open and closed intervals have the same cardinality

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Proposition. *The sets of real numbers* $[0,1]$, $[0,1)$, $(0,1]$, *and* $(0,1)$ *all have the same cardinality.*

We give two proofs of this proposition.

Proof. Define a map $f : [0,1] \to [0,1]$ by $f(x) = (x+1)/3$. The map f is strictly increasing, hence injective. Moreover, the image of f is contained in the interval $[\frac{1}{3}, \frac{2}{3}] \subsetneq (0,1)$, so the maps $f_r : [0,1] \to [0,1)$ and $f_o : [0,1] \to (0,1)$ obtained from f by restricting the codomain are both injective. Since the inclusions into $[0,1]$ are also injective, the `http://planetmath.org/SchroederBernsteinTheor` Schröder-Bernstein theorem can be used to construct bijections $h_r : [0,1] \to [0,1)$ and $h_o : [0,1] \to (0,1)$. Finally, the map $r : (0,1] \to [0,1)$ defined by $r(x) = 1 - x$ is a bijection.

Since having the same cardinality is an equivalence relation, all four intervals have the same cardinality. □

Proof. Since $[0,1] \cap \mathbb{Q}$ is countable, there is a bijection $a : \mathbb{N} \to [0,1] \cap \mathbb{Q}$. We may select a so that $a(0) = 0$ and $a(1) = 1$. The map $f : [0,1] \cap \mathbb{Q} \to (0,1) \cap \mathbb{Q}$ defined by $f(x) = a(a^{-1}(x) + 2)$ is a bijection because it is a composition of bijections. A bijection $h : [0,1] \to (0,1)$ can be constructed by gluing the map f to the identity map on $(0,1) \setminus \mathbb{Q}$. The formula for h is

$$h(x) = \begin{cases} f(x), & x \in \mathbb{Q} \\ x, & x \notin \mathbb{Q}. \end{cases}$$

The other bijections can be constructed similarly. □

The reasoning above can be extended to show that any two arbitrary intervals in $\mathbb{R}$ have the same cardinality.