example of enough injectives

Canonical name	ExampleOfEnoughInjectives
Date of creation	2013-03-22 17:43:38
Last modified on	2013-03-22 17:43:38
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Last modified by	Glotzfrosch (19314)
Numerical id	5
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Entry type	Example
Classification	msc 18E99

The category of Abelian groups has enough injectives.

Proof. First, note that $\mathbb{Q}/\mathbb{Z}$ is an injective Abelian group, since it is divisible. For any Abelian group A, let $A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

We define
$$f : A \to \mathrm{Hom}(A^*, \mathbb{Q}/\mathbb{Z}), a \mapsto f_a,$$
where f_a ist defined as
$$f_a : A^* \to \mathbb{Q}/\mathbb{Z}, \varphi \mapsto \varphi(a).$$

f is one-to-one, for if $f_a = 0$, i.e. $\varphi(a) = 0$ for all $\varphi \in A^*$, it follows $a = 0$. Indeed, if $a \neq 0$, let the order of a be denoted by n, and for any $q \in \mathbb{Q}/\mathbb{Z}$ with order n, the homomorphism defined by $a \mapsto q$ is well-defined on the subgroup generated by a, and since $\mathbb{Q}/\mathbb{Z}$ is injective, it induces a homomorphism $A \to \mathbb{Q}/\mathbb{Z}$ which is different from zero.

Now, if we chose a presentation $\bigoplus_{i \in I} \mathbb{Z} \twoheadrightarrow A^*$, we get an embedding $\mathrm{Hom}(A^*, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}(\bigoplus_{i \in I} \mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, where the latter is clearly isomorphic to the direct product $\prod_{i \in I} \mathbb{Q}/\mathbb{Z}$. This last group is injective as a direct product of injectives. $\square$