growth of exponential function

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Lemma.

$$\lim_{x\to\infty}\frac{x^a}{e^x}=0$$

for all values of a.

Proof. Let ε be any positive number. Then we get:

$$0<\frac{x^a}{e^x}\leqq\frac{x^{\lceil a\rceil}}{e^x}<\frac{x^{\lceil a\rceil}}{\frac{x^{\lceil a\rceil+1}}{(\lceil a\rceil+1)!}}=\frac{(\lceil a\rceil+1)!}{x}<\varepsilon$$

as soon as $x>\max\{1,\,\frac{(\lceil a\rceil+1)!}{\varepsilon}\}$. Here, $\lceil\cdot\rceil$ the ceiling function; e^x has been estimated downwards by taking only one of the all positive

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$$

Theorem. The of the real exponential function $x\mapsto b^x$ exceeds all power functions, i.e.

$$\lim_{x\to\infty}\frac{x^a}{b^x}=0$$

with a and b any , $b>1$.

Proof. Since $\ln b>0$, we obtain by using the lemma the result

$$\lim_{x\to\infty}\frac{x^a}{b^x}=\lim_{x\to\infty}\left(\frac{x^{\frac{a}{\ln b}}}{e^x}\right)^{\ln b}=0^{\ln b}=0.$$

Corollary 1. $\lim\limits_{x\to 0+}x\ln x=0.$

Proof. According to the lemma we get

$$0=\lim_{u\to\infty}\frac{-u}{e^u}=\lim_{x\to 0+}\frac{-\ln\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to 0+}x\ln x.$$

Corollary 2. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}=0.$

Proof. Change in the lemma x to $\ln x$.

Corollary 3. $\lim_{x\to\infty} x^{\frac{1}{x}} = 1$. (Cf. limit of nth root of n.)

Proof. By corollary 2, we can write: $x^{\frac{1}{x}} = e^{\frac{\ln x}{x}} \longrightarrow e^0 = 1$ as $x \to \infty$ (see also theorem 2 in limit rules of functions).