chain conditions in vector spaces

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From the theorem in the parent article - that an A-module M has a composition series if and only if it satisfies both chain conditions - it is easy to see that

Theorem 1. *Let k be a field, V a k-vector space. Then the following are equivalent:*

1. *V is finite-dimensional;*
2. *V has a composition series;*
3. *V satisfies the ascending chain condition (acc);*
4. *V satisfies the descending chain condition (dcc).*

Proof. Clearly (1) $\Rightarrow$ (2), since submodules are just subspaces. (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) from the parent article. So it remains to see that (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1). But if V is infinite-dimensional, we can choose a sequence $\{x_i\}_{i\geq 1}$ of linearly independent elements. Let U_n be the subspace spanned by $x_1,\dots,x_n$ and V_n the subspace spanned by $x_{n+1},x_{n+2},\dots$. Then the U_i form a strictly ascending infinite family of subspaces, so V does not satisfy the ascending chain condition; the V_i form a strictly descending infinite family of subspaces, so V does not satisfy the descending chain condition. □

This easily implies the following:

Corollary 1. *Let A be a ring in which $(0)=\mathfrak{m}_1\dots\mathfrak{m}_n$ where the $\mathfrak{m}_i$ are (not necessarily distinct) maximal ideals. Then A is Noetherian if and only if A is Artinian.*

Proof. We have the sequence of ideals

$$A\supset\mathfrak{m}_1\supset\mathfrak{m}_1\mathfrak{m}_2\supset\cdots\supset\mathfrak{m}_1\dots\mathfrak{m}_n=0$$

Each factor $\mathfrak{m}_1\dots\mathfrak{m}_{i-1}/\mathfrak{m}_1\dots\mathfrak{m}_i$ is a vector space over the field $A/\mathfrak{m}_i$. By the above theorem, each quotient satisfies the acc if and only if it satisfies the dcc. But by repeatedly applying the fact that in a short exact sequence, the middle term satisfies the acc (dcc) if and only if both ends do, we see that A satisfies the acc if and only if it satisfies the dcc. □

References

[1] M.F. Atiyah, I.G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley 1969.