injective and surjective morphisms in concrete categories

Canonical name	InjectiveAndSurjectiveMorphismsInConcreteCategories
Date of creation	2013-03-22 18:47:32
Last modified on	2013-03-22 18:47:32
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	4
Author	joking (16130)
Entry type	Definition
Classification	msc 18A05

Let $(\mathcal{C}, U)$ be concrete category over $\mathcal{SET}$ and let $\alpha : X \to Y$ be a morphism in $\mathcal{C}$.

Definition. Morphism α is called *injective* (resp. *surjective*) if $U(\alpha)$ is an injection (resp. surjection).

Some properties of injective and surjective morphisms:

Proposition. Assume that $\alpha : X \to Y$ is a morphism in $\mathcal{C}$.
i) If α is injective (resp. surjective), then α is a monomorphism (resp. epimorphism);
ii) If α is a section (resp. retraction), then α is injective (resp. surjective).

Proof. i) Assume that α is injective and let $\beta_1, \beta_2 : Z \to X$ be morphisms in $\mathcal{C}$ such that $\alpha \circ \beta_1 = \alpha \circ \beta_2$. Then

$$U(\alpha \circ \beta_1) = U(\alpha \circ \beta_2)$$

and this implies that

$$U(\alpha) \circ U(\beta_1) = U(\alpha) \circ U(\beta_1).$$

Since $U(\alpha)$ is injective, we obtain that $U(\beta_1) = U(\beta_2)$ and since U is faithful, we get that

$$\beta_1 = \beta_2.$$

Analogously we prove, that surjective morphism is an epimorphism.

ii). Assume that $\alpha : X \to Y$ is a section. Then there exists $\beta : Y \to X$ such that $\beta \circ \alpha = \mathrm{id}_X$. Thus we have

$$U(\beta) \circ U(\alpha) = \mathrm{id}_{U(X)},$$

so $U(\alpha)$ is injective. Analogously retractions are surjective. $\square$