properties of the Lebesgue integral of nonnegative measurable functions

Canonical name	PropertiesOfTheLebesgueIntegralOfNonnegativeMeasurableFunctions
Date of creation	2013-03-22 16:13:50
Last modified on	2013-03-22 16:13:50
Owner	Wkbj79 (1863)
Last modified by	Wkbj79 (1863)
Numerical id	22
Author	Wkbj79 (1863)
Entry type	Theorem
Classification	msc 26A42
Classification	msc 28A25
Related topic	PropertiesOfTheLebesgueIntegralOfLebesgueIntegrableFunctions

Theorem. *Let* $(X, \mathfrak{B}, \mu)$ *be a measure space,* $f\colon X \to [0,\infty]$ *and* $g\colon X \to [0,\infty]$ *be measurable functions, and* $A, B \in \mathfrak{B}$. *Then the following properties hold:*

1. $\displaystyle\int_A f\,d\mu \geq 0$
2. *If* $f \leq g$, *then* $\displaystyle\int_A f\,d\mu \leq \int_A g\,d\mu$.
3. $\displaystyle\int_A f\,d\mu = \int_X \chi_A f\,d\mu$, *where* χ_A *denotes the characteristic function of* A
4. *If* $A \subseteq B$, *then* $\displaystyle\int_A f\,d\mu \leq \int_B f\,d\mu$.
5. *If* $c \geq 0$, *then* $\displaystyle\int_A cf\,d\mu = c\int_A f\,d\mu$.
6. *If* $\mu(A) = 0$, *then* $\displaystyle\int_A f\,d\mu = 0$.
7. $\displaystyle\int_A (f+g)\,d\mu = \int_A f\,d\mu + \int_A g\,d\mu$
8. *If* $A \cap B = \emptyset$, *then* $\displaystyle\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu$.
9. *If* $f = g$ *almost everywhere with respect to* μ, *then* $\displaystyle\int_A f\,d\mu = \int_A g\,d\mu$.

Proof. 1. Let s be a simple function with $0 \leq s \leq f$. Let $s = \displaystyle\sum_{k=1}^{n} c_k \chi_{A_k}$ for $c_k \in [0,\infty]$ and $A_k \in \mathfrak{B}$. Then $\displaystyle\int_A s\,d\mu = \sum_{k=1}^{n} c_k \mu(A \cap A_k) \geq 0$. By definition, $\displaystyle\int_A f\,d\mu \geq \int_A s\,d\mu$. It follows that $\displaystyle\int_A f\,d\mu \geq 0$.

2. Let s be a simple function with $0 \leq s \leq f$. Since $f \leq g$, $0 \leq s \leq g$. By definition, $\int_A s\,d\mu \leq \int_A g\,d\mu$. Since this holds for every simple function s with $0 \leq s \leq f$, it follows that $\int_A f\,d\mu \leq \int_A g\,d\mu$.

3. Let s be a simple function with $0 \leq s \leq f$. Then $0 \leq \chi_A s \leq \chi_A f$. Let $s = \sum_{k=1}^{n} c_k \chi_{A_k}$ for $c_k \in [0, \infty]$ and $A_k \in \mathfrak{B}$. Then

$$\begin{aligned}
\int_A s\,d\mu &= \sum_{k=1}^{n} c_k \mu(A \cap A_k) \\
&= \int_X \sum_{k=1}^{n} c_k \chi_{A \cap A_k}\,d\mu \\
&= \int_X \sum_{k=1}^{n} c_k \chi_A \chi_{A_k}\,d\mu \\
&= \int_X \chi_A \sum_{k=1}^{n} c_k \chi_{A_k}\,d\mu \\
&= \int_X \chi_A s\,d\mu \\
&\leq \int_X \chi_A f\,d\mu.
\end{aligned}$$

Thus, $\int_A f\,d\mu \leq \int_X \chi_A f\,d\mu$.

Let t be a simple function with $0 \leq t \leq \chi_A f$. Then $\chi_A t = t$. Thus, $\int_X t\,d\mu = \int_X \chi_A t\,d\mu = \int_A t\,d\mu$. Therefore, $\int_X \chi_A f\,d\mu = \int_A \chi_A f\,d\mu$. Since $\chi_A f \leq f$, $\int_A \chi_A f\,d\mu \leq \int_A f\,d\mu$ by property 2. Hence, $\int_A f\,d\mu \leq \int_X \chi_A f\,d\mu = \int_A \chi_A f\,d\mu \leq \int_A f\,d\mu$. It follows that $\int_A f\,d\mu = \int_X \chi_A f\,d\mu$.

4. Since $A \subseteq B$, $\chi_A \leq \chi_B$. Thus, $\chi_A f \leq \chi_B f$. By property 2, $\int_X \chi_A f\, d\mu \leq \int_X \chi_B f\, d\mu$. By property 3, $\int_A f\, d\mu = \int_X \chi_A f\, d\mu \leq \int_X \chi_B f\, d\mu = \int_B f\, d\mu$.

5. If $c = 0$, then $\int_A cf\, d\mu = \int_A 0\, d\mu = 0 = 0 \int_A f\, d\mu = c \int_A f\, d\mu$.
If $c > 0$, let $S = \{s\colon X \to [0,\infty] \mid s \text{ is simple and } s \leq cf\}$ and $T = \{t\colon X \to [0,\infty] \mid t \text{ is simple and } t \leq f\}$. Then $\int_A cf\, d\mu = \sup_{s\in S} \int_A s\, d\mu = \sup_{s\in S} \int_A c\cdot \frac{s}{c}\, d\mu = c \sup_{s\in S} \int_A \frac{s}{c}\, d\mu = c \sup_{t\in T} \int_A t\, d\mu = c \int_A f\, d\mu$.

6. Let s be a simple function with $0 \leq s \leq f$. Let $s = \sum_{k=1}^{n} c_k \chi_{A_k}$ for $c_k \in [0,\infty]$ and $A_k \in \mathfrak{B}$. Then $\int_A s\, d\mu = \sum_{k=1}^{n} c_k \mu(A \cap A_k) = \sum_{k=1}^{n} c_k \cdot 0 = 0$.
Thus, $\int_A f\, d\mu = 0$.

7. Let $\{s_n\}$ be a nondecreasing sequence of nonnegative simple functions converging pointwise to f and $\{t_n\}$ be a nondecreasing sequence of nonnegative simple functions converging pointwise to g. Then $\{s_n + t_n\}$ is a nondecreasing sequence of nonnegative simple functions converging pointwise to $f + g$. Note that, for every n, $\int_A (s_n + t_n)\, d\mu = \int_A s_n\, d\mu + \int_A t_n\, d\mu$. By Lebesgue's monotone convergence theorem, $\int_A (f+g)\, d\mu = \int_A f\, d\mu + \int_A g\, d\mu$.

8.

$$\begin{aligned}
\int_{A\cup B} f\,d\mu &= \int_X \chi_{A\cup B} f\,d\mu \\
&= \int_X (\chi_A + \chi_B - \chi_{A\cap B})\,f\,d\mu \\
&= \int_X (\chi_A + \chi_B - \chi_{\emptyset})\,f\,d\mu \\
&= \int_X (\chi_A + \chi_B - 0)\,f\,d\mu \\
&= \int_X (\chi_A f + \chi_B f)\,\,d\mu \\
&= \int_X \chi_A f\,d\mu + \int_X \chi_B f\,d\mu \\
&= \int_A f\,d\mu + \int_B f\,d\mu
\end{aligned}$$

9. Let $E = \{x \in A : f(x) = g(x)\}$. Since f and g are measurable functions and $A \in \mathfrak{B}$, it must be the case that $E \in \mathfrak{B}$. Thus, $A \setminus E \in \mathfrak{B}$. By hypothesis, $\mu(A\setminus E) = 0$. Note that $E\cap(A\setminus E) = \emptyset$ and $E\cup(A\setminus E) = A$. Thus, $\int_A f\,d\mu = \int_E f\,d\mu + \int_{A\setminus E} f\,d\mu = \int_E f\,d\mu + 0 = \int_E g\,d\mu + 0 = \int_E g\,d\mu + \int_{A\setminus E} g\,d\mu = \int_A g\,d\mu$.

$\square$