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properties of O and o

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The following properties of Landau notation hold:

1. $o(f)$ and $O(f)$ are vector spaces, i.e. if $g, h \in o(f)$ (resp. in $O(f)$) then $\lambda g + \mu h \in o(f)$ (resp. in $O(f)$) whenever $\lambda, \mu \in \mathbb{R}$; In particular $o(f) + o(f) = o(f)$ and $\lambda o(f) = o(f)$;
2. if $\lambda \neq 0$ then $\lambda o(f) = o(f)$ and $\lambda O(f) = O(f)$;
3. $f o(g) = o(fg)$, $f O(g) = O(fg)$;
4. $o(g)^\alpha = o(g^\alpha)$, $O(g)^\alpha = O(g^\alpha)$;
5. $o(f) \subseteq O(f)$; on the other hand if $f \in o(g)$ then $O(f) \subseteq o(g)$;
6. $o(f) \subseteq o(g)$ if $f \in O(g)$; analogously $O(f) \subseteq O(g)$ if $f \in O(g)$;
7. $o(o(f)) = o(f)$, $O(O(f)) = O(f)$, $o(O(f)) = o(f)$, $O(o(f)) = o(f)$.

Here are some examples. First of all we consider Taylor formula. If $x_0 \in (a, b) \subset \mathbb{R}$ and $f\colon (a, b) \to \mathbb{R}$ has n derivatives, then

$$f(x) \in \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o((x - x_0)^n).$$

As a consequence, if f has $n+1$ derivatives, we can replace $o((x-x_0)^n)$ with $O((x - x_0)^{n+1})$ in the previous formula.

For example:

$$e^x \in 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^4 + O(x^5) \subset 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^4 + o(x^4).$$

Using the properties stated above we can compose and iterate Taylor expansions. For example from the expansions

$$\sin x \in x + \frac{x^3}{3!} + o(x^4), \qquad e^x \in 1 + x + \frac{x^2}{2} + O(x^3),$$

$$\cos x \in 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5) \subseteq 1 - \frac{x^2}{2} + O(x^4), \qquad \log(1+x) \in x - \frac{x^2}{2} + o(x^2)$$

we get

$$
\begin{aligned}
(x\sin x - e^{(x^2)})\log(\cos x) &\in \left(x(x - \frac{x^3}{3!} + o(x^4)) - (1 + x^2 + \frac{x^4}{2} + O((x^2)^3)\right)\log(1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x \\
&= \left(x^2 - \frac{x^4}{3!} + o(x^4) - 1 - x^2 - \frac{x^4}{2} + O(x^6)\right)\left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5) - \frac{(-\frac{x^2}{2}}{} \right. \\
&= (-1 - \frac{2}{3}x^4 + o(x^4) + O(x^6))\left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5) - \frac{\frac{x^4}{4} - 2\frac{x^2}{2}o(x^3) + (o(}{2}\right. \\
&= (-1 - \frac{2}{3}x^4 + o(x^4))(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5) + \frac{x^4}{8} + o(x^5) + o(x^6) + o(x^4)) \\
&= (-1 - \frac{2}{3}x^4 + o(x^4))(-\frac{x^2}{2} + 6x^4 + o(x^4)) \\
&= -\frac{x^2}{2} - 6x^4 + o(x^4) + x^4O(x^2) + o(x^4)O(x^2) \\
&= -\frac{x^2}{2} - 6x^4 + o(x^4) + O(x^6) + o(x^6) \\
&= -\frac{x^2}{2} - 6x^4 + o(x^4)
\end{aligned}
$$