categorical sequence

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Synonym	linear diagrams
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Related topic	SuperdiagramsAsHeterofunctors
Related topic	CategoryTheory
Defines	linear diagram
Defines	(linear) sequence of morphisms
Defines	exact functor
Defines	short exact sequence

Definition 0.1. A *categorical sequence* is a *linear 'diagram' of morphisms, or arrows, in an abstract category*. In a concrete category, such as the category of sets, the categorical sequence consists of sets joined by set-theoretical mappings in linear fashion, such as:

$$\cdots \to A \xrightarrow{f} B \xrightarrow{\phi} Hom_{Set}(A, B),$$

where $Hom_{Set}(A, B)$ is the set of functions from set A to set B.

0.1 Examples

0.1.1 The chain complex is a categorical sequence example:

Consider a ring R and the *chain complex* consisting of a sequence of http://planetmath.org/Module$R$-modules and homomorphisms:

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \cdots$$

(with the additional condition imposed by $d_n \circ d_{n+1} = 0$ for each pair of adjacent homomorphisms (d_{n+1}, d_n); this is equivalent to the condition $\operatorname{im} d_{n+1} \subseteq \ker d_n$ that needs to be satisfied in order to define this categorical sequence completely as a *chain complex*). Furthermore, a sequence of homomorphisms

$$\cdots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \cdots$$

is said to be *exact* if each pair of adjacent homomorphisms (f_{n+1}, f_n) is *exact*, that is, if $\mathrm{im} f_{n+1} = \mathrm{ker} f_n$ for all n. This concept can be then generalized to morphisms in a http://planetmath.org/ExactSequence2categorical exact sequence, thus leading to the corresponding definition of an http://planetmath.org/ExactSequence2exact sequence in an Abelian category.

Remark 0.1. Inasmuch as categorical diagrams can be defined as functors, exact sequences of special types of morphisms can also be regarded as the corresponding, special functors. Thus, exact sequences in Abelian categories can be regarded as certain functors of Abelian categories; the details of such functorial (abelian) constructions are left to the reader as an exercise. Moreover, in (commutative or Abelian) homological algebra, an http://planetmath.org/ExactFunctorexact functor is simply defined as a functor F between two Abelian categories, $\mathcal{A}$ and $\mathcal{B}$, $F : \mathcal{A} \to \mathcal{B}$, which preserves categorical exact sequences, that is, if F carries a short exact sequence $0 \to C \to D \to E \to 0$ (with $0, C, D$ and E objects in $\mathcal{A}$) into the corresponding sequence in the Abelian category $\mathcal{B}$, $(0 \to F(C) \to F(D) \to F(E) \to 0)$, which is also exact (in $\mathcal{B}$).