invariant scalar product

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Author	benjaminfjones (879)
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Defines	invariant scalar product
Defines	associative bilinear form
Defines	Killing form

Let $\mathbb{K}$ be a field and V a vector space over $\mathbb{K}$. Let G be a group with a specified representation on V denoted by $g.v$ for $v \in V$ and $g \in G$.

An *invariant scalar product* (with respect to the action of G) on V is a scalar product $(\cdot|\cdot)$ on V (i.e. a non-degenerate, symmetric $\mathbb{K}$-bilinear form) such that for any $g \in G, u, v \in V$ we have

$$(g.u|g.v) = (u|v)$$

Now let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ with a representation on V denoted by $X.v$ for $X \in \mathfrak{g}, v \in V$. Then an *invariant scalar product* (with respect to the action of $\mathfrak{g}$) is a scalar product on V such that for any $X \in \mathfrak{g}, u, v \in V$ we have

$$(X.u|v) = -(u|X.v)$$

An invariant scalar product on a Lie algebra $\mathfrak{g}$ is by definition an invariant scalar product as above where the representation is the adjoint representation of $\mathfrak{g}$ on itself. In this case invariance is usualy written $([X,Y] \mid Z) = (X \mid [Y,Z])$

1 Examples

For example if $G = O(n)$ the orthogonal subgroup of $n \times n$ real matricies and $\mathbb{R}^n$ is the natural representation for $O(n)$, then the standard Euclidean scalar product on $\mathbb{R}^n$ is an invariant scalar product. Invariance in this example follows from the definition of $O(n)$.

As another example if $\mathfrak{g}$ is a complex semi-simple Lie algebra then the *Killing form* $\kappa(X,Y) := Tr(ad_X \cdot ad_Y)$ is an invariant scalar product on $\mathfrak{g}$ itself via the adjoint representation. Invariance in this example follows from the fact that the trace operator is *associative*, i.e. $Tr([Y,X] \cdot Z) = -Tr([X,Y] \cdot Z) = -Tr(X \cdot [Y,Z])$. Thus an invariant scalar product (with respect to a Lie algebra representation) is sometimes called an *associative scalar product*.