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two improper integrals

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Let us consider first the improper integral

$$I(k) \;:=\; \int_0^\infty \frac{1-\cos kx}{x^2}\,dx.$$

The derivative $I'(k)$ may be formed by http://planetmath.org/DifferentiationUnderIntegral under the integral sign:

$$I'(k) \;=\; \int_0^\infty \left(\frac{\partial}{\partial k}\frac{1-\cos kx}{x^2}\right)dx \;=\; \int_0^\infty \frac{\sin kx}{x}\,dx \;=\; \int_0^\infty \frac{\sin t}{t}\,dt$$

Here, the last form has been gotten by the http://planetmath.org/ChangeOfVariableInDefinit $kx = t$. But since by the http://planetmath.org/SineIntegralInInfinityparent entry we have

$$\int_0^\infty \frac{\sin t}{t}\,dt \;=\; \frac{\pi}{2}$$

and since $I(0) = 0$, we can write

$$I(k) \;=\; \int_0^k \frac{\pi}{2}\,dk \;=\; \frac{\pi k}{2}.$$

Thus we have evaluated the integral $I(k)$:

$$\int_0^\infty \frac{1-\cos kx}{x^2}\,dx \;=\; \frac{\pi k}{2}. \tag{1}$$

The formula (1) gives

$$I(1) \;=\; \int_0^\infty \frac{1-\cos x}{x^2}\,dx \;=\; \frac{\pi}{2}.$$

We use here the consequence formula

$$1-\cos x \;=\; 2\sin^2\frac{x}{2}$$

of the double angle formula $\cos 2\alpha = 1-2\sin^2\alpha$, obtaining

$$\frac{\pi}{2} \;=\; 2\int_0^\infty \frac{\sin^2\frac{x}{2}}{x^2}\,dx \;=\; \int_0^\infty \frac{\sin^2 u}{u^2}\,du,$$

where the substitution $\frac{x}{2} = u$ has produced the last form. Accordingly, we can write as result the formula

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx \;=\; \frac{\pi}{2}. \tag{2}$$