antiderivative

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Defines	constant of integration

Let I be an open interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a real function.

A function $F : I \longrightarrow \mathbb{R}$ is called an *antiderivative* or a *primitive* of f if F is differentiable and its derivative is equal to f, i.e.

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

Note that there are an infinite number of antiderivatives for any function f since any constant can be added or subtracted from any valid antiderivative to yield another equally valid antiderivative.

To account for this, we express the *general antiderivative*, or *indefinite integral*, as follows:

$$\int f(x)\, dx = F + C$$

where C is an arbitrary constant called the *constant of integration*. The dx portion means "with respect to x", because after all, our functions F and f are functions of x.

There is no loss in generality with this notation since in fact *all* antiderivatives of f take this form as the following theorem demonstrates:

Theorem. *Let F, G be two antiderivatives of a given function f defined on an open interval I. Then $F - G = const.$*

Proof. Since $F'(x) = f(x)$ and $G'(x) = f(x)$, we have $F'(x) - G'(x) = 0$ on the whole I. Thus, by the fundamental theorem of integral calculus, $F(x) - G(x) = \text{const}$. $\square$

This is no longer true if the domain of the function f is not an open interval (is not connected). For that scenario, the following more general result holds:

Theorem. *Let $U \subset \mathbb{R}$ be an open set (not necessarily an interval). Suppose F, G are antiderivatives of a given function $f : U \longrightarrow \mathbb{R}$. Then $F - G$ is constant in each connected component of U (each interval in U).*

For example, consider the function $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x}$. Notice that the domain of f is not an interval, but the union of the disjoint intervals $(-\infty, 0)$ and $(0, +\infty)$. Then, all the antiderivatives of f take the form

$$\begin{cases} \log(-x) + C_1, & \text{if } x < 0 \\ \log(x) + C_2, & \text{if } x > 0 \end{cases}$$

0.1 Remarks

- For complex functions, the definition of antiderivative is exactly the same and the above results also hold (one just needs to consider "connected open subsets" instead of "open intervals").