# Hamiltonian quaternions


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**Definition of $\mathbb{H}$**

We define a unital associative algebra $\mathbb{H}$ over $\mathbb{R}$, of dimension 4, by the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the multiplication table

$$\begin{array}{rrrr} 1 & i & j & k \\ i & -1 & k & -j \\ j & -k & -1 & i \\ k & j & -i & -1 \end{array}$$

(where the element in row $x$ and column $y$ is $xy$, not $yx$). Thus an arbitrary element of $\mathbb{H}$ is of the form

$$a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \qquad a, b, c, d \in \mathbb{R}$$

(sometimes denoted by $\langle a, b, c, d\rangle$ or by $a + \langle b, c, d\rangle$) and the product of two elements $\langle a, b, c, d\rangle$ and $\langle \alpha, \beta, \gamma, \delta\rangle$ (order matters) is $\langle w, x, y, z\rangle$ where

$$\begin{array}{rcl} w & = & a\alpha - b\beta - c\gamma - d\delta \\ x & = & a\beta + b\alpha + c\delta - d\gamma \\ y & = & a\gamma - b\delta + c\alpha + d\beta \\ z & = & a\delta + b\gamma - c\beta + d\alpha \end{array}$$

The elements of $\mathbb{H}$ are known as *Hamiltonian quaternions*.

Clearly the subspaces of $\mathbb{H}$ generated by $\{\mathbf{1}\}$ and by $\{\mathbf{1}, \mathbf{i}\}$ are subalgebras isomorphic to $\mathbb{R}$ and $\mathbb{C}$ respectively. $\mathbb{R}$ is customarily identified with the corresponding subalgebra of $\mathbb{H}$. (We shall see in a moment that there are other and less obvious embeddings of $\mathbb{C}$ in $\mathbb{H}$.) The real numbers commute with all the elements of $\mathbb{H}$, and we have

$$\lambda \cdot \langle a, b, c, d\rangle = \langle \lambda a, \lambda b, \lambda c, \lambda d\rangle$$

for $\lambda \in \mathbb{R}$ and $\langle a, b, c, d\rangle \in \mathbb{H}$.

**Norm, conjugate, and inverse of a quaternion**

Like the complex numbers ($\mathbb{C}$), the quaternions have a natural involution called the quaternion conjugate. If $q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, then the quaternion conjugate of $q$, denoted $\overline{q}$, is simply $\overline{q} = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$.

One can readily verify that if $q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, then $q\overline{q} = (a^2 + b^2 + c^2 + d^2)\mathbf{1}$. (See Euler four-square identity.) This product is used to form a norm $\|\cdot\|$ on the algebra (or the ring) $\mathbb{H}$: We define $\|q\| = \sqrt{s}$ where $q\overline{q} = s\mathbf{1}$.

If $v, w \in \mathbb{H}$ and $\lambda \in \mathbb{R}$, then

1. $\|v\| \geq 0$ with equality only if $v = \langle 0,0,0,0\rangle = 0$

2. $\|\lambda v\| = |\lambda| \|v\|$

3. $\|v + w\| \leq \|v\| + \|w\|$

4. $\|v \cdot w\| = \|v\| \cdot \|w\|$

which means that $\mathbb{H}$ qualifies as a normed algebra when we give it the norm $\|\cdot\|$.

Because the norm of any nonzero quaternion $q$ is real and nonzero, we have

$$\frac{q\overline{q}}{\|q\|^2} = \frac{\overline{q}q}{\|q\|^2} = \langle 1,0,0,0\rangle$$

which shows that any nonzero quaternion has an inverse:

$$q^{-1} = \frac{\overline{q}}{\|q\|^2} \ .$$

**Other embeddings of $\mathbb{C}$ into $\mathbb{H}$**

One can use any non-zero $q$ to define an embedding of $\mathbb{C}$ into $\mathbb{H}$. If $\mathbf{n}(z)$ is a natural embedding of $z \in \mathbb{C}$ into $\mathbb{H}$, then the embedding:

$$z \to q\mathbf{n}(z)q^{-1}$$

is also an embedding into $\mathbb{H}$. Because $\mathbb{H}$ is an associative algebra, it is obvious that:

$$(q\mathbf{n}(a)q^{-1})(q\mathbf{n}(b)q^{-1}) = q(\mathbf{n}(a)\mathbf{n}(b))q^{-1}$$

and with the distributive laws, it is easy to check that

$$(q\mathbf{n}(a)q^{-1}) + (q\mathbf{n}(b)q^{-1}) = q(\mathbf{n}(a) + \mathbf{n}(b))q^{-1}$$

**Rotations in 3-space**

Let us write

$$U = \{q \in \mathbb{H} : ||q|| = 1\}$$

With multiplication, $U$ is a group. Let us briefly sketch the relation between $U$ and the group $SO(3)$ of rotations (about the origin) in 3-space.

An arbitrary element $q$ of $U$ can be expressed $\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$, for some real numbers $\theta, a, b, c$ such that $a^2 + b^2 + c^2 = 1$. The permutation $v \mapsto qv$ of $U$ thus gives rise to a permutation of the real sphere. It turns out that that permutation is a rotation. Its axis is the line through $(0, 0, 0)$ and $(a, b, c)$, and the angle through which it rotates the sphere is $\theta$. If rotations $F$ and $G$ correspond to quaternions $q$ and $r$ respectively, then clearly the permutation $v \mapsto qrv$ corresponds to the composite rotation $F \circ G$. Thus this mapping of $U$ onto $SO(3)$ is a group homomorphism. Its kernel is the subset $\{1, -1\}$ of $U$, and thus it comprises a double cover of $SO(3)$. The kernel has a geometric interpretation as well: two unit vectors in opposite directions determine the same axis of rotation.

On the algebraic side, the quaternions provide an example of a division ring that is not a field.