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monomorphisms of category of sets

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Theorem 1. *Every monomorphism in the category of sets is an injection.*

Proof. Assume $f\colon A \to B$ is a monomorphism. Then, by definition of monomorphism, given any two maps $g, h\colon C \to A$, if $f \circ g = f \circ h$, then $g = h$. Suppose x and y are two elements of A such that $f(x) = f(y)$. Let C be a set with one element, let g be the map which sends this one element to x and let h be the map which sends this one element to y. Because $f(x) = f(y)$, we have $f \circ g = f \circ h$. Since f is a monomorphism, $g = h$, so $x = y$. This implies that f is injective. $\square$

Theorem 2. *Every injection is a split monomorphism.*

Proof. Assume $f\colon A \to B$ is injection. If A is empty, the result is trivial, so we assume that A is not empty; let z be an element of A. Set

$$g = \{(f(x), x) \mid x \in A\} \cup \{(x, z) \mid x \in B \land (\forall y \in A) x \neq f(y)\}$$

We claim that g is a function from B into A. Suppose that x is an element of B. If $x \neq f(y)$ for any $y \in A$, then we have exactly one element of g with x as the first element, namely (x, z). If $x = f(y)$ for some $y \in A$, then we the pair (x, y) with x as first element; were there another pair with x as first element, then we would have $(f(x_1), x_1) = (f(x_2), x_2)$ but, as f is an injection, $f(x_1) = f(x_2)$ would imply $x_1 = x_2$, so this would not be a distinct pair. Hence g is a function. Furthermore, by construction $g \circ f(x) = x$ for all $x \in A$, so f is a split monomorphism. $\square$

Note that the second theorem is stronger than a simple converse to the first theorem — it states that an injection is not just a monomorphism, but that it is actually a split monomorphism. In particular, this means that, in the category of sets, all monomorphisms are actually split monomorphisms.