alternative algebra

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Defines	Artin's theorem on alternative algebras
Defines	alternative ring
Defines	left alternative law
Defines	right alternative law

A non-associative algebra A is *alternative* if

1. (left alternative laws) $[\ a,a,b\ ]=0$, and
2. (right alternative laws) $[\ b,a,a\ ]=0$,

for any $a,b\in A$, where $[\ ,\ ,]$ is the associator on A.

Remarks

- Let A be alternative and suppose $\operatorname{char}(A)\neq 2$. From the fact that $[\ a+b,a+b,c\ ]=0$, we can deduce that the associator $[\ ,\ ,]$ is *anti-commutative*, when one of the three coordinates is held fixed. That is, for any $a,b,c\in A$,

  1. $[\ a,b,c\ ]=-[\ b,a,c\ ]$
  2. $[\ a,b,c\ ]=-[\ a,c,b\ ]$
  3. $[\ a,b,c\ ]=-[\ c,b,a\ ]$

  Put more succinctly,

  $$[\ a_1,a_2,a_3\ ]=\operatorname{sgn}(\pi)[\ a_{\pi(1)},a_{\pi(2)},a_{\pi(3)}\ ],$$

  where $\pi\in S_3$, the symmetric group on three letters, and $\operatorname{sgn}(\pi)$ is the `http://planetmath.org/SignatureOfAPermutation`sign of π.

- An alternative algebra is a flexible algebra, provided that the algebra is not `http://planetmath.org/BooleanLattice`Boolean (`http://planetmath.org/Characte` $\neq 2$). To see this, replace c in the first anti-commutative identities above with a and the result follows.

- **Artin's Theorem**: If a non-associative algebra A is not Boolean, then A is alternative iff every subalgebra of A generated by two elements is associative. The proof is clear from the above discussion.

- A commutative alternative algebra A is a Jordan algebra. This is true since $a^2(ba)=a^2(ab)=(ab)a^2=((ab)a)a=(a(ab))a=(a^2b)a$ shows that the Jordan identity is satisfied.

- Alternativity can be defined for a general ring R: it is a non-associative ring such that for any $a,b\in R$, $(aa)b=a(ab)$ and $(ab)b=a(bb)$. Equivalently, an alternative ring is an alternative algebra over $\mathbb{Z}$.