cohomology of small categories

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Defines	derived functors of inverse limit

Let $\mathcal{C}$ be a small category. For $n \geq 0$ we have functors $\Delta_n : \mathcal{C} \to \mathrm{Ab}$ which send an object $X \in \mathcal{C}$ to the free abelian group generated by $n+1$-tuples of morphisms to X. The action of Δ_n on a morphism $f : X \to Y$ is defined by:

$$\Delta_n(f) : (g_0, g_1, \cdots, g_n) \mapsto (fg_0, fg_1, \cdots, fg_n)$$

for any morphisms $g_0, g_1, \cdots, g_n \in \mathcal{C}$ with codomain X.

For $n > 0$ the natural transformation $\partial_n : \Delta_n \to \Delta_{n-1}$ is defined by letting the homomorphism $[\partial_n]_X : \Delta_n(X) \to \Delta_{n-1}(X)$ be given by:

$$[\partial_n]_X(f_0, f_1, \cdots, f_n)$$

$$= (f_1, \cdots f_n) - (f_0, f_2, \cdots, f_n) + \cdots + [-1^n](f_0, f_1, \cdots, f_{n-1})$$

Hence we have a of natural transformations:

$$\cdots \stackrel{\partial_{n+1}}{\to} \Delta_n \stackrel{\partial_n}{\to} \Delta_{n-1} \stackrel{\partial_{n-1}}{\to} \cdots \stackrel{\partial_2}{\to} \Delta_1 \stackrel{\partial_1}{\to} \Delta_0$$

For any functor $F : \mathcal{C} \to \mathrm{Ab}$, let $[\Delta_n, F]$ denote the abelian group of natural transformations $\Delta_n \to F$. Also let $\partial^n : [\Delta_{n-1}, F] \to [\Delta_n, F]$ denote the abelian group homomorphism sending $\eta \to \eta\partial_n$.

We have a chain complex:

$$\cdots \stackrel{\partial^{n+1}}{\leftarrow} [\Delta_n, F] \stackrel{\partial^n}{\leftarrow} [\Delta_{n-1}, F] \stackrel{\partial^{n-1}}{\leftarrow} \cdots \stackrel{\partial^2}{\leftarrow} [\Delta_1, F] \stackrel{\partial^1}{\leftarrow} [\Delta_0, F]$$

It is easily verified that $H_0([\Delta_*, F], \partial^*)$ is just $\lim_{\leftarrow}(F)$, the inverse limit of F. This motivates the definition:

$$\lim\nolimits^n_{\leftarrow}(F) = H_n([\Delta_*, F], \partial^*)$$

Note that if $\mathcal{C}$ is a group G (that is $\mathcal{C}$ has one object and all its morphisms are invertible) then F may be regarded as a module M, over G. In this case $\lim^n_{\leftarrow}(F)$ coincides with group cohomology: $\lim^n_{\leftarrow}(F) = H^n(G; M)$.