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von Neumann regular

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Defines	von Neumann regular ring
Defines	regular ring
Defines	pseudoinverse

An element a of a ring R is said to be *von Neumann regular* if there exists $b \in R$ such that $aba = a$. Such an element b is known as a of a.

For example, any unit in a ring is von Neumann regular. Also, any idempotent element is von Neumann regular. For a non-unit, non-idempotent von Nuemann regular element, take $M_2(\mathbb{R})$, the ring of 2×2 matrices over $\mathbb{R}$. Then

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

is von Neumann regular. In fact, we can replace 2 with any non-zero $r \in \mathbb{R}$ and the resulting matrix is also von Neumann regular. There are several ways to generalize this example. One way is take a central idempotent e in any ring R, and any $rs = f$ with $ef = e$. Then re is von Neumann regular, with s, se and sf all as pseudoinverses. In another generalization, we have two rings R, S where R is an algebra over S. Take any idempotent $e \in R$, and any invertible element $s \in S$ such that s commutes with e. Then se is von Neumann regular.

A ring R is said to be a *von Neumann regular ring* (or simply a *regular ring*, if the is clear from context) if every element of R is von Neumann regular.

For example, any division ring is von Neumann regular, and so is any ring of matrices over a division ring. In general, any semisimple ring is von Neumann regular.

Remark. Note that *regular ring* in the sense of von Neumann should not be confused with *regular ring* in the sense of , which is a Noetherian ring whose localization at every prime ideal is a regular local ring.