illustration of integration techniques

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The following integral is an example that illustrates many integration techniques.

Problem. Determine the antiderivative of $\sqrt{\tan x}$.

. We start with http://planetmath.org/IntegrationBySubstitutionsubstitution:

$$\begin{aligned} u &= \sqrt{\tan x} \\ u^2 &= \tan x \\ 2u\,du &= \sec^2 x\,dx \end{aligned}$$

Using the Pythagorean identity $\tan^2 x + 1 = \sec^2 x$, we obtain:

$$\begin{aligned} 2u\,du &= (\tan^2 x + 1)\,dx \\ 2u\,du &= (u^4 + 1)\,dx \\ \frac{2u}{u^4+1}\,du &= dx \end{aligned}$$

Thus,

$$\begin{aligned} \int \sqrt{\tan x}\,dx &= \int u\,\frac{2u}{u^4+1}\,du \\ &= \int \frac{2u^2}{(u^2 - u\sqrt{2} + 1)(u^2 + u\sqrt{2} + 1)}\,du. \end{aligned}$$

For this last integral, we use the method of http://planetmath.org/ALectureOnThePartialFr fractions:

$$\frac{2u^2}{(u^2 - u\sqrt{2} + 1)(u^2 + u\sqrt{2} + 1)} = \frac{A + Bu}{u^2 - u\sqrt{2} + 1} + \frac{C + Du}{u^2 + u\sqrt{2} + 1}$$

$$\begin{aligned} 2u^2 &= (A + Bu)(u^2 + u\sqrt{2} + 1) + (C + Du)(u^2 - u\sqrt{2} + 1) \\ &= (B{+}D)u^3 + (A{+}C{+}(B{-}D)\sqrt{2})u^2 + (B{+}D{+}(A{-}C)\sqrt{2})u + A \end{aligned}$$

From this, we obtain the following system of equations:

$$\left\{ \begin{array}{lcll} & & B + D & = 0 \\ A + C & + & (B - D)\sqrt{2} & = 2 \\ (A - C)\sqrt{2} & + & B + D & = 0 \\ A + C & & & = 0 \end{array} \right.$$

This can be into two smaller systems of equations:

$$\begin{cases} A + C = 0 \\ A\sqrt{2} - C\sqrt{2} = 0 \end{cases}$$

$$\begin{cases} B + D = 0 \\ B\sqrt{2} - D\sqrt{2} = 2 \end{cases}$$

It is clear that the first system yields $A = C = 0$, and it can easily be verified that $B = \frac{1}{\sqrt{2}}$ and $D = \frac{-1}{\sqrt{2}}$. Therefore,

$$\begin{aligned}\int \sqrt{\tan x}\, dx &= \frac{1}{\sqrt{2}} \int \frac{u}{u^2 - u\sqrt{2} + 1}\, du - \frac{1}{\sqrt{2}} \int \frac{u}{u^2 + u\sqrt{2} + 1}\, du \\ &= \frac{1}{\sqrt{2}} \int \frac{u}{u^2 - u\sqrt{2} + \frac{1}{2} + \frac{1}{2}}\, du - \frac{1}{\sqrt{2}} \int \frac{u}{u^2 + u\sqrt{2} + \frac{1}{2} + \frac{1}{2}}\, du \\ &= \frac{1}{\sqrt{2}} \int \frac{u}{(u - \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}\, du - \frac{1}{\sqrt{2}} \int \frac{u}{(u + \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}\, du.\end{aligned}$$

Now we make the following substitutions:

$$\begin{aligned} v &= u - \tfrac{1}{\sqrt{2}} & \qquad w &= u + \tfrac{1}{\sqrt{2}} \\ dv &= du & dw &= du \end{aligned}$$

Note that we have $v + \frac{1}{\sqrt{2}} = u = w - \frac{1}{\sqrt{2}}$. Therefore,

$$\begin{aligned}\int \sqrt{\tan x}\, dx &= \frac{1}{\sqrt{2}} \int \frac{v + \frac{1}{\sqrt{2}}}{v^2 + \frac{1}{2}}\, dv - \frac{1}{\sqrt{2}} \int \frac{w - \frac{1}{\sqrt{2}}}{w^2 + \frac{1}{2}}\, dw \\ &= \frac{1}{\sqrt{2}} \int \frac{v}{v^2 + \frac{1}{2}}\, dv - \frac{1}{2} \int \frac{dv}{v^2 + \frac{1}{2}} - \frac{1}{\sqrt{2}} \int \frac{w}{w^2 + \frac{1}{2}}\, dw + \frac{1}{2} \int \frac{dw}{w^2 + \frac{1}{2}}.\end{aligned}$$

For the first and third integrals in the last expression, note that the numerator is a of the derivative of the denominator. For these, we use the formula

$$\int \frac{k f'(x)}{f(x)}\, dx = k \ln |f(x)|.$$

For the second and fourth integrals in the last expression, we use the formula

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$$

with $a = \frac{1}{\sqrt{2}}$. Hence,

$$
\begin{aligned}
\int \sqrt{\tan x}\, dx &= \frac{1}{2\sqrt{2}} \ln\left(v^2 + \frac{1}{2}\right) + \frac{1}{\sqrt{2}} \arctan(v\sqrt{2}) - \frac{1}{2\sqrt{2}} \ln\left(w^2 + \frac{1}{2}\right) + \frac{1}{\sqrt{2}} \arctan(w\sqrt{2}) + K \\
&= \frac{1}{2\sqrt{2}} \ln\left(\frac{v^2 + \frac{1}{2}}{w^2 + \frac{1}{2}}\right) + \frac{1}{\sqrt{2}}(\arctan(v\sqrt{2}) + \arctan(w\sqrt{2})) + K \\
&= \frac{1}{2\sqrt{2}} \ln\left(\frac{(u - \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}{(u + \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}\right) + \frac{1}{\sqrt{2}}\left(\arctan\left[\left(u - \frac{1}{\sqrt{2}}\right)\sqrt{2}\right] + \arctan\left[\left(u + \frac{1}{\sqrt{2}}\right)\sqrt{2}\right]\right) + \\
&= \frac{1}{2\sqrt{2}} \ln\left(\frac{u^2 - u\sqrt{2} + 1}{u^2 + u\sqrt{2} + 1}\right) + \frac{1}{\sqrt{2}}[\arctan(u\sqrt{2} - 1) + \arctan(u\sqrt{2} + 1)] + K \\
&= \frac{1}{2\sqrt{2}} \ln\left(\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right) + \frac{1}{\sqrt{2}}[\arctan(\sqrt{2\tan x} - 1) + \arctan(\sqrt{2\tan x} + 1)
\end{aligned}
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(We use K for the constant of integration to avoid confusion with C from the system of equations.)