integrals of even and odd functions

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Theorem. Let the real function f be `http://planetmath.org/RiemannIntegrable`Riemann-integrable on $[-a,a]$. If f is an

- even function, then $\displaystyle\int_{-a}^{a} f(x)\,dx \;=\; 2\int_{0}^{a} f(x)\,dx,$
- odd function, then $\displaystyle\int_{-a}^{a} f(x)\,dx \;=\; 0.$

Of course, both cases concern the *zero map* which is both .

Proof. Since the definite integral is additive with respect to the interval of integration, one has

$$I \;:=\; \int_{-a}^{a} f(x)\,dx \;=\; \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(x)\,dx.$$

Making in the first addend the substitution $t = -x,\; dt = -dx$ and swapping the limits of integration one gets

$$I \;=\; \int_{a}^{0} f(-x)(-dx) + \int_{0}^{a} f(x)\,dx \;=\; \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx.$$

Using then the definitions of `http://planetmath.org/EvenoddFunction`even $(+)$ and `http://planetmath.org/EvenoddFunction`odd $(-)$ function yields

$$I \;=\; \int_{0}^{a} (\pm f(x))\,dx + \int_{0}^{a} f(x)\,dx \;=\; \pm\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx,$$

which settles the equations of the theorem.