# Hessian form

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Given a smooth manifold $M$ and $f\colon M \to \mathbb{R}$ being in $C^2(M)$, if $x$ is a critical point of $f$, that is $df = 0$ at $x$, then we can define a symmetric 2-form

$$H(u_x, v_x) = u(v(f)) = v(u(f)),$$

where $H \in T^{*\otimes 2}$ and $u$ and $v$ are any vector fields that take the values $u_x$ and $v_x$, respectively, at point $x$. Equality of the two defining expressions follows from the fact that $x$ is a critical point of $f$, because then $[u,v](f) = df([u,v]) = 0$, where $[u,v]$ denotes the Lie bracket of the two vector fields. The form $H$ is called the *Hessian form*.

In local coordinates, the Hessian form is given by

$$H = \frac{\partial^2 f}{\partial x^i \partial x^j}\, dx^i \otimes dx^j.$$

Its components are those of the Hessian matrix in the same coordinates. The advantage of the above formulation is coordinate independence. However, the price is that the Hessian form is only defined at critical points. It does not define a tensor field as one would naïvely expect.

Using the Hessian form, it is possible to analyze the critical points of $f$ (determine whether they are local minima, maxima, or saddle points) in a coordinate independent way.