characterizations of majorization

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Defines	Pigou-Dalton transfer

Let $\mathcal{E}_n$ be the set of all $n \times n$ permutation matrices that exchange two components. Such matrices have the form

$$\begin{bmatrix} \ddots & & & & \\ & 0 & & 1 & \\ & & \ddots & & \\ & 1 & & 0 & \\ & & & & \ddots \end{bmatrix}$$

A matrix T is called a *Pigou-Dalton transfer* (PDT) if

$$T = \alpha I + (1-\alpha)E$$

for some α between 0 and 1, and $E \in \mathcal{E}_n$.

The following are equivalent

1. x is `http://planetmath.org/Majorization`majorized by y.
2. $x = Dy$ for a doubly stochastic matrix D.
3. $x = T_1 T_2 \cdots T_k y$ for finitely many PDT $T_1, \dots, T_k$.
4. $\sum_{i=1}^n \theta(x_i) \le \sum_{i=1}^n \theta(y_i)$ for all convex function θ.
5. x lies in the convex hull whose vertex set is
$$\{(y_{\pi(1)}, y_{\pi(2)}, \dots, y_{\pi(n)}) : \pi \text{ is a permutation of } \{1, \dots, n\}\}.$$
6. For any n non-negative real numbers $a_1, \dots, a_n$,
$$\sum_\pi a_1^{x_{\pi(1)}} a_2^{x_{\pi(2)}} \cdots a_n^{x_{\pi(n)}} \le \sum_\pi a_1^{y_{\pi(1)}} a_2^{y_{\pi(2)}} \cdots a_n^{y_{\pi(n)}}$$
where summation is taken over all permutations of $\{1, \dots, n\}$.

The equivalence of the above conditions are due to Hardy, Littlewood, Pólya, Birkhoff, von Neumann and Muirhead.

Reference

- G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd edition, 1952, Cambridge University Press, London.
- A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, 1979, Acadamic Press, New York.