application of fundamental theorem of integral calculus

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We will derive the addition formulas of the sine and the cosine functions supposing known only their derivatives and the chain rule.

Define the function $F:\mathbb{R}\to\mathbb{R}$ through

$$F(x) \;:=\; [\sin x\cos\alpha+\cos x\sin\alpha-\sin(x+\alpha)]^2+[\cos x\cos\alpha-\sin x\sin\alpha-\cos(x+\alpha)]^2$$

where α is, for the , a constant. The derivative of F is easily calculated:
$F'(x) \;=$
$2[\sin x\cos\alpha+\cos x\sin\alpha-\sin(x+\alpha)][\cos x\cos\alpha-\sin x\sin\alpha-\cos(x+\alpha)]$
$+\,2[\cos x\cos\alpha-\sin x\sin\alpha-\cos(x+\alpha)][-\sin x\cos\alpha-\cos x\sin\alpha+\sin(x+\alpha)]$

But this expression is identically 0. By the fundamental theorem of integral calculus, F must be a constant function. Since $F(0)=0$, we have

$$F(x) \;\equiv\; 0$$

for any x and naturally also for any α. Because $F(x)$ is a sum of two squares, the both addends of it have to vanish identically, which yields the equalities

$$\sin x\cos\alpha+\cos x\sin\alpha-\sin(x+\alpha) \;=\; 0, \qquad \cos x\cos\alpha-\sin x\sin\alpha-\cos(x+\alpha) \;=\; 0.$$

These the `http://planetmath.org/GoniometricFormulae`addition formulas

$$\begin{aligned}\sin(x+\alpha) &= \sin x\cos\alpha+\cos x\sin\alpha,\\ \cos(x+\alpha) &= \cos x\cos\alpha-\sin x\sin\alpha.\end{aligned}$$