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n-category

Canonical name	Ncategory
Date of creation	2013-03-22 18:18:33
Last modified on	2013-03-22 18:18:33
Owner	bci1 (20947)
Last modified by	bci1 (20947)
Numerical id	37
Author	bci1 (20947)
Entry type	Definition
Classification	msc 18E05
Classification	msc 18-00
Synonym	higher order categories
Synonym	higher dimensional algebra
Related topic	2Category
Related topic	FunctorCategories
Related topic	RSupercategory
Related topic	Supercategories3
Related topic	AxiomsOfMetacategoriesAndSupercategories
Related topic	HigherDimensionalAlgebraHDA
Related topic	VariableTopology3
Related topic	CategoryTheory
Related topic	OverviewArticleForAlgebraicTopology
Defines	higher order category
Defines	(n-1)-supercategory

Definition 0.1. a small n-category , $\mathcal{C}_n$, is the n-th order category of (small) n-categories n-$\mathcal{C}at$ constructed by induction on n in two main stages:

1. define the category 0-Cat as the category $\mathcal{S}et$ of sets and functions;
2. define the category $(n+1) - \mathcal{C}at$ as the category of (n) categories enriched over the category $\mathcal{C}_n$. The construction is simplified by beginning with the definition of the 2-category.

   The following, more detailed *recursive construction* of $n - \mathcal{C}at$ utilizes the fact that *if a category* $\mathcal{C}$ *has finite products, the category of* $\mathcal{C}$*-enriched categories also has finite products.*

1. define $\mathcal{C}at$, or category $1 - \mathcal{C}at$ as the category of small categories and functors;
2. define a class of objects $A, B, ...$ in $\mathcal{C}at$ called '0-*cells*';
3. for all '0-cells' A, B, consider the set $Hom_{\mathcal{C}_2}(A, B)$, or `http://planetmath.org/FunctorCategories`$\mathcal{C}_2($ organized as a small category, whose 2-morphisms, or '1-cells', are defined as natural transformations called '2-cells', $\eta : F \to G$ for any two 'morphisms' of $\mathcal{C}at$, with F and G being functors between the '0-cells' A and B, $F, G : A \to B$);
4. the 2-categorical composition is denoted as "$\bullet$" and is called the vertical composition;
5. a *horizontal composition*, "$\circ$", is defined for all triples of 0-cells, A, B and C in $\mathcal{C}at$ as the functor $\circ : \mathcal{C}_2(B, C) \times \mathcal{C}_2(A, B) = \mathcal{C}_2(A, C)$; which is *associative*;
6. the identities under horizontal composition are the identities of the 2-cells of 1_X for any X in $\mathcal{C}at$;
7. for any object A in $\mathcal{C}at$ there is a functor from the one-object/one-arrow category 1 (terminal object) to $\mathcal{C}_2(A, A)$.
8. repeat the last $(n-1)$ steps to define '3'-cells, ..., to n-cells; the resulting structure is called an n-category, but it is in fact a metagraph, metacategory, or more generally, a $\S_{n-1}$-supercategory with n *composition laws* and it is also called more recently a *higher order category* or a *higher dimensional algebra.*

Note Because the 2-cells can be considered as 2-morphisms between 1-morphisms, they are also written as: $\eta : F \Rightarrow G$, and are depicted as labelled faces in the plane determined by their domains and codomains.