PID and UFD are equivalent in a Dedekind domain

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This article shows that if A is a Dedekind domain, then A is a UFD if and only if it is a PID. Note that this result implies the more specific result given in the article unique factorization and ideals in ring of integers.

Since any PID is a UFD, we need only prove the other direction. So assume A is a UFD, let $\mathfrak{p}$ be a nonzero (proper) prime ideal, and choose $0 \neq x \in \mathfrak{p}$. Note that x is a nonunit since $\mathfrak{p}$ is a proper ideal. Since A is a UFD, we may write x uniquely (up to units) as $x = p_1^{a_1} \cdots p_k^{a_k}$ where the p_i are distinct irreducibles in A, the a_i are positive integers, and $k > 0$ since x is not a unit. Since $\mathfrak{p}$ is prime and $x \in \mathfrak{p}$, it follows that some p_i, say p_1, is in $\mathfrak{p}$. Then $(p_1) \subset \mathfrak{p}$. But (p_1) is prime since clearly in a UFD any ideal generated by an irreducible is prime. Since A is Dedekind and thus has Krull dimension 1, it must be that $(p_1) = \mathfrak{p}$ and thus $\mathfrak{p}$ is principal.