proof of monotonicity criterion

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Let us start from the implications "$\Rightarrow$".

Suppose that $f'(x) \geq 0$ for all $x \in (a, b)$. We want to prove that therefore f is increasing. So take $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. Applying the mean-value Theorem on the interval $[x_1, x_2]$ we know that there exists a point $x \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$$

and being $f'(x) \geq 0$ we conclude that $f(x_2) \geq f(x_1)$.

This proves the first claim. The other three cases can be achieved with minor modifications: replace all "$\geq$" respectively with $\leq$, $>$ and $<$.

Let us now prove the implication "$\Leftarrow$" for the first and second statement.

Given $x \in (a, b)$ consider the ratio

$$\frac{f(x+h) - f(x)}{h}.$$

If f is increasing the numerator of this ratio is ≥ 0 when $h > 0$ and is ≤ 0 when $h < 0$. Anyway the ratio is ≥ 0 since the denominator has the same sign of the numerator. Since we know by hypothesis that the function f is differentiable in x we can pass to the limit to conclude that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq 0.$$

If f is decreasing the ratio considered turns out to be ≤ 0 hence the conclusion $f'(x) \leq 0$.

Notice that if we suppose that f is strictly increasing we obtain the this ratio is > 0, but passing to the limit as $h \to 0$ we cannot conclude that $f'(x) > 0$ but only (again) $f'(x) \geq 0$.