calculation of Riemann–Stieltjes integral

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- If f is defined on $[a, b]$ and g is a constant function, then

$$\int_a^b f\,dg \;=\; 0.$$

- Let f be continuous on $[a, b]$, $a < c < b$ and g the step function defined as

$$g(x) = k \quad \text{for } x < c, \quad g(x) = k+\alpha \quad \text{for } x > c.$$

Then

$$\int_a^b f\,dg \;=\; f(c)\cdot\alpha.$$

- Let f be continuous on $[a, b]$, $a < c < b$ and the function g be otherwise continuous but have in $x = c$ a step of magnitude α. Then g is sum of a continuous function g^* and a step function

$$h(x) = 0 \quad \text{for } x < c, \quad h(x) = \alpha \quad \text{for } x > c,$$

and one has

$$\int_a^b f\,dg \;=\; \int_a^b f\,d(g^*{+}h) \;=\; \int_a^b f\,dg^* + \int_a^b f\,dh \;=\; \int_a^b f\,dg^* + f(c)\cdot\alpha.$$

- Suppose that g can be expressed in the form $g = g^* + h$ where g^* is continuous and h a step function having an at most denumerable amount of steps α_i in respectively the same points c_i on the interval $[a, b]$ as the function g. If f is Riemann–Stieltjes integrable on $[a, b]$, then

$$\int_a^b f\,dg \;=\; \int_a^b f\,dg^* + \sum_i f(c_i)\cdot\alpha_i. \tag{1}$$

- Suppose that $g = g^* + h$ (as above) has a finite amount of steps α_i in the points c_i of the interval $[a, b]$ but f does not have same-sided discontinuities as g in any of those points. Then f is Riemann–Stieltjes integrable on the interval and the equation (1) is true.

Example. Find the value of the Riemann–Stieltjes integral

$$I \;:=\; \int_{-3}^{6} (x-\lfloor x\rfloor)\, dg(x)$$

where the integrand f is the mantissa function and the integrator g defined by

$$g(x) \;:=\; \begin{cases} -x^2 & \text{for} \quad x \leqq -2, \\ x & \text{for} \quad -2 < x \leqq 3, \\ 2x+1 & \text{for} \quad x > 3. \end{cases}$$

Now, f is from the left discontinuous at every integer, but g is bounded and only discontinuous from the right at -2 and 3. By the above last item, f is Riemann–Stieltjes integrable with respect to g on $[-3,\, 6]$. We can set

$$g \;=\; g^* + h$$

where g^* is continuous and the step function h has the step of 2 at -2 and the step of 4 at 3. Using (1) we get

$$\begin{aligned} I \;&=\; \int_{-3}^{6} f\, dg^* + f(-2)\cdot 2 + f(3)\cdot 4 \;=\; \sum_{i=-3}^{5} \int_{i}^{i+1} f(x) g'(x)\, dx + 0\cdot 2 + 0\cdot 4 \\ &=\; \int_{-3}^{-2} (x+3)(-2x)\, dx + \int_{-2}^{-1} (x+2)\cdot 1\, dx + \int_{-1}^{0} (x+1)\cdot 1\, dx + \int_{0}^{1} x\cdot 1\, dx + \int_{1}^{2} (x-1)\cdot 1\, dx \\ &\qquad + \int_{2}^{3} (x-2)\cdot 1\, dx + \int_{3}^{4} (x-3)\cdot 2\, dx + \int_{4}^{5} (x-4)\cdot 2\, dx + \int_{5}^{6} (x-5)\cdot 2\, dx \\ &=\; \frac{47}{6}. \end{aligned}$$