absolutely continuous on $[0,1]$ versus absolutely continuous on $[\varepsilon,1]$ for every $\varepsilon>0$

Canonical name	AbsolutelyContinuousOn01VersusAbsolutelyContinuousOnvarepsilon1ForEver
Date of creation	2013-03-22 16:12:19
Last modified on	2013-03-22 16:12:19
Owner	Wkbj79 (1863)
Last modified by	Wkbj79 (1863)
Numerical id	11
Author	Wkbj79 (1863)
Entry type	Example
Classification	msc 26A46
Classification	msc 26B30

Lemma. *Define* $f \colon \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0. \end{cases}$$

Then f *is* ***http://planetmath.org/AbsolutelyContinuousFunction2****absolutely continuous on* $[\varepsilon, 1]$ *for every* $\varepsilon > 0$ *but is not absolutely continuous on* $[0, 1]$*.*

Proof. Note that f is continuous on $[0, 1]$ and differentiable on $(0, 1]$ with $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$.

Let $\varepsilon > 0$. Then for all $x \in [\varepsilon, 1]$:

$$\begin{aligned} |f'(x)| &= \left|\sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)\right| \\ &\leq \left|\sin\left(\frac{1}{x}\right)\right| + \left|\frac{1}{x}\right| \cdot \left|\cos\left(\frac{1}{x}\right)\right| \\ &\leq 1 + \frac{1}{\varepsilon} \cdot 1 \\ &= 1 + \frac{1}{\varepsilon} \end{aligned}$$

Since f is continuous on $[\varepsilon, 1]$ and differentiable on $(\varepsilon, 1)$, the http://planetmath.org/MeanValu value theorem can be applied to f. Thus, for every $x_1, x_2 \in (\varepsilon, 1)$ with $x_1 \neq x_2$, $\left|\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right| \leq 1 + \frac{1}{\varepsilon}$. This yields $|f(x_2) - f(x_1)| \leq \left(1 + \frac{1}{\varepsilon}\right)|x_2 - x_1|$, which also holds when $x_1 = x_2$. Thus, f is Lipschitz on $(\varepsilon, 1)$. It follows that f is absolutely continuous on $[\varepsilon, 1]$.

On the other hand, it can be verified that f is not of bounded variation on $[0, 1]$ and thus cannot be absolutely continuous on $[0, 1]$. $\square$