maximal ideal is prime (general case)

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Theorem. In a ring (not necessarily commutative) with unity, any maximal ideal is a prime ideal.

Proof. Let $\mathfrak{m}$ be a maximal ideal of such a ring R and suppose R has ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{m}$, but $\mathfrak{a} \nsubseteq \mathfrak{m}$. Since $\mathfrak{m}$ is maximal, we must have $\mathfrak{a} + \mathfrak{m} = R$. Then,

$$\mathfrak{b} = R\mathfrak{b} = (\mathfrak{a} + \mathfrak{m})\mathfrak{b} = \mathfrak{a}\mathfrak{b} + \mathfrak{m}\mathfrak{b} \subseteq \mathfrak{m} + \mathfrak{m} = \mathfrak{m}.$$

Thus, either $\mathfrak{a} \subseteq \mathfrak{m}$ or $\mathfrak{b} \subseteq \mathfrak{m}$. This demonstrates that $\mathfrak{m}$ is prime.

Note that the condition that R has an identity element is essential. For otherwise, we may take R to be a finite zero ring. Such rings contain no proper prime ideals. As long as the number of elements of R is not prime, R will have a non-zero maximal ideal.