hollow matrix rings

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1 Definition

Definition 1. *Suppose that $R \subseteq S$ are both rings. The* hollow matrix ring *of (R, S) is the ring of matrices:*

$$\begin{bmatrix} S & S \\ 0 & R \end{bmatrix} := \left\{ \begin{bmatrix} s & t \\ 0 & r \end{bmatrix} : s, t \in S, r \in R \right\}.$$

It is easy to check that this forms a ring under the usual matrix addition and multiplication. This definition is slightly simplified from the obvious higher dimensional examples and the transpose of these matrices will also qualify as a hollow matrix ring.

The hollow matrix rings are highly counter-intuitive despite their simple definition. In particular, they can be used to prove that in general a ring's left ideal structure need not relate to its right ideal structure. We highlight a few examples of this.

2 Left/Right Artinian and Noetherian

We specialize to an example with the fields $\mathbb{Q}$ and $\mathbb{R}$, though the same argument can be made in much more general settings.

$$R := \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b \in \mathbb{R}, c \in \mathbb{Q} \right\}. \tag{1}$$

Claim 2. *R is left Artinian and left Noetherian.*

Proof. Let I be a left ideal of R and suppose that $r := \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in I$ for some $x, y \in \mathbb{R}$ and $z \in \mathbb{Q}$.

Suppose that $z \neq 0$. Hence, $s_q := \begin{bmatrix} 0 & 0 \\ 0 & q/z \end{bmatrix} \in R$ for each $q \in \mathbb{Q}$ and so $s_q r = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \in I$ for all $q \in \mathbb{Q}$. In particular, $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = r - s_1 r \in I$. So in all cases it follows that $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \in I$. So now we take $r = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ and assume that I does not contain any r with $z \neq 0$. By observing matrix multiplication it follows that I is now a left $\mathbb{R}$-vector space, and so any chain of left R-modules is a chain of subspaces. As $\dim_{\mathbb{R}} I \leq 2$, it follows that such chains are finite.

Hence, there can be no infinite descending chain of distinct left ideals and so R is left Artinian and Noetherian. $\square$

Claim 3. *R is not right Artinian nor right Noetherian.*

Proof. Using π (the usual $3.14\ldots$), or any other transcendental number, we define

$$I_n := \begin{bmatrix} 0 & \mathbb{Q}[\pi; n] \\ 0 & 0 \end{bmatrix}, \tag{2}$$

where

$$\mathbb{Q}[\pi; n] := \{q(\pi)\pi^n : q(x) \in \mathbb{Q}[x]\}. \tag{3}$$

Since $\mathbb{Q}[\pi; n]$ properly contains $\mathbb{Q}[\pi; n+1]$ for all $n \in \mathbb{Z}$, it follows that $\{I_n : n \in \mathbb{Z}\}$ is an infinite proper ascending and descending chain of right ideals. Therefore, R is neither right Artinian nor right Noetherian. $\square$

Corollary 4. *R does not have a ring anti-isomorphism. Thus R is not an involutory ring.*

Proof. If R is a ring with an anti-isomorphism, then the set of left ideals is mapped to the set of right ideals, bijectively and order preserving. This is not possible with R. $\square$