division algebra

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Let K be a unital ring and A a K-algebra. Defining "division" requires special considerations when the algebras are non-associative so we introduce the definition in stages.

1 Associative division algebras

If A is an associative algebra then we say A is a *division algebra* if

(i) A is unital with identity 1. So for all $a \in A$,

$$a1 = 1a = a.$$

(ii) Also every non-zero element of A has an inverse. That is $a \in A$, $a \neq 0$, then there exists a $b \in A$ such that

$$ab = 1 = ba.$$

We denote b by a^{-1} and we may prove a^{-1} is unique to a.

The standard examples of associative division algebras are fields, which are commutative, and the non-split quaternion algebra: $\alpha, \beta \in K$,

$$\left(\frac{\alpha, \beta}{K}\right) = \left\{a_1 1 + a_2 i + a_3 j + a_4 k : i^2 = \alpha 1, j^2 = \beta 1, k^2 = -\alpha\beta 1, ij = k = -ji.\right\}$$

where $x^2 - \alpha$ and $x^2 - \beta$ are irreducible over K.

2 Non-associative division algebras

For non-associative algebras A, the notion of an inverse is not immediate. We use $x.y$ for the product of $x, y \in A$.

Invertible as endomorphisms: Let $a \in A$. Then define $L_a : x \mapsto a.x$ and $R_a : x \mapsto x.a$. As the product of A is distributive, both L_a an R_a are additive endomorphisms of A. If L_a is invertible then we may call a "left invertible" and similarly, when R_a is invertible we may call a "right invertible" and "invertible" if both L_a and R_a are invertible.

In this model of invertible, A is a *division algebra* if, and only if, for each non-zero $a \in A$, both L_a and R_a invertible. Equivalently: the equations

$a.x = b$ and $y.a = b$ have unique solutions for nonzero $a, b \in A$. However, x and y need not be equal.

A common method to produce non-associative division algebras of this sort is through Schur's Lemma.

Invertible in the product: In some instances, the notion of invertible via endomorphisms is not sufficient. Instead, assume A has an identity, that is, an element $1 \in A$ such that for all $a \in A$,

$$1.a = a = a.1.$$

Next if $a \in A$, we say a is *invertible* if there exists a $b \in A$ such that

$$a.b = 1 = b.a \tag{1}$$

and furthermore that for all $x \in A$,

$$b.(a.x) = x = (x.a).b. \tag{2}$$

Evidently (**??**) can be inferred from (**??**). This added assumption substitutes for the need of associativity in the proofs of uniqueness of inverses and in solving equations with non-associative products.

Proposition 1. *If A is a finite dimensional algebra over a field, then invertible in this sense forces both L_a and R_a to be invertible as well.*

Proof. Let $x \in A$. Then $xL_1 = 1.x = x = b.(a.x) = xL_aL_b$. So $L_1 = L_aL_b$. As L_1 is the identity map, L_a is injective and L_b is surjective. As A is finite dimensional, injective and surjective endomorphisms are bijective. $\square$

In this model, a non-associative algebra is a division algebra A if it is unital and every non-zero element is invertible.

3 Alternative division algebras

The standard examples of non-associative division algebras are actually alternative alegbras, specfically, the composition algebras of fields, non-split quaternions and non-split octonions – only the latter are actually not associative. Invertible in the octonions is interpreted in the second stronger form.

Theorem 2 (Bruck-Klienfeld)**.** *Every alternative division algebra is either associative or a non-split octonion.*

This result is usually followed by two useful results which serve to omit the need to consider non-associative examples.

Theorem 3 (Artin-Zorn, Wedderburn)**.** *A finite alternative division algebra is associative and commutative, so it is a finite field.*

Theorem 4. *An alternative division algebra over an algebraically closed field is the field itself.*