solvable Lie algebra

Canonical name	SolvableLieAlgebra
Date of creation	2013-03-22 12:41:06
Last modified on	2013-03-22 12:41:06
Owner	djao (24)
Last modified by	djao (24)
Numerical id	4
Author	djao (24)
Entry type	Definition
Classification	msc 17B30
Defines	nilpotent Lie algebra
Defines	solvable
Defines	nilpotent
Defines	lower central series
Defines	upper central series

Let $\mathfrak{g}$ be a Lie algebra. The *lower central series* of $\mathfrak{g}$ is the filtration of subalgebras

$$\mathcal{D}_1\mathfrak{g} \supset \mathcal{D}_2\mathfrak{g} \supset \mathcal{D}_3\mathfrak{g} \supset \cdots \supset \mathcal{D}_k\mathfrak{g} \supset \cdots$$

of $\mathfrak{g}$, inductively defined for every natural number k as follows:

$$\begin{aligned} \mathcal{D}_1\mathfrak{g} &:= [\mathfrak{g}, \mathfrak{g}] \\ \mathcal{D}_k\mathfrak{g} &:= [\mathfrak{g}, \mathcal{D}_{k-1}\mathfrak{g}] \end{aligned}$$

The *upper central series* of $\mathfrak{g}$ is the filtration

$$\mathcal{D}^1\mathfrak{g} \supset \mathcal{D}^2\mathfrak{g} \supset \mathcal{D}^3\mathfrak{g} \supset \cdots \supset \mathcal{D}^k\mathfrak{g} \supset \cdots$$

defined inductively by

$$\begin{aligned} \mathcal{D}^1\mathfrak{g} &:= [\mathfrak{g}, \mathfrak{g}] \\ \mathcal{D}^k\mathfrak{g} &:= [\mathcal{D}^{k-1}\mathfrak{g}, \mathcal{D}^{k-1}\mathfrak{g}] \end{aligned}$$

In fact both $\mathcal{D}^k\mathfrak{g}$ and $\mathcal{D}_k\mathfrak{g}$ are ideals of $\mathfrak{g}$, and $\mathcal{D}^k\mathfrak{g} \subset \mathcal{D}_k\mathfrak{g}$ for all k. The Lie algebra $\mathfrak{g}$ is defined to be *nilpotent* if $\mathcal{D}_k\mathfrak{g} = 0$ for some $k \in \mathbb{N}$, and *solvable* if $\mathcal{D}^k\mathfrak{g} = 0$ for some $k \in \mathbb{N}$.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *nilpotent* or *solvable* if $\mathfrak{h}$ is nilpotent or solvable when considered as a Lie algebra in its own right. The terms may also be applied to ideals of $\mathfrak{g}$, since every ideal of $\mathfrak{g}$ is also a subalgebra.