one-sided limit

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Entry type	Definition
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Synonym	limit from below
Synonym	limit from above
Synonym	left-sided limit
Synonym	left-handed limit
Synonym	right-sided limit
Synonym	right-handed limit
Related topic	Limit
Related topic	OneSidedDerivatives
Related topic	IntegratingTanXOver0fracpi2
Related topic	OneSidedContinuity
Defines	Heaviside unit step function

Definition Let f be a real-valued function defined on $S \subseteq \mathbb{R}$. The *left-hand one-sided limit* at $a \in \mathbb{R}$ is defined to be the real number L^- such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L^-| < \epsilon$ whenever $0 < a - x < \delta$.

Analogously, the *right-hand one-sided limit* at $a \in \mathbb{R}$ is the real number L^+ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L^+| < \epsilon$ whenever $0 < x - a < \delta$.

Common notations for the one-sided limits are

$$\begin{aligned} L^+ &= f(x+) = \lim_{x \to a^+} f(x) = \lim_{x \searrow a} f(x), \\ L^- &= f(x-) = \lim_{x \to a^-} f(x) = \lim_{x \nearrow a} f(x). \end{aligned}$$

Sometimes, left-handed limits are referred to as limits *from below* while right-handed limits are *from above.*

Theorem The ordinary limit of a function exists at a point if and only if both one-sided limits exist at this point and are equal (to the ordinary limit).

Example The Heaviside unit step function, sometimes colloquially referred to as the diving board function, defined by

$$H(x) = \begin{cases} 0 & \text{if } \ x < 0 \\ 1 & \text{if } \ x \geq 0 \end{cases}$$

has the simplest kind of discontinuity at $x = 0$, a jump discontinuity. Its ordinary limit does not exist at this point, but the one-sided limits do exist, and are

$$\lim_{x \to 0^-} H(x) = 0 \text{ and } \lim_{x \to 0^+} H(x) = 1.$$