root system underlying a semi-simple Lie algebra

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Crystallographic, reduced root systems are in one-to-one correspondence with semi-simple, complex Lie algebras. First, let us describe how one passes from a Lie algebra to a root system. Let $\mathfrak{g}$ be a semi-simple, complex Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra. Since $\mathfrak{g}$ is semi-simple, $\mathfrak{h}$ is abelian. Moreover, $\mathfrak{h}$ acts on $\mathfrak{g}$ (via the adjoint representation) by commuting, simultaneously diagonalizable linear maps. The simultaneous eigenspaces of this $\mathfrak{h}$ action are called *root spaces*, and the decomposition of $\mathfrak{g}$ into $\mathfrak{h}$ and the root spaces is called a *root decomposition* of $\mathfrak{g}$. To be more precise, for $\lambda \in \mathfrak{h}^*$, set

$$\mathfrak{g}_\lambda = \{a \in \mathfrak{g} \colon [h, a] = \lambda(h)a \text{ for all } h \in \mathfrak{h}\}.$$

We call a non-zero $\lambda \in \mathfrak{h}^*$ a root if $\mathfrak{g}_\lambda$ is non-trivial, in which case $\mathfrak{g}_\lambda$ is called a root space. It is possible to show that that $\mathfrak{g}_0$ is just the Cartan subalgebra $\mathfrak{h}$, and that $\dim \mathfrak{g}_\lambda = 1$ for each root λ. Letting $R \subset \mathfrak{h}^*$ denote the set of all roots, we have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in R} \mathfrak{g}_\lambda.$$

The Cartan subalgebra $\mathfrak{h}$ has a natural inner product, called the Killing form, which in turn induces an inner product on $\mathfrak{h}^*$. It is possible to show that, with respect to this inner product, R is a reduced, crystallographic root system.

Conversely, let $R \subset E$ be a reduced, crystallographic root system. Let Δ be a base of positive roots. We define a Lie algebra by taking generators

$$H_\lambda, X_\lambda, Y_\lambda, \quad \lambda \in \Delta,$$

subject to the following relations:

$$\begin{aligned}
[H_\lambda, H_\mu] &= 0, \\
[H_\mu, X_\lambda] &= (\lambda, \mu) X_\lambda, \\
[H_\mu, Y_\lambda] &= -(\lambda, \mu) Y_\lambda, \\
[X_\lambda, Y_\lambda] &= H_\lambda, \\
[X_\lambda, Y_\mu] &= 0, \quad \lambda \neq \mu; \\
(\operatorname{ad} X_\lambda)^{-(\lambda,\mu)+1}(X_\mu) &= 0, \quad \lambda \neq \mu, \\
(\operatorname{ad} Y_\lambda)^{-(\lambda,\mu)+1}(Y_\mu) &= 0, \quad \lambda \neq \mu,
\end{aligned}$$

The above are known as the Chevalley-Serre relations The resulting Lie algebra turns out to be semi-simple, with a root system isomorphic to the given R.

Thanks to the above isomorphism, to the difficult task of classifying complex semi-simple Lie algebras is transformed into the somewhat easier task of classifying crystallographic, reduced roots systems. Furthermore, a complex Lie algebra is simple if and only if the corresponding root system is indecomposable. Thus, we only need to classify indecomposable root systems, since all other root systems and semi-simple Lie algebras are built out of these.