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Kac-Moody algebra

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Let A be an $n \times n$ generalized Cartan matrix. If $n-r$ is the rank of A, then let $\mathfrak{h}$ be a $n+r$ dimensional complex vector space. Choose n linearly independent elements $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$ (called *roots*), and $\check{\alpha}_1, \ldots, \check{\alpha}_n \in \mathfrak{h}$ (called *coroots*) such that $\langle \alpha_i, \check{\alpha}_j \rangle = a_{ij}$, where $\langle \cdot, \cdot \rangle$ is the natural pairing of $\mathfrak{h}^*$ and $\mathfrak{h}$. This choice is unique up to automorphisms of $\mathfrak{h}$.

Then the Kac-Moody algebra associated to $\mathfrak{g}(A)$ is the Lie algebra generated by elements $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and the elements of $\mathfrak{h}$, with the relations

$$\begin{aligned}
[X_i, Y_i] &= \check{\alpha}_i & [X_i, Y_j] &= 0 \\
[X_i, h] &= \alpha_i(h) X_i & [Y_i, h] &= -\alpha_i(h) Y_i \\
\underbrace{[X_i, [X_i, \cdots, [X_i}_{1-a_{ij} \text{ times}}, X_j] \cdots]] &= 0 & \underbrace{[Y_i, [Y_i, \cdots, [Y_i}_{1-a_{ij} \text{ times}}, Y_j] \cdots]] &= 0
\end{aligned}$$

for any $h \in \mathfrak{h}$.

If the matrix A is positive-definite, we obtain a finite dimensional semisimple Lie algebra, and A is the Cartan matrix associated to a Dynkin diagram. Otherwise, the algebra we obtain is infinite dimensional and has an r-dimensional center.