a group homomorphism is injective iff the kernel is trivial

Canonical name	AGroupHomomorphismIsInjectiveIffTheKernelIsTrivial
Date of creation	2013-03-22 14:10:15
Last modified on	2013-03-22 14:10:15
Owner	alozano (2414)
Last modified by	alozano (2414)
Numerical id	7
Author	alozano (2414)
Entry type	Theorem
Classification	msc 18Exx
Classification	msc 18A30
Classification	msc 20K30
Classification	msc 13B10
Classification	msc 16W20
Classification	msc 15A03
Related topic	KernelOfAGroupHomomorphism

Proposition. *Let G, H be groups, and let $f\colon G \to H$ be a group homomorphism. Then f is injective if and only if $\operatorname{Ker}(f) = \{e_G\}$, where e_G is the identity element of G, and Ker denotes the kernel of f (see also Kernel of a group homomorphism).*

Proof. First assume that f is injective (i.e. $f(g_1) = f(g_2) \Rightarrow g_1 = g_2$). Recall that:

$$\operatorname{Ker}(f) = \{g \in G : f(g) = e_H\}$$

where e_H is the identity element of H. Since f is a group homomorphism, it follows that $f(e_G) = e_H$. Let $g \in \operatorname{Ker}(f)$, then $f(g) = e_H = f(e_G)$, which implies that $g = e_G$, by the injectivity of f. Thus $\operatorname{Ker}(f) = \{e_G\}$.

For the converse, we assume that $\operatorname{Ker}(f) = \{e_G\}$ and suppose that $f(g_1) = f(g_2)$, for some $g_1, g_2 \in G$. Since f is a homomorphism:

$$f(g_1) = f(g_2) \Rightarrow f(g_1) \cdot f(g_2)^{-1} = e_H \Rightarrow f(g_1 \cdot g_2^{-1}) = e_H$$

Thus $g_1 \cdot g_2^{-1} \in \operatorname{Ker}(f)$, and the kernel is trivial so $g_1 \cdot g_2^{-1} = e_G$, therefore $g_1 = g_2$. □