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limit superior

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Let $S \subset \mathbb{R}$ be a set of real numbers. Recall that a limit point of S is a real number $x \in \mathbb{R}$ such that for all $\epsilon > 0$ there exist infinitely many $y \in S$ such that

$$|x - y| < \epsilon.$$

We define $\limsup S = \overline{\lim}$, pronounced the *limit superior* of S, to be the supremum of all the limit points of S. If there are no limit points, we define the limit superior to be $-\infty$.

We can generalize the above definition to the case of a mapping $f : X \to \mathbb{R}$. Now, we define a limit point of f to be an $x \in \mathbb{R}$ such that for all $\epsilon > 0$ there exist infinitely many $y \in X$ such that

$$|x - f(y)| < \epsilon.$$

We then define $\limsup f$, to be the supremum of all the limit points of f, or $-\infty$ if there are no limit points. We recover the previous definition as a special case by considering the limit superior of the inclusion mapping $\iota : S \to \mathbb{R}$.

Since a sequence of real numbers $x_0, x_1, x_2, , \ldots$ is just a mapping from $\mathbb{N}$ to $\mathbb{R}$, we may adapt the above definition to arrive at the notion of the limit superior of a sequence. However for the case of sequences, an alternative, but equivalent definition is available. For each $k \in \mathbb{N}$, let y_k be the supremum of the k^{th} tail,

$$y_k = \sup_{j \geq k} x_j.$$

This construction produces a non-increasing sequence

$$y_0 \geq y_1 \geq y_2 \geq \ldots,$$

which either converges to its infimum, or diverges to $-\infty$. We define the limit superior of the original sequence to be this limit;

$$\limsup_k x_k = \lim_k y_k.$$