proof of convergence criterion for infinite product

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Consider the partial product $P_n = \prod_{i=1}^{n} p_i$.

By definition we say that the infinite product $\prod_{n=1}^{\infty} p_n$ is convergent iff P_n is convergent.

Suppose every $p_n > 0$

ln is a continuous bijection from $\mathbb{R}^+$ to $\mathbb{R}$, therefore $\lim_{n\to\infty} a_n = a \iff \lim_{n\to\infty} \ln(a_n) = \ln(a)$, provided $a_n > 0$ and $a > 0$.

so saying $P_n \to P > 0$ is equivalent to saying that $\ln(P_n)$ converges.

Since $\ln(P_n) = \ln(\prod_{i=1}^{n} p_i) = \sum_{i=1}^{n} \ln(p_i)$, the infinite product converges to a positive value iff the series $\sum_{n=1}^{\infty} \ln(p_n)$ is convergent.

In particular, if the infinite product converges to a positive value, then $\ln(p_n) \to 0 \implies p_n \to 1$.

$P_n \to 0$, is equivalent to saying $\sum_{n=1}^{\infty} \ln(p_n) = -\infty$

For the second part of the theorem:

$\prod_{n=1}^{\infty}(1+p_n)$ converges absolutely to a positive value iff $\sum_{n=1}^{\infty} p_n$ converges absolutely.

as we have seen, $1 + p_n \to 1 \implies p_n \to 0$

consider: $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$ (this is easy to prove since by Taylor's expansion $\ln(1 + x) = x + O(x^2)$)

Since $p_n \to 0$ we can say that $\lim_{n\to\infty} \frac{\ln(1+p_n)}{p_n} = 1$ and by the limit comparison test, either both $\sum_{n=1}^{\infty} \ln(1+p_n)$ and $\sum_{i=1}^{n} p_i$ converge or diverge.