proof of Taylor's Theorem

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Let $f(x),\ a < x < b$ be a real-valued, n-times differentiable function, and let $a < x_0 < b$ be a fixed base-point. We will show that for all $x \neq x_0$ in the domain of the function, there exists a ξ, strictly between x_0 and x such that

$$f(x) = \sum_{k=0}^{n-1} f^{(k)}(x_0)\,\frac{(x-x_0)^k}{k!} + f^{(n)}(\xi)\,\frac{(x-x_0)^n}{n!}.$$

Fix $x \neq x_0$ and let R be the remainder defined by

$$f(x) = \sum_{k=0}^{n-1} f^{(k)}(x_0)\,\frac{(x-x_0)^k}{k!} + R\,\frac{(x-x_0)^n}{n!}.$$

Next, define

$$F(\xi) = \sum_{k=0}^{n-1} f^{(k)}(\xi)\,\frac{(x-\xi)^k}{k!} + R\,\frac{(x-\xi)^n}{n!}, \quad a < \xi < b.$$

We then have

$$\begin{aligned} F'(\xi) &= f'(\xi) + \sum_{k=1}^{n-1}\left(f^{(k+1)}(\xi)\,\frac{(x-\xi)^k}{k!} - f^{(k)}(\xi)\,\frac{(x-\xi)^{k-1}}{(k-1)!}\right) - R\,\frac{(x-\xi)^{n-1}}{(n-1)!} \\ &= f^{(n)}(\xi)\,\frac{(x-\xi)^{n-1}}{(n-1)!} - R\,\frac{(x-\xi)^{n-1}}{(n-1)!} \\ &= \frac{(x-\xi)^{n-1}}{(n-1)!}\,(f^{(n)}(\xi) - R), \end{aligned}$$

because the sum telescopes. Since, $F(\xi)$ is a differentiable function, and since $F(x_0) = F(x) = f(x)$, Rolle's Theorem imples that there exists a ξ lying strictly between x_0 and x such that $F'(\xi) = 0$. It follows that $R = f^{(n)}(\xi)$, as was to be shown.