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ideal included in union of prime ideals

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In the following R is a commutative ring with unity.

Proposition 1. *Let I be an ideal of the ring R and $P_1, P_2, ..., P_n$ be prime ideals of R. If $I \nsubseteq P_i$, for all i, then $I \nsubseteq \cup P_i$.*

Proof. We will prove by induction on n. For $n = 1$ the proof is trivial. Assume now that the result is true for $n - 1$. That implies the existence, for each i, of an element s_i such that $s_i \in I$ and $s_i \notin \bigcup_{j \neq i} P_j$. If for some i, $s_i \notin P_i$ then we are done. Thus, we may consider only the case $s_i \in P_i$, for all i.
Let $a_i = r_1 ... r_{i-1} r_{i+1} ... r_n$. Since P_i is prime then $a_i \notin P_i$, for all i. Moreover, for $j \neq i$, the element $a_i \in P_j$. Consider the element $a = \sum a_j \in I$. Since $a_i = a - \sum_{i \neq j} a_j$ and $\sum_{i \neq j} a_j \ \in P_i$, it follows that $a \notin P_i$, otherwise $a_i \in P_i$, contradiction. The existence of the element a proves the proposition. $\square$

Corollary 1. *Let I be an ideal of the ring R and $P_1, P_2, ..., P_n$ be prime ideals of R. If $I \subseteq \cup P_i$, then $I \subseteq P_i$, for some i.*