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examples of epis

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The entry lists some of the common examples of epis (epimorphisms). The examples also demonstrate some of the techniques used in finding epis.

1. In **Set**, the category of sets, the epis are exactly the onto functions. First, suppose $f : A \to B$ is onto and that $g, h : B \to C$ are functions such that $g \circ f = h \circ f$. Then for any $b \in B$, there is $a \in A$ such that $f(a) = b$ since f is onto. This means that $g(b) = g(f(a)) = h(f(a)) = h(b)$, or $g = h$, showing that f is epi. Conversely, suppose $f : A \to B$ is epi. Define functions $g, h : B \to \{0, 1\}$ as follows: $g(x) = 0$ for all $x \in B$, and $h(x) = 0$ if $x \in f(A)$ and $h(x) = 1$ otherwise. Then $g(f(a)) = 0 = h(f(a))$. This means that $g = h$, or $x \in f(A)$ for all $x \in B$. In other words, f is onto.

2. In **Ab**, the category of abelian groups, the epis are exactly the onto abelian group homomorphisms. If f is onto and $g \circ f = h \circ f$, then for any $b \in B$, there is $a \in A$ such that $f(a) = b$. This means that $g(b) = g(f(a)) = h(f(a)) = h(b)$, or $g = h$, showing that f is epi. On the other hand, suppose $f : A \to B$ is epi. Define $g, h : B \to B/f(A)$ as follows: $g(x) = f(A)$ and $h(x) = x + f(A)$ for all $x \in B$. Then $g(f(a)) = f(A) = f(a) + f(A) = h(f(a))$. This implies that $g = h$, so that $x \in f(A)$ for all $x \in B$, or f is onto.

3. In **Top**, the category of topological spaces, the epis are exactly the surjective continuous functions. If f is onto and $g \circ f = h \circ f$, then then for any $b \in B$, there is $a \in A$ such that $f(a) = b$. This means that $g(b) = g(f(a)) = h(f(a)) = h(b)$, or $g = h$, showing that f is epi. On the other hand, suppose $f : X \to Y$ is epi. Equip $\{0, 1\}$ with the trivial topology. Define $g, h : Y \to \{0, 1\}$ as in example 1 above. Then g and h are both continuous. We also have $g(f(a)) = 0 = h(f(a))$, so that $g = h$, or $x \in f(A)$ for all $x \in B$. Therefore, f is onto.

Not all epimorphisms are surjections. For example, in the category **Comm-Rng** of commutative rings with 1, the natural injection $i : \mathbb{Z} \to \mathbb{Q}$ is clearly not a surjection, and yet it is epimorphic. To see this, let R be any commutative ring with characteristic 0. Suppose $g, h : \mathbb{Q} \to R$ are ring homomorphisms such that $g \circ i = h \circ i$, in other words, $g(n) = h(n)$ for all $n \in \mathbb{Z}$. Set $f := g - h$. Then $f(n) = 0$ for all $n \in \mathbb{Z}$. Then $0 = f(n) = mf(n/m)$, where m is an arbitrary positive integer. Since $\mathrm{char}(R) = 0$, this shows that

$f(n/m) = 0$. Since n/m is an arbitrary rational number, $f = 0$, or $g = h$. Hence i is an epi.

For another counterexample, it can be shown that in **HausTop**, the category of Hausdorff topological spaces and continuous functions, the epimorphisms are precisely the continuous functions with dense images. As such, surjections are not a requirement.