rings whose every module is free

Canonical name	RingsWhoseEveryModuleIsFree
Date of creation	2013-03-22 18:50:20
Last modified on	2013-03-22 18:50:20
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	10
Author	joking (16130)
Entry type	Example
Classification	msc 16D40

Recall that if R is a (nontrivial) ring and M is a R-module, then (nonempty) subset $S \subseteq M$ is called linearly independent if for any $m_1, \ldots, m_n \in M$ and any $r_1, \cdots, r_n \in R$ the equality

$$r_1 \cdot m_1 + \ldots + r_n \cdot m_n = 0$$

implies that $r_1 = \ldots = r_n = 0$. If $S \subseteq M$ is a linearly independent subset of generators of M, then S is called a basis of M. Of course not every module has a basis (it even doesn't have to have linearly independent subsets). R-module is called free, if it has basis. In particular if R is a field, then it is well known that every R-module is free. What about the converse?

Proposition. Let R be a unital ring. Then R is a division ring if and only if every left R-module is free.

Proof. „$\Rightarrow$" First assume that R is a divison ring. Then obviously R has only two (left) ideals, namely 0 and R (because every nontrivial ideal contains invertible element and thus it contains 1, so it contains every element of R). Let M be a R-module and $m \in M$ such that $m \neq 0$. Then we have homomorphism of R-modules $f : R \to M$ such that $f(r) = r \cdot m$. Note that $\ker(f) \neq R$ (because $f(1) \neq 0$) and thus $\ker(f) = 0$ (because $\ker(f)$ is a left ideal). It is clear that this implies that $\{m\}$ is linearly independent subset of M. Now let

$$\Lambda = \{P \subseteq M \mid P \text{ is linearly independent}\}.$$

Therefore we proved that $\Lambda \neq \emptyset$. Note that $(\Lambda, \subseteq)$ is a poset (where „$\subseteq$" denotes the inclusion) in which every chain is bounded. Thus we may apply Zorn's lemma. Let $P_0 \in \Lambda$ be a maximal element in Λ. We will show that P_0 is a basis (i.e. P_0 generates M). Assume that $m \in M$ is such that $m \notin P_0$. Then $P_0 \cup \{m\}$ is linearly dependent (because P_0 is maximal) and thus there exist $m_1, \cdots, m_n \in M$ and $\lambda, \lambda_1, \cdots, \lambda_n \in R$ such that $\lambda \neq 0$ and

$$\lambda \cdot m + \lambda_1 \cdot m_1 + \cdots \lambda_n \cdot m_n = 0.$$

Since $\lambda \neq 0$, then λ is invertible in R (because R is a divison ring) and therefore

$$m = (-\lambda^{-1}\lambda_1) \cdot m_1 + \cdots + (-\lambda^{-1}\lambda_n) \cdot m_n.$$

Thus P_0 generates M, so every R-module is free. This completes this implication.

,,$\Leftarrow$" Assume now that every left R-module is free. In particular every left R-module is projective, thus R is semisimple and therefore R is Noetherian. This implies that R has invariant basis number. Let $I \subseteq R$ be a nontrivial left ideal. Thus I is a R-module, so it is free and since all modules are projective (because they are free), then I is direct summand of R. If I is proper, then we have a decomposition of a R-module

$$R \simeq I \oplus I',$$

but rank of R is 1 and rank of $I \oplus I'$ is at least 2. Contradiction, because R has invariant basis number. Thus the only left ideals in R are 0 and R. Now let $x \in R$. Then $Rx = R$, so there exists $\beta \in R$ such that

$$\beta x = 1.$$

Thus every element is left invertible. But then every element is invertible. Indeed, if $\beta x = 1$ then there exist $\alpha \in R$ such that $\alpha\beta = 1$ and thus

$$1 = \alpha\beta = \alpha(\beta x)\beta = (\alpha\beta)x\beta = x\beta,$$

so x is right invertible. Thus R is a divison ring. $\square$

Remark. Note that this proof can be dualized to the case of right modules and thus we obtained that a unital ring R is a divison ring if and only if every right R-module is free.