Descartes' rule of signs

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Descartes's rule of signs is a method for determining the number of positive or negative roots of a polynomial.

Let $p(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial with real coefficients such that $a_m \neq 0$.

Define v to be the number of *variations in sign* of the sequence of coefficients $a_m, \ldots, a_0$. By 'variations in sign' we mean the number of values of n such that the sign of a_n differs from the sign of a_{n-1}, as n ranges from m down to 1.

For example, consider $p(x) = x^2 - 4x + 4$. The coefficients are $1, -4, 4$, so there are 2 variations in sign (since the sign of 1 differs from that of -4, which in turn differs from that of 4.)

Then the number of positive real roots of $p(x)$ is $v - 2N$ for some integer N satisfying $0 \leq N \leq \frac{v}{2}$.

The number N represents the number of irreducible factors of degree 2 in the factorization of $p(x)$. Thus $N = 0$ if is known that $p(x)$ splits over the real numbers.

The number of negative roots of $p(x)$ may be obtained by the same method by applying the rule of signs to $p(-x)$.

History

This result is believed to have been first described by Réné Descartes in his 1637 work *La Géométrie*. In 1828, Carl Friedrich Gauss improved the rule by proving that when there are fewer roots of polynomials than there are variations of sign, the parity of the difference between the two is even.

Example

Let $p(x) = x^3 + 3x^2 - x - 3$. Looking at the list of coefficients, we have $1, 3, -1, -3$, so there is only one variation in sign (from 3 to -1).

Thus $v = 1$. Since $0 \leq N \leq \frac{1}{2}$ then we must have $N = 0$. Thus $v - 2N = 1$ and so there is exactly one positive real root of $p(x)$.

To find the negative roots, we examine $p(-x) = -x^3 + 3x^2 + x - 3$. The coefficient list is $-1, 3, 1, -3$, so here are are two variations in sign (from -1 to 3 and 1 to -3. Thus $v = 2$ and so $0 \leq N \leq \frac{2}{2} = 1$.

Thus we have two possible solutions, $N = 0$ and $N = 1$, and two possible values of $v - 2N$. Therefore there are either two negative real roots or none at all.

We note that $p(-1) = (-1)^3 + 3 \cdot (-1)^2 - (-1) - 3 = 0$, hence there is at least one negative root. Therefore there must be exactly two.