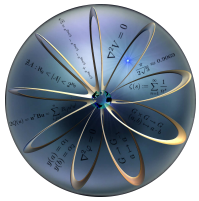

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derivation of half-angle formulae for tangent

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Start with the angle duplication formula

$$\tan(x) = \frac{2\tan(x/2)}{1-\tan^2(x/2)}.$$

Cross-multiply and move terms around:

$$\tan(x)\tan^2(x/2) + 2\tan(x/2) = \tan(x)$$

Divide by $\tan(x)$:

$$\tan^2(x/2) + \frac{2\tan(x/2)}{\tan x} = 1$$

Add $1/\tan^2(x)$ to both sides:

$$\tan^2(x/2) + \frac{2\tan(x/2)}{\tan x} + \frac{1}{\tan^2(x)} = 1 + \frac{1}{\tan^2(x)}$$

`http://planetmath.org/CompletingTheSquare`Complete the square:

$$\left(\tan(x/2) + \frac{1}{\tan(x)}\right)^2 = 1 + \frac{1}{\tan^2(x)}$$

Take a square root and move a term to obtain the half-angle formula:

$$\tan(x/2) = \sqrt{1 + \frac{1}{\tan^2(x)}} - \frac{1}{\tan(x)}$$

To derive the other forms of the formula, we start by substituting $\sin(x)/\cos(x)$ for $\tan(x)$:

$$\tan(x/2) = \sqrt{1 + \frac{\cos^2(x)}{\sin^2(x)}} - \frac{\cos(x)}{\sin(x)}$$

Put the stuff inside the square root over a common denominator:

$$\sqrt{\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)}} - \frac{\cos(x)}{\sin(x)}$$

Recall that $\sin^2(x) + \cos^2(x) = 1$. Hence, we may get rid of the square root:

$$\frac{1}{\sin x} - \frac{\cos(x)}{\sin(x)}$$

Putting the terms over a common denominator, we obtain our formula:

$$\tan(x/2) = \frac{1-\cos(x)}{\sin(x)}$$

To obtain the next formula, multiply both numerator and denominator by $1+\cos(x)$:

$$\frac{(1-\cos(x))(1+\cos(x))}{\sin(x)(1+\cos(x))}$$

Multiply out the numerator and simplify:

$$\frac{1-\cos^2(x)}{\sin(x)(1+\cos(x))}$$

Note that the numerator equals $\sin^2(x)$:

$$\frac{\sin^2(x)}{\sin(x)(1+\cos(x))}$$

Cancel a common factor of $\sin(x)$ to obtain the formula

$$\tan(x/2) = \frac{\sin(x)}{1+\cos(x)}.$$

To obtain the last formula, multiply the previous two formulae:

$$\tan^2(x/2) = \frac{1-\cos(x)}{\sin(x)} \cdot \frac{\sin(x)}{1+\cos(x)}$$

Cancel the common factor of $\sin(x)$:

$$\tan^2(x/2) = \frac{1-\cos(x)}{1+\cos(x)}$$

Take the square root of both sides to obtain the formula

$$\tan\frac{x}{2} \;=\; \pm\sqrt{\frac{1-\cos x}{1+\cos x}};$$

here the sign ($\pm$) has to be chosen according to the quadrant where the angle $\dfrac{x}{2}$ is.