nil is a radical property

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We must show that the nil property, $\mathcal{N}$, is a radical property, that is that it satisfies the following conditions:

1. The class of $\mathcal{N}$-rings is closed under homomorphic images.
2. Every ring R has a largest $\mathcal{N}$-ideal, which contains all other $\mathcal{N}$-ideals of R. This ideal is written $\mathcal{N}(R)$.
3. $\mathcal{N}(R/\mathcal{N}(R)) = 0$.

It is easy to see that the homomorphic image of a nil ring is nil, for if $f\colon R \to S$ is a homomorphism and $x^n = 0$, then $f(x)^n = f(x^n) = 0$.

The sum of all nil ideals is nil (see proof `http://planetmath.org/node/5650`here), so this sum is the largest nil ideal in the ring.

Finally, if N is the largest nil ideal in R, and I is an ideal of R containing N such that I/N is nil, then I is also nil (see proof `http://planetmath.org/node/5650`here). So $I \subseteq N$ by definition of N. Thus R/N contains no nil ideals.