example of differentiation under integral sign

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Differentiation with respect to a parameter under the integral sign may sometimes yield useful formulae. One example is given here.

We know that the equation

$$\int_0^1 x^m \, dx = \frac{1}{m+1}$$

is valid for all $m > -1$. If one differentiates with respect to m `http://planetmath.org/Different` the integral sign in succession, one gets

$$\int_0^1 \frac{\partial}{\partial m} e^{m \ln x} \, dx = \int_0^1 e^{m \ln x} \ln x \, dx = \int_0^1 x^m \ln x \, dx = \frac{-1}{(m+1)^2}$$

$$\int_0^1 \frac{\partial}{\partial m} x^m \ln x \, dx = \int_0^1 x^m (\ln x)^2 \, dx = \frac{+1 \cdot 2}{(m+1)^3}$$

$$\int_0^1 \frac{\partial}{\partial m} x^m (\ln x)^2 \, dx = \int_0^1 x^m (\ln x)^3 \, dx = \frac{-1 \cdot 2 \cdot 3}{(m+1)^4}$$

$$\ldots$$

It's evident that repeating the differentiation n times the final result is the

$$\int_0^1 x^m (\ln x)^n \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}} \qquad (m > -1).$$