# A lecture on trigonometric integrals and trigonometric substitution

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# 1 Trigonometric Integrals

First, we must recall a few trigonometric identities:

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 &(1)\\
\sec^2 x &= 1 + \tan^2 x &(2)\\
\sin^2 x &= \frac{1-\cos(2x)}{2} &(3)\\
\cos^2 x &= \frac{1+\cos(2x)}{2} &(4)\\
\sin(2x) &= 2\sin x\cos x &(5)\\
\cos(2x) &= \cos^2 x - \sin^2 x. &(6)
\end{aligned}$$

The most usual integrals which involve trigonometric functions can be solved using the identities above.

**Example 1.1.** $\int \sin x dx = -\cos x + C$ and $\int \cos x dx = \sin x + C$ are immediate integrals.

**Example 1.2.** For $\int \sin^2 x dx$, $\int \cos^2 x dx$ we use formulas (3) and (4) respectively, e.g.

$$\int \sin^2 x dx = \int \frac{1-\cos(2x)}{2} dx = \frac{1}{2}\int (1-\cos(2x)) dx = \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C.$$

**Example 1.3.** For integrals of the form $\int \cos^m x \sin x \, dx$ or $\int \sin^m x \cos x \, dx$ we use substitution with $u = \cos x$ or $u = \sin x$ respectively, e.g.

$$\int \cos^2 x \sin x dx = \int -u^2 du = -\frac{u^3}{3} + C = -\frac{\cos^3 x}{3} + C. \; [u = \cos x, \; du = -\sin x dx]$$

In the following examples, we use equations (1) in the forms $\sin^2 x = 1 - \cos^2 x$ or $\cos^2 x = 1 - \sin^2 x$ to transform the integral into one of the type described in Example **??**.

**Example 1.4.**

$$\begin{aligned}
\int \sin^3 x dx &= \int \sin^2 x \sin x dx = \int (1-\cos^2 x)\sin x dx\\
&= \int \sin x dx - \int \cos^2 x \sin x dx\\
&= -\cos x + \frac{\cos^3 x}{3} + C.
\end{aligned}$$

Similarly one can solve $\int \cos^3 x dx$.

**Example 1.5.**

$$\begin{aligned}
\int \cos^3 x \sin^2 x dx &= \int \cos^2 x \cos x \sin^2 x dx = \int (1-\sin^2 x)\cos x \sin^2 x dx \\
&= \int \cos x \sin^2 x dx - \int \cos x \sin^4 x dx \\
&= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.
\end{aligned}$$

**Example 1.6.** In order to solve $\int \cos^5 x \sin^3 x dx$ we express it first as $\int \cos^5 x \sin^2 x \sin x = \int \cos^5 x(1-\cos^2 x)\sin x dx$ and then proceed as in the previous example.

One can use similar tricks to solve integrals which involve products of powers of $\sec x$ and $\tan x$, by using Equation (2). Also, recall that the derivative of $\tan x$ is $\sec^2 x$ while the derivative of $\sec x$ is $\sec x \tan x$.

**Example 1.7.**

$$\begin{aligned}
\int \tan^5 x \sec^4 x dx &= \int \tan^5 x \sec^2 x \sec^2 x dx = \int \tan^5 x(1+\tan^2 x)\sec^2 x dx \\
&= \int \tan^5 x \sec^2 x dx + \int \tan^7 x \sec^2 x dx \\
&= \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + C.
\end{aligned}$$

**Example 1.8.**

$$\begin{aligned}
\int \tan^3 x \sec^4 x dx &= \int \tan x \tan^2 x \sec^4 x dx = \int \tan x(\sec^2 x - 1)\sec^4 x dx \\
&= \int \tan x \sec x \sec^5 x dx - \int \tan x \sec x \sec^3 x dx \\
&= \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C.
\end{aligned}$$

# 2 Trigonometric Substitutions

One can easily deduce that $\int_0^1 \sqrt{1-x^2}dx$ has value $\frac{\pi}{4}$. Why? Simply because the graph of the function $y = \sqrt{1-x^2}$ is half a circumference of radius $r = 1$

(because if you square both sides of $y = \sqrt{1-x^2}$ you obtain $x^2 + y^2 = 1$ which is the equation of a circle or radius $r = 1$). Therefore, the area under the graph is a quarter of the area of a circle.

How does one compute $\int_0^1 \sqrt{1-x^2}dx$ without using the geometry of the problem? This is the prototype of integral where a trigonometric substitution will work very nicely. Notice that neither substitution nor integration by parts will work appropriately.

**Example 2.1.** Suppose we want to solve $\int_0^1 \sqrt{1-x^2}dx$ with analytic methods. We will use a substitution $x = \sin\theta$ (so $\theta$ will be our new variable of integration), because, as we know from Equation (1), $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$, thus getting rid of the pesky square root. Notice that $dx = \cos\theta d\theta$. We also need to find the new limits of integration with respect to the new variable of integration, namely $\theta$. When $x = 0 = \sin\theta$ we must have $\theta = 0$. Similarly, when $x = 1 = \sin\theta$ one has $\theta = \pi/2$. We are now ready to integrate:

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}dx &= \int_0^{\pi/2} (\cos\theta)\cos\theta d\theta = \int_0^{\pi/2} \cos^2\theta d\theta \\ &= \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta = \frac{1}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{\pi/2} = \pi/4.\end{aligned}$$

Notice that we made use of Equation (4) in the second line.

**Example 2.2.** Similarly, one can solve $\int_0^r \sqrt{r^2-x^2}dx$ by using a substitution $x = r\sin\theta$. Indeed, $\sqrt{r^2-x^2} = \sqrt{r^2 - r^2\sin^2\theta} = r\cos\theta$ and $dx = r\cos\theta d\theta$. The limits of integration with respect to $\theta$ are again $\theta = 0$ to $\theta = \pi/2$ (check this!). Thus:

$$\begin{aligned}\int_0^r \sqrt{r^2-x^2}dx &= \int_0^{\pi/2} r^2(\cos\theta)\cos\theta d\theta = r^2\int_0^{\pi/2} \cos^2\theta d\theta \\ &= r^2\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta = \frac{r^2}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{\pi/2} = r^2\pi/4.\end{aligned}$$

Thus, we have proved that a quarter of a circle of radius $r$ has area $r^2\pi/4$ which implies that the area of such a circle is $\pi r^2$, as usual.

The trigonometric substitutions *usually* work when expressions like $\sqrt{r^2-x^2}$, $\sqrt{r^2+x^2}$, $\sqrt{x^2-r^2}$ appear in the integral at hand, for some real number $r$. Here is a table of the suggested change of variables in each particular case:

| *If you see...* | *try this...* | *because...* |
|---|---|---|
| $\sqrt{1-x^2}$ | $x=\sin\theta$ | $\sqrt{1-\sin^2\theta}=\cos\theta$ |
| $\sqrt{r^2-x^2}$ | $x=r\sin\theta$ | $\sqrt{r^2-\sin^2\theta}=r\cos\theta$ |
| $\sqrt{1+x^2}$ | $x=\tan\theta$ | $\sqrt{1+\tan^2\theta}=\sec\theta$ |
| $\sqrt{r^2+x^2}$ | $x=r\tan\theta$ | $\sqrt{r^2+\tan^2\theta}=r\sec\theta$ |
| $\sqrt{x^2-1}$ | $x=\sec\theta$ | $\sqrt{\sec^2\theta-1}=\tan\theta$ |
| $\sqrt{x^2-r^2}$ | $x=r\sin\theta$ | $\sqrt{\sec^2\theta-1}=r\tan\theta$ |

**Remark 2.3.** *The above are "suggested" substitutions, they may not be the most ideal choice! For example, for the integral $\int 2x\sqrt{1-x^2}dx$, the change $u=1-x^2$ will work much better than $x=\sin\theta$.*

**Example 2.4.** We would like to find the value of

$$\int_{\sqrt{2}}^{2}\frac{1}{x^3\sqrt{x^2-1}}dx.$$

Since neither a $u$-substitution nor integration by parts seem appropriate, we try $x=\sec\theta$, $dx=\sec\theta\tan\theta d\theta$. When $x=\sqrt{2}=\sec\theta$ one has $\theta=\pi/4$ while $x=2$ implies $\theta=\pi/3$. Hence:

$$\int_{\sqrt{2}}^{2}\frac{1}{x^3\sqrt{x^2-1}}dx = \int_{\pi/4}^{\pi/3}\frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}d\theta=\int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}d\theta=\int_{\pi/4}^{\pi/3}\cos^2\theta d\theta$$

and the last integral is easy to compute using Equation (4).