Verma module

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Defines	highest weight module

Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{b}$ a Borel subalgebra. We work over a field F. Given a weight $\lambda \in \mathfrak{h}^*$, let F_λ be the 1-d dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts by multiplication by λ, and the positive root spaces act trivially. Now, the Verma module M_λ of the weight λ is the $\mathfrak{g}$-module induced from F_λ, i.e.

$$M_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} F_\lambda.$$

Using the Poincar-Birkhoff-Witt theorem we see that as a vector space M_λ is isomorphic to $\mathcal{U}(\overline{\mathfrak{n}})$, where $\overline{\mathfrak{n}}$ is the sum of the negative weight spaces (so $\mathfrak{g} = \mathfrak{b} \oplus \overline{\mathfrak{n}}$). In particular M_λ is infinite dimensional.

We say a $\mathfrak{g}$-module V is a highest weight module if it has a weight $\mu \in \mathfrak{h}^*$ and a non-zero vector $v \in V_\mu$ with $Xv = 0$ for any X in a positive root space and such that V is generated as a $\mathfrak{g}$-module by v. The Verma module M_λ is a highest weight module and we fix a generator $1 \otimes 1$.

The most important property of Verma modules is that they are universal amongst highest weight modules, in the following sense. If V is a highest weight module generated by v which has weight λ then there is a unique surjective homomorphism $M_\lambda \to V$ which sends $1 \otimes 1$ to v. That is, all highest weight modules with highest weight λ are quotients of M_λ. Also, M_λ has a unique maximal submodule, so there is a unique irreducible representation with highest weight λ. If λ is dominant and integral then this module is finite dimensional.