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generator of a category

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Let $\mathcal{C}$ be a category, and $f, g : A \to B$ a pair of distinct morphisms. A morphism $h : X \to A$ is said to *distinguish* or *separate* f and g if $f \circ h \neq g \circ h$. For example, if $f \neq g : A \to B$, then 1_A on A distinguishes f and g.

A set $S = \{X_i \mid i \in I\}$ of objects (indexed by a set I) is called a *generating set* of $\mathcal{C}$ if any pair of distinct morphisms $f, g : A \to B$ can be distinguished by a morphism with domain in S and codomain A. In other words, there is $h : X_i \to A$ for some $i \in I$, such that $f \circ h \neq g \circ h$. If $\{X\}$ is a generating family of $\mathcal{C}$, then X is called a *generator* of $\mathcal{C}$. Any set of morphisms containing a generator is a generating set.

Examples

1. In **Set**, the category of sets, any singleton is a generator. Suppose $f, g : A \to B$ are distinct functions, so that $f(x) \neq g(x)$ for some $x \in A$. Let $\{y\}$ be any singleton. Then $h : \{y\} \to A$ defined by $h(y) = x$ is the function distinguishing f and g: for $f \circ h(y) = f(x) \neq g(x) = g \circ h(y)$.

2. In **Rng**, the category of rings, the ring $\mathbb{Z}$ is a generator. If $f, g : R \to S$ are distinct ring homomorphisms, say, $f(r) \neq g(r)$ for some $r \in R$. Then the ring homomorphism $h : \mathbb{Z} \to R$ given by $h(1) = r$ distinguishes f and g.

Remark. A projective object that is also a generator is called a *progenerator*.

References

[1] F. Borceux *Basic Category Theory, Handbook of Categorical Algebra I*, Cambridge University Press, Cambridge (1994)