integration of fraction power expressions

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The antiderivatives of every expression containing fraction powers can not be expressed by using elementary functions. However, there are after making a substitution.

- $\displaystyle\int R(x,\, x^{r_1},\, \ldots,\, x^{r_m})\, dx$, where R means a rational function of its arguments. If the common denominator of the fraction power exponents r_j is n, the substitution

$$x \;:=\; t^n, \qquad dx \;=\; nt^{n-1}dt$$

changes each exponent to an integer and the whole integrand to a rational function in the variable t.

Example. For $\displaystyle\int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}+1}\, dx$ the least common multiple of the denominators of $\frac{1}{2}$ and $\frac{3}{4}$ is 4, whence we make the substitution $x = t^4$, $dx = 4t^3 dt$. Then we obtain

$$\int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}+1}\, dx \;=\; 4\int \frac{t^5 dt}{t^3+1} \;=\; 4\int \left(t^2 - \frac{t^2}{t^3+1}\right) dt \;=\; 4\left(\frac{t^3}{3} - \frac{1}{3}\ln|t^3+1|\right) + C$$

$$= \frac{4}{3}\left(x^{\frac{3}{4}} - \ln|x^{\frac{3}{4}}+1|\right) + C.$$

- In $\displaystyle\int R\left(x,\, \left(\frac{ax+b}{cx+d}\right)^{r_1},\, \ldots,\, \left(\frac{ax+b}{cx+d}\right)^{r_m}\right) dx$, correspondently the substitution

$$\frac{ax+b}{cx+d} \;:=\; t^n$$

changes the integrand to a rational function.

Example. For $\displaystyle\int \frac{\sqrt{x+4}}{x}\, dx$ we substitute $x+4 = t^2$, $dx = 2t\, dt$, getting

$$\int \frac{\sqrt{x+4}}{x}\, dx \;=\; 2\int \frac{t^2}{t^2-4}\, dt \;=\; 2\int \left(1 + \frac{4}{t^2-4}\right) dt \;=\; 2t + 2\ln\left|\frac{t-2}{t+2}\right| + C$$

$$= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.$$

References

[1] N. Piskunov: *Diferentsiaal- ja integraalarvutus kõrgematele tehnilistele õppeasutustele.* Viies, täiendatud trükk. Kirjastus “Valgus”, Tallinn (1965).