quantum groupoids

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Defines	quantum group
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Defines	quantum spacetime structure

This is a topic entry on quantum groupoids, related mathematical concepts and their applications in modern quantum phyiscs.

0.0.1 Quantum groupoids and related groupoid C*–algebras

Definition 0.1. *quantum groupoids*, Q_{G}' s, are currently defined either as quantized, http://planetmath.org compact groupoids endowed with a left Haar measure system, (G, μ), or as *weak Hopf algebras* (WHA). This concept is also an extension of the notion of quantum 'group', which is sometimes represented by a Hopf algebra, H. *Quantum groupoid representations* define extended quantum symmetries beyond the 'Standard Model' (SUSY) in Mathematical Physics or Noncommutative Geometry.

Quantum groupoid – restricted here to a certain dual of a weak Hopf algebra– and algebroid symmetries figure prominently both in the theory of dynamical deformations of quantum 'groups' (that is, such groups that are the duals of Hopf algebras) and the quantum Yang–Baxter equations (Etingof et al., 1999,2001). On the other hand, one can also consider the natural extension of *locally compact* (quantum) groups to locally compact (proper) *groupoids* equipped with a Haar measure and a corresponding groupoid representation theory (Buneci, 2003) as a major, potentially interesting source for locally compact (but generally *non-Abelian*) *quantum groupoids.* The corresponding quantum groupoid representations on bundles of Hilbert spaces extend quantum symmetries well beyond those of quantum 'groups'/Hopf algebras and simpler operator algebra representations, and are also consistent with the locally compact quantum group representations that were recently studied in some detail by Kustermans and Vaes (2000, and references cited therein). The latter quantum groups are neither Hopf algebras, nor are they equivalent to Hopf algebras or their dual coalgebras. Quantum groupoid representations are, however, the next important step towards unifying quantum field theories with general relativity (GR) in a locally covariant and quantized form. Such representations need not however be restricted to weak Hopf algebra representations, as the latter have no known connection to any type of GR theory and also appear to be inconsistent with GR.

One is also motivated by numerous, important quantum physics examples to introduce a framework for quantum symmetry breaking in terms of either *locally compact quantum groupoid*, or related algebroid, representations, such as those of *weak Hopf C*-algebroids with convolution*; the latter are usually realized in the context of *rigged Hilbert spaces* (Bohm and Gadella, 1989).

Furthermore, with regard to a unified and global framework for symmetry breaking, as well as higher order quantum symmetries, one needs to look towards the *double groupoid* structures of Brown and Spencer (1976), to enable one to introduce the concepts of *quantum and graded Lie bi–algebroids* which are expected to carry a distinctive C*–algebroid convolution structure. The extension to *supersymmetry* leads then naturally to superalgebra, superfield symmetries and their involvement in supergravity or quantum gravity (QG) theories for intense gravitational fields in fluctuating, quantized spacetimes. Their mathematical/quantum algebraic classification then involves superstructures with such supersymmetries that can only be appropriately studied in (quantum) supercategories.

Thus, a natural extension of quantum symmetries leads one to Higher Dimensional Algebra (HDA) and may involve, for example, both *'quantum' double groupoids* defined as 'locally compact' double groupoids equipped with Haar measures via convolution, and an extension of superalgebra to double (super) algebroids, (that are naturally much more general than the Lie double algebroids defined in Mackenzie, 2004).

One can now proceed to formally define several quantum algebraic topology concepts that are needed to express the extended quantum symmetries in terms of proper quantum groupoid and algebroid representations. 'Hidden', higher dimensional quantum symmetries will then also emerge either *via* generalized quantization procedures from higher dimensional algebra representations or be determined as global or local invariants obtainable– at least in principle– through non-Abelian algebraic topology (NAAT) methods.

0.0.2 Weak Hopf algebras

Let us begin by recalling the notion of a quantum 'group' in relation to a Hopf algebra where the former is often realized as an automorphism group for a quantum space, that is, an object in a suitable category of generally noncommutative algebras. One of the most common guises of a quantum 'group' is as the dual of a non-commutative, non-associative Hopf algebra. The Hopf algebras (cf. Chaician and Demichev 1996; Majid,1996), and their generalizations (Karaali, 2007), are some of the fundamental building blocks of quantum operator algebra, even though they cannot be 'integrated' to groups like the 'integration' of Lie algebras to Lie groups; furthermore, the connection of Hopf algebras to quantum symmetries seems to be only indirect.

Definition 0.2. In order to define a *weak Hopf algebra*, one can relax certain axioms of a Hopf algebra as follows :

(1) The comultiplication is not necessarily unit–preserving.

(2) The counit ε is not necessarily a homomorphism of algebras.

(3) The axioms for the antipode map $S : A \longrightarrow A$ with respect to the counit are as follows. For all $h \in H$,

$$\begin{aligned} m(\mathrm{id} \otimes S)\Delta(h) &= (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)) \\ m(S \otimes \mathrm{id})\Delta(h) &= (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)) \\ S(h) &= S(h_{(1)})h_{(2)}S(h_{(3)}) \ . \end{aligned} \tag{0.1}$$

These axioms may be appended by the following commutative diagrams

$$\begin{array}{ccccccc} A \otimes A & \xrightarrow{S \otimes \mathrm{id}} & A \otimes A & \quad & A \otimes A & \xrightarrow{\mathrm{id} \otimes S} & A \otimes A \\ {\scriptstyle\Delta}\big\uparrow & & \big\downarrow{\scriptstyle m} & & {\scriptstyle\Delta}\big\uparrow & & \big\downarrow{\scriptstyle m} \\ A & \xrightarrow{u \circ \varepsilon} & A & & A & \xrightarrow{u \circ \varepsilon} & A \end{array} \tag{0.2}$$

along with the counit axiom:

$$\begin{tikzcd} A \otimes A \arrow[d, "\varepsilon\otimes 1"'] & A \arrow[l, "\Delta"'] \arrow[d, "\Delta"] \arrow[dl, "\mathrm{id}_A"'] \\ A & A \otimes A \arrow[l, "1\otimes\varepsilon"] \end{tikzcd} \tag{0.3}$$

Several mathematicians substitute the term *quantum 'groupoid'* for a weak Hopf algebra, although this algebra in itself is not a proper groupoid, but it may have a component *group* algebra as in the example of the quantum double discussed next; nevertheless, weak Hopf algebras generalize Hopf algebras –that with additional properties– were previously introduced as quantum 'groups' by mathematical physicists. (The latter–as already discussed–are not mathematical groups but algebras). As it will be shown in the next subsection, quasi–triangular quasi–Hopf algebras are directly related to quantum symmetries in conformal (quantum) field theories. Furthermore, weak C*–Hopf quantum algebras lead to weak C*–Hopf algebroids that are linked to quasi–group quantum symmetries, and also to certain Lie algebroids (and their associated Lie–Weinstein groupoids) used to define Hamiltonian (quantum) algebroids over the phase space of (quantum) W_N–gravity.

0.0.3 Examples of weak Hopf algebras.

One can refer here to the example given by Bais et al. (2002). Let G be a non–Abelian group and $H \subset G$ a discrete subgroup. Let *F(H)* denote the space of functions on H and $\mathbb{C}H$ the group algebra (which consists of the linear span of group elements with the group structure). *The quantum double D(H)* (Drinfel'd, 1987) is defined by the eqn :

$D(H) = F(H) \,\widetilde{\otimes}\, \mathbb{C}H$, where, for $x \in H$, the 'twisted tensor product' is specified by the next eqn:

$\widetilde{\otimes} \mapsto \ (f_1 \otimes h_1)(f_2 \otimes h_2)(x) = f_1(x) f_2(h_1 x h_1^{-1}) \otimes h_1 h_2$. The physical interpretation given to this construction usually proceeds by considering H as the 'electric gauge group', and *F(H)* as the 'magnetic symmetry' generated by $\{f \otimes e\}$. In terms of the counit ε, the double *D(H)* has a trivial representation given by $\varepsilon(f \otimes h) = f(e)$. there are several very interesting features of this construction. For the purpose of braiding relations there is available an R matrix, $R \in D(H) \otimes D(H)$, leading to the following operator:

$\mathcal{R} \equiv \sigma \cdot (\Pi^A_\alpha \otimes \Pi^B_\beta)$ to be defined in terms of the Clebsch–Gordan series $\Pi^A_\alpha \otimes \Pi^B_\beta \cong N^{AB\gamma}_{\alpha\beta C}\ \Pi^C_\gamma$, and where σ denotes a flip operator. The operator $\mathcal{R}^2$ is sometimes called the *monodromy* or *Aharanov–Bohm phase factor*. In the case of a condensate in a state $|v\rangle$ in the carrier space of some representation Π^A_α one considers the maximal Hopf subalgebra T of a Hopf algebra A for which $|v\rangle$ is T–invariant; specifically :

$\Pi^A_\alpha(P)\ |v\rangle = \varepsilon(P)|v\rangle\ ,\ \forall P \in T$.

For the second example, consider the example provided by Mack and Schomerus (1992) using a more general notion of the Drinfel'd construction–the notion of a *quasi triangular quasi–Hopf algebra* (QTQHA) which was developed with the aim of studying a range of essential symmetries with special properties, such as the *quantum group algebra* $\mathrm{U}_q(\mathrm{sl}_2)$ with $|q| = 1$. If $q^p = 1$, then it was shown that a QTQHA is canonically associated with $\mathrm{U}_q(\mathrm{sl}_2)$. Such QTQHAs are claimed as the true symmetries of minimal conformal field theories.

0.0.4 Weak Hopf C*–algebras in relation to quantum symmetry breaking.

In our setting,a *weak C*–Hopf algebra* is a weak *–Hopf algebra which admits a faithful *–representation on a Hilbert space. The weak C*–Hopf algebra is therefore much more likely to be closely related to a 'quantum groupoid' representation than any weak Hopf algebra. However, one can argue that locally compact groupoids equipped with a Haar measure (after quantization) come even closer to defining quantum groupoids. There are already several, significant examples that motivate the consideration of weak C*–Hopf algebras which also deserve mentioning in the context of 'standard' quantum theories. Furthermore, notions such as (proper) *weak C*–algebroids* can provide the main framework for symmetry breaking and quantum gravity that we are considering here. Thus, one may consider the quasi-group symmetries constructed by means of special transformations of the 'coordinate space' M. These transformations along with the coordinate space M define certain Lie groupoids, and also their infinitesimal version - the Lie algebroids $\mathbf{A}$, when the former are Weinstein groupoids. If one then lifts the algebroid action from M to the principal homogeneous space $\mathcal{R}$ over the cotangent bundle $T^*M \longrightarrow M$, one obtains a physically significant algebroid structure. The latter was called the Hamiltonian algebroid, $\mathcal{A}^H$, related to the Lie algebroid, $\mathbf{A}$. The Hamiltonian algebroid is an analog of the Lie algebra of symplectic vector fields with respect to the canonical symplectic structure on $\mathcal{R}$ or T^*M. In this recent example, the Hamiltonian algebroid, $\mathcal{A}^H$ over $\mathcal{R}$, was defined over the phase space of W_N–gravity, with the anchor map to Hamiltonians of canonical transformations (Levin and Olshanetsky, 2003,2008). Hamiltonian algebroids thus generalize Lie algebras of canonical transformations; canonical transformations of the Poisson sigma model phase space define a *Hamiltonian algebroid* with the Lie brackets related to such a Poisson structure on the target space. The Hamiltonian algebroid approach was utilized to analyze the symmetries of generalized deformations of complex structures on Riemann surfaces $\sum_{g,n}$ of genus g with n marked points. However, its implicit algebraic connections to von Neumann *–algebras and/or *weak C*–algebroid representations* have not yet been investigated. This example suggests that algebroid (quantum) symmetries are implicated in the foundation of relativistic quantum gravity theories and supergravity.

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