proof of properties of universe

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1. This is the special case of axiom 2 with $x = y$ since $\{x, x\} = \{x\}$. (In other words, in set theory, we do not count duplicate entries twice.)

2. By definition of power set, if $x \subset y$, then $x \in \mathcal{P}(y)$. By axiom 3, $\mathcal{P}(y) \in \mathbf{U}$. By axiom 1, it follows that $x \in \mathbf{U}$.

3. By axiom 2, $\{x, y\} \in \mathbf{U}$. By axiom 2 again, it follows that $\{\{x, y\}, x\} \in \mathbf{U}$.

4. By axiom 2, $\{x, y\} \in \mathbf{U}$. If we set $z_x = x$ and $z_y = y$, then $x \cup y = \bigcup_{i \in \{x,y\}} z_i$, hence, by axiom 4, $x \cup y \in \mathbf{U}$. — If $x \in U$ and $y \in U$ then, by axiom 1, $a \in \mathbf{U}$ for all $a \in x$ and $b \in \mathbf{U}$ for all $b \in y$. By property 3, if $a \in \mathbf{U}$ and $b \in \mathbf{U}$, then $(a, b) \in \mathbf{U}$; further, by property 1, $\{(a, b)\} \in \mathbf{U}$. Hence, by axiom 4, $\{(a, b) \mid b \in y\} = \bigcup_{b \in y} \{(a, b)\} \in \mathbf{U}$ for all $a \in x$. Using axiom 4 again, we conclude that $x \times y = \{(a, b) \mid a \in x \wedge b \in y\} = \bigcup_{a \in y} \{(a, b) \mid b \in y\} \in \mathbf{U}$

5. By axiom 4, $\bigcup_{i \in I} x_i \in \mathbf{U}$. By property 4, $I \times \bigcup_{i \in I} x_i \in \mathbf{U}$. Now, every function from I to $\bigcup_{i \in I} x_i \in \mathbf{U}$ is a subset of $I \times \bigcup_{i \in I} x_i \in \mathbf{U}$. Since $\prod_{i \in I} x_i$ is a set of functions from I to $\bigcup_{i \in I} x_i \in \mathbf{U}$, we have, by defintion of power set, $\prod_{i \in I} x_i \subset (P)(I \times \bigcup_{i \in I} x_i)$. Hence, by axiom 3 and property 2, we conclude that $\prod_{i \in I} x_i \in \mathbf{U}$.

6. Assume the contrary, namely that $x \in \mathbf{U}$ and $\#x \geq \#\mathbf{U}$. By axiom 3, $\mathcal{P}(x) \in \mathbf{U}$ but $\#(\mathcal{P}(x)) = 2^{\#x} \geq 2^{\#\mathbf{U}}$. Since, by axiom 1, every element of $\mathcal{P}(x)$ belongs to $\mathbf{U}$, this would mean that we would have at least $2^{\#\mathbf{U}}$ elements of $\mathbf{U}$, which contradicts the fact that $\#U < 2^{\#\mathbf{U}}$. (This argument is a variation on Cantor's paradox.)