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binomial formula for negative integer powers

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For negative integer powers, the binomial formula can be written in terms of binomial coefficients like so:

$$(1-x)^{-n} = \sum_{m=1}^{\infty} \binom{m+n-1}{n-1} x^m$$

Proof: We shall prove this by induction on n. First, note that, if $n = 1$, then $\binom{m}{0} = 1$, so our formula reduces to

$$(1-x)^{-1} = \sum_{m=1}^{\infty} x^m,$$

which is the formula for the sum of an infinite geometric series.

Next, suppose that the formula is valid for a certain value of n. Then we have

$$(1-x)^{-n-1} = (1-x)^{-1}(1-x)^{-n} = \left(\sum_{k=0}^{\infty} x^k \right) \left(\sum_{m=0}^{\infty} \binom{m+n-1}{n-1} x^m \right)$$

The product of sums can be rewritten as the following double sum:

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{n+k-1}{n-1} x^m$$

The easiest way to see this is by rearranging the double sum as follows and adding columns

$$\begin{array}{lcccccccccccc}
x^0 \sum_{m=0}^{\infty} \binom{m+n-1}{n-1} x^m = & \binom{n-1}{n-1} & + & \binom{n}{n-1} x & + & \binom{n+1}{n-1} x^2 & + & \binom{n+2}{n-1} x^3 & + & \binom{n+3}{n-1} x^4 & + & \cdots \\
x^1 \sum_{m=0}^{\infty} \binom{m+n-1}{n-1} x^m = & & & \binom{n-1}{n-1} x & + & \binom{n}{n-1} x^2 & + & \binom{n+1}{n-1} x^3 & + & \binom{n+2}{n-1} x^4 & + & \cdots \\
x^2 \sum_{m=0}^{\infty} \binom{m+n-1}{n-1} x^m = & & & & & \binom{n-1}{n-1} x^2 & + & \binom{n}{n-1} x^3 & + & \binom{n+1}{n-1} x^4 & + & \cdots \\
x^3 \sum_{m=0}^{\infty} \binom{m+n-1}{n-1} x^m = & & & & & & & \binom{n-1}{n-1} x^3 & + & \binom{n}{n-1} x^4 & + & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

To evaluate the finite sums, we shall use the following identity for binomial coefficients. (See the entry `http://planetmath.org/node/273` "binomial coefficient" for more information about this identity.)

$$\sum_{k=0}^{m} \binom{n+k-1}{n-1} = \binom{m+n}{n}$$

Inserting this result value for the finite sum back into the double sum, we obtain

$$(1-x)^{-n-1} = \sum_{m=0}^{\infty} \binom{m+n}{n} x^m.$$

Q.E.D.