centralizers in algebra

Canonical name	CentralizersInAlgebra
Date of creation	2013-03-22 17:22:30
Last modified on	2013-03-22 17:22:30
Owner	Algeboy (12884)
Last modified by	Algeboy (12884)
Numerical id	14
Author	Algeboy (12884)
Entry type	Definition
Classification	msc 17A01
Synonym	centraliser
Related topic	CentralizerOfASubsetOfAGroup
Related topic	LieAlgebra
Defines	centralizer
Defines	center
Defines	central
Defines	centraliser
Defines	additive commutator

1 Abstract definitions and properties

Definition 1. *Let S be a set with a binary operation $*$. Let T be a subset of S. Then define the* centralizer *in S of T as the subset*

$$C_S(T) = \{s \in S : s * t = t * s, \text{ for all } t \in T\}.$$

The center *of S is defined as $C_S(S)$. This is commonly denoted $Z(S)$ where Z is derived from the German word* zentral. *Subsets and elements of the center are called* central.

If we regard $* : S \times S \to S$ in the of actions we can perscribe a left action $s * t$ and a right action $t * s$. The centralizer is thus the set of elements for which the left regular action and the right regular action agree when to T.

It is generally possible to have $s * t$ not lie in T for $s \in C_S(T)$ and $t \in T$, and likewise, it is also possible that if $s, s' \in C_S(T)$ that $s * s' \neq s' * s$. Therefore it should not be presumed that the centralizer is central.

With further axioms on the of operation we can deduce certain natural for the set $C_S(T)$.

Proposition 2.

1. *If $A \subseteq B$, then $C_S(B) \subseteq C_S(A)$. In particular, $C_S(\varnothing) = S$.*
2. *If S has an identity then $C_S(T)$ is non-empty. In particular, in this case $Z(S)$ is non-empty.* [1]
3. *If S is associative and $s, s' \in C_S(T)$ then $s * s' \in C_S(T)$, we say then that $C_S(T)$ is closed to the binary operation of S.*
4. *If $s \in C_S(T)$ and s has an (strong) inverse s^{-1}, then $s^{-1} \in C_S(T)$.* [2]
5. *If S is commutative then $C_S(T) = S$.*
6. *If T is a subset of the center of S then $C_S(T) = S$.*

Note that it is possible for $C_S(T)$ be a subset closed to the opertaion without the assumption of associativity, as for example, when S is commutative.

[1] An identity of S is an element $e \in S$ such that $e * s = s * e$ for all $s \in S$.

[2] We say an inverse is strong if $s^{-1} * (s*t) = t = (t*s)*s^{-1}$ for all $t \in S$. If the operation is associative then this is given for free. There are natural nonassociative operations with this property, such as alternative algebras.

2 Centralizers in groups

In the category of groups the centralizer in a group G of a subset H can be redefined as:

$$C_G(H) = \{g \in G : g^{-1}hg = h, \text{ for all } h \in H\}.$$

If one regards conjugation as a group action $h^g := g^{-1}hg$ then it follows that the centralizer is the same as the pointwise stabilizer in G of H, where the action is of G on itself by conjugation. Because of this overlap, in some contexts the centralizers is applied to the pointwise stabilizer of a set on which a group acts, though this context no longer refers to the action of conjugation. This is espeically common when there is a need to distinguish between the pointwise stabilizer and the setwise stabilizer.

In this category, the centralizer is always a subgroup of G. Furthermore, if H is a normal subgroup of G, then so too is $C_G(H)$.

3 Centralizers in rings and algebras

For we treat rings as algebras over $\mathbb{Z}$ and now speak only of algebras, which will include nonassociative examples.

In an algebra A there is in fact two binary operations on the set A in question. Thus the abstract definition of the centralizer is ambiguous. However, the additive operation of rings and algebras is always commutative and so any centralizer with respect to this operation is the set A. Thus it is generally accepted practice to assume that centralizers in this context always refer to the multiplicative operation. In this way we have the following properties.

Proposition 3. *Given an algebra A over a commutative unital ring R and a subset B of A, then*

1. *$C_A(B)$ is a submodule of A.*

2. *If A is associative then $C_A(B)$ is a subalgebra.*

3. *$Z(A) \leq C_A(B)$, in particular, if A has a 1 then $1 \in C_A(B)$ and so R embeds in $C_A(B)$.*

Remarks.

- A centralizer in an algebra is also called a commutant. This terminology is mostly used in algebras of operators in functional analysis.

- Let R be a ring (or an algebra). For every ordered pair (a, b) of elements of R, we can define the *additive commutator* of (a, b) to be the element $ab - ba$, written $[a, b]$. With this, one may alternatively define the centralizer of a set $S \subseteq R$ in a ring R as

  $$C_R(S) := \{r \in R \mid [r, s] = 0 \text{ for all } s \in S\}.$$

  Of course, in this definition, two operations (multiplication and subtraction) are needed instead of one. But the nice aspect about this definition is that one can "measure" commutativity of a ring by the additive commutation operation. For example, one can show that, in a division ring, if every element additively commutes with every additive commutator, then the ring must be a field.

4 Centralizers in Lie algebras

Suppose $\mathfrak{g}$ is a Lie aglebra over a commutative ring of characteristic not 2. Given a subset T of $\mathfrak{g}$, then $[s, t] = -[t, s]$ for $s \in \mathfrak{g}$ and $t \in T$ from the axioms of a Lie algebra multiplication. Therefore whenever $[s, t] = [t, s]$ it follows that $-[t, s] = [t, s]$ so that $[t, s] = 0$. This motivates the more common redefinition of the centralizer in a Lie algebra:

$$C_{\mathfrak{g}}(T) = \{s \in \mathfrak{g} : [s, t] = 0, \text{ for all } t \in T\}.$$

Despite the incongruety in characteristic 2, this new definition replaces the original definition of centralizers for Lie algebras. The centralizer of a Lie algebra is a subalgebra.

When the Lie multiplication is regarded as a commutator, so $[a, b] = ab-ba$, for example it it universal enveloping algebra, then $0 = [a, b] = ab-ba$ is the same as $ab = ba$ and so the centralizer of the Lie algebra coincides with the centralizer of the associative envelope.

5 Centralizers in other nonassociative algebras

The centralizer need not be a subalgebra on account of the lack of associativity. There are instances of non-associative algebras where the centralizer is however a subalgebra nontheless, for example, Lie algebras as seen above. In travial fashion, if an algebra is commutative then $C_A(B) = A$ and so the centralizer is a subalgebra but without any useful properties. There is a suitable additional constraint to add to centralizers to force them to be subalgebras and carry with them more useful in the commutative but nonassociative setting.

We write $[a, b]$ for $ab - ba$, called the commutator in A of $a, b \in A$ and also write $/a, b, c/$ for $(ab)c - a(bc)$ and call it the associator in A of $a, b, c \in A$.[3] Then we can redefine the centralizer in A of a subset T of A as

$$C_A(T) = \{s \in A : [s, t] = 0, /s', s, t/ = /s', t, s/ = /t, s', s/ = 0 \text{ for all } t \in T, s' \in A\}.$$

It follows that $C_A(T)$ is a subalgebra of A on account of the added associator condition which forces the subset to be closed to the product.

In alternative algebras, if any one of three associators is 0 then the other three are as well and so the definition reduces to $/s', s, t/ = 0$.   occur of other nonassociative algebras.

[3]This notation for associators is non-standard but the standard $[a, b, c] = (ab)c - a(bc)$ is likely confusing given the usual commutator notation used already.