complexification

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Let G be a real Lie group. Then the *complexification* $G_{\mathbb{C}}$ of G is the unique complex Lie group equipped with a map $\varphi : G \to G_{\mathbb{C}}$ such that any map $G \to H$ where H is a complex Lie group, extends to a holomorphic map $G_{\mathbb{C}} \to H$. If $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ are the respective Lie algebras, $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

For simply connected groups, the construction is obvious: we simply take the simply connected complex group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and φ to be the map induced by the inclusion $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$.

If $\gamma \in G$ is central, then its image is in central in $G_{\mathbb{C}}$ since $g \mapsto \gamma g \gamma^{-1}$ is a map extending φ, and thus must be the identity by uniqueness half of the universal property. Thus, if $\Gamma \subset G$ is a discrete central subgroup, then we get a map $G/\Gamma \to G_{\mathbb{C}}/\varphi(\Gamma)$, which gives a complexification for G/Γ. Since every Lie group is of this form, this shows existence.

Some easy examples: the complexification both of $\mathrm{SL}_n\mathbb{R}$ and $\mathrm{SU}(n)$ is $\mathrm{SL}_n\mathbb{C}$. The complexification of $\mathbb{R}$ is $\mathbb{C}$ and of S^1 is $\mathbb{C}^*$.

The map $\varphi\colon G \to G_{\mathbb{C}}$ is not always injective. For example, if G is the universal cover of $\mathrm{SL}_n\mathbb{R}$ (which has fundamental group $\mathbb{Z}$), then $G_{\mathbb{C}} \cong \mathrm{SL}_n\mathbb{C}$, and φ factors through the covering $G \to \mathrm{SL}_n\mathbb{R}$.