equivalent defining conditions on a Noetherian ring

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Owner	CWoo (3771)
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Author	CWoo (3771)
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Let R be a ring. Then the following are equivalent:

1. every left ideal of R is finitely generated,
2. the ascending chain condition on left ideals holds in R,
3. every non-empty family of left ideals has a maximal element.

Proof. $(1 \Rightarrow 2)$. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of left ideals in R. Let I be the union of all I_j, $j = 1, 2, \ldots$. Then I is a left ideal, and hence finitely generated, by, say, $a_1, \cdots a_n$. Now each a_i belongs to some I_{α_i}. Take the largest of these, say I_{α_k}. Then $a_i \in I_{\alpha_k}$ for all $i = 1, \ldots, n$, and therefore $I \subseteq I_{\alpha_k}$. But $I_{\alpha_k} \subseteq I$ by the definition of I, the equality follows.

$(2 \Rightarrow 3)$. Let $\mathcal{S}$ be a non-empty family of left ideals in R. Since $\mathcal{S}$ is non-empty, take any left ideal $I_1 \in \mathcal{S}$. If I_1 is maximal, then we are done. If not, $\mathcal{S} - \{I_1\}$ must be non-empty, such that pick I_2 from this collection so that $I_1 \subseteq I_2$ (we can find such I_2, for otherwise I_1 would be maximal). If I_2 is not maximal, pick I_3 from $\mathcal{S} - \{I_1, I_2\}$ such that $I_1 \subseteq I_2 \subseteq I_3$, and so on. By assumption, this can not go on indefinitely. So for some positive integer n, we have $I_n = I_m$ for all $m \geq n$, and I_n is our desired maximal element.

$(3 \Rightarrow 1)$. Let I be a left ideal in R. Let $\mathcal{S}$ be the family of all finitely generated ideals of R contained in I. $\mathcal{S}$ is non-empty since (0) is in it. By assumption $\mathcal{S}$ has a maximal element J. If $J \neq I$, then take an element $a \in I - J$. Then $\langle J, a \rangle$ is finitely generated and contained in I, so an element of $\mathcal{S}$, contradicting the maximality of J. Hence $J = I$, in other words, I is finitely generated. $\square$

A ring satisfying any, and hence all three, of the above conditions is defined to be a left Noetherian ring. A right Noetherian ring is similarly defined.