continuous

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Let X and Y be topological spaces. A function $f\colon X \to Y$ is *continuous* if, for every open set $U \subset Y$, the inverse image $f^{-1}(U)$ is an open subset of X.

In the case where X and Y are metric spaces (e.g. Euclidean space, or the space of real numbers), a function $f\colon X \to Y$ is continuous if and only if for every $x \in X$ and every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that whenever a point $z \in X$ has distance less than δ to x, the point $f(z) \in Y$ has distance less than ϵ to $f(x)$.

Continuity at a point

A related notion is that of local continuity, or continuity at a point (as opposed to the whole space X at once). When X and Y are topological spaces, we say f is *continuous at a point* $x \in X$ if, for every open subset $V \subset Y$ containing $f(x)$, there is an open subset $U \subset X$ containing x whose image $f(U)$ is contained in V. Of course, the function $f\colon X \to Y$ is continuous in the first sense if and only if f is continuous at every point $x \in X$ in the second sense (for students who haven't seen this before, proving it is a worthwhile exercise).

In the common case where X and Y are metric spaces (e.g., Euclidean spaces), a function f is continuous at $x \in X$ if and only if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ satisfying the property that $d_Y(f(x), f(z)) < \epsilon$ for all $z \in X$ with $d_X(x, z) < \delta$. Alternatively, the function f is continuous at $a \in X$ if and only if the limit of $f(x)$ as $x \to a$ satisfies the equation

$$\lim_{x \to a} f(x) = f(a).$$