monotone convergence theorem

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Let X be a measure space, and let $0 \leq f_1 \leq f_2 \leq \cdots$ be a monotone increasing sequence of nonnegative measurable functions. Let $f\colon X \to \mathbb{R} \cup \{\infty\}$ be the function defined by $f(x) = \lim_{n\to\infty} f_n(x)$. Then f is measurable, and

$$\lim_{n\to\infty} \int_X f_n = \int_X f.$$

Remark. This theorem is the first of several theorems which allow us to "exchange integration and limits". It requires the use of the Lebesgue integral: with the Riemann integral, we cannot even formulate the theorem, lacking, as we do, the concept of "almost everywhere". For instance, the characteristic function of the rational numbers in $[0, 1]$ is not Riemann integrable, despite being the limit of an increasing sequence of Riemann integrable functions.