preorder as a category

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Defines	preorder category
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Defines	poset category
Defines	lattice category

Every preorder P has an associated structure of a category. Before describing what this category is, we first associate P with a simpler structure, that of a precategory.

Let's call this $\mathrm{PreCat}(P)$. The objects of this precategory are elements of P and for every $a, b \in P$, $\hom(a, b)$ is either a singleton if $a \leq b$, or the empty set otherwise. The category associated with P is the category generated by enlarging $\mathrm{PreCat}(P)$. For now, call this category $\mathrm{Cat}(P)$. Then we see that the objects of $\mathrm{Cat}(P)$ are again elements of P, and for every $a, b \in P$, $\hom(a, b)$ is the set of all finite chains f from a to b.

With this association, we see the following constructs also have the structure of a category:

- a poset: here, a morphism in $\hom(a, b)$ is a finite chain from a to b where successive nodes are related such that the subsequent node covers the prior node

- a partition of a (non-empty) set (a set with an equivalence relation): $\hom(a, b)$ is non-empty iff a and b belong to the same partition

- a lattice: every pair of objects have a product and a coproduct

- a well-ordered set, in particular an ordinal: if $\hom(a, b)$ is non-empty, it is a singleton. For example, $\mathbf{n}$ is the category consisting of objects $0, 1, \ldots, n-1$, and if $a \leq b$, a morphism in $\hom(a, b)$ is the chain $a \to a+1 \to \cdots \to b-1 \to b$.