# dual of a coalgebra is an algebra, the

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| Canonical name | DualOfACoalgebraIsAnAlgebraThe |
| Date of creation | 2013-03-22 16:34:20 |
| Last modified on | 2013-03-22 16:34:20 |
| Owner | mps (409) |
| Last modified by | mps (409) |
| Numerical id | 8 |
| Author | mps (409) |
| Entry type | Derivation |
| Classification | msc 16W30 |
| Related topic | GrassmanHopfAlgebrasAndTheirDualCoAlgebras |
| Related topic | DualityInMathematics |
| Related topic | QuantumGroups |
| Defines | dualities of algebraic structures |

Let $R$ be a commutative ring with unity. Suppose we have a coassociative coalgebra $(C, \Delta)$ and an associative algebra $A$, both over $R$. Since $C$ and $A$ are both $R$-modules, it follows that $\mathrm{Hom}_R(C, A)$ is also an $R$-module. But in fact we can give it the structure of an associative $R$-algebra. To do this, we use the convolution product. Namely, given morphisms $f$ and $g$ in $\mathrm{Hom}_R(C, A)$, we define their product $fg$ by

$$(fg)(x) = \sum_x f(x_{(1)}) \cdot g(x_{(2)}),$$

where we use the Sweedler notation

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$$

for the comultiplication $\Delta$. To see that the convolution product is associative, suppose $f$, $g$, and $h$ are in $\mathrm{Hom}_R(C, A)$. By applying the coassociativity of $\Delta$, we may write

$$((fg)h)(x) = \sum_x (f(x_{(1)}) \cdot g(x_{(2)})) \cdot h(x_{(3)})$$

and

$$(f(gh))(x) = \sum_x f(x_{(1)}) \cdot (g(x_{(2)})) \cdot h(x_{(3)}).$$

Since $A$ has an associative product, it follows that $(fg)h = f(gh)$.

In the foregoing, we have not assumed that $C$ is counitary or that $A$ is unitary. If $C$ is counitary with counit $\varepsilon\colon C \to R$ and $A$ is unitary with identity $1\colon R \to A$, then their composition $1 \circ \varepsilon\colon C \to A$ is the identity for the convolution product.

**Example.** Let $C$ be a coassociative coalgebra over $R$. Then $R$ itself is an associative $R$-algebra. The algebra $\mathrm{Hom}_R(C, R)$ is called the *algebra dual to the coalgebra $C$*.

We have seen that any coalgebra dualizes to give an algebra. One might expect that a similar construction could be performed on $\mathrm{Hom}_R(A, R)$ to give a coalgebra dual to $A$. However, this is not the case. Thus coalgebras (based on "factoring") are more fundamental than algebras (based on "multiplying").

(The proof will be provided at a later stage).

**Remark on Al/gebraic Duality**–*Mirror or tangled 'duality' of algebras and 'gebras':*
An interesting twist to duality was provided in Fauser's publications on al/gebras where mirror or tangled 'duality' has been defined for Grassman-Hopf al/gebras. Thus, an algebra not only has the usual reversed arrow dual coalgebra but a mirror (or tangled) gebra which is quite distinct from the coalgebra.

**Note:** The dual of a quantum group is a Hopf algebra.

# References


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