arithmetic-geometric mean as a product

Canonical name	ArithmeticgeometricMeanAsAProduct
Date of creation	2013-03-22 17:09:59
Last modified on	2013-03-22 17:09:59
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Numerical id	6
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Entry type	Derivation
Classification	msc 26E60
Classification	msc 33E05

Recall that, given two real numbers $0 < x \leq y$, their arithmetic-geometric mean may be defined as $M(x, y) = \lim_{n \to \infty} g_n$, where

$$\begin{aligned} g_0 &= x \\ a_0 &= y \\ g_{n+1} &= \sqrt{a_n g_n} \\ a_{n+1} &= \frac{a_n + g_n}{2}. \end{aligned}$$

In this entry, we will re-express this quantity as an infinite product. We begin by rewriting the recursion for g_n:

$$g_{n+1} = \sqrt{a_n g_n} = \sqrt{\frac{a_n}{g_n} \cdot g_n^2} = g_n \sqrt{\frac{a_n}{g_n}}$$

From this, it follows that

$$g_n = g_0 \prod_{m=0}^{n-1} h_m$$

where $h_n = \sqrt{a_n / g_n}$.

As it stands, this is not so interesting because no way has been given to determine the factors h_n other than first computing a_n and g_n. We shall now correct this defect by deriving a recursion which may be used to compute the h_n's directly:

$$\begin{aligned} h_{n+1} &= \sqrt{\frac{a_{n+1}}{g_{n+1}}} \\ &= \sqrt{\frac{a_n + g_n}{2\sqrt{a_n g_n}}} \\ &= \sqrt{\frac{1}{2}\left(\sqrt{\frac{a_n}{g_n}} + \sqrt{\frac{g_n}{a_n}}\right)} \\ &= \sqrt{\frac{1}{2}\left(h_n + \frac{1}{h_n}\right)} \\ &= \sqrt{\frac{h_n^2 + 1}{2h_n}} \end{aligned}$$

Taking the limit $n \to \infty$, we then have the formula

$$M(x,y) = x \prod_{m=0}^{\infty} h_n$$

where

$$h_0 = \frac{y}{x}$$

and

$$h_{n+1} = \sqrt{\frac{h_n^2 + 1}{2h_n}}.$$