all bases for a vector space have the same cardinality

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In this entry, we want to show the following property of bases for a vector space:

Theorem 1. *All bases for a vector space V have the same cardinality.*

Let B be a basis for V (B exists, see `http://planetmath.org/ZornsLemmaAndBasesForVectorS` link). If B is infinite, then all bases for V have the same cardinality as that of B (`http://planetmath.org/CardinalitiesOfBasesForModules`proof). So all we really need to show is where V has a finite basis.

Before proving this important property, we want to prove something that is almost as important:

Lemma 1. *If A and B are subsets of a vector space V such that A is linearly independent and B spans V, then $|A| \leq |B|$.*

Proof. If A is finite and B is infinite, then we are done. Suppose now that A is infinite. Since A is linearly independent, there is a superset C of A that is a basis for V. Since A is infinite, so is C, and therefore all bases for V are infinite, and have the same cardinality as that of C. Since B spans V, there is a subset D of B that is a basis for V. As a result, we have $|A| \leq |C| = |D| \leq |B|$.

Now, we suppose that A and B are both finite. The case where $A = \varnothing$ is clear. So assume $A \neq \varnothing$. As B spans V, $B \neq \varnothing$. Let $A = \{a_1, \ldots, a_n\}$ and

$$B = \{b_1, \ldots, b_m\}$$

and assume $m < n$. So $a_i \neq 0$ for all $i = 1, \ldots, n$. Since B spans V, a_1 can be expressed as a linear combination of elements of B. In this expression, at least one of the coefficients (in the field k) can not be 0 (or else $a_1 = 0$). Rename the elements if possible, so that b_1 has a non-zero coefficient in the expression of a_1. This means that b_1 can be written as a linear combination of a and the remaining b's. Set

$$B_1 = \{a_1, b_2, \ldots, b_m\}.$$

As every element in V is a linear combination of elements of B, it is therefore a linear combination of elements of B_1. Thus, B_1 spans V. Next, express a_2 as a linear combination of elements in B_1. In this expression, if the only non-zero coefficient is in front of a_1, then a_1 and a_2 would be linearly dependent, a contradiction! Therefore, there must be a non-zero coefficient in front of one of the b's, and after some renaming once more, we have that b_2 is the one with a non-zero coefficient. Therefore, b_2, likewise, can be expressed as a linear combination of a_1, a_2 and the remaining b's. It is easy to see that

$$B_2 = \{a_1, a_2, b_3, \ldots, b_m\}$$

spans V as well. Continue this process until all of the b's have been replaced, which is possible since $m < n$. We have finally arrived at the set

$$B_m = \{a_1, \ldots, a_m\}$$

which is a proper subset of A. In addition, B_m spans V. But this would imply that A is linearly dependent, a contradiction. □

Now we can complete the proof of theorem 1.

Proof. Suppose A and B are bases for V. We apply the lemma. Then $|A| \leq |B|$ since A is linearly independent and B spans V. Similarly, $|B| \leq |A|$ since B is linearly independent and A spans V. An application of Schroeder-Bernstein theorem completes the proof. □