planetmath.org

Math for the people, by the people.

sinc is L^2

Canonical name	SincIsL2
Date of creation	2013-03-22 15:44:44
Last modified on	2013-03-22 15:44:44
Owner	cvalente (11260)
Last modified by	cvalente (11260)
Numerical id	9
Author	cvalente (11260)
Entry type	Result
Classification	msc 26A06

Our objective will be to prove the integral $\int_{\mathbb{R}} f^2(x)dx$ exists in the Lebesgue sense when $f(x) = \text{sinc}(x)$.

The integrand is an even function and so we can restrict our proof to the set $\mathbb{R}^+$.

Since f is a continuous function, so will f^2 be and thus for every $a > 0$, $f \in L^2([0,a])$.

Thus, if we prove $f \in L^2([\pi, \infty[)$, the result will be proved.

Consider the intervals $I_k = [k\pi, (k+1)\pi]$ and $U_k = \bigcup_{i=1}^{k} I_k = [\pi, (k+1)\pi]$.

and the succession of functions $f_n(x) = f^2(x)\chi_{U_n}(x)$, where χ_{U_n} is the characteristic function of the set U_n.

Each f_n is a continuous function of compact support and will thus be integrable in $\mathbb{R}^+$. Furthermore $f_n(x) \nearrow f^2(x)$ (pointwise) in this set.

In each I_k,$0 \leq f^2(x) \leq \frac{\sin^2(x)}{(k\pi)^2}$, for $k > 0$.

So:

$$\int_{x \geq \pi} f_n(x)dx = \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \frac{\sin(x)^2}{x^2} dx \leq \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \frac{\sin(x)^2}{(k\pi)^2} = \sum_{k=1}^{n} \frac{1}{2k^2\pi}$$

[1]

So: $\lim_{n\to\infty} \int_{x\geq\pi} f_n(x)dx \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2k^2\pi}$ and since the series on the right side converges[2] and $f_n \nearrow f^2$ we can use the monotone convergence theorem to state that $f^2 \in L([\pi, \infty[)$.

So we get the result that $\text{sinc} \in L^2(\mathbb{R})$

[1] we have used the well known result $\int_0^\pi \sin^2(x)dx = \frac{\pi}{2}$

[2] asymptotic behaviour as k^{-2}