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subdirectly irreducible ring

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A ring R is said to be *subdirectly irreducible* if every subdirect product of R is trivial.

Equivalently, a ring R is *subdirectly irreducible* iff the intersection of *all* non-zero ideals of R is non-zero.

Proof. Let $\{I_i\}$ be the set of all non-zero ideals of R.

($\Rightarrow$). Suppose first that R is subdirectly irreducible. If $\bigcap I_i = 0$, then R is a subdirect product of $R_i := R/I_i$, for $\epsilon : R \to \prod R_i$ given by $\epsilon(r)(i) = r + I_i$ is injective. If $\epsilon(r) = 0$, then $r \in I_i$ for all i, or $r \in \bigcap I_i = 0$, or $r = 0$. But then $R \to \prod R_i \to R_i$ given by $r \mapsto r + I_i$ is not an isomorphism for any i, contradicting the fact that R is subdirectly irreducible. Therefore, $\bigcap I_i \neq 0$.

($\Leftarrow$). Suppose next that $\bigcap I_i \neq 0$. Let R be a subdirect product of some R_i, and let $J_i := \ker(R \to \prod R_i \to R_i)$. Each J_i is an ideal of R. Let $J = \bigcap J_i$. If $R \to \prod R_i \to R_i$ is not an isomorphism (therefore not injective), J_i is non-zero. This means that if R is not subdirectly irreducible, $J \neq 0$. But $J \subseteq \ker(R \to \prod R_i)$, contradicting the subdirect irreducibility of R. As a result, some $J_i = 0$, or $R \to \prod R_i \to R_i$ is an isomorphism. $\square$

As an application of the above equivalence, we have that a simple ring is subdirectly irreducible. In addition, a commutative subdirectly irreducible reduced ring is a field. To see this, let $\{I_i\}$ be the set of all non-zero ideals of a commutative subdirectly irreducible reduced ring R, and let $I = \bigcap I_i$. So $I \neq 0$ by subdirect irreducibility. Pick $0 \neq s \in I$. Then $s^2R \subseteq sR \subseteq I$. So $s^2R = sR$ since I is minimal. This means $s = s^2t$, or $1 = st \in sR = I$, which means $I = R$. Now, let any $0 \neq r \in R$, then $R = I \subseteq rR$, so $1 = pr$ for some $p \in R$, which means R is a field.