pairwise comaximal ideals property

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Proposition 1. *Let R be a commutative ring with unity. For every pairwise comaximal ideals $I_1, I_2, ..., I_n$, the following holds:*

$$I_1 \cap I_2 \cap ... \cap I_n = I_1 I_2 ... I_n. \tag{1}$$

Proof. We prove by induction on n. For $n = 2$, $I_1 + I_2 = R$ implies:

$$I_1 \cap I_2 = R(I_1 \cap I_2) = (I_1 + I_2)(I_1 \cap I_2) \subseteq I_1 I_2. \tag{2}$$

The converse inclusion is trivial. Assume now that the equality holds for $n \geq 2$: $J := I_1 \cap I_2 \cap ... \cap I_n = I_1 I_2 ... I_n$. Since $I_{n+1} + I_j = R$, for every $j \neq n+1$, there exist the elements $a_j \in I_j$ and $b_j \in I_{n+1}$ such that $a_j + b_j = 1$. The product $c := \prod_{j=1}^{n} a_j = \prod_{j=1}^{n} (1 - b_j) \in 1 + I_{n+1}$. Also $c \in J$, then $1 \in J + I_{n+1}$ or $J + I_{n+1} = R$.
Applying the case 2, the induction step is satisfied:

$$I_1 I_2 ... I_{n+1} = J I_{n+1} = J \cap I_{n+1} = I_1 \cap I_2 \cap ... \cap I_n \cap I_{n+1}. \tag{3}$$

$\square$