kernel pair

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Defines	cokernel pair

Let $f : A \to B$ be a morphism in a category $\mathcal{C}$. The *kernel pair* of f is defined as the pair of morphisms $(k_1 : K \to A, k_2 : K \to A)$ such that

$$\begin{tikzcd} K \arrow[r, "k_1"] \arrow[d, "k_2"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback diagram.

Since

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a commutative diagram, we have a unique morphism $g : A \to K$ such that

$$\begin{tikzcd} A \arrow[dr, "g"] \arrow[drr, bend left, "1_A"] \arrow[ddr, bend right, "1_A"'] & & \\ & K \arrow[r, "k_1"'] \arrow[d, "k_2"] & A \arrow[d, "f"] \\ & A \arrow[r, "f"'] & B. \end{tikzcd}$$

is commutative. As a result, k_1 and k_2 are both monomorphisms: if $k_1 \circ h_1 = k_1 \circ h_2$, then

$$h_1 = 1_A \circ h_1 = (g \circ k_1) \circ h_1 = g \circ (k_1 \circ h_1) = g \circ (k_1 \circ h_2) = (g \circ k_1) \circ h_2 = 1_A \circ h_2 = h_2.$$

For example, in **Set**, the category of sets, the kernel pair of a function $f : A \to B$ is the pair $p_1 : K \to A$ and $p_2 : K \to A$, given by

$$K = \{(a, b) \in A \times A \mid f(a) = f(b)\},$$

and p_1 and p_2 are given by

$$p_1(a,b) = a \qquad \text{and} \qquad p_2(a,b) = b.$$

This is just the kernel of a function, in the sense of universal algebra. Please see `http://planetmath.org/KernelOfAHomomorphismBetweenAlgebraicSystems`this entry for more details.

The notion of *cokernel pair* is dually defined.

Remark. $f : A \to B$ is a monomorphism iff the kernel pair of f is $(1_A, 1_A)$. Dually, f is an epimorphism iff the cokernel pair of f is $(1_A, 1_A)$.

References

[1] F. Borceux *Basic Category Theory, Handbook of Categorical Algebra I*, Cambridge University Press, Cambridge (1994)