angle multiplication and division formulae for tangent

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From the angle addition formula for the tangent, we may derive formulae for tangents of multiples of angles:

$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$
$$\tan(3x) = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}$$
$$\tan(4x) = \frac{4\tan x - 3\tan^3 x}{1-6\tan^2 x + \tan^4 x}$$

These formulae may be derived from a recursion. Write $\tan x = w$ and write $\tan(nx) = u_n/v_n$ where the u's and the v's are polynomials in w. Then we have the initial values $u_1 = w$ and $v_1 = 1$ and the recursions

$$u_{n+1} = u_n + wv_n$$
$$v_{n+1} = v_n - wu_n,$$

which follow from the addition formula. Moreover, if we know the tangent of an angle and are interested in finding the tangent of a multiple of that angle, we may use our recursions directly without first having to derive the multiple angle formulae. From these recursions, one may show that the u's will only involve odd powers of w and the v's will only involve even powers of w.

Proceeding in the opposite direction, one may consider bisecting an angle. Solving for $\tan x$ in the duplication formula above, one arrives at the following half-angle formula:

$$\tan\left(\frac{x}{2}\right) = \sqrt{1+\frac{1}{\tan^2 x}} - \frac{1}{\tan x}$$

Expressing the tangent in terms of sines and cosines and simplifying, one finds the following equivalent formulae:

$$\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$