proof of Barbalat's lemma

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We suppose that $y(t) \not\to 0$ as $t \to \infty$. There exists a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ as $n \to \infty$ and $|y(t_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. By the uniform continuity of y, there exists a $\delta > 0$ such that, for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$,

$$|t_n - t| \leq \delta \;\Rightarrow\; |y(t_n) - y(t)| \leq \frac{\varepsilon}{2}.$$

So, for all $t \in [t_n, t_n + \delta]$, and for all $n \in \mathbb{N}$ we have

$$\begin{aligned} |y(t)| &= |y(t_n) - (y(t_n) - y(t))| \geq |y(t_n)| - |y(t_n) - y(t)| \geq \\ &\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \end{aligned}$$

Therefore,

$$\left| \int_0^{t_n+\delta} y(t)dt - \int_0^{t_n} y(t)dt \right| = \left| \int_{t_n}^{t_n+\delta} y(t)dt \right| = \int_{t_n}^{t_n+\delta} |y(t)|dt \geq \frac{\varepsilon\delta}{2} > 0$$

for each $n \in \mathbb{N}$. By the hypothesis, the impropre Riemann integral $\int_0^\infty y(t)dt$ exists, and thus the left hand side of the inequality converges to 0 as $n \to \infty$, yielding a contradiction.