planetmath.org

Math for the people, by the people.

The Hamiltonian ring is not a complex algebra

Canonical name	TheHamiltonianRingIsNotAComplexAlgebra
Date of creation	2013-03-22 16:01:57
Last modified on	2013-03-22 16:01:57
Owner	Algeboy (12884)
Last modified by	Algeboy (12884)
Numerical id	10
Author	Algeboy (12884)
Entry type	Result
Classification	msc 16W99

The http://planetmath.org/QuaternionAlgebra2Hamiltonian algebra $\mathbb{H}$ contains isomorphic copies of the real $\mathbb{R}$ and complex $\mathbb{C}$ numbers. However, the reals are a central subalgebra of $\mathbb{H}$ which makes $\mathbb{H}$ into a real algebra. This makes identifying $\mathbb{R}$ in $\mathbb{H}$ canonical: $1 \in \mathbb{H}$ determines a unique embedding $\mathbb{R} \to \mathbb{H} : r \mapsto r1$. Yet $\mathbb{H}$ is not a complex algebra. The goal presently is to outline some of the incongruities of $\mathbb{C} = \langle 1, i \rangle$ and $\mathbb{H} = \langle 1, \hat{\imath}, \hat{\jmath}, \hat{k} \rangle$ which may be obscured by the notational overlap of the letter i.

Proposition 1. *There are no proper finite dimensional division rings over algebraically closed fields.*

Proof. Let D be a finite dimensional division ring over an algebraically closed field K. This means that K is a central subalgebra of D. Let $a \in D$ and consider $K(a)$. Since K is central in D, $K(a)$ is commutative, and so $K(a)$ is a field extension of K. But as D is a finite dimensional K space, so is $K(a)$. As any finite dimensional extension of K is algebraic, $K(a)$ is an algebraic extension. Yet K is algebraically closed so $K(a) = K$. Thus $a \in K$ so in fact $D = K$. $\square$

- In particular, this proposition proves $\mathbb{H}$ is not a complex algebra.
- Alternatively, from the Wedderburn-Artin theorem we know the only semisimple complex algebra of dimension 2 is $\mathbb{C} \oplus \mathbb{C}$. This has proper ideals and so it cannot be the division ring $\mathbb{H}$.
- It is also evident that the usual, notationally driven, embedding of $\mathbb{C}$ into $\mathbb{H}$ is non-central. That is, $\mathbb{C}$ embeds as $a + bi \mapsto a + b\hat{i}$, into $\mathbb{H} = \langle 1, \hat{\imath}, \hat{\jmath}, \hat{k} \rangle$. This is not central:

$$(1 + \hat{\imath})\hat{\jmath} = \hat{\jmath} + \hat{k} \neq \hat{\jmath}(1 + \hat{\imath}) = \hat{\jmath} - \hat{k}.$$

- Further evidence of the incompatiblity of $\mathbb{H}$ and $\mathbb{C}$ comes from considering polynomials. If x^2+1 is considered as a polynomial over $\mathbb{C}[x]$ then it has exactly two roots $i, -i$ as expected. However, if it is considered as a polynomial over $\mathbb{H}[x]$ we arrive at 6 obvious roots: $\{\hat{\imath}, -\hat{\imath}, \hat{\jmath}, -\hat{\jmath}, \hat{k}, -\hat{k}\}$. But indeed, given any $q \in \mathbb{H}$, $q \neq 0$, then $q\hat{\imath}q^{-1}$ is also a root. Thus there are an infinite number of roots to $x^2 + 1$. Therefore declaring $\hat{\imath} = \sqrt{-1}$ can be greatly misleading. Such a conflict does not arise for polynomials with real roots since $\mathbb{R}$ is a central subalgebra.