characterization of almost convex functions

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A real function f is almost convex iff it is monotonic or there exists $p \in \mathbb{R}$ such that f is nonincreasing on the half-line $(-\infty, p)$ and nondecreasing on the half-line $(p, +\infty)$

Proof:

The proof is based on some simple observations about the values of an almost convex function. Suppose that $a < b$ and $f(a) \leq f(b)$. Then for any $c > b$, it must be the case that $f(b) \leq f(c)$. This follows from the fact that, by definition of almost convex, either $f(b) \leq f(a)$ or $f(b) \leq f(c)$. Since the first option is excluded by assumption, the second option must be true.

Furthermore, with a, b as above, f is nondecreasing in the half-line $[b, \infty)$. By the result of the last paragraph, it suffices to show that f is non-decreasing in the open half-line (c, ∞). This is tantamount to showing that, if $c < d < e$, then $f(d) \leq f(e)$. From the conlusion of last paragraph, we already know that $f(c) \leq f(d)$. Applying the result shown in the last paragraph to this conclusion, we further conclude that $f(d) \leq f(e)$, as desired.

By replacing "$\leq$" by "$\geq$" in the above two paragraphs suitably, we also can likewise that, if $a < b$ and $f(a) \geq f(b)$, then f is nonincreasing on the half-line $(-\infty, a]$.

Now assume that f is almost convex but not monotonic. By the hypothesis of nonmomotonicity, there must exist $a < b < c$ such that it is the case that neither $f(a) \leq f(b) \leq f(c)$ nor $f(a) \geq f(b) \geq f(c)$. Furthermore, by almost-convexity, it follows that $f(b) \leq f(a)$ and $f(b) \leq f(c)$. This, in turn, implies that f is nonincreasing on $(-\infty, a]$ and nondecreasing on $[c, +\infty)$.

Let L be the set of all real numbers q such that f is nondecreasing on the interval $(q, +\infty)$. This set is not empty because $c \in L$. It is a proper subset of the real line because, for instance, $q \notin L$ whenever $q < a$. This follows from the observation that f cannot be nondecreasing on $(q, +\infty)$ because $f(a) > f(b)$. Also, L must be a proper subset of the real line, because, if it were not, f would be nondecreasing on the whole real line, which is contrary to assumption.

Note that, if $r < q$ and $q \notin L$, then $r \notin L$ as well. This is an expression of the fact that, if a function is not monotonic on a set, it is not monotonic on a superset, which is the contrapositive of the assertion that a the resticition of a function which is monotonic on a set to a subset is still monotonic. Since there exists a real number r such that $r \notin L$, this means that r is a lower bound for L. Since L is bounded from below and not empty, it follows that L has a greatest lower bound, which we shall call p.

By construction, f is non-decreasing on the half-line $(p, +\infty)$. We will now show that f is nonincreasing on the half-line $(-\infty, p)$. Suppose that $q < p$. Then, by the choice of p, the function f is not nondecreasing on the half-line $(q, +\infty)$. This means that there must exist a, b such that $q < a < b$ and $f(a) > f(b)$. By the result demonstrated above, it follows that f is nonincreasing on $(-\infty, a)$, hence, since $q < a$, in particular, f is nononicreasing on $(-\infty, q)$. Since f is nonincreasing on $(-\infty, q)$ for all q, it is the case that f is nonincreasing on $(-\infty, p)$.