least and greatest zero

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Theorem. If a real function f is continuous on the interval $[a,\,b]$ and has zeroes on this interval, then f has a least zero and a greatest zero.

Proof. If $f(a) = 0$ then the assertion concerning the least zero is true. Let's assume therefore, that $f(a) \neq 0$.

The set $A = \{x \in [a,\,b] \colon f(x) = 0\}$ is bounded from below since all numbers of A are greater than a. Let the `http://planetmath.org/InfimumAndSupremumForRealNumbe` of A be ξ. Let us make the antithesis, that $f(\xi) \neq 0$. Then, by the continuity of f, there is a positive number δ such that

$$f(x) \neq 0 \quad \text{always when}\ |x-\xi| < \delta.$$

Chose a number x_1 between ξ and $\xi+\delta$; then $f(x_1) \neq 0$, but this number x_1 is not a lower bound of A. Therefore there exists a member a_1 of A which is less than x_1 ($\xi < a_1 < x_1$). Now $|a_1-\xi| < |x_1-\xi| < \delta$, whence this member of A ought to satisfy that $f(a_1) = 0$. This a contradiction. Thus the antithesis is wrong, and $f(\xi) = 0$.

This that $\xi \in A$ and ξ is the least number of A.

Analogically one shows that the supremum of A is the greatest zero of f on the interval.