Lie algebra cohomology

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1 Definition.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K, and let M be a http://planetmath.org/RepresentationLieAlgebra$\mathfrak{g}$-module. Our goal is to define a cochain complex $C^k(\mathfrak{g}, M)$, whose cohomology is known as the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in M. We define the space of k-cochains to be

$$C^k(\mathfrak{g}, M) = \mathrm{Hom}_K(\Lambda^k \mathfrak{g}, M), \quad k = 0, 1, \ldots, \dim \mathfrak{g},$$

the vector space of multilinear, alternating mappings from $\overbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}^{k \text{ times}}$ to M. [1] The coboundary operator $\delta : C^k(\mathfrak{g}, M) \to C^{k+1}(\mathfrak{g}, M)$ is defined to be

$$(\delta\omega)(a_0, a_1, \ldots, a_k) = \sum_{0 \leq i \leq k} (-1)^k a_i \cdot \omega(\ldots, \widehat{a_i}, \ldots) + \\ + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([a_i, a_j], \ldots, \widehat{a_i}, \ldots \widehat{a_j}, \ldots), \quad \omega \in C^k(\mathfrak{g}, M),$$

where $\widehat{a_i}$ indicates the omission of the argument a_i. We leave it as an exercise for the reader to check that $\delta\omega$, as defined above, is multi-linear, and alternating. Another, only slightly more challenging exercise, is to prove that $\delta^2 = 0$. The proof involves, in an essential way, the Jacobi identity, and the commutator identity for the action of $\mathfrak{g}$ on M, namely

$$a \cdot (b \cdot u) - b \cdot (a \cdot u) = [a, b] \cdot u, \quad a, b \in \mathfrak{g}, \; u \in M.$$

If $\mathfrak{g}$ is the tangent space of a Lie group, and M is the reals with 0 $\mathfrak{g}$-action, then any alternating map on $\mathfrak{g}$ extends to a smooth alternating form on the Lie group, via (left)- translation by the group action. The above formula for the coboundary gives the exterior derivative with respect to this extension.

2 Infinite-dimensional generalizations.

The above definition generalizes readily to infinite-dimensional Lie algebras. The notion of a linear mapping with an infinite-dimensional domain is quite

[1] It should be understood that $C^0(\mathfrak{g}, M) \cong M$.

tricky, and so the key requirement of any such generalization is some kind of restriction on the space of cochains. Thus, de Rham cohomology of a manifold X can be regarded as $H^k(V(X), C^\infty(X))$, the cohomology of the Lie algebra of vector fields with coefficients that are smooth functions, with the caveat that the cochain spaces $C^k(V(X), C^\infty(X))$ are restricted to $\Omega^k(X)$, the smooth differential forms[2]. Another interesting infinite-dimensional generalization is the Gelfand-Fuchs cohomology of $V(X)$. Here we are allowed cochains that are not simply linear combinations of the components of a vector field, but that also include the derivatives of these components.

3 Applications.

Owing to numerous and useful various applications, it's useful to list the formulas for the first few coboundary operators:

$$\begin{aligned}(\delta\alpha)(a) =& a \cdot \alpha;\\ (\delta\beta)(a,b) =& a\cdot\beta(b) - b\cdot\beta(a) - \beta([a,b]);\\ (\delta\gamma)(a,b,c) =& a\cdot\gamma(b,c) + b\cdot\gamma(c,a) + c\cdot\gamma(a,b) - \gamma([a,b],c) - \gamma([b,c],a) - \gamma([c,a],b),\end{aligned}$$

where $a, b, c \in \mathfrak{g}$, and where α, β, γ are $0, 1$, and 2-cochains, respectively. In particular, for small k, the cohomology groups $H^k(\mathfrak{g}, M)$ have certain interesting interpretations.

The first cohomology space, $H^1(\mathfrak{g}, K)$ is isomorphic as a vector space to $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, the abelianization of $\mathfrak{g}$. More generally $H^1(\mathfrak{g}, M)$ classifies, up to natural equivalence, Lie algebras consisting of inhomogeneous operators

$$a + \beta(a), \quad a \in \mathfrak{g},\ \beta \in C^1(\mathfrak{g}, M).$$

In more fancy language, such operators are called derivations of $T(M)$, the tensor algebra of M.

The second cohomology space, $H^2(\mathfrak{g}, M)$, is naturally isomorphic to the vector space of abelian extensions of $\mathfrak{g}$ by M. Thinking of M as an abelian Lie algebra, such an extension is a Lie algebra $\hat{\mathfrak{g}}$ that occurs in the short-exact sequence

$$0 \to M \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

The third cohomology space has an interesting interpretation in terms of deformations of something or other. This is due to Murray Gerstenhaber of U. Penn, but I've forgotten the details.

[2]The evaluation of a cochain on a list of vector fields is given by contraction.

4 Homological algebra.

Generalizing a bit, Lie algebra cohomology is just the cohomology of a particular kind of algebraic theory. There are analogous cohomology theories for groups, associative algebras, and commutative rings. All these theories can be unified by employing the notion of an injective resolution.

Broadening the scope even further, we can employ category theory and re-conceptualize Lie algebra cohomology as a functor from the category of $\mathfrak{g}$-modules to the category of cochain complexes. One begins with the covariant, left-exact functor

$$(-)^{\mathfrak{g}} : M \mapsto M^{\mathfrak{g}} = \{u \in M : \mathfrak{g} \cdot u = 0\}$$

from the category of $\mathfrak{g}$-modules to the category of K-vector spaces. One then defines $H^k(\mathfrak{g}, -)$ to be the right-derived functors $R^k((-)^{\mathfrak{g}})$.

5 Historical notes.

Lie algebra cohomology was first formalized in an influential 1948 paper by C. Chevalley and S. Eilenberg[**?**]. The aim was to calculate the cohomology, in the topological sense, of a compact Lie group by using the finite-dimensional data of the corresponding Lie algebra. In this they were inspired by an even earlier idea of Elie Cartan, who was the first to announce that there was a connection between the topology of a Lie group and the algebraic structure of the underlying Lie algebra [**?**]. What makes this story particularly interesting is that Homological Algebra, as a subject, was launched by the remarkable 1956 book[**?**] by Cartan and Eilenberg called, oddly enough "Homological Algebra". However the Cartan involved this time is not Elie, but Henri, the equally remarkable son of the very remarkable Elie. A survey of the history of homological algebra by Charles Weibel is available at the K-theory archive[**?**].

References

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