Fubini's theorem

Canonical name	FubinisTheorem
Date of creation	2013-03-22 13:39:13
Last modified on	2013-03-22 13:39:13
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	11
Author	mathcam (2727)
Entry type	Theorem
Classification	msc 26B12
Related topic	TonellisTheorem
Related topic	FubinisTheoremForTheLebesgueIntegral
Related topic	IntegrationUnderIntegralSign

Fubini's theorem Let $I \subset \mathbb{R}^N$ and $J \subset \mathbb{R}^M$ be compact intervals, and let $f : I \times J \to \mathbb{R}^K$ be a Riemann integrable function such that, for each $x \in I$ the integral

$$F(x) := \int_J f(x, y)\, d\mu_J(y)$$

exists. Then $F : I \to \mathbb{R}^K$ is Riemann integrable, and

$$\int_I F = \int_{I \times J} f.$$

This theorem effectively states that, given a function of N variables, you may integrate it one variable at a time, and that the order of integration does not affect the result.

Example Let $I := [0, \pi/2] \times [0, \pi/2]$, and let $f : I \to \mathbb{R}, x \mapsto \sin(x)\cos(y)$ be a function. Then

$$\begin{aligned}
\int_I f &= \iint_{[0,\pi/2]\times[0,\pi/2]} \sin(x)\cos(y) \\
&= \int_0^{\pi/2} \left(\int_0^{\pi/2} \sin(x)\cos(y)\, dy \right) dx \\
&= \int_0^{\pi/2} \sin(x)\,(1 - 0)\; dx = (0 - -1) = 1.
\end{aligned}$$

Note that it is often simpler (and no less correct) to write $\int \cdots \int_I f$ as $\int_I f$.