



planetmath.org

Math for the people, by the people.

a lecture on the partial fraction decomposition method

Canonical name	ALectureOnThePartialFractionDecompositionMethod
Date of creation	2013-03-22 15:39:12
Last modified on	2013-03-22 15:39:12
Owner	alozano (2414)
Last modified by	alozano (2414)
Numerical id	5
Author	alozano (2414)
Entry type	Feature
Classification	msc 26A42
Classification	msc 28-00
Related topic	ALectureOnIntegrationBySubstitution
Related topic	ALectureOnIntegrationByParts
Related topic	ALectureOnTrigonometricIntegralsAndTrigonometricSubstitution
Related topic	PartialFractionsOfExpressions
Related topic	PartialFractionsForPolynomials

1 Integrating Rational Functions

A rational function is a function of the form $y = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials (with real coefficients). Here we are interested in how to solve the integral

$$\int \frac{p(x)}{q(x)} dx$$

We already know how to integrate some functions of this type. As we know, the chain rule tells us that the derivative of $y = \ln(g(x))$ is $y' = \frac{g'(x)}{g(x)}$, where $g(x)$ is any other function. Therefore:

$$\int \frac{g'(x)}{g(x)} dx = \ln |g(x)| + C. \quad (1)$$

Example 1.1.

$$\int \frac{2x+1}{x^2+x+1} dx = \ln |x^2+x+1| + C.$$

Example 1.2. Equation (??) may also be used to integrate any function of the form $y = \frac{a}{bx+c}$, for any constants a, b, c . Indeed:

$$\int \frac{a}{bx+c} dx = \frac{a}{b} \int \frac{b}{bx+c} dx = \frac{a}{b} \ln |bx+c| + C$$

or alternatively use a substitution $u = bx + c$.

Example 1.3. Using a substitution we may also integrate any function of the form $y = \frac{a}{(bx+c)^n}$, namely $u = bx + c$ does the job. For example, using $u = 2x + 1$, $du = 2dx$:

$$\int \frac{3}{(2x+1)^4} dx = \frac{3}{2} \int \frac{1}{u^4} du = -\frac{1}{2} \frac{1}{u^3} + C = -\frac{1}{2(2x+1)^3} + C.$$

Example 1.4. The derivative of the arc tangent function, $y = \arctan x$, is $y' = \frac{1}{1+x^2}$. Therefore:

$$\int \frac{1}{1+x^2} dx = \arctan x + C.$$

Thus, for any constant a , using a substitution $x = au$ one may integrate

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C.$$

You may also use the arctangent function to integrate other functions, by completing the square of the denominator. For example, $x^2 + 2x + 2 = 1 + (x + 1)^2$, thus:

$$\int \frac{1}{x^2 + 2x + 2} dx = \int \frac{1}{1 + (x + 1)^2} dx = \arctan(x + 1) + C.$$

Example 1.5. The arctangent also allows us to integrate another family of functions, namely:

$$\int \frac{bx + c}{a^2 + x^2} dx.$$

The trick is to use the favorite strategy of Napoleon: *divide and conquer*, i.e. we break the fraction into a sum of two (here we pick $a = 1$ for simplicity):

$$\int \frac{bx + c}{1 + x^2} dx = \int \frac{bx}{1 + x^2} dx + \int \frac{c}{1 + x^2} dx = \frac{b}{2} \ln |1 + x^2| + c \arctan x + C.$$

2 Partial Fraction Decomposition

The objective of this method is to reduce any integral of the type $\int \frac{p(x)}{q(x)} dx$ to a sum of integrals of the types described in the previous sections. For example:

Example 2.1. We would like to solve the following integral:

$$\int \frac{2}{x^2 + 3x - 4} dx.$$

First, we factor the denominator:

$$x^2 + 3x - 4 = (x + 4)(x - 1)$$

In order to integrate, we are going to express the fraction in the integrand as a sum:

$$\frac{2}{x^2 + 3x - 4} = \frac{A}{x + 4} + \frac{B}{x - 1}$$

for some constants A, B to be determined. The right hand side (after realizing a common denominator) adds up to $\frac{A(x-1)+B(x+4)}{(x+4)(x-1)}$. Therefore, in order to have an equality we need the numerators to be equal:

$$A(x-1) + B(x+4) = 2 \quad (2)$$

for **all** values of x (i.e. this should be an equality of polynomials). Thus, we can substitute values of x and obtain equations relating A and B and hopefully we'll be able to determine the value of the constants. The easiest values to pick are the roots of the denominator of the original fraction. In this case, when we plug $x = 1$ in Eq. (??) we get $5B = 2$, and so $B = 2/5$. When we plug $x = -4$ we obtain $-5A = 2$ and so $A = -2/5$, and we are done! Now we can finish the integral:

$$\begin{aligned} \int \frac{2}{x^2 + 3x - 4} dx &= \int \frac{-2/5}{x+4} dx + \int \frac{2/5}{x-1} dx \\ &= -\frac{2}{5} \ln|x+4| + \frac{2}{5} \ln|x-1| + C = \frac{2}{5} \ln \left| \frac{x-1}{x+4} \right| + C. \end{aligned}$$

The method. The goal of the method, as explained above, is to express any fraction $\frac{p(x)}{q(x)}$ as the sum of *partial fractions* of the types discussed in the previous section.

1. If the degree of $p(x)$ is larger than the degree of $q(x)$ then use polynomial division to divide and obtain a quotient $t(x)$ and remainder $r(x)$ polynomials such that $p(x) = q(x)t(x) + r(x)$. Thus

$$\frac{p(x)}{q(x)} = t(x) + \frac{r(x)}{q(x)}$$

where the degree of r is lower than the degree of q . Now use the partial fraction decomposition with $r(x)/q(x)$.

2. Factor the denominator, $q(x)$, into irreducible polynomials (over \mathbb{R}). Thus, we may express $q(x)$ as a product of linear polynomials (perhaps to a power other than 1) and irreducible quadratic polynomials.
3. If a linear factor $(x-a)$ to the **first** power appears in the denominator of $q(x)$, the partial fraction decomposition should have a term $\frac{A}{(x-a)}$, for some constant A to be determined.

4. If a linear factor to the **n**th power, say $(x - b)^n$ appears in the denominator of $q(x)$, the partial fraction decomposition should have terms

$$\frac{B_1}{(x - b)} + \frac{B_2}{(x - b)^2} + \cdots + \frac{B_n}{(x - b)^n}$$

for some n constants B_1, B_2, \dots, B_n to be determined.

5. If a quadratic polynomial $ax^2 + bx + c$ to the n th power appears in the factorization of $q(x)$, i.e. $(ax^2 + bx + c)^n$ is a factor of $q(x)$, then the partial fraction decomposition should have terms

$$\frac{C_1x + D_1}{ax^2 + bx + c} + \frac{C_2x + D_2}{(ax^2 + bx + c)^2} + \cdots + \frac{C_nx + D_n}{(ax^2 + bx + c)^n}$$

for some constants C_i, D_i to be determined.

6. Once you have the sum of all appropriate partial fractions (see above), group together all these partial fractions into one fraction $\frac{P(x)}{q(x)}$ (use a minimum common multiple for the denominator! The minimum common multiple will actually be $q(x)$). In order to have an equality you need to find appropriate constants A, B, \dots such that $p(x) = P(x)$. For this, plug values of x to obtain equations relating the constants.

Example 2.2. Suppose we want to find the partial fraction decomposition of:

$$\frac{3x + 2}{(x - 1)(x - 2)(x - 3)^2(1 + x^2)}$$

Then, we need constants A, B, C, D, E, F such that

$$\frac{3x + 2}{(x - 1)(x - 2)(x - 3)^2(1 + x^2)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3} + \frac{D}{(x - 3)^2} + \frac{Ex + F}{1 + x^2}.$$

The next step would be to add up all the partial fractions into one big fraction

$$\frac{P(x)}{(x - 1)(x - 2)(x - 3)^2(1 + x^2)}$$

and find constants A, B, C, D, E, F such that $P(x) = 3x + 2$ for all x .