integration of polynomial

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Theorem. *For all nonnegative integers n,*

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

Proof. It will first be proven that, for any nonnegative integer n and any $a \in \mathbb{R}$,

$$\int_0^a x^n\,dx = \frac{1}{n+1}a^{n+1}.$$

If $a = 0$, the above statement is obvious. If $a > 0$, the following computation uses the right hand rule for computing the http://planetmath.org/RiemannIntegralintegral; if $a < 0$, the following computation uses the left hand rule for computing the integral:

$$\begin{aligned}
\int_0^a x^n\,dx &= \lim_{t\to\infty}\sum_{k=1}^{t}\left(\frac{ak}{t}\right)^n\left(\frac{a}{t}\right) \\
&= a^{n+1}\lim_{t\to\infty}\frac{1}{t^{n+1}}\sum_{k=1}^{t}k^n \\
&= a^{n+1}\lim_{t\to\infty}\frac{1}{t^{n+1}}\sum_{l=1}^{n+1}\binom{n+1}{r}\frac{B_{n+1-l}}{n+1}(t+1)^l \text{ by http://planetmath.org/SumOfKthPow} \\
&= a^{n+1}\lim_{t\to\infty}\frac{1}{t^{n+1}}\binom{n+1}{n+1}\frac{B_{n+1-(n+1)}}{n+1}(t+1)^{n+1} \\
&= \frac{B_0}{n+1}a^{n+1}\lim_{t\to\infty}\left(\frac{t+1}{t}\right)^{n+1} \\
&= \frac{1}{n+1}a^{n+1}
\end{aligned}$$

Thus, if $a, b \in \mathbb{R}$, then $\int_a^b x^n\,dx = \int_0^b x^n\,dx - \int_0^a x^n\,dx = \frac{1}{n+1}b^{n+1} - \frac{1}{n+1}a^{n+1}$.

It follows that $\int x^n\,dx = \frac{1}{n+1}x^n + C$. □