relations between Hessian matrix and local extrema

Canonical name	RelationsBetweenHessianMatrixAndLocalExtrema
Date of creation	2013-03-22 12:59:52
Last modified on	2013-03-22 12:59:52
Owner	bshanks (153)
Last modified by	bshanks (153)
Numerical id	14
Author	bshanks (153)
Entry type	Result
Classification	msc 26B12
Related topic	Extrema
Related topic	Extremum
Related topic	HessianForm
Related topic	TestsForLocalExtremaForLagrangeMultiplierMethod
Defines	second derivative test

Let x be a vector, and let $H(x)$ be the Hessian for f at a point x. Let f have continuous partial derivatives of first and second order in a neighborhood of x. Let $\nabla f(x) = 0$.

If $H(x)$ is http://planetmath.org/PositiveDefinitepositive definite, then x is a strict local minimum for f.

If x is a local minimum for x, then $H(x)$ is positive semidefinite.

If $H(x)$ is http://planetmath.org/NegativeDefinitenegative definite, then x is a strict local maximum for f.

If x is a local maximum for x, then $H(x)$ is negative semidefinite.

If $H(x)$ is indefinite, x is a nondegenerate saddle point.

If the case when the dimension of x is 1 (i.e. $f : \mathbb{R} \to \mathbb{R}$), this reduces to the Second Derivative Test, which is as follows:

Let the neighborhood of x be in the domain for f, and let f have continuous partial derivatives of first and second order. Let $f'(x) = 0$. If $f''(x) > 0$, then x is a strict local minimum. If $f''(x) < 0$, then x is a strict local maximum. In the case that $f''(x) = 0$, being $f'''(x) \neq 0$, x is said to be an inflexion point (also called turning point). A typical example is $f(x) = \sin x$, $f''(x) = -\sin x = 0$, $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, $f'''(x) = -\cos x$, $f'''(n\pi) = -\cos n\pi = (-1)^{n+1} \neq 0$.