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solutions of $x^y = y^x$

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The equation

$$x^y \ = \ y^x \tag{1}$$

has trivial solutions on the line $y = x$. For other solutions one has the

Theorem. 1°. The only positive solutions of the equation (1) with $1 < x < y$ are in a parametric form

$$x \ = \ (1+u)^{\frac{1}{u}}, \qquad y \ = \ (1+u)^{\frac{1}{u}+1} \tag{2}$$

where $u > 0$.
2°. The only rational solutions of (1) are

$$x \ = \ \left(1+\frac{1}{n}\right)^n, \qquad y \ = \ \left(1+\frac{1}{n}\right)^{n+1} \tag{3}$$

where $n = 1,\, 2,\, 3,\, \ldots$
3°. Consequently, the only integer solution of (1) is

$$2^4 \ = \ 16 \ = \ 4^2.$$

Proof. 1°. Let $(x,\, y)$ be a solution of (1) with $1 < x < y$. Set $y = x+\delta$ $(\delta > 0)$. Now

$$x^{x+\delta} \ = \ (x+\delta)^x,$$

from which we obtain easily

$$x \ = \ \left(1+\frac{\delta}{x}\right)^{\frac{x}{\delta}} \ := \ (1+u)^{\frac{1}{u}},$$

where $u = \frac{\delta}{x}$. Then

$$y \ = \ x+\delta \ = \ x\left(1+\frac{\delta}{x}\right) \ = \ (1+u)^{\frac{1}{u}}(1+u) \ = \ (1+u)^{\frac{1}{u}+1}.$$

2°. The unit fractions $u = \frac{1}{n}$ yield from (2) rational solutions (3). Further, no irrational value of u cannot make both x and y of (2) rational, since otherwise the ratio $1+u$ of the latter numbers would be irrational (cf. rational and irrational). Accordingly, for other rational solutions than (3), we must consider the values

$$u := \frac{m}{n}$$

with coprime positive integers m, n where $m > 1$. Make the antithesis that

$$x = \left(1+\frac{m}{n}\right)^{\frac{n}{m}} \in \mathbb{Q}.$$

Because the integers coprime with m form a group with respect to the multiplication modulo m (cf. prime residue classes), the congruence

$$nz \equiv 1 \pmod{m}$$

has a solution z. Thus we may write $nz = km+1$ and rewrite the rational number

$$\left[\left(1+\frac{m}{n}\right)^{\frac{n}{m}}\right]^z = \left(1+\frac{m}{n}\right)^{\frac{nz}{m}} = \left(1+\frac{m}{n}\right)^{\frac{km+1}{m}} = \left(1+\frac{m}{n}\right)^k \left(1+\frac{m}{n}\right)^{\frac{1}{m}}. \tag{4}$$

This product form tells that $\left(1+\frac{m}{n}\right)^{\frac{1}{m}}$ is rational. But the number

$$\left(1+\frac{m}{n}\right)^{\frac{1}{m}} = \sqrt[m]{\frac{m+n}{n}}$$

cannot be rational without the coprime integers $m+n$ and n both being m^{th} `http://planetmath.org/GeneralAssociativity`powers. If we had $n = v^m$, then by Bernoulli inequality,

$$(v+1)^m > v^m+mv \geqq n+m,$$

i.e. $m+n$ could not be a m^{th} power. The contradictory situation means, by (4), that the antithesis is wrong. Therefore, the numbers (3) give the only rational solutions of (1).

Note. The value $n = 2$ in (3) produces $x = \frac{9}{4}$, $y = \frac{27}{8}$, whence (1) reads

$$\left(\frac{9}{4}\right)^{\frac{27}{8}} = \left(\frac{27}{8}\right)^{\frac{9}{4}}. \tag{5}$$

The truth of the equality (5) may also be checked by the calculation

$$\left(\frac{9}{4}\right)^{\frac{27}{8}} = \left[\left(\frac{9}{4}\right)^{\frac{1}{2}}\right]^{\frac{27}{4}} = \left(\frac{3}{2}\right)^{\frac{27}{4}} = \left[\left(\frac{3}{2}\right)^{3}\right]^{\frac{9}{4}} = \left(\frac{27}{8}\right)^{\frac{9}{4}}.$$

References

[1] P. Hohler & P. Gebauer: Kann man ohne Rechner entscheiden, ob e^π oder π^e grösser ist? – *Elemente der Mathematik* **36** (1981).

[2] J. Sondow & D. Marques: Algebraic and transcendental solutions of some exponential equations. – *Annales Mathematicae et Informaticae* **37** (2010); available directly at `http://arxiv.org/pdf/1108.6096.pdf`arXiv.