coalgebra

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Entry type	Definition
Classification	msc 16W30
Defines	comultiplication
Defines	counit
Defines	coassociative
Defines	cocommutative

A **coalgebra** is a vector space A over a field $\mathbb{K}$ with a $\mathbb{K}$-linear map $\Delta\colon A \to A \otimes A$, called the **comultiplication**, and a (non-zero) $\mathbb{K}$-linear map $\varepsilon\colon A \to \mathbb{K}$, called the **counit**, such that

$$\begin{aligned}
(\Delta \otimes \mathrm{id}) \circ \Delta \quad &= \quad (\mathrm{id} \otimes \Delta) \circ \Delta \quad \text{(coassociativity)},\\
(\varepsilon \otimes \mathrm{id}) \circ \Delta \quad &= \mathrm{id} = \quad (\mathrm{id} \otimes \varepsilon) \circ \Delta.
\end{aligned}$$

In of commutative diagrams:

$$\begin{tikzcd}
& A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
A \otimes A \arrow[dr, "\Delta\otimes\mathrm{id}"'] & & A \otimes A \arrow[dl, "\mathrm{id}\otimes\Delta"] \\
& A \otimes A \otimes A &
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[dd, "\mathrm{id}"] & \\
A \otimes A \arrow[dr, "\varepsilon\otimes\mathrm{id}"'] & & A \otimes A \arrow[dl, "\mathrm{id}\otimes\varepsilon"] \\
& A &
\end{tikzcd}$$

Let $\sigma\colon A \otimes A \to A \otimes A$ be the flip map $\sigma(a \otimes b) = b \otimes a$. A coalgebra is said to be **cocommutative** if $\sigma \circ \Delta = \Delta$.

Let A and B be two coalgebras over a field $\mathbb{K}$. A coalgebra homomorphism is a $\mathbb{K}$-linear map $f\colon A \to B$ such that $\Delta_B \circ f = (f \otimes f) \circ \Delta_A$ and $\varepsilon_B \circ f = \varepsilon_A$.

Example 1 (Coalgebra of a set)
Let S be a set. The free vector space $\mathbb{K}S$, with basis given by the elements of S, is a coalgebra with comultiplication $\Delta(s) = s \otimes s$ and counit $\varepsilon(s) = 1$.