example of algebras and coalgebras which cannot be turned into Hopf algebras

Canonical name	ExampleOfAlgebrasAndCoalgebrasWhichCannotBeTurnedIntoHopfAlgebras
Date of creation	2013-03-22 18:58:45
Last modified on	2013-03-22 18:58:45
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Last modified by	joking (16130)
Numerical id	4
Author	joking (16130)
Entry type	Example
Classification	msc 16W30

Let $H = \mathbb{M}_n(k)$ be a matrix algebra over a field k with standard multiplication and assume that $n > 1$. Assume that H can be turned into a Hopf algebra. In particular, there is $\varepsilon : H \to k$ such that ε is a morphism of algebras. It can be shown that matrix algebra is simple, i.e. if $I \subseteq H$ is a two-sided ideal, then $I = 0$ or $I = H$. Thus we have that $\ker\varepsilon = 0$ (because $\varepsilon(1) = 1$). Contradiction, because $\dim_k H > 1 = \dim_k k$.

Now consider $H = \mathbb{M}^c(n, k)$ a vector space of all $n \times n$ matrices over k. We introduce coalgebra structure on H. Let E_{ij} be a matrix in H with 1 in (i, j) place and 0 everywhere else. Of course $\{E_{ij}\}$ forms a basis of H and it is sufficient to define comultiplication and counit on it. Define

$$\Delta(E_{ij}) = \sum_{p=1}^{n} E_{ip} \otimes E_{pj};$$

$$\varepsilon(E_{ij}) = \delta_{ij},$$

where δ_{ij} denotes Kronecker delta. It can be easily checked, that $\big(\mathbb{M}^c(n, k), \Delta, \varepsilon\big)$ is a coalgebra known as the matrix coalgebra. Also, is well known that the dual algebra $\mathbb{M}^c(n, k)^*$ is isomorphic to the standard matrix algebra.

Now assume that matrix coalgebra $H = \mathbb{M}^c(n, k)$ (where $n > 1$) can be turned into a Hopf algebra. Since H is finite dimensional, then we can take dual Hopf algebra H^*. But the underlaying algebra structure of H^* is isomorphic to a matrix algebra (as we remarked earlier), which we've already shown to be impossible. Thus matrix coalgebra cannot be turned into a Hopf algebra.