antiderivative of rational function

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The most notable real functions, which can be integrated in a closed form, are the rational functions:

Theorem. The antiderivative of a rational function is always expressible in a closed form, which only can comprise, except a rational expression summand, summands of logarithms and arcustangents of rational functions.

One can justify the theorem by using the general form of the (unique) partial fraction decomposition

$$\begin{aligned} R(x) \;=\; & H(x)+\sum_{i=1}^{m}\left(\frac{A_{i1}}{x-a_i}+\frac{A_{i2}}{(x-a_i)^2}+\ldots+\frac{A_{i\mu_i}}{(x-a_i)^{\mu_i}}\right) \\ & +\sum_{j=1}^{n}\left(\frac{B_{j1}x+C_{j1}}{x^2+2p_jx+q_j}+\frac{B_{j2}x+C_{j2}}{(x^2+2p_jx+q_j)^2}+\ldots+\frac{B_{j\nu_j}x+C_{j\nu_j}}{(x^2+2p_jx+q_j)^{\nu_j}}\right), \end{aligned}$$

of the rational function $R(x)$; here, $H(x)$ is a polynomial, the first sum expression is determined by the real zeroes a_i of the denominator of $R(x)$, the second sum is determined by the real quadratic prime factors $x^2+2p_jx+q_j$ of the denominator (which have no real zeroes).

The addends of the form $\dfrac{A}{(x-a)^r}$ in the first sum are integrated directly, giving

$$\int\frac{A}{x-a}\,dx \;=\; A\ln|x-a|+\text{constant} \qquad (r=1) \tag{1}$$

and

$$\int\frac{A}{(x-a)^r}\,dx \;=\; -\frac{A}{r-1}\cdot\frac{1}{(x-a)^{r-1}}+\text{constant} \qquad (r>1). \tag{2}$$

The remaining partial fractions are of the form $\dfrac{Bx+C}{(x^2+2px+q)^s}$ where $p^2<q$ and s is a positive integer. Now we may write

$$x^2+2px+q \;=\; (x+p)^2+q-p^2 \;=\; (q-p^2)\left[1+\left(\frac{x+p}{\sqrt{q-p^2}}\right)^2\right]$$

and make the substitution

$$\frac{x+p}{\sqrt{q-p^2}} \;=\; t, \tag{3}$$

i.e. $x = t\sqrt{q-p^2}-p$, getting

$$\int \frac{Bx+C}{(x^2+2px+q)^s}\,dx = \int \frac{Et+F}{(1+t^2)^s}\,dt = E\int \frac{t\,dt}{(1+t^2)^s} + F\int \frac{dt}{(1+t^2)^s} \tag{4}$$

where E and F are certain constants. In the case $s = 1$ we have

$$\int \frac{t\,dt}{1+t^2} = \frac{1}{2}\ln(1+t^2) + \text{constant} \tag{5}$$

and in the case $s > 1$

$$\int \frac{t\,dt}{(1+t^2)^s} = -\frac{1}{2(s-1)}\cdot\frac{1}{(1+t^2)^{s-1}} + \text{constant.} \tag{6}$$

The latter addend of the right hand side of (4) is for $s = 1$ got from

$$\int \frac{dt}{1+t^2} = \arctan t + \text{constant} \tag{7}$$

and for the cases $s > 1$ on may first write

$$\int \frac{dt}{(1+t^2)^s} = \int \frac{(1+t^2)-t^2}{(1+t^2)^s}\,dt = \int \frac{dt}{(1+t^2)^{s-1}} - \int t\cdot\frac{t\,dt}{(1+t^2)^s}.$$

Using integration by parts in the last integral, this equation can be converted into the reduction formula

$$\int \frac{dt}{(1+t^2)^s} = \frac{1}{2s-2}\cdot\frac{t}{(1+t^2)^{s-1}} + \frac{2n-3}{2n-2}\int \frac{dt}{(1+t^2)^{s-1}}. \tag{8}$$

The assertion of the theorem follows from (1), …, (8).

Example.

$$\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\ln\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\arctan\frac{x\sqrt{2}}{1-x^2} + C$$