Sophomore's dream

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The integral

$$I := \int_0^1 x^x\,dx \tag{1}$$

may be expanded to a rapidly converging series as follows.

Changing the integrand to a power of e and using the power series expansion of the exponential function gives us

$$I = \int_0^1 e^{x\ln x}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(x\ln x)^n}{n!}\,dx. \tag{2}$$

Here the series is uniformly convergent on $[0,1]$ and may be integrated termwise:

$$I = \sum_{n=0}^{\infty} \int_0^1 \frac{x^n(\ln x)^n}{n!} dx. \tag{3}$$

The last equation of the `http://planetmath.org/ExampleOfDifferentiationUnderIntegralSig` entry then gives in the case $m = n$ from (3) the result

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}}, \tag{4}$$

i.e.,

$$\int_0^1 x^x\,dx \;=\; 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + - \ldots \tag{5}$$

Cf. the `http://planetmath.org/FunctionXX`function x^x.

Since the series (5) satisfies the conditions of `http://planetmath.org/LeibnizEstimateForAl` theorem for alternating series, one may easily estimate the error made when a partial sum of (5) is used for the exact value of the integral I. If one for example takes for I the sum of nine first terms, the first omitted term is $-\frac{1}{10^{10}}$; thus the error is negative and its absolute value less than 10^{-10}.