concavity of sine function

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Theorem 1. *The sine function is concave on the interval* $[0, \pi]$.

Proof. Suppose that x and y lie in the interval $[0, \pi/2]$. Then $\sin x$, $\sin y$, $\cos x$, and $\cos y$ are all non-negative. Subtracting the identities

$$\sin^2 x + \cos^2 x = 1$$

and

$$\sin^2 y + \cos^2 y = 1$$

from each other, we conclude that

$$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x.$$

This implies that $\sin^2 x - \sin^2 y \geq 0$ if and only if $\cos^2 y - \cos^2 x \geq 0$, which is equivalent to stating that $\sin^2 x \geq \sin^2 y$ if and only if $\cos^2 x \leq \cos^2 y$. Taking square roots, we conclude that $\sin x \leq \sin y$ if and only if $\cos x \geq \cos y$.

Hence, we have

$$(\sin x - \sin y)(\cos x - \cos y) \leq 0.$$

Multiply out both sides and move terms to conclude

$$\sin x \cos x + \sin y \cos y \leq \sin x \cos y + \sin y \cos x.$$

Applying the angle addition and double-angle identities for the sine function, this becomes

$$\frac{1}{2}\left(\sin(2x) + \sin(2y)\right) \leq \sin(x + y).$$

This is equivalent to stating that, for all $u, v \in [0, \pi]$,

$$\frac{1}{2}\left(\sin u + \sin v\right) \leq \sin\left(\frac{u+v}{2}\right),$$

which implies that sin is concave in the interval $[0, \pi]$. $\square$