Schur's inequality

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Owner	rspuzio (6075)
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Author	rspuzio (6075)
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If a, b, and c are non-negative real numbers and $k \geq 1$ is real, then the following inequality holds:

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0$$

Proof. We can assume without loss of generality that $c \leq b \leq a$ via a permutation of the variables (as both sides are symmetric in those variables). Then collecting terms, we wish to show that

$$(a-b)\left(a^k(a-c) - b^k(b-c)\right) + c^k(a-c)(b-c) \geq 0$$

which is clearly true as every term on the left is positive. □

There are a couple of special cases worth noting:

- Taking $k = 1$, we get the well-known

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + ac(a+c) + bc(b+c)$$

- If $c = 0$, we get $(a-b)(a^{k+1} - b^{k+1}) \geq 0$.
- If $b = c = 0$, we get $a^{k+2} \geq 0$.
- If $b = c$, we get $a^k(a-c)^2 \geq 0$.