precategory

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A *precategory* $\mathcal{B}$ consists of the following

1. a class of objects, called objects of $\mathcal{B}$, written $Ob(\mathcal{B})$
2. a set of elements, called arrows or morphisms, for *each* ordered pair (A, B) of objects in $\mathcal{B}$, usually written $\hom(A, B)$. For any arrow $f \in \hom(A, B)$, A is called the *domain* of f and B is the *codomain* of f. It is required that $\hom(A, B) \cap \hom(C, D) = \varnothing$ if $(A, B) \neq (C, D)$.

If $Ob(\mathcal{B})$ is a set, then we say that $\mathcal{B}$ is *small*. A small precategory is just a directed pseudograph (a digraph allowing multiple edges between pairs of vertices), indeed, for the collection of all arrows in $\mathcal{B}$ is a set, written $Mor(\mathcal{B})$. In addition, there are two functions

$$\mathrm{dom}, \mathrm{codom} : Mor(\mathcal{B}) \to Obj(\mathcal{B})$$

such that $\mathrm{dom}(f)$ is the domain of f and $\mathrm{codom}(f)$ is the codomain of f. Note that both dom and codom are well-defined functions because if $f \in \hom(A, B) \cap \hom(C, D)$, then $A = B$ and $C = D$, so that both dom and codom map f to unique objects A and B respectively.

With the realization that a precategory is essentially a directed graph, we may use the language of graph theory to define concepts such as paths and loops in a precategory. This will allow us to enlarge any precategory to a category. We will carry out the construction below.

Paths Defined

Let $\mathcal{B}$ be a precategory. A *path* p (in $\mathcal{B}$) is a finite sequence of arrows $f_1, \ldots, f_n$ such that the codomain of f_i is the domain of f_{i+1}. Note that the definition here does not parallel the one given for a graph (as in graph theory), since we allow vertices (domains and codomains), as well as edges (arrows or morphisms) to coincide. The *length* of a path $p = (p_1, \ldots, p_n)$ is defined to be the non-negative integer n.

Given a path $p = (f_1, \ldots, f_n)$, we may set the domain of p, written $\mathrm{dom}(p)$, to be $\mathrm{dom}(f_1)$, and codomain of p, written $\mathrm{codom}(p)$, to be $\mathrm{codom}(f_n)$. A *loop* is a path p where $\mathrm{dom}(p) = \mathrm{codom}(p)$.

Next, for each ordered pair of objects (A, B) in a precategory $\mathcal{B}$, the collection of paths with with domain A and codomain B is a set, and we denote it by $\mathrm{Hom}(A, B)$.

Composition of Paths Defined

Now, let $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$. So $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_m)$. Since $\mathrm{codom}(f_n) = B = \mathrm{dom}(g_1)$, we can "concatenate" the two paths and form a new path

$$(f_1, \ldots, f_n, g_1, \ldots, g_m),$$

and we write $g \circ f$ for this new path. It is clear that $g \circ f \in \mathrm{Hom}(A, C)$. It is also easy to see that $\circ$ is a function from $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C)$ to $\mathrm{Hom}(A, C)$, if we set $\circ(f, g) := g \circ f$. As the "concatenation" operation is evidently associative, $(h \circ g) \circ f = h \circ (g \circ f)$.

Empty Paths Defined

Finally, for each object A in $Ob(\mathcal{B})$, we can artificially associate an *empty path* 1_A with A, with the following properties

- 1_A is a path with length 0
- $\mathrm{dom}(1_A) = \mathrm{codom}(1_A) := A$; in other words, $1_A \in \mathrm{Hom}(A, A)$
- for any $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(C, A)$, $f \circ 1_A := f$ and $1_A \circ g := g$.

The class of all paths, including every empty path for each object, in $\mathcal{B}$ is written $Path(\mathcal{B})$.

Precategory Enlarged to a Category

So if we start out with a precategory $\mathcal{B}$, we end up with a category $\overline{\mathcal{B}}$ such that

1. $Ob(\overline{\mathcal{B}}) = Ob(\mathcal{B})$
2. $Mor(\overline{\mathcal{B}}) = Path(\mathcal{B})$, such that
    - domain and codomain of each morphism are defined to be the domain and codomain of the underlying path
    - for each ordered pair (A, B) of objects in $\overline{\mathcal{B}}$, the collection of morphisms with domain A and codomain B is a set, and is denoted by $\mathrm{Hom}(A, B)$

- for every triple of objects A, B, C, a function $\circ$ is defined to be the "concatenation" of a path from A to B and a path from B to C
- the identity morphism 1_A each object A is just the empty path associated with A.

We may embed $\mathcal{B}$ in $\overline{\mathcal{B}}$ so that $\mathcal{B}$ is just a diagram of $\overline{\mathcal{B}}$. Because of this, $\mathcal{B}$ is also known as a *diagram scheme*. $\overline{\mathcal{B}}$, also written $F(\mathcal{B})$, is known as the *free category* freely generated by $\mathcal{B}$.