near-ring

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Defines	commutative near-ring
Defines	commutative near ring
Defines	commutative nearring
Defines	distributative near-ring
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Defines	distributative nearring
Defines	near field
Defines	nearfield

Definitions

A *near-ring* is a http://planetmath.org/Setset N together with two binary operations, denoted $+\colon N \times N \to N$ and $\cdot\colon N \times N \to N$, such that

1. $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in N$ (associativity of both operations)
2. There exists an element $0 \in N$ such that $a + 0 = 0 + a = a$ for all $a \in N$ (additive identity)
3. For all $a \in N$, there exists $b \in N$ such that $a + b = b + a = 0$ (additive inverse)
4. $(a+b) \cdot c = (a \cdot c) + (b \cdot c)$ for all $a, b, c \in N$ (right distributive law)

Note that the axioms of a near-ring differ from those of a ring in that they do not require addition to be http://planetmath.org/Commutativecommutative, and only require distributivity on one side.

A *near-field* is a near-ring N such that $(N \setminus \{0\}, \cdot)$ is a group.

Notes

Every element a in a near-ring has a unique additive inverse, denoted $-a$.

We say N has an *identity element* if there exists an element $1 \in N$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in N$. We say N is *distributive* if $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ holds for all $a, b, c \in N$. We say N is *commutative* if $a \cdot b = b \cdot a$ for all $a, b \in N$.

Every commutative near-ring is distributive. Every distributive near-ring with an identity element is a unital ring (see the http://planetmath.org/ConditionOnANearRingT
proof).

Example

A natural example of a near-ring is the following. Let $(G, +)$ be a group (not necessarily http://planetmath.org/AbelianGroup2abelian), and let M be the set of all functions from G to G. For two functions f and g in M define $f + g \in M$ by $(f+g)(x) = f(x) + g(x)$ for all $x \in G$. Then $(M, +, \circ)$ is a near-ring with identity, where $\circ$ denotes composition of functions.

References

[1] Günter Pilz, *Near-Rings*, North-Holland, 1983.