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grouplike elements in Hopf algebras

Canonical name	GrouplikeElementsInHopfAlgebras
Date of creation	2013-03-22 18:58:39
Last modified on	2013-03-22 18:58:39
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	5
Author	joking (16130)
Entry type	Definition
Classification	msc 16W30

Recall, that if k is a field and G is a group, then the group algebra kG can be turned into a Hopf algebra, by defining comultiplication $\Delta(g) = g \otimes g$, counit $\varepsilon(g) = 1$ and antipode $S(g) = g^{-1}$.

Now let H be a Hopf algebra over a field k, with identity 1, comultiplication Δ, counit ε and antipode S. Recall that element $g \in H$ is called grouplike iff $g \neq 0$ and $\Delta(g) = g \otimes g$. The set of all grouplike elements $G(H)$ is nonempty, because $1 \in G(H)$. Also, since comultiplication is an algebra morphism, then $G(H)$ is multiplicative, i.e. if $g, h \in G(H)$, then $gh \in G(H)$. Furthermore, it can be shown that for any $g \in G(H)$ we have $S(g) \in G(H)$ and $S(g)g = gS(g) = 1$. Thus $G(H)$ is a group under multiplication inherited from H.

It is easy to see, that the vector subspace spanned by $G(H)$ is a Hopf subalgebra of H isomorphic to $kG(H)$. It can be shown that $G(H)$ is always linearly independent, so if H is finite dimensional, then $G(H)$ is a finite group. Also, if H is finite dimensional, then it follows from the Nichols-Zoeller Theorem, that the order of $G(H)$ divides $\dim_k H$.

From these observations it follows that if $\dim_k H = p$ is a prime number, then $G(H)$ is either trivial or the order of $G(H)$ is equal to p (i.e. $G(H)$ is cyclic of order p). The second case implies that H is isomorphic to $k\mathbb{Z}_p$ and it can be shown that the first case cannot occur.