proof of arithmetic-geometric-harmonic means inequality

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Let M be $\max\{x_1, x_2, x_3, \ldots, x_n\}$ and let m be $\min\{x_1, x_2, x_3, \ldots, x_n\}$. Then

$$M = \frac{M + M + M + \cdots + M}{n} \geq \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

$$m = \frac{n}{\frac{n}{m}} = \frac{n}{\frac{1}{m} + \frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}} \leq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}$$

where all the summations have n terms. So we have proved in this way the two inequalities at the extremes.

Now we shall prove the inequality between arithmetic mean and geometric mean.

1 Case $n = 2$

We do first the case $n = 2$.

$$\begin{aligned}
(\sqrt{x_1} - \sqrt{x_2})^2 &\geq 0 \\
x_1 - 2\sqrt{x_1 x_2} + x_2 &\geq 0 \\
x_1 + x_2 &\geq 2\sqrt{x_1 x_2} \\
\frac{x_1 + x_2}{2} &\geq \sqrt{x_1 x_2}
\end{aligned}$$

2 Case $n = 2^k$

Now we prove the inequality for any power of 2 (that is, $n = 2^k$ for some integer k) by using mathematical induction.

$$\begin{aligned}
&\frac{x_1 + x_2 + \cdots + x_{2^k} + x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^{k+1}} \\
&= \frac{\left(\frac{x_1 + x_2 + \cdots + x_{2^k}}{2^k}\right) + \left(\frac{x_{2^k+1} + x_{2^k+2} + \cdots + x_{2^{k+1}}}{2^k}\right)}{2}
\end{aligned}$$

and using the case $n = 2$ on the last expression we can state the following inequality

$$\frac{x_1 + x_2 + \cdots + x_{2^k} + x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^{k+1}}$$

$$
\begin{aligned}
&\geq \sqrt{\left(\frac{x_1 + x_2 + \cdots + x_{2^k}}{2^k}\right)\left(\frac{x_{2^k+1} + x_{2^k+2} + \cdots + x_{2^{k+1}}}{2^k}\right)} \\
&\geq \sqrt{\sqrt[2^k]{x_1 x_2 \cdots x_{2^k}}\ \sqrt[2^k]{x_{2^k+1} x_{2^k+2} \cdots x_{2^{k+1}}}}
\end{aligned}
$$

where the last inequality was obtained by applying the induction hypothesis with $n = 2^k$. Finally, we see that the last expression is equal to $\sqrt[2^{k+1}]{x_1 x_2 x_3 \cdots x_{2^{k+1}}}$ and so we have proved the truth of the inequality when the number of terms is a power of two.

3 Inequality for n numbers implies inequality for $n-1$

Finally, we prove that if the inequality holds for any n, it must also hold for $n-1$, and this proposition, combined with the preceding proof for powers of 2, is enough to prove the inequality for any positive integer.

Suppose that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

is known for a given value of n (we just proved that it is true for powers of two, as example). Then we can replace x_n with the average of the first $n-1$ numbers. So

$$
\begin{aligned}
&\frac{x_1 + x_2 + \cdots + x_{n-1} + \left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)}{n} \\
&= \frac{(n-1)x_1 + (n-1)x_2 + \cdots + (n-1)x_{n-1} + x_1 + x_2 + \cdots + x_{n-1}}{n(n-1)} \\
&= \frac{nx_1 + nx_2 + \cdots + nx_{n-1}}{n(n-1)} \\
&= \frac{x_1 + x_2 + \cdots + x_{n-1}}{(n-1)}
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\sqrt[n]{x_1 x_2 \cdots x_{n-1}\left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)} \\
&= \sqrt[n]{x_1 x_2 \cdots x_{n-1}}\ \sqrt[n]{\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}}
\end{aligned}
$$

which, by hypothesis (the inequality holding for n numbers) and the observations made above, leads to:

$$\left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)^n \geq (x_1 x_2 \cdots x_n) \left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)$$

and so

$$\left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)^{n-1} \geq x_1 x_2 \cdots x_n$$

from where we get that

$$\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1 x_2 \cdots x_n}.$$

So far we have proved the inequality between the arithmetic mean and the geometric mean. The geometric-harmonic inequality is easier. Let t_i be $1/x_i$.

From

$$\frac{t_1 + t_2 + \cdots + t_n}{n} \geq \sqrt[n]{t_1 t_2 t_3 \cdots t_n}$$

we obtain

$$\frac{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}{n} \geq \sqrt[n]{\frac{1}{x_1}\frac{1}{x_2}\frac{1}{x_3}\cdots\frac{1}{x_n}}$$

and therefore

$$\sqrt[n]{x_1 x_2 x_3 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}$$

and so, our proof is completed.