Algebraic K-theory

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Algebraic K-theory is a series of functors on the category of rings. Broadly speaking, it classifies ring invariants, i.e. ring properties that are Morita invariant.

The functor K_0

Let R be a ring and denote by $\mathrm{M}_\infty(R)$ the algebraic direct limit of matrix algebras $\mathrm{M}_n(R)$ under the embeddings $\mathrm{M}_n(R) \to \mathrm{M}_{n+1}(R) : a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. The zeroth K-group of R, $K_0(R)$, is the Grothendieck group (abelian group of formal differences) of idempotents in $\mathrm{M}_\infty(R)$ up to similarity transformations. Let $p \in \mathrm{M}_m(R)$ and $q \in \mathrm{M}_n(R)$ be two idempotents. The sum of their equivalence classes $[p]$ and $[q]$ is the equivalence class of their direct sum: $[p]+[q] = [p \oplus q]$ where $p \oplus q = \mathrm{diag}(p,q) \in \mathrm{M}_{m+n}(R)$. Equivalently, one can work with finitely generated projective modules over R.

The functor K_1

Denote by $\mathrm{GL}_\infty(R)$ the direct limit of general linear groups $\mathrm{GL}_n(R)$ under the embeddings $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R) : g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Give $\mathrm{GL}_\infty(R)$ the direct limit topology, i.e. a subset U of $\mathrm{GL}_\infty(R)$ is open if and only if $U \cap \mathrm{GL}_n(R)$ is an open subset of $\mathrm{GL}_n(R)$, for all n. The first K-group of R, $K_1(R)$, is the abelianisation of $\mathrm{GL}_\infty(R)$, i.e.

$$K_1(R) = \mathrm{GL}_\infty(R)/[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)].$$

Note that this is the same as $K_1(R) = H_1(\mathrm{GL}_\infty(R), \mathbb{Z})$, the first group homology group (with integer coefficients).

The functor K_2

Let $\mathrm{E}_n(R)$ be the elementary subgroup of $\mathrm{GL}_n(R)$. That is, the group generated by the elementary $n \times n$ matrices $e_{ij}(r)$, $r \in R$, where $e_{ij}(r)$ is the matrix with ones on the diagonals, the value r in row i, column j and zeros elsewhere. Denote by $\mathrm{E}_\infty(R)$ the direct limit of the $\mathrm{E}_n(R)$ using the construction above (note $\mathrm{E}_\infty(R)$ is a subgroup of $\mathrm{GL}_\infty(R)$). The second K-group of R, $K_2(R)$, is the second group homology group (with integer coefficients) of $\mathrm{E}_\infty(R)$,

$$K_2(R) = H_2(\mathrm{E}_\infty(R), \mathbb{Z}).$$

Higher K-functors

Higher K-groups are defined using the Quillen plus construction,

$$K_n^{\mathrm{alg}}(R) = \pi_n(B\mathrm{GL}_\infty(R)^+), \tag{1}$$

where $B\mathrm{GL}_\infty(R)$ is the classifying space of $\mathrm{GL}_\infty(R)$.

Rough sketch of suspension:

$$\Sigma R = \Sigma\mathbb{Z} \otimes_{\mathbb{Z}} R \tag{2}$$

where $\Sigma\mathbb{Z} = C\mathbb{Z}/J\mathbb{Z}$. The cone, $C\mathbb{Z}$, is the set of infinite matrices with integral coefficients that have a finite number of non-trivial elements on each row and column. The ideal $J\mathbb{Z}$ consists of those matrices that have only finitely many non-trivial coefficients.

$$K_i(R) \cong K_{i+1}(\Sigma R) \tag{3}$$

Algebraic K-theory has a product structure,

$$K_i(R) \otimes K_j(S) \to K_{i+j}(R \otimes S). \tag{4}$$

References

[1] H. Inassaridze, *Algebraic K-theory.* Kluwer Academic Publishers, 1994.

[2] Jean-Louis Loday, *Cyclic Homology.* Springer-Verlag, 1992.