proof of Leray's Theorem (via spectral sequences)

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Let's consider an resolution (i.e. a http://planetmath.org/ResolutionOfASheafresolution by http://planetmath.org/AcyclicSheafacyclic sheaves) $\mathcal{F}^\bullet$ of $\mathcal{F}$ and the double complex $K^{\bullet,\bullet} = \check{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ endowed with the differentials

$$\begin{aligned}\partial &\colon \check{C}^p(\mathcal{U}, \mathcal{F}^\bullet) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F}^\bullet),\\ \bar{\partial} &\colon \check{C}^\bullet(\mathcal{U}, \mathcal{F}^q) \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}^{q+1}),\end{aligned}$$

which are respectively the http://planetmath.org/CechCohomologyGroup2Cech one and the one by the resolution.

The first filtration of this complex gives rise to a spectral sequence whose of the first generation are:

$${}'E_r^{p,q} = H^q_{\bar{\partial}}(\check{C}^p(\mathcal{U}, \mathcal{F}^\bullet)) = \prod_{i_0,\dots,i_p \in I} H^q(U_{i_0\cdots i_p}, \mathcal{F}),$$

where I is the set indexing the covering and the last equality is obtained in virtue of the De Rham-Weil theorem. Thanks to the hypothesis on $\mathcal{U}$, we have

$${}'E_1^{p,q} = \begin{cases} \check{C}^p(\mathcal{U}, \mathcal{F}) & \text{if } q = 0 \\ 0 & \text{if } q \geq 1. \end{cases}$$

Then

$$\begin{aligned}{}'E_2^{p,q} &= H^p_\partial(H^q_{\bar{\partial}}(\check{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)))\\ &= \begin{cases} H^p_\partial(\check{C}^\bullet(\mathcal{U}, \mathcal{F})) & \text{if } q = 0 \\ 0 & \text{if } q \geq 1. \end{cases}\end{aligned}$$

So ${}'E_2^{p,0} = \check{H}^p(\mathcal{U}, \mathcal{F})$ and ${}'E_2^{p,q} = 0$ if $q \neq 0$. The general of spectral sequences of a double complex now yields:

$$H^l(K^\bullet) = {}'E_2^{l,0} = \check{H}^l(\mathcal{U}, \mathcal{F}).$$

Now we consider the second filtration to obtain a spectral sequence whose of the first generations are:

$${}''E_1^{p,q} = H^q_\partial(\check{C}^\bullet(\mathcal{U}, \mathcal{F}^p)) = \check{H}^q(\mathcal{U}, \mathcal{F}^p).$$

The resolution being , we obtain:

$${}''E_1^{p,q} = \begin{cases} \mathcal{F}^p(X) & \text{if } q = 0 \\ 0 & \text{if } q \geq 1, \end{cases}$$

and so

$$''E_2^{p,q} = H^p_{\bar\partial}(H^q_{\partial}(\check{C}^\bullet(\mathcal{U},\mathcal{F}^\bullet))) \quad = \begin{cases} H^p(\mathcal{F}^\bullet(X)) & \text{if } q = 0 \\ 0 & \text{if } q \geq 1. \end{cases}$$

But then $''E_2^{p,0} = H^p(X,\mathcal{F})$ thanks to the De Rham-Weil theorem and $''E_2^{p,q} = 0$ if $q \neq 0$ and so we get:

$$H^l(K^\bullet) = ''E_2^{l,0} = H^l(X,\mathcal{F}),$$

which leads to the claim.