semicontinuous

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Defines	lower semicontinuous
Defines	upper semicontinuous
Defines	lower semi-continuous
Defines	upper semi-continuous

Suppose X is a topological space, and f is a function from X into the extended real numbers $\mathbb{R}^*$; $f : X \to \mathbb{R}^*$. Then:

1. If $f^{-1}((\alpha, \infty]) = \{x \in X \mid f(x) > \alpha\}$ is an open set in X for all $\alpha \in \mathbb{R}$, then f is said to be **lower semicontinuous**.

2. If $f^{-1}([-\infty, \alpha)) = \{x \in X \mid f(x) < \alpha\}$ is an open set in X for all $\alpha \in \mathbb{R}$, then f is said to be **upper semicontinuous**.

In other words, f is lower semicontinuous, if f is continuous with respect to the topology for $\mathbb{R}^*$ containing $\emptyset$ and open sets

$$U(\alpha) = (\alpha, \infty], \qquad \alpha \in \mathbb{R} \cup \{-\infty\}.$$

It is not difficult to see that this is a topology. For example, for a union of sets $U(\alpha_i)$ we have $\cup_i U(\alpha_i) = U(\inf \alpha_i)$. Obviously, this topology is much coarser than the usual topology for the extended numbers. However, the sets $U(\alpha)$ can be seen as neighborhoods of infinity, so in some sense, semicontinuous functions are "continuous at infinity" (see example 3 below).

0.0.1 Examples

1. A function $f \colon X \to \mathbb{R}^*$ is continuous if and only if it is lower and upper semicontinuous.

2. Let f be the characteristic function of a set $\Omega \subseteq X$. Then f is lower (upper) semicontinuous if and only if Ω is open (closed). This also holds for the function that equals ∞ in the set and 0 outside.

   It follows that the characteristic function of $\mathbb{Q}$ is not semicontinuous.

3. On $\mathbb{R}$, the function $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$, is not semicontinuous. This example illustrate how semicontinuous "at infinity".

0.0.2 Properties

Let $f \colon X \to \mathbb{R}^*$ be a function.

1. Restricting f to a subspace preserves semicontinuity.

2. Suppose f is upper (lower) semicontinuous, A is a topological space, and $\Psi \colon A \to X$ is a homeomorphism. Then $f \circ \Psi$ is upper (lower) semicontinuous.

3. Suppose f is upper (lower) semicontinuous, and $S\colon \mathbb{R}^* \to \mathbb{R}^*$ is a sense preserving homeomorphism. Then $S \circ f$ is upper (lower) semicontinuous.

4. f is lower semicontinuous if and only if $-f$ is upper semicontinuous.

References

[1] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Inc., 1987.

[2] D.L. Cohn, *Measure Theory*, Birkhäuser, 1980.