properties of the Lebesgue integral of Lebesgue integrable functions

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Theorem. *Let $(X, \mathfrak{B}, \mu)$ be a measure space, $f\colon X \to [-\infty, \infty]$ and $g\colon X \to [-\infty, \infty]$ be Lebesgue integrable functions, and $A, B \in \mathfrak{B}$. Then the following properties hold:*

1. $\left|\int_A f\,d\mu\right| \le \int_A |f|\,d\mu$
2. *If* $f \le g$, *then* $\int_A f\,d\mu \le \int_A g\,d\mu$.
3. $\int_A f\,d\mu = \int_X \chi_A f\,d\mu$, *where* χ_A *denotes the characteristic function of* A
4. *If* $c \in \mathbb{R}$, *then* $\int_A cf\,d\mu = c\int_A f\,d\mu$.
5. *If* $\mu(A) = 0$, *then* $\int_A f\,d\mu = 0$.
6. $\int_A (f+g)\,d\mu = \int_A f\,d\mu + \int_A g\,d\mu$.
7. *If* $A \cap B = \emptyset$, *then* $\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu$.
8. *If* $f = g$ *almost everywhere with respect to* μ, *then* $\int_A f\,d\mu = \int_A g\,d\mu$.

Proof. 1.

$$\left|\int_A f\,d\mu\right| = \left|\int_A f^+\,d\mu - \int_A f^-\,d\mu\right| \text{ by definition}$$
$$\le \left|\int_A f^+\,d\mu\right| + \left|\int_A f^-\,d\mu\right| \text{ by the triangle inequality}$$
$$= \int_A f^+\,d\mu + \int_A f^-\,d\mu \text{ by the}$$

properties of the Lebesgue integral of nonnegative measurable functions (proper

$$= \int_A (f^+ + f^-)\,d\mu \text{ by the}$$

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$$= \int_A |f|\,d\mu$$

2. Since $f \leq g$, the following must hold:

   - $f^+ = \max\{0, f\} \leq \max\{0, g\} = g^+$;
   - $-f \geq -g$;
   - $f^- = \max\{0, -f\} \geq \max\{0, -g\} = g^-$.

   Thus, by the http://planetmath.org/PropertiesOfTheLebesgueIntegralOfNonnegative of the Lebesgue integral of nonnegative measurable functions (property 2), $\displaystyle\int_A f^+ \, d\mu \leq \int_A g^+ \, d\mu$ and $\displaystyle\int_A f^- \, d\mu \geq \int_A g^- \, d\mu$. Therefore, $\displaystyle -\int_A f^- \, d\mu \leq -\int_A g^- \, d\mu$. Hence, $\displaystyle\int_A f^+ \, d\mu - \int_A f^- \, d\mu \leq \int_A g^+ \, d\mu - \int_A f^- \, d\mu \leq \int_A g^+ \, d\mu - \int_A g^- \, d\mu$. It follows that $\displaystyle\int_A f \, d\mu \leq \int_A g \, d\mu$.

3.

$$\begin{aligned}
\int_A f \, d\mu &= \int_A f^+ \, d\mu - \int_A f^- \, d\mu \text{ by definition} \\
&= \int_X \chi_A f^+ \, d\mu - \int_X \chi_A f^- \, d\mu \text{ by the} \\
&\texttt{http://planetmath.org/PropertiesOfTheLebesgueIntegralOfNonnegativeMe} \\
&= \int_X (\chi_A f)^+ \, d\mu - \int_X (\chi_A f)^- \, d\mu \\
&= \int_X \chi_A f \, d\mu \text{ by definition}
\end{aligned}$$

4. If $c \geq 0$, then

$$\begin{aligned}
\int_A cf \, d\mu &= \int_A (cf)^+ \, d\mu - \int_A (cf)^- \, d\mu \text{ by definition} \\
&= \int_A cf^+ \, d\mu - \int_A cf^- \, d\mu \\
&= c \int_A f^+ \, d\mu - c \int_A f^- \, d\mu \text{ by the} \\
&\texttt{http://planetmath.org/PropertiesOfTheLebesgueIntegralOfNonnegativeM} \\
&= c \left(\int_A f^+ \, d\mu - \int_A f^- \, d\mu \right) \\
&= c \int_A f \, d\mu \text{ by definition.}
\end{aligned}$$

If $c < 0$, then

$$\begin{aligned}
\int_A cf\,d\mu &= \int_A (cf)^+\,d\mu - \int_A (cf)^-\,d\mu \text{ by definition} \\
&= \int_A (-c)f^-\,d\mu - \int_A (-c)f^+\,d\mu \\
&= -c\int_A f^-\,d\mu + c\int_A f^+\,d\mu \text{ by the}
\end{aligned}$$

http://planetmath.org/PropertiesOfTheLebesgueIntegralOfNonnegativeM

$$\begin{aligned}
&= c\left(-\int_A f^-\,d\mu + \int_A f^+\,d\mu\right) \\
&= c\int_A f\,d\mu \text{ by definition.}
\end{aligned}$$

5. Note that $\int_A f^+\,d\mu = 0$ and $\int_A f^-\,d\mu = 0$ by the http://planetmath.org/PropertiesOfThe of the Lebesgue integral of nonnegative measurable functions (property 6). It follows that $\int_A f\,d\mu = 0$.

6. Let $\{s_n\}$ be a nondecreasing sequence of nonnegative simple functions converging pointwise to $f^+ + g^+$ and $\{t_n\}$ be a nondecreasing sequence of nonnegative simple functions converging pointwise to $f^- + g^-$. Note that, for every n, $\int_A s_n\,d\mu - \int_A t_n\,d\mu = \int_A (s_n - t_n)\,d\mu$.

   Since f and g are integrable and $|f+g| \leq |f| + |g|$, $f+g$ is integrable. Thus,

$$
\begin{aligned}
\int_A f\,d\mu + \int_A g\,d\mu &= \int_A f^+\,d\mu - \int_A f^-\,d\mu + \int_A g^+\,d\mu - \int_A g^-\,d\mu \text{ by definition}\\
&= \int_A f^+\,d\mu + \int_A g^+\,d\mu - \left(\int_A f^-\,d\mu + \int_A g^- d\mu\right)\\
&= \int_A (f^+ + g^+)\,d\mu - \left(\int_A (f^- + g^-)\,d\mu\right) \text{ by the}\\
&\texttt{http://planetmath.org/PropertiesOfTheLebesgueIntegralOfNonne}\\
&= \lim_{n\to\infty}\int_A s_n\,d\mu - \left(\lim_{n\to\infty}\int_A t_n\,d\mu\right) \text{ by Lebesgue's monotone convergence}\\
&= \lim_{n\to\infty}\left(\int_A s_n\,d\mu - \int_A t_n\,d\mu\right)\\
&= \lim_{n\to\infty}\int_A (s_n - t_n)\,d\mu\\
&= \int_A (f^+ + g^+ - (f^- + g^-))\,d\mu \text{ by Lebesgue's dominated convergence th}\\
&= \int_A (f^+ - f^- + g^+ - g^-)\,d\mu\\
&= \int_A (f+g)\,d\mu \text{ by definition.}
\end{aligned}
$$

7.

$$
\begin{aligned}
\int_{A\cup B} f\,d\mu &= \int_{A\cup B} f^+\,d\mu - \int_{A\cup B} f^-\,d\mu \text{ by definition}\\
&= \int_A f^+\,d\mu + \int_B f^+\,d\mu - \left(\int_A f^-\,d\mu + \int_B f^-\,d\mu\right) \text{ by the}\\
&\texttt{http://planetmath.org/PropertiesOfTheLebesgueIntegralOfNonnegative}\\
&= \int_A f^+\,d\mu - \int_A f^-\,d\mu + \int_B f^+\,d\mu - \int_B f^-\,d\mu\\
&= \int_A f\,d\mu + \int_B f\,d\mu \text{ by definition}
\end{aligned}
$$

8. Let $E = \{x \in A : f(x) = g(x)\}$. Since f and g are measurable functions and $A \in \mathfrak{B}$, it must be the case that $E \in \mathfrak{B}$. Thus, $A - E \in \mathfrak{B}$. By hypothesis, $\mu(A\setminus E) = 0$. Note that $E\cap(A\setminus E) = \emptyset$ and $E\cup(A\setminus E) = A$. Thus, $\int_A f\,d\mu = \int_E f\,d\mu + \int_{A\setminus E} f\,d\mu = \int_E f\,d\mu + 0 = \int_E g\,d\mu + 0 = \int_E g\,d\mu + \int_{A\setminus E} g\,d\mu = \int_A g\,d\mu$.

$\square$