proof of the fundamental theorem of calculus

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Recall that a continuous function is Riemann integrable on every interval $[c, x]$, so the integral

$$F(x) = \int_c^x f(t)\,dt$$

is well defined.

Consider the increment of F:

$$F(x+h) - F(x) = \int_c^{x+h} f(t)\,dt - \int_c^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

(we have used the linearity of the integral with respect to the function and the additivity with respect to the domain).

Since f is continuous, by the mean-value theorem, there exists $\xi_h \in [x, x+h]$ such that $f(\xi_h) = \frac{F(x+h)-F(x)}{h}$ so that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} f(\xi_h) = f(x)$$

since $\xi_h \to x$ as $h \to 0$. This proves the first part of the theorem.

For the second part suppose that G is any antiderivative of f, i.e. $G' = f$. Let F be the integral function

$$F(x) = \int_a^x f(t)\,dt.$$

We have just proven that $F' = f$. So $F'(x) = G'(x)$ for all $x \in [a, b]$ or, which is the same, $(G - F)' = 0$. This means that $G - F$ is constant on $[a, b]$ that is, there exists k such that $G(x) = F(x) + k$. Since $F(a) = 0$ we have $G(a) = k$ and hence $G(x) = F(x) + G(a)$ for all $x \in [a, b]$. Thus

$$\int_a^b f(t)\,dt = F(b) = G(b) - G(a).$$