local ring

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Commutative case

A commutative ring with multiplicative identity is called *local* if it has exactly one maximal ideal. This is the case if and only if $1 \neq 0$ and the sum of any two non-http://planetmath.org/unitunits in the ring is again a non-unit; the unique maximal ideal consists precisely of the non-units.

The name comes from the fact that these rings are important in the study of the local behavior of http://planetmath.org/varietyvarieties and manifolds: the ring of function germs at a point is always local. (The reason is simple: a germ f is invertible in the ring of germs at x if and only if $f(x) \neq 0$, which implies that the sum of two non-invertible elements is again non-invertible.) This is also why schemes, the generalizations of varieties, are defined as certain locally ringed spaces. Other examples of local rings include:

- All fields are local. The unique maximal ideal is (0).
- Rings of formal power series over a field are local, even in several variables. The unique maximal ideal consists of those without .
- if R is a commutative ring with multiplicative identity, and $\mathfrak{p}$ is a prime ideal in R, then the localization of R at $\mathfrak{p}$, written as $R_{\mathfrak{p}}$, is always local. The unique maximal ideal in this ring is $\mathfrak{p}R_{\mathfrak{p}}$.
- All discrete valuation rings are local.

A local ring R with maximal ideal $\mathfrak{m}$ is also written as $(R, \mathfrak{m})$.

Every local ring $(R, \mathfrak{m})$ is a topological ring in a natural way, taking the powers of $\mathfrak{m}$ as a neighborhood base of 0.

Given two local rings $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$, a *local ring homomorphism* from R to S is a ring homomorphism $f : R \to S$ (respecting the multiplicative identities) with $f(\mathfrak{m}) \subseteq \mathfrak{n}$. These are precisely the ring homomorphisms that are continuous with respect to the given topologies on R and S.

The *residue field* of the local ring $(R, \mathfrak{m})$ is the field $R/\mathfrak{m}$.

General case

One also considers non-commutative local rings. A http://planetmath.org/ringring with multiplicative identity is called *local* if it has a unique maximal left ideal.

In that case, the ring also has a unique maximal right ideal, and the two coincide with the ring's Jacobson radical, which in this case consists precisely of the non-units in the ring.

A ring R is local if and only if the following condition holds: we have $1 \neq 0$, and whenever $x \in R$ is not invertible, then $1 - x$ is invertible.

All skew fields are local rings. More interesting examples are given by endomorphism rings: a finite-length module over some ring is indecomposable if and only if its endomorphism ring is local, a consequence of Fitting's lemma.