example of free module with bases of diffrent cardinality

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Let k be a field and V be an infinite dimensional vector space over k. Let $\{e_i\}_{i\in I}$ be its basis. Denote by $R = \mathrm{End}(V)$ the ring of endomorphisms of V with standard addition and composition as a multiplication.

Let J be any set such that $|J| \leq |I|$.

Proposition. R and $\prod_{j\in J} R$ are isomorphic as a R-modules.

Proof. Let $\alpha : I \to J \times I$ be a bijection (it exists since $|I| \geq |J|$ and I is infinite) and denote by $\pi_1 : J \times I \to J$ and $\pi_2 : J \times I \to I$ the projections. Moreover let $\delta_1 = \pi_1 \circ \alpha$ and $\delta_2 = \pi_2 \circ \alpha$.

Recall that $\prod_{j\in J} R = \{f : J \to R\}$ (with obvious R-module structure) and define a map $\phi : \prod_{j\in J} R \to R$ by defining the endomorphism $\phi(f) \in R$ for $f \in \prod_{j\in J} R$ as follows:

$$\phi(f)(e_i) = f(\delta_1(i))(e_{\delta_2(i)}).$$

We will show that ϕ is an isomorphism. It is easy to see that ϕ is a R-module homomorphism. Therefore it is enough to show that ϕ is injective and surjective.

1) Recall that ϕ is injective if and only if $\ker(\phi) = 0$. So assume that $\phi(f) = 0$ for $f \in \prod_{j\in J} R$. Note that $f = 0$ if and only if $f(j) = 0$ for all $j \in J$ and this is if and only if $f(j)(e_i) = 0$ for all $j \in J$ and $i \in I$. So take any $(j, i) \in J \times I$. Then (since α is bijective) there exists $i_0 \in I$ such that $\alpha(i_0) = (j, i)$. It follows that $\delta_1(i_0) = j$ and $\delta_2(i_0) = i$. Thus we have

$$0 = \phi(f)(e_{i_0}) = f(\delta_1(i_0))(e_{\delta_2(i_0)}) = f(j)(e_i).$$

Since j and i were arbitrary, then $f = 0$ which completes this part.

2) We wish to show that ϕ is onto, so take any $h \in R$. Define $f \in \prod_{j\in J} R$ by the following formula:

$$f(j)(e_i) = h(e_{\alpha^{-1}(j,i)}).$$

It is easy to see that $\phi(f) = h$. □

Corollary. For any two numbers $n, m \in \mathbb{N}$ there exists a ring R and a free module M such that M has two bases with cardinality n, m respectively.

Proof. It follows from the proposition, that for $R = \mathrm{End}(V)$ we have

$$R^n \simeq R \simeq R^m.$$

For finite set J module $\prod_{j \in J} R$ is free with basis consisting $|J|$ elements (product is the same as direct sum). Therefore (due to existence of previous isomorphisms) R-module R has two bases, one of cardinality n and second of cardinality m. $\square$