proof of properties of derivatives by pure algebra

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Theorem 1. *The derivative satisfies the following rules:*

Linearity

$$\frac{d}{dx}(f(x)+g(x)) = \frac{df}{dx}+\frac{dg}{dx}, \qquad \frac{d}{dx}(af(x)) = a\frac{df}{dx},$$

for $f(x), g(x) \in R[x]$ and $a \in R$.

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Product Rule

$$\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}.$$

Remark 2. *The following proofs apply to* `http://planetmath.org/DerivativesByPureAlgebra` *by pure algebra. While the nature of the proofs are similar to the usual proofs, the notion of a limit is replaced by modular arithmetic in $R[x,h]/(h)$.*

Proof. *Power rule.*

$$\begin{aligned} \frac{d}{dx}(x^n) &\equiv \frac{(x+h)^n - x^n}{h} \\ &= \sum_{j=1}^{n} \binom{i}{j} x^{n-j}h^{j-1} \\ &\equiv \binom{n}{1} x^{n-1} = nx^{n-1}. \end{aligned}$$

Linearity rule. For all $f(x), g(x) \in R[x] \cong R[x,h]/(h)$, it follows

$$\frac{(f+g)(x+h)-(f+g)(x)}{h} \equiv \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \equiv \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-}{h}$$

Furthermore, for all $a \in R$

$$\frac{(af)(x+h)-(af)(x)}{h} \equiv \frac{af(x+h)-af(x)}{h} = a\frac{f(x+h)-f(x)}{h}.$$

Product rule. In $R[x,h]$ modulo (h) we have:

$$\begin{aligned}
\frac{d}{dx}(fg) &\equiv \frac{f(x+h)g(x+h)) - f(x)g(x)}{h} \\
&\equiv \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&\equiv \frac{(f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x))}{h} \\
&\equiv \frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h} \\
&\equiv \frac{df}{dx} g(x) + f(x)\frac{dg}{dx}.
\end{aligned}$$

$\square$