composition algebras over finite fields

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Theorem 1. *There are 5 non-isomorphic composition algebras over a finite field k of characteristic not 2, 2 division algebras and 3 split algebras.*

1. *The field k.*
2. *The unique quadratic extension field K/k.*
3. *The* exchange *algebra: $k \oplus k$.*
4. 2×2 *matrices over k: $M_2(k)$.*
5. *The split Cayley algebra.*

Proof. Following Hurwitz's theorem every composition algebra is given by the Cayley-Dickson construction and has dimension 1,2, 4 or 8. Now we consider the possible non-degenerate quadratic forms of these dimensions.

Since every anisotropic 2 space corresponds to a quadratic field extension, and our field is finite, it follows that there is at most one anisotropic 2 subspace of our quadratic form. Therefore if $\dim C > 2$ then the quadratic form is isotropic and so the algebra is a split. Therefore in the Cayley-Dickson construction over a finite field there every quaternion algebra is split, thus $M_2(k)$. To build the non-associative division Cayley algebra of dimension 8 requires we start the Cayley-Dickson construction with a division ring which is not a field, and thus there are no Cayley division algebras over finite fields. □

This result also can be seen as a consequence of Wedderburn's theorem that every finite division ring is a field. Likewise, a theorem of Artin and Zorn asserts that every finite alternative division ring is in fact associative, thus excluding the Cayley algebras in a fashion similar to how Wedderburn's theorem excludes division quaternion algebras.