if $d(x_i, x_{i+1}) < 1/2^i$ then x_i is a Cauchy sequence

Canonical name	IfDxiXi112iThenXiIsACauchySequence
Date of creation	2013-03-22 14:37:31
Last modified on	2013-03-22 14:37:31
Owner	matte (1858)
Last modified by	matte (1858)
Numerical id	5
Author	matte (1858)
Entry type	Result
Classification	msc 26A03
Classification	msc 54E35

Lemma 1. *Suppose $x_1, x_2, \dots,$ is a sequence in a metric space. If for some $N \geq 1$, we have $d(a_i, a_{i+1}) < 1/2^i$ for all $i \geq N$, then $\{x_i\}$ is a Cauchy sequence.*

Proof. Let us denote by d the metric function. If $\varepsilon > 0$, then for some $N \in \mathbb{N}$ we have $1/2^N < \varepsilon$. Thus, if $N < m < n$ we have

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m+1}) + \cdots + d(x_{n-1}, x_n) \\
&= \left(\frac{1}{2}\right)^m + \cdots + \left(\frac{1}{2}\right)^{n-1} \\
&= \left(\frac{1}{2}\right)^{m-1} \sum_{i=1}^{n-m} \left(\frac{1}{2}\right)^i \\
&= \left(\frac{1}{2}\right)^{m-1} \frac{1 - \left(\frac{1}{2}\right)^{n-m}}{1 - \frac{1}{2}} \\
&< \left(\frac{1}{2}\right)^m \\
&< \left(\frac{1}{2}\right)^N \\
&< \varepsilon,
\end{aligned}$$

where we have used the triangle inequality and the `http://planetmath.org/GeometricSeries`geom sum formula. □