relation between positive function and its gradient when its Hessian matrix is bounded

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Let $f : R^n \rightarrow R$ a positive function, twice differentiable everywhere. Furthermore, let $\|\mathbf{H}_f(\mathbf{x})\|_2 \leq M, M > 0 \ \forall \mathbf{x} \in R^n$, where $\mathbf{H}_f(\mathbf{x})$ is the Hessian matrix of $f(\mathbf{x})$. Then, for any $\mathbf{x} \in R^n$,

$$\|\nabla f(\mathbf{x})\|_2 \leq \sqrt{2Mf(\mathbf{x})}$$

Proof. Let $\mathbf{x}$, $\mathbf{x}_0 \in R^n$ be arbitrary points. By positivity of $f(\mathbf{x})$, writing Taylor expansion of $f(\mathbf{x})$ with Lagrange error formula around $\mathbf{x}_0$, a point $\mathbf{c} \in R^n$ exists such that:

$$\begin{aligned}
0 &\leq f(\mathbf{x}) \\
&= f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T \cdot \mathbf{H}_f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{x}_0) \\
&= \left| f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T \cdot \mathbf{H}_f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{x}_0) \right| \\
&\leq f(\mathbf{x}_0) + |\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)| + \frac{1}{2} \left| (\mathbf{x} - \mathbf{x}_0)^T \cdot \mathbf{H}_f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{x}_0) \right| \\
&\leq f(\mathbf{x}_0) + \|\nabla f(\mathbf{x}_0)\|_2 \|\mathbf{x} - \mathbf{x}_0\|_2 + \frac{1}{2} \|\mathbf{H}_f(\mathbf{c})\|_2 \|\mathbf{x} - \mathbf{x}_0\|_2^2 \qquad \text{(by Cauchy-Schwartz inequality)} \\
&\leq f(\mathbf{x}_0) + \|\nabla f(\mathbf{x}_0)\|_2 \|\mathbf{x} - \mathbf{x}_0\|_2 + \frac{1}{2} M \|\mathbf{x} - \mathbf{x}_0\|_2^2
\end{aligned}$$

The rightest side is a second degree polynomial in variable $\|\mathbf{x} - \mathbf{x}_0\|_2$; for it to be positive for any choice of $\|\mathbf{x} - \mathbf{x}_0\|_2$ (that is, for any choice of $\mathbf{x}$), the discriminant

$$\|\nabla f(\mathbf{x}_0)\|_2^2 - 4 \cdot \frac{1}{2} M f(\mathbf{x}_0)$$

must be negative, whence the thesis. □

Note: The condition on the boundedness of the Hessian matrix is actually needed. In fact, in the Lagrange form remainder, the constant $\mathbf{c}$ depends upon the point $\mathbf{x}$. Thus, if we couldn't rely on the condition $\|\mathbf{H}_f(\mathbf{x})\|_2 \leq M$, we could only state $f(\mathbf{x}_0) + \|\nabla f(\mathbf{x}_0)\|_2 \|\mathbf{x} - \mathbf{x}_0\|_2 + \frac{1}{2} \|\mathbf{H}_f(\mathbf{c}(\mathbf{x}))\|_2 \|\mathbf{x} - \mathbf{x}_0\|_2^2 \geq 0$ which, not being a second degree polynomial, wouldn't imply any particular further condition. Moreover, in the case $n = 1$, the lemma assumes the simpler form: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ a positive function, twice differentiable everywhere. Furthermore, let $f''(x) \leq M, M > 0 \ \forall x \in \mathbb{R}$. Then, for any $x \in \mathbb{R}$, $|f'(x)| \leq \sqrt{2Mf(x)}$.