# subcategory

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Let $\mathcal{C}$ be a (small) category. If $\mathcal{S}$ is a collection of both a subset, call it $\mathrm{Ob}(\mathcal{S})$, of objects of $\mathcal{C}$ and a subset, call it $\mathrm{Mor}(\mathcal{S})$, of morphisms of $\mathcal{C}$ such that

1. For each $S \in \mathrm{Ob}(\mathcal{S})$, the identity morphism of $S$, $id_S \in \mathrm{Mor}(\mathcal{S})$;
2. For each $f \in \mathrm{Mor}(\mathcal{S})$, domain$(f)$ and codomain$(f) \in \mathrm{Ob}(\mathcal{S})$;
3. For every pair $f, g \in \mathrm{Mor}(\mathcal{S})$ such that $f \circ g$ exists, then $f \circ g \in \mathrm{Mor}(\mathcal{S})$.

Then $\mathcal{S}$ is readily seen to be a category. It is called a *subcategory* of the category $\mathcal{C}$.

Given a category $\mathcal{C}$ and a subcategory $\mathcal{S}$ of $\mathcal{C}$, a map

$$\mathrm{Incl} : \mathcal{S} \hookrightarrow \mathcal{C}$$

that sends each object of $\mathcal{S}$ to itself (in $\mathcal{C}$), and each morphism of $\mathcal{S}$ to itself (in $\mathcal{C}$), is a functor. Incl is called the *inclusion functor*, or an *embedding*. This inclusion functor is a faithful functor. If it is also `http://planetmath.org/fullfunctor`full, then we call the corresponding subcategory $\mathcal{S}$ a *full subcategory* of $\mathcal{C}$. In other words, if $\mathcal{S}$ is a full subcategory of $\mathcal{C}$, then

$$\hom_{\mathcal{C}}(S_1, S_2) = \hom_{\mathcal{S}}(S_1, S_2)$$

for pair of $S_1, S_2 \in \mathrm{Ob}(\mathcal{S})$.

**Remarks**

1. Let $T : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor. Then $T(\mathcal{C})$ is a full subcategory of $\mathcal{D}$.
2. Again, let $T : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor. If $\mathcal{S}$ is a full subcategory of $\mathcal{D}$, then $T^{-1}(\mathcal{S})$ defined by:
   - $\mathrm{Ob}(T^{-1}(\mathcal{S})) := \{C \in \mathrm{Ob}(\mathcal{C}) \mid T(C) \in \mathrm{Ob}(\mathcal{S})\}$
   - $\mathrm{Mor}(T^{-1}(\mathcal{S})) := \{f \in \mathrm{Mor}(\mathcal{C}) \mid T(f) \in \mathrm{Mor}(\mathcal{S})\}$

   is a subcategory of $\mathcal{C}$.

**Examples of Subcategories**

1. In **Set**, the category of finite sets is a full subcategory, and so is the category of $k$-element sets, where $k$ is any (possibly infinite) cardinality. If $k$ is finite, then every morphism in the subcategory is invertible.

2. In **Top**, we have the full subcategories whose objects are Euclidean spaces, compact spaces, or Hausdorff spaces.

3. In **Grp**, there is the full subcategory whose objects are abelian groups with additive homomorphisms.

4. **Grp** is in fact a subcategory of the category of topological groups, since every group may be viewed as a topological group with the discrete topology.

5. In , there are the subcategories of commutative rings, matrix rings, or fields. Note that **Field** is not a full subcategory of **Ring**, since the ring homomorphism that maps every element to 0 is not a field homomorphism.

# References

[1] S. Mac Lane, *Categories for the Working Mathematician* (2nd edition), Springer-Verlag, 1997.