# octonion

| | |
|---|---|
| Canonical name | Octonion |
| Date of creation | 2013-03-22 15:21:42 |
| Last modified on | 2013-03-22 15:21:42 |
| Owner | CWoo (3771) |
| Last modified by | CWoo (3771) |
| Numerical id | 15 |
| Author | CWoo (3771) |
| Entry type | Definition |
| Classification | msc 17A75 |
| Classification | msc 17D05 |
| Synonym | Cayley algebra |
| Related topic | TheoremsOnSumsOfSquares |
| Related topic | DivisionAlgebra |
| Defines | octonion algebra |

Let $\mathbb{H}$ be the quaternions over the reals $\mathbb{R}$. Apply the Cayley-Dickson construction to $\mathbb{H}$ once, and we obtain an algebra, variously called *Cayley algebra*, *the octonion algebra*, or simply *the octonions*, over $\mathbb{R}$. Specifically the construction is carried out as follows:

1. Form the vector space $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\mathbf{k}$; any element of $\mathbb{O}$ can be written as $a + b\mathbf{k}$, where $a, b \in \mathbb{H}$;

2. Define a binary operation on $\mathbb{O}$ called *the multiplication in* $\mathbb{O}$ by

$$(a + b\mathbf{k})(c + d\mathbf{k}) := (ac - \overline{d}b) + (da + b\overline{c})\mathbf{k},$$

   where $a, b, c, d \in \mathbb{H}$, and $\overline{c}$ is the quaternionic conjugation of $c \in \mathbb{H}$. When $b = d = 0$, the multiplication is reduced the multiplication in $\mathbb{H}$. In addition, the multiplication rule above imply the following:

$$a(d\mathbf{k}) = (da)\mathbf{k} \tag{1}$$
$$(b\mathbf{k})c = (b\overline{c})\mathbf{k} \tag{2}$$
$$(b\mathbf{k})(d\mathbf{k}) = -\overline{d}b. \tag{3}$$

   In particular, in the last equation, if $b = d = 1$, $\mathbf{k}^2 = -1$.

3. Define a unary operation on $\mathbb{O}$ called *the octonionic conjugation in* $\mathbb{O}$ by

$$\overline{a + b\mathbf{k}} := \overline{a} - b\mathbf{k},$$

   where $a, b \in \mathbb{H}$. Clearly, the octonionic conjugation is an `http://planetmath.org/Involutic` ($\overline{\overline{x}} = x$).

4. Finally, define a unary operation $N$ on $\mathbb{O}$ called *the norm in* $\mathbb{O}$ by $N(x) := x\overline{x}$, where $x \in \mathbb{O}$. Write $x = a + b\mathbf{k}$, then

$$N(x) = (a + b\mathbf{k})(\overline{a} - b\mathbf{k}) = (a\overline{a} + \overline{b}b) + (-ba + b\overline{\overline{a}})\mathbf{k} = a\overline{a} + b\overline{b} \geq 0.$$

   It is not hard to see that $N(x) = 0$ iff $x = 0$.

The above four (actually, only the first two suffice) steps makes $\mathbb{O}$ into an 8-dimensional algebra over $\mathbb{R}$ such that $\mathbb{H}$ is embedded as a subalgebra.

With the last two steps, one can define the inverse of a non-zero element $x \in \mathbb{O}$ by

$$x^{-1} := \frac{\overline{x}}{N(x)}$$

so that $xx^{-1} = x^{-1}x = 1$. Since $x$ is arbitrary, $\mathbb{O}$ has no zero divisors. Upon checking that $x^{-1}(xy) = y = (yx)x^{-1}$, the non-associative algebra $\mathbb{O}$ is turned into a division algebra.

Since $N(x) \geq 0$ for any $x \in \mathbb{O}$, we can define a non-negative real-valued function $\|\cdot\|$ on $\mathbb{O}$ by $\|x\| = \sqrt{N(x)}$. This is clearly well-defined and $\|x\| = 0$ iff $x = 0$. In addition, it is not hard to see that, for any $r \in \mathbb{R}$ and $x \in \mathbb{O}$, $\|rx\| = |r|\|x\|$, and that $\|\cdot\|$ satisfies the triangular inequality. This makes $\mathbb{O}$ into a normed division algebra.

Since the multiplication in $\mathbb{H}$ is noncommutative, $\mathbb{O}$ is noncommutative. In fact, if we write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbf{j}$, where $\mathbb{C}$ are the complex numbers and $\mathbf{j}^2 = -1$, then $B = \{1, \mathbf{i}, \mathbf{j}, \mathbf{ij}\}$ is a basis for the vector space $\mathbb{H}$ over $\mathbb{R}$. With the introduction of $\mathbf{k} \in \mathbb{O}$, we quickly check that $\mathbf{k}$ anti-commute with the non-real basis elements in $B$:

$$\mathbf{ik} = -\mathbf{ki}, \qquad \mathbf{jk} = -\mathbf{kj}, \qquad (\mathbf{ij})\mathbf{k} = -\mathbf{k}(\mathbf{ij}).$$

Furthermore, one checks that $\mathbf{i}(\mathbf{jk}) = (\mathbf{ji})\mathbf{k} = -(\mathbf{ij})\mathbf{k}$, so that $\mathbb{O}$ is not associative.

Since $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\mathbf{k}$, the set $\{1, \mathbf{i}, \mathbf{j}, \mathbf{ij}, \mathbf{k}, \mathbf{ik}, \mathbf{jk}, (\mathbf{ij})\mathbf{k}\}(= B \cup B\mathbf{k})$ is a basis for $\mathbb{O}$ over $\mathbb{R}$. A less messy way to represent these basis elements is done the following assignment:

| basis element | $1$ | $\mathbf{i}$ | $\mathbf{j}$ | $\mathbf{ij}$ | $\mathbf{k}$ | $\mathbf{ik}$ | $\mathbf{jk}$ | $(\mathbf{ij})\mathbf{k}$ |
|---|---|---|---|---|---|---|---|---|
| basis element rewritten | $\mathbf{i_0}$ | $\mathbf{i_1}$ | $\mathbf{i_2}$ | $\mathbf{i_3}$ | $\mathbf{i_4}$ | $\mathbf{i_5}$ | $\mathbf{i_6}$ | $\mathbf{i_7}$ |

Any element $x$ of $\mathbb{O}$ can thus be expressed uniquely as $\sum_{n=0}^{7} r_n \mathbf{i_n}$, where $r_n \in \mathbb{R}$. Using Equations (1)-(3) above, one can form a multiplication table for these basis elements $i_n$'s:

| row×column | $\mathbf{i_1}$ | $\mathbf{i_2}$ | $\mathbf{i_3}$ | $\mathbf{i_4}$ | $\mathbf{i_5}$ | $\mathbf{i_6}$ | $\mathbf{i_7}$ |
|---|---|---|---|---|---|---|---|
| $\mathbf{i_1}$ | -1 | $\mathbf{i_3}$ | $\mathbf{-i_2}$ | $\mathbf{i_5}$ | $\mathbf{-i_4}$ | $\mathbf{-i_7}$ | $\mathbf{i_6}$ |
| $\mathbf{i_2}$ | $\mathbf{-i_3}$ | -1 | $\mathbf{i_1}$ | $\mathbf{i_6}$ | $\mathbf{i_7}$ | $\mathbf{-i_4}$ | $\mathbf{-i_5}$ |
| $\mathbf{i_3}$ | $\mathbf{i_2}$ | $\mathbf{-i_1}$ | -1 | $\mathbf{i_7}$ | $\mathbf{-i_6}$ | $\mathbf{i_5}$ | $\mathbf{-i_4}$ |
| $\mathbf{i_4}$ | $\mathbf{-i_5}$ | $\mathbf{-i_6}$ | $\mathbf{-i_7}$ | -1 | $\mathbf{i_1}$ | $\mathbf{i_2}$ | $\mathbf{i_3}$ |
| $\mathbf{i_5}$ | $\mathbf{i_4}$ | $\mathbf{-i_7}$ | $\mathbf{i_6}$ | $\mathbf{-i_1}$ | -1 | $\mathbf{-i_3}$ | $\mathbf{i_2}$ |
| $\mathbf{i_6}$ | $\mathbf{i_7}$ | $\mathbf{i_4}$ | $\mathbf{-i_5}$ | $\mathbf{-i_2}$ | $\mathbf{i_3}$ | -1 | $\mathbf{-i_1}$ |
| $\mathbf{i_7}$ | $\mathbf{-i_6}$ | $\mathbf{i_5}$ | $\mathbf{i_4}$ | $\mathbf{-i_3}$ | $\mathbf{-i_2}$ | $\mathbf{i_1}$ | -1 |

Other well known properties of the octonions are

1. $\overline{xy} = \overline{y}\,\overline{x}$ for any $x, y \in \mathbb{O}$.
2. $N(xy) = N(x)N(y)$ so that $\mathbb{O}$ is a composition algebra. It also proves that the product of sums of eight squares is a sum of eight squares.
3. $\mathbb{O}$ is an alternative algebra. As a result, any two elements of $\mathbb{O}$ generate an associative algebra. If fact, the algebra is isomorphic of one of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. This is the consequence of `http://planetmath.org/ArtinsTheoremOnAlternativeA` Theorem.