Laplace transform of convolution

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Owner	pahio (2872)
Last modified by	pahio (2872)
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Author	pahio (2872)
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Theorem. If

$$\mathcal{L}\{f_1(t)\} \;=\; F_1(s) \quad \text{and} \quad \mathcal{L}\{f_2(t)\} \;=\; F_2(s),$$

then

$$\mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right\} \;=\; F_1(s)F_2(s).$$

Proof. According to the definition of Laplace transform, one has

$$\mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right\} \;=\; \int_0^\infty e^{-st}\left(\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right)dt,$$

where the right hand side is a double integral over the angular region bounded by the lines $\tau = 0$ and $\tau = t$ in the first quadrant of the $t\tau$-plane. Changing the of integration, we write

$$\mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right\} \;=\; \int_0^\infty \left(f_1(\tau)\int_\tau^\infty e^{-st}f_2(t-\tau)\,dt\right)d\tau.$$

Making in the inner integral the substitution $t-\tau := u$, we obtain

$$\int_\tau^\infty e^{-st}f_2(t-\tau)\,dt \;=\; \int_0^\infty e^{-(u+\tau)s}f_2(u)\,du = e^{-\tau s}\int_0^\infty e^{-su}f_2(u)\,du = e^{-\tau s}F_2(s),$$

whence

$$\mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right\} \;=\; \int_0^\infty f_1(\tau)e^{-\tau s}F_2(s)\,d\tau \;=\; F_2(s)\int_0^\infty f_1(\tau)e^{-\tau s}\,d\tau = F_1(s)F_2(s),$$

Q.E.D.