rational sine and cosine

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Theorem. The only acute angles, whose sine and cosine are rational, are those determined by the Pythagorean triplets (a, b, c).

Proof. 1º. When the catheti a, b and the hypotenuse c of a right triangle are integers, i.e. they form a Pythagorean triplet, then the sine $\frac{a}{c}$ and the cosine $\frac{b}{c}$ of one of the acute angles of the triangle are rational numbers.

2º. Let the sine and the cosine of an acute angle ω be rational numbers

$$\sin\omega = \frac{a}{c}, \quad \cos\omega = \frac{b}{d},$$

where the integers a, b, c, d satisfy

$$\gcd(a, c) = \gcd(b, d) = 1. \tag{1}$$

Since the square sum of sine and cosine is always 1, we have

$$\frac{a^2}{c^2} + \frac{b^2}{d^2} = 1. \tag{2}$$

By removing the denominators we get the Diophantine equation

$$a^2d^2 + b^2c^2 = c^2d^2.$$

Since two of its terms are divisible by c^2, also the third term a^2d^2 is divisible by c^2. But because by (1), the integers a^2 and c^2 are coprime, we must have $c^2 \mid d^2$ (see the corollary of Bézout's lemma). Similarly, we also must have $d^2 \mid c^2$. The last divisibility relations mean that $c^2 = d^2$, whence (2) may be written

$$a^2 + b^2 = c^2,$$

and accordingly the sides a, b, c of a corresponding right triangle are integers.