example of Riemann double integral

Canonical name	ExampleOfRiemannDoubleIntegral
Date of creation	2013-03-22 19:12:22
Last modified on	2013-03-22 19:12:22
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Last modified by	pahio (2872)
Numerical id	11
Author	pahio (2872)
Entry type	Example
Classification	msc 26A42
Classification	msc 28-00
Related topic	SubstitutionNotation
Related topic	ChangeOfVariablesInIntegralOnMathbbRn
Related topic	ExampleOfRiemannTripleIntegral

Let us determine the value of the double integral

$$I \;:=\; \iint_D \frac{dx\,dy}{(1+x^2+y^2)^2} \tag{1}$$

where D is the triangle by the lines $x = 0,\;\; y = 0$ and $x+y = 1$.

Since the triangle is defined by the inequalities $0 \leqq x \leqq 1,\;\; 0 \leqq y \leqq 1-x$, one can write

$$\begin{aligned}
I \;&=\; \int_0^1\int_0^{1-x} \frac{dx\,dy}{(1+x^2+y^2)^2} \;=\; \int_0^1 \frac{dx}{(1+x^2)^2}\int_0^{1-x} \frac{dy}{\left[1+\left(\frac{y}{\sqrt{1+x^2}}\right)^2\right]^2} \\
&= \int_0^1 \frac{1}{(1+x^2)^2}\cdot\frac{\sqrt{1+x^2}}{2} \;\;\Big/_{y=0}^{1-x} \left(\arctan\frac{y}{\sqrt{1+x^2}} + \frac{\frac{y}{\sqrt{1+x^2}}}{1+\frac{y^2}{1+x^2}}\right) dx \\
&= \int_0^1 \left(\frac{1}{2}(1+x^2)^{-\frac{3}{2}}\arctan\frac{1-x}{\sqrt{1+x^2}} + \frac{1-x}{(1-x+x^2)(1+x^2)}\right) dx.
\end{aligned}$$

The last expression seems quite difficult to calculate to a closed form ...

Some appropriate http://planetmath.org/ChangeOfVariablesInIntegralOnMathbbRnsubsti

$$x \;:=\; x(u,\,v), \quad y \;:=\; y(u,\,v)$$

directly to the form (1) could offer a better is

$$\iint_D f(x,\,y)\,dx\,dy \;=\; \iint_\Delta f(x(u,\,v),\,y(u,\,v))\left|\frac{\partial(x,\,y)}{\partial(u,\,v)}\right| du\,dv. \tag{2}$$

What kind a change of variables would be good? One idea were to use some "natural substitution", i.e. such one that would give constant http://planetmath.org/DefiniteIntegrallimits. For example, the equations

$$x+y \;:=\; u, \quad \frac{y}{x} \;:=\; v,$$

map the triangular http://planetmath.org/Domain2domain D to the "rectangle"

$$\Delta:\;\; 0 \leqq u \leqq 1, \quad 0 \leqq v < \infty.$$

Then we need the Jacobian

$$\frac{\partial(x,\, y)}{\partial(u,\, v)} \;=\; \frac{u+v^2}{(v+1)^3}.$$

By (2), we have

$$I \;=\; \int_0^1\!\int_0^\infty \frac{(v+1)^4}{u^2+2v^2+2v+1}\cdot\frac{u+v^2}{(v+1)^3}\,du\,dv \;=\; \int_0^\infty (v{+}1)\,dv \int_0^1 \frac{u+v^2}{u^2+2v^2+2v+1}\,du.$$

But here after integrating with respect to u, one obtains a difficult single integral. Thus, when the , the integrand may become awkward.

A second idea would be to try to make the integrand simpler. For this end, the transition to the polar coordinates

$$x \;:=\; r\cos\varphi, \quad y \;:=\; r\sin\varphi$$

in (1) is more suitable. We have

$$\frac{\partial(x,\, y)}{\partial(r,\, \varphi)} \;=\; \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} \;\equiv\; r.$$

The Pythagorean theorem gives the equation $r^2 \;=\; x^2{+}y^2 \;=\; (r\cos\varphi)^2 + (1-r\cos\varphi)^2$, i.e.

$$r^2\cos 2\varphi - 2r\cos\varphi + 1 \;=\; 0,$$

from which we get the upper limit

$$r \;=\; \frac{2\cos\varphi \pm \sqrt{4\cos^2\varphi - 4\cos 2\varphi}}{2\cos 2\varphi} \;=\; \frac{\cos\varphi \pm \sin\varphi}{\cos^2\varphi - \sin^2\varphi};$$

this is $\dfrac{1}{\cos\varphi + \sin\varphi}$, since the "+" alternative can be excluded by choosing e.g. $\varphi = \frac{\pi}{2}$. Thus

$$\Delta\colon \quad 0 \leqq \varphi \leqq \frac{\pi}{2}, \quad 0 \leqq r \leqq \frac{1}{\cos\varphi + \sin\varphi}$$

and

$$I \;=\; \frac{1}{2}\int_0^{\frac{\pi}{2}}\!\int_0^{\frac{1}{\cos\varphi+\sin\varphi}} \frac{2r\,dr}{(1{+}r^2)^2}\,d\varphi \;=\; \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{2+\sin 2\varphi}.$$

Here, the `http://planetmath.org/node/9380`Weierstrass substitution $\tan\varphi := t$ easily yields the final result

$$I \;=\; \frac{2\pi\sqrt{3}}{9}. \tag{3}$$