direct limit of sets

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Let $\mathcal{A} = \{A_i \mid i \in I\}$ be a family of sets indexed by a non-empty set I. $\mathcal{A}$ is said to be a *direct family* if

1. I is a directed set,
2. whenever $i \leq j$ in I, there is a function $\phi_{ij} : A_i \to A_j$,
3. ϕ_{ii} is the identity function on A_i,
4. if $i \leq j \leq k$, then $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$.

In the last condition, if we write $a\phi_{ij} := \phi_{ij}(a)$ for $a \in A_i$, then the equation can be rewritten as $\phi_{ij}\phi_{jk} = \phi_{ik}$.

For example, the natural numbers $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$ can be regarded as a direct family. Here, for any $i \leq j$, $\phi_{ij} : i \to j$ is given by the natural injection $\phi_{ij}(\ell) := \ell$ for any $\ell \in i$.

Let $\mathcal{A}$ be a direct family of sets, indexed by I. Take the disjoint union of the members of $\mathcal{A}$ and call it A (this can be achieved even when the members themselves have non-empty intersections, simply form the product $A_i \times \{i\}$ first before taking the union). Therefore, A has the properties that

- for any $a \in A$, $a \in A_i$ for some $i \in I$, and
- if $a \in A_i$ and $b \in A_j$ and $i \neq j$, then $a \neq b$.

Define a binary relation $\sim$ on A as follows: given that $a \in A_i$ and $b \in A_j$, $a \sim b$ iff there is A_k such that $\phi_{ik}(a) = \phi_{jk}(b)$.

Proposition 1. *$\sim$ on A is an equivalence relation.*

Proof. Clearly, $\sim$ is symmetric. By condition 2 of a direct family, $\sim$ is also reflexive. Now, suppose $a \sim b$ and $b \sim c$ with $a \in A_i$, $b \in A_j$ and $c \in A_k$. So there are $p, q \in I$ such that $\phi_{ip}(a) = \phi_{jp}(b)$ and $\phi_{jq}(b) = \phi_{kq}(c)$. Since I is directed, there is $r \in I$ such that $p, q \leq r$. From this, we have $\phi_{ir}(a) = \phi_{pr}(\phi_{ip}(a)) = \phi_{pr}(\phi_{jp}(b)) = \phi_{jr}(b)$. Similarly, $\phi_{kr}(c) = \phi_{qr}(\phi_{kq}(c)) = \phi_{qr}(\phi_{jq}(b))$. Hence $a \sim c$. $\square$

Definition. Let $\mathcal{A}$ be a direct family of sets indexed by I. Let A and $\sim$ be defined as above. Then the quotient $A/\sim$ is called the *direct limit* of the sets in $\mathcal{A}$. The direct limit of sets A_i is sometimes written A_∞, or $\varinjlim A_i$. Elements of A_∞ are sometimes denoted by $[a]_I$ or $[a]$ whenever there is no confusion.

Remarks.

- This definition is consistent with the formal definition of direct limits in a category. The index I, being a directed set, can be viewed as a category whose objects are elements of I and morphisms defined by the partial order on I.

- The notation A_∞ comes from the following fact: if $|I| = n < \infty$, then $\varinjlim A_i \cong A_n$. Here, $\cong$ stands for bijection.

- For every $i \in I$, there is a natural mapping $A_i \to A_\infty$, given by $a \mapsto [a]_I$. This map may be variously denoted by ϕ_i, $\phi_{i\infty}$, or ϕ_{iI}.

- Let J be a subset of a directed set I. Let $\mathcal{A}$ be a direct family indexed by I and $\mathcal{A}' \subseteq \mathcal{A}$ indexed by J. Form the direct limit A'_∞ of sets in $\mathcal{A}'$. Then there is a natural mapping $\phi_{JI} : A'_\infty \to A_\infty$ such that for any $j \in J$, $\phi_{JI} \circ \phi_{jJ} = \phi_{jI}$.

The dual notion of a direct limit of sets is that of an inverse limit. Instead of starting from a direct family of sets, we start with an *inverse family* of sets, which is defined similarly to that to of a direct family, except I is a filtered set, and the mappings $\phi_{ij} : A_i \to A_j$ is defined whenever $j \leq i$. An inverse family is also known as an *inverse system*, or a *projective system*.