cohomology of semi-simple Lie algebras

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There are some important facts that make the cohomology of semi-simple Lie algebras easier to deal with than general Lie algebra cohomology. In particular, there are a number of vanishing theorems.

First of all, let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over a field $\mathbb{K}$ of characteristic 0.

Theorem [Whitehead] - Let M be an irreducible `http://planetmath.org/RepresentationI` module of dimension greater than 1. Then all the cohomology groups with coefficients in M are trivial, i.e. $H^n(\mathfrak{g}, M) = 0$ for all $n \in \mathbb{N}$.

Thus, the only interesting cohomology groups with coefficients in an irreducible $\mathfrak{g}$-module are $H^n(\mathfrak{g}, \mathbb{K})$. For arbitrary $\mathfrak{g}$-modules there are still two vanishing results, which are usually called *Whitehead's lemmas*.

Whitehead's Lemmas - Let M be a finite-dimensional $\mathfrak{g}$-module. Then

- **First Lemma :** $H^1(\mathfrak{g}, M) = 0$.
- **Second Lemma :** $H^2(\mathfrak{g}, M) = 0$.

Whitehead's lemmas lead to two very important results. From the vanishing of H^1, we can derive Weyl's theorem, the fact that representations of semi-simple Lie algebras are completely reducible, since extensions of M by N are classified by $H^1(\mathfrak{g}, \mathrm{Hom}(M, N))$. And from the vanishing of H^2, we obtain Levi's theorem, which that every Lie algebra is a split extension of a semi-simple algebra by a solvable algebra since $H^2(\mathfrak{g}, M)$ classifies extensions of $\mathfrak{g}$ by M with a specified action of $\mathfrak{g}$ on M.