product of categories

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There are occasions when we need to consider objects from different categories being paired up. For example, if $\mathcal{C}$ is a category, then $\hom(A,B)$ where A,B are objects of $\mathcal{C}$ is a set, and we can think of hom as a functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to the category of sets. But what is $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ exactly? We will give this a formal definition presently.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Define the *Cartesian product* $\mathcal{C} \times \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ as the following pair (O, M), where

- O is the class consisting of ordered pairs (X, Y), where X is an object in $\mathcal{C}$ and Y is an object in $\mathcal{D}$
- M is the class consisting of ordered pairs (f, g), where f is a morphism in $\mathcal{C}$ and g is a morphism in $\mathcal{D}$.

There is a category structure on $\mathcal{C} \times \mathcal{D}$. But several things need to be defined first.

1. Elements of O are called the *objects* of $\mathcal{C} \times \mathcal{D}$ and elements of M are called the *morphisms* of $\mathcal{C} \times \mathcal{D}$. For each morphism $(f,g) \in M$, we define the domain and codomain operations

   $$\mathrm{dom}(f,g) := (\mathrm{dom}(f), \mathrm{dom}(g)) \quad \text{and} \quad \mathrm{cod}(f,g) := (\mathrm{cod}(f), \mathrm{cod}(g)).$$

   Note that for simplicity, we have used the same symbol dom and cod for $\mathcal{C}, \mathcal{D}$, and $\mathcal{C} \times \mathcal{D}$.

2. Next, for each pair of objects $A, B \in \mathcal{C} \times \mathcal{D}$, we have a set $\hom(A,B)$ consisting of all morphisms in $\mathcal{C} \times \mathcal{D}$ whose domain is A and codomain is B. Note that $\hom(A,B)$ is a set because it is $\hom(X,Z) \times \hom(Y,T)$, where $A = (X,Y)$ and $B = (Z,T)$ and each component in the product is assumed to be a set.

3. Finally, for objects A, B, C in $\mathcal{C} \times \mathcal{D}$, we have a function $\circ$ called composition:

   $$\circ : \hom(A,B) \times \hom(B,C) \to \hom(A,C).$$

   To define $\circ$, write each object A, B, C as ordered pairs: $A = (X,Y)$, $B = (Z,T)$, $C = (U,V)$. In addition, let $\alpha = (f,g) \in \hom(A,B)$ and $\beta = (p,q) \in \hom(B,C)$. Then

   $$\circ(\alpha, \beta) := (\circ_1(f,p), \circ_2(g,q)),$$

where $\circ_1$ and $\circ_2$ are compositions defined in $\mathcal{C}$ and $\mathcal{D}$ respectively, such that

$$\circ_1 : \hom(X,Z)\times\hom(Z,U) \to \hom(X,U) \text{ and } \circ_2 : \hom(Y,T)\times\hom(T,V) \to \hom(Y,V).$$

As usual, we write $\beta \circ \alpha$ for $\circ(\alpha, \beta)$.

4. Now, it is not hard to see that $\mathcal{C} \times \mathcal{D}$ with $\circ$ is a category. For example, let us verify that $(A, B) \neq (C, D)$ implies $\hom(A, B)\cap\hom(C, D) = \varnothing$. Write $A = (X, Y)$, $B = (Z, S)$, $C = (T, U)$ and $D = (V, W)$. Suppose $\alpha = (f, g) \in \hom(A, B)\cap\hom(C, D)$. Then $f \in \hom(X, Z)\cap\hom(T, V)$ and $g \in \hom(Y, S) \cap \hom(U, W)$. But this implies $X = T$, $Z = V$, $Y = U$, and $S = W$. So $A = (X, Y) = (T, U) = C$ and $B = (Z, S) = (V, W) = D$.

Remarks.

- The above construction can be generalized to n-fold Cartesian products. If $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be categories. Then $\mathcal{C} := \mathcal{C}_1 \times \cdots \mathcal{C}_n$ can be defined much the same way as in the case $n = 2$. $\mathcal{C}$ is a category and is sometimes written $\prod_{i=1}^n \mathcal{C}_i$.
- Associated with this product, we can form n (covariant) functors called *projection functors* $\Pi_i : \mathcal{C} \to \mathcal{C}_i$, given by $\Pi_i(A) = A_i$ and $\Pi_i(\alpha) = \alpha_i$, where $A = (A_1, \ldots, A_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$.
- The product $\mathcal{C}$ of $\mathcal{C}_i$ also enjoys the universal property that for every category $\mathcal{D}$ and functors $F_i : \mathcal{D} \to \mathcal{C}_i$, there is a unique functor $F : \mathcal{D} \to \mathcal{C}$ such that $\Pi_i \circ F = F_i$ (in other words, F_i factors through F).
- If fact, any category that enjoys the universal property described above is naturally equivalent to the product of $\mathcal{C}_i$. We may actually define product category this way, and then prove its existence using the construction that is given as the definition at the beginning of this article.
- More generally, we can define arbitrary (direct) product of categories. The definition is completely similar to the one above. If $\{\mathcal{C}_i \mid i \in I\}$ is a family of categories indexed by a set I, we often write $\prod_{i\in I} \mathcal{C}_i$ as the product category. Objects and morphisms are written $(A_i)_{i\in I}$ and $(\alpha_i)_{i\in I}$ respectively. When all the $\mathcal{C}_i$ are identical, say, equal to $\mathcal{C}$, we also write the product as $\mathcal{C}^I$, and call it the *I-fold direct product of $\mathcal{C}$*.

- The existence of the product of categories indexed by an arbitrary set shows that the category of (small) categories **Cat** has products.

- Let $\mathcal{C} = \mathcal{D} \times \mathcal{E}$. Then we may identify $\mathcal{D}$ as a subcategory of $\mathcal{C}$: for each object E in $\mathcal{E}$, define $F_E : \mathcal{D} \to \mathcal{C}$, by $F_E(A) := (A, E)$ and $F(\alpha) = (\alpha, 1_E)$. Then F_E is a faithful functor. The image $\mathcal{C}_E$ of F_E (with objects (A, E) and morphisms $(\alpha, 1_E)$) is a subcategory of $\mathcal{C}$. It is not hard to see that $\mathcal{D}$ and $\mathcal{C}_E$ are `http://planetmath.org/CategoryIsomorphism`isomorphic as categories.

- The above also shows that for any objects A, B in $\mathcal{E}$, $\mathcal{C}_A$ and $\mathcal{C}_B$ are isomorphic.