proof of characterizations of the Jacobson radical

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First, note that by definition a left primitive ideal is the annihilator of an irreducible left R-module, so clearly characterization 1) is equivalent to the definition of the Jacobson radical.

Next, we will prove cyclical containment. Observe that 5) follows after the equivalence of 1) - 4) is established, since 4) is independent of the choice of left or right ideals.

1) $\subset$ 2) We know that every left primitive ideal is the largest ideal contained in a maximal left ideal. So the intersection of all left primitive ideals will be contained in the intersection of all maximal left ideals.

2) $\subset$ 3) Let $S = \{M : M \text{ a maximal left ideal of } R\}$ and take $r \in R$. Let $t \in \cap_{M \in S} M$. Then $rt \in \cap_{M \in S} M$.

Assume $1 - rt$ is not left invertible; therefore there exists a maximal left ideal M_0 of R such that $R(1 - rt) \subseteq M_0$.

Note then that $1 - rt \in M_0$. Also, by definition of t, we have $rt \in M_0$. Therefore $1 \in M_0$; this contradiction implies $1 - rt$ is left invertible.

3) $\subset$ 4) We claim that 3) satisfies the condition of 4).

Let $K = \{t \in R : 1 - rt \text{ is left invertible for all } r \in R\}$.

We shall first show that K is an ideal.

Clearly if $t \in K$, then $rt \in K$. If $t_1, t_2 \in K$, then

$$1 - r(t_1 + t_2) = (1 - rt_1) - rt_2$$

Now there exists u_1 such that $u_1(1 - rt_1) = 1$, hence

$$u_1((1 - rt_1) - rt_2) = 1 - u_1 r t_2$$

Similarly, there exists u_2 such that $u_2(1 - u_1 r t_2) = 1$, therefore

$$u_2 u_1 (1 - r(t_1 + t_2)) = 1$$

Hence $t_1 + t_2 \in K$.

Now if $t \in K, r \in R$, to show that $tr \in K$ it suffices to show that $1 - tr$ is left invertible. Suppose $u(1 - rt) = 1$, hence $u - urt = 1$, then $tur - turtr = tr$.

So $(1 + tur)(1 - tr) = 1 + tur - tr - turtr = 1$.

Therefore K is an ideal.

Now let $v \in K$. Then there exists u such that $u(1-v) = 1$, hence $1 - u = -uv \in K$, so $u = 1 - (1-u)$ is left invertible.

So there exists w such that $wu = 1$, hence $wu(1-v) = w$, then $1 - v = w$. Thus $(1-v)u = 1$ and therefore $1 - v$ is a unit.

Let J be the largest ideal such that, for all $v \in J$, $1 - v$ is a unit. We claim that $K \subseteq J$.

Suppose this were not true; in this case $K + J$ strictly contains J. Consider $rx + sy \in K + J$ with $x \in K, y \in J$ and $r, s \in R$. Now $1 - (rx + sy) = (1 - rx) - sy$, and since $rx \in K$, then $1 - rx = u$ for some unit $u \in R$.

So $1 - (rx + sy) = u - sy = u(1 - u^{-1}sy)$, and clearly $u^{-1}sy \in J$ since $y \in J$. Hence $1 - u^{-1}sy$ is also a unit, and thus $1 - (rx + sy)$ is a unit.

Thus $1 - v$ is a unit for all $v \in K + J$. But this contradicts the assumption that J is the largest such ideal. So we must have $K \subseteq J$.

4) $\subset$ 1) We must show that if I is an ideal such that for all $u \in I$, $1 - u$ is a unit, then $I \subset \text{ann}({}_RM)$ for every irreducible left R-module ${}_RM$.

Suppose this is not the case, so there exists ${}_RM$ such that $I \not\subset \text{ann}({}_RM)$. Now we know that $\text{ann}({}_RM)$ is the largest ideal inside some maximal left ideal J of R. Thus we must also have $I \not\subset J$, or else this would contradict the maximality of $\text{ann}({}_RM)$ inside J.

But since $I \not\subset J$, then by maximality $I + J = R$, hence there exist $u \in I$ and $v \in J$ such that $u + v = 1$. Then $v = 1 - u$, so v is a unit and $J = R$. But since J is a proper left ideal, this is a contradiction.