trace forms on algebras

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Given an finite dimensional algebra A over a field k we define the left(right) regular representation of A as the map $L : A \to \operatorname{End}_k A$ given by $L_a b := ab$ ($R_a b := ba$).

Example 1. *In a Lie algebra the left representation is called the* adjoint *representation and denoted* ad x *and defined* $(\text{ad}\quad x)(y) = [x, y]$*. Because* $[x, y] = -[y, x]$ *in characteristic not 2, there is generally no distinction of left/right adjoint representations.*

The *trace form* of A is defined as $\langle , \rangle : T \times T \to k$:

$$\langle a, b\rangle := \text{tr } (L_a L_b).$$

Proposition 2. *The trace form is a symmetric bilinear form.*

Proof. Given $a, b, x \in A$ and $l \in k$ then $L_{a+lb}x = (a + lb)x = ax + lbx = L_a x + lL_b x$. So $L_{a+lb} = L_a + lL_b$. So we have

$$\langle a+lb, x\rangle = \text{tr } (L_{a+lb}L_x) = \text{tr } (L_aL_x+lL_bL_x) = \text{tr } (L_aL_x)+l\,\text{tr } (L_bL_x) = \langle a, x\rangle+l\langle b, x\rangle.$$

Furthermore, $\text{tr } (fg) = \text{tr } (gf)$ is general property of traces, thus

$$\langle a, b\rangle = \text{tr } (L_aL_b) = \text{tr } (L_bL_a) = \langle b, a\rangle.$$

So the trace form is a symmetric bilinear form. □

The symmetric property can be interpreted as a weak form of commutativity of the product: $a, b \in A$ commute within their trace from. A more essential property arises for certain algebras and can be interpreted as "the product is associative within the trace" and written as

$$\langle ab, c\rangle = \langle a, bc\rangle. \tag{1}$$

We shall call such an algebra *weakly associative* though the term is not standard.

This property is clear for all associative algebras as:

$$L_{ab}L_c(x) = ((ab)c)x = (a(bc))x = L_aL_{bc}x.$$

When we use a Lie algebra, the trace form is commonly called the *Killing form* which has property (**??**). A result of Koecher shows that Jordan algebras also have this property.

Proposition 3. *Given a weakly associative algebra, then the radical of the trace form is an ideal of the algebra.*

Proof. We know the radical of form R is a subspace so we must simply show that R is an ideal. Given $x \in R$ and $y \in A$ then for all $z \in A$, $\langle xy, z\rangle = \langle x, yz\rangle = 0$. Thus $xy \in R$. Likewise $yz \in R$ so R is a two-sided ideal of A. $\square$

From this result many authors define an algebra to be semi-simple if its trace form is non-degenerate. In this way, A/R, R the radical of A, is semi-simple. [Some variations on this definition are often required over small fields/characteristics, especially when characteristic is 2.]

More can be said when ideals are considered.

Proposition 4. *Given a weakly associative algebra A, then if I is an ideal of A then so is $I^\perp$.*

Proof. Given $a \in I^\perp$, then for all $b \in A$ and $c \in I$, then $bc \in I$ as I is an ideal and so $\langle ab, c\rangle = \langle a, bc\rangle = 0$ as $a \in I^\perp$. This makes $ab \in I^\perp$ so I is a right ideal. Likewise $\langle c, ba\rangle = \langle cb, a\rangle = 0$ so $ba \in I^\perp$ and thus $I^\perp$ is an ideal of A. $\square$

To proceed one factors out the radical so that A is semisimple. Then given an ideal I of A, if $I \cap I^\perp = 0$ then as the trace form is a non-degenerate bilinear from, $A = I \oplus I^\perp$, and so by iterating we produce a decomposition of A into minimal ideals:

$$A = A_1 \oplus \cdots \oplus A_s.$$

Hence we arrive at the alternative definition of a semisimple algebra: that the algebra be a direct product of simple algebras. To obtain the property $I \cap I^\perp = 0$ it is sufficient to assume A has not ideal I such that $I^2 = 0$. This is the content of the proof in

Theorem 5. *[**?**, Thm III.3] Let A be a finite-dimensional weakly associative (trace) semisimple algebra over a field k in which no ideal $I \neq 0$ of A has $I^2 = 0$, then A is a direct product of minimal ideals, that is, of simple algebras.*

Alternatively any bilinear form with (**??**) can be used. However, the trace form is always definable and the desired properties are easily translated into implications about the multiplication of the algebra.

References

[1] Jacobson, Nathan *Lie Algebras*, Interscience Publishers, New York, 1962.

[2] Koecher, Max, *The Minnesota notes on Jordan algebras and their applications.* Edited and annotated by Aloys Krieg and Sebastian Walcher. [B] Lecture Notes in Mathematics 1710. Berlin: Springer. (1999).