the connection between Lie groups and Lie algebras

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Given a finite dimensional Lie group G, it has an associated Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The Lie algebra encodes a great deal of information about the Lie group. I've collected a few results on this topic:

Theorem 1 *(Existence) Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. Then there exists a finite dimensional real or complex Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}$.*

Theorem 2 *(Uniqueness) There is a unique connected simply-connected Lie group G with any given finite-dimensional Lie algebra. Every connected Lie group with this Lie algebra is a quotient G/Γ by a discrete central subgroup Γ.*

Even more important, is the fact that the correspondence $G \mapsto \mathfrak{g}$ is functorial: given a homomorphism $\varphi : G \to H$ of Lie groups, there is natural homomorphism defined on Lie algebras $\varphi_* : \mathfrak{g} \to \mathfrak{h}$, which just the derivative of the map φ at the identity (since the Lie algebra is canonically identified with the tangent space at the identity).

There are analogous existence and uniqueness theorems for maps:

Theorem 3 *(Existence) Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Lie algebras. Then if G is the unique connected, simply-connected group with Lie algebra $\mathfrak{g}$, and H is any Lie group with Lie algebra $\mathfrak{h}$, there exists a homomorphism of Lie groups $\varphi : G \to H$ with $\varphi_* = \psi$.*

Theorem 4 *(Uniqueness) Let G be connected Lie group and H an arbitrary Lie group. Then if two maps $\varphi, \varphi' : G \to H$ induce the same maps on Lie algebras, then they are equal.*

Essentially, what these theorems tell us is the correspondence $\mathfrak{g} \mapsto G$ from Lie algebras to simply-connected Lie groups is functorial, and right `http://planetmath.org/AdjointFunctor`adjoint to the functor $H \mapsto \mathrm{Lie}(H)$ from Lie groups to Lie algebras.