surface normal

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Defines	parameter plane
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Defines	Gaussian coordinates

Let S be a smooth surface in $\mathbb{R}^3$. The *surface normal* of S at a point P of S is the line passing through P and perpendicular to the tangent plane τ of S at the point P, i.e. perpendicular to all lines in τ.

If the surface S is given in a parametric form

$$x = x(u,\, v), \quad y = y(u,\, v), \quad z = z(u,\, v),$$

it is useful to interpret the parameters u and v as the rectangular coordinates of a point in a plane, the so-called *parameter plane*. We can consider on S the so-called *parameter curves*, namely the *u-curves* which correspond the lines parallel to the u-axis and the *v-curves* which correspond the lines parallel to the v-axis in the parameter plane. One u-curve and one v-curve passes through every point on the surface (the values of u and v in a point of S are the *Gaussian coordinates* of this point). The surface normal at any point of S is perpendicular to both parameter curves, and thus its direction cosines a, b, c satisfy the equations

$$\begin{cases} a\dfrac{\partial x}{\partial u} + b\dfrac{\partial y}{\partial u} + c\dfrac{\partial z}{\partial u} = 0, \\[2ex] a\dfrac{\partial x}{\partial v} + b\dfrac{\partial y}{\partial v} + c\dfrac{\partial z}{\partial v} = 0. \end{cases}$$

This homogeneous pair of linear equations determines the ratio of the direction cosines

$$a : b : c = \frac{\partial(y,\, z)}{\partial(u,\, v)} : \frac{\partial(z,\, x)}{\partial(u,\, v)} : \frac{\partial(x,\, y)}{\partial(u,\, v)}$$

via the Jacobians.

Example. Determine the direction cosines of the normal of the helicoid

$$x = u\cos v, \quad y = u\sin v, \quad z = cv.$$

We have the Jacobians

$$\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} \sin v & 0 \\ u\cos v & c \end{vmatrix} = c\sin v, \quad \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & \cos v \\ c & -u\sin v \end{vmatrix} = -c\cos v, \quad \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos \end{vmatrix}$$

These are the components of the normal vector of the helicoid surface in the point with the Gaussian coordinates u and v. The length of the vector is

$\sqrt{(c\sin v)^2+(-c\cos v)^2+u^2} = \sqrt{u^2+c^2}$. If we http://planetmath.org/Divisiondivide the vector by its length, we obtain a unit vector, the components of which are the direction cosines of the surface normal:

$$\frac{c\sin v}{\sqrt{u^2+c^2}}, \quad -\frac{c\cos v}{\sqrt{u^2+c^2}}, \quad \frac{u}{\sqrt{u^2+c^2}}.$$