2-category

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Related topic	CategoryTheory
Related topic	FunctorCategory2
Related topic	2CCategory
Related topic	IndexOfCatego
Defines	2-morphism
Defines	small Cat-category
Defines	0-cell
Defines	2-cell
Defines	3-cell
Defines	2-morphism
Defines	2-categorical composition
Defines	horizontal composition
Defines	vertical composition
Defines	2-morphism
Defines	0-cell
Defines	multifunctor

Definition 0.1. A small 2-category, $\mathcal{C}_2$, is the first of *higher-order* n-categories constructed as follows.

1. Define $\mathcal{C}at$ as the category of small categories and functors
2. Define a class of objects $A, B, ...$ in $\mathcal{C}at$ called '0- *cells*'
3. For all '0-cells' A, B, consider a set denoted as "$\mathcal{C}_2(A, B)$" that is defined as `http://planetmath.org/Mul` with the elements of the latter set being the functors between the 0-cells A and B; the latter is then organized as a small category whose `http://planetmath.org/FunctorCategories2-` 'morphisms', or '1-cells' are defined by the natural transformations $\eta : F \to G$ for any two morphisms of $\mathcal{C}at$, (with F and G being functors between the '0-cells' A and B, that is, $F, G : A \to B$); as the '2-cells' can be considered as '2-morphisms' between 1-morphisms, they are also written as: $\eta : F \Rightarrow G$, and are depicted as labelled faces in the plane determined by their domains and codomains
4. The 2-categorical composition of 2-morphisms is denoted as "$\bullet$" and is called the *vertical composition*
5. A *horizontal composition*, "$\circ$", is also defined for all triples of 0-cells, A, B and C in $\mathcal{C}at$ as the functor
$$\circ : \mathcal{C}_2(B, C) \times \mathcal{C}_2(A, B) = \mathcal{C}_2(A, C),$$
which is *associative*
6. The identities under horizontal composition are the identities of the 2-cells of 1_X for any X in $\mathcal{C}at$
7. For any object A in $\mathcal{C}at$ there is a functor from the one-object/one-arrow category **1** (terminal object) to $\mathcal{C}_2(A, A)$.

0.1 Examples of 2-categories

1. The 2-category $\mathcal{C}at$ of small categories, functors, and natural transformations;
2. The 2-category $\mathcal{C}at(\mathcal{E})$ of *internal categories in any category $\mathcal{E}$ with finite limits*, together with the internal functors and the internal natural transformations between such internal functors;
3. When $\mathcal{E} = \mathcal{S}et$, this yields again the category $\mathcal{C}at$, but if $\mathcal{E} = \mathcal{C}at$, then one obtains the 2-category of small *double categories*;
4. When $\mathcal{E} = \mathbf{Group}$, one obtains the 2-*category of crossed modules.*

0.2 Remarks

- In a manner similar to the (alternative) definition of small categories, one can describe 2-categories in terms of 2-arrows. Thus, let us consider a set with two defined operations $\otimes$, $\circ$, and also with units such that each operation endows the set with the structure of a (strict) category. Moreover, one needs to assume that all $\otimes$-units are also $\circ$-units, and that an associativity relation holds for the two products:

$$(S \otimes T) \circ (S \otimes T) = (S \circ S) \otimes (T \circ T);$$

- A 2-category is an example of a supercategory with just two composition laws, and it is therefore an $\S_1$-supercategory, because the $\S_0$ supercategory is defined as a standard '1'-category subject only to the ETAC axioms.