quotient ring

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Definition. Let R be a ring and let I be a `http://planetmath.org/Ideal`two-sided ideal of R. To define the quotient ring R/I, let us first define an equivalence relation in R. We say that the elements $a, b \in R$ are equivalent, written as $a \sim b$, if and only if $a - b \in I$. If a is an element of R, we denote the corresponding equivalence class by $[a]$. Thus $[a] = [b]$ if and only if $a - b \in I$. The *quotient ring* of R modulo I is the set $R/I = \{[a] \,|\, a \in R\}$, with a ring structure defined as follows. If $[a], [b]$ are equivalence classes in R/I, then

- $[a] + [b] = [a + b]$,
- $[a] \cdot [b] = [a \cdot b]$.

Here a and b are some elements in R that represent $[a]$ and $[b]$. By construction, every element in R/I has such a representative in R. Moreover, since I is closed under addition and multiplication, one can verify that the ring structure in R/I is well defined.

A common notation is $a + I = [a]$ which is consistent with the notion of classes $[a] = aH \in G/H$ for a group G and a normal subgroup H.

Properties

1. If R is commutative, then R/I is commutative.
2. The mapping $R \to R/I$, $a \mapsto [a]$ is a homomorphism, and is called the `http://planetmath.org/NaturalHomomorphism`natural homomorphism.

Examples

1. For a ring R, we have $R/R = \{[0]\}$ and $R/\{0\} = R$.
2. Let $R = \mathbb{Z}$, and let $I = 2\mathbb{Z}$ be the set of even numbers. Then R/I contains only two classes; one for even numbers, and one for odd numbers. Actually this quotient ring is a field. It is the only field with two elements (up to isomorphy) and is also denoted by $\mathbb{F}_2$.
3. One way to construct complex numbers is to consider the field $\mathbb{R}[T]/(T^2 + 1)$. This field can viewed as the set of all polynomials of degree 1 with normal addition and $(a + bT)(c + dT) = ac - bd + (ad + bc)T$, which is like complex multiplication.