span

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The *span* of a set of vectors $\overrightarrow{\mathbf{v}}_1, \ldots, \overrightarrow{\mathbf{v_n}}$ of a vector space V over a field K is the set of linear combinations $a_1\overrightarrow{\mathbf{v}}_1 + \cdots + a_n\overrightarrow{\mathbf{v}}_n$ with $a_i \in K$. It is denoted $\mathrm{Sp}(\overrightarrow{\mathbf{v}}_1, \ldots, \overrightarrow{\mathbf{v}}_n)$. More generally, the span of a set S (not necessarily finite) of vectors is the collection of all (finite) linear combinations of elements of S. The span of the empty set is defined to be the singleton consisting of the zero vector $\overrightarrow{\mathbf{0}}$.

For example, the standard basis vectors $\hat{\imath}$ and $\hat{\jmath}$ span $\mathbb{R}^2$ because every vector of $\mathbb{R}^2$ can be represented as a linear combination of $\hat{\imath}$ and $\hat{\jmath}$.

$\mathrm{Sp}(\overrightarrow{\mathbf{v}}_1, \ldots, \overrightarrow{\mathbf{v}}_n)$ is a subspace of V and is the smallest subspace containing $\overrightarrow{\mathbf{v}}_1, \ldots, \overrightarrow{\mathbf{v}}_n$.

Span is both a noun and a verb; a *set of vectors* can span a vector space, and a vector can be *in the span* of a set of vectors.

Checking span: To see whether a vector is *in the span* of other vectors, one can set up an augmented matrix, since if $\overrightarrow{\mathbf{u}}$ is in the span of $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2$, then $\overrightarrow{\mathbf{u}} = x_1\overrightarrow{\mathbf{v}}_1 + x_2\overrightarrow{\mathbf{v}}_2$. This is a system of linear equations. Thus, if it has a solution, $\overrightarrow{\mathbf{u}}$ is in the span of $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2$. Note that the solution does not have to be unique for $\overrightarrow{\mathbf{u}}$ to be in the span.

To see whether a set of vectors *spans* a vector space, you need to check that there are at least as many linearly independent vectors as the dimension of the space. For example, it can be shown that in $\mathbb{R}^n$, $n+1$ vectors are never linearly independent, and $n-1$ vectors never span.

Remark. We can define the concept of span also for a module M over a ring R. Given a subset $X \subset M$ we define the module generated by X as the set of all finite linear combinations of elements of X. Be aware that in general there does not exist a linearly independent subset which generates the whole module, i.e. there does not have to exist a basis. Also, even if M is generated by n elements, it is in general not true that any other set of n linearly independent elements of M spans M. For example $\mathbb{Z}$ is generated by 1 as a $\mathbb{Z}$-module but not by 2.