existence of nth root

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Theorem. *If $a \in \mathbb{R}$ with $a > 0$ and n is a positive integer, then there exists a unique positive real number u such that $u^n = a$.*

Proof. The statement is clearly true for $n = 1$ (let $u = a$). Thus, it will be assumed that $n > 1$.

Define $p\colon \mathbb{R} \to \mathbb{R}$ by $p(x) = x^n - a$. Note that a positive real root of $p(x)$ corresponds to a positive real number u such that $u^n = a$.

If $a = 1$, then $p(1) = 1^n - 1 = 0$, in which case the existence of u has been established.

Note that $p(x)$ is a polynomial function and thus is continuous. If $a < 1$, then $p(1) = 1^n - a > 1 - 1 = 0$. If $a > 1$, then $p(a) = a^n - a = a(a^{n-1} - 1) > 0$. Note also that $p(0) = 0^n - a = -a < 0$. Thus, if $a \neq 1$, then the intermediate value theorem can be applied to yield the existence of u.

For uniqueness, note that the function $p(x)$ is strictly increasing on the interval $(0, \infty)$. It follows that u as described in the statement of the theorem exists uniquely. $\square$