examples of lamellar field

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In the examples that follow, show that the given vector field $\vec{U}$ is lamellar everywhere in $\mathbb{R}^3$ and determine its scalar potential u.

Example 1. Given

$$\vec{U} \;:=\; y\,\vec{i}+(x+\sin z)\,\vec{j}+y\cos z\,\vec{k}.$$

For the `http://planetmath.org/NablaNabla`rotor (curl) of the we obtain
$$\nabla\times\vec{U} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x{+}\sin z & y\cos z \end{vmatrix}$$
$$= \left(\frac{\partial(y\cos z)}{\partial y}-\frac{\partial(x{+}\sin z)}{\partial z}\right)\vec{i}+\left(\frac{\partial y}{\partial z}-\frac{\partial(y\cos z)}{\partial x}\right)\vec{j}+\left(\frac{\partial(x{+}\sin z)}{\partial x}-\frac{\partial y}{\partial y}\right)\vec{k},$$
which is identically $\vec{0}$ for all $x,\,y,\,z$. Thus, by the definition given in the `http://planetmath.org/LaminarField`parent entry, $\vec{U}$ is lamellar.
Since $\nabla u = \vec{U}$, the scalar potential $u = u(x,\,y,\,z)$ must satisfy the conditions
$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x{+}\sin z, \quad \frac{\partial u}{\partial z} = y\cos z.$$
Thus we can write
$$u = \int y\,dx = xy + C_1,$$
where C_1 may depend on y or z. Differentiating this result with respect to y and comparing to the second condition, we get

$$\frac{\partial u}{\partial y} = x + \frac{\partial C_1}{\partial y} = x + \sin z.$$

Accordingly,
$$C_1 = \int \sin z\,dy = y\sin z + C_2,$$
where C_2 may depend on z. So

$$u = xy + y\sin z + C_2.$$

Differentiating this result with respect to z and comparing to the third condition yields
$$\frac{\partial u}{\partial z} \;=\; y\cos z + \frac{\partial C_2}{\partial z} \;=\; y\cos z.$$

This means that C_2 is an arbitrary . Thus the form

$$u = xy + y \sin z + C$$

expresses the required potential function.

Example 2. This is a particular case in $\mathbb{R}^2$:

$$\vec{U}(x,\, y,\, 0) \;:=\; \omega y\,\vec{i} + \omega x\,\vec{j}, \quad \omega = \text{constant}$$

Now, $\nabla \times \vec{U} \;=\; \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega y & \omega x & 0 \end{vmatrix} \;=\; \left(\dfrac{\partial(\omega x)}{\partial x} - \dfrac{\partial(\omega y)}{\partial y} \right) \vec{k} \;=\; \vec{0}$, and so $\vec{U}$ is lamellar.

Therefore there exists a potential u with $\vec{U} = \nabla u$. We deduce successively:

$$\frac{\partial u}{\partial x} = \omega y; \quad u(x,y,0) = \omega xy + f(y); \quad \frac{\partial u}{\partial y} = \omega x + f'(y) \equiv \omega x; \quad f'(y) = 0; \quad f(y) = C$$

Thus we get the result

$$u(x,\, y,\, 0) = \omega xy + C,$$

which corresponds to a particular case in $\mathbb{R}^2$.

Example 3. Given

$$\vec{U} \;:=\; ax\vec{i} + by\vec{j} - (a+b)z)\vec{k}.$$

The rotor is now $\nabla \times \vec{U} \;=\; \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax & by & -(a+b)z \end{vmatrix} \;=\; \vec{0}$. From $\nabla u \;=\; \vec{U}$ we obtain

$$\frac{\partial u}{\partial x} = ax \implies u = \frac{ax^2}{2} + f(y,z) \quad (1)$$

$$\frac{\partial u}{\partial y} = by \implies u = \frac{by^2}{2} + g(z,x) \quad (2)$$

$$\frac{\partial u}{\partial z} = -(a+b)z \implies u = -(a+b)\frac{z^2}{2} + h(x,y) \quad (3)$$

Differentiating (1) and (2) with respect to z and using (3) give

$$-(a+b)z = \frac{\partial f(y,z)}{\partial z} \quad \Longrightarrow \quad f(y,z) = -(a+b)\frac{z^2}{2} + F(y) \quad (1');$$

$$-(a+b)z = \frac{\partial g(z,x)}{\partial z} \quad \Longrightarrow \quad g(z,x) = -(a+b)\frac{z^2}{2} + G(x) \quad (2').$$

We substitute $(1')$ and $(2')$ again into (1) and (2) and deduce as follows:

$$u = \frac{ax^2}{2} - (a+b)\frac{z^2}{2} + F(y); \;\; \frac{\partial u}{\partial y} = F'(y) = by; \;\; F(y) = \frac{by^2}{2} + C_1; \;\; f(y,z) = \frac{by^2}{2} - (a+b)\frac{z^2}{2} + C_1 \quad (1'')$$

$$u = \frac{by^2}{2} - (a+b)\frac{z^2}{2} + G(x); \;\; \frac{\partial u}{\partial x} = G'(x) = ax; \;\; G(x) = \frac{ax^2}{2} + C_2; \;\; g(z,x) = \frac{ax^2}{2} - (a+b)\frac{z^2}{2} + C_2 \quad (2'')$$

putting $(1'')$, $(2'')$ into (1), (2) then gives us

$$u = \frac{ax^2}{2} + \frac{by^2}{2} - (a+b)\frac{z^2}{2} + C_1, \quad u = \frac{ax^2}{2} + \frac{by^2}{2} - (a+b)\frac{z^2}{2} + C_2,$$

whence, by comparing, $C_1 = C_2 = C$, so that by (3), the expression $h(x,y)$ and u itself have been found, that is,

$$u = \frac{ax^2}{2} + \frac{by^2}{2} - (a+b)\frac{z^2}{2} + C.$$

Unlike Example 1, the last two examples are also solenoidal, i.e. $\nabla \cdot \vec{U} = 0$, which physically may be interpreted as the continuity equation of an incompressible fluid flow.

Example 4. An additional example of a lamellar field would be

$$\vec{U} \;:=\; -\frac{ay}{x^2+y^2}\vec{i} + \frac{ax}{x^2+y^2}\vec{j} + v(z)\vec{k}$$

with a differentiable function $v : \mathbb{R} \to \mathbb{R}$; if v is a constant, then $\vec{U}$ is also solenoidal.