maximal ideal is prime

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Theorem. In a commutative ring with non-zero unity, any maximal ideal is a prime ideal.

Proof. Let $\mathfrak{m}$ be a maximal ideal of such a ring R and let the ring product rs belong to $\mathfrak{m}$ but e.g. $r \notin \mathfrak{m}$. The maximality of $\mathfrak{m}$ implies that $\mathfrak{m}+(r) = R = (1)$. Thus there exists an element $m \in \mathfrak{m}$ and an element $x \in R$ such that $m+xr = 1$. Now m and rs belong to $\mathfrak{m}$, whence

$$s = 1s = (m+xr)s = sm+x(rs) \in \mathfrak{m}.$$

So we can say that along with rs, at least one of its `http://planetmath.org/Product`factors belongs to $\mathfrak{m}$, and therefore $\mathfrak{m}$ is a prime ideal of R.