Jacobson radical of a module category and its power

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Assume that k is a field and A is a k-algebra. The category of (left) A-modules will be denoted by $\mathrm{Mod}(A)$ and $\mathrm{Hom}_A(X,Y)$ will denote the set of all A-homomorphisms between A-modules X and Y. Of course $\mathrm{Hom}_A(X,Y)$ is an A-module itself and $\mathrm{End}_A(X) = \mathrm{Hom}_A(X,X)$ is a k-algebra (even A-algebra) with composition as a multiplication.

Let X and Y be A-modules. Define

$$\mathrm{rad}_A(X,Y) = \{f \in \mathrm{Hom}_A(X,Y) \mid \forall_{g\in \mathrm{Hom}_A(Y,X)}\ 1_X - gf \text{ is invertible in } \mathrm{End}_A(X)\}.$$

Definition. *The Jacobson radical* of a category $\mathrm{Mod}(A)$ is defined as a class

$$\mathrm{rad}\ \mathrm{Mod}(A) = \bigcup_{X,Y\in \mathrm{Mod}(A)} \mathrm{rad}_A(X,Y).\square$$

Properties. 1) The Jacobson radical is an ideal in $\mathrm{Mod}(A)$, i.e. for any $X, Y, Z \in \mathrm{Mod}(A)$, for any $f \in \mathrm{rad}_A(X,Y)$, any $h \in \mathrm{Hom}_A(Y,Z)$ and any $g \in \mathrm{Hom}_A(Z,X)$ we have $hf \in \mathrm{rad}_A(X,Z)$ and $fg \in \mathrm{rad}_A(Z,X)$. Additionaly $\mathrm{rad}_A(X,Y)$ is an A-submodule of $\mathrm{Hom}_A(X,Y)$.

2) For any A-module X we have $\mathrm{rad}_A(X,X) = \mathrm{rad}(\mathrm{End}_A(X))$, where on the right side we have the classical Jacobson radical.

3) If X, Y are both indecomposable A-modules such that both $\mathrm{End}_A(X)$ and $\mathrm{End}_A(Y)$ are local algebras (in particular, if X and Y are finite dimensional), then

$$\mathrm{rad}_A(X,Y) = \{f \in \mathrm{Hom}_A(X,Y) \mid f \text{ is not an isomorphism}\}.$$

In particular, if X and Y are not isomorphic, then $\mathrm{rad}_A(X,Y) = \mathrm{Hom}_A(X,Y)$. $\square$

Let $n \in \mathbb{N}$ and let $f \in \mathrm{Hom}_A(X,Y)$. Assume there is a sequence of A-modules $X = X_0, X_1, \dots, X_{n-1}, X_n = Y$ and for any $0 \le i \le n-1$ we have an A-homomorphism $f_i \in \mathrm{rad}_A(X_i, X_{i+1})$ such that $f = f_{n-1}f_{n-2}\cdots f_1 f_0$. Then we will say that f is *n-factorizable through Jacobson radical.*

Definition. *The n-th power of a Jacobson radical* of a category $\mathrm{Mod}(A)$ is defined as a class

$$\mathrm{rad}^n\ \mathrm{Mod}(A) = \bigcup_{X,Y\in \mathrm{Mod}(A)} \mathrm{rad}^n_A(X,Y),$$

where $\mathrm{rad}^n_A(X,Y)$ is an A-submodule of $\mathrm{Hom}_A(X,Y)$ generated by all homomorphisms n-factorizable through Jacobson radical. Additionaly define

$$\mathrm{rad}^\infty_A(X,Y) = \bigcap_{n=1}^{\infty} \mathrm{rad}^n_A(X,Y)\ \square$$

Properties. 0) Obviously $\operatorname{rad}_A(X,Y) = \operatorname{rad}^1_A(X,Y)$ and for any $n \in \mathbb{N}$ we have

$$\operatorname{rad}^n_A(X,Y) \supseteq \operatorname{rad}^\infty_A(X,Y).$$

1) Of course each $\operatorname{rad}^n_A(X,Y)$ is an A-submodule of $\operatorname{Hom}_A(X,Y)$ and we have following sequence of inclusions:

$$\operatorname{Hom}_A(X,Y) \supseteq \operatorname{rad}^1_A(X,Y) \supseteq \operatorname{rad}^2_A(X,Y) \supseteq \operatorname{rad}^3_A(X,Y) \supseteq \cdots$$

2) If both X and Y are finite dimensional, then there exists $n \in \mathbb{N}$ such that

$$\operatorname{rad}^\infty_A(X,Y) = \operatorname{rad}^n_A(X,Y).$$