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examples of equalizers

Canonical name	ExamplesOfEqualizers
Date of creation	2013-03-22 18:23:52
Last modified on	2013-03-22 18:23:52
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Example
Classification	msc 18A30
Classification	msc 18A20

This entry illustrates some common examples of equalizers and coequalizers.

Examples of Equalizers

- In **Set**, the category of sets, the equalizer of a pair of functions $f, g : A \to B$ is given by the following: a set

  $$C = \{x \in A \mid f(x) = g(x)\},$$

  and function $i : C \to A$ the canonical injection. Clearly i equalizes f and g by construction: $f \circ i = g \circ i$. Now, if $j : D \to A$ also equalizes f and g, then define $k : D \to C$ by $k(d) = j(d)$. To see that this is well-defined, we need to show that $j(d) \in C$. Since j equalizes f and g, we have $f(j(d)) = g(j(d))$, so that $j(d) \in C$. Therefore k is a well-defined function from D into C. In addition, $i \circ k(d) = i(j(d)) = j(d)$. Finally, it is easy to see that if $i \circ t = j$, then $t = k$. Therefore, (C, i) is the equalizer of f and g.
- In fact, most concrete categories (concrete over **Sets**), the equalizer of a pair of morphisms is given by the object C above with i the corresponding injective mapping.
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Examples of Coequalizers

- In **Set**, the coequalizer of a pair of functions $f, g : A \to B$ can be found as follows: define a binary relation $\sim$ on B such that for any $x, y \in B$, $x \sim y$ iff either $x = y$, or there is an $a \in A$ such that $h_1(a) = x$ and $h_2(a) = y$, where $h_1, h_2 \in \{f, g\}$. Then $\sim$ is easily seen to be a reflexive symmetric relation. Now, take the transitive closure $\sim^*$ of $\sim$. So $\sim^*$ is an equivalence relation on B. Let $B/\sim^*$ be the set of all the equivalence classes, and $p : B \to B/\sim^*$ the canonical projection. Then $(B/\sim^*, p)$ is the coequalizer of f and g. First, $p \circ f(a) = [f(a)] = [g(a)] = p \circ g(a)$, since $f(a) \sim g(a)$. Suppose now that $q : B \to D$ is another function that coequalizes f and g. Define $r : B/\sim^* \to D$ by $r([b]) = q(b)$. We want to show that r is well-defined. In other words, if $b \sim^* c$, then $q(b) = q(c)$. First, assume $b \sim c$. Then either $b = c$ (in which case, $q(b) = q(c)$ is immediate), or there is $a \in A$ such that $h_1(a) = b$ and $h_2(a) = c$, with $h_1, h_2 \in \{f, g\}$. In this case, $q(b) = q(h_1(a)) = q(h_2(a)) = q(c)$ since $q \circ f = q \circ g$. Now, if $b \sim^* c$, then there are $d_1, \dots, d_n \in B$ such that $b = d_1 \sim d_2 \sim \cdots \sim d_n = c$. As a result, $q(b) = q(d_1) = q(d_2) = \cdots = q(d_n) = q(c)$. In addition, $r \circ p(b) = r([b]) = q(b)$, and that r is uniquely determined this way. Therefore, $(B/\sim^*, p)$ is the coequalizer of f and g.