a lecture on integration by parts

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The Method of Integration by Parts

This method is used to find indefinite integrals that look like the result of a product rule.

- *When to use it:* Use this method when the integrand is a product of two functions (and when the method of substitution clearly does not work).
- *How to use it:* The method is based in the following formula:

$$\int U \cdot V' \, dx = U \cdot V - \int U' \cdot V \, dx$$

  Suppose we want to solve $\int f(x)\, dx$:

  1. Find functions U and V' such that $U \cdot V' = f(x)$. There are many possible choices.
  2. U should be a function *easy* to derive and such that the derivative of U is *easier, less complicated* than U itself. For example good choices for U are $U = x, x^2, x^3, e^x, \ln x$.
  3. V' should be a function which is *easy* to integrate, such that we can find $\int V' dx$ easily and the integral is less complicated than V' itself. Good choices for V' are $V' = e^x, \sin x, \cos x$. The functions $x, x^2, x^3, \ln x$ are **bad** choices.
  4. Once the functions U and V' are chosen, find $U' = \frac{d}{dx}U$ and $V = \int V'\, dx$.
  5. Plug in the formula.
  6. Solve the new integral $\int U' \cdot V\, dx$, which if the choices of U and V' were good, should be easy.
  7. If the new integral is hard, the choices of U and V' might be wrong. So repeat the choice.

Again, the method is best explained through examples:

Example 0.1. Find $\int xe^x\, dx$. This is a product of two functions and there is no composition visible, so we will be using integration by parts. We need two functions U and V' such that $U \cdot V' = xe^x$. A good choice for U is x because

$U' = (x)' = 1$ and the derivative is simpler than x itself. A good choice for $V' = e^x$ because it is easy to integrate, $\int e^x \, dx = e^x$ (and $U \cdot V' = xe^x$). Therefore we apply the formula:

$$\int xe^x \, dx = UV - \int U'V \, dx = xe^x - \int 1 \cdot e^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C.$$

Example 0.2. Find:

$$\int x \cos(x) \, dx.$$

We choose $U = x$ and $V' = \cos(x)$. Then $U' = 1$ and $V = \int V' \, dx = \sin(x)$. Apply the formula:

$$\int x \cos(x) \, dx = x \sin(x) - \int \sin(x) \, dx = x \sin(x) - (-\cos(x)) + C = x \sin(x) + \cos(x) + C.$$

Example 0.3. Find $\int \ln x \, dx$. Although this is not the product of two functions, $\ln x$ is a typical example of a function to be integrated by parts. Here is how, let $U = \ln x$ and $V' = 1$ so that $U' = 1/x$ and $V = x$. Look what happens when we use the formula:

$$\int \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C.$$

Example 0.4. In some cases we have to use the method of integration by parts twice to get an answer. For example find $\int x^2 \sin x \; dx$. We put $U = x^2$ and $V' = \sin x$, and so $U' = 2x$ and $V = \int \sin x \; dx = -\cos x$. Thus:

$$\int x^2 \sin x \; dx = -x^2 \cos x + \int 2x \cos x \; dx = -x^2 \cos x + 2 \int x \cos x \; dx$$

and above, we have seen that in order to find $\int x \cos x \; dx$ we use integration by parts. Therefore the final anwser is:

$$\int x^2 \sin x \; dx = -x^2 \cos x + 2(x \sin x + \cos x) + C.$$

Example 0.5. Finally, in some other cases, after we do integration by parts twice, we get to the same integral we wanted to solve. Although it would seem we are stuck, no no! we will be able to find a solution right away.

Find $\int e^x \sin x \; dx$. Let $I = \int e^x \sin x \; dx$. We start with integration by parts, taking

$$U = e^x, V' = \sin x, U' = e^x, V = -\cos x$$

Thus:

$$I = -e^x \cos x + \int e^x \cos x \; dx$$

Hmmm...the integral $\int e^x \cos x \; dx$ looks like the one we started with...we use integration by parts to solve this one. Take $U = e^x$ and $V' = \cos x$ so $U' = e^x$ and $V = \sin x$. Thus:

$$\int e^x \cos x \; dx = e^x \sin x - \int e^x \sin x \; dx.$$

Therefore:

$$I = -e^x \cos x + (e^x \sin x - \int e^x \sin x \; dx) = -e^x \cos x + e^x \sin x - I$$

Remember that we want to find I, so if we solve for I above, we obtain:

$$2I = -e^x \cos x + e^x \sin x$$

and so $I = 1/2(-e^x \cos x + e^x \sin x) + C$.