function continuous at only one point

Canonical name	FunctionContinuousAtOnlyOnePoint
Date of creation	2013-03-22 14:56:19
Last modified on	2013-03-22 14:56:19
Owner	Andrea Ambrosio (7332)
Last modified by	Andrea Ambrosio (7332)
Numerical id	7
Author	Andrea Ambrosio (7332)
Entry type	Example
Classification	msc 26A15
Classification	msc 54C05
Related topic	DirichletsFunction
Related topic	FunctionDifferentiableAtOnlyOnePoint

Let us show that the function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x, & \text{when } x \text{ is rational,} \\ -x, & \text{when } x \text{ is irrational,} \end{cases}$$

is continuous at $x = 0$, but discontinuous for all $x \in \mathbb{R} \setminus \{0\}$ [**?**].

We shall use the following characterization of continuity for f: f is continuous at $a \in \mathbb{R}$ if and only if $\lim_{k\to\infty} f(x_k) = f(a)$ for all sequences $(x_k) \subset \mathbb{R}$ such that $\lim_{k\to\infty} x_k = a$.

It is not difficult to see that f is continuous at $x = 0$. Indeed, if x_k is a sequence converging to 0. Then

$$\begin{aligned} \lim_{k\to\infty} |f(x_k)| &= \lim_{k\to\infty} |f(x_k)| \\ &= \lim_{k\to\infty} |x_k| \\ &= 0. \end{aligned}$$

Suppose $a \neq 0$. Then there exists a sequence of irrational numbers $x_1, x_2, \ldots$ converging to a. For instance, if a is irrational, we can take $x_k = a + 1/k$, and if a is rational, we can take $x_k = a + \sqrt{2}/k$. For this sequence we have

$$\begin{aligned} \lim_{k\to\infty} f(x_k) &= -\lim_{k\to\infty} x_k \\ &= -a. \end{aligned}$$

On the other hand, we can also construct a sequence of rational numbers $y_1, y_2, \ldots$ converging to a. For example, if a is irrational, this follows as the rational numbers are dense in $\mathbb{R}$, and if a is rational, we can set $y_k = x_k + 1/k$. For this sequence we have

$$\begin{aligned} \lim_{k\to\infty} f(y_k) &= \lim_{k\to\infty} y_k \\ &= a. \end{aligned}$$

In conclusion f is not continuous at a.

References

[1] Homepage of Thomas Vogel, `http://www.math.tamu.edu/ tom.vogel/gallery/node3.htmlA` function which is continuous at only one point.