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finite dimensional modules over algebra

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Assume that k is a field, A is a k-algebra and M is a A-module over k. In particular M is a A-module and a vector space over k, thus we may speak about M being finitely generated as A-module and finite dimensional as a vector space. These two concepts are related as follows:

Proposition. Assume that A and M are both unital and additionaly A is finite dimensional. Then M is finite dimensional vector space if and only if M is finitely generated A-module.

Proof. ,,$\Rightarrow$" Of course if M is finite dimensional, then there exists basis

$$\{x_1, \ldots, x_n\} \subset M.$$

Thus every element of M can be (uniquely) expressed in the form

$$\sum_{i=1}^{n} \lambda_i \cdot x_i$$

which is equal to

$$\sum_{i=1}^{n} (\lambda_i \cdot 1) \cdot x_i$$

since M and A are unital. This completes this implication, because $\lambda_i \cdot 1 \in A$ for all i.

,,$\Leftarrow$" Assume that M is finitely generated A-module. In particular there is a subset

$$\{x_1, \ldots, x_n\} \subset M$$

such that every element of M is of the form

$$\sum_{i=1}^{n} a_i \cdot x_i$$

with all $a_i \in A$. Let $m \in M$ be with the decomposition as above. Now A is finite dimensional, so there is a subset

$$\{y_1, \ldots, y_t\} \subset A$$

which is a k-basis of A. In particular for each i we have

$$a_i = \sum_{j=1}^{t} \lambda_{ij} \cdot y_j$$

with $\lambda_{ij} \in k$. Thus we obtain

$$m = \sum_{i=1}^{n} a_i \cdot x_i = \sum_{i=1}^{n} \big(\sum_{j=1}^{t} \lambda_{ij} \cdot y_j \big) \cdot x_i =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{t} \lambda_{ij} \cdot (y_j \cdot x_i)$$

which shows, that all $y_j \cdot x_i \in M$ together make a set of generators of M over k (note that y_j and x_i are independent on m). Since it is finite, then M is finite dimensional and the proof is complete. $\square$