fraction power

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Let m be an integer and n a positive factor of m. If x is a positive real number, we may write the identical equation

$$(x^{\frac{m}{n}})^n \;=\; x^{\frac{m}{n}\cdot n} \;=\; x^m$$

and therefore the definition of http://planetmath.org/NthRootn^{th} root gives the

$$\sqrt[n]{x^m} \;=\; x^{\frac{m}{n}}. \tag{1}$$

Here, the exponent $\frac{m}{n}$ is an integer. For enabling the validity of (1) for the cases where n does not divide m we must set the following

Definition. Let $\frac{m}{n}$ be a fractional number, i.e. an integer m not divisible by the integer n, which latter we assume to be positive. For any positive real number x we define the *fraction power* $x^{\frac{m}{n}}$ as the n^{th}

$$x^{\frac{m}{n}} \;:=\; \sqrt[n]{x^m}. \tag{2}$$

Remarks

1. The existence of the in the right hand side of (2) is proved http://planetmath.org/existenc

2. The defining equation (2) is independent on the form of the exponent $\frac{m}{n}$: If $\frac{k}{l} = \frac{m}{n}$, then we have $(\sqrt[n]{x^m})^{ln} = [(\sqrt[n]{x^m})^n]^l = x^{lm} = x^{kn} = [(\sqrt[l]{x^k})^l]^n = (\sqrt[l]{x^k})^{ln}$, and because the mapping $y \mapsto y^{ln}$ is injective in $\mathbb{R}_+$, the positive numbers $\sqrt[l]{x^k}$ and $\sqrt[n]{x^m}$ must be equal.

3. The fraction power function $x \mapsto x^{\frac{m}{n}}$ is a special case of power function.

4. The presumption that x is positive signifies that one can not identify all n^{th} http://planetmath.org/NthRootroots $\sqrt[n]{x}$ and the powers $x^{\frac{1}{n}}$. For example, $\sqrt[3]{-8}$ equals -2 and $\frac{2}{6} = \frac{1}{3}$, but one **must not**

$$(-8)^{\frac{1}{3}} \;=\; (-8)^{\frac{2}{6}} \;=\; \sqrt[6]{(-8)^2} \;=\; \sqrt[6]{64} \;=\; 2.$$

The point is that $(-8)^{\frac{1}{3}}$ is not defined in $\mathbb{R}$. Here we have $l = 6$ and the mapping $y \mapsto y^{ln}$ is not injective in $\mathbb{R}_- \cup \mathbb{R}_+$. — Nevertheless, some people and books may use also for negative x the equality $\sqrt[n]{x} = x^{\frac{1}{n}}$ and more generally $\sqrt[n]{x^m} = x^{\frac{m}{n}}$ where one then insists that $\gcd(m,\, n) = 1$.

5. According to the preceding item, for the negative values of x the derivative of http://planetmath.org/NthRootodd roots, e.g. $\sqrt[3]{x}$, ought to be calculated as follows:

$$\frac{d\sqrt[3]{x}}{dx} = \frac{d(-\sqrt[3]{-x})}{dx} = -\frac{d(-x)^{\frac{1}{3}}}{dx} = -\frac{1}{3}(-x)^{-\frac{2}{3}}(-1) = \frac{1}{3\sqrt[3]{(-x)^2}} = \frac{1}{3\sqrt[3]{x^2}}$$

The result is similar as $\frac{d\sqrt[3]{x}}{dx}$ for positive x's, although the root functions are not special cases of the power function.