sinc is not L^1

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The main results used in the proof will be that $f \in L^1(A) \iff |f| \in L^1(A)$ and the dominated convergence theorem.

Proof by contradiction:

Let $f(x) = |\operatorname{sinc}(x)|$ and suppose it's Lebesgue integrable in $\mathbb{R}^+$.

Consider the intervals $I_k = [k\pi, (k+1)\pi]$ and $U_k = \bigcup_{i=0}^{k} I_k = [0, (k+1)\pi]$.

and the succession of functions $f_n(x) = f(x)\chi_{U_n}(x)$, where χ_{U_n} is the characteristic function of the set U_n.

Each f_n is a continuous function of compact support and will thus be integrable in $\mathbb{R}^+$. Furthermore $f_n(x) \nearrow f(x)$ (pointwise)

in each I_k, $f(x) \geq \frac{|\sin(x)|}{(k+1)\pi}$.

So

$$\int_{\mathbb{R}^+} f_n = \sum_{k=0}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin(x)|}{x} dx \geq \sum_{k=0}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|sin(x)|}{(k+1)\pi} = \sum_{k=0}^{n} \frac{2}{(k+1)\pi}.$$

Suppose f is integrable in $\mathbb{R}^+$. Then by the dominated convergence theorem $\lim_{n\to\infty} \int_{\mathbb{R}^+} f_n = \int_{\mathbb{R}^+} f$.

But $\lim_{n\to\infty} \int_{\mathbb{R}^+} f_n \geq \lim_{n\to\infty} \sum_{k=0}^{n} \frac{2}{(k+1)\pi} = +\infty$ and we get the contradiction $\int_{\mathbb{R}^+} f \geq +\infty$.

So f cannot be integrable in $\mathbb{R}^+$. This implies that f cannot be integrable in $\mathbb{R}$ and since a function is integrable in a set iff its absolute value is

$\operatorname{sinc}(x) \notin L^1(\mathbb{R})$