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fractional integration

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The basic idea of "Riemann-Liouville" type fractional integration comes from the following observation:

Given any integrable function $f : \mathbb{R} \mapsto \mathbb{R}$ in one variable, we have the following Cauchy Integration Formula:

$$D^{-n}(f)(x) = \int_{t_n=0}^{x} dt_n \dots \int_{t_1=0}^{t_2} f(t_1)\, dt_1 = \frac{1}{(n-1)!} \int_{t=0}^{x} f(t)(x-t)^{n-1}\, dt$$

when switching the index from integer n to non-integer α gives the ideas of the following definitions:

Definition 1: Left-Hand Riemann-Liouville Integration

$$I_L^{\alpha}(f)(s,t) = \frac{1}{\Gamma(\alpha)} \int_{u=s}^{t} f(u)(t-u)^{\alpha-1}\, du = \int_{u=s}^{t} f(u)\, dg_t^{\alpha}(u)$$

where

$$g_t^{\alpha}(u) = \frac{t^{\alpha} - (t-u)^{\alpha}}{\Gamma(\alpha+1)}$$

Definition 2: Right-Hand Riemann-Liouville Integration

$$I_R^{\alpha}(f)(s,t) = \frac{1}{\Gamma(\alpha)} \int_{u=s}^{t} f(u)(u-s)^{\alpha-1}\, du = \int_{u=s}^{t} f(u)\, dh_t^{\alpha}(u)$$

where

$$h_t^{\alpha}(u) = \frac{s^{\alpha} + (u-s)^{\alpha}}{\Gamma(\alpha+1)}$$

Definition 3: Riesz Potential

$$I_C^{\alpha}(f)(s,t;p) = \frac{1}{\Gamma(\alpha)} \int_{u=s}^{t} f(u)|u-p|^{\alpha-1}\, du = \int_{u=s}^{t} f(u)\, dr_p^{\alpha}(u)$$

where

$$r_p^{\alpha}(u) = \frac{p^{\alpha} + \operatorname{sign}(u-p)|u-p|^{\alpha}}{\Gamma(\alpha+1)}$$

,

$\operatorname{sign}(x) = 1$ for $x > 0$, $\operatorname{sign}(x) = 0$ for $x = 0$, $\operatorname{sign}(x) = -1$ for $x < 0$ and $\Gamma(x)$ is the gamma function of x