Fredholm module

Canonical name	FredholmModule
Date of creation	2013-03-22 12:57:43
Last modified on	2013-03-22 12:57:43
Owner	mhale (572)
Last modified by	mhale (572)
Numerical id	6
Author	mhale (572)
Entry type	Definition
Classification	msc 19K33
Classification	msc 46L87
Classification	msc 47A53
Related topic	KHomology

Fredholm modules represent abstract elliptic pseudo-differential operators.

Definition 1. An **odd Fredholm module** $(\mathcal{H}, F)$ over a C^*-algebra A is given by an involutive representation π of A on a Hilbert space $\mathcal{H}$, together with an operator F on $\mathcal{H}$ such that $F = F^*$, $F^2 = \mathbb{1}$ and $[F, \pi(a)] \in \mathbb{K}(\mathcal{H})$ for all $a \in A$.

Definition 2. An **even Fredholm module** $(\mathcal{H}, F, \Gamma)$ is given by an odd Fredholm module $(\mathcal{H}, F)$ together with a $\mathbb{Z}_2$-grading Γ on $\mathcal{H}$, $\Gamma = \Gamma^*$, $\Gamma^2 = \mathbb{1}$, such that $\Gamma\pi(a) = \pi(a)\Gamma$ and $\Gamma F = -F\Gamma$.

Definition 3. A Fredholm module is called **degenerate** if $[F, \pi(a)] = 0$ for all $a \in A$. Degenerate Fredholm modules are homotopic to the 0-module.

Example 1 (Fredholm modules over $\mathbb{C}$)
An even Fredholm module $(\mathcal{H}, F, \Gamma)$ *over* $\mathbb{C}$ *is given by*

$$\begin{aligned}
\mathcal{H} &= \mathbb{C}^k \oplus \mathbb{C}^k \quad \textit{with } \pi(a) = \begin{pmatrix} a\mathbb{1}_k & 0 \\ 0 & 0 \end{pmatrix}, \\
F &= \begin{pmatrix} 0 & \mathbb{1}_k \\ \mathbb{1}_k & 0 \end{pmatrix}, \\
\Gamma &= \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & -\mathbb{1}_k \end{pmatrix}.
\end{aligned}$$