$\sim$ is an equivalence relation

Canonical name	simIsAnEquivalenceRelation
Date of creation	2013-03-22 16:13:16
Last modified on	2013-03-22 16:13:16
Owner	Wkbj79 (1863)
Last modified by	Wkbj79 (1863)
Numerical id	9
Author	Wkbj79 (1863)
Entry type	Result
Classification	msc 26A12

Note that $\sim$ as defined in the entry Landau notation is an equivalence relation on the set of all functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. This set of functions will be denoted in this entry as F.

http://planetmath.org/ReflexiveReflexive: For any $f \in F$, $\displaystyle\lim_{x\to\infty} \frac{f(x)}{f(x)} = 1$, and $f \sim f$.

Symmetric: If $f, g \in F$ with $f \sim g$, then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Thus:

$$\begin{aligned}
\lim_{x\to\infty} \frac{g(x)}{f(x)} &= \lim_{x\to\infty} \frac{1}{\left(\frac{f(x)}{g(x)}\right)} \\
&= \frac{1}{1} \\
&= 1
\end{aligned}$$

Therefore, $g \sim f$.

`http://planetmath.org/Transitive3`*Transitive:* If $f, g, h \in F$ with $f \sim g$ and $g \sim h$, then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\displaystyle\lim_{x\to\infty} \frac{g(x)}{h(x)} = 1$. Thus:

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{h(x)} &= \lim_{x\to\infty} \left(\frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)}\right) \\
&= 1 \cdot 1 \\
&= 1
\end{aligned}$$

Therefore, $f \sim h$.