Jordan triple product

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Defines	Jordan triple product

Given a Jordan algebra J we define a ternary product on J by

$$\{xyz\} = (x.y).z + (y.z).x - (z.x).y.$$

When J is a special Jordan algebra of characteristic not 2, we know the product $x.y = \frac{1}{2}(xy + yx)$. In this context a computation shows

$$\{xyz\} = \frac{1}{2}(xyz + zyx).$$

This gives a simple method by which to compute the triple product in a special Jordan algebra.

A key instance of the Jordan triple product is the case when $x = z$ (setting $x^{\cdot 2} = x.x$ for notation). Here we get

$$\{xyx\} = 2(x.y).x - x^{\cdot 2}.y.$$

In a special Jordan algebra this becomes $\{xyx\} = xyx$ in the associative product. To treat a Jordan algebra as a quadratic Jordan algebra this product plays the role of one of the two quadratic operations: $U_y : y \mapsto \{xyx\}$. The other is the usual squaring product $y \mapsto y^{\cdot 2}$.

To establish uniform proof for Jordan algebra in characteristic 2 and also include exceptional Jordan algebras it is often preferable to encode computations using the unary product: $x^{\cdot 2}$ and the triple product $\{xyz\}$. The connection between the triple product and the quadratic unary product is found in the Jordan identity:

$$(a^{\cdot 2}.b).a = a^{\cdot 2}.(b.a)$$

This idea was exploited by McCrimmon to establish quadratic Jordan algebras and many uniform and previously unknown results on Jordan algebras.

The triple product can be compared to the Jacobi product on an algebra with a $[,]$ multiplication

$$\#xyz\# = [[x, y], z] + [[y, z], x] + [[z, x], y].$$

To make a closer parallel use the typical assumption that $[z, x] = -[x, z]$ (outside of characteristic 0) then we can write:

$$\#xyz\# = [[x, y], z] + [[y, z], x] - [[x, z], y].$$

Since Jordan algebras are commutative we can also write

$$\{xyz\} = (x.y).z + (y.z).x - (x.z).y.$$

However, unlike Lie algebras where $\#xyz\# = 0$, in Jordan algebras the triple product is almost never 0.