testing continuity via filters

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Proposition 1. *Let X, Y be topological spaces. Then a function $f : X \to Y$ is continuous iff it sends converging filters to converging filters.*

Proof. Suppose first f is continuous. Let $\mathbb{F}$ be a filter in X converging to x. We want to show that $f(\mathbb{F}) := \{f(F) \mid F \in \mathbb{F}\}$ converges to $f(x)$. Let N be a neighborhood of $f(x)$. So there is an open set U such that $f(x) \in U \subseteq N$. So $f^{-1}(U)$ is open and contains x, which means that $f^{-1}(U) \in \mathbb{F}$ by assumption. This means that $ff^{-1}(U) \in f(\mathbb{F})$. Since $ff^{-1}(U) \subseteq U \subseteq N$, we see that $N \in f(\mathbb{F})$ as well.

Conversely, suppose f preserves converging filters. Let V be an open set in Y containing $f(x)$. We want to find an open set U in X containing x, such that $f(U) \subseteq V$. Let $\mathbb{F}$ be the neighborhood filter of x. So $\mathbb{F} \to x$. By assumption, $f(\mathbb{F}) \to f(x)$. Since V is an open neighborhood of $f(x)$, we have $V \in f(\mathbb{F})$, or $f(F) \subseteq V$ for some $F \in \mathbb{F}$. Since F is a neighborhood of x, it contains an open neighborhood U of x. Furthermore, $f(U) \subseteq f(F) \subseteq V$. Since x is arbitrary, f is continuous. $\square$