proof of continuous functions are Riemann integrable

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Recall the definition of Riemann integral. To prove that f is integrable we have to prove that $\lim_{\delta\to 0^+} S^*(\delta) - S_*(\delta) = 0$. Since $S^*(\delta)$ is decreasing and $S_*(\delta)$ is increasing it is enough to show that given $\epsilon > 0$ there exists $\delta > 0$ such that $S^*(\delta) - S_*(\delta) < \epsilon$.

So let $\epsilon > 0$ be fixed.

By Heine-Cantor Theorem f is uniformly continuous i.e.

$$\exists \delta > 0\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Let now P be any partition of $[a, b]$ in $C(\delta)$ i.e. a partition $\{x_0 = a, x_1, \ldots, x_N = b\}$ such that $x_{i+1} - x_i < \delta$. In any small interval $[x_i, x_{i+1}]$ the function f (being continuous) has a maximum M_i and minimum m_i. Since f is uniformly continuous and $x_{i+1} - x_i < \delta$ we have $M_i - m_i < \epsilon/(b - a)$. So the difference between upper and lower Riemann sums is

$$\sum_i M_i(x_{i+1} - x_i) - \sum_i m_i(x_{i+1} - x_i) \leq \frac{\epsilon}{b-a} \sum_i (x_{i+1} - x_i) = \epsilon.$$

This being true for every partition P in $C(\delta)$ we conclude that $S^*(\delta) - S_*(\delta) < \epsilon$.