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Peirce decomposition

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Let e be an idempotent of a ring R, not necessarily with an identity. For any subset X of R, we introduce the notations:

$$(1-e)X = \{x - ex \mid x \in X\}$$

and

$$X(1-e) = \{x - xe \mid x \in X\}.$$

If it happens that R has an identity element, then $1-e$ is a legitimate element of R, and this notation agrees with the usual product of an element and a set.

It is easy to see that $Xe \cap X(1-e) = 0 = eX \cap (1-e)X$ for any set X which contains 0.

Applying this first on the right with $X = R$ and then on the left with $X = Re$ and $X = R(1-e)$, we obtain:

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e).$$

This is called the *Peirce Decompostion* of R with respect to e.

Note that eRe and $(1-e)R(1-e)$ are subrings, $eR(1-e)$ is an eRe-$(1-e)R(1-e)$-bimodule, and $(1-e)Re$ is a $(1-e)R(1-e)$-eRe-bimodule.

This is an example of a generalized matrix ring:

$$R \cong \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$$

More generally, if R has an identity element, and $e_1, e_2, \ldots, e_n$ is a complete set of orthogonal idempotents, then

$$R \cong \begin{pmatrix} e_1Re_1 & e_1Re_2 & \ldots & e_1Re_n \\ e_2Re_1 & e_2Re_2 & \ldots & e_2Re_n \\ \vdots & \vdots & \ddots & \vdots \\ e_nRe_1 & e_nRe_2 & \ldots & e_nRe_n \end{pmatrix}$$

is a generalized matrix ring.