properties of Riemann–Stieltjes integral

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Denote by $R(g)$ the set of bounded real functions which are http://planetmath.org/node/3187
Stieltjes integrable with respect to a given monotonically nondecreasing function g on a given interval.

The http://planetmath.org/node/3187Riemann–Stieltjes integral is a generalisation of the Riemann integral, and both have properties; N.B. however the items 5, 7 and 9.

1. If $f_1,\, f_2 \in R(g)$ on $[a,\, b]$, then also $f_1{+}f_2,\, cf_1 \in R(g)$ on $[a,\, b]$ and $\int_a^b (f_1{+}f_2)dg = \int_a^b f_1\,dg + \int_a^b f_2\,dg, \quad \int_a^b cf_1\,dg = c\int_a^b f_1\,dg.$
2. If $f_1,\, f_2 \in R(g)$ on $[a,\, b]$, then also $f_1 f_2 \in R(g)$ on $[a,\, b]$.
3. If $f_1,\, f_2 \in R(g)$ on $[a,\, b]$ and $\inf\limits_{x\in[a,b]} |f_2(x)| > 0$, then also $\frac{f_1}{f_2} \in R(g)$ on $[a,\, b]$.
4. If $f_1,\, f_2 \in R(g)$ and $f_1 \leq f_2$ on $[a,\, b]$, then $\int_a^b f_1\,dg \leq \int_a^b f_2\,dg.$
5. If $f \in R(g)$ on $[a,\, b]$, and V_g is the total variation of g on $[a,\, b]$, then $\left|\int_a^b f dg\right| \leq \sup\limits_{x\in[a,b]} f(x) \cdot V_g.$
6. If $f \in R(g)$ on $[a,\, b]$, then also $|f| \in R(g)$ on $[a,\, b]$ and $\left|\int_a^b f\,dg\right| \leq \int_a^b |f|\,dg.$
7. If $f \in R(g)$ and $m \leq f(x) \leq M$ on $[a,\, b]$, then $m[g(b)-g(a)] \leq \int_a^b f\,dg \leq M[g(b)-g(a)].$
8. If $f \in R(g)$ on $[a,\, b]$ and on $[b,\, c]$, then also $f \in R(g)$ on $[a,\, c]$ and $\int_a^c f\,dg = \int_a^b f\,dg + \int_b^c f\,dg.$
9. If $f \in R(g_1),\, R(g_2)$ on $[a,\, b]$, then $f \in R(g_1{+}g_2)$ on the same interval and
$\int_a^b f\,d(g_1{+}g_2) = \int_a^b f\,dg_1 + \int_a^b f\,dg_2.$
10. If $f \in R(g)$ on $[a,\, b]$, then $g \in R(f)$ on the same interval and one can integrate by parts:
$\int_a^b f\,dg \;=\; f(b)g(b){-}f(a)g(a){-}\int_a^b g\,df.$