a lecture on integration by substitution

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The Method of Substitution (or Change of Variables)

The following is a general method to find indefinite integrals that look like the result of a chain rule.

- *When to use it:* We use the method of substitution for indefinite integrals which look like the result of a chain rule. In particular, try to use this method when you see a **composition of two functions**.

- *How to use it:* In this method, we go from integrating with respect to x to integrating with respect to a new variable, u, which makes the integral much easier.

  1. Find inside the integral the composition of two functions and set $u =$ "the inner function".
  2. We also write $du = \frac{du}{dx} dx$.
  3. Substitute everything in the integral that depends on x in terms of u.
  4. Integrate with respect to u.
  5. Once we have the result of integration in terms of u $(+C)$, substitute back in terms of x.

The method is best explained through examples:

Example 0.1. We want to find $\int e^{2x} dx$. The integrand is e^{2x}, which is a composition of two functions. The inner function is $2x$ so we set:

$$u = 2x, \quad du = 2dx$$

Thus,

$$x = u/2, \quad dx = du/2$$

Substitute into the integral:

$$\int e^{2x} dx = \int e^u \frac{du}{2} = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{2x} + C$$

The following are typical examples where we use the subsitution method:

Example 0.2.

$$\int xe^{3x^2+7}dx$$

The inner function is $u = 3x^2 + 7$ and $du = 6xdx$. Thus $dx = du/(6x)$. Substitute:

$$\int xe^{3x^2+7}dx = \int \frac{xe^u}{6x}du = \int \frac{e^u}{6}du = \frac{e^u}{6} + C = \frac{e^{3x^2+7}}{6} + C.$$

Example 0.3.

$$\int \sin(3x+7)dx$$

The inner function is $u = 3x + 7$ and $du = 3dx$. Therefore:

$$\int \sin(3x+7)dx = \int \frac{\sin(u)}{3}du = -\frac{\cos(u)}{3} + C = -\frac{\cos(3x+7)}{3} + C.$$

Example 0.4.

$$\int (2x+3)\sqrt{x^2+3x+20}\ dx$$

Inner $u = x^2 + 3x + 20$ and $du = (2x+3)dx$. Thus:

$$\int (2x+3)\sqrt{x^2+3x+20}\ dx = \int \sqrt{u}du = \int u^{1/2}du = \frac{2u^{3/2}}{3} + C = \frac{2(x^2+3x+20)^{3/2}}{3} + C.$$

Now another integral which is a little more difficult:

Example 0.5.

$$\int \frac{\cos(\ln x)}{x}dx$$

The inner function here is $u = \ln x$ and $du = \frac{1}{x}dx$.

$$\int \frac{\cos(\ln x)}{x}dx = \int \cos(u)\cdot\frac{1}{x}dx = \int \cos(u)du = \sin(u) + C = \sin(\ln x) + C.$$

Example 0.6.

$$\int \frac{3x^2+14x+1}{x^3+7x^2+x+115}dx$$

This function is also a typical example of integration with substitution. Whenever there is a fraction, and the numerator looks like the derivative of the denominator, we set u to be the denominator:

$$u = x^3 + 7x^2 + x + 115, \quad du = (3x^2 + 14x + 1)dx$$

Thus:

$$\int \frac{3x^2+14x+1}{x^3+7x^2+x+115}dx = \int \frac{1}{u}du = \ln u + C = \ln(x^3+7x^2+x+115) + C.$$

Example 0.7.

$$\int \frac{7}{1+3x}dx$$

As in the example above, we set $u = 1 + 3x$, $du = 3dx$:

$$\int \frac{7}{1+3x}dx = \int \frac{7}{u}\frac{du}{3} = \frac{7}{3}\int \frac{1}{u}du = \frac{7}{3}\ln u + C = \frac{7}{3}\ln(1+3x) + C.$$

Example 0.8.

$$\int t^3(t^4-50)^{700}dt$$

Here the inner function is $u = t^4 - 50$ and $du = 4t^3dt$. Thus

$$\int t^3(t^4-50)^{700}dt = \int \frac{u^{700}}{4}du = \frac{1}{4}\frac{u^{701}}{701} + C = \frac{(t^4-50)^{701}}{4\cdot 701} + C.$$

Some other examples (solve them!):

$$\int e^x \sin(e^x)dx, \quad \int \frac{e^x}{e^x+1}dx, \quad \int \frac{1}{x\ln x}dx$$