Lipschitz condition and differentiability result

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About lipschitz continuity of differentiable functions the following holds.

Theorem 1. *Let X, Y be Banach spaces and let A be a convex (see convex set), open subset of X. Let $f\colon \overline{A} \to Y$ be a function which is continuous in $\overline{A}$ and differentiable in A. Then f is lipschitz continuous on $\overline{A}$ if and only if the derivative Df is bounded on A i.e.*

$$\sup_{x\in A} \|Df(x)\| < +\infty.$$

Proof. Suppose that f is lipschitz continuous:

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

Then given any $x \in A$ and any $v \in X$, for all small $h \in \mathbb{R}$ we have

$$\|\frac{f(x + hv) - f(x)}{h}\| \leq L.$$

Hence, passing to the limit $h \to 0$ it must hold $\|Df(x)\| \leq L$.

On the other hand suppose that Df is bounded on A:

$$\|Df(x)\| \leq L, \qquad \forall x \in A.$$

Given any two points $x, y \in \overline{A}$ and given any $\alpha \in Y^*$ consider the function $G : [0, 1] \to \mathbb{R}$

$$G(t) = \langle \alpha, f((1 - t)x + ty)\rangle.$$

For $t \in (0, 1)$ it holds

$$G'(t) = \langle \alpha, Df((1 - t)x + ty)[y - x]\rangle$$

and hence

$$|G'(t)| \leq L\|\alpha\| \, \|y - x\|.$$

Applying Lagrange mean-value theorem to G we know that there exists $\xi \in (0, 1)$ such that

$$|\langle \alpha, f(y) - f(x)\rangle| = |G(1) - G(0)| = |G'(\xi)| \leq \|\alpha\|L\|y - x\|$$

and since this is true for all $\alpha \in Y^*$ we get

$$\|f(y) - f(x)\| \leq L\|y - x\|$$

which is the desired claim. □