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modular function

Canonical name	ModularFunction
Date of creation	2013-03-22 17:58:18
Last modified on	2013-03-22 17:58:18
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	8
Author	asteroid (17536)
Entry type	Definition
Classification	msc 22D05
Classification	msc 28C10
Synonym	Haar modulus
Synonym	modular character
Synonym	modular homomorphism

Let G be a locally compact Hausdorff topological group and μ a left Haar measure. Although left and right Haar measures in G always exist, they generally do not coincide, i.e. a left Haar measure is usually not invariant under right translations. Nevertheless, the right translations of a left Haar measure can be easily described as explained in the following theorem.

Theorem - Let G be a locally compact Hausdorff topological group and μ a left Haar measure in G. Then, there exists a continuous homomorphism $\Delta : G \longrightarrow \mathbb{R}^+$ such that, for every $t \in G$ and every measurable subset A

$$\mu(At) = \Delta(t^{-1})\mu(A)$$

Moreover, if $f : G \longrightarrow \mathbb{C}$ is an integrable function then

$$\Delta(t) \int_G f(st)\mu(s) = \int_G f(s)\mu(s)$$

The function Δ is called the **modular function** of G (notice that, by uniqueness up to scalar multiple of left Haar measures, Δ only depends on G). Other names for Δ that can be found are: *Haar modulus*, or *modular character* or *modular homomorphism*.

We now prove the above theorem, except the continuity of Δ (which is slightly harder to obtain).

Proof (except continuity of Δ):

Let $t \in G$. The function ν, defined on measurable subsets A by

$$\nu(A) := \mu(At)$$

is easily seen to be a measure in G. Moreover, ν is left invariant (since μ is left invariant) and satisfies the additional conditions to be a left Haar measure. By the uniqueness of left Haar measures, μ must be a multiple of ν, i.e. $\mu = \Delta(t)\nu$ for some positive scalar $\Delta(t) \in \mathbb{R}^+$. Thus, we have proven that for every measurable subset A

$$\mu(At) = \Delta(t)^{-1}\mu(A)$$

Now for $s, t \in G$ we have that $\mu(Ast) = \Delta(st)^{-1}\mu(A)$, but also

- $\mu(Ast) = \Delta(t)^{-1}\mu(As)$, and

- $\mu(As) = \Delta(s)^{-1}\mu(A)$

So, we can see that, for every measurable subset A,

$$\Delta(st)^{-1}\mu(A) = \Delta(t)^{-1}\Delta(s)^{-1}\mu(A)$$

Hence, $\Delta(st) = \Delta(s)\Delta(t)$. Thus, Δ is an homomorphism.

The statement about integrals of functions follows easily by approximation by simple functions. For simple functions it is easy to see it is true using the now established condition $\mu(At) = \Delta(t^{-1})\mu(A)$. $\square$