proof of general means inequality

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Let w_1, w_2, ..., w_n be positive real numbers such that $w_1+w_2+\cdots+w_n = 1$. For any real number $r \neq 0$, the weighted power mean of degree r of n positive real numbers x_1, x_2, ..., x_n (with respect to the weights w_1, ..., w_n) is defined as

$$M_w^r(x_1, x_2, \ldots, x_n) = (w_1 x_1^r + w_2 x_2^r + \cdots + w_n x_n^r)^{1/r}.$$

The definition is extended to the case $r = 0$ by taking the limit $r \to 0$; this yields the weighted geometric mean

$$M_w^0(x_1, x_2, \ldots, x_n) = x_1^{w_1} x_2^{w_2} \ldots x_n^{w_n}$$

(see derivation of zeroth weighted power mean). We will prove the weighted power means inequality, which states that for any two real numbers $r < s$, the weighted power means of orders r and s of n positive real numbers x_1, x_2, ..., x_n satisfy the inequality

$$M_w^r(x_1, x_2, \ldots, x_n) \leq M_w^s(x_1, x_2, \ldots, x_n)$$

with equality if and only if all the x_i are equal.

First, let us suppose that r and s are nonzero. We distinguish three cases for the signs of r and s: $r < s < 0$, $r < 0 < s$, and $0 < r < s$. Let us consider the last case, i.e. assume r and s are both positive; the others are similar. We write $t = \frac{s}{r}$ and $y_i = x_i^r$ for $1 \leq i \leq n$; this implies $y_i^t = x_i^s$. Consider the function

$$\begin{aligned} f\colon (0,\infty) &\to (0,\infty) \\ x &\mapsto x^t. \end{aligned}$$

Since $t > 1$, the second derivative of f satisfies $f''(x) = t(t-1)x^{t-2} > 0$ for all $x > 0$, so f is a strictly convex function. Therefore, according to Jensen's inequality,

$$\begin{aligned} (w_1 y_1 + w_2 y_2 + \cdots + w_n y_n)^t &= f(w_1 y_1 + w_2 y_2 + \cdots + w_n y_n) \\ &\leq w_1 f(y_1) + w_2 f(y_2) + \cdots + w_n f(y_n) \\ &= w_1 y_1^t + w_2 y_2^t + \cdots + w_n y_n^t, \end{aligned}$$

with equality if and only if $y_1 = y_2 = \cdots = y_n$. By substituting $t = \frac{s}{r}$ and $y_i = x_i^r$ back into this inequality, we get

$$(w_1 x_1^r + w_2 x_2^r + \cdots + w_n x_n^r)^{s/r} \leq w_1 x_1^s + w_2 x_2^s + \cdots + w_n x_n^s$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. Since s is positive, the function $x \mapsto x^{1/s}$ is strictly increasing, so raising both sides to the power $1/s$ preserves the inequality:

$$(w_1 x_1^r + w_2 x_2^r + \cdots + w_n x_n^r)^{1/r} \le (w_1 x_1^s + w_2 x_2^s + \cdots + w_n x_n^s)^{1/s},$$

which is the inequality we had to prove. Equality holds if and only if all the x_i are equal.

If $r = 0$, the inequality is still correct: M_w^0 is defined as $\lim_{r \to 0} M_w^r$, and since $M_w^r \le M_w^s$ for all $r < s$ with $r \ne 0$, the same holds for the limit $r \to 0$. The same argument shows that the inequality also holds for $s = 0$, i.e. that $M_w^r \le M_w^0$ for all $r < 0$. We conclude that for all real numbers r and s such that $r < s$,

$$M_w^r(x_1, x_2, \ldots, x_n) \le M_w^s(x_1, x_2, \ldots, x_n).$$