proof of Bernoulli's inequality employing the mean value theorem

Canonical name	ProofOfBernoullisInequalityEmployingTheMeanValueTheorem
Date of creation	2013-03-22 15:49:53
Last modified on	2013-03-22 15:49:53
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Numerical id	10
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Entry type	Proof
Classification	msc 26D99

Let us take as our assumption that $x \in I = (-1, \infty)$ and that $r \in J = (0, \infty)$. Observe that if $x = 0$ the inequality holds quite obviously. Let us now consider the case where $x \neq 0$. Consider now the function $f : I\text{x}J \to \mathbb{R}$ given by

$$f(x, r) = (1 + x)^r - 1 - rx$$

Observe that for all r in J fixed, f is, indeed, differentiable on I. In particular,

$$\frac{\partial}{\partial x} f(x, r) = r(1 + x)^{r-1} - r$$

Consider two points $a \neq 0$ in I and 0 in I. Then clearly by the mean value theorem, for any arbitrary, fixed α in J, there exists a c in I such that,

$$f'_x(c, \alpha) = \frac{f(a, \alpha) - f(0, \alpha)}{a}$$

$$\Leftrightarrow f'_x f(c, \alpha) = \frac{(1 + a)^\alpha - 1 - \alpha a}{a} \tag{1}$$

Since α is in J, it is clear that if $a < 0$, then

$$f'_x(a, \alpha) < 0$$

and, accordingly, if $a > 0$ then

$$f'_x(a, \alpha) > 0$$

Thus, in either case, from **??** we deduce that

$$\frac{(1 + a)^\alpha - 1 - \alpha a}{a} < 0$$

if $a < 0$ and

$$\frac{(1 + a)^\alpha - 1 - \alpha a}{a} > 0$$

if $a > 0$. From this we conclude that, in either case,$(1 + a)^\alpha - 1 - \alpha a > 0$. That is,

$$(1 + a)^\alpha > 1 + \alpha a$$

for all choices of a in $I - \{0\}$ and all choices of α in J. If $a = 0$ in I, we have

$$(1 + a)^\alpha = 1 + \alpha a$$

for all choices of α in J. Generally, for all x in I and all r in J we have:

$$(1+x)^r \geq 1+rx$$

This completes the proof.

Notice that if r is in $(-1,0)$ then the inequality would be reversed. That is:

$$(1+x)^r \leq 1+rx$$

. This can be proved using exactly the same method, by fixing α in the proof above in $(-1,0)$.