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Laguerre polynomial

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1 Definition

The *Laguerre polynomials* are orthogonal polynomials with respect to the weighting function e^{-x} on the half-line $[0, \infty)$. They are denoted by the letter "L" with the order as subscript and are normalized by the condition that the coefficient of the highest order term of L_n is $(-1)^n/n!$.

The first few Laguerre poynomials are as follows:

$$
\begin{aligned}
L_0(x) &= 1 \\
L_1(x) &= -x + 1 \\
L_2(x) &= \frac{1}{2}x^2 - 2x + 1 \\
L_3(x) &= -\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1
\end{aligned}
$$

A generalization is given by the *associated Laguerre polynomials* which depends on a parameter (traditionally denoted "α"). As it turns out, they are polynomials of the argument α as as well, so they are polynomials of two variables. They are defined over the same interval with the same normalization condition, but the weight function is generalized to $x^\alpha e^{-x}$. They are notated by including the parameter as a parenthesized superscript (not all authors use the parentheses).

The ordinary Laguerre polynomials are the special case of the generalized Laguerre polynomials when the parameter goes to zero. When some result holds for generalized Laguerre polynomials which is not more complicated than that for ordinary Laguerre polynomials, we shall only provide the more general result and leave it to the reader to send the parameter to zero to recover the more specific result.

The first few generalized Laguerre polynomials are as follows:

$$
\begin{aligned}
L_0^{(\alpha)}(x) &= 1 \\
L_0^{(\alpha)}(x) &= -x + \alpha + 1 \\
L_0^{(\alpha)}(x) &= \frac{1}{2}x^2 - (\alpha + 2)x + \frac{1}{2}(\alpha + 2)(\alpha + 1) \\
L_0^{(\alpha)}(x) &= -\frac{1}{6}x^3 + \frac{1}{2}(\alpha + 3)x^2 - \frac{1}{2}(\alpha + 2)(\alpha + 3)x + \frac{1}{6}(\alpha + 1)(\alpha + 2)(\alpha + 3)
\end{aligned}
$$

2 Formulae for these polynomials

The Laguerre polynomials may be exhibited explicitly as a sum in terms of factorials, which may also be written using binomial coefficients:

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{(k!)^2(n-k)!}(-x)^k = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!}$$

The generalization may be expressed in terms of gamma functions or falling factorials:

$$L_n^{(\alpha)} = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!} = \sum_{k=0}^{n} \frac{(n+\alpha)^{\underline{n-k}}}{k!(n-k)!}(-x)^k$$

They can be computed from a Rodrigues formula:

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n}\left(e^{-x} x^{n+\alpha}\right)$$

They have several integral representations. They can be expressed in terms of a countour integral

$$L_n(x) = \frac{1}{2\pi i} \oint \frac{e^{-\frac{xt}{1-t}}}{(1-t)t^{n+1}}\, dt,$$

where the origin is enclosed by the contour, but not $z = 1$.

3 Equations they satisfy

The Laguerre polynomials satisfy the orthogonality relation

$$\int_0^\infty e^{-x} x^\alpha L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = \frac{(n+\alpha)!}{n!}\delta_{nm}.$$

The Laguerre polynomials satisfy the differential equation

$$x\frac{d^2}{dx^2}L_n^{(\alpha)}(x) + (\alpha+1-x)\frac{d}{dx}L_n^{(\alpha)}(x) + (n-\alpha)L_n^{(\alpha)}(x) = 0$$

This equation arises in many contexts such as in the quantum mechanical treatment of the hydrogen atom.