properties of vector-valued functions

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Owner	pahio (2872)
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If $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_n)$ are vector-valued and u a real-valued function of the real variable t, one defines the vector-valued functions $F{+}G$ and uF componentwise as

$$F{+}G \ := \ (f_1{+}g_1, \ldots, f_n{+}g_n), \quad uF \ := \ (uf_1, \ldots, uf_n)$$

and the real valued dot product as

$$F \cdot G \ := \ f_1 g_1 + \ldots + f_n g_n.$$

If $n = 3,$ one my define also the vector-valued cross product function as

$$F{\times}G \ := \ \left(\begin{vmatrix} f_2 & f_3 \\ g_2 & g_3 \end{vmatrix}, \begin{vmatrix} f_3 & f_1 \\ g_3 & g_1 \end{vmatrix}, \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} \right).$$

It's not hard to verify, that if F, G and u are differentiable on an interval, so are also $F{+}G$, uF and $F \cdot G$, and the formulae

$$(F{+}G)' \ = \ F'{+}G', \quad (uF)' \ = \ u'F{+}uF', \quad (F \cdot G)' \ = \ F' \cdot G + F \cdot G'$$

are valid, in $\mathbb{R}^3$ additionally

$$(F{\times}G)' \ = \ F'{\times}G + F{\times}G'.$$

Likewise one can verify the following theorems.

Theorem 1. If u is continuous in the point t and F in the point $u(t)$, then

$$H \ = \ F{\circ}u \ := \ (f_1{\circ}u, \ldots, f_n{\circ}u)$$

is continuous in the point t. If u is differentiable in the point t and F in the point $u(t)$, then the composite function H is differentiable in t and the chain rule

$$H'(t) \ = \ F'(u(t))\, u'(t)$$

is in .

Theorem 2. If F and G are integrable on $[a,\, b]$, so is also c_1F+c_2G, where $c_1,\, c_2$ are real constants, and

$$\int_a^b (c_1F+c_2G)\,dt \;=\; c_1\int_a^b F\,dt + c_2\int_a^b G\,dt.$$

Theorem 3. Suppose that F is continuous on the interval I and $c \in I$. Then the vector-valued function

$$t \;\mapsto\; \int_c^t F(\tau)\,d\tau \;:=\; G(t) \quad \forall\, t \in I$$

is differentiable on I and satisfies $G' = F$.

Theorem 4. Suppose that F is continuous on the interval $[a,\, b]$ and G is an arbitrary function such that $G' = F$ on this interval. Then

$$\int_a^b F(t)\,dt \;=\; G(b)-G(a).$$

Theorem 2 may be generalised to

Theorem 5. If F is integrable on $[a,\, b]$ and $C = (c_1,\, \ldots,\, c_n)$ is an arbitrary vector of $\mathbb{R}^n$, then dot product $C \cdot F$ is integrable on this interval and

$$\int_a^b C \cdot F(t)\,dt \;=\; C \cdot \int_a^b F(t)\,dt.$$