nested ideals in von Neumann regular ring

Canonical name	NestedIdealsInVonNeumannRegularRing
Date of creation	2013-03-22 14:48:24
Last modified on	2013-03-22 14:48:24
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	12
Author	CWoo (3771)
Entry type	Theorem
Classification	msc 16E50

Theorem. Let $\mathfrak{a}$ be an ideal of the von Neumann regular ring R. Then $\mathfrak{a}$ itself is a von Neumann regular ring and any ideal $\mathfrak{b}$ of $\mathfrak{a}$ is likewise an ideal of R.

Proof. If $a \in \mathfrak{a}$, then $asa = a$ for some $s \in R$. Setting $t = sas$ we see that t belongs to the ideal $\mathfrak{a}$ and

$$ata = a(sas)a = (asa)sa = asa = a.$$

Secondly, we have to show that whenever $b \in \mathfrak{b} \subseteq \mathfrak{a}$ and $r \in R$, then both br and rb lie in $\mathfrak{b}$. Now, $br \in \mathfrak{a}$ because $\mathfrak{a}$ is an ideal of R. Thus there is an element x in $\mathfrak{a}$ satisfying $brxbr = br$. Since $rxbr$ belongs to $\mathfrak{a}$ and $\mathfrak{b}$ is assumed to be an ideal of $\mathfrak{a}$, we conclude that the product $b \cdot rxbr$ must lie in $\mathfrak{b}$, i.e. $br \in \mathfrak{b}$. Similarly it can be shown that $rb \in \mathfrak{b}$. $\square$

References

[1] David M. Burton: *A first course in rings and ideals*. Addison-Wesley. Reading, Massachusetts (1970).