intermediate value theorem for extended real numbers

Canonical name	IntermediateValueTheoremForExtendedRealNumbers
Date of creation	2013-03-22 15:35:15
Last modified on	2013-03-22 15:35:15
Owner	matte (1858)
Last modified by	matte (1858)
Numerical id	6
Author	matte (1858)
Entry type	Theorem
Classification	msc 26A06
Related topic	ExtendedRealNumbers

Theorem 1. *Let $\overline{\mathbb{R}}$ be the extended real numbers, and suppose $f\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a continuous function. Suppose $x_1 < x_2 \in \overline{\mathbb{R}}$ are such that $f(x_1) \neq f(x_2)$. If $y \in (f(x_1), f(x_2))$, then for some $c \in (x_1, x_2)$ we have*

$$f(c) = y.$$

Proof. As $\overline{\mathbb{R}}$ is homeomorphic to $[0,1]$, we can assume that f is a function $f\colon [0,1] \to \overline{\mathbb{R}}$. For simplicity, let us also assume that $x_1 = 0$,$x_2 = 1$, and $f(0) < f(1)$. Then for some $\varepsilon > 0$ we have

$$f(0) < y - \varepsilon < y < y + \varepsilon < f(1).$$

Let $g\colon [0,1] \to \mathbb{R}$ be the continuous function

$$g(x) = \max\{\min\{f(x), y + \varepsilon\}, y - \varepsilon\}.$$

Now $g(0) = y - \varepsilon$ and $g(1) = y + \varepsilon$, so for some $c \in (0,1)$, we have $g(c) = y$, and thus $f(c) = y$. $\square$