proof of 9-lemma

Canonical name	ProofOf9lemma
Date of creation	2013-03-22 16:42:49
Last modified on	2013-03-22 16:42:49
Owner	rm50 (10146)
Last modified by	rm50 (10146)
Numerical id	4
Author	rm50 (10146)
Entry type	Proof
Classification	msc 18G35

As in the proof of the 5-lemma, we assume without loss of generality that we are working in modules over a ring. In keeping with the notion that the maps between the A's (as well as between the B's and the C's) are cohomology sequences, we denote all vertical maps by d. The map $A_i \to B_i$ is denoted α_i, and the map $B_i \to C_i$ is denoted β_i. We must show that

1. β_1 is surjective;

2. α_1 is injective;

3. $\ker \beta_1 \subset \text{im } \alpha_1$;

4. $\beta_1 \circ \alpha_1 = 0$ (i.e. $\ker \beta_1 \supset \text{im } \alpha_1$)

β_1 is surjective: Choose $c \in C_1$. Then $dc = \beta_2 b$, and $\beta_3 db = d\beta_2 b = d^2 c = 0$, so $db = \alpha_3 a = \alpha_3 da'$. Thus $d(b - \alpha_2 a') = 0$, so $db' = b - \alpha_2 a'$. Finally, $d\beta_1 b' = \beta_2 db' = \beta_2(b - \alpha_2 a') = \beta_2 b = dc$. But d is injective, so $c = \beta_1 b'$.

α_1 is injective: This is clear, since $d\alpha_1 = \alpha_2 d$, and α_2 and both d's are injective.

$\ker \beta_1 \subset \text{im } \alpha_1$: Suppose $\beta_1(b) = 0$. Then $\beta_2 db = d\beta_1 b = 0$, so $db = \alpha_2 a$. But then $\alpha_3 da = d\alpha_2 a = d^2 b = 0$, and α_3 is injective, so $a \in \ker d$ and $da' = a$. Finally, $d\alpha_1 a' = \alpha_2 da' = \alpha_2 a = db$. d is injective and thus $b = \alpha_1 a'$.

$\beta_1 \circ \alpha_1 = 0$: $d\beta_1\alpha_1 = \beta_2\alpha_2 d = 0$. But d is injective, so $\beta_1\alpha_1 = 0$.

Similar diagram chasing can be used to prove that if the top two rows are exact then so is the bottom row.