properties of regular and extremal monomorphisms

Canonical name	PropertiesOfRegularAndExtremalMonomorphisms
Date of creation	2013-03-22 16:03:48
Last modified on	2013-03-22 16:03:48
Owner	kompik (10588)
Last modified by	kompik (10588)
Numerical id	12
Author	kompik (10588)
Entry type	Theorem
Classification	msc 18A05
Related topic	ExtremalMonomorphism
Related topic	RegularMonomorphism
Related topic	Equalizer
Related topic	StrongMonomorphism

We will denote the equalizer of f and g by $e = \mathrm{Eq}(f, g)$.

Proposition 1. *Every regular monomorphism is a monomorphism. (Every regular epimorphism is an epimorphism.)*

Proof. Let $f = \mathrm{Eq}(r, s)$. Let $f \circ g = f \circ h$. Then $r \circ (f \circ g) = s \circ (f \circ g)$ and by the definition of the equalizer there exists a unique morphism h such that $f \circ g = f \circ h$, thus $g = h$. $\square$

Proposition 2. *If $g \circ f$ is an extremal monomorphism, then f is an extremal monomorphism.*
If $g \circ f$ is an extremal epimorphism, then g is an extremal epimorphism.

Proof. Since $g \circ f$ is a monomorphism, f is a monomorphism too. Let $f = h \circ e$ and e be an epimorphism. Then $g \circ f = g \circ h \circ e$, but $g \circ f$ is an extremal monomorphism, thus e is an isomorphism.

The second part of the proposition is `http://planetmath.org/DualityPrinciple`dual to the first part. $\square$

Proposition 3. *If $f : X \to Y$ is a morphism then each of the following conditions implies the next one:*

(i) f is an isomorphism

(ii) f is a section

(iii) f is a regular monomorphism

(iv) f is an extremal monomorphism

(v) f is a monomorphism.

(Dual claim: f is an isomorphism $\Rightarrow$ retraction $\Rightarrow$ regular epimorphism $\Rightarrow$ extremal epimorphism $\Rightarrow$ epimorphism.)

Proof. (i)$\Rightarrow$(ii) straightforward from the definition.

(ii)$\Rightarrow$(iii) Let $g \circ f = id_A$, we will show that $f = \mathrm{Eq}(id_B, f \circ g)$. It holds $(f \circ g) \circ f = f \circ (g \circ f) = f \circ id_A = f = id_B \circ f$. If $(f \circ g) \circ h = h$ then $h = f \circ (g \circ h)$ and there is unique such morphism, since f is a monomorphism (every section is a monomorphism).

(iii)$\Rightarrow$(iv) Let $f = \mathrm{Eq}(r, s)$ and $f = g \circ e$ with e an epimorphism. It holds: $(r \circ g) \circ e = r \circ (g \circ e) = r \circ f = s \circ f = s \circ (g \circ e) = (s \circ g) \circ e$, thus it holds $r \circ g = s \circ g$ as well (since e is an epimorphism). By the universal property in the definition of equalizer there exists a unique morphism e' such that $g = f \circ e'$. Thus we get $f \circ id_A = f = g \circ e = f \circ e' \circ e$ and f is a monomorphism, hence $e' \circ e = id_A$, i.e., e is a section. Moreover $id_E \circ e = e = e \circ id_A = e \circ (e' \circ e) = (e \circ e') \circ e$ and e is an epimorphism, hence $id_E = e \circ e'$, i.e., e is a section. The morphism e is a retraction and a section too, thus e is an isomorphism.

(iv)⇒(v) Follows easily from the definition. □

The implication retraction ⇒ regular epimorphism can be interpreted in the category of topological spaces **Top** as the well-known fact that each retraction is a quotient map.

Proposition 4. *Let $f : A \to B$ be a morphism. The following conditions are equivalent:*

(i) f is an isomorphism

(ii) f is an epimorphism and a section

(iii) f is an epimorphism and an extremal monomorphism

(iv) f is a monomorphism and a retraction

(v) f is a monomorphism and an extremal epimorphism.

Proof. Thanks to the duality principle, it suffices to prove the equivalence of the first three conditions.

(i) ⇒ (ii) follows directly from the definition and (ii) ⇒ (iii) is an easy consequence of the above proposition. (iii) ⇒ (i): $f = id_B \circ f$ and f an epimorphism and extremal monomorphism. This implies that f is an isomorphism. □