exact sequence

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Let $\mathcal{A}$ be an abelian category. We begin with a preliminary definition.

Definition 1. For any morphism $f : A \longrightarrow B$ in $\mathcal{A}$, let $m : X \longrightarrow B$ be the morphism equal to $\ker(\operatorname{cok}(f))$. Then the object X is called the *image* of f, and denoted $\operatorname{Im}(f)$. The morphism m is called the *image morphism* of f, and denoted $\operatorname{im}(f)$.

Note that $\operatorname{Im}(f)$ is not the same as $\operatorname{im}(f)$: the former is an object of $\mathcal{A}$, while the latter is a morphism of $\mathcal{A}$. We note that f factors through $\operatorname{im}(f)$:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[rr, bend right, "f"'] & \operatorname{Im}(f) \arrow[r, "m"] & B \end{tikzcd}$$

The proof is as follows: by definition of cokernel, $\operatorname{cok}(f)f = 0$; therefore by definition of kernel, the morphism f factors through $\ker(\operatorname{cok}(f)) = \operatorname{im}(f) = m$, and this factor is the morphism e above. Furthermore m is a monomorphism and e is an epimorphism, although we do not prove these facts.

Definition 2. A sequence

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots$$

of morphisms in $\mathcal{A}$ is *exact* at B if $\ker(g) = \operatorname{im}(f)$.