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proof of rearrangement inequality

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We first prove the rearrangement inequality for the case $n = 2$. Let x_1, x_2, y_1, y_2 be real numbers such that $x_1 \le x_2$ and $y_1 \le y_2$. Then

$$(x_2 - x_1)(y_2 - y_1) \ge 0,$$

and therefore

$$x_1 y_1 + x_2 y_2 \ge x_1 y_2 + x_2 y_1.$$

Equality holds iff $x_1 = x_2$ or $y_1 = y_2$.

For the general case, let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers such that $x_1 \le x_2 \le \cdots \le x_n$. Suppose that $(z_1, z_2, \ldots, z_n)$ is a permutation (rearrangement) of $\{y_1, y_2, \ldots, y_n\}$ such that the sum

$$x_1 z_1 + x_2 z_2 + \cdots + x_n z_n$$

is maximized. If there exists a pair $i < j$ with $z_i > z_j$, then $x_i z_j + x_j z_i \ge x_i z_i + x_j z_j$ (the $n = 2$ case); equality holds iff $x_i = x_j$. Therefore, $x_1 z_1 + x_2 z_2 + \cdots + x_n z_n$ is not maximal unless $z_1 \le z_2 \le \cdots \le z_n$ or $x_i = x_j$ for all pairs $i < j$ such that $z_i > z_j$. In the latter case, we can consecutively interchange these pairs until $z_1 \le z_2 \le \cdots \le z_n$ (this is possible because the number of pairs $i < j$ with $z_i > z_j$ decreases with each step). So $x_1 z_1 + x_2 z_2 + \cdots + x_n z_n$ is maximized if

$$z_1 \le z_2 \le \cdots \le z_n.$$

To show that $x_1 z_1 + x_2 z_2 + \cdots + x_n z_n$ is minimal for a permutation $(z_1, z_2, \ldots, z_n)$ of $\{y_1, y_2, \ldots, y_n\}$ if $z_1 \ge z_2 \ge \cdots \ge z_n$, observe that $-(x_1 z_1 + x_2 z_2 + \cdots + x_n z_n) = x_1(-z_1) + x_2(-z_2) + \cdots + x_n(-z_n)$ is maximized if $-z_1 \le -z_2 \le \cdots \le -z_n$. This implies that $x_1 z_1 + x_2 z_2 + \cdots + x_n z_n$ is minimized if

$$z_1 \ge z_2 \ge \cdots \ge z_n.$$