a characterization of the radical of an ideal

Canonical name	ACharacterizationOfTheRadicalOfAnIdeal
Date of creation	2013-03-22 18:04:50
Last modified on	2013-03-22 18:04:50
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	7
Author	CWoo (3771)
Entry type	Derivation
Classification	msc 16N40
Classification	msc 13-00
Classification	msc 14A05

Proposition 1. *Let I be an ideal in a ring R, and $\sqrt{I}$ be its radical. Then $\sqrt{I}$ is the intersection of all prime ideals containing I.*

Proof. Suppose $x \in \sqrt{I}$, and P is a prime ideal containing I. Then $R - P$ is an `http://planetmath.org/MSystem`m-system. If $x \in R - P$, then $(R - P) \cap I \neq \varnothing$, contradicting the assumption that $I \subseteq P$. Therefore $x \notin R - P$. In other words, $x \in P$, and we have one of the inclusions.

Conversely, suppose $x \notin \sqrt{I}$. Then there is an m-system S containing x such that $S \cap I = \varnothing$. Enlarge I to a prime ideal P disjoint from S, so that $x \notin P$ (we can do this; for a proof, see the second remark in `http://planetmath.org/MSystem`this entry). By contrapositivity, we have the other inclusion. □

Remark. This shows that every prime ideal is a radical ideal: for $\sqrt{P}$ is the intersection of all prime ideals containing P, and if P is itself prime, then $P = \sqrt{P}$.