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proof of Bernoulli's inequality

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Let I be the interval $(-1, \infty)$ and $f : I \to \mathbb{R}$ the function defined as:

$$f(x) = (1+x)^\alpha - 1 - \alpha x$$

with $\alpha \in \mathbb{R} \setminus \{0, 1\}$ fixed. Then f is differentiable and its derivative is

$$f'(x) = \alpha(1+x)^{\alpha-1} - \alpha, \text{ for all } x \in I,$$

from which it follows that $f'(x) = 0 \Leftrightarrow x = 0$.

1. If $0 < \alpha < 1$ then $f'(x) < 0$ for all $x \in (0, \infty)$ and $f'(x) > 0$ for all $x \in (-1, 0)$ which means that 0 is a global maximum point for f. Therefore $f(x) < f(0)$ for all $x \in I \setminus \{0\}$ which means that $(1+x)^\alpha < 1 + \alpha x$ for all $x \in (-1, 0)$.

2. If $\alpha \notin [0, 1]$ then $f'(x) > 0$ for all $x \in (0, \infty)$ and $f'(x) < 0$ for all $x \in (-1, 0)$ meaning that 0 is a global minimum point for f. This implies that $f(x) > f(0)$ for all $x \in I \setminus \{0\}$ which means that $(1+x)^\alpha > 1 + \alpha x$ for all $x \in (-1, 0)$.

Checking that the equality is satisfied for $x = 0$ or for $\alpha \in \{0, 1\}$ ends the proof.