existence and uniqueness of compact real form

Canonical name	ExistenceAndUniquenessOfCompactRealForm
Date of creation	2013-03-22 13:23:37
Last modified on	2013-03-22 13:23:37
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Last modified by	bwebste (988)
Numerical id	5
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Entry type	Theorem
Classification	msc 22E10

Let G be a semisimple complex Lie group. Then there exists a unique (up to isomorphism) real Lie group K such that K is compact and a real form of G. Conversely, if K is compact, semisimple and real, it is the real form of a unique semisimple complex Lie group G. The group K can be realized as the set of fixed points of a special involution of G, called the *Cartan involution*.

For example, the compact real form of $\mathrm{SL}_n\mathbb{C}$, the complex special linear group, is $\mathrm{SU}(n)$, the special unitary group. Note that $\mathrm{SL}_n\mathbb{R}$ is also a real form of $\mathrm{SL}_n\mathbb{C}$, but is *not* compact.

The compact real form of $\mathrm{SO}_n\mathbb{C}$, the complex special orthogonal group, is $\mathrm{SO}_n\mathbb{R}$, the real orthogonal group. $\mathrm{SO}_n\mathbb{C}$ also has other, non-compact real forms, called the pseudo-orthogonal groups.

The compact real form of $\mathrm{Sp}_{2n}\mathbb{C}$, the complex symplectic group, is less well-known. It is (unfortunately) also usually denoted $\mathrm{Sp}(2n)$, and consists of $n \times n$ "unitary" quaternion matrices, that is,

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}_n\mathbb{H} | MM^* = I\}$$

where M^* denotes M conjugate transpose. This different from the real symplectic group $\mathrm{Sp}_{2n}\mathbb{R}$.