Weierstrass substitution formulas

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Defines	Weierstrass substitution
Defines	Weierstaß substitution
Defines	universal trigonometric substitution

The *Weierstrass substitution formulas* for $-\pi < x < \pi$ are:

$$\begin{aligned} \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \\ dx &= \frac{2}{1+t^2}\,dt \end{aligned}$$

They can be obtained in the following manner:

Make the *Weierstrass substitution* $t = \tan\left(\frac{x}{2}\right)$. (This substitution is also known as the *universal trigonometric substitution*.) Then we have

$$\begin{aligned} \cos\left(\frac{x}{2}\right) &= \frac{1}{\sec\left(\frac{x}{2}\right)} \\ &= \frac{1}{\sqrt{1+\tan^2\left(\frac{x}{2}\right)}} \\ &= \frac{1}{\sqrt{1+t^2}} \end{aligned}$$

and

$$\begin{aligned} \sin\left(\frac{x}{2}\right) &= \cos\left(\frac{x}{2}\right)\cdot\tan\left(\frac{x}{2}\right) \\ &= \frac{t}{\sqrt{1+t^2}}. \end{aligned}$$

Note that these are just the "formulas involving `http://planetmath.org/Radical6`radicals" as designated in the entry goniometric formulas; however, due to the restriction on x, the $\pm$'s are unnecessary.

Using the above formulas along with the double angle formulas, we obtain

$$\begin{aligned}
\sin x &= 2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right) \\
&= 2\cdot\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}} \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

and

$$\begin{aligned}
\cos x &= \cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right) \\
&= \left(\frac{1}{\sqrt{1+t^2}}\right)^2-\left(\frac{t}{\sqrt{1+t^2}}\right)^2 \\
&= \frac{1}{1+t^2}-\frac{t^2}{1+t^2} \\
&= \frac{1-t^2}{1+t^2}.
\end{aligned}$$

Finally, since $t=\tan\left(\frac{x}{2}\right)$, solving for x yields that $x=2\arctan t$. Thus, $dx=\dfrac{2}{1+t^2}\,dt$.

The Weierstrass substitution formulas are most useful for `http://planetmath.org/Integratio` rational functions of sine and cosine.