Jacobson's theorem on composition algebras

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Recall that composition algebra C over a field k is specified with a quadratic form $q : C \to k$. Furthermore, two quadratic forms $q : C \to k$ and $r : D \to k$ are isometric if there exists an invertible linear map $f : C \to D$ such that $r(f(x)) = q(x)$ for all $x \in C$.

Theorem 1 (Jacobson)**.** *[**?**, Theorem 3.23] Two unital Cayley-Dickson algebras C and D over a field k of characteristic not 2 are isomorphic if, and only if, their quadratic forms are isometric.*

A Cayley-Dickson algebra is split if the algebra has non-trivial zero-divisors.

Corollary 2. *[**?**, Corollary 3.24] Upto isomorphism there is only one split Cayley-Dickson algebra and the quadratic form has Witt index 4.*

Over the real numbers instead of Witt index, we say the signature of the quadratic form is $(4, 4)$.

This result is often used together with a theorem of Hurwitz which limits the dimensions of composition algebras to dimensions 1,2, 4 or 8. Thus to classify the composition algebras over a given field k of characteristic not 2, it suffices to classify the non-degenerate quadratic forms $q : k^n \to k$ with $n = 1, 2, 4$ or 8.

References

[1] Richard D. Schafer, *An introduction to nonassociative algebras*, Pure and Applied Mathematics, Vol. 22, Academic Press, New York, 1966.