polynomial identity algebra

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Defines	Hall identity

Let R be a commutative ring with 1. Let X be a countable set of *variables*, and let $R\langle X\rangle$ denote the free associative algebra over R. If X is finite, we can also write $R\langle X\rangle$ as $R\langle x_1, \ldots x_n\rangle$, where the $x_i's \in X$. Because of the freeness condition on the algebra, the variables are non-commuting among themselves. However, the variables do commute with elements of R. A typical element f of $R\langle X\rangle$ is a polynomial over R in n (finite) non-commuting variables of X.

Definition. Let A be a R-algebra and $f = f(x_1, \ldots, x_n) \in R\langle X\rangle$. For any $a_1, \ldots, a_n \in A$, $f(a_1, \ldots, a_n) \in A$ is called an *evaluation of f at n-tuple* $(a_1, \ldots, a_n)$. If the evaluation vanishes (=0) for all n-tuples of $\Pi_{i=1}^n A$, then f is called a *polynomial identity for A*.

A polynomial $f \in R\langle X\rangle$ is *proper*, or *monic*, if, in the homogeneous component of the highest degree in f, one of its monomials has coefficient = 1.

Definition. An algebra A over a commutative ring R is said to be a *polynomial identity algebra over* R, or a PI-*algebra* over R, if there is a proper polynomial $f \in R\langle x_1, \ldots, x_n\rangle$, such that f is a polynomial identity for A. A *polynomial identity ring*, or PI-*ring*, R is a polynomial identity $\mathbb{Z}$-algebra.

Examples

1. A commutative ring is a PI-ring, satisfying the polynomial $[x, y] = xy - yx$.

2. A finite field (with q elements) is a PI-ring, satisfying $x^q - x$.

3. The ring T of upper triangular $n \times n$ matrices over a field is a PI-ring. This is true because for any $a, b \in T$, $ab - ba$ is strictly upper triangular (zeros along the diagonal). Any product of n strictly upper triangular matrices in T is 0. Therefore, T satisfies $[x_1, y_1][x_2, y_2] \cdots [x_n, y_n]$.

4. The ring S of 2×2 matrices over a field is a PI-ring. One can show that S satisfies $[[x_1, x_2]^2, x_3]$. This identity is called the *Hall identity.*

5. A subring of a PI-ring is a PI-ring. A homomorphic image of a PI-ring is a PI-ring.

6. One can show that a ring R with polynomial identity $x^n - x$ is commutative. Thus, one sees that $x^n - x$ and $xy - yx$, although very different (one is homogeneous of degree 2 in 2 variables, the other one is not

even homogeneous, in one variable of degree n), are both polynomial identities for R.