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## Hopf algebra

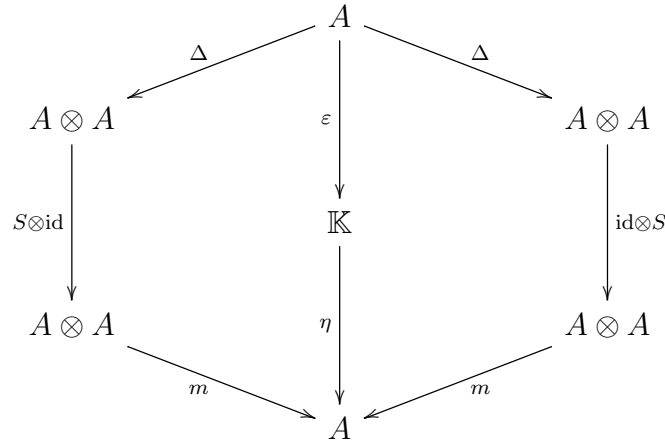
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Defines	antipode

A **Hopf algebra** is a bialgebra  $A$  over a field  $\mathbb{K}$  with a  $\mathbb{K}$ -linear map  $S : A \rightarrow A$ , called the **antipode**, such that

$$m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\text{id} \otimes S) \circ \Delta, \quad (1)$$

where  $m : A \otimes A \rightarrow A$  is the multiplication map  $m(a \otimes b) = ab$  and  $\eta : \mathbb{K} \rightarrow A$  is the unit map  $\eta(k) = k\mathbb{1}$ .

In of a commutative diagram:



The category of commutative Hopf algebras is anti-equivalent to the category of affine group schemes. The prime spectrum of a commutative Hopf algebra is an affine group scheme of multiplicative units. And going in the opposite direction, the algebra of natural transformations from an affine group scheme to its <http://planetmath.org/AffineSpace> affine 1-space is a commutative Hopf algebra, with coalgebra structure given by dualising the group structure of the affine group scheme. Further, a commutative Hopf algebra is a cogroup object in the category of commutative algebras.

**Example 1** (Algebra of functions on a finite group)

Let  $A = C(G)$  be the algebra of complex-valued functions on a finite group  $G$  and identify  $C(G \times G)$  with  $A \otimes A$ . Then,  $A$  is a Hopf algebra with comultiplication  $(\Delta(f))(x, y) = f(xy)$ , counit  $\varepsilon(f) = f(e)$ , and antipode  $(S(f))(x) = f(x^{-1})$ .

**Example 2** (Group algebra of a finite group)

Let  $A = \mathbb{C}G$  be the complex group algebra of a finite group  $G$ . Then,  $A$  is

a Hopf algebra with comultiplication  $\Delta(g) = g \otimes g$ , counit  $\varepsilon(g) = 1$ , and antipode  $S(g) = g^{-1}$ .

The above two examples are dual to one another. Define a bilinear form  $C(G) \otimes \mathbb{C}G \rightarrow \mathbb{C}$  by  $\langle f, x \rangle = f(x)$ . Then,

$$\begin{aligned}\langle fg, x \rangle &= \langle f \otimes g, \Delta(x) \rangle, \\ \langle 1, x \rangle &= \varepsilon(x), \\ \langle \Delta(f), x \otimes y \rangle &= \langle f, xy \rangle, \\ \varepsilon(f) &= \langle f, e \rangle, \\ \langle S(f), x \rangle &= \overline{\langle f, S(x) \rangle}, \\ \langle f^*, x \rangle &= \overline{\langle f, S(x)^* \rangle}.\end{aligned}$$

**Example 3** (Polynomial functions on a Lie group)

Let  $A = \text{Poly}(G)$  be the algebra of complex-valued polynomial functions on a complex Lie group  $G$  and identify  $\text{Poly}(G \times G)$  with  $A \otimes A$ . Then,  $A$  is a Hopf algebra with comultiplication  $(\Delta(f))(x, y) = f(xy)$ , counit  $\varepsilon(f) = f(e)$ , and antipode  $(S(f))(x) = f(x^{-1})$ .

**Example 4** (Universal enveloping algebra of a Lie algebra)

Let  $A = \mathcal{U}(\mathfrak{g})$  be the universal enveloping algebra of a complex Lie algebra  $\mathfrak{g}$ . Then,  $A$  is a Hopf algebra with comultiplication  $\Delta(X) = X \otimes 1 + 1 \otimes X$ , counit  $\varepsilon(X) = 0$ , and antipode  $S(X) = -X$ .

The above two examples are dual to one another (if  $\mathfrak{g}$  is the Lie algebra of  $G$ ). Define a bilinear form  $\text{Poly}(G) \otimes \mathcal{U}(\mathfrak{g}) \rightarrow \mathbb{C}$  by  $\langle f, X \rangle = \left. \frac{d}{dt} \right|_{t=0} f(\exp(tX))$ .