construction of a Brandt groupoid

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In the parent entry, we give an example of a Brandt groupoid. In the example, we started with a non-empty set I and a group G, and showed that $I \times G \times I$ has the structure of a Brandt groupoid. In this entry, we show that every Brandt groupoid may be constructed this way.

Proposition 1. *If B is a Brandt groupoid, then there is a non-empty set I, and a group G, such that B is isomorphic to $I \times G \times I$. In other words, there is a bijection $\phi : B \to I \times G \times I$ such that ab is defined in B iff $\phi(a)\phi(b)$ is defined in $I \times G \times I$, and $\phi(ab) = \phi(a)\phi(b)$ whenever the multiplication is defined.*

To prove this, let us observe the following series of facts: given a Brandt groupoid B, let I be the set of idempotents in B.

Lemma 1. *Let $H(e, f)$ be the set consisting of all isomorphisms with source e and target f. Then the set $K = \{H(e, f) \mid e, f \in I\}$ partitions B.*

Proof. This is clear from the previous discussion, as B can be thought of as a category. Another way to see this is to define a binary relation R on B so that aRb iff a, b have the same source and target. Then R is an equivalene relation, and its equivalence classes have the form $H(e, f)$. □

Lemma 2. *The cardinality of $H(e, f)$ is independent of e and f.*

Proof. Define $\phi : H(e, f) \to H(e', f')$, by $\phi(a) = uav$, where $u \in H(f, f')$ and $v \in H(e', e)$. Notice that u, v exist by condition 6 above. First, ϕ is well-defined, because both ua and av are defined by condition 3, and hence $uav = (ua)v = u(av)$ is defined. In addition, ϕ is a bijection, whose inverse is the map $b \mapsto u^{-1}bv^{-1}$. □

Lemma 3. *$H(e, e)$ is a group for every $e \in I$.*

Proof. The multiplication on $H(e, e)$ is just the multiplication on B restricted to $H(e, e)$, which is total (defined for all of $H(e, e)$), and associative, with e its multiplicative identity. For $a \in H(e, e)$, its inverse is guaranteed by condition 5 above. □

Lemma 4. *$H(e, e)$ is group isomorphic to $H(f, f)$ for every $e, f \in I$.*

Proof. The function $\phi : H(e, e) \to H(f, f)$ given by $\phi(a) = uau^{-1}$, where $u \in H(e, f)$, is a well-defined bijection according to the proof of the first observation. Furthermore, $\phi(ab) = u(ab)u^{-1} = u((ae)b)u^{-1} = u(a(u^{-1}u)b)u^{-1} = u(((au^{-1})u)b)u^{-1} = u((au^{-1})(ub))u^{-1} = (uau^{-1})(ubu^{-1}) = \phi(a)\phi(b)$, hence ϕ is a group isomorphism. □

Set $G = H(e,e)$ for some $e \in I$. We are now ready to prove the proposition. Notice that the proof involves the axiom of choice.

Proof of Proposition 1. By the axiom of choice, there is a function $\alpha : I \to B$ such that $\alpha(f) \in H(e,f)$ and $\alpha(e) = e$. For any $a \in B$, set

$$\overline{a} := \alpha(t(a))^{-1} a \alpha(s(a)) \in G.$$

If ab is defined, then $s(a) = t(b)$, so that

$$\begin{aligned} \overline{ab} &= \alpha(t(ab))^{-1} ab \alpha(s(ab)) \\ &= \alpha(t(a))^{-1} ab \alpha(s(b)) \\ &= \alpha(t(a))^{-1} a \alpha(s(a)) \alpha(s(a))^{-1} b \alpha(s(b) \\ &= \alpha(t(a))^{-1} a \alpha(s(a)) \alpha(t(b))^{-1} b \alpha(s(b) \\ &= \overline{a}\overline{b}. \end{aligned}$$

Now, define $\phi : B \to I \times G \times I$ by

$$\phi(a) = (t(a), \overline{a}, s(a)).$$

This is clearly a well-defined function. In addition, it is one-to-one: if $\phi(a) = \phi(b)$, then $s(a) = s(b) := f$, $t(a) = t(b) := g$ and $\alpha(g)^{-1} a \alpha(f) = \overline{a} = \overline{b} = \alpha(g)^{-1} b \alpha(f)$. As a result, $a = \alpha(g) \overline{a} \alpha(f)^{-1} = \alpha(g) \overline{b} \alpha(f)^{-1} = b$. It is also onto: given $(g, c, f) \in I \times G \times I$, then $\phi(d) = c$, where $d = \alpha(g) c \alpha(f)^{-1}$.

Finally, for $a, b \in B$, the multiplication ab is defined in B iff $s(a) = t(b)$ iff the multiplication

$$\phi(a)\phi(b), \quad \text{or} \quad (t(a), \overline{a}, s(a))(t(b), \overline{b}, s(b))$$

is defined in $I \times G \times I$, which is equal to

$$(t(a), \overline{a}\overline{b}, s(b)) = (t(a), \overline{ab}, s(b)) = (t(ab), \overline{ab}, s(ab)) = \phi(ab),$$

showing that ϕ preserves partial multiplications. $\square$