proof of Fermat's Theorem (stationary points)

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Suppose that x_0 is a local maximum (a similar proof applies if x_0 is a local minimum). Then there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and such that we have $f(x_0) \geq f(x)$ for all x with $|x - x_0| < \delta$. Hence for $h \in (0, \delta)$ we notice that it holds

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

Since the limit of this ratio as $h \to 0^+$ exists and is equal to $f'(x_0)$ we conclude that $f'(x_0) \leq 0$. On the other hand for $h \in (-\delta, 0)$ we notice that

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

but again the limit as $h \to 0^+$ exists and is equal to $f'(x_0)$ so we also have $f'(x_0) \geq 0$.

Hence we conclude that $f'(x_0) = 0$.

To prove the second part of the statement (when x_0 is equal to a or b), just notice that in such points we have only one of the two estimates written above.