subdirect product of rings

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Synonym	subdirect sum
Defines	trivial subdirect product

A ring R is said to be (represented as) a *subdirect product* of a family of rings $\{R_i : i \in I\}$ if:

1. there is a monomorphism $\varepsilon : R \longrightarrow \prod R_i$, and
2. given 1., $\pi_i \circ \varepsilon : R \longrightarrow R_i$ is surjective for each $i \in I$, where $\pi_i : \prod R_i \longrightarrow R_i$ is the canonical projection map.

A *subdirect product* () of R is said to be *trivial* if one of the $\pi_i \circ \varepsilon : R \longrightarrow R_i$ is an isomorphism.

Direct products and direct sums of rings are all examples of subdirect products of rings. $\mathbb{Z}$ does not have non-trivial direct product nor non-trivial direct sum of rings. However, $\mathbb{Z}$ can be represented as a non-trivial subdirect product of $\mathbb{Z}/({p_i}^{n_i})$.

As an application of subdirect products, it can be shown that any ring can be represented as a subdirect product of subdirectly irreducible rings. Since a subdirectly commutative reduced ring is a field, a Boolean ring B can be represented as a subdirect product of $\mathbb{Z}_2$. Furthermore, if this Boolean ring B is finite, the subdirect product becomes a direct product . Consequently, B has 2^n elements, where n is the number of copies of $\mathbb{Z}_2$.