F-isomorphisms in categories

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Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F : \mathcal{C} \to \mathcal{D}$ be a (covariant or contravariant) functor and let $\alpha \in \mathrm{Hom}(A, B)$ be a morphism, where $A, B \in \mathrm{Ob}(\mathcal{C})$.

Definition. A morphism $\alpha : A \to B$ is an *F-isomorphism* if $F(\alpha)$ is an isomorphism in $\mathcal{D}$.

Note that each isomorphism in $\mathcal{C}$ is an F-isomorphism for each functor F. The converse is true in the following sense: if α is an F-isomorphism for each functor F then α is an isomorphism. On the other hand there are F-isomorphisms which are not isomorphisms.

Also note, that if $\alpha : X \to Y$ is an F-isomorphism, then there does not have to exist morphism $\beta : Y \to X$ which is „F-inverse" to α. Indeed, there may be no F-isomorphism from Y to X, see examples:

Example. 1) Let X be an object in $\mathcal{D}$ and define $F_X : \mathcal{C} \to \mathcal{D}$ as follows: for $A \in \mathrm{Ob}(\mathcal{C})$ put $F_X(A) = X$ and for $\alpha \in \mathrm{Hom}(A, B)$ put $F_X(\alpha) = \mathrm{id}_X$. This is the constant functor and every morphism in $\mathcal{C}$ is an F_X-isomorphism (although it does not have to be an isomorphism). In particular, it may happen that for some objects X and Y in $\mathcal{C}$ there is a morphism from X to Y but no morphism from Y to X. In this case there is an F_X-isomorphism from X to Y but not vice versa.

2) Let $\mathcal{T}\mathrm{op}^*$ be the category of pointed topological spaces and continous maps preserving based point, $\mathcal{S}\mathrm{et}$ be the category of sets and functions, $\mathcal{G}\mathrm{r}$ be the category of groups and homomorphisms. Consider the functor $\pi : \mathcal{T}\mathrm{op}^* \to \mathcal{S}\mathrm{et} \times \mathcal{G}\mathrm{r} \times \mathcal{G}\mathrm{r} \times \cdots$ defined by:

$$\pi(X, x_0) = (\pi_0(X, x_0), \pi_1(X, x_0), \pi_2(X, x_0), \pi_3(X, x_0), \ldots);$$

$$\pi(f) = (\pi_0(f), \pi_1(f), \pi_2(f), \pi_3(f), \ldots),$$

where π_n is the n-th homotopy group functor. Then π-isomorphism is a weak homotopy equivalence and it is known (due to Whitehead) that each weak homotopy equivalence between pointed CW-complexes is the homotopy equivalance. On the other hand there are weak homotopy equivalences which are not homotopy equivalences.

The concept of F-isomorphism is especially important in representation theory, where F is the homology functor from category of complexes over an abelian category $\mathcal{C}$ to $\mathcal{C}$.