integration of $\sqrt{x^2+1}$

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The integral

$$I := \int\sqrt{x^2+1}\,dx$$

can be found by using the first http://planetmath.org/EulersSubstitutionsForIntegrationEu
substitution

$$\sqrt{x^2+1} := -x+t,$$

but another possibility is to use http://planetmath.org/ASpecialCaseOfPartialIntegrationpa
integration if one knows the integral $\int\frac{dx}{\sqrt{x^2+1}}$. The corresponding may be said of the more general

$$\int\sqrt{x^2+c}\,dx.$$

We think that the integrand of I has the other factor 1 and integrate partially:

$$I = \int 1\cdot\sqrt{x^2+1}\,dx = x\sqrt{x^2+1}-\int x\cdot\frac{1}{2\sqrt{x^2+1}}\cdot 2x\,dx+C' = x\sqrt{x^2+1}-\int\frac{x^2}{\sqrt{x^2+1}}\,dx+C'.$$

Writing the numerator as $(x^2+1)-1$ and dividing its minuend and subtrahend separately, we can write

$$I = x\sqrt{x^2+1}-\left(\int\sqrt{x^2+1}\,dx-\int\frac{1}{\sqrt{x^2+1}}\,dx\right)+C' = x\sqrt{x^2+1}-I+\int\frac{dx}{\sqrt{x^2+1}}+C'.$$

Having I in two , we solve it from these equalities, obtaining

$$I = \frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\int\frac{dx}{\sqrt{x^2+1}}+C,$$

i.e.,

$$\int\sqrt{x^2+1}\,dx = \frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\operatorname{arsinh}x+C$$