Grothendieck category theorems

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1 Two New Theorems for Grothendieck categories

1.1 Introduction

The theory has its origin in the work of Grothendieck [**?**] who introduced the following notation of properties of abelian categories:

Ab3. An abelian category with coproducts or equivalently, a cocomplete abelian category.

Ab5. Ab3-category, in which for any directed family $\{A_i\}_{i\in I}$ of subobjects of an arbitrary object X and for any subobject B of X the following relation holds:

$$(\sum_{i\in I} A_i) \cap B = \sum_{i\in I}(A_i \cap B)$$

Ab5-categories possessing a family of generators are called *Grothendieck categories*. They constitute a natural extension of the class of module categories, with which they share a great number of important properties.

The **Popescu-Gabriel Theorem** is generalized as follows.

Theorem[Popescu and Gabriel] *Let* $\mathbf{G}$ *be a Grothendieck category with a family of generators* $\{U_i\}_{i\in I}$ *and* $T = (-,?) : \mathbf{G} \to mcAb$ *be the representation functor that takes each* $X \in \mathbf{G}$ *to* $(-, X)$*, where* $mcAb = \{h_{U_i} = (-, U_i)\}_{i\in I}$*. Then:*

(1.) T *is full and faithful.*

(2.) T *induces an equivalence between* $\mathbf{G}$ *and the quotient category* $mcAb\mathbf{S}$*, where* $\mathbf{S}$ *denotes the largest localizing subcategory in* $mcAb$ *for which all modules* $TX = (-, X)$ *are* $\mathbf{S}$*-closed.*

This extension of the **Popescu-Gabriel Theorem** is due to *Grigory Garkusha* from the Saint-Petersburg State University, Higher Algebra and Number Theory Department, School of Mathematics and Mechanics, Bibliotechnaya Sq. 2, 198904 (Russia).

The advantage of this Theorem is that we can freely choose a family of generators U of $\mathbf{G}$. To be precise, if M is an arbitrary family of objects of $\mathbf{G}$, then the family: $\{U_i\}_{i\in I} = U \cup M$ is also a family of generators.

We say that an object C of $\mathbf{G}$ is *U-finitely generated* (or respectively *U-finitely presented*) if there is an epimorphism $\eta : \psi^n_{i=1} U_i \to C$ (if there is an exact sequence $\psi^n_{i=1} U_i \to \psi^m_{j=1} U_i \to C$) where $U_i \in U$. The full subcategory of U-finitely generated (U-finitely presented) objects of $\mathbf{G}$ is denoted by $\mathbf{fg}_U\mathbf{G}$ ($\mathbf{fp}_U\mathbf{G}$). When every $U_i \in U$ is finitely generated (finitely presented), that is the functor $(U_i, -)$ preserves direct unions (limits), we write $\mathbf{fg}_{\mathbf{G}} = \mathbf{fg} \in \mathbf{G}$ ($\mathbf{fp}_{U(\mathbf{G})} = \mathbf{fp} \in \mathbf{G}$). Then every Grothendieck category is locally U-finitely generated (locally U-finitely presented) which means that every object C of $\mathbf{G}$ is a direct union (limit)

$$C = \sum_{i \in I} C_i$$

,

$(C = \mathbf{lp}_{i \in I} C_i)$

of U-finitely generated (U-finitely presented) objects C_i.

Recall also that a localizing subcategory $\mathbf{S}$ of $\mathbf{G}$ is of prefinite (finite) type provided that the inclusion functor $\mathbf{J} : \mathbf{S} \to \mathbf{G}$ commutes with direct unions (limits). So the following proposition holds.

Theorem[Breitsprecher] Let $\mathbf{G}$ be a Grothendieck category with a family of generators $U = \{U_i\}_{i \in I}$. Then the representation functor

$T = (-, ?) : \mathbf{G} \to \mathbf{fp}_U\mathbf{G}(\mathrm{op}, \mathrm{Ab})$

defines an equivalence between $\mathbf{G}$ and $(\mathbf{fp}_U\mathbf{G})^{\mathrm{op}}, \mathrm{Ab})/\mathbf{S}$, where $\mathbf{S}$ is some localizing subcategory of $\mathbf{fp}_U\mathbf{G}^{(\mathrm{op}, \mathrm{Ab})}$.

Moreover, $\mathbf{S}$ is of finite type if and only if $\mathbf{fp}_U\mathbf{G} = \mathbf{fp}\mathbf{G}$. In this case, $\mathbf{G}$ is equivalent to the category

$\mathbf{Lex}((\mathbf{fp}_U\mathbf{G})^{(\mathrm{op}, \mathrm{Ab})})$ of contravariant left exact functors from $\mathbf{fp}_U\mathbf{G}$ to Ab.

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more to come...