proof of extended mean-value theorem

Canonical name	ProofOfExtendedMeanvalueTheorem
Date of creation	2013-03-22 13:09:00
Last modified on	2013-03-22 13:09:00
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Last modified by	pbruin (1001)
Numerical id	6
Author	pbruin (1001)
Entry type	Proof
Classification	msc 26A06
Synonym	proof of Cauchy's mean-value theorem
Related topic	MeanValueTheorem

Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Define the function

$$h(x) = f(x)\left(g(b) - g(a)\right) - g(x)\left(f(b) - f(a)\right) - f(a)g(b) + f(b)g(a).$$

Because f and g are continuous on $[a, b]$ and differentiable on (a, b), so is h. Furthermore, $h(a) = h(b) = 0$, so by Rolle's theorem there exists a $\xi \in (a, b)$ such that $h'(\xi) = 0$. This implies that

$$f'(\xi)\left(g(b) - g(a)\right) - g'(\xi)\left(f(b) - f(a)\right) = 0$$

and, if $g(b) \neq g(a)$,

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$