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Cayley-Dickson construction

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In the foregoing discussion, an algebra shall mean a non-associative algebra.

Let A be a normed $*$-algebra, an algebra admitting an http://planetmath.org/Involution2in $*$, over a commutative ring R with $1 \neq 0$. The Cayley-Dickson construction is a way of enlarging A to a new algebra, $KD(A)$, extending the $*$ as well as the norm operations in A, such that A is a subalgebra of $KD(A)$.

Define $KD(A)$ to be the module (external) direct sum of A with itself:

$$KD(A) := A \oplus A.$$

Therefore, addition in $KD(A)$ is defined by addition componentwise in each copy of A. Next, let λ be a unit in R and define three additional operations:

1. (Multiplication) $(a \oplus b)(c \oplus d) := (ac + \lambda d^* b) \oplus (da + bc^*)$, where $*$ is the involution on A,
2. (Extended involution) $(a \oplus b)^* := a^* \oplus (-b)$, and
3. (Extended Norm) $N(a \oplus b) := (a \oplus b)(a \oplus b)^*$.

One readily checks that the multiplication is bilinear, since the involution $*$ (on A) is linear. Therefore, $KD(A)$ is an algebra.

Furthermore, since the extended involution $*$ is clearly bijective and linear, and that

$$(a \oplus b)^{**} = (a^* \oplus (-b))^* = a^{**} \oplus b = a \oplus b,$$

this extended involution is well-defined and so $KD(A)$ is in addition a $*$-algebra.

Finally, to see that $KD(A)$ is a normed $*$-algebra, we identify A as the first component of $KD(A)$, then A becomes a subalgebra of $KD(A)$ and elements of the form $a \oplus 0$ can now be written simply as a. Now, the extended norm

$$N(a \oplus b) = (a \oplus b)(a^* \oplus (-b)) = (aa^* - \lambda b^* b) \oplus 0 = N(a) - \lambda N(b) \in A,$$

where N in the subsequent terms of the above equation array is the norm on A given by $N(a) = aa^*$. The fact that the $N\colon KD(A) \to A$, together with the equality $N(0 \oplus 0) = 0$ show that the extended norm N on $KD(A)$ is well-defined. Thus, $KD(A)$ is a normed $*$-algebra.

The normed $*$-algebra $KD(A)$, together with the invertible element $\lambda \in R$, is called the *Cayley-Dickson algebra*, $KD(A, \lambda)$, *obtained from* A.

If A has a unity 1, then so does $KD(A, \lambda)$ and its unity is $1 \oplus 0$. Furthermore, write $i = 0 \oplus 1$, we check that, $ia = (0 \oplus 1)(a \oplus 0) = 0 \oplus a^* = (a^* \oplus 0)(0 \oplus 1) = a^* i$. Therefore, $iA = Ai$ and we can identify the second component of $KD(A, \lambda)$ with Ai and write elements of Ai as ai for $a \in A$.

It is not hard to see that $A(Ai) = (Ai)A \subseteq Ai$ and $(Ai)(Ai) \subseteq A$. We are now able to write

$$KD(A, \lambda) = A \oplus Ai,$$

where each element $x \in KD(A, \lambda)$ has a unique expression $x = a + bi$.

Properties. Let x, y, z will be general elements of $KD(A, \lambda)$.

1. $(xy)^* = y^* x^*$,
2. $x + x^* \in A$,
3. $N(xy) = N(x)N(y)$.

Examples. All examples considered below have ground ring the reals $\mathbb{R}$.

- $KD(\mathbb{R}, -1) = \mathbb{C}$, the complex numbers.
- $KD(\mathbb{C}, -1) = \mathbb{H}$, the quaternions.
- $KD(\mathbb{H}, -1) = \mathbb{O}$, the octonions.
- $KD(\mathbb{O}, -1) = \mathbb{S}$, which are called the *sedenions*, an algebra of dimension 16 over $\mathbb{R}$.

Remarks.

1. Starting from $\mathbb{R}$, notice each stage of Cayley-Dickson construction produces a new algebra that loses some intrinsic properties of the previous one: $\mathbb{C}$ is no longer orderable (or formally real); commutativity is lost in $\mathbb{H}$; associativity is gone from $\mathbb{O}$; and finally, $\mathbb{S}$ is not even a division algebra anymore!

2. More generally, given any field k, any algebra obtained by applying the Cayley-Dickson construction twice to k is called a *quaternion algebra* over k, of which $\mathbb{H}$ is an example. In other words, a quaternion algebra has the form

$$KD(KD(k, \lambda_1), \lambda_2),$$

where each $\lambda_i \in k^* := k - \{0\}$. Any algebra obtained by applying the Cayley-Dickson construction three times to k is called a *Cayley algebra*, of which $\mathbb{O}$ is an example. In other words, a Cayley algebra has the form

$$KD(KD(KD(k, \lambda_1), \lambda_2), \lambda_3),$$

where each $\lambda_i \in k^*$. A Cayley algebra is an *octonion algebra* when $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

References

[1] Richard D. Schafer, *An Introduction to Nonassociative Algebras*, Dover Publications, (1995).