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radical theory

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Let $\mathcal{X}$ represent a property which a ring may or may not have. This property may be anything at all: what is important is that for any ring R, the statement "R has property $\mathcal{X}$" is either true or false.

We say that a ring which has the property $\mathcal{X}$ is an $\mathcal{X}$-ring. An ideal I of a ring R is called an $\mathcal{X}$-ideal if, as a ring, it is an $\mathcal{X}$-ring. (Note that this definition only makes sense if rings are not required to have identity elements; otherwise and ideal is not, in general, a ring. Rings are not required to have an identity element in radical theory.)

The property $\mathcal{X}$ is a *radical property* if it satisfies:

1. The class of $\mathcal{X}$-rings is closed under homomorphic images.
2. Every ring R has a largest $\mathcal{X}$-ideal, which contains all other $\mathcal{X}$-ideals of R. This ideal is written $\mathcal{X}(R)$.
3. $\mathcal{X}(R/\mathcal{X}(R)) = 0$.

The ideal $\mathcal{X}(R)$ is called the *$\mathcal{X}$-radical* of R. A ring is called *$\mathcal{X}$-radical* if $\mathcal{X}(R) = R$, and is called *$\mathcal{X}$-semisimple* if $\mathcal{X}(R) = 0$.

If $\mathcal{X}$ is a radical property, then the class of $\mathcal{X}$-rings is also called the class of *$\mathcal{X}$-radical rings.*

The class of $\mathcal{X}$-radical rings is closed under ideal extensions. That is, if A is an ideal of R, and A and R/A are $\mathcal{X}$-radical, then so is R.

Radical theory is the study of radical properties and their interrelations. There are several well-known radicals which are of independent interest in ring theory (See examples – to follow).

The class of all radicals is however very large. Indeed, it is possible to show that any partition of the class of simple rings into two classes $\mathcal{R}$ and $\mathcal{S}$ such that isomorphic simple rings are in the same class, gives rise to a radical $\mathcal{X}$ with the property that all rings in $\mathcal{R}$ are $\mathcal{X}$-radical and all rings in $\mathcal{S}$ are $\mathcal{X}$-semisimple. In fact, there are at least two distinct radicals for each such partition.

A radical $\mathcal{X}$ is *hereditary* if every ideal of an $\mathcal{X}$-radical ring is also $\mathcal{X}$-radical.

A radical $\mathcal{X}$ is *supernilpotent* if the class of $\mathcal{X}$-rings contains all nilpotent rings.

1 Examples

Nil is a radical property. This property defines the nil radical, $\mathcal{N}$.

Nilpotency is not a radical property.

Quasi-regularity is a radical property. The associated radical is the Jacobson radical, $\mathcal{J}$.