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testing for continuity via basic open sets

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Proposition 1. *Let X, Y be topological spaces, and $f : X \to Y$ a function. The following are equivalent:*

1. *f is continuous;*
2. *$f^{-1}(U)$ is open for any U in a basis (U called a basic open set) for the topology of Y;*
3. *$f^{-1}(U)$ is open for any U is a subbasis for the topology of Y.*

Proof. First, note that (1) ⇒ (2) ⇒ (3), since every basic open set is open, and every element in a subbasis is in the basis it generates. We next prove (3) ⇒ (2) ⇒ (1).

- (2) ⇒ (1). Suppose $\mathcal{B}$ is a basis for the topology of Y. Let U be an open set in Y. Then U is the union of elements in $\mathcal{B}$. In other words,

$$U = \bigcup \{U_i \in \mathcal{B} \mid i \in I\},$$

  for some index set I. So

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\bigcup \{U_i \in \mathcal{B} \mid i \in I\}) \\ &= \bigcup \{f^{-1}(U_i) \mid i \in I\}. \end{aligned}$$

  By assumption, each $f^{-1}(U_i)$ is open, so is their union $f^{-1}(U)$.

- (3) ⇒ (2). Suppose now that $\mathcal{S}$ is a subbasis, which generates the basis $\mathcal{B}$ for the topology of Y. If U is a basic open set, then

$$U = \bigcap_{i=1}^{n} U_i,$$

  where each $U_i \in \mathcal{S}$. Then

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\bigcap_{i=1}^{n} U_i) \\ &= \bigcap_{i=1}^{n} f^{-1}(U_i). \end{aligned}$$

  By assumption, each $f^{-1}(U_i)$ is open, so is their (finite) intersection $f^{-1}(U)$.

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