integration of differential binomial

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Theorem. Let a, b, c, α, β be given real numbers and $\alpha\beta \neq 0$. The antiderivative

$$I = \int x^a(\alpha + \beta x^b)^c \, dx$$

is expressible by of the elementary functions only in the three cases: (1) $\frac{a+1}{b} + c \in \mathbb{Z}$, (2) $\frac{a+1}{b} \in \mathbb{Z}$, (3) $c \in \mathbb{Z}$

In accordance with P. L. Chebyshev (1821–1894), who has proven this theorem, the expression $x^a(\alpha + \beta x^b)^c \, dx$ is called a *differential binomial.*

It may be worth noting that the differential binomial may be expressed in terms of the incomplete beta function and the hypergeometric function. Define $y = \beta x^b/\alpha$. Then we have

$$I = \frac{1}{b}\alpha^{\frac{a+1}{b}+c}\beta^{-\frac{a+1}{b}} B_y\left(\frac{1+a}{b}, c-1\right)$$

$$= \frac{1}{1+a}\alpha^{\frac{a+1}{b}+c}\beta^{-\frac{a+1}{b}} y^{\frac{1+a}{b}} F\left(\frac{a+1}{b}, 2-c; \frac{1+a+b}{b}; y\right)$$

Chebyshev's theorem then follows from the theorem on elementary cases of the hypergeometric function.