derivation of zeroth weighted power mean

Canonical name	DerivationOfZerothWeightedPowerMean
Date of creation	2013-03-22 13:10:29
Last modified on	2013-03-22 13:10:29
Owner	pbruin (1001)
Last modified by	pbruin (1001)
Numerical id	6
Author	pbruin (1001)
Entry type	Derivation
Classification	msc 26B99
Related topic	PowerMean
Related topic	GeometricMean
Related topic	GeneralMeansInequality
Related topic	DerivationOfHarmonicMeanAsTheLimitOfThePowerMean

Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let $w_1, w_2, \ldots, w_n$ be positive real numbers such that $w_1 + w_2 + \cdots + w_n = 1$. For $r \neq 0$, the r-th weighted power mean of $x_1, x_2, \ldots, x_n$ is

$$M_w^r(x_1, x_2, \ldots, x_n) = (w_1 x_1^r + w_2 x_2^r + \cdots + w_n x_n^r)^{1/r}.$$

Using the Taylor series expansion $e^t = 1 + t + \mathcal{O}(t^2)$, where $\mathcal{O}(t^2)$ is Landau notation for terms of order t^2 and higher, we can write x_i^r as

$$x_i^r = e^{r \log x_i} = 1 + r \log x_i + \mathcal{O}(r^2).$$

By substituting this into the definition of M_w^r, we get

$$\begin{aligned} M_w^r(x_1, x_2, \ldots, x_n) &= \left[w_1(1 + r \log x_1) + \cdots + w_n(1 + r \log x_n) + \mathcal{O}(r^2)\right]^{1/r} \\ &= \left[1 + r(w_1 \log x_1 + \cdots + w_n \log x_n) + \mathcal{O}(r^2)\right]^{1/r} \\ &= \left[1 + r \log(x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}) + \mathcal{O}(r^2)\right]^{1/r} \\ &= \exp\left\{\frac{1}{r} \log\left[1 + r \log(x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}) + \mathcal{O}(r^2)\right]\right\}. \end{aligned}$$

Again using a Taylor series, this time $\log(1 + t) = t + \mathcal{O}(t^2)$, we get

$$\begin{aligned} M_w^r(x_1, x_2, \ldots, x_n) &= \exp\left\{\frac{1}{r}\left[r \log(x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}) + \mathcal{O}(r^2)\right]\right\} \\ &= \exp\left[\log(x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}) + \mathcal{O}(r)\right]. \end{aligned}$$

Taking the limit $r \to 0$, we find

$$\begin{aligned} M_w^0(x_1, x_2, \ldots, x_n) &= \exp\left[\log(x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n})\right] \\ &= x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}. \end{aligned}$$

In particular, if we choose all the weights to be $\frac{1}{n}$,

$$M^0(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n},$$

the geometric mean of $x_1, x_2, \ldots, x_n$.