Poincaré-Birkhoff-Witt theorem

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Let $\mathfrak{g}$ be a Lie algebra over a field k, and let B be a k-basis of $\mathfrak{g}$ equipped with a linear order $\leq$. The *Poincaré-Birkhoff-Witt-theorem* (often abbreviated to *PBW-theorem*) states that the monomials

$$x_1 x_2 \cdots x_n \text{ with } x_1 \leq x_2 \leq \cdots \leq x_n \text{ elements of } B$$

constitute a k-basis of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Such monomials are often called *ordered monomials* or *PBW-monomials*.

It is easy to see that they span $U(\mathfrak{g})$: for all $n \in \mathbb{N}$, let M_n denote the set

$$M_n = \{(x_1, \ldots, x_n) \mid x_1 \leq \cdots \leq x_n\} \subset B^n,$$

and denote by $\pi : \bigcup_{n=0}^{\infty} B^n \to U(\mathfrak{g})$ the multiplication map. Clearly it suffices to prove that

$$\pi(B^n) \subseteq \sum_{i=0}^{n} \pi(M_i)$$

for all $n \in \mathbb{N}$; to this end, we proceed by induction. For $n = 0$ the statement is clear. Assume that it holds for $n-1 \geq 0$, and consider a list $(x_1, \ldots, x_n) \in B^n$. If it is an element of M_n, then we are done. Otherwise, there exists an index i such that $x_i > x_{i+1}$. Now we have

$$\begin{aligned} \pi(x_1, \ldots, x_n) = {} & \pi(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n) \\ & + x_1 \cdots x_{i-1} [x_i, x_{i+1}] x_{i+1} \cdots x_n. \end{aligned}$$

As B is a basis of $\mathfrak{k}$, $[x_i, x_{i+1}]$ is a linear combination of B. Using this to expand the second term above, we find that it is in $\sum_{i=0}^{n-1} \pi(M_i)$ by the induction hypothesis. The argument of π in the first term, on the other hand, is lexicographically smaller than $(x_1, \ldots, x_n)$, but contains the same entries. Clearly this rewriting proces must end, and this concludes the induction step.

The proof of linear independence of the PBW-monomials is slightly more difficult, but can be found in most introductory texts on Lie algebras, such as the classic below.

References

[1] N. Jacobson. . Dover Publications, New York, 1979