directional derivative

Canonical name	DirectionalDerivative
Date of creation	2013-03-22 11:58:37
Last modified on	2013-03-22 11:58:37
Owner	matte (1858)
Last modified by	matte (1858)
Numerical id	15
Author	matte (1858)
Entry type	Definition
Classification	msc 26B12
Classification	msc 26B10
Synonym	derivative with respect to a vector
Synonym	partial derivative with respect to a vector
Related topic	PartialDerivative
Related topic	Derivative
Related topic	DerivativeNotation
Related topic	JacobianMatrix
Related topic	Gradient
Related topic	FixedPointsOfNormalFunctions
Related topic	HessianMatrix

Let U be an open set in $\mathbb{R}^n$ and $f\colon U \to \mathbb{C}$ is a differentiable function. If $u \in U$ and $v \in \mathbb{R}^n$, then the *directional derivative* of f in the direction of v is

$$(D_v f)(u) = \frac{d}{ds} f(u + sv)\Big|_{s=0}.$$

In other words, $(D_v f)(u)$ measures how f changes in the direction of v from u.

Alternatively,

$$\begin{aligned} (D_v f)(u) &= \lim_{h \to 0} \frac{f(u + hv) - f(u)}{h} \\ &= Df(u) \cdot v, \end{aligned}$$

where Df is the Jacobian matrix of f.

Properties

Let $u \in U$.

1. $D_v f$ is linear in v. If $v, w \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$, then

   $$D_{\lambda v + \mu w} f(u) = \lambda D_v f(u) + \mu D_w f(u).$$

   In particular, $D_0 f = 0$.

2. If f is twice differentiable and $v, w \in \mathbb{R}^n$, then

   $$\begin{aligned} D_v D_w f(u) &= \frac{\partial^2}{\partial s \partial t} f(u + sv + tw)\Big|_{s=0}, \\ &= v^T \cdot \operatorname{Hess} f(u) \cdot w, \end{aligned}$$

   where Hess is the Hessian matrix of f.

Example

For example, if $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + 3y^2z$, and we wanted to find the derivative at the point $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in the direction $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, our equation would be

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h}\big((1+h)^2 + 3(2+h)^2(3+h) - 37\big) &= \lim_{h\to 0} \frac{1}{h}(3h^3 + 37h^2 + 50h) \\ &= \lim_{h\to 0} 3h^2 + 37h + 50 = 50 \end{aligned}$$