Hardy's inequality

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Suppose $p > 1$ and $\{a_n\}$ is a sequence of nonnegative real numbers. Let $A_n = \sum_{i=1}^{n} a_i$. Then

$$\sum_{n \geq 1} \left(\frac{A_n}{n} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n \geq 1} {a_n}^p,$$

unless all the a_n are zero. The constant is best possible.

This theorem has an integral analogue: Suppose that $p > 1$ and $f \geq 0$ on $(0, \infty)$. Let $F(x) = \int_0^x f(t) dt$. Then

$$\int_0^\infty \left(\frac{F}{x} \right)^p dx < \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx,$$

unless $f \equiv 0$. The constant is best possible.

References

[1] G.H. Hardy, J.E. Littlewood and G.Pólya, *Inequalities*, Cambridge University Press, Cambridge, 2nd edition, 1952, pp. 239-240.