exponential

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Preamble. We use $\mathbb{R}^+ \subset \mathbb{R}$ to denote the set of positive real numbers. Our aim is to define the exponential, or the generalized power operation,

$$x^p, \quad x \in \mathbb{R}^+,\ p \in \mathbb{R}.$$

The power index p in the above expression is called the exponent. We take it as proven that $\mathbb{R}$ is a complete, ordered field. No other properties of the real numbers are invoked.

Definition. For $x \in \mathbb{R}^+$ and $n \in \mathbb{Z}$ we define x^n in terms of repeated multiplication. To be more precise, we inductively characterize natural number powers as follows:

$$x^0 = 1, \quad x^{n+1} = x \cdot x^n, \quad n \in \mathbb{N}.$$

The existence of the reciprocal is guaranteed by the assumption that $\mathbb{R}$ is a field. Thus, for negative exponents, we can define

$$x^{-n} = (x^{-1})^n, \quad n \in \mathbb{N},$$

where x^{-1} is the reciprocal of x.

The case of arbitrary exponents is somewhat more complicated. A possible strategy is to define roots, then rational powers, and then extend by continuity. Our approach is different. For $x \in \mathbb{R}^+$ and $p \in \mathbb{R}$, we define the set of all reals that one would want to be smaller than x^p, and then define the latter as the least upper bound of this set. To be more precise, let $x > 1$ and define

$$L(x,p) = \{z \in \mathbb{R}^+ : z^n < x^m \text{ for all } m \in \mathbb{Z},\ n \in \mathbb{N} \text{ such that } m < pn\}.$$

We then define x^p to be the least upper bound of $L(x,p)$. For $x < 1$ we define

$$x^p = (x^{-1})^{-p}.$$

The exponential operation possesses a number of important `http://planetmath.org/Properti` some of which characterize it up to uniqueness.

Note. It is also possible to define the exponential operation in terms of the exponential function and the natural logarithm. Since these concepts require the context of differential theory, it seems preferable to give a basic definition that relies only on the foundational property of the reals.