area of spherical zone

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Let us consider the circle

$$(x-r)^2+y^2 \;=\; r^2$$

with radius r and centre $(r,\,0)$. A spherical zone may be thought to be formed when an arc of the circle rotates around the x-axis. For finding the are of the zone, we can use the formula

$$A \;=\; 2\pi\int_a^b y\,\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{1}$$

of the entry area of surface of revolution. Let the ends of the arc correspond the values a and b of the abscissa such that $b-a = h$ is the of the spherical zone. In the formula, we must use the solved form

$$y \;=\; (\pm)\sqrt{rx-x^2}$$

of the equation of the circle. The formula then yields

$$A \;=\; 2\pi\int_a^b \sqrt{rx-x^2}\,\sqrt{1+\left(\frac{r-x}{\sqrt{rx-x^2}}\right)^2}\,dx \;=\; 2\pi\int_a^b r\,dx \;=\; 2\pi r(b-a).$$

Hence the area of a spherical zone (and also of a spherical calotte) is

$$A \;=\; 2\pi rh. \tag{2}$$

From here one obtains as a special case $h = 2r$ the area of the whole sphere:

$$A \;=\; 4\pi r^2. \tag{3}$$

Remark. The formula (2) implies that the centre of mass of a half-sphere is at the halfway point of the axis of symmetry ($h = \frac{r}{2}$).