Taylor series of arcus sine

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We give an example of obtaining the Taylor series of an elementary function by integrating the Taylor series of its derivative.

For $-1 < x < 1$ we have the derivative of the principal of the `http://planetmath.org/Cyclom`
sine function:

$$\frac{d\,\arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}.$$

Using the generalized binomial coefficients $\binom{-\frac{1}{2}}{r}$ we thus can form the Taylor series for it as `http://planetmath.org/BinomialFormula`Newton's binomial series:

$$(1-x^2)^{-\frac{1}{2}} = \sum_{r=0}^{\infty}\binom{-\frac{1}{2}}{r}(-x^2)^r = 1+\binom{-\frac{1}{2}}{1}(-x^2)+\binom{-\frac{1}{2}}{2}(-x^2)^2+\binom{-\frac{1}{2}}{3}(-x^2)^3+\cdots =$$

$$= 1-\frac{-\frac{1}{2}}{1!}x^2+\frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!}x^4-\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}x^6+-\cdots =$$

$$= 1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\cdots \qquad \text{for } -1 < x < 1$$

Because $\arcsin 0 = 0$ for the principal `http://planetmath.org/GeneralPower`branch of the function, we get, by `http://planetmath.org/SumFunctionOfSeries`integrating the series termwise, the

$$\arcsin x = \int_0^x \frac{dx}{\sqrt{1-x^2}} = x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\cdots,$$

the validity of which is true for $|x| < 1$. It can be proved, in addition, that it is true also when $x = \pm 1$.