properties of tensor product

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Theorem 1. *Let V_1, V_2 be two vector spaces over a field F. If $x_1, \dots, x_n \in V_1$ are linearly independent and $y_1, \dots, y_n \in V_2$ then*

$$\sum x_i \otimes y_i = 0 \Rightarrow y_i = 0, \text{ for all } i$$

Proof. Take the dual vectors x_i^* to the vectors x_i, i.e. $x_i^*(x_j) = \delta_{i,j}$. Given arbitrary linear functionals $f_i : V_2 \to F$, define a bilinear form $f : V_1 \times V_2 \to F$ by

$$f(x, y) = \sum_{j=1}^{n} x_j^*(x) f_j(y)$$

By the definition of tensor product there exists a unique linear functional $\phi : V_1 \otimes V_2 \to F$ such that $\phi \circ \iota = f$. Therefore

$$\begin{aligned} 0 &= \phi\left(\sum_i x_i \otimes y_i\right) \\ &= \sum_i \phi \circ \iota(x_i, y_i) \\ &= \sum_i f(x_i, y_i) \\ &= \sum_i \sum_j x_j^*(x_i) f_j(y_i) \\ &= \sum_i f_i(y_i). \end{aligned}$$

Since the f_i are arbitrary, it follows that $y_i = 0$ for all i.

$\square$