idempotent semiring

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A semiring S is called an *idempotent semiring*, or *i-semiring* for short, if, addition $+$ is an idempotent binary operation:

$$a + a = a, \qquad \text{for all } a \in S.$$

Some properties of an i-semiring S.

1. If we define a binary relation $\leq$ on S by

   $$a \leq b \qquad \text{iff} \qquad a + b = b$$

   then $\leq$ becomes a partial order on S. Indeed, for $a + a = a$ implies $a \leq a$; if $a \leq b$ and $b \leq a$, then $b = a + b = a$; and finally, if $a \leq b$ and $b \leq c$, then $a + c = a + (b + c) = (a + b) + c = b + c = c$ so $a \leq c$.

2. $0 \leq a$ for any $a \in S$, because $0 + a = a$.

3. Define $a \vee b$ as the supremum of a and b (with respect to $\leq$). Then $a \vee b$ exists and

   $$a \vee b = a + b.$$

   To see this, we have $a+(a+b) = (a+a)+b = a+b$, so $a \leq a+b$. Similarly $b \leq a+b$. If $a \leq c$ and $b \leq c$, then $(a+b)+c = a+(b+c) = a+c = c$. So $a + b \leq c$.

4. Collecting all the information above, we see that $(S, +)$ is an upper semilattice with $+$ as the join operation on S and 0 the bottom element.

5. Additon and multiplication respect partial ordering: suppose $a \leq b$, then for any $c \in S$, $(c + a) + (c + b) = (c + c) + (a + b) = c + b$, hence $c + a \leq c + b$; also, $cb = c(a + b) = ca + cb$ implies $ca \leq cb$.

Remark. S in general is not a lattice, and 1 is not the top element of S.

The main example of an i-semiring is a Kleene algebra used in the theory of computations.