proof of Green's theorem

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Consider the region R bounded by the closed curve P in a simply connected space. P can be given by a vector valued function $\vec{F}(x,y) = (f(x,y), g(x,y))$. The region R can then be described by

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R \frac{\partial g}{\partial x}\, dA - \iint_R \frac{\partial f}{\partial y}\, dA$$

The double integrals above can be evaluated separately. Let's look at

$$\iint_R \frac{\partial g}{\partial x}\, dA = \int_a^b \int_{A(y)}^{B(y)} \frac{\partial g}{\partial x}\, dxdy$$

Evaluating the above double integral, we get

$$\int_a^b (g(A(y),y) - g(B(y),y))\, dy = \int_a^b g(A(y),y)\, dy - \int_a^b g(B(y),y)\, dy$$

According to the fundamental theorem of line integrals, the above equation is actually equivalent to the evaluation of the line integral of the function $\vec{F_1}(x,y) = (0, g(x,y))$ over a path $P = P_1 + P_2$, where $P_1 = (A(y), y)$ and $P_2 = (B(y), y)$.

$$\int_a^b g(A(y),y)\, dy - \int_a^b g(B(y),y)\, dy = \int_{P_1} \vec{F_1} \cdot d\vec{t} + \int_{P_2} \vec{F_1} \cdot d\vec{t} = \oint_P \vec{F_1} \cdot d\vec{t}$$

Thus we have

$$\iint_R \frac{\partial g}{\partial x}\, dA = \oint_P \vec{F_1} \cdot d\vec{t}$$

By a similar argument, we can show that

$$\iint_R \frac{\partial f}{\partial y}\, dA = -\oint_P \vec{F_2} \cdot d\vec{t}$$

where $\vec{F_2} = (f(x,y), 0)$. Putting all of the above together, we can see that

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \oint_P \vec{F_1} \cdot d\vec{t} + \oint_P \vec{F_2} \cdot d\vec{t} = \oint_P (\vec{F_1} + \vec{F_2}) \cdot d\vec{t} = \oint_P (f(x,y), g(x,y)) \cdot d\vec{t}$$

which is Green's theorem.