characterization of Alexandroff groups

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Topological group G is called *Alexandroff* if G is an Alexandroff space as a topological space. For example every finite topological group is Alexandroff. We wish to characterize them. First recall, that if A is a subset of a topological space, then A^o denotes an intersection of all open neighbourhoods of A.

Lemma. Let X be an Alexandroff space, $f : X \times \cdots \times X \to X$ be a continuous map and $x \in X$ such that $f(x, \ldots, x) = x$. Then $f(A \times \cdots \times A) \subseteq A$, where $A = \{x\}^o$.

Proof. Let $A = \{x\}^o$. Of course A is open (because X is Alexandroff). Therefore $f^{-1}(A)$ is open in $X \times \cdots \times X$. Thus (from the definition of product topology and continuous map), there are open subsetes $V_1, \ldots, V_n \subseteq X$ such that each V_i is an open neighbourhood of x and

$$f(V_1 \times \cdots \times V_n) \subseteq A.$$

Now let $U_i = V_i \cap A$. Of course $x \in U_i$, so U_i is nonempty and U_i is open. Furthermore $U_i \subseteq V_i$ and thus

$$f(U_1 \times \cdots \times U_n) \subseteq A.$$

On the other hand $U_i \subseteq A$ and U_i is open neighbourhood of x. Thus $U_i = A$, because A is minimal open neighbourhood of x. Therefore

$$f(A \times \cdots \times A) = f(U_1 \times \cdots \times U_n) \subseteq A,$$

which completes the proof. $\square$

Proposition. Let G be an Alexandroff group. Then there exists open, normal subgroup H of G such that for every open subset $U \subseteq G$ there exist $\{g_i\}_{i \in I} \subseteq G$ such that

$$U = \bigcup_{i \in I} g_i H.$$

Proof. Let $H = \{e\}^o$ be an intersection of all open neighbourhoods of the identity $e \in G$. Let U be an open subset of G. If $g \in U$, then $g^{-1}U$ is an open neighbourhood of e. Thus $H \subseteq g^{-1}U$ and therefore $gH \subseteq U$. Thus

$$U = \bigcup_{g \in U} gH.$$

To complete the proof we need to show that H is normal subgroup of G. Consider the following mappings:

$$M : G \times G \to G \text{ is such that } M(x, y) = xy;$$

$$\psi : G \to G \text{ is such that } \psi(x) = x^{-1};$$

$$\varphi_g : G \to G \text{ is such that } \varphi_g(x) = gxg^{-1} \text{ for any } g \in G.$$

Of course each of them is continuous (because G is a topological group). Furthermore each of them satisfies Lemma's assumptions (for $x = e$). Thus we have:

$$HH = M(H \times H) \subseteq H;$$

$$H^{-1} = \psi(H) \subseteq H;$$

$$gHg^{-1} = \varphi_g(H) \subseteq H \text{ for any } g \in G.$$

This shows that H is a normal subgroup, which completes the proof. $\square$

Corollary. Let G be a topological group such that G is finite and simple. Then G is either discrete or antidiscrete.

Proof. Of course finite topological groups are Alexandroff. Since G is simple, then there are only two normal subgroups of G, namely the trivial group and entire G. Therfore (due to proposition) the topology on G is ,,generated" by either the trivial group or entire G. In the first case we gain the discrete topology and in the second the antidiscrete topology. $\square$