K-theory

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Topological K-theory is a generalised cohomology theory on the category of compact Hausdorff spaces. It classifies the vector bundles over a space X up to stable equivalences. Equivalently, via the Serre-Swan theorem, it classifies the finitely generated projective modules over the C^*-algebra $C(X)$.

Let A be a unital C^*-algebra over $\mathbb{C}$ and denote by $\mathrm{M}_\infty(A)$ the algebraic direct limit of matrix algebras $\mathrm{M}_n(A)$ under the embeddings $\mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A) : a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Identify the completion of $\mathrm{M}_\infty(A)$ with the stable algebra $A \otimes \mathbb{K}$ (where $\mathbb{K}$ is the compact operators on $l_2(\mathbb{N})$), which we will continue to denote by $\mathrm{M}_\infty(A)$. The $K_0(A)$ group is the Grothendieck group (abelian group of formal differences) of the homotopy classes of the projections in $\mathrm{M}_\infty(A)$. Two projections p and q are homotopic if there exists a norm continuous path of projections from p to q. Let $p \in \mathrm{M}_m(A)$ and $q \in \mathrm{M}_n(A)$ be two projections. The sum of their homotopy classes $[p]$ and $[q]$ is the homotopy class of their direct sum: $[p] + [q] = [p \oplus q]$ where $p \oplus q = \mathrm{diag}(p, q) \in \mathrm{M}_{m+n}(A)$. Alternatively, one can consider equivalence classes of projections up to unitary transformations. Unitary equivalence coincides with homotopy equivalence in $\mathrm{M}_\infty(A)$ (or $\mathrm{M}_n(A)$ for n large enough).

Denote by $\mathrm{U}_\infty(A)$ the direct limit of unitary groups $\mathrm{U}_n(A)$ under the embeddings $\mathrm{U}_n(A) \to \mathrm{U}_{n+1}(A) : u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. Give $\mathrm{U}_\infty(A)$ the direct limit topology, i.e. a subset U of $\mathrm{U}_\infty(A)$ is open if and only if $U \cap \mathrm{U}_n(A)$ is an open subset of $\mathrm{U}_n(A)$, for all n. The $K_1(A)$ group is the Grothendieck group (abelian group of formal differences) of the homotopy classes of the unitaries in $\mathrm{U}_\infty(A)$. Two unitaries u and v are homotopic if there exists a norm continuous path of unitaries from u to v. Let $u \in \mathrm{U}_m(A)$ and $v \in \mathrm{U}_n(A)$ be two unitaries. The sum of their homotopy classes $[u]$ and $[v]$ is the homotopy class of their direct sum: $[u] + [v] = [u \oplus v]$ where $u \oplus v = \mathrm{diag}(u, v) \in \mathrm{U}_{m+n}(A)$. Equivalently, one can work with invertibles in $\mathrm{GL}_\infty(A)$ (an invertible g is connected to the unitary $u = g|g|^{-1}$ via the homotopy $t \to g|g|^{-t}$).

Higher K-groups can be defined through repeated suspensions,

$$K_n(A) = K_0(S^n A). \tag{1}$$

But, the Bott periodicity theorem means that

$$K_1(SA) \cong K_0(A). \tag{2}$$

The main properties of K_i are:

$$\begin{aligned}
K_i(A \oplus B) &= K_i(A) \oplus K_i(B), & (3)\\
K_i(\mathrm{M}_n(A)) &= K_i(A) \quad \text{(Morita invariance)}, & (4)\\
K_i(A \otimes \mathbb{K}) &= K_i(A) \quad \text{(stability)}, & (5)\\
K_{i+2}(A) &= K_i(A) \quad \text{(Bott periodicity)}. & (6)
\end{aligned}$$

There are three flavours of topological K-theory to handle the cases of A being complex (over $\mathbb{C}$), real (over $\mathbb{R}$) or Real (with a given real structure).

$$\begin{aligned}
K_i(C(X,\mathbb{C})) &= KU^{-i}(X) \quad \text{(complex/unitary)}, & (7)\\
K_i(C(X,\mathbb{R})) &= KO^{-i}(X) \quad \text{(real/orthogonal)}, & (8)\\
KR_i(C(X),J) &= KR^{-i}(X,J) \quad \text{(Real)}. & (9)
\end{aligned}$$

Real K-theory has a Bott period of 8, rather than 2.

References

[1] N. E. Wegge-Olsen, *K-theory and C^*-algebras.* Oxford science publications. Oxford University Press, 1993.

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[3] M. Rørdam, F. Larsen and N. J. Laustsen, *An Introduction to K-Theory for C^*-Algebras.* Cambridge University Press, 2000.