preadditive category

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0.1 Ab-Category

A category $\mathcal{C}$ is an *ab-category* or ***ab**-category* if

1. for every pair of objects A, B of $\mathcal{C}$, there is a binary operation called addition, written $+_{(A,B)}$ or simply $+$, defined on $\hom(A, B)$,

2. the set $\hom(A, B)$, together with $+$ is an abelian group,

3. (left distributivity) if $f, g \in \hom(A, B)$ and $h \in \hom(B, C)$, then $h(f + g) = hf + hg$,

4. (right distributivity) if $f, g \in \hom(A, B)$ and $h \in \hom(C, A)$, then $(f + g)h = fh + gh$.

In a nutshell, an *ab-category* is a category in which every hom set in $\mathcal{C}$ is an abelian group such that morphism composition distributes over addition. Ab in the name stands for abelian, clearly indicative of the second condition above.

Since a group has a multiplicative (or additive if abelian) identity, $\hom(A, B) \neq \varnothing$ for every pair of objects A, B in $\mathcal{C}$. Furthermore, each $\hom(A, B)$ contains a unique morphism, written $0_{(A,B)}$, as the additive identity of $\hom(A, B)$. Because the subset

$$\{f \cdot 0_{(A,B)} \mid f \in \hom(B, C)\}$$

of $\hom(A, C)$ is also a subgroup by right distributivity, and the additive identity of a subgroup coincides with the additive identity of the group, we have the following identity

$$0_{(B,C)}0_{(A,B)} = 0_{(A,C)}.$$

There are many examples of ab-categories, including the category of abelian groups, the category of R-modules (R a ring), the category of chain complexes, and the category of rings (not necessarily containing a multiplicative identity). However, the category of rings with 1 is not an ab-category (see below for more detail). Nevertheless, a unital ring R itself considered as a category is an ab-category, as the ring of endomorphisms clearly forms an abelian group. It is in fact a ring! This can be seen as a special case of the fact that, in an ab-category, $\mathrm{End}(A) = \hom(A, A)$ is always a ring (with 1). So, conversely, an ab-category with one object is a ring with 1, whose morphisms are elements of the ring.

0.2 Preadditive Category

If an ab-category has an initial object, that object is also a terminal object. By duality, the converse is also true. Therefore, in an ab-category, initial object, terminal object, and zero object are synonymous. In the category $\mathcal{R}$ of unital rings, $\mathbb{Z}$ is an initial object, but it has no terminal object, therefore $\mathcal{R}$ is not an ab-category.

An ab-category with a zero object O is called a *preadditive category*.

In a preadditive category, the groups $\hom(A, O)$ and $\hom(O, B)$ are trivial groups by the definition of the zero object O. Therefore, the zero morphism in $\hom(A, B)$ is also the additive identity of $\hom(A, B)$:

$$0_{(A,B)} = 0_{(O,B)}0_{(A,O)} = A \longrightarrow O \longrightarrow B.$$

Most of the examples of ab-categories are readily seen to be preadditive. If a preadditive category R has only one object, we see from above that it must be a ring. But this object must also be a zero object, so that $\mathrm{End}(R)$ must be trivial, which means R itself must be trivial too, $R = 0$!

Remark. In some literature, a preadditive category is an ab-category, and some do not insist that a preadditive category contains a zero object. Here, we choose to differentiate the two.