limit

Canonical name	Limit
Date of creation	2013-03-22 12:28:25
Last modified on	2013-03-22 12:28:25
Owner	djao (24)
Last modified by	djao (24)
Numerical id	12
Author	djao (24)
Entry type	Definition
Classification	msc 26A06
Classification	msc 26B12
Classification	msc 54E35
Related topic	Continuous

Let X and Y be metric spaces and let $a \in X$ be a limit point of X. Suppose that $f\colon X \setminus \{a\} \to Y$ is a function defined everywhere except at a. For $L \in Y$, we say the *limit* of $f(x)$ as x approaches a is equal to L, or

$$\lim_{x \to a} f(x) = L$$

if, for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, whenever $x \in X$ with $0 < d_X(x, a) < \delta$, then $d_Y(f(x), L) < \varepsilon$.

The formal definition of limit as given above has a well–deserved reputation for being notoriously hard for inexperienced students to master. There is no easy fix for this problem, since the concept of a limit is inherently difficult to state precisely (and indeed wasn't even accomplished historically until the 1800's by Cauchy, well after the development of calculus in the 1600's by Newton and Leibniz). However, there are number of related definitions, which, taken together, may shed some light on the nature of the concept.

- The notion of a limit can be generalized to mappings between arbitrary topological spaces, under some mild restrictions. In this context we say that $\lim_{x \to a} f(x) = L$ if a is a limit point of X and, for every neighborhood V of L (in Y), there is a deleted neighborhood U of a (in X) which is mapped into V by f. One also requires that the range Y be Hausdorff (or at least T_1) in order to ensure that limits, when they exist, are unique.

- Let $a_n, n \in \mathbb{N}$ be a sequence of elements in a metric space X. We say that $L \in X$ is the limit of the sequence, if for every $\varepsilon > 0$ there exists a natural number N such that $d(a_n, L) < \varepsilon$ for all natural numbers $n > N$.

- The definition of the limit of a mapping can be based on the limit of a sequence. To wit, $\lim_{x \to a} f(x) = L$ if and only if, for every sequence of points x_n in X converging to a (that is, $x_n \to a$, $x_n \neq a$), the sequence of points $f(x_n)$ in Y converges to L.

In calculus, X and Y are frequently taken to be Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, in which case the distance functions d_X and d_Y cited above are just Euclidean distance.