integral of limit function

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Theorem. If a sequence $f_1,\,f_2,\,\ldots$ of real functions, continuous on the interval $[a,\,b]$, converges uniformly on this interval to the limit function f, then

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\int_a^b f_n(x)\,dx. \tag{1}$$

Proof. Let $\varepsilon > 0$. The uniform continuity implies the existence of a positive integer n_ε such that

$$|f_n(x)-f(x)| \;<\; \frac{\varepsilon}{b-a} \quad \forall x \in [a,\,b] \qquad \text{when } \; n \;>\; n_\varepsilon.$$

The function f is continuous (see http://planetmath.org/node/7191this) and thus http://planetmath.org/RiemannIntegralRiemann integrable (see http://planetmath.org/node/4461this) on the interval. Utilising the estimation theorem of integral, we obtain

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \;=\; \left|\int_a^b (f_n(x)-f(x))\,dx\right| \;\leqq\; \int_a^b |f_n(x){-}f(x)|\,dx \;<\; \frac{\varepsilon}{b-a}(b{-}a) \;=\; \varepsilon$$

as soon as $n > n_\varepsilon$. Consequently, (1) is true.

Remark 1. The equation (1) may be written in the form

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx \;=\; \lim_{n\to\infty}\int_a^b f_n(x)\,dx, \tag{2}$$

i.e. under the assumptions of the theorem, the integration and the limit process can be interchanged.

Remark 2. Considering the partial sums of a series $\sum_{n=1}^{\infty} f_n(x)$ with continuous terms and converging uniformly on $[a,\,b]$, one gets from the theorem the result analogous to (2):

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx \;=\; \sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx. \tag{3}$$