properties of direct product

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Let $\mathcal{C}$ be a category. This entry lists some of the basic properties of categorical direct product:

Proposition 1. *(uniqueness of products) A product $(C, \{\pi_i\}_{i\in I})$ of objects $\{C_i\}_{i\in I}$, if it exists, is unique up to isomorphism.*

Before proving this, let us observe first that if $f : C \to C$ is a morphism such that

$$\pi_i = \pi_i \circ f \tag{1}$$

then $f = 1_C$ necessarily, since the universal property of product C, f is the unique morphism such that (1) holds, but then $\pi_1 = \pi_1 \circ 1_C$ as well, and this forces $f = 1_C$.

Proof. If $(D, \{g_i\}_{i\in I})$ is another product of $\{C_i\}_{i\in I}$, we get two unique morphisms $x : D \to C$ and $y : C \to D$ such that $\pi_i = g_i \circ y$ and $g_i = \pi_i \circ x$ for all $i \in I$. So $\pi_i = (\pi_i \circ x) \circ y = \pi_i \circ (x \circ y)$. From the previous paragraph, we see that $x \circ y = 1_C$. Similarly, $g_i = g_i \circ (y \circ x)$, so that $y \circ x = 1_D$. This shows that C is isomorphic to D. □

This justifies writing $\prod_{i\in I} C_i$ (with π_i) as *the* product of $\{C_i\}_{i\in I}$. In case I has cardinality 2, we write $C_1 \times C_2$ as the product of C_1 and C_2. Also, when $I = \varnothing$, we set the product as any terminal object T in $\mathcal{C}$.

Proposition 2. *If I is the disjoint union of J and K, then*

$$\prod_{i\in I} C_i \cong \prod_{j\in J} C_j \times \prod_{k\in K} C_k,$$

assuming all products exist.

Proof. Let

$$C = \prod_{i\in I} C_i, \quad D = \prod_{j\in J} C_j, \quad \text{and} \quad E = \prod_{k\in K} C_k.$$

We break down the proof into two cases:

- Suppose one of J and K is the empty set, say, $J = \varnothing$. Then D is a terminal object, and $K = I$, so that $E = C$. In other words, we want to show that

$$C \cong D \times C.$$

First, notice that we have morphisms $1_C : C \to C$ and $e_C : C \to D$ (where e_C is unique since D is terminal). If A is any object with morphisms $f : A \to C$ and $e_A : A \to D$. Any $g : A \to C$ with $f = 1_C \circ g$ and $e_A = e_C \circ g$ must result in $f = g$. This shows that C may be viewed as the product of D and C, or $C \cong D \times C$.

- Now, suppose neither J nor K is empty. We have projection morphisms $f_i : C \to C_i$ for all $i \in I$, $g_j : D \to C_j$ for all $j \in J$, and $h_k : E \to C_k$ for all $k \in K$. Write $F = D \times E$ with projections $p_1 : F \to D$ and $p_2 : F \to E$.

  For every $i \in I$, define morphisms $x_i : F \to C_i$ as follows: if $i \in J$, then $x_i = g_i \circ p_1$. Otherwise, $x_i = h_i \circ p_2$. Since J and K are disjoint, this I-indexed set of morphisms is well-defined. By the universality of the product C, we get a unique morphism $x : F \to C$ such that $x_i = f_i \circ x$.

  Next, from the universal properties of the products D and E, we have two unique morphisms $y : C \to D$ and $z : C \to E$ such that $f_j = g_j \circ y$ and $f_k = h_k \circ z$ for any $j \in J$ and $k \in K$. From the morphisms $y : C \to D$ and $z : C \to E$ and the universality of the product F, we have another unique morphism $f : C \to F$ such that $y = p_1 \circ f$ and $z = p_2 \circ f$.

  Then $p_1 \circ (f \circ x) = (p_1 \circ f) \circ x = y \circ x$. Since $g_j \circ y \circ x = f_j \circ x = x_j = g_j \circ p_1$ for any $j \in J$, we have $y \circ x = p_1$, so that $p_1 \circ (f \circ x) = p_1$. Similarly, $p_2 \circ (f \circ x) = p_2$. This shows that $f \circ x = 1_F$. Also, $f_i \circ (x \circ f) = (f_i \circ x) \circ f = x_i \circ f$. Now, if $i \in J$, then $x_i \circ f = g_i \circ p_1 \circ f = g_i \circ y = f_i$, and if $i \in K$, then $x_i \circ f = h_i \circ p_2 \circ f = h_i \circ z = f_i$. As a result, $f_i \circ (x \circ f) = f_i$ for all $i \in I$, which implies $x \circ f = 1_C$. This shows that $C \cong F = D \times E$.

This completes the proof. □

Corollary 1. *(commutativity of products) $A \times B \cong B \times A$, if one (and hence the other) exists.*

This shows that it does not matter whether we say $A \times B$ as the product of A and B, or the product of B and A.

Corollary 2. *(associativity of products) $A \times (B \times C) \cong A \times B \times C \cong (A \times B) \times C$, whenever the products are defined.*

Remarks. All of the properties can be dualized, so that coproducts are unique up to isomorphism, and commutativity and associativity laws hold as well.