alternating harmonic series

Canonical name	AlternatingHarmonicSeries
Date of creation	2013-03-22 17:53:25
Last modified on	2013-03-22 17:53:25
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	13
Author	CWoo (3771)
Entry type	Example
Classification	msc 26A06
Classification	msc 40A05
Synonym	alternating p-series
Related topic	AbsoluteConvergence
Related topic	MultiplicationOfSeries
Related topic	SumOfSeriesDependsOnOrder
Defines	alternating p-series

The *alternating harmonic series* is given by the following infinite series:

$$\sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{n}. \tag{1}$$

The series converges to $\ln 2$ and it is the prototypical example of a conditionally convergent series.

First, notice that the series is not absolutely convergent. By taking the absolute value of each term, we get the harmonic series, which is divergent. There are several ways to show this, and we invite the reader to the entry on harmonic series for further exploration.

Next, to show that the series (1) converges, we use the `http://planetmath.org/AlternatingSe` series test: since

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

the alternating series (1) converges.

Remarks.

- Other examples of conditionally convergent series can be discovered using variants of the alternating harmonic series. For instance, the following series

  $$\sum_{i=1}^{\infty} \frac{(-1)^n n^2 + \cos n}{n^3 - n^2 + e^{-n}}.$$

  can easily be shown to be conditionally convergent. Here is another example, more of a generalization, called the :

  $$\sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{n^p}, \tag{2}$$

  where p is non-negative real number. The convergence of the is tabulated below:

p	convergence
$(1,\infty)$	absolutely convergent
$(0,1]$	conditionally convergent
0	divergent

- Using Riemann series theorem, one easily sees that not every conditionally convergent series is alternating. By appropriately rearranging the alternating harmonic series, one gets a conditionally convergent series that is not alternating: define $\sigma : \mathbb{N} \to \mathbb{N}$ as follows:

$$\sigma(i) := \begin{cases} \dfrac{2i+1}{3} & \text{if } 2i \equiv -1 \pmod 3 \\ \dfrac{4i-2}{3} & \text{if } 2i \equiv 1 \pmod 3 \\ \dfrac{4i}{3} & \text{otherwise} \end{cases}$$

  It can be shown that σ is a bijection. Now, let a_n be the nth term of alternating harmonic series. Then it is not hard to see that

$$\sum_{i=1}^{n} a_{\sigma(i)} \longrightarrow \frac{\ln 2}{2} \qquad \text{as} \qquad n \longrightarrow \infty.$$

  Thus, it is conditionally convergent and yet it is not alternating (the first three terms are 1, $-\frac{1}{2}$, $-\frac{1}{4}$).