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proof of 5-lemma

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First, assume that we are in modules over a ring (this is the most commonly used setting anyways).

The method of proof is what is usually called diagram-chasing.

Let a be in the kernel of γ_3. Then $d(\gamma_3(a)) = \gamma_4(da) = 0$, and γ_4 is injective, so $da = 0$. By exactness, $a = da'$ for some $a' \in A_4$. Now, $d(\gamma_2 a') = \gamma_3 a = 0$, so $\gamma_2 a' = db$, and by the surjectivity of γ_1, $b = \gamma_1 a''$. $da''' = \gamma_2^{-1} d\gamma_1(a'') = a'$. Thus, $a = d^2 a'' = 0$. So, γ_3 is an injection.

Now, assume b is not in the image of γ_3. $db \neq 0$, so $a' = \gamma_4^{-1} db \neq 0$. $\gamma_5 da' = d^2 b = 0$, and γ_5 is injective, so $da' = 0$, and there exists an a'' such that $da'' = a'$. Thus, $d(b - \gamma_3 a'') = 0$. So there is an α such that $d\gamma_2\alpha = b - \gamma_3 a''$. Thus, $\gamma_3(a'' + d\alpha) = b$. Thus, γ_3 is surjective.

This actually implies the result for all abelian categories, since by the `http://planetmath.org/MitchellsEmbeddingTheorem`Freyd embedding theorem, any abelian category is equivalent to a subcategory of modules over a ring. This trick is necessary since the trick above required us to have a notion of elements in the objects of our category, one which doesn't always make sense. The 5-lemma can be proved directly, but the proof is just less enlightening than the one above.