)_{i \in I}$ generate $\mathfrak{a}$ (this always exists, as we can let a_i include every element of $\mathfrak{a}$). Then let M be a module with free basis $(e_i)_{i \in I}$ and define $f\colon M \to \mathfrak{a}$ by $f(e_i) = a_i$. As $\mathfrak{a}$ is projective, f has a right inverse $g\colon \mathfrak{a} \to M$. As e_i freely generate M, we can uniquely define $g_i\colon \mathfrak{a} \to R$ by

$$g(x) = \sum_{i \in I} g_i(x) e_i,$$

noting that all but finitely many $g_i(x)$ must be zero for any given x. Choosing any fixed nonzero $a \in \mathfrak{a}$, we can set $b_i = a^{-1} g_i(a)$ so that

$$g_i(x) = a^{-1} g_i(ax) = a^{-1} x g_i(a) = b_i x$$

for all $x \in \mathfrak{a}$, and b_i must equal zero for all but finitely many i. So, we can let $\mathfrak{b}$ be the fractional ideal generated by the b_i and, noting that $x b_i = g_i(x) \in R$ we get $\mathfrak{a}\mathfrak{b} \subseteq R$. Furthermore, for any $x \in R$,

$$x = a^{-1} f \circ g(ax) = \sum_i a^{-1} g_i(ax) f(e_i) = \sum_i x b_i f(e_i) \in \mathfrak{b}\mathfrak{a}$$

so that $R \subseteq \mathfrak{a}\mathfrak{b}$, and $\mathfrak{b}$ is the inverse of $\mathfrak{a}$ as required.