Riemann multiple integral

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Defines	Peano Jordan
Defines	measurable
Defines	area
Defines	volume
Defines	Jordan content

We are going to extend the concept of Riemann integral to functions of several variables.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a bounded function with compact support. Recalling the definitions of polyrectangle and the definitions of upper and lower Riemann sums on polyrectangles, we define

$$S^*(f) := \inf\{S^*(f,P)\colon P \text{ is a polyrectangle, } f(x)=0 \text{ for every } x \in \mathbb{R}^n \setminus \cup P\},$$

$$S_*(f) := \sup\{S_*(f,P)\colon P \text{ is a polyrectangle, } f(x)=0 \text{ for every } x \in \mathbb{R}^n \setminus \cup P\}.$$

If $S^*(f) = S_*(f)$ we say that f is *Riemann-integrable* on $\mathbb{R}^n$ and we define the Riemann integral of f:

$$\int f(x)\,dx := S^*(f) = S_*(f).$$

Clearly one has $S^*(f,P) \geq S_*(f,P)$. Also one has $S^*(f,P) \geq S_*(f,P')$ when P and P' are any two polyrectangles containing the support of f. In fact one can always find a common refinement P'' of both P and P' so that $S^*(f,P) \geq S^*(f,P'') \geq S_*(f,P'') \geq S_*(f,P')$. So, to prove that a function is Riemann-integrable it is enough to prove that for every $\epsilon > 0$ there exists a polyrectangle P such that $S^*(f,P) - S_*(f,P) < \epsilon$.

Next we are going to define the integral on more general domains. As a byproduct we also define the measure of sets in $\mathbb{R}^n$.

Let $D \subset \mathbb{R}^n$ be a bounded set. We say that D is *Riemann measurable* if the characteristic function

$$\chi_D(x) := \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$

is Riemann measurable on $\mathbb{R}^n$ (as defined above). Moreover we define the *Peano-Jordan measure* of D as

$$\mathbf{meas}(D) := \int \chi_D(x)\,dx.$$

When $n = 3$ the Peano Jordan measure of D is called the *volume* of D, and when $n = 2$ the Peano Jordan measure of D is called the *area* of D.

Let now $D \subset \mathbb{R}^n$ be a Riemann measurable set and let $f\colon D \to \mathbb{R}$ be a bounded function. We say that f is *Riemann measurable* if the function $\bar{f}\colon \mathbb{R}^n \to \mathbb{R}$

$$\bar{f}(x) := \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable as defined before. In this case we denote with

$$\int_D f(x)\,dx := \int \bar{f}(x)\,dx$$

the *Riemann integral* of f on D.