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gradient in curvilinear coordinates

Canonical name	GradientInCurvilinearCoordinates
Date of creation	2013-03-22 15:27:32
Last modified on	2013-03-22 15:27:32
Owner	stevecheng (10074)
Last modified by	stevecheng (10074)
Numerical id	5
Author	stevecheng (10074)
Entry type	Result
Classification	msc 26B12
Classification	msc 26B10
Related topic	gradient
Related topic	Gradient

We give the formulas for the gradient expressed in various curvilinear coordinate systems. We also show the metric tensors g_{ij} so that the reader may verify the results by working from the basic formulas for the gradient.

Contents

1 Cylindrical coordinate system

In the cylindrical system of coordinates (r, θ, z) we have

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

So that

$$\nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{k}\,,$$

where

$$\begin{aligned} \mathbf{e}_r &= \frac{\partial}{\partial r} = \frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} \\ \mathbf{e}_\theta &= \frac{1}{r}\frac{\partial}{\partial \theta} = -\frac{y}{r}\,\mathbf{i} + \frac{x}{r}\,\mathbf{j} \end{aligned}$$

are the unit vectors in the direction of increase of r and θ. Of course, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the unit vectors along the positive x, y, z axes respectively.

The notations $\partial/\partial r, \partial/\partial\theta$, etc., denote the tangent vectors corresponding to infinitesimal changes in r, θ, etc. respectively. Concretely, in terms of Cartesian coordinates, $\partial/\partial r$ is the vector $\mathbf{i}\,\partial x/\partial r + \mathbf{j}\,\partial y/\partial r + \mathbf{k}\,\partial z/\partial r$. And similarly for the other variables. (There is a deep reason for using the seemingly strange notation: see Leibniz notation for vector fields for details.)

2 Polar coordinate system

This is just the special case of the cylindrical coordinate system where we chop off the z coordinate. Thus

$$\nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta\,.$$

3 Spherical coordinate system

To stave off confusion, note that this is the "mathematicians' " convention for the spherical coordinate system (ρ, ϕ, θ). That is, ϕ is the co-latitude angle, and θ is the longitudinal angle.

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin^2 \phi \end{pmatrix}.$$

$$\nabla f = \frac{\partial f}{\partial \rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\,\mathbf{e}_\phi + \frac{1}{\rho \sin \phi}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta\,,$$

where

$$\begin{aligned} \mathbf{e}_\rho &= \frac{\partial}{\partial \rho} = \frac{x}{\rho}\,\mathbf{i} + \frac{y}{\rho}\,\mathbf{j} + \frac{z}{\rho}\,\mathbf{k} \\ \mathbf{e}_\phi &= \frac{1}{\rho}\frac{\partial}{\partial \phi} = \frac{zx}{r\rho}\,\mathbf{i} + \frac{zy}{r\rho}\,\mathbf{j} - \frac{r}{\rho}\,\mathbf{k} \\ \mathbf{e}_\theta &= \frac{1}{\rho \sin \theta}\frac{\partial}{\partial \theta} = -\frac{y}{r}\,\mathbf{i} + \frac{x}{r}\,\mathbf{j} \end{aligned}$$

are the unit vectors in the direction of increase of ρ, ϕ, θ, respectively.