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categorical pullback

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Defines pullback square

Defines pushout

Defines categorical pushout
Defines pushout square
Defines generalized pullback
Defines generalized pushout
Defines have pullbacks
Defines have pushouts

Pullbacks

Let $f: X \to B$ and $g: Y \to B$ be morphisms in a category \mathcal{C} . Then a pullback diagram, or pullback square of f and g is a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{q} & Y \\
\downarrow p & & \downarrow g \\
X & \xrightarrow{f} & B.
\end{array}$$

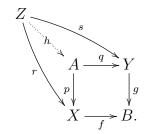
such that if we have another commutative diagram

$$Z \xrightarrow{s} Y$$

$$r \downarrow \qquad \qquad \downarrow g$$

$$X \xrightarrow{f} B.$$

then there is a unique morphism $h: Z \to A$ with the commutative diagram



A pullback of (f, g) is the ordered triple (A, q, p). We also say that p is a pullback of g along f, and q a pullback of f along g. When a pullback of (f, g) exists, it is unique up to isomorphism. The object A, and the morphisms p and q in the diagram above are often denoted by

$$X \times_B Y$$
, $1_X \times_B g$ and $f \times_B 1_Y$

respectively, and the uniquely determined morphism h by

$$\binom{r}{s}$$
.

It is easy to see that

$$X \times_B Y \cong Y \times_B X$$

whenever one (and hence the other) exists.

Remarks.

• The pullback of f and g can be equivalently defined as a limiting cone over the diagram $X \to B \leftarrow Y$. In other words, a pullback diagram is a terminal object in the category of commutative squares of the form

$$Z \longrightarrow Y$$

$$\downarrow \qquad \qquad \downarrow g$$

$$X \longrightarrow B.$$

- A category \mathcal{C} is said to have pullbacks if every diagram $X \to B \leftarrow Y$ can be completed into a pullback diagram.
- The notion of pullbacks can be generalized: let $\{x_i: C_i \to C \mid i \in I\}$ be a collection of morphisms indexed by set I, considered as a small diagram. The *generalized pullback* of the x_i 's is just the limiting cone of the diagram. Using this definition, the generalized pullback of one morphism f is the identity morphism of dom(f), the domain of f, and the generalized pullback of the empty set is a terminal object.
- A pullback is sometimes known as an amalgamated sum.

Pushouts

Dually, given morphisms $f: B \to X$ and $g: B \to Y$, a pushout square of f and g is a commutative diagram

$$B \xrightarrow{g} Y$$

$$f \downarrow s$$

$$X \xrightarrow{t} A$$

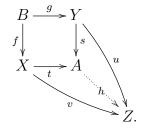
such that if we have another commutative diagram

$$B \xrightarrow{g} Y$$

$$f \downarrow \qquad \qquad \downarrow u$$

$$X \xrightarrow{v} Z$$

Then there is a unique morphism $h:A\to Z$ with the following commutative diagram:



The pair (s,t) of morphisms is a *pushout* of (f,g). s is the the *pushout* of f along g, and t the *pushout* of g along f. Like pullbacks, pushouts are unique up to unique isomorphism when they exist. The object A and the morphisms s and t are typically written

$$X \coprod_B Y$$
, $f \coprod_B 1_Y$ and $1_X \coprod_B g$,

and the unique morphism h is denoted by

Remark. The pushout of f and g can be thought of as the limiting cocone under the diagram $X \leftarrow B \rightarrow Y$. Equivalently, they are initial objects in the category of commutative squares whose top edge is $B \rightarrow Y$ and left edge is $B \rightarrow X$. A category is said to have pushouts if every diagram $X \leftarrow B \rightarrow Y$ can be completed to a pushout diagram. The generalized pushout is defined as the limiting cocone under the diagram consisting of morphisms $y_i : B \rightarrow B_i$, where i belongs to some set I.