category of small categories

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Defines	Cat

The category **Cat** of small categories consists of all small categories as objects, and, functors between small categories as morphisms. The composition of morphisms in **Cat** is the functor composition, and, associated with each small category, the identity functor acts as the identity morphism. Now, **Cat** is indeed a category, since $\hom(\mathcal{C}, \mathcal{D})$, the class of all functors from $\mathcal{C}$ to $\mathcal{D}$ is a set. The proof of this fact can be found `http://planetmath.org/FunctorCategory`here.

Here are some of the basic properties of **Cat**:

1. It has `http://planetmath.org/ProductOfCategories`arbitrary products
2. It has `http://planetmath.org/DisjointUnionOfCategories`arbitrary coproducts
3. Initial object exists: the initial object is the empty category and the associated empty functor.
4. Terminal object exists: the terminal object is any trivial category and the associated constant functor into the trival category.
5. It has pullbacks. See `http://planetmath.org/ExamplesOfPullbacks`this entry. So it has equalizers, and therefore, it is complete.
6. however, it does have coequalizers. This, together with 2 above, shows that it is cocomplete.

Remarks.

- If we replace functors in $\hom(\mathcal{C}, \mathcal{D})$ by natural transformations between pairs of functors from $\mathcal{C}$ to $\mathcal{D}$, and composition of morphisms the horizontal composition $\circ$ of natural transformations, then we again end up with a category (provided that both $\mathcal{C}$ and $\mathcal{D}$ are small). Indeed, every natural transformation η between two functors from $\mathcal{C}$ to $\mathcal{D}$ is a set function from the *set* of objects of $\mathcal{C}$ to the *set* of morphisms of $\mathcal{D}$. As a result, $\hom(\mathcal{C}, \mathcal{D})$ is a subcollection of the *set* of *all* functions from $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Mor}(\mathcal{D})$, and hence a set. For more detail, please see `http://planetmath.org/CompositionsOfNaturalTransformations`this entry.
- In fact, **Cat** has the structure of a 2-category, where the small categories are the 0-cells, the functors between them are the 1-cells, and the natural transformations between parallel functors are the 2-cells.

- If we remove the requirement that each object in **Cat** be small, then $\hom(\mathcal{C}, \mathcal{D})$ may no longer be a set, and we end up with a large category.