version of the fundamental lemma of calculus of variations

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Lemma. If a real function f is continuous on the interval $[a,\, b]$ and if

$$\int_a^b f(x)\varphi(x)\, dx \;=\; 0$$

for all functions φ continuously differentiable on the interval and vanishing at its end points, then $f(x) \;\equiv\; 0$ on the whole interval.

Proof. We make the antithesis that f does not vanish identically. Then there exists a point x_0 of the open interval $(a,\, b)$ such that $f(x_0) \neq 0$; for example $f(x_0) > 0$. The continuity of f implies that there are the numbers α and β such that $a < \alpha < x_0 < \beta < b$ and $f(x) > 0$ for all $x \in [\alpha,\, \beta]$. Now the function φ_0 defined by

$$\varphi_0(x) \;:=\; \begin{cases} (x-\alpha)^2(x-\beta)^2 & \text{for } \alpha \leqq x \leqq \beta, \\ 0 & \text{otherwise} \end{cases}$$

fulfils the requirements for the functions φ. Since both f and φ_0 are positive on the open interval $(\alpha,\, \beta)$, we however have

$$\int_a^b f(x)\varphi_0(x)\, dx \;=\; \int_\alpha^\beta f(x)\varphi_0(x)\, dx \;>\; 0.$$

Thus the antithesis causes a contradiction. Consequently, we must have $f(x) \;\equiv\; 0$.