connected category

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Let $\mathcal{C}$ be a category. Two objects A, B in $\mathcal{C}$ are said to be *joined* if there is a morphism with domain one object and codomain the other. In other words, $\hom(A,B) \cup \hom(B,A) \neq \varnothing$. Two objects A, B are said to be *connected* if there is a finite sequence of objects in $\mathcal{C}$

$$A = C_1, C_2, \ldots, C_n = B$$

such that C_i, C_{i+1} are joined for $i = 1, \ldots, n-1$.

A category is said to be *connected* if every pair of objects are connected, and *strongly connected* if every pair of objects are joined.

For example, every category with either an initial object or a terminal object is connected. If a category has a zero object, it is strongly connected.

A small category may be viewed as a graph or a digraph. Then the underlying graph of a small connected category is connected, and the underlying digraph of a small strongly connected category is strongly connected. Conversely, the free category freely generated a connected graph is connected, and the free category freely generated by a strongly connected digraph is strongly connected.

The relation (on objects of $\mathcal{C}$) of being joined is in general not an equivalence relation (it is reflexive and symmetric, but not transitive). Let us call this relation R. The relation of being connected, on the other hand, is an equivalence relation, and is the transitive closure R^* of R. Therefore, we may partition the class of objects in $\mathcal{C}$ by R^*. Furthermore, R^* induces an equivalence relation R' on the class of all morphisms in $\mathcal{C}$: for morphisms f, g, set

$$fR'g \qquad \text{iff} \qquad \operatorname{dom}(f) R^* \operatorname{dom}(g).$$

If A is an object of $\mathcal{C}$, denote $[A]$ the equivalence class containing A under R^*, together with the equivalence class containing 1_A under R'. Then $[A]$ is a connected full subcategory of $\mathcal{C}$. $[A]$ is called a connected component of $\mathcal{C}$. Every small category can be expressed as the disjoint union of its connected components.

References

[1] S. Mac Lane, *Categories for the Working Mathematician*, Springer, New York (1971).