integral over plane region

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The integrals over a planar region are generalisations of usual Riemann integrals, but special cases of http://planetmath.org/node/6660surface integrals.

0.1 Integral over a rectangle

Let R be the rectangle of xy-plane defined by

$$a \leqq x \leqq b, \quad c \leqq y \leqq d \tag{1}$$

and the function f be defined and bounded in R. Let

$$D : \begin{cases} x_0 = a,\, x_1,\, \ldots,\, x_m = b \\ y_0 = c,\, y_1,\, \ldots,\, y_n = d \end{cases} \tag{2}$$

a of R into the rectangular parts Δ_i with areas $\Delta_i A$ $(i = 1,\, \ldots,\, mn)$. Denote

$$m_i \;:=\; \inf_{\Delta_i} f(x,\, y), \qquad M_i \;:=\; \sup_{\Delta_i} f(x,\, y)$$

and

$$s_D \;:=\; \sum_D m_i \Delta_i A, \qquad S_D \;:=\; \sum_D M_i \Delta_i A.$$

Definition 1. If $\sup_D\{s_D\} = \inf_D\{S_D\}$, then we say that f is *integrable over* R and call the common value the (*Riemann*) *integral of* f *over the rectangle* R and denote it by

$$\int_R f, \quad \int_R f(x,\, y)\, dx\, dy \quad \text{or} \quad \iint_R f(x,\, y)\, dx\, dy.$$

Let then f be defined in a region A of xy-plane such that that it can be enclosed in a rectangle R defined by (1). Define the new function f_1 through

$$f_1(x,\, y) \;:=\; \begin{cases} f(x,\, y) & \text{when } (x,\, y) \in A, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Definition 2. If f_1 is integrable over the rectangle R, we say that f is *integrable over* A and define

$$\int_A f \;:=\; \int_R f_1. \tag{4}$$

It's apparent that (4) is on the choice of R since the points of $\mathbb{R}^2 \smallsetminus A$ give zero-terms to the lower and upper sums.

0.2 Double integrals

Definition 3. Let f be bounded in R as before. Suppose that

$$\varphi(x) \;:=\; \int_c^d f(x,\, y)\, dy$$

is defined on $[a,\, b]$. If also the integral

$$\int_a^b \varphi(x)\, dx \;=\; \int_a^b \left[\int_c^d f(x,\, y)\, dy\right] dx \tag{5}$$

exists, it is called a *double integral* or *iterated integral* and denoted by

$$\int_a^b dx \int_c^d f(x,\, y)\, dy.$$

One may prove the

Theorem. $\displaystyle\int_R f(x,\, y)\, dx\, dy \;=\; \int_a^b dx \int_c^d f(x,\, y)\, dy,$ provided that the integral of the left side exists and that the inner integral $\int_c^d f(x,\, y)\, dy$ of the right side exists for every x in $[a,\, b]$.

It's clear that $\displaystyle\int_a^b dx \int_c^d f(x,\, y)\, dy \;=\; \int_c^d dy \int_a^b f(x,\, y)\, dx$ if also the integral $\int_a^b f(x,\, y)\, dx$ exists for every y in $[c,\, d]$. If especially the function f is continuous in the rectangle R, then surely

$$\int_R f(x,\, y)\, dx\, dy \;=\; \int_a^b dx \int_c^d f(x,\, y)\, dy \;=\; \int_c^d dy \int_a^b f(x,\, y)\, dx.$$

Assume now, that f is defined and bounded in the region

$$A \;:=\; \{(x,\,y) \in \mathbb{R}^2\colon\, a \leqq x \leqq b,\; h_1(x) \leqq y \leqq h_2(x)\}$$

where A is contained in the rectangle R determined by (1). Then the planar integral $\displaystyle\int_A f$ is defined as

$$\int_A f \;=\; \int_R f$$

if the integral of right side exists. For this, he continuity of f in A does not necessarily suffice, because f_1 may have a jump discontinuity on the border of A whence the integrability of f_1 needs not be guaranteed. One case where the integrability is true is that the graphs of the functions h_1 and h_2 are rectifiable (i.e. the functions have continuous derivatives). For a continuous f, we then have

$$\int_c^d f_1(x,\,y)\,dy \;=\; \int_c^{h_1(x)} 0\,dy + \int_{h_1(x)}^{h_2(x)} f(x,\,y)\,dy + \int_{h_2(x)}^{d} 0\,dy \;=\; \int_{h_1(x)}^{h_2(x)} f(x,\,y)\,dy.$$

Thus

$$\int_A f \;=\; \int_R f_1 \;=\; \int_a^b dx \int_{h_1(x)}^{h_2(x)} f(x,\,y)\,dy, \tag{6}$$

i.e. the planar integral has been expressed as a double integral.

[Not ready ...]