exact sequences for modules with finite projective dimension

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Proposition. Let R be a ring and M be a (left) R-module, such that proj $\dim(M) = n < \infty$. If

$$0 \to K \to P_n \to \cdots \to P_0 \to M \to 0$$

is an exact sequence of R-modules, such that each P_i is projective, then K is projective.

Proof. Since proj $\dim(M) = n < \infty$, then there exists exact sequence of R-modules

$$0 \to P'_n \to \cdots \to P'_0 \to M \to 0,$$

Note that sequences

$$P_n \to \cdots \to P_0 \to M \to 0;$$

$$P'_n \to \cdots \to P_0 \to M \to 0,$$

are projective resolutions of M. Let $\delta : P_n \to P_{n-1}$ and $\beta : P'_n \to P'_{n-1}$ be maps take from these resolutions. Then generalized Schanuel's lemma implies that $\ker\delta$ and $\ker\beta$ are projectively equivalent. But $\ker\delta \simeq K$ and $\ker\beta = 0$. This means, that there are projective modules P, Q such that

$$K \oplus P \simeq Q.$$

Therefore K is a direct summand of a free module (since Q is), which completes the proof. $\square$