example of computing limits using Taylor expansion

Canonical name	ExampleOfComputingLimitsUsingTaylorExpansion
Date of creation	2013-03-22 15:39:48
Last modified on	2013-03-22 15:39:48
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Numerical id	15
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Entry type	Example
Classification	msc 26A06

Problem. *Evaluate*

$$\lim_{x\to-\infty}\left(x+\sqrt{x^2+23x}\right).$$

In beginners' courses in calculus, one is usually told to rationalize the above expression by multiplying by $(x-\sqrt{x^2+23x})/(x-\sqrt{x^2+23x})$. This approach is somewhat dissatisfying, because it depends on a specific *algebraic* trick that works only for square roots — for example, it would have been more difficult or impossible to rationalize, if instead, we had a cube root or even a transcendental function — and this trick does not appeal to our intuition that $\sqrt{x^2+23x}$ should be approximately $|x|$ for large $|x|$. Fortunately, there is another approach that exploits the *analytic* properties of the functions involved.

http://planetmath.org/LHpitalsRuleL'Hôpital's Rule is one analytic approach, but in many cases, using Taylor expansion is even easier and straightforward. Essentially, Taylor expansion approximates complicated functions by polynomials, whose limits are easy to evaluate. We illustrate the method below.

First rewrite

$$\lim_{x\to-\infty}\left(x+\sqrt{x^2+23x}\right)=\lim_{x\to+\infty}\left(\sqrt{x^2-23x}-x\right),$$

so that we do not have to worry about the pesky negatives any more. Then, with the help of the binomial formula:

$$(1+y)^{1/2}=1+\frac{1}{2}y+o(y)\,,\quad \text{as } y\to 0,$$

("o" is Landau notation) we obtain:

$$\begin{aligned}
\sqrt{x^2-23x}-x &= x\left(\sqrt{1-\frac{23}{x}}-1\right)\\
&= x\left(1-\frac{1}{2}\frac{23}{x}+o\left(\frac{23}{x}\right)-1\right)\,,\quad \text{as } x\to\infty \text{ (so } y=-\frac{23}{x}\to 0)\\
&= -\frac{23}{2}+x\,o\left(\frac{23}{x}\right)\\
&= -\frac{23}{2}+o(1)\,.
\end{aligned}$$

Therefore

$$\lim_{x\to-\infty}\left(x+\sqrt{x^2+23x}\right)=\lim_{x\to\infty}\left(\sqrt{x^2-23x}-x\right)=-\frac{23}{2}\,.$$

Problem. *Evaluate*

$$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{\tan x}\right).$$

This example is admittedly artificial; it was made to be annoying to solve using L'Hôpital's Rule alone, but much simpler if one knows how to use the Taylor expansions:

$$\begin{aligned}
\ln(1+x) &= x-\frac{1}{2}x^2+o(x^2)\,, && \text{as } x\to 0, \text{ and}\\
\tan x &= x+o(x^2)\,, && \text{as } x\to 0.
\end{aligned}$$

So we compute:

$$\begin{aligned}
\frac{1}{\ln(1+x)}-\frac{1}{\tan x}=\frac{\tan x-\ln(1+x)}{\ln(1+x)\tan x} &= \frac{\bigl(x+o(x^2)\bigr)-\bigl(x-\frac{1}{2}x^2+o(x^2)\bigr)}{\ln(1+x)\tan x}\\
&= \frac{\frac{1}{2}x^2+o(x^2)}{\bigl(x+o(x)\bigr)\bigl(x+o(x)\bigr)}\\
&= \frac{\frac{1}{2}+o(1)}{\bigl(1+o(1)\bigr)\bigl(1+o(1)\bigr)}
\end{aligned}$$

Therefore,

$$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{\tan x}\right)=\frac{1}{2}\,.$$

The reader might reasonably ask how did we know the right number of terms to use in the Taylor expansions. The answer is to guess. This is not as problematic as it sounds. First, there is no harm in using more terms than necessary in the expansion (only that there is more writing). And if we used too few terms, we would know when we later encounter indeterminate forms such as $o(1)/x$ (as $x\to 0$) in our derivations. If that happens, it is not hard to go back and add the needed terms.

Notice that all the essential information to evaluate the limit is contained in the first few derivatives of the functions involved *at particular points* — in the above example, only at $x=0$. This information can be obtained by manipulating series, unlike L'Hôpital's Rule which necessitates computing the derivative *functions* at all points. So even monstrous expressions like this one is tractable with Taylor expansion:

$$\lim_{x\to 0}\left(\frac{1}{\ln(1+\tan(\sin x))}-\frac{1}{e^{\sin(\tan x)}-1}\right).$$

On the other hand, there definitely are situations where L'Hôpital's Rule works but Taylor expansion does not: for instance,

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{x}{1/\ln x},$$

because $1/\ln x$ cannot be expanded in a Taylor series about $x = 0$.

Problem. *Here is a problem involving a different sort of limit: does the following series converge?*

$$\sum_{n=1}^{\infty}\arctan(n^{-1})$$

Our intuition suggests that it does not, because $\arctan(n^{-1})$ should be approximately n^{-1}, and $\sum_n n^{-1}$ diverges. However, the standard comparison test does not work because $\arctan x \le x$ (for $x \ge 0$) has the inequality in the wrong direction. But with Taylor expansion the solution is a snap. By expanding

$$\arctan x = x + O(x^3)\,, \quad \text{as } x \to 0.$$

and summing both sides, we get

$$\sum_{n=1}^{\infty}\arctan(n^{-1}) = \sum_{n=1}^{\infty} n^{-1} + \sum_{n=1}^{\infty} O(n^{-3})\,.$$

As $\sum_n O(n^{-3})$ converges (being dominated by $C\sum_n n^{-3}$ for some constant C), $\sum_n \arctan(n^{-1})$ must diverge (to ∞).

(Of course, this problem could be solved by using the integral test, but who really wants to integrate $\int \arctan(1/x)\,dx$?)