regular element of a Lie algebra

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An element $X \in \mathfrak{g}$ of a Lie algebra is called *regular* if the dimension of its centralizer $\zeta_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \mid [X, Y] = 0\}$ is minimal among all centralizers of elements in $\mathfrak{g}$.

Regular elements clearly exist and moreover they are Zariski dense in $\mathfrak{g}$. The function $X \mapsto \dim \zeta_{\mathfrak{g}}(X)$ is an upper semi-continuous function $\mathfrak{g} \to \mathbb{Z}_{\geq 0}$. Indeed, it is a constant minus $\operatorname{rank}(ad_X)$ and $X \mapsto \operatorname{rank}(ad_X)$ is lower semi-continuous. Thus the set of elements whose centralizer dimension is (greater than or) equal to that of any given regular element is Zariski open and non-empty.

If $\mathfrak{g}$ is reductive then the minimal centralizer dimension is equal to the rank of $\mathfrak{g}$.

More generally if V is a representation for a Lie algebra $\mathfrak{g}$, an element $v \in V$ is called *regular* if the dimension of its stabilizer is minimal among all stabilizers of elements in V.

Examples

1. In $\mathfrak{sl}_n\mathbb{C}$ a diagonal matrix $X = \operatorname{diag}(s_1, \ldots, s_n)$ is regular iff $(s_i - s_j) \neq 0$ for all pairs $1 \leq i < j \leq n$. Any conjugate of such a matrix is also obviously regular.

2. In $\mathfrak{sl}_n\mathbb{C}$ the nilpotent matrix

$$\begin{pmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & 1 \\ 0 & & \cdots & & 0 \end{pmatrix}$$

   is regular. Moreover, it's adjoint orbit contains the set of all regular nilpotent elements. The centralizer of this matrix is the full subalgebra of trace zero, diagonal matricies.