integration under integral sign

Canonical name	IntegrationUnderIntegralSign
Date of creation	2013-03-22 18:46:27
Last modified on	2013-03-22 18:46:27
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Last modified by	pahio (2872)
Numerical id	5
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Entry type	Theorem
Classification	msc 26A42
Related topic	FubinisTheorem
Related topic	DifferentiationUnderIntegralSign
Related topic	RelativeOfExponentialIntegral

Let

$$I(\alpha) \;=\; \int_a^b f(x,\,\alpha)\,dx.$$

where $f(x,\,\alpha)$ is continuous in the rectangle

$$a \leqq x \leqq b, \qquad \alpha_1 \leqq \alpha \leqq \alpha_2.$$

Then $\alpha \mapsto I(\alpha)$ is continuous and hence http://planetmath.org/RiemannIntegrableintegrable on the interval $\alpha_1 \leqq \alpha \leqq \alpha_2$; we have

$$\int_{\alpha_1}^{\alpha_2} I(\alpha)\,d\alpha \;=\; \int_{\alpha_1}^{\alpha_2} \left(\int_a^b f(x,\,\alpha)\,dx\right) d\alpha.$$

This is a double integral over a in the $x\alpha$-plane, whence one can change the http://planetmath.org/FubinisTheoremorder of integration and accordingly write

$$\int_{\alpha_1}^{\alpha_2} \left(\int_a^b f(x,\,\alpha)\,dx\right) d\alpha \;=\; \int_a^b \left(\int_{\alpha_1}^{\alpha_2} f(x,\,\alpha)\,d\alpha\right) dx.$$

Thus, a definite integral depending on a parametre may be integrated with respect to this parametre by performing the integration under the integral sign.

Example. For being able to evaluate the improper integral

$$I \;=\; \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx \qquad (a>0,\; b>0),$$

we may interprete the integrand as a definite integral:

$$\frac{e^{-ax}-e^{-bx}}{x} \;=\; \Big/_{\!\!\alpha=b}^{\,a} \frac{e^{-\alpha x}}{x} \;=\; \int_a^b e^{-\alpha x}\,d\alpha.$$

Accordingly, we can calculate as follows:

$$
\begin{aligned}
I &= \int_0^\infty \left(\int_a^b e^{-\alpha x} \, d\alpha \right) dx \\
&= \int_a^b \left(\int_0^\infty e^{-\alpha x} \, dx \right) d\alpha \\
&= \int_a^b \left(\Big/_{x=0}^{\infty} - \frac{e^{-\alpha x}}{\alpha} \right) d\alpha \\
&= \int_a^b \frac{1}{\alpha} \, d\alpha \;=\; \Big/_a^b \ln \alpha \\
&= \ln \frac{b}{a}
\end{aligned}
$$