groupoid (category theoretic)

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A *groupoid*, also known as a *virtual group*, is a small category where every morphism is invertible. We can give a more explicit, algebraic definition: start with a set G, and a partial binary operation $\circ$ on G. Call a pair (x, y) of elements of G a *composable pair* if $(x, y) \in \operatorname{dom}(\circ)$. A *groupoid* is the pair $(G, \circ)$, together with two unary operations e_L and e_R on it, satisfying the following conditions:

1. (x, y) is a composable pair iff $e_R(x) = e_L(y)$.
2. (x, y) and $(x \circ y, z)$ are composable pairs iff (y, z) and $(x, y \circ z)$ are, and if one of these is true, then $(x \circ y) \circ z = x \circ (y \circ z)$.
3. $(e_L(x), x)$ and $(x, e_R(x))$ are composable pairs and $x = e_L(x) \circ x = x \circ e_R(x)$.
4. for each $x \in G$, there exists $y \in G$ such that (x, y) and (y, x) are composable pairs, and $e_L(x) = x \circ y$ and $e_R(x) = y \circ x$.

Below are some properties:

1. In condition 4 above, $e_L(x) = e_R(y)$ and $e_R(x) = e_L(y)$. This is true by condition 1, since both (x, y) and (y, x) are composable pairs.
2. Again, in condition 4, y is unique. To see this, suppose $z \in G$ satisfies condition 4 (in place of y). Then $y = y \circ e_R(y) = y \circ e_L(x) = y \circ (x \circ y) = y \circ (x \circ z) = (y \circ x) \circ z = (z \circ x) \circ z = e_R(x) \circ z = e_L(z) \circ z = z$. Notice property 1 is used in the proof. We call y the *inverse* of x, and write x^{-1}.
3. In view of condition 4, both e_L and e_R are unique. In other words, if $f_L, f_R : G \to G$ are unary operators on G satisfying conditions 3 and 4 above (in place of e_L and e_R), then $f_L = e_L$ and $f_R = e_R$. In fact, $e_L(x) = x \circ x^{-1}$ and $e_R(x) = x^{-1} \circ x$.
4. Since $x = e_L(x) \circ x = e_L(x) \circ (e_L(x) \circ x) = (e_L(x) \circ e_L(x)) \circ x$, we see that $e_L(x)$ is composable with itself, and that $e_L(x) \circ e_L(x) = e_L(x)$ by the previous property. Similarly, $e_R(x) \circ e_R(x) = e_R(x)$. This shows that $e_R(x)$ and $e_L(x)$ are idempotent with respect to $\circ$ for every $x \in G$.
5. Since $(e_L(x), x)$ is a composable pair, $e_R(e_L(x)) = e_L(x)$ for any $x \in G$. Similarly, $e_L(e_R(x)) = e_R(x)$. Hence $e_R(e_R(x)) = e_R(e_L(e_R(x))) =$

$e_L(e_R(x)) = e_R(x)$. Similarly, $e_L(e_L(x)) = e_L(x)$. This shows that e_R and e_L are idempotent with respect to functional compositions.

6. (Cancellation property): if $x \circ y = x \circ z$, then $y = z$; if $y \circ x = z \circ x$, then $y = z$.

   Proof. Since (x, y) is a composable pair, $e_R(x) = e_L(y)$. But $e_R(e_R(x)) = e_R(x)$, we have $e_R(e_R(x)) = e_L(y)$ so that $(e_R(x), y) = (x^{-1} \circ x, y)$ is a composable pair, hence $(x^{-1}, x \circ y)$ is a composable pair and $x^{-1} \circ (x \circ y) = (x^{-1} \circ x) \circ y = e_R(x) \circ y$. Since $(e_R(x), y)$ is a composable pair, $e_R(x) = e_R(e_R(x)) = e_L(y)$. As a result, $x^{-1} \circ (x \circ y) = e_L(y) \circ y = y$. Similarly $x^{-1} \circ (x \circ z) = z$. By assumption, we deduce that $y = z$. The other statement is proved similarly. $\square$

7. The algebraic definition given can be easily turned into a categorical definition (using objects and morphisms). The details are left for the reader.

If e_R and e_L are constant functions, then G is a group.

Remark. There is also a `http://planetmath.org/Groupoid`group-theoretic concept with the same name.