Lie group

Canonical name	LieGroup
Date of creation	2013-05-19 19:12:53
Last modified on	2013-05-19 19:12:53
Owner	rmilson (146)
Last modified by	jocaps (12118)
Numerical id	21
Author	rmilson (12118)
Entry type	Definition
Classification	msc 22E10
Classification	msc 22E15
Related topic	Group
Related topic	LieAlgebra
Related topic	SimpleAndSemiSimpleLieAlgebras2
Defines	left-invariant
Defines	right-invariant

A Lie group is a group endowed with a compatible http://planetmath.org/ComplexAnalyticM
structure. To be more precise, Lie group structure consists of two kinds of data

- a finite-dimensional, real-analytic manifold G, and
- two analytic maps, one for multiplication $G \times G \to G$ and one for inversion $G \to G$, which obey the appropriate group axioms.

Thus, a homomorphism in the category of Lie groups is a group homomorphism that is simultaneously an analytic mapping between two real-analytic manifolds.

One can equivalently define a Lie group G using the following easy characterization

Proposition 1 *A finite-dimensional real analytic manifold G is a Lie algebra iff the map*

$$G \times G \to G \qquad (x, y) \mapsto x^{-1}y \quad \forall x, y \in G$$

is analytic.

Next, we describe a natural construction that associates a certain Lie algebra $\mathfrak{g}$ to every Lie group G. Let $e \in G$ denote the identity element of G. For $g \in G$ let $\lambda_g \colon G \to G$ denote the diffeomorphisms corresponding to left multiplication by g.

Definition 2 *A vector field V on G is called left-invariant if V is invariant with respect to all left multiplications. To be more precise, V is left-invariant if and only if*

$$(\lambda_g)_*(V) = V$$

(see push-forward of a vector-field) for all $g \in G$.

Proposition 3 *The Lie bracket of two left-invariant vector fields is again, a left-invariant vector field.*

Proof. Let V_1, V_2 be left-invariant vector fields, and let $g \in G$. The bracket operation is covariant with respect to diffeomorphism, and in particular

$$(\lambda_g)_*[V_1, V_2] = [(\lambda_g)_*V_1, (\lambda_g)_*V_2] = [V_1, V_2].$$

Q.E.D.

Definition 4 *The Lie algebra of G, denoted hereafter by $\mathfrak{g}$, is the vector space of all left-invariant vector fields equipped with the vector-field bracket.*

Now a right multiplication is invariant with respect to all left multiplications, and it turns out that we can characterize a left-invariant vector field as being an infinitesimal right multiplication.

Proposition 5 *Let $a \in T_eG$ and let V be a left-invariant vector-field such that $V_e = a$. Then for all $g \in G$ we have*

$$V_g = (\lambda_g)_*(a).$$

The intuition here is that a gives an infinitesimal displacement from the identity element and that V_g gives a corresponding infinitesimal right displacement away from g. Indeed consider a curve

$$\gamma\colon (-\epsilon, \epsilon) \to G$$

passing through the identity element with velocity a; i.e.

$$\gamma(0) = e, \quad \gamma'(0) = a.$$

The above proposition is then saying that the curve

$$t \mapsto g\gamma(t), \quad t \in (-\epsilon, \epsilon)$$

passes through g at $t = 0$ with velocity V_g.

Thus we see that a left-invariant vector field is completely determined by the value it takes at e, and that therefore $\mathfrak{g}$ is isomorphic, as a vector space to T_eG.

Of course, we can also consider the Lie algebra of right-invariant vector fields. The resulting Lie-algebra is anti-isomorphic (the order in the bracket is reversed) to the Lie algebra of left-invariant vector fields. Now it is a general principle that the group inverse operation gives an anti-isomorphism between left and right group actions. So, as one may well expect, the anti-isomorphism between the Lie algebras of left and right-invariant vector fields can be realized by considering the linear action of the inverse operation on T_eG.

Finally, let us remark that one can induce the Lie algebra structure directly on T_eG by considering adjoint action of G on T_eG.

History and motivation.

Examples.

Notes.

1. No generality is lost in assuming that a Lie group has analytic, rather than C^∞ or even C^k, $k = 1, 2, \ldots$ structure. Indeed, given a C^1 differential manifold with a C^1 multiplication rule, one can show that the exponential mapping endows this manifold with a compatible real-analytic structure.

   Indeed, one can go even further and show that even C^0 suffices. In other words, a topological group that is also a finite-dimensional topological manifold possesses a compatible analytic structure. This result was formulated by Hilbert as his `http://www.reed.edu/ wieting/essays/LieHilbert.pdf`fifth problem, and proved in the 50's by Montgomery and Zippin.

2. One can also speak of a complex Lie group, in which case G and the multiplication mapping are both complex-analytic. The theory of complex Lie groups requires the notion of a holomorphic vector-field. Not withstanding this complication, most of the essential features of the real theory carry over to the complex case.

3. The name "Lie group" honours the Norwegian mathematician Sophus Lie who pioneered and developed the theory of continuous transformation groups and the corresponding theory of Lie algebras of vector fields (the group's infinitesimal generators, as Lie termed them). Lie's original impetus was the study of continuous symmetry of geometric objects and differential equations.

   The scope of the theory has grown enormously in the 100+ years of its existence. The contributions of Elie Cartan and Claude Chevalley figure prominently in this evolution. Cartan is responsible for the celebrated ADE classification of simple Lie algebras, as well as for charting the essential role played by Lie groups in differential geometry and mathematical physics. Chevalley made key foundational contributions to the analytic theory, and did much to pioneer the related theory of algebraic groups. Armand Borel's book "Essays in the History of Lie groups and algebraic groups" is the definitive source on the evolution of the Lie group concept. Sophus Lie's contributions are the subject of a number of excellent articles by T. Hawkins.