connected locally compact topological groups are σ-compact

Canonical name	ConnectedLocallyCompactTopologicalGroupsAresigmacompact
Date of creation	2013-03-22 17:37:12
Last modified on	2013-03-22 17:37:12
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	9
Author	asteroid (17536)
Entry type	Theorem
Classification	msc 22A05
Classification	msc 22D05

The main result of this entry is the following theorem (whose proof is given below). The result expressed in the title then follows as a corollary.

Theorem - Every locally compact topological group G has an open `http://planetmath.org/SigmaCompact`σ-compact subgroup H.

Corollary 1 - Every locally compact topological group is the topological disjoint union of σ-compact spaces.

Corollary 2 - Every connected locally compact topological group is σ-compact.

We first outline the proofs of the above corollaries:

Proof (Corollaries 1 and 2) : Let G be a locally compact topological group. The main theorem implies that there is an open σ-compact subgroup H.

It is known that every open subgroup of G is also closed (see this `http://planetmath.org/Close`
Therefore, each gH is a clopen σ-compact subset of G, and G is the topological disjoint union $\bigcup_{g \in G} gH$.

Of course, if G is connected then H must be all of G. Hence, G is σ-compact. $\square$

Proof (Theorem) : Let us fix some notation first. If A is a subset of G we use the notation $A^{-1} := \{a^{-1} : a \in A\}$, $A^n := \{a_1 \dots a_n : a_1, \dots, a_n \in A\}$ and $\overline{A}$ denotes the closure of A.

Pick a neighborhood W of e (the identity element of G) with compact closure. Then $V := W \cap W^{-1}$ is a neighborhood of e with compact closure such that $V = V^{-1}$.

Let $H := \bigcup_{n=1}^{\infty} V^n$. H is clearly a subgroup of G. We now only have to prove that H is open and σ-compact.

We have that (see this `http://planetmath.org/BasicResultsInTopologicalGroups`entry - 3, 4 and 5)

- V^n is open
- $\overline{V}^n$ is compact
- $\overline{V}^n \subset V^{2n}$

So H is open and also $H = \bigcup_{n=1}^{\infty} \overline{V}^n$, which implies that H is σ-compact. $\square$