connected topological group is generated by any neighborhood of identity

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Owner	asteroid (17536)
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Author	asteroid (17536)
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Theorem - Let G be a connected topological group and e its identity element. If U is any open neighborhood of e, then G is generated by U.

Proof: Let U be an open neighborhood of e. For each $n \in \mathbb{N}$ we denote by U^n the set of elements of the form $u_1 \dots u_n$, where each $u_i \in U$. Let $W := \bigcup_{n \in \mathbb{N}} U^n$.

Since each U^n is open (see `http://planetmath.org/BasicResultsInTopologicalGroups`this entry - 3), we have that W is an open set. We now see that it is also closed.

Let $g \in \overline{W}$, the closure of W. Since gU^{-1} is an open neighborhood of g, it must intersect W. Thus, let $h \in W \cap gU^{-1}$.

- Since $h \in gU^{-1}$, then $h = gu^{-1}$ for some element $u \in U$.
- Since $h \in W$, then $h \in U^n$ for some $n \in \mathbb{N}$, i.e. $h = u_1 \dots u_n$ with each $u_i \in U$.

We then have $g = u_1 \dots u_n u$, i.e. $g \in U^{n+1} \subseteq W$. Hence, W is closed.

Since G is connected and W is open and closed, we must have $W = G$. This means that G is generated by U. $\square$