proof of intermediate value theorem

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We first prove the following lemma.

If $f : [a, b] \to \mathbb{R}$ is a continuous function with $f(a) \leq 0 \leq f(b)$ then there exists a $c \in [a, b]$ such that $f(c) = 0$.

Define the sequences (a_n) and (b_n) inductively, as follows.

$$a_0 = a \quad b_0 = b$$
$$c_n = \frac{a_n + b_n}{2}$$
$$(a_n, b_n) = \begin{cases} (a_{n-1}, c_{n-1}) & f(c_{n-1}) \geq 0 \\ (c_{n-1}, b_{n-1}) & f(c_{n-1}) < 0 \end{cases}$$

We note that

$$a_0 \leq a_1 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_1 \leq b_0$$
$$(b_n - a_n) = 2^{-n}(b_0 - a_0) \tag{1}$$
$$f(a_n) \leq 0 \leq f(b_n) \tag{2}$$

By the fundamental axiom of analysis $(a_n) \to \alpha$ and $(b_n) \to \beta$. But $(b_n - a_n) \to 0$ so $\alpha = \beta$. By continuity of f

$$(f(a_n)) \to f(\alpha) \quad (f(b_n)) \to f(\alpha)$$

But we have $f(\alpha) \leq 0$ and $f(\alpha) \geq 0$ so that $f(\alpha) = 0$. Furthermore we have $a \leq \alpha \leq b$, proving the assertion.

Set $g(x) = f(x) - k$ where $f(a) \leq k \leq f(b)$. g satisfies the same conditions as before, so there exists a c such that $f(c) = k$. Thus proving the more general result.