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Cartesian closed category

Canonical name	CartesianClosedCategory
Date of creation	2013-03-22 16:36:23
Last modified on	2013-03-22 16:36:23
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	9
Author	CWoo (3771)
Entry type	Definition
Classification	msc 18D15
Related topic	CompleteCategory
Related topic	IndexOfCategories
Defines	Cartesian closed

A category $\mathcal{C}$ with finite products is said to be *Cartesian closed* if each of the following functors has a right adjoint

1. $\mathbf{0} : \mathcal{C} \to \mathbf{1}$, where $\mathbf{1}$ is the trivial category with one object 0, and $\mathbf{0}(A) = 0$
2. the diagonal functor $\delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, where $\delta(A) = (A, A)$, and
3. for any object B, the functor $(- \times B) : \mathcal{C} \to \mathcal{C}$, where $(- \times B)(A) = A \times B$, the product of A and B.

Furthermore, we require that the corresponding right adjoints for these functors to be

1. any functor $\mathbf{1} \to \mathcal{C}$, where 0 is mapped to an object T in $\mathcal{C}$. T is necessarily a terminal object of $\mathcal{C}$.
2. the product (`http://planetmath.org/Bifunctor`bifunctor) $(- \times -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ given by $(- \times -)(A, B) \mapsto A \times B$, the product of A and B.
3. for any object B, the exponential functor $(-^B) : \mathcal{C} \to \mathcal{C}$ given by $(-^B)(A) = A^B$, the exponential object from B to A.

In other words, a Cartesian closed category $\mathcal{C}$ is a category with finite products, has a terminal objects, and has exponentials. It can be shown that a Cartesian closed category is the same as a finitely complete category having exponentials.

Examples of Cartesian closed categories are the category of sets **Set** (terminal object: any singleton; product: any Cartesian product of a finite number of sets; exponential object: the set of functions from one set to another) the category of small categories **Cat** (terminal object: any trivial category; product object: any finite product of categores; exponential object: any functor category), and every elementary topos.

References

[1] S. Mac Lane, *Categories for the Working Mathematician*, Springer, New York (1971).