proof of Bolzano's theorem

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Consider the compact interval $[a, b], a < b$ and a continuous real valued function f. If $f(a).f(b) < 0$ then there exists $c \in (a, b)$ such that $f(c) = 0$

WLOG consider $f(a) < 0$ and $f(b) > 0$. The other case can be proved using $-f(x)$ which will also verify the theorem's conditions.

consider $a_1 = \frac{a+b}{2}$, three cases can occur:

- $f(a_1) = 0$, in this case the theorem is proved $c = a_1$
- $f(a_1) > 0$, in this case consider the interval $I_1 = (a, a_1)$
- $f(a_1) < 0$, in this case consider the interval $I_1 = (a_1, b)$

so starting with an open interval $I_0 = (a, b)$ we get another open interval $I_1 \subset I_0$ with length half of the original $|I_1| = \frac{|I_0|}{2}$.

Repeat the procedure to the interval I_n and get another interval I_{n+1}.

We can thus define a succession of open intervals I_n such that $I_{n+1} \subset I_n$, $|I_n| = 2^{-n}|I_0|$, such that $I_n = (a_n, b_n)$ and $f(a_n) < 0 < f(b_n)$.

The succession $c_{2n} = a_n, c_{2n+1} = b_n$ is Cauchy by construction since $m > n \implies |c_m - c_n| < 2^{-[n/2]}|I_0|$.

c_n is therefore convergent $c_n \to c \in [a, b]$, and since a_n and b_n are sub-successions, they converge to the same limit.

f is continuous in $[a, b]$ so $x_n \to x \implies f(x_n) \to f(x)$

By construction

$f(a_n) < 0$ and $f(b_n) > 0$ so in the limit $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(c) \leq 0$ and $\lim_{n\to\infty} f(b_n) = f(c) \geq 0$.

So there exists $c \in [a, b]$ such that $0 \leq f(c) \leq 0 \implies f(c) = 0$.

But since $f(a).f(b) < 0$, neither $f(a) = 0$ nor $f(b) = 0$ and since $f(c) = 0$, $c \in (a, b)$