Hopf algebra

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A **Hopf algebra** is a bialgebra A over a field $\mathbb{K}$ with a $\mathbb{K}$-linear map $S : A \to A$, called the **antipode**, such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta, \tag{1}$$

where $m : A \otimes A \to A$ is the multiplication map $m(a \otimes b) = ab$ and $\eta : \mathbb{K} \to A$ is the unit map $\eta(k) = k\mathbb{1}$.

In of a commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "\Delta"'] \arrow[dd, "\varepsilon"] \arrow[dr, "\Delta"] & \\
A \otimes A \arrow[dd, "S\otimes\mathrm{id}"'] & & A \otimes A \arrow[dd, "\mathrm{id}\otimes S"] \\
 & \mathbb{K} \arrow[dd, "\eta"] & \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
 & A &
\end{tikzcd}$$

The category of commutative Hopf algebras is anti-equivalent to the category of affine group schemes. The prime spectrum of a commutative Hopf algebra is an affine group scheme of multiplicative units. And going in the opposite direction, the algebra of natural transformations from an affine group scheme to its `http://planetmath.org/AffineSpace`affine 1-space is a commutative Hopf algebra, with coalgebra structure given by dualising the group structure of the affine group scheme. Further, a commutative Hopf algebra is a cogroup object in the category of commutative algebras.

Example 1 (Algebra of functions on a finite group)
Let $A = C(G)$ be the algebra of complex-valued functions on a finite group G and identify $C(G \times G)$ with $A \otimes A$. Then, A is a Hopf algebra with comultiplication $(\Delta(f))(x, y) = f(xy)$, counit $\varepsilon(f) = f(e)$, and antipode $(S(f))(x) = f(x^{-1})$.

Example 2 (Group algebra of a finite group)
Let $A = \mathbb{C}G$ be the complex group algebra of a finite group G. Then, A is

a Hopf algebra with comultiplication $\Delta(g) = g \otimes g$*, counit* $\varepsilon(g) = 1$*, and antipode* $S(g) = g^{-1}$*.*

The above two examples are dual to one another. Define a bilinear form $C(G) \otimes \mathbb{C}G \to \mathbb{C}$ by $\langle f, x\rangle = f(x)$. Then,

$$\begin{aligned}
\langle fg, x\rangle &= \langle f \otimes g, \Delta(x)\rangle, \\
\langle 1, x\rangle &= \varepsilon(x), \\
\langle \Delta(f), x \otimes y\rangle &= \langle f, xy\rangle, \\
\varepsilon(f) &= \langle f, e\rangle, \\
\langle S(f), x\rangle &= \langle f, S(x)\rangle, \\
\langle f^*, x\rangle &= \overline{\langle f, S(x)^*\rangle}.
\end{aligned}$$

Example 3 (Polynomial functions on a Lie group)
Let $A = \mathrm{Poly}(G)$ *be the algebra of complex-valued polynomial functions on a complex Lie group* G *and identify* $\mathrm{Poly}(G \times G)$ *with* $A \otimes A$*. Then,* A *is a Hopf algebra with comultiplication* $(\Delta(f))(x, y) = f(xy)$*, counit* $\varepsilon(f) = f(e)$*, and antipode* $(S(f))(x) = f(x^{-1})$*.*

Example 4 (Universal enveloping algebra of a Lie algebra)
Let $A = \mathcal{U}(\mathfrak{g})$ *be the universal enveloping algebra of a complex Lie algebra* $\mathfrak{g}$*. Then,* A *is a Hopf algebra with comultiplication* $\Delta(X) = X \otimes 1 + 1 \otimes X$*, counit* $\varepsilon(X) = 0$*, and antipode* $S(X) = -X$*.*

The above two examples are dual to one another (if $\mathfrak{g}$ is the Lie algebra of G). Define a bilinear form $\mathrm{Poly}(G) \otimes \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ by $\langle f, X\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(\exp(tX))$.