functor

Canonical name	Functor
Date of creation	2013-03-22 12:02:50
Last modified on	2013-03-22 12:02:50
Owner	nerdy2 (62)
Last modified by	nerdy2 (62)
Numerical id	7
Author	nerdy2 (62)
Entry type	Definition
Classification	msc 18-00
Synonym	covariant functor
Synonym	contravariant functor
Related topic	Endofunctor
Related topic	Monad

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a covariant *functor* $T : \mathcal{C} \to \mathcal{D}$ consists of an assignment for each object X of $\mathcal{C}$ an object $T(X)$ of $\mathcal{D}$ (i.e. a "function" $T : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$) together with an assignment for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, to a morphism $T(f) \in \mathrm{Hom}_{\mathcal{D}}(T(A), T(B))$, such that:

- $T(1_A) = 1_{T(A)}$ where 1_X denotes the identity morphism on the object X (in the respective category).
- $T(g \circ f) = T(g) \circ T(f)$, whenever the composition $g \circ f$ is defined.

A contravariant functor $T : \mathcal{C} \to \mathcal{D}$ is just a covariant functor $T : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ from the opposite category. In other words, the assignment reverses the direction of maps. If $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, then $T(f) \in \mathrm{Hom}_{\mathcal{D}}(T(B), T(A))$ and $T(g \circ f) = T(f) \circ T(g)$ whenever the composition is defined (the domain of g is the same as the codomain of f).

Given a category $\mathcal{C}$ and an object X we always have the functor $T : \mathcal{C} \to$ **Sets** to the category of sets defined on objects by $T(A) = \mathrm{Hom}(X, A)$. If $f : A \to B$ is a morphism of $\mathcal{C}$, then we define $T(f) : \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$ by $g \mapsto f \circ g$. This is a covariant functor, denoted by $\mathrm{Hom}(X, -)$.

Similarly, one can define a contravariant functor $\mathrm{Hom}(-, X) : \mathcal{C} \to$ **Sets**.