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categorical pullback

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Pullbacks

Let $f : X \to B$ and $g : Y \to B$ be morphisms in a category $\mathcal{C}$. Then a *pullback diagram*, or *pullback square* of f and g is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

such that if we have another commutative diagram

$$\begin{tikzcd} Z \arrow[r, "s"] \arrow[d, "r"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

then there is a unique morphism $h : Z \to A$ with the commutative diagram

$$\begin{tikzcd} Z \arrow[dr, dotted, "h"] \arrow[drr, bend left, "s"] \arrow[ddr, bend right, "r"'] & & \\ & A \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & B. \end{tikzcd}$$

A *pullback* of (f, g) is the ordered triple (A, q, p). We also say that p is a *pullback of* g *along* f, and q a *pullback of* f *along* g. When a pullback of (f, g) exists, it is unique up to isomorphism. The object A, and the morphisms p and q in the diagram above are often denoted by

$$X \times_B Y, \qquad 1_X \times_B g \qquad \text{and} \qquad f \times_B 1_Y$$

respectively, and the uniquely determined morphism h by

$$\begin{pmatrix} r \\ s \end{pmatrix}.$$

It is easy to see that

$$X \times_B Y \cong Y \times_B X,$$

whenever one (and hence the other) exists.

Remarks.

- The pullback of f and g can be equivalently defined as a limiting cone over the diagram $X \to B \leftarrow Y$. In other words, a pullback diagram is a terminal object in the category of commutative squares of the form

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

- A category $\mathcal{C}$ is said to *have pullbacks* if every diagram $X \to B \leftarrow Y$ can be completed into a pullback diagram.

- The notion of pullbacks can be generalized: let $\{x_i : C_i \to C \mid i \in I\}$ be a collection of morphisms indexed by set I, considered as a small diagram. The *generalized pullback* of the x_i's is just the limiting cone of the diagram. Using this definition, the generalized pullback of one morphism f is the identity morphism of $\mathrm{dom}(f)$, the domain of f, and the generalized pullback of the empty set is a terminal object.

- A pullback is sometimes known as an *amalgamated sum*.

Pushouts

Dually, given morphisms $f : B \to X$ and $g : B \to Y$, a *pushout square* of f and g is a commutative diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "s"] \\ X \arrow[r, "t"'] & A \end{tikzcd}$$

such that if we have another commutative diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "u"] \\ X \arrow[r, "v"'] & Z \end{tikzcd}$$

Then there is a unique morphism $h : A \to Z$ with the following commutative diagram:

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "s"] \arrow[ddr, bend left, "u"] & \\
X \arrow[r, "t"'] \arrow[drr, bend right, "v"'] & A \arrow[dr, dotted, "h"] & \\
& & Z.
\end{tikzcd}$$

The pair (s, t) of morphisms is a *pushout* of (f, g). s is the the *pushout of f along g*, and t the *pushout of g along f*. Like pullbacks, pushouts are unique up to unique isomorphism when they exist. The object A and the morphisms s and t are typically written

$$X \amalg_B Y, \qquad f \amalg_B 1_Y \qquad \text{and} \qquad 1_X \amalg_B g,$$

and the unique morphism h is denoted by

$$(f \quad g).$$

Remark. The pushout of f and g can be thought of as the limiting cocone under the diagram $X \leftarrow B \to Y$. Equivalently, they are initial objects in the category of commutative squares whose top edge is $B \to Y$ and left edge is $B \to X$. A category is said to *have pushouts* if every diagram $X \leftarrow B \to Y$ can be completed to a pushout diagram. The *generalized pushout* is defined as the limiting cocone under the diagram consisting of morphisms $y_i : B \to B_i$, where i belongs to some set I.