Jordan algebra

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Synonym	Jordan isomorphism
Defines	Jordan identity
Defines	special Jordan algebra
Defines	exceptional Jordan algebra
Defines	Jordan algebra homomorphism
Defines	Jordan subalgebra
Defines	Jordan algebra isomorphism

Let R be a commutative ring with $1 \neq 0$. An R-algebra A with multiplication not assumed to be associative is called a (commutative) *Jordan algebra* if

1. A is commutative: $ab = ba$, and
2. A satisfies the *Jordan identity*: $(a^2b)a = a^2(ba)$,

for any $a, b \in A$.

The above can be restated as

1. $[A, A] = 0$, where $[\ ,\ ]$ is the commutator bracket, and
2. for any $a \in A$, $[a^2, A, a] = 0$, where $[\ ,\ ,\ ]$ is the associator bracket.

If A is a Jordan algebra, a subset $B \subseteq A$ is called a *Jordan subalgebra* if $BB \subseteq B$. Let A and B be two Jordan algebras. A *Jordan algebra homomorphism*, or simply *Jordan homomorphism*, from A to B is an algebra homomorphism that respects the above two laws. A *Jordan algebra isomorphism* is just a bijective Jordan algebra homomorphism.

Remarks.

- If A is a Jordan algebra such that $\mathrm{char}(A) \neq 2$, then A is http://planetmath.org/PowerAsso associative.
- If in addition $2 = 1+1 \neq \mathrm{char}(A)$, then by replacing a with $a+1$ in the Jordan identity and simplifying, A is http://planetmath.org/FlexibleAlgebraflexible.
- Given any associative algebra A, we can define a Jordan algebra A^+. To see this, let A be an associative algebra with associative multiplication $\cdot$ and suppose $2 = 1+1$ is invertible in R. Define a new multiplication given by
$$ab = \frac{1}{2}(a \cdot b + b \cdot a). \qquad (1)$$
It is readily checked that this new multiplication satisifies both the commutative law and the Jordan identity. Thus A with the new multiplication is a Jordan algebra and we denote it by A^+. However, unlike Lie algebras, not every Jordan algebra is embeddable in an associative algebra. Any Jordan algebra that is isomorphic to a Jordan subalgebra of A^+ for some associative algebra A is called a *special Jordan algebra*. Otherwise, it is called an *exceptional Jordan algebra*. As a side note, the right hand side of Equation (1) is called the *Jordan product*.

- An example of an exceptional Jordan algebra is $H_3(\mathbb{O})$, the algebra of 3×3 Hermitian matrices over the octonions.