examples of radicals of ideals in commutative rings

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Owner	joking (16130)
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Let R be a commutative ring. Recall, that ideals I, J in R are called *coprime* iff $I + J = R$. It can be shown, that if I, J are coprime, then $IJ = I \cap J$. Elements $x_1, \dots, x_n \in R$ are called *pairwise coprime* iff $(x_i) + (x_j) = R$ for $i \neq j$. It follows by induction, that for pairwise coprime $x_1, \dots, x_n \in R$ we have $(x_1 \cdots x_n) = (x_1) \cap \cdots \cap (x_n)$,

Let $x \in R$ be such that

$$x = p_1^{\alpha_1} \cdots p_n^{\alpha_n},$$

for some prime elements $p_i \in R$, $\alpha_i \in \mathbb{N}$ and assume that $p_1, \dots, p_n$ are coprime. Denote by

$$\overline{x} = p_1 \cdots p_n.$$

We shall denote by $r(I)$ the radical of an ideal $I \subseteq R$.

Proposition. $r\big((x)\big) = (\overline{x})$.

Proof. ,,$\supseteq$" Let $\alpha = \max(\alpha_1, \dots, \alpha_n)$. Then we have

$$\overline{x}^\alpha = (p_1 \cdots p_n)^\alpha = p_1^\alpha \cdots p_n^\alpha = p_1^{\alpha-\alpha_1} \cdots p_n^{\alpha-\alpha_n} p_1^{\alpha_1} \cdots p_n^{\alpha_n} = yx$$

and thus $\overline{x}^\alpha \in (x)$. This shows the first inclusion.

,,$\subseteq$" Assume that $y \in r\big((x)\big)$ and $y \neq 0$. Then there is $n \in \mathbb{N}$ such that $y^n \in (x)$. Thus x divides y^n. Of course for any $i \in \{1, \dots, n\}$ we have that p_i divides x. Thus p_i divides y^n and since p_i is prime, we obtain that p_i divides y. Now for $i \neq j$ elements p_i and p_j are coprime, thus $\overline{x}$ divides y and therefore $y \in (\overline{x})$, which completes the proof. □

Remark. If we assume that R is a PID (and thus UFD), then the previous proposition gives us the full characterization of radicals of ideals in R. In particular an ideal in PID is radical if and only if it is generated by an element of the form $p_1 \cdots p_n$, where for $i \neq j$ elements p_i and p_j are not associated primes.

Examples. Consider ring of integers $\mathbb{Z}$. Then we have:

$$r\big((12)\big) = (6);$$

$$r\big((9)\big) = (3);$$

$$r\big((7)\big) = (7);$$

$$r\big((1125)\big) = (15).$$