loop algebra

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Defines	loop algebra

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. The **loop algebra** based on $\mathfrak{g}$ is defined to be $\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[t, t^{-1}]$ as a vector space over $\mathbb{K}$. The Lie bracket is determined by

$$\left[X \otimes t^k, Y \otimes t^l\right] = [X, Y]_{\mathfrak{g}} \otimes t^{k+l}$$

where $[\ ,\ ]_{\mathfrak{g}}$ denotes the Lie bracket from $\mathfrak{g}$.

This clearly determines a Lie bracket. For instance the three term sum in the Jacobi identity (for elements which are homogeneous in t) simplifies to the three term sum for the Jacobi identity in $\mathfrak{g}$ tensored with a power of t and thus is zero in $\mathcal{L}(\mathfrak{g})$.

The name "loop algebra" comes from the fact that this Lie algebra arises in the study of Lie algebras of loop groups. For the time being, assume that $\mathbb{K}$ is the real or complex numbers so that the familiar structures of analysis and topology are available. Consider the set of all mappings from the circle S^1 (we may think of this circle more concretely as the unit circle of the complex plane) to a finite-dimensional Lie group G with Lie algebra is $\mathfrak{g}$. We may make this set into a group by defining multiplication pointwise: given $a, b \colon S^1 \to G$, we define $(a \cdot b)(x) = a(x) \cdot b(x)$.

References

[1] Victor Kac, *Infinite Dimensional Lie Algebras*, Third edition. Cambridge University Press, Cambridge, 1990.