vector-valued function

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Let n be a positive integer greater than 1. A function F from a subset T of $\mathbb{R}$ to the Cartesian product $\mathbb{R}^n$ is called a *vector-valued function* of one real variable. Such a function to any real number t of T a coordinate vector

$$F(t) \;=\; (f_1(t),\, \ldots,\, f_n(t)).$$

Hence one may say that the vector-valued function F is composed of n real functions $t \mapsto f_i(t)$, the values of which at t are the components of $F(t)$. Therefore the function F itself may be written in the component form

$$F \;=\; (f_1,\, \ldots,\, f_n). \tag{1}$$

Example. The ellipse

$$\{(a\cos t,\, b\sin t) \vdots\; t \in \mathbb{R}\}$$

is the value set of a vector-valued function $\mathbb{R} \to \mathbb{R}^2$ (t is the eccentric anomaly).

Limit, derivative and integral of the function (1) are defined componentwise through the equations

- $\displaystyle\lim_{t\to t_0} F(t) \;:=\; \left(\lim_{t\to t_0} f_1(t),\, \ldots,\, \lim_{t\to t_0} f_n(t)\right)$
- $F'(t) \;:=\; (f_1'(t),\, \ldots,\, f_n'(t))$
- $\displaystyle\int_a^b F(t)\,dt \;:=\; \left(\int_a^b f_1(t)\,dt,\, \ldots,\, \int_a^b f_n(t)\,dt\right)$

The function F is said to be *continuous, differentiable or integrable* on an interval $[a,\, b]$ if every component of F has such a property.

Example. If F is continuous on $[a,\, b]$, the set

$$\gamma \;:=\; \{F(t) \vdots\;\; t \in [a,\, b]\} \tag{2}$$

is a (continuous) curve in $\mathbb{R}^n$. It follows from the above definition of the derivative $F'(t)$ that $F'(t)$ is the limit of the expression

$$\frac{1}{h}[F(t+h) - F(t)] \tag{3}$$

as $h \to 0$. Geometrically, the vector (3) is parallel to the line segment connecting (the end points of the position vectors of) the points $F(t+h)$ and $F(t)$. If F is differentiable in t, the direction of this line segment then tends infinitely the direction of the tangent line of γ in the point $F(t)$. Accordingly, the direction of the tangent line is determined by the derivative vector $F'(t)$.