Euler's theorem on homogeneous functions

Canonical name	EulersTheoremOnHomogeneousFunctions
Date of creation	2013-03-22 15:18:58
Last modified on	2013-03-22 15:18:58
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Theorem
Classification	msc 26B12
Classification	msc 26A06
Classification	msc 15-00
Defines	Euler operator

Theorem 1 (Euler)**.** *Let $f(x_1, \dots, x_k)$ be a smooth homogeneous function of degree n. That is,*

$$f(tx_1, \dots, tx_k) = t^n f(x_1, \dots, x_k). \tag{*}$$

Then the following identity holds

$$x_1 \frac{\partial f}{\partial x_1} + \cdots + x_k \frac{\partial f}{\partial x_k} = nf.$$

Proof. By homogeneity, the relation (**??**) holds for all t. Taking the t-derivative of both sides, we establish that the following identity holds for all t:

$$x_1 \frac{\partial f}{\partial x_1}(tx_1, \dots, tx_k) + \cdots + x_k \frac{\partial f}{\partial x_k}(tx_1, \dots, tx_k) = nt^{n-1} f(x_1, \dots, x_k).$$

To obtain the result of the theorem, it suffices to set $t = 1$ in the previous formula. $\square$

Sometimes the differential operator $x_1 \dfrac{\partial}{\partial x_1} + \cdots + x_k \dfrac{\partial}{\partial x_k}$ is called the *Euler operator*. An equivalent way to state the theorem is to say that homogeneous functions are eigenfunctions of the Euler operator, with the degree of homogeneity as the eigenvalue.