example of a BV function which is not $W^{1,1}$

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The following example presents a function $u \in BV(\Omega) \setminus W^{1,1}(\Omega)$.

Example 1. Let $\Omega := (-1,1) \times (-1,1) \subset \mathbb{R}^2$. We will show that the function

$$u(x,y) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

belongs to $BV(\Omega)$. Given $\phi \in C^1_c(\Omega, \mathbb{R}^2)$, $\phi = (\phi^1, \phi^2)$, one has

$$\begin{aligned} \iint_\Omega u(x,y)\mathrm{div}\phi(x,y)\,dxdy &= \int_{-1}^1 \left[\int_0^1 \phi^1_x(x,y)\,dx\right] dy + \int_0^1 \left[\int_{-1}^1 \phi^2_y(x,y)\,dy\right] dx \\ &= \int_{-1}^1 \phi^1(1,y) - \phi^1(0,y)\,dy + \int_0^1 \phi^2(x,1) - \phi^2(x,-1)\,dx \\ &= -\int_{-1}^1 \phi^1(0,y) + 0 = -\int \phi(x,y)\,d\mu(x,y) \end{aligned}$$

if we choose $\mu := (\mu^1, \mu^2) := (\mathcal{H}^1 \llcorner (\{0\} \times (-1,1)), 0)$. So we notice that $u \in BV(\Omega)$ and $Du = \mu$ is singular with respect to the Lebesgue measure $\mathcal{L}$.