total variation

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Defines	bounded variation
Defines	rectifiable path

Let $\gamma : [a,b] \to X$ be a function mapping an interval $[a,b]$ to a metric space (X,d). We say that γ is of *bounded variation* if there is a constant M such that, for each partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$,

$$v(\gamma, P) = \sum_{k=1}^{n} d(\gamma(t_k), \gamma(t_{k-1})) \leq M.$$

The *total variation* V_γ of γ is defined by

$$V_\gamma = \sup\{v(\gamma, P) : P \text{ is a partition of } [a,b]\}.$$

It can be shown that, if X is either $\mathbb{R}$ or $\mathbb{C}$, every continuously differentiable (or piecewise continuously differentiable) function $\gamma : [a,b] \to X$ `http://planetmath.org/ContinuousDerivativeImpliesBoundedVariation`is of bounded variation, and

$$V_\gamma = \int_a^b |\gamma'(t)| dt.$$

Also, if γ is of bounded variation and $f : [a,b] \to X$ is continuous, then the Riemann-Stieltjes integral $\int_a^b f d\gamma$ is finite.

If γ is also continuous, it is said to be a *rectifiable path*, and $V(\gamma)$ is the length of its trace.

If $X = \mathbb{R}$, it can be shown that γ is of bounded variation if and only if it is the difference of two monotonic functions.