function of not bounded variation

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Example. We show that the function

$$f\colon\ x \mapsto \begin{cases} x\cos\frac{\pi}{x} & \text{when } x \neq 0,\\ 0 & \text{when } x = 0,\end{cases}$$

which is continuous in the whole $\mathbb{R}$, is not of bounded variation on any interval containing the zero.

Let us take e.g. the interval $[0,\,a]$. Chose a positive integer m such that $\frac{1}{m} < a$ and the partition of the interval with the points $\frac{1}{m},\,\frac{1}{m+1},\,\frac{1}{m+2},\,\ldots,\,\frac{1}{n}$ into the subintervals $[0,\,\frac{1}{n}],\ [\frac{1}{n},\,\frac{1}{n-1}],\ \ldots,\ [\frac{1}{m+1},\,\frac{1}{m}],\ [\frac{1}{m},\,a]$. For each positive integer ν we have (see `http://planetmath.org/CosineAtMultiplesOfStraightAngle`this)

$$f\left(\frac{1}{\nu}\right) = \frac{1}{\nu}\cos\nu\pi = \frac{(-1)^\nu}{\nu}.$$

Thus we see that the total variation of f in all partitions of $[0,\,a]$ is at least

$$\frac{1}{n}+\left(\frac{1}{n}+\frac{1}{n-1}\right)+\ldots+\left(\frac{1}{m+1}+\frac{1}{m}\right) = \frac{1}{m}+2\sum_{\nu=m+1}^{n}\frac{1}{\nu}.$$

Since the harmonic series diverges, the above sum increases to ∞ as $n \to \infty$. Accordingly, the total variation must be infinite, and the function f is not of bounded variation on $[0,\,a]$.

It is not difficult to justify that f is of bounded variation on any finite interval that does not contain 0.

References

[1] E. Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset III. Toinen osa.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1940).