flux of vector field

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Defines	flux

Let

$$\vec{U} = U_x\vec{i} + U_y\vec{j} + U_z\vec{k}$$

be a vector field in $\mathbb{R}^3$ and let a be a portion of some surface in the vector field. Define one ; if a is a closed surface, then the of it. For any surface element da of a, the corresponding *vectoral surface element* is

$$d\vec{a} = \vec{n}\,da,$$

where $\vec{n}$ is the unit normal vector on the of da.

The *flux* of the vector $\vec{U}$ through the surface a is the

$$\int_a \vec{U} \cdot d\vec{a}.$$

Remark. One can imagine that $\vec{U}$ represents the velocity vector of a flowing liquid; suppose that the flow is , i.e. the velocity $\vec{U}$ depends only on the location, not on the time. Then the scalar product $\vec{U} \cdot d\vec{a}$ is the volume of the liquid flown per time-unit through the surface element da; it is positive or negative depending on whether the flow is from the negative to the positive or contrarily.

Example. Let $\vec{U} = x\vec{i} + 2y\vec{j} + 3z\vec{k}$ and a be the portion of the plane $x + y + x = 1$ in the first octant ($x \geqq 0,\ y \geqq 0,\ z \geqq 0$) with the away from the origin.

One has the constant unit normal vector:

$$\vec{n} = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}.$$

The flux of $\vec{U}$ through a is

$$\varphi = \int_a \vec{U} \cdot d\vec{a} = \frac{1}{\sqrt{3}} \int_a (x + 2y + 3z)\,da.$$

However, this surface integral may be converted to one in which a is replaced by its `http://planetmath.org/ProjectionOfPoint`projection A on the xy-plane, and da is then similarly replaced by its projection dA;

$$dA = \cos\alpha\,da$$

where α is the angle between the normals of both surface elements, i.e. the angle between $\vec{n}$ and $\vec{k}$:

$$\cos\alpha \;=\; \vec{n}\cdot\vec{k} \;=\; \frac{1}{\sqrt{3}}.$$

Then we also express z on a with the coordinates x and y:

$$\varphi \;=\; \frac{1}{\sqrt{3}}\int_A (x{+}2y{+}3(1{-}x{-}y))\,\sqrt{3}\,dA \;=\; \int_0^1\left(\int_0^{1-x}(3-2x-y)\,dy\right)dx \;=\; 1$$