# quotient group of a topological group by its identity component is totally disconnected

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Assume that $G$ is a topological group and $G_e$ is the identity component. It is well known, that $G_e$ is a normal subgroup of $G$, thus we may speak about quotient group.

**Proposition.** The quotient group $G/G_e$ is totally disconnected.

*Proof.* First of all note that connected components of $G$ are of the form $gG_e$. Indeed, $G_e$ is a connected component of $e \in G$ and for any $g \in G$ we have a homeomorphism $f_g : G \to G$ such that $f_g(x) = gx$. Thus $f_g(G_e) = gG_e$ is a connected component of $g \in G$ (please, see `http://planetmath.org/HomeomorphismsPres` entry for more details).

Now let $\pi : G \to G/G_e$ be the quotient map (which is open and onto) and $A \subseteq G/G_e$ be an arbitrary, connected subset of $G/G_e$. Assume that there are at least two points in $A$. Consider the subset $\pi^{-1}(A) \subseteq G$ (which is the union of some cosets). Since $A$ has at least two points, then $\pi^{-1}(A)$ contains at least two cosets, which are connected components of $G$. Thus $\pi^{-1}(A)$ is not connected. Therefore there exist $U, V \subseteq \pi^{-1}(A)$ such that $U, V$ are open (in $\pi^{-1}(A)$), disjoint and $U \cup V = \pi^{-1}(A)$.

Note that if $x \in U$, then the connected component of $x$ (which is equal to $xG_e$) is contained in $U$. Indeed, assume that $xG_e \not\subseteq U$. Then there is $h \in xG_e$ such that $h \notin U$. Then, since $U \cup V = \pi^{-1}(A)$ we have that $h \in V$. But then $U \cap xG_e$ and $V \cap xG_e$ are nonempty open and disjoint subsets of $xG_e$ such that $(U \cap xG_e) \cup (V \cap xG_e) = xG_e$. Contradiction, because $xG_e$ is connected. Analogusly, whenever $x \in V$, then $xG_e \subseteq V$.

Therefore both $U$ and $V$ are unions of cosets. Thus $\pi(U)$ and $\pi(V)$ are disjoint. Furthermore $\pi(U) \cup \pi(V) = A$ and both $\pi(U)$, $\pi(V)$ are open in $A$ (because $\pi$ is an open map). This means that $A$ is not connected. Contradiction. Thus $A$ has at most one element, which completes the proof. $\square$

**Remark.** This proposition can be easily generalized as follows: assume that $X$ is a topological space, $X = \bigcup X_i$ is a decomposition of $X$ into connected components and $R$ is an equivalence relation associated to this decomposition (i.e. $xRy$ if and only if there exists $i$ such that $x, y \in X_i$). Then, if the quotient map $\pi : X \to X/R$ is open, then $X/R$ is totally disconnected.