Euler's substitutions for integration

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In the integration task

$$\int R(x,\, \sqrt{ax^2+bx+c})\, dx,$$

where the integrand is a rational function of x and $\sqrt{ax^2+bx+c}$, the integrand can be changed to a rational function of a new variable t by using the following substitutions of Euler.

- **The first substitution of Euler.** If $a > 0$, we may write

$$\sqrt{ax^2+bx+c} \;=\; \pm x\sqrt{a}+t. \tag{1}$$

  When we take $\sqrt{a}$ with the minus sign, then

$$ax^2+bx+c \;=\; ax^2-2xt\sqrt{a}+t^2,$$

  from which we get the expression

$$x \;=\; \frac{t^2-c}{b+2t\sqrt{a}};$$

  thus also dx is expressible rationally via t. We have

$$\sqrt{ax^2+bx+c} \;=\; -x\sqrt{a}+t \;=\; \frac{c-t^2}{b+2t\sqrt{a}}\sqrt{a}+t.$$

- **The second substitution of Euler.** If $c > 0$, we take

$$\sqrt{ax^2+bx+c} \;=\; xt\pm\sqrt{c}. \tag{2}$$

  With the minus sign we obtain, similarly as above,

$$x \;=\; \frac{2t\sqrt{c}+b}{t^2-a}.$$

- **The third substitution of Euler.** If the polynomial ax^2+bx+c has the real zeros α and β, we may chose

$$\sqrt{ax^2+bx+c} \;=\; (x-\alpha)t. \tag{3}$$

Now

$$ax^2+bx+c \;=\; a(x-\alpha)(x-\beta) \;=\; (x-\alpha)^2t^2,$$

whence $a(x-\beta) = (x-\alpha)t^2$. This gives the expression

$$x \;=\; \frac{a\beta-\alpha t^2}{a-t^2}.$$

As in the preceding cases, we can express dx and $\sqrt{ax^2+bx+c}$ rationally via t.

Examples.

1. In the integral $\int\frac{dx}{\sqrt{x^2+c}}$ we can use the first substitution: $\sqrt{x^2+c} = -x+t$; then $x^2+c = x^2-2xt+t^2$ and thus

$$x \;=\; \frac{t^2-c}{2t}, \quad dx \;=\; \frac{t^2+c}{2t^2}\,dt, \quad \sqrt{x^2+c} \;=\; -\frac{t^2-c}{2t}+t \;=\; \frac{t^2+c}{2t}.$$

Accordingly we obtain

$$\int\frac{dx}{\sqrt{x^2+c}} \;=\; \int\frac{\frac{t^2+c}{2t^2}dt}{\frac{t^2+c}{2t}} \;=\; \int\frac{dt}{t} \;=\; \ln|t|+C \;=\; \ln|x+\sqrt{x^2+c}|+C.$$

Especially the cases $c=\pm1$ give the formulas

$$\int\frac{dx}{\sqrt{x^2+1}} \;=\; \operatorname{arsinh}x+C, \qquad \int\frac{dx}{\sqrt{x^2-1}} \;=\; \operatorname{arcosh}x+C \quad (x>1).$$

2. The integral $\int\frac{\sqrt{c^2-x^2}}{x}\,dx$ is needed in deriving the equation of the tractrix. We use for integrating the second substitution $\sqrt{c^2-x^2} = xt-c$; then $c^2-x^2 = x^2t^2-2cxt+c^2$, which implies

$$x \;=\; \frac{2ct}{t^2+1}, \quad dx \;=\; \frac{2c(1-t^2)dt}{(1+t^2)^2}, \quad \sqrt{c^2-x^2} \;=\; \frac{2ct^2}{t^2+1}-c \;=\; \frac{c(t^2-1)}{t^2+1}.$$

We then obtain

$$\int\frac{\sqrt{c^2-x^2}}{x}\,dx \;=\; -c\int\frac{(1-t^2)^2}{t(1+t^2)^2}\,dt \;=\; c\int\left(\frac{4t}{(1+t^2)^2}-\frac{1}{t}\right)dt \;=\; -\frac{2c}{1+t^2}-c\ln|t|+C_1.$$

The equation tying x and t gives $\frac{2c}{1+t^2} = \frac{x}{t}$ and $t = \frac{c+\sqrt{c^2-x^2}}{x}$, whence

$$\int \frac{\sqrt{c^2-x^2}}{x}\,dx \;=\; -\frac{x^2}{c+\sqrt{c^2-x^2}} - c\ln\frac{c+\sqrt{c^2-x^2}}{x} + C_1 \;=\; -c+\sqrt{c^2-x^2} - c\ln\frac{c+\sqrt{c^2-x^2}}{x} + C_1$$

i.e.

$$\int \frac{\sqrt{c^2-x^2}}{x}\,dx \;=\; \sqrt{c^2-x^2} - c\ln\frac{c+\sqrt{c^2-x^2}}{x} + C.$$

3. In the integral $\displaystyle\int \frac{dx}{\sqrt{x^2+3x-4}}$, the radicand is $(x+4)(x-1)$. Using the third substitution of Euler, we take $\sqrt{x^2+4x-3} = (x+4)t$. This simplifies to $x-1 = (x+4)t^2$. Then we get

$$x \;=\; \frac{1+4t^2}{1-t^2}, \quad dx \;=\; \frac{10t}{(1-t^2)^2}\,dt, \quad \sqrt{x^2+3x-4} \;=\; \left(\frac{1+4tr}{1-t^2}+4\right)t \;=\; \frac{5t}{1-t^2}.$$

And we obtain

$$\int \frac{dx}{\sqrt{x^2+3x-4}} \;=\; \int \frac{10t(1-t^2)}{(1-t^2)^2\cdot 5t}\,dt \;=\; \int \frac{2}{1-t^2}\,dt = \ln\left|\frac{1+t}{1-t}\right| + C \;=\; \ln\left|\frac{1+\sqrt{\frac{x-1}{x+4}}}{1-\sqrt{\frac{x-1}{x+4}}}\right| + C$$

$$= \ln\left|\frac{\sqrt{x+4}+\sqrt{x-1}}{\sqrt{x+4}-\sqrt{x-1}}\right| + C.$$

References

[1] N. Piskunov: *Diferentsiaal- ja integraalarvutus kõrgematele tehnilistele õppeasutustele.* Viies, täiendatud trükk. Kirjastus “Valgus”, Tallinn (1965).