ostensibly discontinuous antiderivative

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The real function

$$x \;\mapsto\; \frac{1}{5-3\cos x} \tag{1}$$

is continuous for any x (the denominator is always positive) and therefore it has an antiderivative, defined for all x. Using the universal trigonometric substitution

$$\cos x \;:=\; \frac{1-t^2}{1+t^2}, \qquad dx \;=\; \frac{2dt}{1+t^2}, \qquad t \;=\; \tan\frac{x}{2},$$

we obtain

$$5-3\cos x \;=\; \frac{5(1+t^2)-3(1-t^2)}{1+t^2} \;=\; \frac{2(1+4t^2)}{1+t^2},$$

whence

$$\int\frac{dx}{5-3\cos x} \;=\; \int\frac{dt}{1+4t^2} \;=\; \frac{1}{2}\arctan 2t + C \;=\; \frac{1}{2}\arctan\!\left(2\tan\frac{x}{2}\right) + C.$$

This result is not defined in the odd multiples of π, and it seems that the function (1) does not have a continuous antiderivative.

However, one can check that the function

$$x \;\mapsto\; \frac{x}{4} + \frac{1}{2}\arctan\frac{\sin x}{3-\cos x} + C \tag{2}$$

is everywhere continuous and has as its derivative the function (1); one has

$$\left|\frac{\sin x}{3-\cos x}\right| \;\leqq\; \frac{1}{3-1} \;=\; \frac{1}{2} \;<\; \frac{\pi}{2}.$$

References

[1] Ernst Lindelöf: *Johdatus korkeampaan analyysiin.* Fourth edition. Werner Söderström Osakeyhtiö, Porvoo ja Helsinki (1956).