composition algebra

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1 Definition

The classical definition of a composition algebra is a non-associative algebra C over a field k where

1. C admits a non-degenerate quadratic form $q : C \to k$, such that
2. q is multiplicative: $q(xy) = q(x)q(y)$.

We also say q permits composition or that it obeys the composition law.

This definition is geometric in that quadratic forms give rise to geometric atributes for a vector space such as length, distance and orthogonality. Indeed, originally created over the real numbers such properties seem appropriate for an algebra; however, concepts of length and distance are less appropriate over arbitrary fields and encourage a second equivalent definition based solely on the algebraic aspect of such algebras.

Alternatively, a composition algebra can be defined as a unital alternative algebra C over a field k with an involution $x \mapsto \bar{x}$, that is an anti-isomorphism of order at most 2, such that:

1. C has no non-zero absolute zero divisors (that is, $(xa)x = 0$ for all $a \in C$ implies $x = 0$);
2. the norm $N(x) := x\bar{x}$ is a scalar multiple of 1, that is, $N : C \to k1$.

The first definition makes the composition property part of the definition but obscures the alternative multiplication as well as the existence of an involutary anti-isomorphism for the algebra. The second definition makes both of these properties evident but obscures the composition property of the norm, and also hides the property that N is a quadratic form. However both definitions have merit, the first captures the classical view of an algebra respecting a certain geometric condition while the second, introduced by Jacobson, promotes a purely algebraic treatment. In our examples and constructions to follows we attempt to exhibit both aspects by supplying the norm, the involution, and the product.

Both definitions can be generalized to algebras over commutative unital rings k.

Recall that a quadratic form gives rise to a symmetric bilinear from $b : C \times C \to k$ by $b(x, y) = q(x + y) - q(x) - q(y)$, for all $x, y \in C$. Some of the immediate properties include:

1. $b(x, x) = 2q(x)$,
2. $b(xz, yz) = b(x, y)q(z)$,
3. $b(xy, zw) + b(xw, zy) = b(x, y)b(z, w)$.

These strongly limit the structure of composition algebras and leads to the celebrated theorem of Hurwitz (see Theorem **??**) which suitably classifies the composition algebras over $\mathbb{R}$. The work of many others including Albert, Dickson, Jacobson, and Kaplansky extended the essential conclussion of Hurwitz to all fields and the resulting generalization is still refered to as Hurwitz's theorem.

There are other algebras A with norms $q : A \to k$ which permit composition in the sense that $q(xy) = q(x)q(y)$. For example, alternative algebras with involutions. However, the distinguishing property of composition algebras is that q is a quadratic form. Classifications for such norms have been caried out by Schafer and McCrimmon.

2 Norms

Originally, composition algebras were created over the real numbers $k = \mathbb{R}$. Here the usual positive definite norm on the real vector space was used instead of the quadratic form (the square of the norm is the quadratic form).

The first non-trivial example is the set of complex numbers $\mathbb{C}$ with where $z = a + bi \in \mathbb{C}$ is assigned:

$$\begin{aligned} \bar{z} &= a - bi; \\ |z| &= |a + bi| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}. \end{aligned}$$

More interesting is the non-commutative algebra of Hamiltonians $\mathbb{H}$, created by Hamilton, where each $x \in \mathcal{H}$ has the form $x = a + bi + cj + dk$ and

$$\begin{aligned} \bar{x} &= a - bi - cj - dk; \\ |x| &= |a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{x\bar{x}}. \end{aligned}$$

The last addition to the list was the non-associative algebra of *octonions* initially created by Cayley and the norm is simply

$$\begin{aligned}\bar{x} &= a - bi - cj - dk - fil - gjl - hkl; \\ |x| &= |a + bi + cj + dk + el + fil + gjl + hkl| \\ &= \sqrt{a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2} = \sqrt{x\bar{x}}.\end{aligned}$$

Because general fields do not sufficient squareroots, the use of norms in the classical Euclidean sense is replaced by the use of quadratic forms. Furthermore, the lack of ordering a field, such as a finite field, introduces the need to use non-degenerate rather than positive definite conditions. Under these generalizations composition algebras can be redefined form the classical context of composition algebras over $\mathbb{R}$ to general composition algebras over arbitrary fields, as done by our original definitions above. In this context, there are three further composition algebras over $\mathbb{R}$.

Example 1. *Let $C = \mathbb{R} \oplus \mathbb{R}$ with $q(x, y) = xy$ for all $(x, y) \in C$. Then C is a composition algebra.*

Proof. Evidently $q(ax, ay) = a^2 q(x, y)$ and the polarization of q is the symmetric bilinear form $b((x, y), (z, w)) = xz - yw$ for all $(x, y), (z, w) \in C$ (so the signature is $(1, -1)$). Thus q is a quadratic form.

To check that q has the compositional property let $(x, y), (z, w) \in C$. Then

$$q((x, y)(z, w)) = q(xz, yw) = (xz)(yw) = (xy)(zw) = q(x, y)q(z, w).$$

Note also that by defining $\overline{(x, y)} = (y, x)$ then $(x, y)\overline{(x, y)} = (xy, yx) = q(x, y)(1, 1)$ and $b((x, y), (z, w))(1, 1) = (x, y)\overline{(z, w)} + \overline{(x, y)(z, w)}$. $\square$

Example 2. *Let $C = M_2(\mathbb{R})$ with $q(X) = \det X$ for all $X \in C$. Then C is a composition algebra.*

Proof. Let $X \in C$ and $a \in \mathbb{R}$. Then $q(aX) = \det(aX) = \det(aI_2) \det X = a^2 \det X = a^2 q(X)$. It is also evident that if $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then setting $\bar{X} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ makes $(\det X)I_2 = X\bar{X}$ and also $\mathrm{T}(X)I_2 = X + \bar{X}$, where $\mathrm{T}(X)$ is the trace of X. Hence

$$\begin{aligned}b(X, Y) = (q(X+Y) - q(X) - q(Y))I_2 &= (\det(X+Y))I_2 - (\det X)I_2 - (\det Y)I_2 \\ &= (X + Y)\overline{(X + Y)} - X\bar{X} - Y\bar{Y} \\ &= Y\bar{X} + X\bar{Y} = \mathrm{T}(X\bar{Y})I_2.\end{aligned}$$

Therefore, $b(X,Y) = \mathrm{T}(X\bar{Y})$. Since $\mathrm{T}(X\bar{Y}) = \mathrm{T}(\bar{Y}X) = \mathrm{T}(X\bar{Y})$, it follows that b is a symmetric bilinear form and so q is quadratic form.

Finally, for composition note

$$q(XY) = \det(XY) = detX \det Y = q(X)q(Y).$$

Therefore C is a composition algebra. $\square$

This gives two new composition algebras over $\mathbb{R}$ and indeed there is a third, constructed below as the algebra $\left(\frac{1,1,1}{\mathbb{R}}\right)$, which is 8-dimensional and non-associative but unlike the octonions, it has non-trivial zero-divisors.

Definition 3. *A composition algebra is* split *if the quadratic form is isotropic.*

The example of $\mathbb{R} \oplus \mathbb{R}$ and $M_2(\mathbb{R})$ just given are both examples of split composition algebras.

3 Involution

Define

$$\bar{x} := b(x,1)1 - x, \qquad \forall x \in C.$$

Immediately it follows that: for all $x, y \in C$,

1. $\bar{\bar{x}} = x$,
2. $\overline{x+y} = \bar{x} + \bar{y}$,
3. $\bar{1} = 1$.

Define the *trace* of x as $T(x) = x + \bar{x}$ and the *norm* of x as $N(x) = \bar{x}x$. Then it follows that:

$$x^2 - T(x)x + N(x) = 0, \qquad \forall x \in C.$$

So C is a *quadratic* algebra since every element in x has at most a quadratic minimal polynomial. In fact $N(x)$ is a quadratic form allowing composition.

4 Constructing composition algebras

All of the following are composition algebras. [**?**, III.4]

dim 1**:** k, with trivial involution $x = \bar{x}$ for all x in k.

dim 2**:** For any $\alpha \in k$, a *quadratic* extension of k, that is

$$\left(\frac{\alpha}{k}\right) = \langle 1, i | i^2 = \alpha \rangle.$$

Here $\{1, i\}$ is a basis and has an involution defined by $\bar{1} = 1$ and $\bar{i} = -i$.

dim 4**:** For any $\alpha, \beta \in k$, a *quaternion* algebra over k defined as

$$\left(\frac{\alpha, \beta}{k}\right) = \langle 1, i, j | i^2 = \alpha, j^2 = \beta, ij = -ji \rangle$$

Then $\{1, i, j, ij\}$ forms a basis.[1] An involution is defined by $\bar{1} = 1$, $\bar{i} = -i$, $\bar{j} = -j$ and extended linearly.

dim 8**:** For any $\alpha, \beta, \gamma \in k$, an *octonion* algebra over k:

$$\left(\frac{\alpha, \beta, \gamma}{k}\right) = \langle 1, i, j, l | i^2 = \alpha, j^2 = \beta, l^2 = \gamma, \quad ij = -ji, il = -li, jl = -lj, \\ i(lj) = -l(ij), (li)j = l(ji), (li)(lj) = -\gamma ji \rangle.$$

The set $\{1, i, j, ij, l, il, jl, ijl\}$ is a basis. An involution is defined by $\bar{1} = 1$, $\bar{i} = -i$, $\bar{j} = -j$, $\bar{l} = -l$ and extended linearly.

Each of these algebras can be realized by the Cayley-Dickson method which takes C an associative k-algebra with involution and produces for each $\alpha \in C - \{0\}$ a new algebra $\left(\frac{\alpha}{C}\right)$ on the vector space $C \oplus C$ with product

$$(a, b)(c, d) = (ac + \alpha d\bar{b}, \bar{a}d + cb).$$

Set the involution on $\left(\frac{\alpha}{C}\right)$ to be $\overline{(a, b)} = (\bar{a}, -b)$.

The algebras are equipped with a *trace* $Tr(x) = x + \bar{x}$, and *norm* $N(x) = x\bar{x}$. This norm serves as the quadratic map to establish these algebras a composition algebras. The images of the trace and norm lie in k.

[1] It is common to use k for ij, but k here is used exclusively for the underlying field.

The new algebra is associative only if C is commutative, otherwise it is alternative. This means that $k, \left(\dfrac{\alpha}{k}\right), \left(\dfrac{\alpha,\beta}{k}\right)$ are the associative composition algebras.

An algebra is a *division* algebra if the only zero-divisor is 0 [**?**, II.2]. A central simple composition algebra with a non-trivial zero-divisor is called a *split* composition algebra. Finite dimensional split central simple composition algebras are unique up to isomorphism to one of

$$k, \quad \left(\frac{1}{k}\right) \cong k \oplus k, \quad \left(\frac{1,1}{k}\right) \cong M_2(k), \quad \left(\frac{1,1,1}{k}\right).$$

5 Classification theorem

Theorem 4. *[**?**, Theorem 6.2.3] A composition algebra C over a field k with quadratic form $q(x) = x\bar{x}$ is isomorphic to one of the following:*

(i) A purely inseparable extension field E/F of characteristic 2 and exponent 1 (trivial involution) so $q(x) = x^2$.

(ii) k with trivial involution, so $q(x) = x^2$,

(iii) Quadratic composition algebra: $\left(\frac{\alpha}{k}\right)$ for $\alpha \in k$,

(iv) Quaternion algebra: $\left(\frac{\alpha,\beta}{k}\right)$ for $\alpha, \beta \in k$,

(v) Octonion algebra: $\left(\frac{\alpha,\beta,\gamma}{k}\right)$ for $\alpha, \beta, \gamma \in k$.

In particular, all composition algebras over k, save perhaps those of type (i), are finite dimensional and of dimension 1, 2, 4 or 8.

References

[1] T.Y. Lam: *Introduction to Quadratic Forms over Fields*, AMS, Providence (2004).

[2] N. Jacobson *Structure theory of Jordan algebras*, The University of Arkansas lecture notes in mathematics, vol. 5, Fayetteville, 1981.

[3] K. McCrimmon: *A Taste of Jordan Algebras*, Springer, New York (2004).

[4] J.H. Conway, D.A. Smith: *On Quaternions and Octonions, Their Geometry, Arithmetic, and Symmetry*, AK Peters, Natick, Mass (2003)

[5] Richard D. Schafer, *An introduction to nonassociative algebras*, Pure and Applied Mathematics, Vol. 22, Academic Press, New York, 1966.