examples of pullbacks

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This entry shows some examples of categorical pullbacks.

1. In the category of sets, the pullback of a pair of functions $f : A \to C$ and $g : B \to C$ is given by the set $D := \{(a, b) \in A \times B \mid f(a) = g(b)\}$, along with the projections $r : D \to A$ and $s : D \to B$. Here's a sketch of the proof: first, $f \circ r = g \circ s$, and if there are functions $u : E \to A$ and $v : E \to B$ with $f \circ u = g \circ v$, then define a function $w : E \to D$ by $w(e) = (u(e), v(e))$. As $f(u(e)) = g(v(e))$, we have that $(u(e), v(e)) \in D$, so that w is a well-defined function. Furthermore, $r \circ w(e) = r(u(e), v(e)) = u(e)$ and $s \circ w(e) = s(u(e), v(e)) = v(e)$. Finally, this w is easily seen to be unique. Therefore, $(D, r : D \to A, s : D \to B)$ is the pullback of f and g.

2. In the category of groups, the pullback of a pair of group homomorphisms $f : A \to C$ and $g : B \to C$ is again the group $D = \{(a, b) \in A \times B \mid f(a) = g(b)\}$, where the product is defined componentwise, along with the usual projections. The verification that this is indeed the pullback of f and g is almost like the one above. The only thing that needs to be verified is that D is indeed a group. If $(a, b), (c, d) \in D$, then $f(ac) = f(a)f(c) = g(b)g(d) = g(bd)$, so that $(ac, bd) \in D$. Also, $f(1_A) = 1_C = g(1_B)$, so that $(1_A, 1_B) \in D$. Finally, if $(x, y) \in D$, then $f(x^{-1}) = f(x)^{-1} = g(y)^{-1} = g(y^{-1})$, or $(x^{-1}, y^{-1}) \in D$. Therefore, D is a group (a subgroup of $A \times B$).

3. In fact, both of the examples above can be obtained by finding the equalizer of $f \circ p_A$ and $g \circ p_B$, where p_A and p_B are projections from $A \times B$ to A and B respectively. This is the consequence of the fact that a category with finite products and equalizers also has pullbacks, and the pullbacks are obtained in the manner just described (see proof `http://planetmath.org/PropertiesOfPullback`here).

4. The category of small categories has pullbacks. Given small categories $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, and functors $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$, consider the subcategory $\mathcal{D}$ of the comma category $(F \downarrow G)$, where

   - objects are (A, B, f) where $F(A) = G(B)$ and $f = 1_{F(A)}$, and
   - morphisms are $(x, y) : (A, B, 1_{F(A)}) \to (C, D, 1_{F(C)})$ where $F(x) = G(y)$.

   Then it can be shown that $\mathcal{D}$, along with the the functors

- $H_{\mathcal{A}} : \mathcal{D} \to \mathcal{A}$ with $H_{\mathcal{A}}(A, B, f) = A$ and $H_{\mathcal{A}}(x, y) = x$, and
- $H_{\mathcal{B}} : \mathcal{D} \to \mathcal{B}$ with $H_{\mathcal{B}}(A, B, f) = B$ and $H_{\mathcal{B}}(x, y) = y$

is the pullback of F and G. The proof is similar to the proof on the http://planetmath.org/PropertiesOfACommaCategoryuniversal property of a comma category.