Heronian mean is between geometric and arithmetic mean

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Theorem. For non-negative numbers x and y, the inequalities

$$\sqrt{xy} \;\leqq\; \frac{x+\sqrt{xy}+y}{3} \;\leqq\; \frac{x+y}{2}$$

are in , i.e. the Heronian mean is always at least equal to the geometric mean and at most equal to the arithmetic mean. The equality signs are true if and only if $x = y$.

Proof.
1°.

$$\begin{aligned}
\sqrt{xy} \;\leqq\; \frac{x+\sqrt{xy}+y}{3} \quad & \Leftrightarrow \quad 3\sqrt{xy} \leqq x+\sqrt{xy}+y \\
& \Leftrightarrow \quad 2\sqrt{xy} \leqq x+y \\
& \Leftrightarrow \quad 4xy \leqq x^2+2xy+y^2 \\
& \Leftrightarrow \quad 0 \leqq x^2-2xy+y^2 \\
& \Leftrightarrow \quad 0 \leqq (x-y)^2
\end{aligned}$$

2°.

$$\begin{aligned}
\frac{x+\sqrt{xy}+y}{3} \leqq \frac{x+y}{2} \quad & \Leftrightarrow \quad 2x+2\sqrt{xy}+2y \leqq 3x+3y \\
& \Leftrightarrow \quad 2\sqrt{xy} \leqq x+y \\
& \Leftrightarrow \quad 4xy \leqq x^2+2xy+y^2 \\
& \Leftrightarrow \quad 0 \leqq (x-y)^2
\end{aligned}$$

All inequalities of both chains are `http://planetmath.org/Equivalent3`equivalent since x and y are non-negative. As for the equalities, the chains are valid with the mere equality signs.