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one-parameter subgroup

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Let G be a Lie Group. A *one-parameter subgroup* of G is a group homomorphism

$$\phi\colon \mathbb{R} \to G$$

that is also a differentiable map at the same time. We view $\mathbb{R}$ additively and G multiplicatively, so that $\phi(r+s) = \phi(r)\phi(s)$.

Examples.

1. If $G = \mathrm{GL}(n,k)$, where $k = \mathbb{R}$ or $\mathbb{C}$, then any one-parameter subgroup has the form
$$\phi(t) = e^{tA},$$
where $A = \frac{d\phi}{dt}(0)$ is an $n \times n$ matrix over k. The matrix A is just a tangent vector to the Lie group $\mathrm{GL}(n,k)$. This property establishes the fact that there is a one-to-one correspondence between one-parameter subgroups and tangent vectors of $\mathrm{GL}(n,k)$. The same relationship holds for a general Lie group. The one-to-one correspondence between tangent vectors at the identity (the Lie algebra) and one-parameter subgroups is established via the exponential map instead of the matrix exponential.

2. If $G = \mathrm{O}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$, the orthogonal group over R, then any one-parameter subgroup has the same form as in the example above, except that A is skew-symmetric: $A^{\mathrm{T}} = -A$.

3. If $G = \mathrm{SL}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$, the special linear group over R, then any one-parameter subgroup has the same form as in the example above, except that $\mathrm{tr}(A) = 0$, where tr is the trace operator.

4. If $G = \mathrm{U}(n) = \mathrm{O}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$, the unitary group over C, then any one-parameter subgroup has the same form as in the example above, except that A is `http://planetmath.org/SkewHermitianMatrix`skew-Hermitian: $A = -A^* = -\overline{A}^{\mathrm{T}}$ and $\mathrm{tr}(A) = 0$.