IBN

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Related topic	ExampleOfFreeModuleWithBasesOfDiffrentCardinality
Defines	basis of a module
Defines	finite rank
Defines	rank of a module

Bases of a Module

Like a vector space over a field, one can define a basis of a module M over a general ring R with 1. To simplify matter, suppose R is commutative with 1 and M is unital. A basis of M is a subset $B = \{b_i \mid i \in I\}$ of M, where I is some ordered index set, such that every element $m \in M$ can be uniquely written as a linear combination of elements from B:

$$m = \sum_{i \in I} r_i b_i$$

such that all but a finite number of $r_i = 0$.

As the above example shows, the commutativity of R is not required, and M can be assumed either as a left or right module of R (in the example above, we could take M to be the left R-module).

However, unlike a vector space, a module may not have a basis. If it does, it is a called a *free module*. Vector spaces are examples of free modules over fields or division rings. If a free module M (over R) has a finite basis with cardinality n, we often write R^n as an isomorphic copy of M.

Suppose that we are given a free module M over R, and two bases $B_1 \neq B_2$ for M, is

$$|B_1| = |B_2|?$$

We know that this is true if R is a field or even a division ring. But in general, the equality fails. Nevertheless, it is a fact that if B_1 is finite, so is B_2. So the finiteness of basis in a free module M over R is preserved when we go from one basis to another. When M has a finite basis, we say that M has *finite rank* (without saying what rank is!).

Now, even if M has finite rank, the cardinality of one basis may still be different from the cardinality of another. In other words, R^m may be isomorphic to R^n without m and n being equal.

Invariant Basis Number

A ring R is said to have *IBN*, or *invariant basis number* if whenever $R^m \cong R^n$ where $m, n < \infty$, $m = n$. The positive integer n in this case is called the *rank* of module M. To rephrase, when F is a free R-module of finite rank, then R has IBN iff F has unique finite rank. Also, R has IBN iff all finite dimensional invertible matrices over R are square matrices.

Examples

1. If R is commutative, then R has IBN.
2. If R is a division ring, then R has IBN.
3. An example of a ring R not having IBN can be found as follows: let V be a countably infinite dimensional vector space over a field. Let R be the endomorphism ring over V. Then $R = R \oplus R$ and thus $R^m = R^n$ for any pairs of (m, n).