properties of monomorphisms and epimorphisms

Canonical name	PropertiesOfMonomorphismsAndEpimorphisms
Date of creation	2013-03-22 16:03:41
Last modified on	2013-03-22 16:03:41
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Last modified by	kompik (10588)
Numerical id	7
Author	kompik (10588)
Entry type	Theorem
Classification	msc 18A05

This entry deals with basic properties of monomorphisms and related notions (as extremal and regular monomorphisms, retractions etc) as well as the dual notions.

Monomorphisms (epimorphisms, bimorphisms) are closed under composition.

Proposition 1. *If $f : A \to B$, $g : B \to C$ are monomorphisms (epimorphisms, bimorphisms) then $g \circ f$ is a monomorphism (epimorphism, bimorphism).*

Proof. a) $g \circ f \circ h = g \circ f \circ k \Rightarrow f \circ h = f \circ k \Rightarrow h = k$
b) $h \circ g \circ f = k \circ g \circ f \Rightarrow h \circ g = k \circ g \Rightarrow h = k$
c) An easy corollary of a) and b). $\square$

Proposition 2. *Let $f \circ g$ be a monomorphism. Then g is a monomorphism.*

`http://planetmath.org/DualityPrinciple`*Dual claim: If $g \circ f$ is an epimorphism, then g is an epimorphism.*

Proof. $g \circ h = g \circ k \Rightarrow f \circ g \circ h = f \circ g \circ k \Rightarrow h = k$ $\square$

Retractions and sections are closed under composition.

Proposition 3. *If $f : A \to B$, $g : B \to C$ are retractions (sections, isomorphisms), then $g \circ f$ is a retraction (section, isomorphism).*

Proof. Suppose we are given $h : B \to A$, $k : C \to B$ such that $f \circ h = id_B$, $g \circ k = id_C$. Then $(g \circ f) \circ (h \circ k) = g \circ (f \circ h) \circ k = g \circ id_B \circ k = g \circ k = id_C$. Thus we have shown the first part of the claim. The second part is dual to the first one and the third one follows from the first two. $\square$

Proposition 4. *Let $f : A \to B$, $g : B \to C$ be morphisms. If $g \circ f$ is a section then f is a section. If $g \circ f$ is a retraction then g is a retraction.*

Proof. If $g \circ f$ is a section then there exists a morphism h such that $h \circ (g \circ f) = (h \circ g) \circ f = id_A$, thus f is a section as well. $\square$

Proposition 5. *Every section is a monomorphism. Every retraction is an epimorphism.*

Proof. Let $f : A \to B$ be a section and $g : B \to A$ be the left inverse to f, i.e., $g : B \to A$, $g \circ f = id_A$. If $f \circ h = f \circ k$ then $h = id_A \circ h = g \circ f \circ h = g \circ f \circ k = id_A \circ k = k$. The duality principle yields the second part of the claim. $\square$

Recall that a morphism is called an isomorphism if it is a section and a retraction at the same time.

Lemma 1. *If $f : A \to B$, $g, h : B \to A$ are morphisms such that $g \circ f = id_A$ and $f \circ h = id_B$ then $g = h$.*

Proof. $h = id_A \circ h = (g \circ f) \circ h = g \circ (f \circ h) = g \circ id_B = g$ $\square$

Proposition 6. *A morphism $f : A \to B$ is an isomorphism if and only if there exists a morphism $g : B \to A$ such that $g \circ f = id_A$, $f \circ g = id_B$. The morphism g is determined uniquely.*

Proof. The implication $\boxed{\Leftarrow}$ is obvious. The implication $\boxed{\Rightarrow}$ follows from the above lemma. $\square$

The morphism g from the above proposition is called the inverse of f and denoted by f^{-1}.

As an easy corollary we get:

Proposition 7. *If f is an isomorphism then also f^{-1} is an isomorphism.*