Weyl algebra

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Abstract definition

Let F be a field and V be an F-vector space with basis $\{P_i\}_{i\in I} \cup \{Q_i\}_{i\in I}$, where I is some non-empty index set. Let T be the tensor algebra of V and let J be the ideal in T generated by the set $\{P_i \otimes Q_j - Q_j \otimes P_i - \delta_{ij}\}_{i,j\in I}$ where δ is the Kronecker delta symbol. Then the quotient T/J is the $|I|$*-th Weyl algebra*.

A more concrete definition

If the field F has characteristic zero we have the following more concrete definition. Let $R := F[\{X_i\}_{i\in I}]$ be the polynomial ring over F in indeterminates X_i labeled by I. For any $i \in I$, let ∂_i denote the partial differential operator with respect to X_i. Then the $|I|$-th Weyl algebra is the set W of all differential operators of the form

$$D = \sum_{|\alpha|\leq n} f_\alpha \partial^\alpha$$

where the summation variable α is a multi-index with $|I|$ entries, n is the degree of D, and $f_\alpha \in R$. The algebra structure is defined by the usual operator multiplication, where the coefficients $f_\alpha \in R$ are identified with the operators of left multiplication with them for conciseness of notation. Since the derivative of a polynomial is again a polynomial, it is clear that W is closed under that multiplication.

The equivalence of these definitions can be seen by replacing the generators Q_i with left multiplication by the indeterminates X_i, the generators P_i with the partial differential operator ∂_i, and the tensor product with operator multiplication, and observing that $\partial_i X_j - X_j \partial_i = \delta_{ij}$. If, however, the characteristic p of F is positive, the resulting homomorphism to W is not injective, since for example the expressions ∂_i^p and X_i^n commute, while $P_i^{\otimes p}$ and $Q_i^{\otimes n}$ do not.