involutory ring

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Synonym	ring admitting an involution
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Related topic	HollowMatrixRings
Defines	involution
Defines	adjoint element
Defines	self-adjoint element
Defines	projection element
Defines	norm element
Defines	trace element
Defines	skew symmetric element
Defines	*-homomorphism
Defines	normal element
Defines	unitary element

General Definition of a Ring with Involution

Let R be a ring. An $*$ on R is an anti-endomorphism whose square is the identity map. In other words, for $a, b \in R$:

1. $(a+b)^* = a^* + b^*$,
2. $(ab)^* = b^* a^*$,
3. $a^{**} = a$

A ring admitting an involution is called an *involutory ring*. a^* is called the *adjoint* of a. By (3), a is the adjoint of a^*, so that every element of R is an adjoint.

Remark. Note that the traditional definition of an `http://planetmath.org/Involution`invol on a vector space is different from the one given here. Clearly, $*$ is bijective, so that it is an anti-automorphism. If $*$ is the identity on R, then R is commutative.

Examples. Involutory rings occur most often in rings of endomorphisms over a module. When V is a finite dimensional vector space over a field k with a given basis $\boldsymbol{b}$, any linear transformation over T (to itself) can be represented by a square matrix M over k via $\boldsymbol{b}$. The map taking M to its transpose M^T is an involution. If k is $\mathbb{C}$, then the map taking M to its conjugate transpose $\overline{M}^T$ is also an involution. In general, the composition of an isomorphism and an involution is an involution, and the composition of two involutions is an isomorphism.

***-Homomorphisms**

Let R and S be involutory rings with involutions $*_R$ and $*_S$. A **-homomorphism* $\phi : R \to S$ is a ring homomorphism which respects involutions. More precisely,

$$\phi(a^{*_R}) = \phi(a)^{*_S}, \qquad \text{for any } a \in R.$$

By abuse of notation, if we use $*$ to denote both $*_R$ and $*_S$, then we see that any *-homomorphism ϕ commutes with $*$: $\phi * = * \phi$.

Special Elements

An element $a \in R$ such that $a = a^*$ is called a *self-adjoint*. A ring with involution is usually associated with a ring of square matrices over a field, as such, a self-adjoint element is sometimes called a *Hermitian element*, or a *symmetric element*. For example, for any element $a \in R$,

1. aa^* and a^*a are both self-adjoint, the first of which is called the *norm* of a. A *norm element* b is simply an element expressible in the form aa^* for some $a \in R$, and we write $b = \mathrm{n}(a)$. If $aa^* = a^*a$, then a is called a *normal element*. If a^* is the multiplicative inverse of a, then a is a *unitary element*. If a is unitary, then it is normal.

2. With respect to addition, we can also form self-adjoint elements $a+a^* = a^* + a$, called the *trace* of a, for any $a \in R$. A *trace element* b is an element expressible as $a + a^*$ for some $a \in R$, and written $b = \mathrm{tr}(a)$.

Let S be a subset of R, write $S^* := \{a^* \mid a \in S\}$. Then S is said to be *self-adjoint* if $S = S^*$.

A self-adjoint that is also an idempotent in R is called a *projection*. If e and f are two projections in R such that $eR = fR$ (principal ideals generated by e and f are equal), then $e = f$. For if $ea = ff = f$ for some $a \in R$, then $f = ea = eea = ef$. Similarly, $e = fe$. Therefore, $e = e^* = (fe)^* = e^*f^* = ef = f$.

If the characteristic of R is not 2, we also have a companion concept to self-adjointness, that of skew symmetry. An element a in R is skew symmetric if $a = -a^*$. Again, the name of this is borrowed from linear algebra.