Lie algebra

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A *Lie algebra* over a field k is a vector space $\mathfrak{g}$ with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket* and denoted $(x, y) \mapsto [x, y]$. It is required to satisfy:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$.
2. The *Jacobi identity*: $[x, [y, z]]+[y, [z, x]]+[z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

1 Subalgebras & Ideals

A vector subspace $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ is a *subalgebra* if $\mathfrak{h}$ is closed under the Lie bracket operation, or, equivalently, if $\mathfrak{h}$ itself is a Lie algebra under the same bracket operation as $\mathfrak{g}$. An *ideal* of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ for which $[x, y] \in \mathfrak{h}$ whenever either $x \in \mathfrak{h}$ or $y \in \mathfrak{h}$. Note that every ideal is also a subalgebra.

Some general examples of subalgebras:

- The *center* of $\mathfrak{g}$, defined by $Z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x, y] = 0 \text{for all } y \in \mathfrak{g}\}$. It is an ideal of $\mathfrak{g}$.
- The *normalizer* of a subalgebra $\mathfrak{h}$ is the set $N(\mathfrak{h}) := \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subset \mathfrak{h}\}$. The Jacobi identity guarantees that $N(\mathfrak{h})$ is always a subalgebra of $\mathfrak{g}$.
- The *centralizer* of a subset $X \subset \mathfrak{g}$ is the set $C(X) := \{x \in \mathfrak{g} \mid [x, X] = 0\}$. Again, the Jacobi identity implies that $C(X)$ is a subalgebra of $\mathfrak{g}$.

2 Homomorphisms

Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ over the field k, a *homomorphism* from $\mathfrak{g}$ to $\mathfrak{g}'$ is a linear transformation $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$. An injective homomorphism is called a *monomorphism*, and a surjective homomorphism is called an *epimorphism*.

The *kernel* of a homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ (considered as a linear transformation) is denoted $\ker(\phi)$. It is always an ideal in $\mathfrak{g}$.

3 Examples

- Any vector space can be made into a Lie algebra simply by setting $[x, y] = 0$ for all vectors x, y. The resulting Lie algebra is called an *abelian* Lie algebra.
- If G is a Lie group, then the tangent space at the identity forms a Lie algebra over the real numbers.
- $\mathbb{R}^3$ with the cross product operation is a nonabelian three dimensional Lie algebra over $\mathbb{R}$.

4 Historical Note

Lie algebras are so-named in honour of Sophus Lie, a Norwegian mathematician who pioneered the study of these mathematical objects. Lie's discovery was tied to his investigation of continuous transformation groups and symmetries. One joint project with Felix Klein called for the classification of all finite-dimensional groups acting on the plane. The task seemed hopeless owing to the generally non-linear nature of such group actions. However, Lie was able to solve the problem by remarking that a transformation group can be locally reconstructed from its corresponding "infinitesimal generators", that is to say vector fields corresponding to various 1-parameter subgroups. In terms of this geometric correspondence, the group composition operation manifests itself as the bracket of vector fields, and this is very much a linear operation. Thus the task of classifying group actions in the plane became the task of classifying all finite-dimensional Lie algebras of planar vector field; a project that Lie brought to a successful conclusion.

This "linearization trick" proved to be incredibly fruitful and led to great advances in geometry and differential equations. Such advances are based, however, on various results from the theory of Lie algebras. Lie was the first to make significant contributions to this purely algebraic theory, but he was surely not the last.