supernumber

Canonical name	Supernumber
Date of creation	2013-03-22 13:03:27
Last modified on	2013-03-22 13:03:27
Owner	mhale (572)
Last modified by	mhale (572)
Numerical id	12
Author	mhale (572)
Entry type	Definition
Classification	msc 16W55
Related topic	SuperAlgebra
Defines	body
Defines	soul

Supernumbers are the generalisation of complex numbers to a commutative superalgebra of commuting and anticommuting "numbers". They are primarily used in the description of in .

Let Λ_N be the Grassmann algebra generated by θ^i, $i = 1 \dots N$, such that $\theta^i \theta^j = -\theta^j \theta^i$ and $(\theta^i)^2 = 0$. Denote by Λ_∞, the Grassmann algebra of an infinite number of generators θ^i. A **supernumber** is an element of Λ_N or Λ_∞.

Any supernumber z can be expressed uniquely in the form

$$z = z_0 + z_i \theta^i + \frac{1}{2} z_{ij} \theta^i \theta^j + \dots + \frac{1}{n!} z_{i_1 \dots i_n} \theta^{i_1} \dots \theta^{i_n} + \dots,$$

where the coefficients $z_{i_1 \dots i_n} \in \mathbb{C}$ are antisymmetric in their indices.

1 Body and soul

The **body** of a supernumber z is defined as $z_\mathrm{B} = z_0$, and its **soul** is defined as $z_\mathrm{S} = z - z_\mathrm{B}$. If $z_\mathrm{B} \neq 0$ then z has an inverse given by

$$z^{-1} = \frac{1}{z_\mathrm{B}} \sum_{k=0}^{\infty} \left(-\frac{z_\mathrm{S}}{z_\mathrm{B}} \right)^k .$$

2 Odd and even

A supernumber can be decomposed into the even and odd parts:

$$\begin{aligned}
z_\mathrm{even} &= z_0 + \frac{1}{2} z_{ij} \theta^i \theta^j + \dots + \frac{1}{(2n)!} z_{i_1 \dots i_{2n}} \theta^{i_1} \dots \theta^{i_{2n}} + \dots, \\
z_\mathrm{odd} &= z_i \theta^i + \frac{1}{6} z_{ijk} \theta^i \theta^j \theta^k + \dots + \frac{1}{(2n+1)!} z_{i_1 \dots i_{2n+1}} \theta^{i_1} \dots \theta^{i_{2n+1}} + \dots .
\end{aligned}$$

Even supernumbers commute with each other and are called **c-numbers**, while odd supernumbers anticommute with each other and are called **a-numbers**. Note, the product of two c-numbers is even, the product of a c-number and an a-number is odd, and the product of two a-numbers is even. The superalgebra Λ_N has the vector space decomposition $\Lambda_N = \mathbb{C}_c \oplus \mathbb{C}_a$, where $\mathbb{C}_c$ is the space of c-numbers, and $\mathbb{C}_a$ is the space of a-numbers.

3 Conjugation and involution

There are two ways, one can define a complex conjugation for supernumbers. The first is to define a linear conjugation in complete analogy with complex numbers:

$$\overline{(z_1 z_2)} = \overline{z_1}\;\overline{z_2}.$$

The second way is to define an anti-linear involution:

$$(z_1 z_2)^* = z_2^* z_1^*.$$

The comes down to whether the product of two real odd supernumbers is real or imaginary.