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rigorous definition of the logarithm

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In this entry, we shall construct the logarithm as a Dedekind cut and then demonstrate some of its basic properties. All that is required in the way of background material are the properties of integer powers of real numbers.

Theorem 1. *Suppose that a, b, c, d are positive integers such that $a/b = c/d$ and that $x > 0$ and $y > 0$ are real numbers. Then $x^a \leq y^b$ if and only if $x^c \leq y^d$.*

Proof. Cross multiplying, the condition $a/b = c/d$ is equivalent to $ad = bc$. By elementary properties of powers, $x^a \leq y^b$ if and only if $x^{ad} \leq y^{bd}$. Likewise, $x^c \leq x^d$ if and only if $x^{bc} \leq y^{bd}$ which, since $bc = ad$, is equivalent to $x^{ad} \leq y^{bd}$. Hence, $x^a \leq y^b$ if and only if $x^c \leq x^d$. □

Theorem 2. *Suppose that a, b, c, d are positive integers such that $a/b \leq c/d$ and that $x > 1$ and $y > 0$ are real numbers. If $x^c \leq y^d$ then $x^a \leq y^b$.*

Proof. Since we assumed that $b > 0$, we have that $x^c \leq y^d$ is equivalent to $x^{bc} \leq y^{bd}$. Likewise, since $d > 0$, we have that $x^a \leq y^b$ is equivalent to $x^{ad} \leq y^{bd}$. Cross-multiplying, $a/b \leq c/d$ is equivalent to $ad \leq bc$. Since $x > 1$, we have $x^{ad} \leq x^{bc}$. Combining the above statements, we conclude that $x^c \leq y^d$ implies $x^a \leq y^b$. □

Theorem 3. *Suppose that a, b, c, d are positive integers such that $a/b > c/d$ and that $x > 1$ and $y > 0$ are real numbers. If $x^a > y^b$ then $x^c > y^d$.*

Proof. Since we assumed that $b > 0$, we have that $x^c > y^d$ is equivalent to $x^{bc} > y^{bd}$. Likewise, since $d > 0$, we have that $x^a > y^b$ is equivalent to $x^{ad} > y^{bd}$. Cross-multiplying, $a/b > c/d$ is equivalent to $ad > bc$. Since $x > 1$, we have $x^{ad} > x^{bc}$. Combining the above statements, we conclude that $x^c > y^d$ implies $x^a > y^b$. □

Theorem 4. *Let $x > 1$ and y be real numbers. Then there exists an integer n such that $x^n > y$.*

Proof. Write $x = 1 + h$. Then we have $(1 + h)^n \geq 1 + nh$ for all positive integers n. This fact is easily proved by induction. When $n = 1$, it reduces to the triviality $1 + h \geq h$. If $(1 + h)^n \geq 1 + nh$, then

$$(1+h)^{n+1} = (1+h)(1+h)^n \geq (1+h)(1+nh) = 1+(n+1)h+nh^2 \geq 1+(n+1)h.$$

By the Archimedean property, there exists an integer n such that $1+nh > y$, so $x^n > y$. □

Theorem 5. *Let $x > 1$ and y be real numbers. Then the pair of sets (L, U) where*

$$L = \{r \in \mathbb{Q} \mid (\exists a, b \in \mathbb{Z}) \quad b > 0 \ \wedge \ r = a/b \ \wedge \ x^a \leq y^b\} \tag{1}$$
$$U = \{r \in \mathbb{Q} \mid (\exists a, b \in \mathbb{Z}) \quad b > 0 \ \wedge \ r = a/b \ \wedge \ x^a > y^b\} \tag{2}$$

forms a Dedekind cut.

Proof. Let r be any rational number. Then we have $r = a/b$ for some integers a and b such that $b > 0$. The possibilities $x^a \leq y^b$ and $x^a > y^b$ are exhaustive so r must belong to at least one of U and L. By theorem 1, it cannot belong to both. By theorem 2, if $r \in L$ and $s \leq r$, then $s \in L$ as well. By theorem 3, if $r \in U$ and $s > r$, then $s \in U$ as well. By theorem 4, neither L nor U are empty. Hence, (L, U) is a Dedekind cut and defines a real number. $\square$

Definition 1. *Suppose $x > 1$ and $y > 0$ are real numbers. Then, we define $\log_x y$ to be the real number defined by the cut (L, U) of the above theorem.*