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example of an Alexandroff space which cannot be turned into a topological group

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Let $\mathbb{R}$ denote the set of real numbers and $\tau = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{(b, \infty) \mid b \in \mathbb{R}\}$. One can easily verify that $(\mathbb{R}, \tau)$ is an Alexandroff space.

Proposition. The Alexandroff space $(\mathbb{R}, \tau)$ cannot be turned into a topological group.

Proof. Assume that $\mathbb{R} = (\mathbb{R}, \tau, \circ)$ is a topological group. It is well known that this implies that there is $H \subseteq \mathbb{R}$ which is open, normal subgroup of $\mathbb{R}$. This subgroup ,,generates" the topology (see the parent object for more details). Thus $H \neq \mathbb{R}$ because τ is not antidiscrete. Let $g \in \mathbb{R}$ such that $g \notin H$ (and thus $gH \cap H = \emptyset$). Then gH is again open (because the mapping $f(x) = g \circ x$ is a homeomorphism). But since both H and gH are open, then $gH \cap H \neq \emptyset$. Indeed, every two open subsets in τ have nonempty intersection. Contradiction, because diffrent cosets are disjoint. $\square$