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proof of generalized Leibniz rule

Canonical name	ProofOfGeneralizedLeibnizRule
Date of creation	2013-03-22 14:34:14
Last modified on	2013-03-22 14:34:14
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Numerical id	7
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Entry type	Proof
Classification	msc 26A06

The generalized Leibniz rule can be derived from the plain Leibniz rule by induction on r.

If $r = 2$, the generalized Leibniz rule reduces to the plain Leibniz rule. This will be the starting point for the induction. To complete the induction, assume that the generalized Leibniz rule holds for a certain value of r; we shall now show that it holds for $r + 1$.

Write $\prod_{i=1}^{r+1} f_i(t) = (f_{r+1}(t)) \left(\prod_{i=1}^{r+1} f_i(t)\right)$. Applying the plain Leibniz rule,

$$\frac{d^n}{dt^n} (f_{r+1}(t)) \left(\prod_{i=1}^{r+1} f_i(t)\right) = \sum_{n_{r+1}=0}^{n} \binom{n}{n_{r+1}} \left(\frac{d^{n_{r+1}}}{dn^{n_{r+1}}} f_{r+1}(t)\right) \left(\frac{d^{n-n_{r+1}}}{dn^{n-n_{r+1}}} \prod_{i=1}^{r+1} f_i(t)\right)$$

By the generalized Leibniz rule for r (assumed to be true as the induction hypothesis), this equals

$$\sum_{n_{r+1}=0}^{n} \sum_{n_1+\cdots+n_r=n-n_{r+1}} \binom{n-n_{r+1}}{n_1, n_2, \ldots n_r} \binom{n}{n_{r+1}} \left(\frac{d^{n_{r+1}}}{dn^{n_{r+1}}} f_{r+1}(t)\right) \left(\prod_{i=1}^{r} \frac{d^{n_i}}{dt^{n_i}} f_i(t)\right)$$

Note that

$$\binom{n-n_{r+1}}{n_1, n_2, \ldots n_r} \binom{n}{n_{r+1}} = \binom{n-n_{r+1}}{n_1, n_2, \ldots n_r, n_{r+1}}$$

This is an immediate consequence of the expression for multinomial coefficients as quotients of factorials. Using this identity, the quantity can be written as

$$\sum_{n_1+\cdots+n_r+n_{r+1}=n} \binom{n-n_{r+1}}{n_1, n_2, \ldots n_r, n_{r+1}} \prod_{i=1}^{r+1} \frac{d^{n_i}}{dt^{n_i}} f_i(t)$$

which is the generalized Leibniz rule for the case of $r + 1$.