quantum symmetry

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1 Quantum Symmetry

Often quantum symmetry is understood in terms of properties of symmetry groups, their representations and related algebras. Quantum groups also possess quantum symmetries which are distinct from those exhibited by classical Lie groups, groups of rotations and Poisson or Lorentz transformation groups. Extended quantum symmetries are also encountered for quantum groupoids, quantum categories, Hamilton algebroids, graded Lie super-algebras, Lie algebroids and quantum systems with topological order.

1.1 Paracrystal Theory and Convolution Algebra

As reported in a recent publication [?], the general theory of scattering by partially ordered, atomic or molecular, structures in terms of *paracrystals* and *lattice convolutions* was formulated by Hosemann and Bagchi in [?] using basic techniques of Fourier analysis and convolution products. A natural generalization of such molecular, partial symmetries and their corresponding analytical versions involves convolution algebras – a functional/distribution [?, ?] based theory that we will discuss in the context of a more general and original concept of a *convolution-algebroid of an extended symmetry groupoid of a paracrystal*, of any molecular or nuclear system, or indeed, any quantum system in general; such applications also include quantum fields theories, and local quantum net configurations that are endowed with either partially disordered or 'completely' ordered structures, as well as in the graded, or super–algelbroid extension of these concepts for very massive structures such as stars and black holes treated by quantum gravity theories.

A statistical analysis linked to structural symmetry and scattering theory considerations shows that a real paracrystal can be defined by a three dimensional convolution polynomial with a semi-empirically derived composition law, $*$, [?]. As was shown in [?, ?] – supported with computed specific examples – several systems of convolution can be expressed analytically, thus allowing the numerical computation of X-ray, or neutron, scattering by partially disordered layer lattices via complex Fourier transforms of one-dimensional structural models using fast digital computers. The range of paracrystal theory applications is however much wider than the one-dimensional lattices with disorder, thus spanning very diverse non-crystalline systems, from metallic glasses and spin glasses to superfluids, high-temperature superconductors, and extremely hot anisotropic plasmas such as those encountered in controlled nuclear fusion (for example, JET) experiments. Other applications – as previously suggested in [?] – may also include novel designs of ‘fuzzy’ quantum machines and quantum computers with extended symmetries of quantum state spaces.

1.2 Convolution product of groupoids and the convolution algebra of functions

From a purely mathematical perspective, Alain Connes introduced the concept of a C^*-algebra of a (discrete) group (see, e.g., [?]). The underlying vector space is that of complex valued functions with finite support, and the multiplication of the algebra is the fundamental *convolution product* which it is convenient for our purposes to write slightly differently from the common formula as:

$$(f * g)(z) = \sum_{xy=z} f(x)g(y), Eq.1.1.$$

and $*$-operation
Eq. 1.2.

$$f^*(x) = \overline{f(x^{-1})}$$

The more usual expression of these formulas has a sum over the elements of a selected group. For topological groups, where the underlying vector space consists of continuous complex valued functions, this product requires the availability of some structure of measure and of measurable functions, with the sum replaced by an integral. Notice also that this algebra has an identity, the distribution function δ_1, which has value 1 on the identity 1 of the group, and has zero value elsewhere. Currently, however, there are several important aspects of quantum dynamics left out of the invariant, simplified picture provided by group symmetries and their corresponding representations of quantum operator algebras [?]. An alternative approach proposed in [?] employs differential forms to find symmetries. Physicists deal often with such problems in terms of either spontaneous symmetry breaking or approximate symmetries that require underlying assumptions or ad-hoc dynamic restrictions that have a phenomenological basisl. A well-studied example of this kind is that of the dynamic Jahn–Teller effect and the corresponding 'theorem' (Chapter 21 on pp. 807–831, as well as p. 735 of [?]) which in its simplest form stipulates that *a quantum state with electronic non-Kramers degeneracy may be unstable against small distortions of the surroundings, that would lower the symmetry of the crystal field and thus lift the degeneracy* (i.e., cause an observable splitting of the corresponding energy levels). This effect occurs in certain paramagnetic ion systems *via* dynamic distortions of the crystal field symmetries around paramagnetic or high-spin centers by moving ligands that are diamagnetic. The established physical explanation is that the Jahn–Teller coupling replaces a purely electronic degeneracy by a vibronic degeneracy (of *exactly the same* symmetry!). The dynamic, or spontaneous breaking of crystal field symmetry (for example, distortions of the octahedral or cubic symmetry) results in certain systems in the appearance of doublets of symmetry γ_3 or singlets of symmetry γ_1 or γ_2. Such dynamic systems could be locally expressed in terms of symmetry representations of a Lie algebroid, or globally in terms of a special Lie (or Lie–Weinstein) symmetry groupoid representations that can also take into account the spin exchange interactions between the Jahn–Teller centers exhibiting such quantum dynamic effects. Unlike the simple symmetries expressed by group representations, the latter can accommodate a much wider range of possible or approximate symmetries that are indeed characteristic of real, molecular systems with varying crystal field symmetry, as for example around certain transition ions dynamically bound to ligands in liquids where motional narrowing becomes very important. This well known example illustrates the importance of the interplay between symmetry and dynamics in quantum processes which is undoubtedly involved in many other instances including: *quantum chromodynamics (QCD), superfluidity, spontaneous symmetry breaking (SSB), quantum gravity and Universe dynamics* (i.e., the inflationary Universe), some of which will be discussed in further detail in Section 5. Physicists deal often with such problems in terms of either spontaneous symmetry breaking or approximate symmetries that require underlying assumptions or ad-hoc dynamic restrictions that have a phenomenological basisl. A well-studied example of this kind is that of the dynamic Jahn–Teller effect and the corresponding 'theorem' (Chapter 21 on pp. 807–831, as well as p. 735 of [?]) which in its simplest form stipulates that *a quantum state with electronic non-Kramers degeneracy may be unstable against small distortions of the surroundings, that would lower the symmetry of the crystal field and thus lift the degeneracy* (i.e., cause an observable splitting of the corresponding energy levels). This effect occurs

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Therefore, the various interactions and interplay between the symmetries of quantum operator state space geometry and quantum dynamics at various levels leads to both algebraic and topological structures that are variable and complex, well beyond symmetry groups and well-studied group algebras (such as Lie algebras, see for example [?]). A unified treatment of quantum phenomena/dynamics and structures may thus become possible with the help of algebraic topology, non-Abelian treatments; such powerful mathematical tools are capable of revealing novel, fundamental aspects related to extended symmetries and quantum dynamics through a detailed analysis of the variable geometry of (quantum) operator algebra state spaces. At the center stage of non-Abelian algebraic topology are groupoid and algebroid structures with their internal and external symmetries [?] that allow one to treat physical spacetime structures and dynamics within an unified categorical, higher dimensional algebra framework [?]. As already suggested in our recent report [?], the interplay between extended symmetries and dynamics generates higher dimensional structures of quantized spacetimes that exhibit novel properties not found in lower dimensional representations of groups, group algebras or Abelian groupoids.

It is also our intention here to explore new links between several important but seemingly distinct mathematical approaches to extended quantum symmetries that were not considered in previous reports. An important example example is the general theory of scattering by partially ordered, atomic or molecular, structures in terms of *paracrystals* and *lattice convolutions* that was formulated in [?] using basic techniques of Fourier analysis and convolution products. Further specific applications of the paracrystal theory to X-ray scattering, based on computer algorithms, programs and explicit numerical computations, were subsequently developed by the first author [?] for one-dimensional paracrystals, partially ordered membrane lattices [?] and other biological structures with partial structural disorder [?]. Such biological structures, 'quasi-crystals', and the paracrystals, in general, provide rather interesting physical examples of extended symmetries (cf. [?]). Moreover, the quantum inverse scattering problem and the treatment of nonlinear dynamics in ultra-hot plasmas of white stars and nuclear fusion reactors requires the consideration of quantum doubles, or respectively, quantum double groupoids and graded double algebroid representations [?].

1.3 Group and Groupoid Representations

Whereas group representations of quantum unitary operators are extensively employed in standard quantum mechanics, the quantum applications of *groupoid representations* are still under development. For example, a description of stochastic quantum mechanics in curved spacetime [?] involving a Hilbert bundle is possible in terms of groupoid representations which can indeed be defined on such a Hilbert bundle $(X * H, \pi)$, but cannot be expressed as the simpler group representations on a Hilbert space H. On the other hand, as in the case of group representations, unitary groupoid representations induce associated C^*-algebra representations. In the next subsection we recall some of the basic results concerning groupoid representations and their associated groupoid *-algebra representations. For further details and recent results in the mathematical theory of groupoid representations one has also available a succint monograph [?] (and references cited therein).

Let us consider first the relationships between these mainly algebraic concepts and their extended quantum symmetries. Then we introducer several extensions of symmetry and algebraic topology in the context of local quantum physics, ETQFT, spontaneous symmetry breaking, QCD and the development of novel supersymmetry theories of quantum gravity. In this respect one can also take spacetime 'inhomogeneity' as a criterion for the comparisons between physical, partial or local, symmetries: on the one hand, the example of paracrystals reveals thermodynamic disorder (entropy) within its own spacetime framework, whereas in spacetime itself, whatever the selected model, the inhomogeneity arises through (super) gravitational effects. More specifically, in the former case one has the technique of the generalized Fourier–Stieltjes transform (along with convolution and Haar measure), and in view of the latter, we may compare the resulting 'broken'/paracrystal–type symmetry with that of the supersymmetry predictions for weak gravitational fields, as well as with the spontaneously broken global supersymmetry in the presence of intense gravitational fields.

Another significant extension of quantum symmetries may result from the superoperator algebra and/or algebroids of Prigogine's quantum *superoperators* which are defined only for irreversible, infinite-dimensional systems [?]. The latter extension is also incompatible with a commutative logic al-

gebra such as the Heyting algebraic logic currently utilized to define topoi [**?**].
`http://planetphysics.us/encyclopedia/QuantumSymmetryBibliography.html`Quantum Symmetry Bibliography

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