boundedly homogeneous function

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Defines	set of homogeneity
Defines	degree of homogeneity

A function $f\colon \mathbb{R}^n \to \mathbb{R}$, where n is a positive integer, is called *boundedly homogeneous* with respect to a set Λ of positive reals and a real number r, if the equation

$$f(\lambda\vec{x}) \;=\; \lambda^r f(\vec{x})$$

is true for all and $\vec{x} \in \mathbb{R}^n$ and $\lambda \in \Lambda$. Then Λ is the *set of homogeneity* and r the *degree of homogeneity* of f.

Example. The function $x \mapsto x^r \sin(\ln x)$ is boundedly homogeneous with respect to the set $\Lambda = \{e^{2\pi\nu} \vdots\ \nu \in \mathbb{Z}\}$ and with degree of homogeneity r.

Theorem. Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a boundedly homogeneous function with the degree of homogeneity r and the set of homogeneity $\Lambda \supset \{1\}$. Then f is of the form

$$f(x) \;=\; x^r f_1(\ln x) \tag{1}$$

where $f_1\colon \mathbb{R} \to \mathbb{R}$ is a periodic real function depending on f.

Proof. Defining $g(x) := \frac{f(x)}{x^r}$, we obtain

$$g(\lambda x) \;=\; \frac{f(\lambda x)}{(\lambda x)^r} \;=\; \frac{\lambda^r f(x)}{\lambda^r x^r} \;=\; \lambda^0 g(x) \quad \forall \lambda \in \Lambda.$$

Thus g is a boundedly homogeneous function with the set of homogeneity Λ and the degree of homogeneity 0. Moreover, define $f_1(x) := g(e^x)$. If $\lambda \in \Lambda \setminus \{1\}$ and $p := \ln\lambda$, we see that

$$f_1(x+p) \;=\; g(e^x e^p) \;=\; g(e^x\lambda) \;=\; g(e^x) \;=\; f_1(x) \quad \forall x \in \mathbb{R}_+.$$

Therefore, f_1 is periodic and (1) is in .

References

[1] Konrad Schlude: “Bemerkung zu beschränkt homogenen Funktionen”. – *Elemente der Mathematik* **54** (1999).