index of a Lie algebra

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Entry type	Definition
Classification	msc 17B05
Defines	index of a Lie algebra
Defines	Frobenius Lie algebra
Defines	Kirillov form

Let $\mathfrak{q}$ be a Lie algebra over $\mathbb{K}$ and $\mathfrak{q}^*$ its vector space dual. For $\xi \in \mathfrak{q}^*$ let $\mathfrak{q}_\xi$ denote the stabilizer of ξ with respect to the co-adjoint representation.

The *index* of $\mathfrak{q}$ is defined to be

$$\operatorname{ind}\,\mathfrak{q} := \min_{\xi \in \mathfrak{g}^*} \dim \mathfrak{q}_\xi$$

Examples

1. If $\mathfrak{q}$ is reductive then $\operatorname{ind}\,\mathfrak{q} = \operatorname{rank}\,\mathfrak{q}$. Indeed, $\mathfrak{q}$ and $\mathfrak{q}^*$ are isomorphic as representations for $\mathfrak{q}$ and so the index is the minimal dimension among stabilizers of elements in $\mathfrak{q}$. In particular the minimum is realized in the stabilizer of any *regular* element of $\mathfrak{q}$. These elemtents have stabilizer dimension equal to the rank of $\mathfrak{q}$.

2. If $\operatorname{ind}\,\mathfrak{q} = 0$ then $\mathfrak{q}$ is called a *Frobenius Lie algebra*. This is equivalent to condition that the *Kirillov form* $K_\xi \colon \mathfrak{q} \times \mathfrak{q} \to \mathbb{K}$ given by $(X, Y) \mapsto \xi([X, Y])$ is non-singular for some $\xi \in \mathfrak{q}^*$. Another equivalent condition when $\mathfrak{q}$ is the Lie algebra of an algebraic group Q is that $\mathfrak{q}$ is Frobenius if and only if Q has an open orbit on $\mathfrak{q}^*$.