site

Canonical name	Site
Date of creation	2013-03-22 12:16:46
Last modified on	2013-03-22 12:16:46
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	19
Author	mathcam (2727)
Entry type	Definition
Classification	msc 18F10
Classification	msc 14F20
Classification	msc 18F20
Synonym	Grothendieck topology
Related topic	EtaleMorphism
Related topic	Cover
Related topic	TopologicalSpace
Related topic	Sheaf2
Related topic	Sheafification2
Related topic	EtaleCohomology
Related topic	CoveringSpace
Related topic	SomethingRelatedToSheaf
Defines	cover
Defines	covering
Defines	morphism of sites

Definition

A *site* is a generalization of a topology, designed to address the problem that in the algebraic category, the only reasonable topology is the Zariski topology, in which the open sets are much too large. In order to obtain a well-behaved cohomology theory (and an algebraic version of the fundamental group) one needs to find open sets that are "finer" than the Zariski open sets.

Using the machinery of sites, one can construct étale (or l-adic) cohomology, and one can construct crystalline cohomology, both of which can be used to prove the Weil conjectures, and both of which serve as generalizations of the familiar cohomology from topology and complex analysis.

Fix a universe $\mathcal{U}$.

Definition 1. *A* site *is a $\mathcal{U}$-category $\mathcal{C}$ whose objects we call "open sets" and a set S of collections of maps we call "coverings". A covering of an object U of $\mathcal{C}$ is a* **http://planetmath.org/Small***small set of morphisms $\{p_\alpha : U_\alpha \to U\}$ in $\mathcal{C}$. These objects must satisfy the following:*

1. *If $p : U' \to U$ is an isomorphism, then $\left\{U' \xrightarrow{p} U\right\}$ is a covering.*
2. *If*
$$\left\{U_\alpha \xrightarrow{p_\alpha} U\right\}$$
is a covering, and for all α
$$\left\{U_{\alpha,\beta} \xrightarrow{q_{\alpha,\beta}} U_\alpha\right\}$$
is also a covering, then
$$\left\{U_{\alpha,\beta} \xrightarrow{p_\alpha \circ q_{\alpha,\beta}} U\right\}$$
is a covering.
3. *If $\left\{U_\alpha \xrightarrow{p_\alpha} U\right\}$ is a covering, and $V \to U$ is a morphism, then the fibred products $U_\alpha \times_U V$ exist for all α, and we can produce a covering of V:*
$$\left\{V \times_U U_\alpha \xrightarrow{q_\alpha} V\right\}$$
where q_α is the projection onto the first factor of the fibre product.

Given a site, it is very natural to construct presheaves and sheaves on it; the category of sheaves on a site is called a topos. This category is (under some technical assumptions) rich enough to allow a cohomology theory.

The reference to universes and small sets in the definition may be safely ignored for most purposes; they exist to deal with set-theoretic difficulties one can encounter when dealing with certain sites (such as the crystalline site or the big étale site).

References

[1] Grothendieck et al., *Séminaires en Gèometrie Algèbrique 4*, tomes 1, 2, and 3, available on the web at `http://www.math.mcgill.ca/ archibal/SGA/SGA.html`http://www.math.mcgill.ca/ archibal