equivalent definition of a representable functor

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We provide an equivalent, motivating, way of defining a representable functor.

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to Set$ be a covariant functor and $A \in \mathcal{C}$. Then the following are equivalent

1. $\mathcal{C}(A, -)$ is naturally isomorphic to F (or, isomorphic in the appropriate category of functors)

2. There exists an element $i \in F(A)$ such that for every $B \in \mathcal{C}, r \in F(B)$ there exists a unique $f \in \mathcal{C}(A, B)$ such that $F(f)(i) = r$

To illustrate the significance of this, consider the category $\mathcal{C} = \mathbf{Vect_k}$ of vector spaces over a field k. For arbitrary vector spaces V, W consider the functor $F : \mathcal{C} \to Set$ determined by

$$F(U) = \mathrm{Bilin}(V \times W, U)$$

Where this denotes the set of maps which are linear in both entries. This is a covariant functor in the obvious way. Then one may define $V \otimes W$ as the object which represents F (if it exists). The significance of the result is it shows this is equivalent to the 'usual' definition: there is a bilinear map $i : V \times W \to V \otimes W$ through which all bilinear maps from $V \times W$ (these are quantified by r in the theorem) factor uniquely. This is because $r : V \times W \to U$ factors through i exactly when there is an $f \in \mathcal{C}(V \otimes W, U)$ such that $F(f)(i) = r$.

Such universal constructions can be shown to be functorial in the basic objects. For instance the tensor product may be shown to be a functor

$$\mathbf{Vect_k} \times \mathbf{Vect_k} \to \mathbf{Vect_k}$$

To generalise this suppose that $\mathcal{D}$ is a category (roughly representing $\mathbf{Vect_k} \times \mathbf{Vect_k}$ in our case) and we have a functor

$$F : \mathcal{D}^{op} \times \mathcal{C} \to Set$$

such that $F(d, -) : \mathcal{C} \to Set$ is isomorphic to $\mathcal{C}(G(d), -)$ for some object $G(d)$. Then one may show that G extends to a functor in such a way that $F(-, -)$ is naturally isomorphic to $\mathcal{C}(G(-), -)$.

We may show further that if F, F' are isomorphic functors and G, G' are functors which represent them respectively, then there is a natural isomorphism between G and G'.