product of injective modules is injective

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Proposition. Let R be a ring and $\{Q_i\}_{i\in I}$ a family of injective R-modules. Then the product

$$Q = \prod_{i\in I} Q_i$$

is injective.

Proof. Let B be an arbitrary R-module, $A \subseteq B$ a submodule and $f : A \to Q$ a homomorphism. It is enough to show that f can be extended to B. For $i \in I$ denote by $\pi_i : Q \to Q_i$ the projection. Since Q_i is injective for any i, then the homomorphism $\pi_i \circ f : A \to Q_i$ can be extended to $f'_i : B \to Q_i$. Then we have

$$f' : B \to Q;$$

$$f'(b) = \big(f'_i(b)\big)_{i\in I}.$$

It is easy to check, that if $a \in A$, then $f'(a) = f(a)$, so f' is an extension of f. Thus Q is injective. $\square$

Remark. Unfortunetly direct sum of injective modules need not be injective. Indeed, there is a theorem which states that direct sums of injective modules are injective if and only if ring R is Noetherian. Note that the proof presented above cannot be used for direct sums, because $f'(b)$ need not be an element of the direct sum, more precisely, it is possible that $f'_i(b) \neq 0$ for infinetly many $i \in I$. Nevertheless products are always injective.