example of reducible and irreducible G-modules

Canonical name	ExampleOfReducibleAndIrreducibleGmodules
Date of creation	2013-03-22 16:37:50
Last modified on	2013-03-22 16:37:50
Owner	rm50 (10146)
Last modified by	rm50 (10146)
Numerical id	6
Author	rm50 (10146)
Entry type	Example
Classification	msc 16D60
Defines	augmentation

Let $G = S_r$, the permutation group on r elements, and $N = k^r$ where k is an arbitrary field. Consider the permutation representation of G on N given by

$$\sigma(a_1, \ldots, a_r) = (a_{\sigma(1)}, \ldots, a_{\sigma(r)}), \ \sigma \in S_r, a_i \in k$$

If $r > 1$, we can define two submodules of N, called the *trace* and *augmentation*, as

$$N' = \{(a, a, \ldots, a)\}$$
$$N'' = \{(a_1, a_2, \ldots, a_r) \mid \sum a_i = 0\}$$

Clearly both N' and N'' are stable under the action of G and thus in fact form submodules of N.

If the characteristic of k divides r, then obviously $N'' \supset N'$. Otherwise, N'' is a simple (irreducible) G-module. For suppose N'' has a nontrivial submodule M, and choose a nonzero $u \in M$. Then some pair of coordinates of u are unequal, for if not, then $u = (a, \ldots, a)$ and then $u \notin N''$ because of the restriction on the characteristic of k forces $ra \neq 0$. So apply a suitable element of G to get another element of M, $v = (b_1, b_2, \ldots, b_r)$ where $b_1 \neq b_2$ (note here that we use the fact that M is a submodule and thus is stable under the action of G).

But now $(12)v - ev = (b_1 - b_2, b_2 - b_1, 0, \ldots, 0)$ is also in M, so $w = (1, -1, 0, \ldots, 0) \in M$. It is obvious that by multiplying w by elements of k and by permuting, we can obtain any element of N'' and thus $M = N''$. Thus N'' is simple.

It is also obvious that $N = N' \oplus N''$.