subgroup of topological group is either clopen or has empty interior

Canonical name	SubgroupOfTopologicalGroupIsEitherClopenOrHasEmptyInterior
Date of creation	2013-03-22 18:01:29
Last modified on	2013-03-22 18:01:29
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	6
Author	asteroid (17536)
Entry type	Theorem
Classification	msc 22A05

Theorem - Every subgroup of a topological group is either clopen or has empty interior.

Proof: Let G be a topological group and $H \subseteq G$ a subgroup. Suppose the interior of H is nonempty, i.e. there is a non-empty open set U of G such that $U \subseteq H$. Translating U around H we can see that H is open: if $u \in U$ then for every $h \in H$ the set $hu^{-1}U$ is open in G, is contained in H and contains h, which implies that H is open in G.

Let us now see that H is closed. Let $\overline{H}$ denote the closure of H and let H^2 be the set of elements of the form $h_1 h_2$ where $h_1, h_2 \in H$. Of course, since H is a subgroup of G, we have that $H^2 = H$. Also, since H is open we know that$H \subseteq \overline{H} \subseteq H^2$ (see `http://planetmath.org/BasicResultsInTopologicalGroups`this entry - 5). Hence $\overline{H} = H$, i.e. H is closed.

We have proven that a subgroup of a topological group must be clopen or it must have empty interior. Since this two topological properties can never be satisfied simultaneously, we have that every subgroup of a topological group is either clopen or it has empty interior. $\square$