derivations on a ring of continuous functions

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Owner	joking (16130)
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Let X be a topological space and denote by $\mathbb{R}$ the set of reals. Of course the set of all continuous functions $C(X,\mathbb{R})$ is a $\mathbb{R}$-algebra. Let $c \in \mathbb{R}$. By the symbol $\bar{c}$ we will denote constant function at c, i.e. $\bar{c} : X \to \mathbb{R}$ is defined by $\bar{c}(x) = c$.

Proposition. If $D : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ is a $\mathbb{R}$-derivation, then $D(x) = \bar{0}$ for any $x \in C(X,\mathbb{R})$.

Proof. Step one. We will prove that $D(\bar{c}) = \bar{0}$ for any $c \in \mathbb{R}$. Indeed

$$D(\bar{1}) = D(\bar{1} \cdot \bar{1}) = \bar{1} \cdot D(\bar{1}) + \bar{1} \cdot D(\bar{1}) = D(\bar{1}) + D(\bar{1}) = 2 \cdot D(\bar{1})$$

and thus $D(\bar{1}) = \bar{0}$. Now from linearity of D we obtain that

$$D(\bar{c}) = D(c \cdot \bar{1}) = c \cdot D(\bar{1}) = c \cdot \bar{0} = \bar{0}.$$

Step two. If $f : X \to \mathbb{R}$ is continuous and $c \in \mathbb{R}$, then $f + \bar{c}$ is continuous and obviously $D(f) = D(f+\bar{c})$. Moreover, if $x \in X$ then $D(f - \overline{f(x)}) = D(f)$, but $(f - \overline{f(x)})(x) = 0$. Thus we may assume that $f(x) = 0$ for fixed $x \in X$.

Let $x_0 \in X$. Now we will restrict only to such maps $f : X \to \mathbb{R}$ that $f(x_0) = 0$.

Step three. We now decompose f into sum of two nonnegative functions. Indeed, if $f : X \to \mathbb{R}$ is continuous, then define $f^+, f^- : X \to \mathbb{R}$ by the formula:

$$f^+(x) = \max(f(x), 0); \quad f^-(x) = \max(-f(x), 0).$$

Of course both f^- and f^+ are continous, nonnegative and $f = f^+ - f^-$. Thus

$$D(f) = D(f^+) - D(f^-),$$

so it is enough to show that $D(f) = \bar{0}$ only for nonnegative and continuous functions.

Step four. Assume that $f : X \to \mathbb{R}$ is nonnegative, continuous and $f(x_0) = 0$. Then there exists $g : X \to \mathbb{R}$ continuous such that $g^2 = f$ (indeed $g = \sqrt{f}$ and it is well defined, continuous map, because f was nonnegative). Then we have

$$D(f) = D(g^2) = g \cdot D(g) + g \cdot D(g) = 2 \cdot g \cdot D(g).$$

Now we have $g(x_0) = \sqrt{f(x_0)} = 0$ and thus

$$D(f)(x_0) = 2 \cdot g(x_0) \cdot D(g)(x_0) = 0.$$

Now we can take any $x \in X$ and repeat steps two, three and four to get that for any $x \in X$ we have

$$D(f)(x) = 0$$

and thus

$$D(f) = \bar{0},$$

which completes the proof. $\square$

Remark. Note that this proof cannot be repeated if we (for example) consider the set of all smooth functions $C^\infty(M, \mathbb{R})$ on a smooth manifold M, because f^+, f^- and $\sqrt{f}$ need not be smooth.