BV function

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Functions of bounded variation, BV functions, are functions whose distributional derivative is a finite Radon measure. More precisely:

Definition 1 (functions of bounded variation)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be an open set. We say that a function* $u \in L^1(\Omega)$ *has* bounded variation*, and write* $u \in BV(\Omega)$*, if there exists a finite Radon vector measure* $Du \in \mathcal{M}(\Omega, \mathbb{R}^n)$ *such that*

$$\int_\Omega u(x)\, \mathrm{div}\phi(x)\, dx = -\int_\Omega \langle \phi(x), Du(x) \rangle$$

for every function $\phi \in C^1_c(\Omega, \mathbb{R}^n)$*. The measure* Du*, represents the distributional derivative of* u *since the above equality holds true for every* $\phi \in C^\infty_c(\Omega, \mathbb{R}^n)$*.*

Notice that $W^{1,1}(\Omega) \subset BV(\Omega)$. In fact if $u \in W^{1,1}(\Omega)$ one can choose $\mu := \nabla u \mathcal{L}$ (where $\mathcal{L}$ is the Lebesgue measure on Ω). The equality $\int u \mathrm{div}\phi = -\int \phi\, d\mu = -\int \phi \nabla u$ is nothing else than the definition of weak derivative, and hence holds true. One can easily find an example of a BV functions which is not $W^{1,1}$.

An equivalent definition can be given as follows.

Definition 2 (variation)**.** *Given* $u \in L^1(\Omega)$ *we define the* variation *of* u *in* Ω *as*

$$V(u, \Omega) := \sup\{\int_\Omega u \mathrm{div}\phi \colon \phi \in \mathcal{C}^1_c(\Omega, \mathbb{R}^n),\ \|\phi\|_{L^\infty(\Omega)} \leq 1\}.$$

We define $BV(\Omega) = \{u \in L^1(\Omega) \colon V(u, \Omega) < +\infty\}$*.*

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