continuity of natural power

Canonical name	ContinuityOfNaturalPower
Date of creation	2013-03-22 15:39:25
Last modified on	2013-03-22 15:39:25
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	7
Author	pahio (2872)
Entry type	Theorem
Classification	msc 26C05
Classification	msc 26A15
Related topic	Exponentiation2

Theorem. Let n be arbitrary positive integer. The power function $x \mapsto x^n$ from $\mathbb{R}$ to $\mathbb{R}$ (or $\mathbb{C}$ to $\mathbb{C}$) is continuous at each point x_0.

Proof. Let ε be any positive number. Denote $x_0+h=x$ and $x^n-x_0^n=\Delta$. Then identically

$$\Delta \;=\; (x-x_0)(x^{n-1}+x^{n-2}x_0+...+x_0^{n-1}).$$

Taking the absolute value and using the triangle inequality give

$$|\Delta| \;=\; |h|\cdot|x^{n-1}+x^{n-2}x_0+...+x_0^{n-1}| \;\leqq\; |h|\cdot(|x^{n-1}|+|x^{n-2}x_0|+...+|x_0^{n-1}|).$$

But since $|x| = |x_0+h| \leqq |x_0|+|h|$ and also $|x_0| \leqq |x_0|+|h|$, so each summand in the parentheses is at most equal to $(|x_0|+|h|)^{n-1}$, and since there are n summands, the sum is at most equal to $n(|x_0|+|h|)^{n-1}$. Thus we get

$$|\Delta| \;\leqq\; n|h|(|x_0|+|h|)^{n-1}.$$

We may choose $|h|<1$; this implies

$$|\Delta| \;\leqq\; n|h|(|x_0|+1)^{n-1}.$$

The right hand side of this inequality is less than ε as soon as we still require

$$|h| \;<\; \frac{\varepsilon}{n(|x_0|+1)^{n-1}}.$$

This means that the power function $x \mapsto x^n$ is continuous at the point x_0.

Note. Another way to prove the theorem is to use induction on n and the rule 2 in limit rules of functions.