area of plane region

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Let the contour of the region in the xy-plane be a closed curve P. Then the area of the region equals to the path integral

$$A \;=\; \frac{1}{2}\oint_P (x\,dy - y\,dx) \tag{1}$$

taken in the positive (i.e. anticlockwise) circling direction.

Remarks

1. The (1) can be gotten as a special case of Green's theorem by setting $\vec{F} := \frac{1}{2}(-y,\, x)$.

2. Because $x\,dy + y\,dx = d(xy)$, we have

   $$0 \;=\; \frac{1}{2}\oint_P (x\,dy + y\,dx).$$

   This equation may be added to or subtracted from (1), giving the alternative forms

   $$A \;=\; \oint_P x\,dy \;=\; -\oint_P y\,dx. \tag{2}$$

3. The formulae (1) and (2) all other formulae concerning the planar area computing, e.g.

   $$A \;=\; \int_a^b f(x)\,dx,$$
   $$A \;=\; \frac{1}{2}\int_{\varphi_1}^{\varphi_2} [r(\varphi)]^2\,d\varphi,$$

   the former of which is factually same as the latter form of (2).

Example. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has the parametric $x = a\cos t$, $y = b\sin t$ $(0 \leqq t < 2\pi)$. We have

$$x\,dy - y\,dx \;=\; [a\cos t \cdot b\cos t + b\sin t \cdot a\sin t]\,dt \;=\; ab\,dt,$$

and hence (1) gives for the area of the ellipse

$$A \;=\; \frac{1}{2}ab\int_0^{2\pi} dt \;=\; \pi ab.$$