even and odd functions

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Definition

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. If $f(-x) = f(x)$ for all $x \in \mathbb{R}$, then f is an *even function*. Similarly, if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, then f is an *odd function*.

Although this entry is mainly concerned with functions $\mathbb{R} \to \mathbb{R}$, the definition can be generalized to other types of function.

Notes

A real function is even if and only if it is symmetric about the y-axis. It is odd if and only if symmetric about the origin.

Examples

1. The function $f(x) = x$ is odd.
2. The function $f(x) = |x|$ is even.
3. The sine and cosine functions are odd and even, respectively.

Properties

1. The only function that is both even and odd is the function defined by $f(x) = 0$ for all real x.
2. A sum of even functions is even, and a sum of odd functions is odd. In fact, the even functions form a real vector space, as do the odd functions.
3. Every real function can be expressed in a unique way as the sum of an odd function and an even function.
4. From the above it follows that the vector space of real functions is the direct sum of the vector space of even functions and the vector space of odd functions. See the entry `http://planetmath.org/DirectSumOfEvenoddFunctionsExamp` sum of even/odd functions (example).)

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function.

   (a) If f is an even function, then the derivative f' is an odd function.

   (b) If f is an odd function, then the derivative f' is an even function.

   (For a proof, see the entry `http://planetmath.org/DerivativeOfEvenoddFunctionProofde` of even/odd function (proof).)

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. Then there exist smooth functions $g, h\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = g(x^2) + xh(x^2)$$

   for all $x \in \mathbb{R}$. Thus, if f is even, we have $f(x) = g(x^2)$, and if f is odd, we have $f(x) = xh(x^2)$. ([**?**], Exercise 1.2)

7. The Fourier transform of a real even function is purely real and even. The Fourier transform of a real odd function is purely imaginary and odd.

References

[1] L. Hörmander, *The Analysis of Linear Partial Differential Operators I, (Distribution theory and Fourier Analysis)*, 2nd ed, Springer-Verlag, 1990.