alternate statement of Bolzano-Weierstrass theorem

Canonical name	AlternateStatementOfBolzanoWeierstrassTheorem
Date of creation	2013-03-22 16:40:13
Last modified on	2013-03-22 16:40:13
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	7
Author	mathcam (2727)
Entry type	Theorem
Classification	msc 26A06
Related topic	BolzanoWeierstrassTheorem
Related topic	LimitPoint
Related topic	Bounded
Related topic	Infinite

Theorem. *Every bounded, infinite set of real numbers has a limit point.*

Proof. Let $S \subset \mathbb{R}$ be bounded and infinite. Since S is bounded there exist $a, b \in \mathbb{R}$, with $a < b$, such that $S \subset [a, b]$. Let $b - a = l$ and denote the midpoint of the interval $[a, b]$ by m. Note that at least one of $[a, m], [m, b]$ must contain infinitely many points of S; select an interval satisfying this condition, denoting its left endpoint by a_1 and its right endpoint by b_1. Continuing this process inductively, for each $n \in \mathbb{N}$, we have an interval $[a_n, b_n]$ satisfying

$$[a_n, b_n] \subset [a_{n-1}, b_{n-1}] \subset \cdots \subset [a_1, b_1] \subset [a, b], \tag{1}$$

where, for each $i \in \mathbb{N}$ such that $1 \leq i \leq n$, the interval $[a_i, b_i]$ contains infinitely many points of S and is of length $l/2^i$. Next we note that the set $A = \{a_1, a_2 \ldots, a_n\}$ is contained in $[a, b]$, hence is bounded, and as such, has a supremum which we denote by x. Now, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x - \epsilon < a_N \leq x$. Furthermore, for every $m \geq N$, we have $x - \epsilon < a_N \leq a_m \leq x$. In particular, if we select $m \geq N$ such that $l/2^m < \epsilon$, then we have

$$x - \epsilon < a_n \leq a_m \leq x \leq b_m = a_m + \frac{l}{2^m} < x + \epsilon. \tag{2}$$

Since $[a_m, b_m] \subset (x - \epsilon, x + \epsilon)$ contains infinitely many points of S, we may conclude that x is a limit point of S. $\square$