algebraic category

Canonical name	AlgebraicCategory
Date of creation	2013-03-22 19:19:43
Last modified on	2013-03-22 19:19:43
Owner	bci1 (20947)
Last modified by	bci1 (20947)
Numerical id	14
Author	bci1 (20947)
Entry type	Definition
Classification	msc 18-00
Defines	monadic functor
Defines	adjoint functor
Defines	T-algebra
Defines	monad on a category
Defines	structure map
Defines	monadic adjunction
Defines	adjoint functor pair

1 Algebraic Category

1.1 Introduction

Classes of algebras can be categorized at least in two types: either classes of *specific algebras*, such as: group algebras, K-algebras, groupoid algebras, logic algebras, and so on, or *general* ones, such as general classes of: categorical algebras, higher dimensional algebra (HDA), supercategorical algebras, universal algebras, and so on.

1.1.1 Equivalence classes

Although all classes can be regarded as equivalence, weak equivalence, relations etc., classes of algebras (either specific or general ones), these classes do not define identical, or even isomorphic structures, as the notion of 'equivalence' can have more than one meaning even in the algebraic case.

1.1.2 Basic concepts and definitions

Monad on a category $\mathcal{C}$, and a T-algebra in $\mathcal{C}$

Definition 1.1. Let us consider a category $\mathcal{C}$, two functors: $T : \mathcal{C} \to \mathcal{C}$ (called the monad functor) and $T^2 : \mathcal{C} \to \mathcal{C} = T \circ T$, and two natural transformations: $\eta : 1_{\mathcal{C}} \to T$ and $\mu : T^2 \to T$. The triplet $(\mathcal{C}, \eta, \mu)$ is called a *monad on the category* $\mathcal{C}$. Then, a *T-algebra* (Y, h) is defined as an object Y of a category $\mathcal{C}$ together with an arrow $h : TY \to Y$ called the *structure map* in $\mathcal{C}$ such that:

1.
$$Th : T^2 \to TY,$$

2.
$$h \circ Th = h \circ \mu_Y,$$
where: $\mu_Y : T^2Y \to TY$; and

3.
$$h \circ \eta_Y = 1_Y.$$

Category of Eilenberg-Moore algebras of a monad T

An important definition related to abstract classes of algebras and universal algebras is that of the category of Eilenberg-Moore algebras of a monad T:

Definition 1.2. The category $\mathcal{C}^T$ of T-algebras and their morphisms is called the *Eilenberg-Moore category* or *category of Eilenberg-Moore algebras* of the monad T.

1.2 Algebraic Category Definition

With the above definition, one can also define a *category of classes of algebras and their associated groupoid homomorphisms* which is then an algebraic category.

Another example of algebraic category is that of the category of C*-algebras.

Generally, a category $\mathcal{A}_C$ is called *algebraic* if it is monadic over the category of sets and set-theoretical mappings, *Set*; thus, a functor $G : \mathcal{D} \to \mathcal{C}$ is called *monadic* if it has a left adjoint $F : \mathcal{C} \to \mathcal{D}$ forming a *monadic adjunction* (F, G, η, ϵ) with G and η, ϵ being, respectively, the unit and counit; such a *monadic adjunction* between categories $\mathcal{C}$ and $\mathcal{D}$ is defined by the condition that category $\mathcal{D}$ is equivalent to the to the Eilenberg-Moore category $\mathcal{C}^T$ for the monad

$$T = GF.$$