fractional differentiation

Canonical name	FractionalDifferentiation
Date of creation	2013-03-22 16:18:46
Last modified on	2013-03-22 16:18:46
Owner	Wkbj79 (1863)
Last modified by	Wkbj79 (1863)
Numerical id	21
Author	Wkbj79 (1863)
Entry type	Definition
Classification	msc 26A06
Synonym	Grunwald-Letnikov differentiation
Related topic	HigherOrderDerivativesOfSineAndCosine
Defines	fractional derivative
Defines	left-hand Grunwald-Letnikov derivative
Defines	left hand Grundwald Letnikov derivative
Defines	right-hand Grundwald-Letnikov derivative
Defines	right hand Grundwald-Letnikov derivative

The idea of Grunwald-Letnikov differentiation comes from the following formulas of `http://planetmath.org/BackwardDifference`backward and forward difference . Within this entry, $[\cdot]$ will be used to denote the greatest integer function and Γ will be used to denote the gamma function.

Backward difference

$$D_{-}(f)(x)=\lim_{h\to 0}\frac{f(x)-f(x-h)}{h} \tag{1}$$

$$D_{-}^{n}(f)(x)=\lim_{h\to 0}\frac{1}{h^{n}}\sum_{k=0}^{n}\frac{(-1)^{k}n!}{k!(n-k)!}f(x-kh) \tag{2}$$

For derivatives of integer orders, we only requires to specifies one point $x\in\mathbb{R}$. Fractional derivatives, like fractional definite integrals, require an interval $[a,b]$ to be specified for the function $f:\mathbb{R}\to\mathbb{R}$ we are talking about.

Definition 1: Left-hand Grunwald-Letnikov derivative

$$D_{-}^{p}(f)(x)=\lim_{h\to 0}\frac{1}{h^{p}}\sum_{k=0}^{\left[\frac{b-a}{h}\right]}\frac{(-1)^{k}\Gamma(p+1)}{k!\Gamma(p-k+1)}f(x-kh) \tag{3}$$

Forward difference

$$D_{+}(f)(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \tag{4}$$

$$D_{+}^{n}(f)(x)=\lim_{h\to 0}\frac{1}{h^{n}}\sum_{k=0}^{n}\frac{(-1)^{k}n!}{k!(n-k)!}f(x+(n-k-1)h) \tag{5}$$

Definition 2: Right-hand Grunwald-Letnikov derivative

$$D_{+}^{p}(f)(x)=\lim_{h\to 0}\frac{1}{h^{p}}\sum_{k=0}^{\left[\frac{b-a}{h}\right]}\frac{(-1)^{k}\Gamma(p+1)}{k!\Gamma(p-k+1)}f(x+(m-k-1)h) \tag{6}$$

Theorem 1: Properties of fractional derivatives

- Linearity: $D_{\pm}^{p}(af+bg)(x)=aD_{\pm}^{p}(f)(x)+bD_{\pm}^{p}(g)(x)$ where $a,b\in\mathbb{R}$ are any real constants
- Iteration: $D_{\pm}^{p}D_{\pm}^{q}(f)(x)=D_{\pm}^{p+q}(f)(x)$

- Chain rule: $\dfrac{d^\beta f(g(x))}{dx^\beta} = \displaystyle\sum_{k=0}^{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+k)\Gamma(1-k+\beta)} \frac{d^{\beta-k}1}{dx^{\beta-k}} \frac{d^k f(g(x))}{dx^k}$

- Leibniz Rule: $\dfrac{d^\beta (f(x)g(x))}{dx^\beta} = \displaystyle\sum_{k=0}^{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+k)\Gamma(1-k+\beta)} \frac{d^k f(x)}{dx^k} \frac{d^{\beta-k} g(x)}{dx^{\beta-k}}$

Theorem 2: Table of fractional derivatives

- $D^\alpha_\pm(x^p) = \dfrac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}$ where $\alpha, p \in \mathbb{R}$ and $\Gamma(x)$

- $D^\alpha_\pm(e^{\lambda x}) = \lambda^\alpha e^{\lambda x}$ for all $\lambda \in \mathbb{R}$

- $D^\alpha_\pm(\sin x) = \sin\left(x + \dfrac{\alpha\pi}{2}\right)$

- $D^\alpha_\pm(\cos x) = \cos\left(x + \dfrac{\alpha\pi}{2}\right)$

- $D^\alpha_\pm(e^{ix}) = \cos\left(x + \dfrac{\pi\alpha}{2}\right) + i\sin\left(x + \dfrac{\pi\alpha}{2}\right)$