hyperreal

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Defines	nonprincipal ultrafilter
Defines	infinitesimal
Defines	hypernatural
Defines	hyperinteger
Defines	hyperrational
Defines	hyperfinite

An ultrafilter $\mathcal{F}$ on a set I is called *nonprincipal* if no finite subsets of I are in $\mathcal{F}$.

Fix once and for all a nonprincipal ultrafilter $\mathcal{F}$ on the set $\mathbb{N}$ of natural numbers. Let $\sim$ be the equivalence relation on the set $\mathbb{R}^{\mathbb{N}}$ of sequences of real numbers given by

$$\{a_n\} \sim \{b_n\} \iff \{n \in \mathbb{N} \mid a_n = b_n\} \in \mathcal{F}$$

Let ${}^*\mathbb{R}$ be the set of equivalence classes of $\mathbb{R}^{\mathbb{N}}$ under the equivalence relation $\sim$. The set ${}^*\mathbb{R}$ is called the set of *hyperreals*. It is a field under coordinatewise addition and multiplication:

$$\begin{aligned} \{a_n\} + \{b_n\} &= \{a_n + b_n\} \\ \{a_n\} \cdot \{b_n\} &= \{a_n \cdot b_n\} \end{aligned}$$

The field ${}^*\mathbb{R}$ is an ordered field under the ordering relation

$$\{a_n\} \leq \{b_n\} \iff \{n \in \mathbb{N} \mid a_n \leq b_n\} \in \mathcal{F}$$

The real numbers embed into ${}^*\mathbb{R}$ by the map sending the real number $x \in \mathbb{R}$ to the equivalence class of the constant sequence given by $x_n := x$ for all n. In what follows, we adopt the convention of treating $\mathbb{R}$ as a subset of ${}^*\mathbb{R}$ under this embedding.

A hyperreal $x \in {}^*\mathbb{R}$ is:

- *limited* if $a < x < b$ for some real numbers $a, b \in \mathbb{R}$
- *positive unlimited* if $x > a$ for all real numbers $a \in \mathbb{R}$
- *negative unlimited* if $x < a$ for all real numbers $a \in \mathbb{R}$
- *unlimited* if it is either positive unlimited or negative unlimited
- *positive infinitesimal* if $0 < x < a$ for all positive real numbers $a \in \mathbb{R}^+$
- *negative infinitesimal* if $a < x < 0$ for all negative real numbers $a \in \mathbb{R}^-$
- *infinitesimal* if it is either positive infinitesimal or negative infinitesimal

For any subset A of $\mathbb{R}$, the set ${}^*A$ is defined to be the subset of ${}^*\mathbb{R}$ consisting of equivalence classes of sequences $\{a_n\}$ such that

$$\{n \in \mathbb{N} \mid a_n \in A\} \in \mathcal{F}.$$

The sets ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, and ${}^*\mathbb{Q}$ are called *hypernaturals*, *hyperintegers*, and *hyperrationals*, respectively. An element of ${}^*\mathbb{N}$ is also sometimes called *hyperfinite.*