Grothendieck group

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Let S be an abelian semigroup. The **Grothendieck group** of S is $K(S) = S \times S/\sim$, where $\sim$ is the equivalence relation: $(s,t) \sim (u,v)$ if there exists $r \in S$ such that $s + v + r = t + u + r$. This is indeed an abelian group with zero element (s,s) (any $s \in S$), inverse $-(s,t) = (t,s)$ and addition given by $(s,t)+(u,v) = (s+u, t+v)$. It is common to use the suggestive notation $t - s$ for (t,s).

The Grothendieck group construction is a functor from the category of abelian semigroups to the category of abelian groups. A morphism $f\colon S \to T$ induces a morphism $K(f)\colon K(S) \to K(T)$ which sends an element $(s^+, s^-) \in K(S)$ to $(f(s^+), f(s^-)) \in K(T)$.

Example 1
Let $(\mathbb{N}, +)$ *be the semigroup of natural numbers with composition given by addition. Then,* $K(\mathbb{N}, +) = \mathbb{Z}$.

Example 2
Let $(\mathbb{Z} - \{0\}, \times)$ *be the semigroup of non-zero integers with composition given by multiplication. Then,* $K(\mathbb{Z} - \{0\}, \times) = (\mathbb{Q} - \{0\}, \times)$.

Example 3
Let G *be an abelian group, then* $K(G) \cong G$ *via the identification* $(g,h) \leftrightarrow g - h$ *(or* $(g,h) \leftrightarrow gh^{-1}$ *if* G *is multiplicative).*

Let C be a (essentially small) symmetric monoidal category. Its Grothendieck group is $K([C])$, i.e. the Grothendieck group of the isomorphism classes of objects of C.