simplicial category

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The **simplicial category** Δ is defined as the small category whose objects are the totally ordered finite sets

$$[n] = \{0 < 1 < 2 < \ldots < n\}, \quad n \geq 0, \tag{1}$$

and whose morphisms are monotonic non-decreasing (order-preserving) maps. It is generated by two families of morphisms:

$$\begin{array}{lll} \delta_i^n & : & [n-1] \to [n] \quad \text{is the injection missing } i \in [n], \\ \sigma_i^n & : & [n+1] \to [n] \quad \text{is the surjection such that } \sigma_i^n(i) = \sigma_i^n(i+1) = i \in [n]. \end{array}$$

The δ_i^n morphisms are called **face maps**, and the σ_i^n morphisms are called **degeneracy maps**. They satisfy the following relations,

$$\delta_j^{n+1}\,\delta_i^n = \delta_i^{n+1}\,\delta_{j-1}^n \quad \text{for } i < j, \tag{2}$$

$$\sigma_j^{n-1}\,\sigma_i^n = \sigma_i^{n-1}\,\sigma_{j+1}^n \quad \text{for } i \leq j, \tag{3}$$

$$\sigma_j^n\,\delta_i^{n+1} = \begin{cases} \delta_i^n\,\sigma_{j-1}^{n-1} & \text{if } i < j, \\ \mathrm{id}_n & \text{if } i = j \text{ or } i = j+1, \\ \delta_{i-1}^n\,\sigma_j^{n-1} & \text{if } i > j+1. \end{cases} \tag{4}$$

All morphisms $[n] \to [0]$ factor through σ_0^0, so $[0]$ is terminal.

There is a bifunctor $+ \colon \Delta \times \Delta \to \Delta$ defined by

$$[m] + [n] = [m+n+1], \tag{5}$$

$$(f+g)(i) = \begin{cases} f(i) & \text{if } 0 \leq i \leq m, \\ g(i-m-1) + m' + 1 & \text{if } m < i \leq (m+n+1), \end{cases} \tag{6}$$

where $f \colon [m] \to [m']$ and $g \colon [n] \to [n']$. Sometimes, the simplicial category is defined to include the empty set $[-1] = \emptyset$, which provides an initial object for the category. This makes Δ a strict monoidal category as $\emptyset$ is a unit for the bifunctor: $\emptyset + [n] = [n] = [n] + \emptyset$ and $\mathrm{id}_\emptyset + f = f = f + \mathrm{id}_\emptyset$. Further, Δ is then the free monoidal category on a monoid object (the monoid object being $[0]$, with product $\sigma_0^0 \colon [0] + [0] \to [0]$).

There is a fully faithful functor from Δ to **Top**, which sends each object $[n]$ to an oriented n-simplex. The face maps then embed an $(n-1)$-simplex in an n-simplex, and the degeneracy maps collapse an $(n+1)$-simplex to an n-simplex. The bifunctor forms a simplex from the disjoint union of two simplicies by joining their vertices together in a way compatible with their orientations.

There is also a fully faithful functor from Δ to **Cat**, which sends each object $[n]$ to a pre-order $\mathbf{n+1}$. The pre-order $\mathbf{n}$ is the category consisting of n partially-ordered objects, with one morphism $a \to b$ if and only if $a \leq b$.