sheafification

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Let F be a presheaf over a topological space X with values in a category $\mathcal{A}$ for which sheaves are defined. The *sheafification* of F, if it exists, is a sheaf F' over X together with a morphism $\theta : F \longrightarrow F'$ satisfying the following universal property:

> For any sheaf G over X and any morphism of presheaves $\phi : F \longrightarrow G$ over X, there exists a unique morphism of sheaves $\psi : F' \longrightarrow G$ such that the diagram
>
> $$\begin{tikzcd} F \arrow[r, "\theta"] \arrow[rr, bend right, "\phi"'] & F' \arrow[r, "\psi"] & G \end{tikzcd}$$
>
> commutes.

In light of the universal property, the sheafification of F is uniquely defined up to canonical isomorphism whenever it exists. In the case where $\mathcal{A}$ is a concrete category (one consisting of sets and set functions), the sheafification of any presheaf F can be constructed by taking $F'(U)$ to be the set of all functions $s : U \longrightarrow \bigcup_{p \in U} F_p$ such that

1. $s(p) \in F_p$ for all $p \in U$
2. For all $p \in U$, there is a neighborhood $V \subset U$ of p and a section $t \in F(V)$ such that, for all $q \in V$, the induced element $t_q \in F_q$ equals $s(q)$

for all open sets $U \subset X$. Here F_p denotes the stalk of the presheaf F at the point p.

The following quote, taken from [**?**], is perhaps the best explanation of sheafification to be found anywhere:

> F' is "the best possible sheaf you can get from F". It is easy to imagine how to get it: first identify things which have the same restrictions, and then add in all the things which can be patched together.

References

[1] David Mumford, *The Red Book of Varieties and Schemes*, Second Expanded Edition, Springer–Verlag, 1999 (LNM **1358**)