trigonometric formulas from series

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One may define the sine and the cosine functions for real (and complex) arguments using the power series

$$\sin x \;=\; x-\frac{x^3}{3!}+\frac{x^5}{5!}-+\ldots, \tag{1}$$

$$\cos x \;=\; 1-\frac{x^2}{2!}+\frac{x^4}{4!}-+\ldots, \tag{2}$$

and using only the properties of power series, easily derive most of the goniometric formulas, without any geometry. For example, one gets instantly from (1) and (2) the values

$$\sin 0 \;=\; 0, \qquad \cos 0 \;=\; 1$$

and the `http://planetmath.org/EvenoddFunction`parity relations

$$\sin(-x) \;=\; -\sin x, \qquad \cos(-x) \;=\; \cos x.$$

Using the Cauchy multiplication rule for series one can obtain the addition formulas

$$\begin{cases}\sin(x+y) \;=\; \sin x\cos y+\cos x\sin y,\\ \cos(x+y) \;=\; \cos x\cos y-\sin x\sin y.\end{cases} \tag{3}$$

These produce straightforward many other important formulae, e.g.

$$\sin 2x \;=\; 2\sin x\cos x, \qquad \cos 2x \;=\; \cos^2 x-\sin^2 x \qquad (y \;=:\; x) \tag{4}$$

and

$$\cos^2 x+\sin^2 x \;=\; 1 \qquad\qquad (y \;=:\; -x). \tag{5}$$

The value $\cos\frac{\pi}{2}=0$, as well as the formulae expressing the periodicity of sine and cosine, cannot be directly obtained from the series (1) and (2) — in fact, one must define the number π by using the function properties of the and its `http://planetmath.org/PowerSeries`derivative series.

The equation

$$\cos x \;=\; 0$$

has on the interval $(0,\,2)$ exactly one http://planetmath.org/Equationroot. Actually, as sum of a power series, $\cos x$ is continuous, $\cos 0 = 1 > 0$ and $\cos 2 < 1-\frac{2^2}{2!}+\frac{2^4}{4!} < 0$ (see http://planetmath.org/LeibnizEstimateForAlternatingSeriesLei estimate for alternating series), whence there is at least one root. If there were more than one root, then the derivative

$$-\sin x \;=\; -x+\frac{x^3}{3!}-+\ldots \;=\; -x(1-\frac{x^2}{3!}+-\ldots)$$

would have at least one zero on the interval; this is impossible, since by Leibniz the series in the parentheses does not change its sign on the interval:

$$1-\frac{x^2}{3!}+-\ldots \;>\; 1-\frac{2^2}{3!} \;>\; 0$$

Accordingly, we can define the number pi to be the least positive solution of the equation $\cos x = 0$, multiplied by 2.

Thus we have $0 < \pi < 4$ and $\cos\frac{\pi}{2} = 0$. Furthermore, by (5),

$$\sin\frac{\pi}{2} \;=\; 1,$$

and by (4),

$$\sin\pi \;=\; 0, \qquad \cos\pi \;=\; -1, \qquad \sin 2\pi \;=\; 0, \qquad \cos 2\pi \;=\; 1.$$

Consequently, the addition formulas (3) yield the http://planetmath.org/PeriodicFunctionsper

$$\sin(x+2\pi) \;=\; \sin x, \qquad \cos(x+2\pi) \;=\; \cos x.$$