# Gauss Green theorem

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**Theorem 1 (Gauss-Green)** *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with $C^1$ boundary, let $\nu_\Omega \colon \partial\Omega \to \mathbf{R}^n$ be the exterior unit normal vector to $\Omega$ in the point $x$ and let $f \colon \overline{\Omega} \to \mathbf{R}^n$ be a vector function in $C^0(\overline{\Omega}, \mathbf{R}^n) \cap C^1(\Omega, \mathbf{R}^n)$. Then*

$$\int_\Omega \mathrm{div} f(x)\, dx = \int_{\partial\Omega} \langle f(x), \nu_\Omega(x) \rangle \, d\sigma(x).$$

Some remarks on notation. The operator $\mathrm{div} f$ is the divergence of the vector field $f$, which is sometimes written as $\nabla \cdot f$. In the right-hand side we have a surface integral, $d\sigma$ is the corresponding area measure on $\partial\Omega$. The scalar product in the second integral is sometimes written as $f_n(x)$ and represents the *normal component* of $f$ with respect to $\partial\Omega$; hence the whole integral represents the *flux* of the vector field $f$ through $\partial\Omega$;

This theorem can be easily extended to *piecewise* regular domains. However the more general statement of this Theorem involves the theory of *perimeters* and $BV$ functions.

**Theorem 2 (generalized Gauss-Green)** *Let $E \subset \mathbf{R}^n$ be any measurable set. Then*

$$\int_E \mathrm{div} f(x)\, dx = \int_{\partial^* E} \langle \nu_E(x), f(x) \rangle \, d\mathcal{H}^{n-1}(x)$$

*holds for every continuously differentiable function $f \colon \mathbf{R}^n \to \mathbf{R}^n$ with compact support (i.e. $f \in \mathcal{C}^1_c(\mathbf{R}^n, \mathbf{R}^n)$) where*

- *$\partial^* E$ is the* essential boundary *of $E$ which is a subset of the topological boundary $\partial E$;*
- *$\nu_E(x)$ is the exterior normal vector to $E$, which is defined when $x \in \mathcal{F}E$;*
- *$\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.*