proof of inequalities for difference of powers

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1 First Inequality

We have the factorization

$$u^n - v^n = (u-v)\sum_{k=0}^{n-1} u^k v^{n-k-1}.$$

Since the largest term in the sum is is u^{n-1} and the smallest is v^{n-1}, and there are n terms in the sum, we deduce the following inequalities:

$$n(u-v)v^{n-1} < u^n - v^n < n(u-v)u^{n-1}$$

2 Second Inequality

This inequality is trivial when $x = 0$. We split the rest of the proof into two cases.

2.1 $-1 < x < 0$

In this case, we set $u = 1$ and $v = 1 + x$ in the second inequality above:

$$1 - (1+x)^n < n(-x)$$

Reversing the signs of both sides yields

$$nx < (1+x)^n - 1$$

2.2 $0 < x$

In this case, we set $u = 1 + x$ and $v = 1$ in the first inequality above:

$$nx < (1+x)^n - 1$$

3 Third Inequality

This inequality is trivial when $x = 0$. We split the rest of the proof into two cases.

3.1 $-1 < x < 0$

Start with the first inequality for differences of powers, expand the left-hand side,

$$nuv^{n-1} - nv^n < u^n - v^n,$$

move the v^n to the other side of the inequality,

$$nuv^{n-1} - (n-1)v^n < u^n,$$

and divide by v^n to obtain

$$n\frac{u}{v} - n + 1 < \left(\frac{u}{v}\right)^n.$$

Taking the reciprocal, we obtain

$$\left(\frac{v}{u}\right)^n < \frac{v}{v + n(u-v)} = 1 - \frac{n(u-v)}{v + n(u-v)}$$

Setting $u = 1$ and $v = 1 + x$, and moving a term from one side to the other, this becomes

$$(1+x)^n - 1 < \frac{nx}{1-(n-1)x}.$$

3.2 $0 < x < 1/(n-1)$

Start with the second inequality for differences of powers, expand the right-hand side,

$$u^n - v^n < nu^n - nu^{n-1}v$$

move terms from one side of the inequality to the other,

$$nu^{n-1}v - (n-1)u^n < v^n$$

and divide by u^n to obtain

$$n\frac{v}{u} - n + 1 < \left(\frac{v}{u}\right)^n$$

When the left-hand side is positive, (i.e. $nv > (n-1)u$) we can take the reciprocal:

$$\left(\frac{u}{v}\right)^n < \frac{u}{u - n(u-v)} = 1 + \frac{n(u-v)}{u - n(u-v)}$$

Setting $u = 1 + x$ and $v = 1$, and moving a term from one side to the other, this becomes

$$(1 + x)^n - 1 < \frac{nx}{1 - (n-1)x}$$

and the positivity condition mentioned above becomes $(n - 1)x < 1$.