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polyrectangle

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Defines	Riemann sums on polyrectangles
Defines	compact rectangle

A *polyrectangle* P in $\mathbb{R}^n$ is a finite collection $P = \{R_1, \dots, R_N\}$ of compact rectangles $R_i \subset \mathbb{R}^n$ with disjoint interior. A *compact rectangle* R_i is a Cartesian product of compact intervals: $R_i = [a^i_1, b^i_1] \times \cdots \times [a^i_n, b^i_n]$ where $a^i_j < b^i_j$ (these are also called *n-dimensional intervals*).

The union of the compact rectangles of a polyrectangle P is denoted by

$$\cup P := \bigcup_{R \in P} R = R_1 \cup \cdots \cup R_N.$$

It is a compact subset of $\mathbb{R}^n$.

We can define the (n-dimensional) measure of $\cup P$ in a way. If $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a rectangle we define the measure of R as

$$\operatorname{meas}(R) := (b_1 - a_1) \cdots (b_n - a_n)$$

and define the measure of the polyrectangle P as:

$$\operatorname{meas}(P) := \sum_{R \in P} \operatorname{meas}(R).$$

Moreover if we are given a bounded function $f\colon \cup P \to \mathbb{R}$ we can define the *upper* and *lower Riemann sums* of f over $\cup P$ by

$$S^*(f, P) := \sum_{R \in P} \operatorname{meas}(R) \sup_{x \in R} f(x), \qquad S_*(f, P) := \sum_{R \in P} \operatorname{meas}(R) \inf_{x \in R} f(x).$$

Polyrectangles are then used to define the Peano Jordan measure of subsets of $\mathbb{R}^n$ and to define Riemann multiple integrals. To achieve this, it is useful to introduce the so called *refinements*. The family of rectangles R_i which appear in the definition **??** are called a *partition* of $\overline{\cup P}$ in rectangles. It is clear that the set $\cup P$ can be represented by different polyrectangles. For example any rectangle R can be split in 2^n smaller rectangles by dividing in two parts each of the n intervals defining R. We claim that given two polyrectangles P and Q there exists a polyrectangle S such that $(\cup P) \cup (\cup Q) \subset \cup S$ and such that given any rectangle R in P or Q, R is the union of rectangles in S.