double dual embedding

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Let V be a vector space over a field K. Recall that V^*, the dual space, is defined to be the vector space of all linear forms on V. There is a natural embedding of V into V^{**}, the dual of its dual space. In the language of categories, this embedding is a natural transformation between the identity functor and the double dual functor, both endofunctors operating on $\mathcal{V}_K$, the category of vector spaces over K.

Turning to the details, let

$$\mathrm{I}, \mathrm{D} : \mathcal{V}_K \to \mathcal{V}_K$$

denote the identity and the dual functors, respectively. Recall that for a linear mapping $L : U \to V$ (a morphism in $\mathcal{V}_K$), the dual homomorphism $D[L] : V^* \to U^*$ is defined by

$$D[L](\alpha) : u \mapsto \alpha(Lu), \quad u \in U,\ \alpha \in V^*.$$

The double dual embedding is a natural transformation

$$\delta : \mathrm{I} \to \mathrm{D}^2,$$

that associates to every $V \in \mathcal{V}_K$ a linear homomorphism $\delta_V \in \mathrm{Hom}(V, V^{**})$ described by

$$\delta_V(v) : \alpha \mapsto \alpha(v), \quad v \in V,\ \alpha \in V^*$$

To show that this transformation is natural, let $L : U \to V$ be a linear mapping. We must show that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "\delta_U"] \arrow[d, "L"] & U^{**} \arrow[d, "D^2[L]"] \\ V \arrow[r, "\delta_V"] & V^{**} \end{tikzcd}$$

Let $u \in U$ and $\alpha \in V^*$ be given. Following the arrows down and right we have that

$$(\delta_V \circ L)(u) : \alpha \mapsto \alpha(Lu).$$

Following the arrows right, then down we have that

$$\begin{aligned} (\mathrm{D}[\mathrm{D}[L]] \circ \delta_U)(u) : \alpha &\mapsto (\delta_U u)(\mathrm{D}[L]\alpha) \\ &= (\mathrm{D}[L]\alpha)(u) \\ &= \alpha(Lu), \end{aligned}$$

as desired.

Let us also note that for every non-zero $v \in V$, there exists an $\alpha \in V^*$ such that $\alpha(v) \neq 0$. Hence $\delta_V(v) \neq 0$, and hence δ_V is an embedding, i.e. it is one-to-one. If V is finite dimensional, then V^* has the same dimension as V. Consequently, for finite-dimensional V, the natural embedding δ_V is, in fact, an isomorphism.