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Lipschitz condition and differentiability

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Author	Mathprof (13753)
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If X and Y are Banach spaces, e.g. $\mathbb{R}^n$, one can inquire about the relation between differentiability and the Lipschitz condition. If f is Lipschitz, the ratio

$$\frac{\|f(q) - f(p)\|}{\|q - p\|}, \quad p, q \in X$$

is bounded but is not assumed to converge to a limit.

Proposition 1 *Let $f : X \to Y$ be a* `http://planetmath.org/DifferentiableMapping`*continuo differentiable mapping between Banach spaces. If $K \subset X$ is a compact subset, then the restriction $f : K \to Y$ satisfies the Lipschitz condition.*

Proof. Let $\operatorname{lin}(X, Y)$ denote the Banach space of bounded linear maps from X to Y. Recall that the norm $\|T\|$ of a linear mapping $T \in \operatorname{lin}(X, Y)$ is defined by

$$\|T\| = \sup\{\frac{\|Tu\|}{\|u\|} : u \neq 0\}.$$

Let $\mathrm{D} f : X \to \operatorname{lin}(X, Y)$ denote the derivative of f. By definition $\mathrm{D} f$ is continuous, which really means that $\|\mathrm{D} f\| : X \to \mathbb{R}$ is a continuous function. Since $K \subset X$ is compact, there exists a finite upper bound $B_1 > 0$ for $\|\mathrm{D} f\|$ restricted to K. In particular, this means that

$$\|\mathrm{D} f(p)u\| \leq \|\mathrm{D} f(p)\|\|u\| \leq B_1\|u\|,$$

for all $p \in K,\ u \in X$.

Next, consider the secant mapping $s : X \times X \to \mathbb{R}$ defined by

$$s(p, q) = \begin{cases} \dfrac{\|f(q) - f(p) - \mathrm{D} f(p)(q - p)\|}{\|q - p\|} & q \neq p \\ 0 & p = q \end{cases}$$

This mapping is continuous, because f is assumed to be continuously differentiable. Hence, there is a finite upper bound $B_2 > 0$ for s restricted to the compact set $K \times K$. It follows that for all $p, q \in K$ we have

$$\begin{aligned} \|f(q) - f(p)\| &\leq \|f(q) - f(p) - \mathrm{D} f(p)(q - p)\| + \|\mathrm{D} f(p)(q - p)\| \\ &\leq B_2\|q - p\| + B_1\|q - p\| \\ &= (B_1 + B_2)\|q - p\| \end{aligned}$$

Therefore $B_1 + B_2$ is the desired Lipschitz constant. QED

Neither condition is stronger. For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is differentiable but not Lipschitz.