using convolution to find Laplace transform

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We start from the (see the table of Laplace transforms)

$$e^{\alpha t} \curvearrowleft \frac{1}{s-\alpha}, \quad \frac{1}{\sqrt{t}} \curvearrowleft \sqrt{\frac{\pi}{s}} \qquad (s > \alpha) \tag{1}$$

where the curved from the Laplace-transformed functions to the original functions. Setting $\alpha = a^2$ and dividing by $\sqrt{\pi}$ in (1), the convolution property of Laplace transform yields

$$\frac{1}{(s-a^2)\sqrt{s}} \curvearrowright e^{a^2t} * \frac{1}{\sqrt{\pi t}} = \int_0^t e^{a^2(t-u)} \frac{1}{\sqrt{\pi u}}\,du.$$

The http://planetmath.org/ChangeOfVariableInDefiniteIntegralsubstitution $a^2u = x^2$ then gives

$$\frac{1}{(s-a^2)\sqrt{s}} \curvearrowright \frac{e^{a^2t}}{\sqrt{pi}} \int_0^{a\sqrt{t}} e^{-x^2} \frac{a}{x} \cdot \frac{2x}{a^2}\,dx = \frac{e^{a^2t}}{a} \cdot \frac{2}{\sqrt{\pi}} \int_0^{a\sqrt{t}} e^{-x^2}\,dx = \frac{e^{a^2t}}{a} \operatorname{erf} a\sqrt{t}.$$

Thus we may write the formula

$$\mathcal{L}\{e^{a^2t} \operatorname{erf} a\sqrt{t}\} = \frac{a}{(s-a^2)\sqrt{s}} \qquad (s > a^2). \tag{2}$$

Moreover, we obtain

$$\frac{1}{(\sqrt{s}+a)\sqrt{s}} = \frac{\sqrt{s}-a}{(s-a^2)\sqrt{s}} = \frac{1}{s-a^2} - \frac{a}{(s-a^2)\sqrt{s}} \curvearrowright e^{a^2t} - e^{a^2t} \operatorname{erf} a\sqrt{t} = e^{a^2t}(1-\operatorname{erf} a\sqrt{t}),$$

whence we have the other formula

$$\mathcal{L}\{e^{a^2t} \operatorname{erfc} a\sqrt{t}\} = \frac{1}{(a+\sqrt{s})\sqrt{s}}. \tag{3}$$

0.1 An improper integral

One can utilise the formula (3) for evaluating the improper integral

$$\int_0^\infty \frac{e^{-x^2}}{a^2+x^2}\,dx.$$

We have

$$e^{-tx^2} \curvearrowleft \frac{1}{s+x^2}$$

(see the `http://planetmath.org/TableOfLaplaceTransforms`table of Laplace transforms). Dividing this by a^2+x^2 and integrating from 0 to ∞, we can continue as follows:

$$
\begin{aligned}
\int_0^\infty \frac{e^{-tx^2}}{a^2+x^2}\,dx \;\curvearrowleft\; \int_0^\infty \frac{dx}{(a^2+x^2)(s+x^2)} \;&=\; \frac{1}{s-a^2}\int_0^\infty \left(\frac{1}{a^2+x^2}-\frac{1}{s+x^2}\right)dx \\
&= \frac{1}{s-a^2}\Big/_{x=0}^{\infty}\left(\frac{1}{a}\arctan\frac{x}{a}-\frac{1}{\sqrt{s}}\arctan\frac{x}{\sqrt{s}}\right) \\
&= \frac{1}{s-a^2}\cdot\frac{\pi}{2}\left(\frac{1}{a}-\frac{1}{\sqrt{s}}\right) \;=\; \frac{\pi}{2a}\cdot\frac{1}{(a+\sqrt{s})\sqrt{s}} \\
&\curvearrowright\; \frac{\pi}{2a}e^{a^2t}\,\mathrm{erfc}\,a\sqrt{t}
\end{aligned}
$$

Consequently,

$$\int_0^\infty \frac{e^{-tx^2}}{a^2+x^2}\,dx \;=\; \frac{\pi}{2a}e^{a^2t}\,\mathrm{erfc}\,a\sqrt{t},$$

and especially

$$\int_0^\infty \frac{e^{-x^2}}{a^2+x^2}\,dx \;=\; \frac{\pi}{2a}e^{a^2}\,\mathrm{erfc}\,a.$$