a ring modulo its Jacobson radical is semiprimitive

Canonical name	ARingModuloItsJacobsonRadicalIsSemiprimitive
Date of creation	2013-03-22 12:49:34
Last modified on	2013-03-22 12:49:34
Owner	yark (2760)
Last modified by	yark (2760)
Numerical id	13
Author	yark (2760)
Entry type	Theorem
Classification	msc 16N20

Let R be a ring. Then $J(R/J(R)) = (0)$.

Proof:

We will only prove this in the case where R is a unital ring (although it is true without this assumption).

Let $[u] \in J(R/J(R))$. By one of the characterizations of the Jacobson radical, $1 - [r][u]$ is left invertible for all $r \in R$, so there exists $v \in R$ such that $[v](1 - [r][u]) = 1$.

Then $v(1 - ru) = 1 - a$ for some $a \in J(R)$. There is a $w \in R$ such that $w(1 - a) = 1$, and we have $wv(1 - ru) = 1$.

Since this holds for all $r \in R$, it follows that $u \in J(R)$, and therefore $[u] = 0$.