# construction of tangent function from addition formula



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It is possible to define trigonometric functions rigorously using a procedure based upon the addition formula for the tangent function. The idea is to first note a few purely algebraic facts and then use these to show that a certain limiting process converges to a function which satisfies the properties of the tangent function, from which the remaining trigonometric functions may be defined by purely algebraic operations.

**Theorem 1.** *If $x$ is a positive real number, then*

$$0 < \sqrt{1 + \frac{1}{x^2}} - \frac{1}{x} < 1$$

*(Here and henceforth, the square root sign denotes the positive square root.)*

*Proof.* Let $y = 1/x$. Then $y$ is also a positive real number. We have the following inequalities:

$$y^2 < 1 + y^2 < 1 + 2y + y^2$$

Taking square roots:

$$y < \sqrt{1 + y^2} < 1 + y$$

Subtracting $y$:

$$0 \leq \sqrt{1 + y^2} - y < 1$$

Remembering the definition of $y$, this is the inequality which we set out to demonstrate. ☐

**Definition 1.** *Define the algebraic functions $s\colon \{(x, y) \in \mathbb{R}^2 \mid xy \neq 1\} \to \mathbb{R}$ and $h\colon (0, \infty) \to (0, 1)$ and $g\colon (0, 1) \to (0, 1)$ as follows:*

$$s(x, y) = \frac{x + y}{1 - xy} \tag{1}$$

$$h(x) = \sqrt{1 + \frac{1}{x^2}} - \frac{1}{x} \tag{2}$$

$$g(x) = h\left(\frac{1 + x}{1 - x}\right) = \frac{\sqrt{x^2 - 2x + 2} + x - 1}{x + 1} \tag{3}$$

**Theorem 2.** $s(s(x, y), z) = s(x, s(y, z))$

*Proof.* Calculemus! On the one hand,

$$s(s(x,y),z) = \frac{\frac{x+y}{1-xy} + z}{1 - \frac{x+y}{1-xy} z} = \frac{x+y+z-xyz}{1-xy-yz-zx}$$

On the other hand,

$$s(x,s(y,z)) = \frac{x + \frac{y+z}{1-yz}}{1 - x\frac{y+z}{1-yz}} = \frac{x+y+z-xyz}{1-xy-yz-zx}$$

These quantities are equal. □

**Theorem 3.** $s(h(x), h(x)) = x$

*Proof.* Calculemus rursum!

$$\begin{aligned} s(h(x),h(x)) &= \frac{2\sqrt{1+\frac{1}{x^2}} - \frac{2}{x}}{1 - \left(\sqrt{1+\frac{1}{x^2}} - \frac{1}{x}\right)^2} \\ &= \frac{2\sqrt{1+\frac{1}{x^2}} - \frac{2}{x}}{1 - \left(1 + \frac{2}{x^2} - \frac{2}{x}\sqrt{1+\frac{1}{x^2}}\right)} \\ &= \frac{2\sqrt{1+\frac{1}{x^2}} - \frac{2}{x}}{-\frac{1}{x}\left(\frac{2}{x} - 2\sqrt{1+\frac{1}{x^2}}\right)} = x \end{aligned}$$

□

**Theorem 4.** $s(h(x), h(y)) = h(s(x,y))$

**Theorem 5.** *For all* $x > 0$, *we have* $h(x) < x$.

*Proof.* Since $x > 0$, we have

$$x^2 + 1 < x^4 + 2x^2 + 1.$$

By the binomial identity, the right-hand side equals $(x+1)^2$. Taking square roots of both sides,

$$\sqrt{x^2+1} < x^2 + 1.$$

Subtracting 1 from both sides,

$$\sqrt{x^2+1}-1<x^2.$$

Dividing by $x$ on both sides,

$$\sqrt{1+\frac{1}{x^2}}-\frac{1}{x}<x,$$

or $h(x)<x$. ∎

**Theorem 6.** *Let $a$ be a positive real number. Then the sequence*

$$a, h(a), h(h(a)), h(h(h(a))), h(h(h(h(a)))), h(h(h(h(h(a))))), \ldots$$

*converges to* $0$.

*Proof.* By the foregoing theorem, this sequence is decreasing. Hence, it must converge to its infimum. Call this infimum $b$. Suppose that $b>0$. Then, since $h$ is continuous, we must have $h(b)=b$, which is not possible by the foregoing theorem. Hence, we must have $b=0$, so the sequence converges to 0. ∎

Having made these preliminary observations, we may now begin making the construction of the trigonometric function. We begin by defining the tangent function for successive bisections of a right angle.

**Definition 2.** *Define the sequence $\{t_n\}_{n=0}^{\infty}$ as follows:*

$$\begin{aligned} t_0 &= 1 \\ t_{n+1} &= h(t_n) \end{aligned}$$

By the forgoing theorem, this is a decreasing sequence which tends to zero. These will be the values of the tangent function at successive bisections of the right angle. We now use our function $s$ to construct other values of the tangent function.

**Definition 3.** *Define the sequence $\{r_{mn}\}$ by the following recursions:*

$$\begin{aligned} r_{m0} &= 0 \\ r_{m\,n+1} &= s(r_{mn}, t_m) \end{aligned}$$

There is a subtlety involved in this definition (which is why we did not specify the range of $m$ and $n$). Since $s(x, y)$ is only well-defined when $xy \neq 1$, we do not know that $r_{mn}$ is well defined for all $m$ and $n$. In particular, if it should happen that $r_{mn}$ is well defined for some $m$ and $n$ but that $r_{mn} t_m = 1$, then $r_{mk}$ will be undefined for all $k > m$.

**Theorem 7.** *Suppose that $r_{mn}$, $r_{mn'}$, and $r_{m\,n+n'}$ are all well-defined. Then $r_{m\,n+n'} = s(r_{mn}, r_{mn'})$.*

*Proof.* We proceed by induction on $n'$. If $n' = 0$, then $r_{m0}$ is defined to be 0, and it is easy to see that $s(r_{mn}, 0) = r_{mn}$.

Suppose, then, that we know that $r_{m\,n+n'-1} = s(r_{mn}, r_{m\,n'-1})$. By definition, $r_{mn'} = s(r_{m\,n'-1}, t_m)$ and, by theorem 2, we have

$$\begin{aligned} s(r_{mn}, s(r_{m\,n'-1}, t_m)) &= s(s(r_{mn}, r_{m\,n'-1}), t_m) \\ &= s(r_{m\,n+n'-1}, t_m) \\ &= r_{m\,n+n'} \end{aligned}$$

□

**Theorem 8.** *If $n \leq 2^m$, then $r_{mn}$ is well-defined, $r_{mn} \leq 1$, and $r_{m-1\,n} = r_{m\,2n}$.*

*Proof.* We shall proceed by induction on $m$. To begin, we note that $r_{00} \leq 1$ because $r_{00} = 0$. Also note that, if $m = 0$, then $n = 0$ is the only value for which the condition $n \leq 2^m$ happens to be satisfied. The condition $r_{m-1\,n} = r_{m\,2n}$ is not relevant when $n = 0$.

Suppose that we know that, for a certain $m$, when $n \leq 2^m$, then $r_{mn}$ is well-defined and $r_{mn} \leq 1$. We will now make an induction on $n$ to show that if $n \leq 2^{m+1}$, then $r_{m+1\,n}$ is well-defined, $r_{m\,n} \leq 1$ and $r_{mn} = r_{m+1\,2n}$. When $n = 0$, we have, by definition, $r_{m+1\,0} = 0$ so the quantity is defined and it is obvious that $r_{m\,n} \leq 1$ and $r_{mn} = r_{m+1\,2n}$.

Suppose we know that, for some number $n < 2^m$, we find that $r_{m+1\,2n}$ is well-defined, strictly less than 1 and equals $r_{m+1\,2n}$. By theorem 4, since $r_{mn} \leq 1$ and $r_{m\,n+1} \leq 1$, we may conclude that $h(r_{mn}) < 1$ and $h(r_{m\,n+1}) < 1$, which implies that $h(r_{mn}) h(r_{m\,n+1}) \neq 1$, so $s(h(r_{mn}), h(r_{m\,n+1}))$ is well-defined. By definition, $r_{m\,n+1} = s(r_{mn}, t_m)$, so $h(r_{m\,n+1}) = s(h(r_{mn}), h(t_m))$. Recall that $h(t_m) = t_{m+1}$. By theorem 1, we have

$$s(h(r_{mn}), s(h(r_{mn}), t_{m+1})) = s(s(h(r_{mn}), h(r_{mn})), t_{m+1})).$$

By theorem 2, $s(h(r_{mn}), h(r_{mn}))$ equals $r_{mn}$ which, in turn, by our induction hypothesis, equals $r_{m+1\,n}$. Combining the results of this paragraph, we may conclude that:

$$s(h(r_{mn}), h(r_{m\,n+1})) = s(r_{m+1\,2n}, t_{m+1}),$$

which means that $r_{m+1\,2n+1}$ is defined and equals $s(h(r_{mn}), h(r_{m\,n+1}))$.

Moreover, by definition,

$$s(h(r_{mn}), h(r_{m\,n+1})) = \frac{h(r_{mn}) + h(r_{m\,n+1})}{1 - h(r_{mn})h(r_{m\,n+1})}$$

Since $r_{m\,n+1} > r_{mn}$, we have $h(r_{m\,n+1}) > h(r_{mn})$ as well. This implies that the numerator is less than $2h(r_{m\,n+1})$ and that the denominator is greater than $1 - h(r^2_{m\,n+1}$. Hence, we have $r_{m+1\,2n+1} < s(h(r_{m\,n+1}, h(r_{m\,n+1}) = h(r_{m\,n+1} < 1$.

Since, as we have just shown, $r_{m+1\,2n+1} < 1$ and, as we already know, $t_{m+1} < 1$, we have $r_{m+1\,2n+1}t_{m+1} < 1$, so $r_{m+1\,2n+2}$ is well-defined. Furthermore, we may evaluate this quantity using theorem 1:

$$\begin{aligned} s(r_{m+1\,2n+1}, t_{m+1}) &= s(s(r_{m\,n}, t_{m+1}), t_{m+1}) \\ &= s(r_{m\,n}, s(t_{m+1}, t_{m+1})) \\ &= s(r_{m\,n}, t_m) \\ &= r_{m\,n+1} \end{aligned}$$

Hence, we have $r_{m+12m+2} = r_{m\,n+1}$.

$\square$