proof of Chebyshev's inequality

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Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Write the product $(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n)$ as

$$\begin{aligned}
& (x_1y_1 + x_2y_2 + \cdots + x_ny_n) \\
+ \quad & (x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) \\
+ \quad & (x_1y_3 + x_2y_4 + \cdots + x_{n-2}y_n + x_{n-1}y_1 + x_ny_2) \\
+ \quad & \cdots \\
+ \quad & (x_1y_n + x_2y_1 + x_3y_2 + \cdots + x_ny_{n-1}).
\end{aligned} \tag{1}$$

- If $y_1 \leq y_2 \leq \cdots \leq y_n$, each of the n terms in parentheses is less than or equal to $x_1y_1 + x_2y_2 + \cdots + x_ny_n$, according to the rearrangement inequality. From this, it follows that

  $$(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) \leq n(x_1y_1 + x_2y_2 + \cdots + x_ny_n)$$

  or (dividing by n^2)

  $$\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right) \leq \frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{n}.$$

- If $y_1 \geq y_2 \geq \cdots \geq y_n$, the same reasoning gives

  $$\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right) \geq \frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{n}.$$

It is clear that equality holds if $x_1 = x_2 = \cdots = x_n$ or $y_1 = y_2 = \cdots = y_n$. To see that this condition is also necessary, suppose that not all y_i's are equal, so that $y_1 \neq y_n$. Then the second term in parentheses of (**??**) can only be equal to $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ if $x_{n-1} = x_n$, the third term only if $x_{n-2} = x_{n-1}$, and so on, until the last term which can only be equal to $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ if $x_1 = x_2$. This implies that $x_1 = x_2 = \cdots = x_n$. Therefore, Chebyshev's inequality is an equality if and only if $x_1 = x_2 = \cdots = x_n$ or $y_1 = y_2 = \cdots = y_n$.