fundamental theorem of coalgebras

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Fundamental Theorem of Coalgebras. Let (C, Δ, ε) be a coalgebra over a field k and $x \in C$. Then there exists subcoalgebra $D \subseteq C$ such that $x \in D$ and $\dim_k D < \infty$.

Proof. Let

$$\Delta(x) = \sum_i b_i \otimes c_i.$$

Consider the element

$$\Delta_2(x) = \sum_i \Delta(b_i) \otimes c_i = \sum_{i,j} a_j \otimes b_{ij} \otimes c_i.$$

Note that we may assume that (a_j) are linearly independent and so are (c_i). Let D be a subspace spanned by (b_{ij}). Of course $\dim_k D < \infty$. Furthermore $x \in D$, because

$$x = \sum_{i,j} \varepsilon(a_j)\varepsilon(c_i) b_{ij}.$$

We will show that D is a subcoalgebra, i.e. $\Delta(D) \subseteq D \otimes D$. Indeed, note that

$$\sum_{i,j} \Delta(a_j) \otimes b_{ij} \otimes c_i = \sum_{i,j} a_j \otimes \Delta(b_{ij}) \otimes c_i$$

and since c_i are linearly independent we obtain that

$$\sum_j \Delta(a_j) \otimes b_{ij} = \sum_j a_j \otimes \Delta(b_{ij})$$

for all i. Thus

$$\sum_j a_j \otimes \Delta(b_{ij}) \in C \otimes C \otimes D$$

and since a_j are linearly independent, we obtain that $\Delta(b_{ij}) \in C \otimes D$ for all i, j. Analogously we show that $\Delta(b_{ij}) \in D \otimes C$, thus

$$\Delta(b_{ij}) \in C \otimes D \cap D \otimes C = D \otimes D,$$

(please, see `http://planetmath.org/TensorProductOfSubspacesOfVectorSpaces`this entry for last equality) which completes the proof. □

Remark. The category of finite dimensional coalgebras is dual to the category of finite dimensional algebras (via dual space functor), so one could think that generally they are similar. Unfortunetly Fundamental Theorem of

Coalgebras is major diffrence between algebras and coalgebras. For example consider a field k and its polynomial algebra $k[X]$. Then whenever $f \in k[X]$ is such that $\deg(f) > 0$, then a subalgebra generated by f is always infinite dimensional (if $\deg(f) = 0$ then subalgebra generated by f is k). This can never occur in coalgebras.