O(2)

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still being written

An elementary example of a Lie group is afforded by O(2), the orthogonal group in two dimensions. This is the set of transformations of the plane which fix the origin and preserve the distance between points. It may be shown that a transform has this property if and only if it is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto M \begin{pmatrix} x \\ y \end{pmatrix},$$

where M is a 2×2 matrix such that $M^T M = I$. (Such a matrix is called orthogonal.)

It is easy enough to check that this is a group. To see that it is a Lie group, we first need to make sure that it is a manifold. To that end, we will parameterize it. Calling the entries of the matrix a, b, c, d, the condition becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$$

which is equivalent to the following system of equations:

$$\begin{aligned} a^2 + c^2 &= 1 \\ ab + cd &= 0 \\ b^2 + d^2 &= 1 \end{aligned}$$

The first of these equations can be solved by introducing a parameter θ and writing $a = \cos\theta$ and $c = \sin\theta$. Then the second equation becomes $b\cos\theta + d\sin\theta = 0$, which can be solved by introducing a parameter r:

$$\begin{aligned} b &= -r\sin\theta \\ d &= r\cos\theta \end{aligned}$$

Substituting this into the third equation results in $r^2 = 1$, so $r = -1$ or $r = +1$. This means we have two matrices for each value of θ:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Since more than one value of θ will produce the same matrix, we must restrict the range in order to obtain a bona fide coordinate. Thus, we may cover $O(2)$ with an atlas consisting of four neighborhoods:

$$\left\{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \mid -\frac{3}{4}\pi < \theta < \frac{3}{4}\pi\right\}$$
$$\left\{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \mid \frac{1}{4}\pi < \theta < \frac{7}{4}\pi\right\}$$
$$\left\{\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \mid -\frac{3}{4}\pi < \theta < \frac{3}{4}\pi\right\}$$
$$\left\{\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \mid \frac{1}{4}\pi < \theta < \frac{7}{4}\pi\right\}$$

Every element of $O(2)$ must belong to at least one of these neighborhoods. It its trivial to check that the transition functions between overlapping coordinate patches are