generalisation of Gaussian integral

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The integral

$$\int_0^\infty e^{-x^2}\cos tx\,dx \;:=\; w(t)$$

is a generalisation of the Gaussian integral $w(0) = \frac{\sqrt{\pi}}{2}$. For evaluating it we first form its derivative which may be done by http://planetmath.org/DifferentiationUnderIn under the integral sign:

$$w'(t) \;=\; \int_0^\infty e^{-x^2}(-x)\sin tx\,dx \;=\; \frac{1}{2}\int_0^\infty e^{-x^2}(-2x)\sin tx\,dx$$

Using integration by parts this yields

$$w'(t) \;=\; \frac{1}{2}\Big/_{x=0}^{\infty} e^{-x^2}\sin tx - \frac{t}{2}\int_0^\infty e^{-x^2}\cos tx\,dx \;=\; \frac{1}{2}(0-0) - \frac{t}{2}\int_0^\infty e^{-x^2}\cos tx\,dx \;=\; -\frac{t}{2}w(t).$$

Thus $w(t)$ satisfies the linear differential equation

$$\frac{dw}{dt} \;=\; -\frac{1}{2}tw,$$

where one can http://planetmath.org/SeparationOfVariablesseparate the variables and integrate:

$$\int\frac{dw}{w} \;=\; -\frac{1}{2}\int t\,dt.$$

So, $\ln w \;=\; -\frac{1}{4}t^2 + \ln C$, i.e. $w \;=\; w(t) \;=\; Ce^{-\frac{1}{4}t^2}$, and since there is the initial condition $w(0) = \frac{\sqrt{\pi}}{2}$, we obtain the result

$$w(t) \;=\; \frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}t^2}.$$