rigorous definition of trigonometric functions

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It is possible to define the trigonometric functions rigorously by means of a process based upon the angle addition identities. A sketch of how this is done is provided below.

To begin, define a sequence $\{c_n\}_{n=1}^{\infty}$ by the initial condition $c_1 = 1$ and the recursion

$$c_{n+1} = 1 - \sqrt{1 - \frac{c_n}{2}}.$$

Likewise define a sequence $\{s_n\}_{n=1}^{\infty}$ by the conditions $s_1 = 1$ and

$$s_{n+1} = \sqrt{\frac{c_n}{2}}.$$

(In both equations above, we take the positive square root.) It may be shown that both of these sequences are strictly decreasing and approach 0.

Next, define a sequence of 2×2 matrices as follows:

$$m_n = \begin{pmatrix} 1 - c_n & s_n \\ -s_n & 1 - c_n \end{pmatrix}$$

Using the recursion relations which define c_n and s_n, it may be shown that $m_{n+1}^2 = m_n$, More grenerally, using induction, this can be generalised to $m_{n+k}^{2^k} = m_n$.

It is easy to check that the product of any two matrices of the form

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

is of the same form. Hence, for any integers k and n, the matrix m_n^k will be of this form. We can therefore define functions S and C from rational numbers whose denominator is a power of two to real numbers by the following equation:

$$\begin{pmatrix} C\left(\frac{n}{2^k}\right) & S\left(\frac{n}{2^k}\right) \\ -S\left(\frac{n}{2^k}\right) & C\left(\frac{n}{2^k}\right) \end{pmatrix} = \begin{pmatrix} 1 - c_k & s_k \\ -s_k & 1 - c_k \end{pmatrix}^n.$$

From the recursion relations, we may prove the following identities:

$$\begin{aligned} S^2(r) + C^2(r) &= 1 \\ S(p+q) &= S(p)C(q) + S(q)C(p) \\ C(p+q) &= C(p)C(q) - S(p)S(q) \end{aligned}$$

From the fact that $c_n \to 0$ and $s_n \to 0$ as $n \to \infty$, it follows that, if $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ are two sequences of rational numbers whose denominators are powers of two such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n$, then $\lim_{n\to\infty} C(p_n) = \lim_{n\to\infty} C(q_n)$ and $\lim_{n\to\infty} S(p_n) = \lim_{n\to\infty} S(q_n)$. Therefore, we may define functions by the conditions that, for any convergent series of rational numbers $\{r_n\}_{n=0}^{\infty}$ whose denominators are powers of two,

$$\cos\left(\pi \lim_{n\to\infty} r_n\right) = \lim_{n\to\infty} C(r_n)$$

and

$$\sin\left(\pi \lim_{n\to\infty} r_n\right) = \lim_{n\to\infty} S(r_n).$$

By continuity, we see that these functions satisfy the angle addition identities.

Application. Let us use the definitions above to find $\sin(\frac{\pi}{2})$ and $\cos(\frac{\pi}{2})$. Let $r_i := \frac{1}{2}$ for every positive integer i. Then we need to find $C(\frac{1}{2})$ and $S(\frac{1}{2})$. We use the matrix above defining C and S, and set $n = k = 1$:

$$\begin{pmatrix} C\left(\frac{1}{2}\right) & S\left(\frac{1}{2}\right) \\ -S\left(\frac{1}{2}\right) & C\left(\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1-c_1 & s_1 \\ -s_1 & 1-c_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As a result, $\cos(\frac{\pi}{2}) = \cos(\pi \lim_{i\to\infty} \frac{1}{2}) = \lim_{i\to\infty} C(\frac{1}{2}) = C(\frac{1}{2}) = 0$. Similarly, $\sin(\frac{\pi}{2}) = 1$.