example of Riemann triple integral

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Determine the volume of the solid in $\mathbb{R}^3$ by the part of the surface

$$(x^2+y^2+z^2)^3 \;=\; 3a^3xyz$$

being in the first octant ($a > 0$).

Since $x^2+y^2+z^2$ is the squared distance of the point $(x,\,y,\,z)$ from the origin, the solid is apparently defined by

$$D \;:=\; \{(x,\,y,\,z)\in\mathbb{R}^3 \vdots\;\, x\geqq 0,\;\; y\geqq 0,\;\; z\geqq 0,\;\; (x^2+y^2+z^2)^3 \leqq 3a^3xyz\}.$$

By the definition

$$\mathbf{meas}(D) \;:=\; \int \chi_D(v)\,dv$$

in the `http://planetmath.org/RiemannMultipleIntegral`parent entry, the volume in the question is

$$V \;=\; \int_D 1\,dv \;=\; \iiint_D dx\,dy\,dz. \tag{1}$$

For calculating the integral (1) we express it by the (geographic) spherical coordinates through

$$\begin{cases} x \;=\; r\cos\varphi\cos\lambda \\ y \;=\; r\cos\varphi\sin\lambda \\ z \;=\; r\sin\varphi \end{cases}$$

where the latitude angle φ of the position vector $\vec{r}$ is measured from the xy-plane (not as the colatitude ϕ from the positive z-axis); λ is the longitude. For the change of coordinates, we need the Jacobian determinant

$$\frac{\partial(x,\,y,\,z)}{\partial(r,\,\varphi,\,\lambda)} \;=\; \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \lambda} & \frac{\partial y}{\partial \lambda} & \frac{\partial z}{\partial \lambda} \end{vmatrix} \;=\; \begin{vmatrix} \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \\ -r\sin\varphi\cos\lambda & -r\sin\varphi\sin\lambda & r\cos\varphi \\ -r\cos\varphi\sin\lambda & r\cos\varphi\cos\lambda & 0 \end{vmatrix},$$

which is simplified to $r^2\cos\varphi$. The equation of the surface attains the form

$$r^6 \;=\; 3a^3r^3\cos^2\varphi\sin\varphi\cos\lambda\sin\lambda,$$

or

$$r \;=\; \sqrt[3]{3a^3\cos^2\varphi\sin\varphi\cos\lambda\sin\lambda} \;:=\; r(\varphi,\,\lambda).$$

In the solid, we have $0 \leqq r \leqq r(\varphi,\, \lambda)$ and

$$r \;=\; 0 \quad \text{if only if} \quad \cos^2\varphi \sin\varphi \cos\lambda \sin\lambda \;=\; 0.$$

Thus we can write

$$V \;=\; \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{r(\varphi,\,\lambda)} r^2 \cos\varphi \; d\varphi \; d\lambda \; dr \;=\; \frac{1}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(\Big/_{r=0}^{r(\varphi,\,\lambda)} r^3 \right) \cos\varphi \; d\varphi \; d\lambda,$$

getting then

$$V \;=\; a^3 \int_0^{\frac{\pi}{2}} (\cos^3\varphi)(-\sin\varphi)\, d\varphi \cdot \int_0^{\frac{\pi}{2}} (\cos\lambda)(-\sin\lambda)\, d\lambda \;=\; a^3 \Big/_{\varphi=0}^{\frac{\pi}{2}} \frac{\cos^4\varphi}{4} \cdot \Big/_{\lambda=0}^{\frac{\pi}{2}} \frac{\cos^2\lambda}{2} \;=\; \frac{a^3}{8}.$$

Remark. The general for variable changing in a triple integral is

$$\iiint_D f(x,\, y,\, z)\, dx\, dy\, dz = \iiint_\Delta f(x(\xi,\, \eta,\, \zeta),\, y(\xi,\, \eta,\, \zeta),\, z(\xi,\, \eta,\, \zeta)) \left| \frac{\partial(x,\, y,\, z)}{\partial(\xi,\, \eta,\, \zeta)} \right| d\xi\, d\eta\, d\zeta.$$