universal enveloping algebra

Canonical name	UniversalEnvelopingAlgebra
Date of creation	2013-03-22 13:03:35
Last modified on	2013-03-22 13:03:35
Owner	draisma (715)
Last modified by	draisma (715)
Numerical id	7
Author	draisma (715)
Entry type	Definition
Classification	msc 17B35
Classification	msc 16S30
Related topic	LieAlgebra
Related topic	PoincareBirkhoffWittTheorem
Related topic	WeylAlgebra
Related topic	FreeLieAlgebra

A *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ over a field k is an associative `http://planetmath.org/node/Algebra`algebra U (with unity) over k, together with a Lie algebra homomorphism $\iota : \mathfrak{g} \to U$ (where the Lie algebra structure on U is given by the commutator), such that if A is a another associative algebra over k and $\phi : \mathfrak{g} \to A$ is another Lie algebra homomorphism, then there exists a unique homomorphism $\psi : U \to A$ of associative algebras such that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\phi"'] & U \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

commutes. Any $\mathfrak{g}$ has a universal enveloping algebra: let T be the associative tensor algebra generated by the vector space $\mathfrak{g}$, and let I be the two-sided ideal of T generated by elements of the form

$$xy - yx - [x, y] \text{ for } x, y \in \mathfrak{g};$$

then $U = T/I$ is a universal enveloping algebra of $\mathfrak{g}$. Moreover, the universal property above ensures that all universal enveloping algebras of $\mathfrak{g}$ are canonically isomorphic; this justifies the standard notation $U(\mathfrak{g})$.

Some remarks:

1. By the Poincaré-Birkhoff-Witt theorem, the map ι is injective; usually $\mathfrak{g}$ is identified with $\iota(\mathfrak{g})$. From the construction above it is clear that this space generates $U(\mathfrak{g})$ as an associative algebra with unity.

2. By definition, the (left) representation theory of $U(\mathfrak{g})$ is identical to that of $\mathfrak{g}$. In particular, any irreducible $\mathfrak{g}$-module corresponds to a maximal left ideal of $U(\mathfrak{g})$.

Example: let $\mathfrak{g}$ be the Lie algebra generated by the elements p, q, and e with Lie bracket determined by $[p, q] = e$ and $[p, e] = [q, e] = 0$. Then $U(g)/(e-1)$ (where $(e-1)$ denotes the two-sided ideal generated by $e-1$) is isomorphic to the skew polynomial algebra $k[x, \frac{\partial}{\partial x}]$, the isomorphism being determined by

$$\begin{aligned} &p + (e-1) \mapsto \frac{\partial}{\partial x} \text{ and} \\ &q + (e-1) \mapsto x. \end{aligned}$$