quantum topos

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Preliminary Data.

There are several distinct definitions of *quantum topos* in the Mathematical Physics literature attempting to redefine the quantum logic that was first introduced by von Neumann and Birkhoff for the foundation of Quantum Mechanics. The definitions of quantum topoi published so far are not, however, those of 'quantum' categories (previously introduced as rigid monoidal categories) - with finite limits and power objects.

Definition 0.1. A *quantum topos* was defined as a *general model, or representation of quantum state spaces (QST) in a topos* with a `http://planetmath.org/Commutative`(commutative) Heyting logic algebra as a subobject *(quantum logic)* classifier.The differences between the several published definitions of a quantum topos differ in the categorical representation in QST' s, and in the choice of category, but not in the choice of quantum logic algebra that was selected as a standard, Heyting logic algebra (or *Heyting algebra*) which has a commutative *Heyting lattice* structure; this choice is at variance with the original quantum logic introduced by von Neumann and Birkhoff. Thus instead of the orthomodular lattice of Birkhoff and von Neumann, the recent definitions of quantum topoi postulate an intuitionistic- Brouwer logic corresponding to a pseudocomplemented and rel. pseudocomplemented lattice structure, as further explained in the next section.

0.1 Heyting Logic Concept and Algebraic Structure

Definition 0.2. A *Heyting lattice* L is a Brouwer-intuitionistic logic lattice with a bottom, or lowest element 0. In the more technical classification it is a `http://planetmath.org/Commutative`commutativ lattice which is both 'pseudocomplemented and also relatively pseudocomplemented'. The concept of `http://planetmath.org/RelativelyPseudocomplemented`relative pseudocomplementation coincides with the material implication operator, $\Rightarrow$, in symbolic propositional logic based on chryssippian or Boolean logic.

Definition 0.3. A *Heyting algebra* is a *p-algebra (as defined next in* ***Definition 1.3*** *) with the relative pseudocomplentation operation* $\rightarrow$ (which replaces the propositional implication $\Rightarrow$).

Given an element a in a bounded lattice L, a *complement* of a is defined to be an element $b \in L$, if such an element exists, such that

$$a \wedge b = 0, \qquad and \qquad a \vee b = 1.$$

To surmount the non-uniqueness of the complement, an alternative to the latter was defined– the *pseudocomplement* of an element.

An element b in a lattice L with 0 is a *pseudocomplement* of $a \in L$ if

1. $b \wedge a = 0$
2. for any c such that $c \wedge a = 0$ then $c \leq b$.

In other words, b is the maximal element in the set $\{c \in L \mid c \wedge a = 0\}$.

Definition 0.4. A convenient modification of the pseudocomplemented (pc) lattice concept is a *p-algebra* (or pseudocomplemented algebra) which is a pc-lattice where * is regarded as an algebraic operator. Thus, a morphism of pc–lattices is a proper lattice homomorphism, whereas a morphism between two p-algebras is a lattice homomorphism f that also preserves the pc-algebraic operation *, i.e., $f(a^*) = f(a)^*$. One can therefore define a *category of p-algebras* by specifying the morphism between any pair of p-algebras (considered as objects of this algebraic logic category) as the $\{0, 1\}$-lattice homomorphism, with the following condition $f(1) = f(0^*) = f(0)^* = 0^* = 1$ being also satisfied.

Remark Unlike the Heyting lattice, an LM_n-logic algebra has a *non-commutative* lattice structure and is therefore considered as a stronger candidate for quantum logics, including those based on the orthomodular lattices of the original quantum logic of Birkhoff and von Neumann. Thus, a generalized topos defined with a subobject classifier based on LM_n-logic algebra may provide suitable representations of arbitrary quantum state spaces.

References

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