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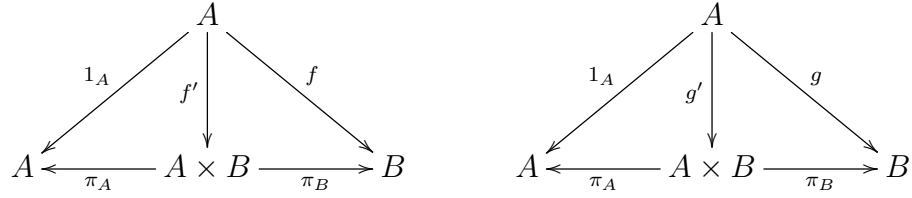
category with arbitrary products and  
pullbacks is complete

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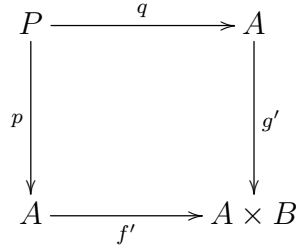
In the parent entry, it is stated that a complete category can be characterized as being a category with arbitrary products and equalizers. In this entry, we show, as a corollary, that every category with arbitrary products and pullbacks is complete. We begin with the following observation:

**Lemma 1.** *If a category has finite products and pullbacks, it has equalizers.*

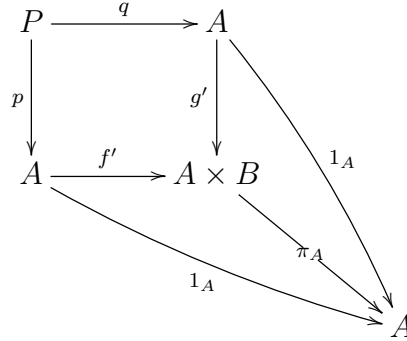
*Proof.* Suppose we have a pair of morphisms  $f, g : A \rightarrow B$ . Given the product  $A \times B$ , there are unique morphisms  $f', g' : A \rightarrow A \times B$  with the following commutative diagrams



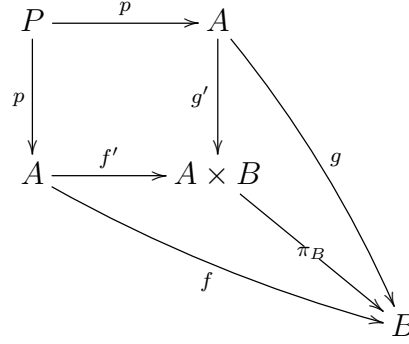
For the pair  $f', g' : A \rightarrow A \times B$ , let



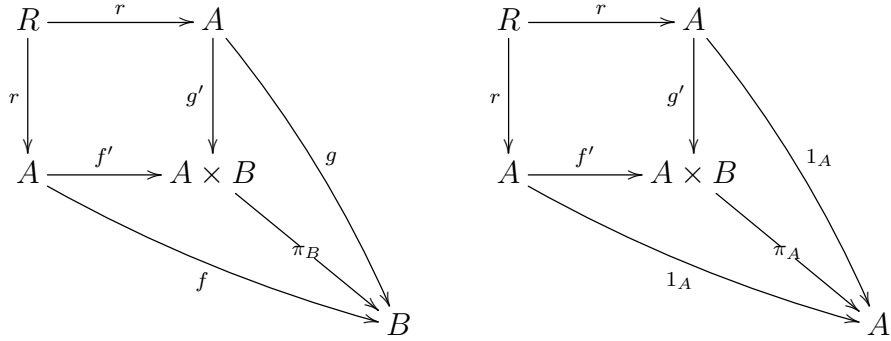
be the pullback diagram, which, after combining with the two small commutative triangles containing the edge  $\pi_A$  above, produces the following commutative diagram



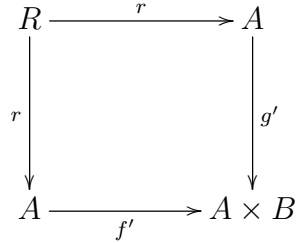
This implies that  $p = q$ . This result, together with the pullback diagram combined with the remaining commutative triangles (containing the edge  $\pi_B$ )



we see that  $p$  equalizes  $f$  and  $g$ . Suppose now that  $r : R \rightarrow A$  also equalizes  $f$  and  $g$ :  $f \circ r = g \circ r$ . Then we get two commutative diagrams



first of which comes from the equation  $f \circ r = g \circ r$  and the second one is obvious. By the universality of the product  $A \times B$ , we have the commutative diagram



By the universality of the pullback diagram, there is a unique morphism  $s : R \rightarrow P$  so that  $r = p \circ s$ , which implies that  $p$  is the equalizer of  $f$  and  $g$ .  $\square$

The following corollary is now immediate:

**Corollary 1.** *A category  $\mathcal{C}$  with arbitrary products and pullbacks is a complete category.*