determination of Fourier coefficients

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Suppose that the real function f may be presented as sum of the Fourier series:

$$f(x) \;=\; \frac{a_0}{2} + \sum_{m=0}^{\infty}(a_m \cos mx + b_m \sin mx) \tag{1}$$

Therefore, f is periodic with period 2π. For expressing the Fourier coefficients a_m and b_m with the function itself, we first multiply the series (1) by $\cos nx$ ($n \in \mathbb{Z}$) and integrate from $-\pi$ to π. Supposing that we can integrate termwise, we may write

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx \;=\; \frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx\,dx + \sum_{m=0}^{\infty}\left(a_m\int_{-\pi}^{\pi}\cos mx\cos nx\,dx + b_m\int_{-\pi}^{\pi}\sin mx\cos nx\,dx\right). \tag{2}$$

When $n = 0$, the equation (2) reads

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\cdot 2\pi = \pi a_0, \tag{3}$$

since in the sum of the right hand side, only the first addend is distinct from zero.

When n is a positive integer, we use the product formulas of the trigonometric identities, getting

$$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m-n)x + \cos(m+n)x]\,dx,$$

$$\int_{-\pi}^{\pi}\sin mx\cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\sin(m-n)x + \sin(m+n)x]\,dx.$$

The latter expression vanishes always, since the sine is an odd function. If $m \neq n$, the former equals zero because the antiderivative consists of sine terms which vanish at multiples of π; only in the case $m = n$ we obtain from it a non-zero result π. Then (2) reads

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \pi a_n \tag{4}$$

to which we can include as a special case the equation (3).

By multiplying (1) by $\sin nx$ and integrating termwise, one obtains similarly

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \pi b_n. \tag{5}$$

The equations (4) and (5) imply the formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad (n = 0,\, 1,\, 2,\, \ldots)$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (n = 1,\, 2,\, 3,\, \ldots)$$

for finding the values of the Fourier coefficients of f.