category with arbitrary products and pullbacks is complete

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In the parent entry, it is stated that a complete category can be characterized as being a category with arbitrary products and equalizers. In this entry, we show, as a corollary, that every category with arbitrary products and pullbacks is complete. We begin with the following observation:

Lemma 1. *If a category has finite products and pullbacks, it has equalizers.*

Proof. Suppose we have a pair of morphisms $f, g : A \to B$. Given the product $A \times B$, there are unique morphisms $f', g' : A \to A \times B$ with the following commutative diagrams

$$\begin{tikzcd}
 & A \arrow[dl, "1_A"'] \arrow[d, "f'"'] \arrow[dr, "f"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl, "1_A"'] \arrow[d, "g'"'] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

For the pair $f', g' : A \to A \times B$, let

$$\begin{tikzcd}
P \arrow[r, "q"] \arrow[d, "p"'] & A \arrow[d, "g'"] \\
A \arrow[r, "f'"'] & A \times B
\end{tikzcd}$$

be the pullback diagram, which, after combining with the two small commutative triangles containing the edge π_A above, produces the following commutative diagram

$$\begin{tikzcd}
P \arrow[r, "q"] \arrow[d, "p"'] & A \arrow[d, "g'"'] \arrow[ddr, bend left, "1_A"] & \\
A \arrow[r, "f'"] \arrow[drr, bend right, "1_A"'] & A \times B \arrow[dr, "\pi_A"] & \\
 & & A
\end{tikzcd}$$

This implies that $p = q$. This result, together with the pullback diagram combined with the remaining commutative triangles (containing the edge π_B)

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "p"'] & A \arrow[d, "g'"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "f'"] \arrow[drr, bend right, "f"'] & A \times B \arrow[dr, "\pi_B"] & \\
& & B
\end{tikzcd}$$

we see that p equalizes f and g. Suppose now that $r : R \to A$ also equalizes f and g: $f \circ r = g \circ r$. Then we get two commutative diagrams

$$\begin{tikzcd}
R \arrow[r, "r"] \arrow[d, "r"'] & A \arrow[d, "g'"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "f'"] \arrow[drr, bend right, "f"'] & A \times B \arrow[dr, "\pi_B"] & \\
& & B
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "r"] \arrow[d, "r"'] & A \arrow[d, "g'"] \arrow[ddr, bend left, "1_A"] & \\
A \arrow[r, "f'"] \arrow[drr, bend right, "1_A"'] & A \times B \arrow[dr, "\pi_A"] & \\
& & A
\end{tikzcd}$$

first of which comes from the equation $f \circ r = g \circ r$ and the second one is obvious. By the universality of the product $A \times B$, we have the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "r"] \arrow[d, "r"'] & A \arrow[d, "g'"] \\
A \arrow[r, "f'"'] & A \times B
\end{tikzcd}$$

By the universality of the pullback diagram, there is a unique morphism $s : R \to P$ so that $r = p \circ s$, which implies that p is the equalizer of f and g. $\square$

The following corollary is now immediate:

Corollary 1. *A category $\mathcal{C}$ with arbitrary products and pullbacks is a complete category.*