interchange law

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Let S be a set and $\circ$ and $\bullet$ two http://planetmath.org/PartialFunctionpartial binary operations on S. Then $\circ$ and $\bullet$ are said to satisfy the *interchange law* if

$$(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d),$$

provided that the operations are defined on both sides of the equation.

An element $e \in S$ is a $\circ$-identity, or an identity with respect to $\circ$, if $e \circ a = a \circ e = a$ provided the operations are defined.

Proposition 1. *If $\circ$ is a total function (defined for all of $S \times S$), then there is at most one $\circ$-identity.*

Proof. If e and f are both $\circ$-idenities, then $e = e \circ f = f$. $\square$

Proposition 2. *If both $\circ$ and $\bullet$ are total functions, and identities exist and the same with respect to both operations, then $\circ = \bullet$ and is commutative.*

Proof. Suppose that e is both the $\circ$-identity and the $\bullet$-identity. Then, according to the interchange law, $a \bullet d = (a \circ e) \bullet (e \circ d) = (a \bullet e) \circ (e \bullet d) = a \circ d$, showing that $\bullet = \circ$. Again, using the interchange law, $a \bullet d = (e \circ a) \bullet (d \circ e) = (e \bullet d) \circ (a \bullet e) = d \circ a = d \bullet a$, showing that $\bullet$ is commutative. $\square$