Prosthaphaeresis formulas

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The Prosthaphaeresis formulas convert sums of sines or cosines to products of them:

$$\begin{aligned}
\sin A + \sin B &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\sin A - \sin B &= 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \\
\cos A + \cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\cos A - \cos B &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)
\end{aligned}$$

We prove the first two using the sine of a sum and sine of a difference formulas:

$$\begin{aligned}
\sin(X+Y) &= \sin X\cos Y + \cos X \sin Y \\
\sin(X-Y) &= \sin X\cos Y - \cos X \sin Y
\end{aligned}$$

Adding or subtracting the two equations yields

$$\begin{aligned}
\sin(X+Y) + \sin(X-Y) &= 2\sin X\cos Y \\
\sin(X+Y) - \sin(X-Y) &= 2\sin Y\cos X
\end{aligned}$$

If we let $X = \frac{A+B}{2}$ and $Y = \frac{A-B}{2}$, then $X+Y = \frac{2A}{2} = A$ and $X - Y = \frac{2B}{2} = B$, and the last two equations become

$$\begin{aligned}
\sin A + \sin B &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\sin A - \sin B &= 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)
\end{aligned}$$

as desired.

The last two can be proven similarly, this time using the cosine of a sum and cosine of a difference formulas:

$$\begin{aligned}
\cos(X+Y) &= \cos X\cos Y - \sin X \sin Y \\
\cos(X-Y) &= \cos X\cos Y + \sin X \sin Y
\end{aligned}$$

Adding or subtracting the two equations yields

$$\begin{aligned}\cos(X+Y)+\cos(X-Y) &= 2\cos X\cos Y\\ \cos(X+Y)-\cos(X-Y) &= -2\sin Y\sin X\end{aligned}$$

Again, if we let $X=\frac{A+B}{2}$ and $Y=\frac{A-B}{2}$, then $X+Y=\frac{2A}{2}=A$ and $X-Y=\frac{2B}{2}=B$, and the last two equations become

$$\begin{aligned}\cos A+\cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\ \cos A-\cos B &= -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)\end{aligned}$$

as desired.

Notes

'Prosthaphaeresis' comes from the Greek: "prosthesi" = addition + "afairo" = subtraction.

The Prosthaphaeresis formula $\cos x\cos y=\frac{\cos(x+y)+\cos(x-y)}{2}$ was used by scientists to transform multiplication into addition. For example, to calculate the product ab, where $0<a,b<1$ (for a and b outside of this range, it is a simple matter to multiply or divide by a factor of 10 and divide or multiply this back in later), one would let $\cos x=a$ and $\cos y=b$. Using a table of cosines, one could then find an approximate value for x and y, then find $x+y$ and $x-y$, and look up the cosines of the resulting two quantities (that is, $\cos(x+y)$ and $\cos(x-y)$). The average of these numbers is the desired product ab. This technique was used by Tycho Brahe to perform astronomical calculations.