Lie algebras from other algebras

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1 Lie algebras from associative algebras

Given an associative (unital) algebra A over a commutative ring R, we define A^- as the R-module A together with a new multiplication $[,] : A \times A \to A$ derived from the associative multiplication as follows:

$$[a, b] = ab - ba.$$

This operation is commonly called the *commutator bracket* on A.

Proposition 1. *A^- is a Lie algebra.*

Proof. We already know A^- is a module so we need simply to confirm that the commutator bracket is a bilinear mapping and then demonstrate that it is alternating and satisfies the Jacobi identity.

Given $a, b, c \in A$, and $l \in R$ then

$$[la + b, c] = (la + b)c - c(la + b) = l(ac - ca) + (bc - cb) = l[a, c] + [b, c].$$

The similar argument in the second variable shows that the operation is bilinear.

Next, $[x, x] = xx - xx = 0$ so $[,]$ is alternating. Finally for the Jacobi identity we compute directly.

$$[[a, b], c]+[[b, c], a]+[[c, a], b] = (ab-ba)c-c(ab-ba)+(bc-cb)a-a(bc-cb)+(ca-ac)b-b(ca-ac)$$
$$= abc - bac - cab + cba + bca - cba - abc + acb + cab - acb - bca + bac = 0.$$

□

We notice this produces a functor from the category of associative algebras to the category of Lie algebras. However, to every commutative algebra A, A^- is a trivial Lie algebra, and so this functor is not faithful. More generally, the center of an arbitrary associative algebra A is lost to the Lie algebra structure A^-.

We do observe some relationships between the algebraic structure of A and that of A^-.

Theorem 2. *If $I \trianglelefteq A$ then $I \triangleleft A^-$.*

Proof. We observe that a submodule of A is a submodule of A^- as the two are identitcal as modules. It remains to show $[I, A^-] \leq I$. So given $a \in I$ and $b \in A$, then $[a, b] = ab - ba$ and as $ab, ba \in I$ we conclude $[a, b] \in I$. □

2 Associative envelopes

Given a Lie algebra $\mathfrak{g}$ it is often desirable to reverse the process described above, that is, to provide an associative algebra A for which $\mathfrak{g} = A^-$. In general this is impossible as we will now explain.

Let V be a vector space and A the endomorphism algebra on V. Then we give the name $\mathfrak{gl}(V)$ to the Lie algebra A^- (noting that A is associative under the composition of functions operation.) Then we can also define a subalgebra $\mathfrak{sl}(V)$ as the set of linear transformations with trace 0.

Now we claim that $\mathfrak{sl}_2(\mathbb{C})$ is not equal to B^- for any associative (unital) algebra B. For it is easy to see $\mathfrak{sl}_2(\mathbb{C})$ has a basis of three elements:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Therefore B would also be 3-dimensional. We also know that $\mathfrak{sl}_2(\mathbb{C})$ is a simple Lie algebra, that is, it has no proper ideals. Therefore by Theorem **??**, B can have no ideals either, so B must be simple. However the finite dimensional simple rings over $\mathbb{C}$ are isomorphic to matrix rings (by the Wedderburn-Artin theorem) $M_n(\mathbb{C})$ and thus cannot have dimension 3.

This forces the weaker question as to whether a Lie algebra can be embedded in A^- for some associative algebra A. We call such embeddings *associative envelopes* of the Lie aglebra. The existence of associative envelopes of arbitrary Lie algebras is answered by a corollary to the Poincare-Birkhoff-Witt theorem.

Theorem 3. *Every Lie algebra $\mathfrak{g}$ embeds in the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})^-$, where $\mathfrak{U}(\mathfrak{g})$ is an associative algebra.*

Finite dimensional analogues also exist, some of which are simpler to observe. For instance, a Lie aglebra $\mathfrak{g}$ can be represented in $\mathfrak{gl}(\mathfrak{g})$ by the adjoint representation. The representation is not faithful unless the center of $\mathfrak{g}$ is trivial. However, for semi-simple Lie algebras, the adjoint representation thus suffices as an associative envelope.

Remark 4. *This result is in contrast to Jordan algebras where there are isomorphism types (for example 3×3 matrices over the octonions) which cannot be embedded in A^+ for any associative algebra A. [A^+ is the derived algebra of A under the product $a.b = ab + ba$.]*

2.1 Lie algebra from non-associative algebras

If A is not an associative algebra to begin with then we may still determine the commutator bracket is bilinear and alternating. However, the Jacobi identity is in question. If we define the *associator bracket* as $[a, b, c] = (ab)c - a(bc)$ then we can write the computation for the Jacobi identity as:

$$\begin{aligned}
&[[a, b], c]+[[b, c], a]+[[c, a], b] = (ab-ba)c-c(ab-ba)+(bc-cb)a-a(bc-cb)+(ca-ac)b-b(ca-ac) \\
&= (ab)c-(ba)c-c(ab)+c(ba)+(bc)a-(cb)a-a(bc)+a(cb)+(ca)b-(ac)b-b(ca)+b(ac) \\
&= ((ab)c-a(bc))+((bc)a-b(ca))+((ca)b-c(ab))-((ba)c-b(ac))-((cb)a-c(ba))-((ac)b-a(cb)) \\
&= [a, b, c] + [b, c, a] + [c, a, b] - [b, a, c] - [c, b, a] - [a, c, b].
\end{aligned}$$

We can write this right hand side using permutations on the set $\{a, b, c\}$ as:

$$\sum_{\sigma\in \mathrm{Alt}(\{a,b,c\})} [[a\sigma, b\sigma], c\sigma] = [a, b], c]+[[b, c], a]+[[c, a], b] = \sum_{\sigma\in \mathrm{Sym}(\{a,b,c\})} \mathrm{sign}(\sigma)[a\sigma, b\sigma, c\sigma].$$

That is, in a non-associative algebra the corresponding Jacobi identity is the possibly non-trivial sum over all permutations of associators. We consider a few non-associative examples.

- If A is a commutative non-associative algebra (perhaps a Jordan algebra) then

  $$[a, b, c] = (ab)c - a(bc) = (ba)c - (bc)a = c(ba) - (cb)a = [c, b, a]$$

  so the Jacobi identity holds. However, if A is commutative then $[a, b] = 0$ to begin with so the associated Lie algebra product is trivial.

- If A is an alternative algebra, so $[a, b, c] = -[b, a, c]$, then again the Jacobi identity holds. So A^- is a Lie algebra. The typicall non-associative examples of an alternative algebra are the octonion algebras. These produce a non-trivial Lie algebra.

- We can also consider beginning with a Lie algebra A and producing A^-. To avoid confusing the bracket of A and that of A^- we let the multiplication of A be denoted by juxtaposition, ab, $a, b \in A$. Recall that in a Lie algebra of characteristic 0 or odd then $ab = -ba$ so that

$[a, b] = ab - ba = 2ab$ in A^-. So we have simply scaled the original product of A by 2. To see the Jacobi identity still holds we note

$$[a, b, c] = (ab)c - a(bc) = -(ba)c + (bc)a = c(ba) - (cb)a = [c, b, a].$$

So once again the associators cancel.