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example of tests for local extrema in Lagrange multiplier method

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Let $n \in \mathbb{N}^+$ and $c \in \mathbb{R}$. We want to find the local extrema of the function

$$f\colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \sum_{1\le i<j\le n} x_i x_j$$

subject to the condition $g = 0$, where

$$g\colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \sum_{1\le i\le n} x_i - c.$$

The first and second order partial derivatives are for all $i, j \in \{1, \dots, n\}$

$$\partial_i f(x) = \sum_{k\ne i} x_k, \quad \partial_i g(x) = 1,$$

$$\partial_i \partial_j f(x) = 1 - \delta_{i,j}, \quad \partial_i \partial_j g(x) = 0,$$

where $\delta_{i,j}$ is the Kroenecker-delta. Thus the necessary condition $f'(x) = \lambda g'(x)$ together with $g(x) = 0$ gives the system of equations

$$\sum_{j\ne i} x_j = \lambda, \quad i \in \{1, \dots, n\},$$

$$\sum_{1\le j\le n} x_j = c.$$

By summing the first n equations and then substituting in the last we get

$$(n-1)c = n\lambda,$$

$$x_i = \sum_{1\le j\le n} x_j - \sum_{j\ne i} x_j = c - \lambda = \frac{c}{n}, \quad i \in \{1, \dots, n\}.$$

Thus there is only one point, where local extremum is possible. We apply the test in the parent entry to the matrix

$$D^2(f - \lambda g)(x) = [1 - \delta_{i,j}]_{i,j=1}^n = nP - I,$$

where P is the matrix containing $1/n$ in all entries, and I is the identity matrix. P is a rank one projection. Therefore the second derivative has spectrum $\sigma(nP - I) = \{n-1, -1\}$, where -1 has multiplicity $n-1$, and $n-1$ has multiplicity 1. Thus the second derivative of $f - \lambda g$ is indefinit, so it has no local extrema. However the nullspace of $g'(x)$ is precisely the nullspace of P, thus the second derivative is strictly negative on the tangent space $T_x(M)$, so the vector $(c/n, \dots, c/n)$ is a local maximum of f subject to $g = 0$.