proof of De l'Hôpital's rule

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Owner	paolini (1187)
Last modified by	paolini (1187)
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Author	paolini (1187)
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Let $x_0 \in \mathbb{R}$, I be an interval containing x_0 and let f and g be two differentiable functions defined on $I \setminus \{x_0\}$ with $g'(x) \neq 0$ for all $x \in I$. Suppose that

$$\lim_{x \to x_0} f(x) = 0, \quad \lim_{x \to x_0} g(x) = 0$$

and that

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = m.$$

We want to prove that hence $g(x) \neq 0$ for all $x \in I \setminus \{x_0\}$ and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = m.$$

First of all (with little abuse of notation) we suppose that f and g are defined also in the point x_0 by $f(x_0) = 0$ and $g(x_0) = 0$. The resulting functions are continuous in x_0 and hence in the whole interval I.

Let us first prove that $g(x) \neq 0$ for all $x \in I \setminus \{x_0\}$. If by contradiction $g(\bar{x}) = 0$ since we also have $g(x_0) = 0$, by Rolle's Theorem we get that $g'(\xi) = 0$ for some $\xi \in (x_0, \bar{x})$ which is against our hypotheses.

Consider now any sequence $x_n \to x_0$ with $x_n \in I \setminus \{x_0\}$. By Cauchy's mean value Theorem there exists a sequence x'_n such that

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f'(x'_n)}{g'(x'_n)}.$$

But as $x_n \to x_0$ and since $x'_n \in (x_0, x_n)$ we get that $x'_n \to x_0$ and hence

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = \lim_{n \to \infty} \frac{f'(x_n)}{g'(x_n)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = m.$$

Since this is true for any given sequence $x_n \to x_0$ we conclude that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = m.$$