Steinberg group

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Given an associative ring R with identity, the Steinberg group $St(R)$ describes the minimal amount of relations between elementary matrices in R.

For $n \geq 3$, define $St_n(R)$ to be the free abelian group on symbols $x_{ij}(r)$ for i, j distinct integers between 1 and n, and $r \in R$, subject to the following relations:

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$$

$$[x_{ij}, x_{kl}] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l \\ x_{il}(rs) & \text{if } j = k \text{ and } i \neq l \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = l. \end{cases}$$

Note that if $e_{ij}(r)$ denotes the elementary matrix with one along the diagonal, and r in the (i, j) entry, then the $e_{ij}(r)$ also satisfy the above relations, giving a well defined morphism $St_n(R) \to E_n(R)$, where the latter is the group of elementary matrices.

Taking a colimit over n gives the Steinberg group $St(R)$. The importance of the Steinberg group is that the kernel of the map $St(R) \to E(R)$ is the second algebraic K-group of the ring R, $K_2(R)$. This also coincides with the kernel of the Steinberg group. One can also show that the Steinberg group is the universal central extension of the group $E(R)$.