classical Stokes' theorem

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Let M be a compact, oriented two-dimensional differentiable manifold (surface) with boundary in $\mathbb{R}^3$, and $\mathbf{F}$ be a C^2-smooth vector field defined on an open set in $\mathbb{R}^3$ containing M. Then

$$\iint_M (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \int_{\partial M} \mathbf{F} \cdot d\mathbf{s}\,.$$

Here, the boundary of M, ∂M (which is a curve) is given the induced orientation from M. The symbol $\nabla \times \mathbf{F}$ denotes the curl of $\mathbf{F}$. The symbol $d\mathbf{s}$ denotes the line element ds with a direction parallel to the unit tangent vector $\mathbf{t}$ to ∂M, while $d\mathbf{A}$ denotes the area element dA of the surface M with a direction parallel to the unit outward normal $\mathbf{n}$ to M. In precise terms:

$$d\mathbf{A} = \mathbf{n}\, dA\,, \quad d\mathbf{s} = \mathbf{t}\, ds\,.$$

The classical Stokes' theorem reduces to Green's theorem on the plane if the surface M is taken to lie in the xy-plane.

The classical Stokes' theorem, and the other "Stokes' type" theorems are special cases of the general Stokes' theorem involving differential forms. In fact, in the proof we present below, we appeal to the general Stokes' theorem.

Physical interpretation

(To be written.)

Proof using differential forms

The proof becomes a triviality once we express $(\nabla \times \mathbf{F}) \cdot d\mathbf{A}$ and $\mathbf{F} \cdot d\mathbf{s}$ in terms of differential forms.

Proof. Define the differential forms η and ω by

$$\begin{aligned} \eta_p(\mathbf{u}, \mathbf{v}) &= \langle \operatorname{curl} \mathbf{F}(p), \mathbf{u} \times \mathbf{v} \rangle\,, \\ \omega_p(\mathbf{v}) &= \langle \mathbf{F}(p), \mathbf{v} \rangle\,. \end{aligned}$$

for points $p \in \mathbb{R}^3$, and tangent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The symbol $\langle , \rangle$ denotes the dot product in $\mathbb{R}^3$. Clearly, the functions η_p and ω_p are linear and alternating in $\mathbf{u}$ and $\mathbf{v}$.

We claim

$$\eta = \nabla \times \mathbf{F} \cdot d\mathbf{A} \qquad \text{on } M. \tag{1}$$
$$\omega = \mathbf{F} \cdot d\mathbf{s} \qquad \text{on } \partial M. \tag{2}$$

To prove (1), it suffices to check it holds true when we evaluate the left- and right-hand sides on an orthonormal basis $\mathbf{u}, \mathbf{v}$ for the tangent space of M corresponding to the orientation of M, given by the unit outward normal $\mathbf{n}$. We calculate

$$\begin{aligned} \nabla \times \mathbf{F} \cdot d\mathbf{A}(\mathbf{u}, \mathbf{v}) &= \langle \operatorname{curl} \mathbf{F}, \mathbf{n} \rangle \, dA(\mathbf{u}, \mathbf{v}) && \text{definition of } d\mathbf{A} = \mathbf{n}\, dA \\ &= \langle \operatorname{curl} \mathbf{F}, \mathbf{n} \rangle && \text{definition of volume form } dA \\ &= \langle \operatorname{curl} \mathbf{F}, \mathbf{u} \times \mathbf{v} \rangle && \text{since } \mathbf{u} \times \mathbf{v} = \mathbf{n} \\ &= \eta(\mathbf{u}, \mathbf{v}) \,. \end{aligned}$$

For equation (2), similarly, we only have to check that it holds when both sides are evaluated at $\mathbf{v} = \mathbf{t}$, the unit tangent vector of ∂M with the induced orientation of ∂M. We calculate again,

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{s}(\mathbf{t}) &= \langle \mathbf{F}, \mathbf{t} \rangle \, ds(\mathbf{t}) && \text{definition of } d\mathbf{s} = \mathbf{t}\, ds \\ &= \langle \mathbf{F}, \mathbf{t} \rangle && \text{definition of volume form } ds \\ &= \omega(\mathbf{t}) \,. \end{aligned}$$

Furthermore, $d\omega = \eta$. (This can be checked by a calculation in Cartesian coordinates, but in fact this equation is one of the coordinate-free *definitions* of the curl.)

The classical Stokes' Theorem now follows from the general Stokes' Theorem,

$$\int_M \eta = \int_M d\omega = \int_{\partial M} \omega \,. \qquad \square$$

References

[1] Michael Spivak. *Calculus on Manifolds*. Perseus Books, 1998.