continuity of convex functions, alternate proof

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Let f be convex and $y \in (a,b)$ be arbitrary but fixed. Then

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda) f(y) & (1) \\ f(\lambda x + (1-\lambda)y) - f(y) &\leq \lambda(f(x) - f(y)) \leq \lambda |f(x) - f(y)|. & (2) \end{aligned}$$

Fix a number $c > \sup\{|f(u) - f(v)| : u, v \in (a,b)\}$. Then

$$|f(\lambda x + (1-\lambda)y) - f(y)| \leq \lambda |f(x) - f(y)| < \lambda c. \quad (3)$$

Given $\epsilon > 0$, let λ range over $(0, \epsilon/c)$ if $\epsilon/c < 1$, or $\lambda = 1$ otherwise. Then it is easy to see that $f(\lambda x + (1-\lambda)y)$ and $f(y)$ lie within ϵ distance of each other when λ varies as specified.

Continuity of f now follows–for $x < y$, the left-hand limit equals $f(y)$ and for $y < x$, the right-hand limit also equals $f(y)$, hence the limit is $f(y)$.