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proof of Rolle's theorem

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Because f is continuous on a compact (closed and bounded) interval $I = [a, b]$, it attains its maximum and minimum values. In case $f(a) = f(b)$ is both the maximum and the minimum, then there is nothing more to say, for then f is a constant function and $f' \equiv 0$ on the whole interval I. So suppose otherwise, and f attains an extremum in the open interval (a, b), and without loss of generality, let this extremum be a maximum, considering $-f$ in lieu of f as necessary. We claim that at this extremum $f(c)$ we have $f'(c) = 0$, with $a < c < b$.

To show this, note that $f(x) - f(c) \leq 0$ for all $x \in I$, because $f(c)$ is the maximum. By definition of the derivative, we have that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Looking at the one-sided limits, we note that

$$R = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

because the numerator in the limit is nonpositive in the interval I, yet $x - c > 0$, as x approaches c from the right. Similarly,

$$L = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Since f is differentiable at c, the left and right limits must coincide, so $0 \leq L = R \leq 0$, that is to say, $f'(c) = 0$.