commutator bracket

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Let A be an associative algebra over a field K. For $a, b \in A$, the element of A defined by

$$[a, b] = ab - ba$$

is called the *commutator* of a and b. The corresponding bilinear operation

$$[-, -] : A \times A \to A$$

is called the commutator bracket.

The commutator bracket is bilinear, skew-symmetric, and also satisfies the Jacobi identity. To wit, for $a, b, c \in A$ we have

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

The proof of this assertion is straightforward. Each of the brackets in the left-hand side expands to 4 terms, and then everything cancels.

In categorical terms, what we have here is a functor from the category of associative algebras to the category of Lie algebras over a fixed field. The action of this functor is to turn an associative algebra A into a Lie algebra that has the same underlying vector space as A, but whose multiplication operation is given by the commutator bracket. It must be noted that this functor is right-adjoint to the universal enveloping algebra functor.

Examples

- Let V be a vector space. Composition endows the vector space of endomorphisms $\mathrm{End}V$ with the structure of an associative algebra. However, we could also regard $\mathrm{End}V$ as a Lie algebra relative to the commutator bracket:

$$[X, Y] = XY - YX, \quad X, Y \in \mathrm{End}V.$$

- The algebra of differential operators has some interesting properties when viewed as a Lie algebra. The fact is that even though the composition of differential operators is a non-commutative operation, it is commutative when restricted to the highest order terms of the involved operators. Thus, if X, Y are differential operators of order p and q, respectively, the compositions XY and YX have order $p + q$. Their highest order term coincides, and hence the commutator $[X, Y]$ has order $p + q - 1$.

- In light of the preceding comments, it is evident that the vector space of first-order differential operators is closed with respect to the commutator bracket. Specializing even further we remark that, a vector field is just a homogeneous first-order differential operator, and that the commutator bracket for vector fields, when viewed as first-order operators, coincides with the usual, geometrically motivated vector field bracket.