direct sum of even/odd functions (example)

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Example. Direct sum of even and odd functions
Let us define the sets

$$\begin{aligned}
F &= \{f \,|\, f \text{ is a function from } \mathbb{R} \text{ to } \mathbb{R}\}, \\
F_+ &= \{f \in F \,|\, f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}, \\
F_- &= \{f \in F \,|\, f(x) = -f(-x) \text{ for all } x \in \mathbb{R}\}.
\end{aligned}$$

In other words, F contain all functions from $\mathbb{R}$ to $\mathbb{R}$, $F_+ \subset F$ contain all even functions, and $F_- \subset F$ contain all odd functions. All of these spaces have a natural vector space structure: for functions f and g we define $f + g$ as the function $x \mapsto f(x) + g(x)$. Similarly, if c is a real constant, then cf is the function $x \mapsto cf(x)$. With these operations, the zero vector is the mapping $x \mapsto 0$.

We claim that F is the direct sum of F_+ and F_-, i.e., that

$$F = F_+ \oplus F_-. \tag{1}$$

To prove this claim, let us first note that $F_\pm$ are vector subspaces of F. Second, given an arbitrary function f in F, we can define

$$\begin{aligned}
f_+(x) &= \frac{1}{2}\big(f(x) + f(-x)\big), \\
f_-(x) &= \frac{1}{2}\big(f(x) - f(-x)\big).
\end{aligned}$$

Now f_+ and f_- are even and odd functions and $f = f_+ + f_-$. Thus any function in F can be split into two components f_+ and f_-, such that $f_+ \in F_+$ and $f_- \in F_-$. To show that the sum is direct, suppose f is an element in $F_+ \cap F_-$. Then we have that $f(x) = -f(-x) = -f(x)$, so $f(x) = 0$ for all x, i.e., f is the zero vector in F. We have established equation **??**.