continuous functions of several variables are Riemann summable

Canonical name	ContinuousFunctionsOfSeveralVariablesAreRiemannSummable
Date of creation	2013-03-22 15:07:56
Last modified on	2013-03-22 15:07:56
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Last modified by	paolini (1187)
Numerical id	9
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Entry type	Theorem
Classification	msc 26A42

Theorem 1. *Continuous functions defined on compact subsets of $\mathbb{R}^n$ are Riemann integrable.*

Proof. Let $D \subset \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ and let $f\colon D \to \mathbb{R}$ be a continuous function. Since f is defined on a compact set, f is uniformly continuous i.e. given $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon$. Let $R > 0$ be large enough so that $D \subset (-R, R)^n$ (such an R exists because D is bounded). Let P be a polyrectangle such that $D \subset \cup P \subset (-R, R)^n$ and such that every rectangle R in P has diameter which is less then δ. So one has $\sup_R f(x) - \inf_R f(x) \leq \epsilon$ and hence

$$S^*(f, P) - S_*(f, P) \leq \epsilon \sum_{Q \in P} \text{meas}(Q) \leq \epsilon \text{meas}(P) \leq \epsilon \text{meas}[-R, R]^n = \epsilon 2^n R^n.$$

Letting $\epsilon \to 0$ one concludes that $S^*(f) = S_*(f)$. □