Heisenberg algebra

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Let R be a commutative ring. Let M be a http://planetmath.org/node/5420module over R http://planetmath.org/node/5420freely generated by two sets $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$, where I is an index set, and a further element c. Define a product $[\cdot,\cdot]\colon M\times M\to M$ by bilinear extension by setting

$$[c,c]=[c,P_i]=[P_i,c]=[c,Q_i]=[Q_i,c]=[P_i,P_j]=[Q_i,Q_j]=0 \text{ for all } i,j\in I,$$
$$[P_i,Q_j]=[Q_i,P_j]=0 \text{ for all distinct } i,j\in I,$$
$$[P_i,Q_i]=-[Q_i,P_i]=c \text{ for all } i\in I.$$

The module M together with this product is called a *Heisenberg algebra*. The element c is called the *central element*.

It is easy to see that the product $[\cdot,\cdot]$ also fulfills the Jacobi identity, so a Heisenberg algebra is actually a Lie algebra of rank $|I|+1$ (opposed to the rank of M as free module, which is $2|I|+1$) with one-dimensional center generated by c.

Heisenberg algebras arise in quantum mechanics with $R=\mathbb{C}$ and typically $I=\{1,2,3\}$, but also in the theory of vertex with $I=\mathbb{Z}$.

In the case where R is a field, the Heisenberg algebra is related to a Weyl algebra: let U be the universal enveloping algebra of M, then the quotient $U/\langle c-1\rangle$ is isomorphic to the $|I|$-th Weyl algebra over R.