support of integrable function is σ-finite

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Theroem - Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ a measurable function. If f is integrable, then the support of f is http://planetmath.org/SigmaFiniteσ-finite.

It follows easily from this result that any function in an http://planetmath.org/LpSpaceL^p-space, with $1 \leq p < \infty$, must have σ-finite support.

: Let $A_0 := [1, \infty[$, and for each $n \in \mathbb{N}$ let $A_n := [\frac{1}{n+1}, \frac{1}{n}[$. Since f is integrable, we must necessarily have $\mu\big(|f|^{-1}(A_n)\big) < \infty$ for each $n \in \mathbb{N} \cup \{0\}$, because

$$\mu\big(|f|^{-1}(A_n)\big) \cdot \frac{1}{n+1} \leq \int_{|f|^{-1}(A_n)} |f| \, d\mu \leq \int_X |f| \, d\mu < \infty.$$

Since f and $|f|$ have the same support, and the the support of the latter is $\operatorname{supp}|f| = \bigcup_{n=0}^{\infty} |f|^{-1}(A_n)$, it follows that the support of f is σ-finite. $\square$