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free objects in concrete categories

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By concrete category we will understand pair $(\mathcal{C}, U)$, where C is a category and $U : \mathcal{C} \to \mathcal{SET}$ is a faithful (covariant) functor. Assume that $(\mathcal{C}, U)$ is a concrete category.

Definition 1. Let X be an object in $\mathcal{C}$. Subset $B \subseteq U(X)$ (possibly empty) is called a *basis of* X if for any object Y in $\mathcal{C}$ and any function $g : B \to U(Y)$ there exists exactly one morphism $\alpha : X \to Y$ such that $U(\alpha)(x) = g(x)$ for any $x \in B$. In this case we will say that g *lifts* to α.

Definition 2. Object X will be called *free* if there exists basis of X.

Free objects generalize the notion of free modules over a ring. Some of the properties of free modules can be easily generalized to free objects in arbitrary concrete category. For example:

Proposition. Let X and Y be free objects with bases B and B' respectively and let $f : B \to B'$ be a function. The following statements hold:
i) If f is an injection, then there exists a section $\alpha : X \to Y$ in $\mathcal{C}$;
ii) If f is a surjection, then there exists a retraction $\beta : X \to Y$ in $\mathcal{C}$;
iii) If f is a bijection, then X and Y are isomorphic.

Proof. i) Assume that $f : B \to B'$ is an injection. Let $f_1 : B \to U(Y)$ be defined as

$$f_1(x) = f(x)$$

for all $x \in B$. Now, since $f : B \to B'$ is an injection, then there exists a surjection $f' : B' \to B$ such that

$$f'(f(x)) = x$$

for all $x \in B$. Let $f_2 : B' \to U(X)$ be defined by

$$f_2(y) = f'(y)$$

for all $y \in B'$. Now both X and Y are free and thus there are morphism $\alpha : X \to Y$ and $\beta : Y \to X$ such that

$$U(\alpha)(x) = f_1(x) \text{ and } U(\beta)(y) = f_2(y)$$

for all $x \in B$ and $y \in B'$. It is easy to check, that this implies that

$$U(\beta \circ \alpha)(x) = x$$

for all $x \in B$. But $U(\mathrm{id}_X)(x) = x$ for all $x \in B$ and thus canonical injection $i : B \to U(X)$ lifts to both $\beta \circ \alpha$ and id_X. Since lift is unique, then $\beta \circ \alpha = \mathrm{id}_X$, so α is a section.

ii) Note that if $f : B \to B'$ is a surjection, then there exists an injection $g : B' \to B$ such that $f(g(y)) = y$ for all $y \in B'$. Thus, from i) we obtain that $\beta \circ \alpha = \mathrm{id}_Y$ for $\alpha : Y \to X$ and $\beta : X \to Y$ constructed as in i). Therefore $\beta : X \to Y$ is a retraction.

iii) If f is a bijection, then proof of i) and ii) shows that there are two morphisms $\alpha : X \to Y$ and $\beta : Y \to X$ such that $\beta \circ \alpha = \mathrm{id}_X$ and $\alpha \circ \beta = \mathrm{id}_Y$. Thus X and Y are isomorphic. $\square$

Remark 1. Free objects does not have to exist. For example, the category of finite groups (without the trivial group) and group homomorphisms (where U is a forgetful functor) does not have free objects (this is because there are no nontrivial group homomorphisms between groups with relatively prime orders).

Remark 2. Note that, if there is a free object X in a concrete category $(\mathcal{C}, U)$ such that $\emptyset$ is a basis of X, then X is an initial object. This follows directly from the definition, since any morphism $\alpha : X \to Y$ is a lift of $f : \emptyset \to U(Y)$, thus it has to be unique. Conversly one can easily show, that initial object is always free with $\emptyset$ as a basis.