integration of rational function of sine and cosine

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The integration task

$$\int R(\sin x,\, \cos x)\, dx, \tag{1}$$

where the integrand is a rational function of $\sin x$ and $\cos x$, changes via the Weierstrass substitution

$$\tan\frac{x}{2} \;=\; t \tag{2}$$

to a form having an integrand that is a rational function of t. Namely, since $x = 2\arctan t$, we have

$$dx \;=\; 2\cdot\frac{1}{1+t^2}\, dt, \tag{3}$$

and we can substitute

$$\sin x \;=\; \frac{2t}{1+t^2}, \quad \cos x \;=\; \frac{1-t^2}{1+t^2}, \tag{4}$$

getting

$$\int R(\sin x,\, \cos x)\, dx \;=\; 2\int R\left(\frac{2t}{1+t^2},\, \frac{1-t^2}{1+t^2}\right)\frac{dt}{1+t^2}.$$

Proof of the formulae (4): Using the double angle formulas of sine and cosine and then dividing the numerators and the denominators by $\cos^2\frac{x}{2}$ we obtain

$$\sin x \;=\; \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \;=\; \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \;=\; \frac{2t}{1+t^2},$$

$$\cos x \;=\; \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \;=\; \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} \;=\; \frac{1-t^2}{1+t^2}.$$

Example. The above formulae give from $\displaystyle\int\frac{dx}{\sin x}$ the result

$$\int\frac{dx}{\sin x} \;=\; \int\frac{1+t^2}{2t}\cdot 2\cdot\frac{1}{1+t^2}\, dt = \int\frac{dt}{t} \;=\; \ln|t|+C \;=\; \ln\left|\tan\frac{x}{2}\right|+C$$

(which can also be expressed in the form $-\ln|\csc x+\cot x|+C$; see the goniometric formulas).

Note 1. The substitution (2) is sometimes called the `http://planetmath.org/UniversalTrig`
trigonometric substitution". In practice, it often gives rational functions that are too complicated. In many cases, it is more profitable to use other substitutions:

- In the case $\int R(\sin x)\cos x\,dx$ the substitution $\sin x = t$ is simpler.
- Similarly, in the case $\int R(\cos x)\sin x\,dx$ the substitution $\cos x = t$ is simpler.
- If the integrand depends only on $\tan x$, the substitution $\tan x = t$ is simpler.
- If the integrand is of the form $R(\sin^2 x,\,\cos^2 x)$, one can use the substitution $\tan x = t$; then $\cos^2 x = \dfrac{1}{1+\tan^2 x} = \dfrac{1}{1+t^2}$, $\sin^2 x = 1-\cos^2 x = \dfrac{t^2}{1+t^2}$, $dx = \dfrac{dt}{1+t^2}$.

Example. The integration of $\displaystyle\int \frac{dx}{\cos^4 x}\,dx$ is of the last case:

$$\int \frac{dx}{\cos^4 x}\,dx = \int \frac{1}{(\cos^2 x)^2}\,dx = \int (1+t^2)^2 \cdot \frac{dt}{1+t^2} = \int (1+t^2)\,dt = \frac{t^3}{3}+t+C = \frac{1}{3}\tan^3 x+\tan x+C.$$

Example. The integral $I = \displaystyle\int \frac{dx}{\cos^3 x}\,dx = \int \sec^3 x\,dx$ is a peculiar case in which one does not have to use the substitutions mentioned above, as integration by parts is a simpler method for evaluating this integral. Thus,

$$u = \sec x \;\Rightarrow\; du = \sec x\,\tan x\,dx; \qquad dv = \sec^2 x\,dx \;\Rightarrow\; v = \tan x.$$

Therefore,

$$\begin{aligned} I &= \int \sec^3 x\,dx \\ &= \sec x\,\tan x - \int \sec x\,\tan^2 x\,dx \\ &= \sec x\,\tan x - \int \sec x\,(\sec^2 x - 1)\,dx \\ &= \sec x\,\tan x - I + \int \sec x\,dx, \end{aligned}$$

and consequently

$$\int \frac{dx}{\cos^3 x}\, dx \;=\; \frac{1}{2}\big(\sec x\;\tan x\; + \ln\;|\sec x + \tan x|\big) + C.$$

Note 2. There is also the "universal hyperbolic substitution" for integrating rational functions of hyperbolic sine and cosine:

$$\tanh\frac{x}{2} \;=\; t, \quad dx \;=\; \frac{2dt}{1-t^2}, \quad \sinh x \;=\; \frac{2t}{1-t^2}, \quad \cosh x \;=\; \frac{1+t^2}{1-t^2}$$

References

[1] Л. Д. Кдрячев: *Математичецкии анализ*. Издательство "Вүсшая-Школа". Москва (1970).