nichols-zoeller theorem

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Let H be a Hopf algebra over a field k with an antipode S. We will say that $K \subseteq H$ is a *Hopf subalgebra* if K is both subalgebra and subcoalgebra of underlaying algebra and coalgebra structures of H, and additionaly $S(K) \subseteq K$. In particular a Hopf subalgebra $K \subseteq H$ is an algebra over k, so H may be regarded as a K-module.

The Nichols-Zoeller Theorem. If $K \subseteq H$ is a Hopf subalgebra of a Hopf algebra H, then H is free as a K-module. In particular, if H is finite dimensional, then $\dim_k K$ divides $\dim_k H$.

Remark 1. This theorem shows that Hopf algebras are very similar to groups, because this is a Hopf analogue of the Lagrange Theorem.

Remark 2. Generally this theorem does not need to hold if H is only an algebra. For example, consider $H = \mathbb{M}_n(k)$ the matrix algebra, where $n \geq 2$ and let $T \subseteq H$ be the upper triangular matrix subalgebra. It is well known that $\dim_k H = n^2$ and $\dim_k T = \frac{n(n+1)}{2}$. Of course $\frac{n(n+1)}{2}$ does not divide n^2 for $n \geq 2$. Thus the Nichols-Zoeller Theorem does not hold for algebras.