proof of chain rule

Canonical name	ProofOfChainRule
Date of creation	2013-03-22 12:41:48
Last modified on	2013-03-22 12:41:48
Owner	n3o (216)
Last modified by	n3o (216)
Numerical id	6
Author	n3o (216)
Entry type	Proof
Classification	msc 26A06

Let's say that g is differentiable in x_0 and f is differentiable in $y_0 = g(x_0)$. We define:

$$\varphi(y) = \begin{cases} \frac{f(y)-f(y_0)}{y-y_0} & \text{if } y \neq y_0 \\ f'(y_0) & \text{if } y = y_0 \end{cases}$$

Since f is differentiable in y_0, φ is continuous. We observe that, for $x \neq x_0$,

$$\frac{f(g(x)) - f(g(x_0))}{x - x_0} = \varphi(g(x))\frac{g(x) - g(x_0)}{x - x_0},$$

in fact, if $g(x) \neq g(x_0)$, it follows at once from the definition of φ, while if $g(x) = g(x_0)$, both members of the equation are 0.

Since g is continuous in x_0, and φ is continuous in y_0,

$$\lim_{x\to x_0} \varphi(g(x)) = \varphi(g(x_0)) = f'(g(x_0)),$$

hence

$$\begin{aligned} (f \circ g)'(x_0) &= \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} \\ &= \lim_{x\to x_0} \varphi(g(x))\frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(g(x_0))g'(x_0). \end{aligned}$$