another proof of Jensen's inequality

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First of all, it's clear that defining

$$\lambda_k = \frac{\mu_k}{\sum_{k=1}^{n} \mu_k}$$

we have

$$\sum_{k=1}^{n} \lambda_k = 1$$

so it will we enough to prove only the simplified version.

Let's proceed by induction.

1) $n = 2$; we have to show that, for any $x_1$and x_2 in $[a, b]$,

$$f\left(\lambda_1 x_1 + \lambda_2 x_2\right) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

But, since $\lambda_1 + \lambda_2$ must be equal to 1, we can put $\lambda_2 = 1 - \lambda_1$, so that the thesis becomes

$$f\left(\lambda_1 x_1 + (1 - \lambda_1)\, x_2\right) \leq \lambda_1 f(x_1) + (1 - \lambda_1)\, f(x_2),$$

which is true by definition of a convex function.

2) Taking as true that $f\left(\sum_{k=1}^{n-1} \mu_k x_k\right) \leq \sum_{k=1}^{n-1} \mu_k f\left(x_k\right)$, where $\sum_{k=1}^{n-1} \mu_k = 1$, we have to prove that

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f\left(x_k\right),$$

where $\sum_{k=1}^{n} \lambda_k = 1$.

First of all, let's observe that

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{1 - \lambda_n} = \frac{\left(\sum_{k=1}^{n} \lambda_k\right) - \lambda_n}{1 - \lambda_n} = \frac{1 - \lambda_n}{1 - \lambda_n} = 1$$

and that if all $x_k \in [a, b]$, $\sum_{k=1}^{n-1} \frac{\lambda_k}{1-\lambda_n} x_k$ belongs to $[a, b]$ as well. In fact, $\frac{\lambda_k}{1-\lambda_n}$ being non-negative,

$$a \leq x_k \leq b \Rightarrow \frac{\lambda_k}{1 - \lambda_n} a \leq \frac{\lambda_k}{1 - \lambda_n} x_k \leq \frac{\lambda_k}{1 - \lambda_n} b,$$

and, summing over k,

$$a \sum_{k=1}^{n-1} \frac{\lambda_k}{1 - \lambda_n} \leq \sum_{k=1}^{n-1} \frac{\lambda_k}{1 - \lambda_n} x_k \leq b \sum_{k=1}^{n-1} \frac{\lambda_k}{1 - \lambda_n},$$

that is

$$a \leq \sum_{k=1}^{n-1} \frac{\lambda_k}{1-\lambda_n} x_k \leq b.$$

We have, by definition of a convex function:

$$\begin{aligned}
f\left(\sum_{k=1}^{n} \lambda_k x_k\right) &= f\left(\sum_{k=1}^{n-1} \lambda_k x_k + \lambda_n x_n\right) \\
&= f\left((1-\lambda_n)\sum_{k=1}^{n-1} \frac{\lambda_k}{1-\lambda_n} x_k + \lambda_n x_n\right) \\
&\leq (1-\lambda_n) f\left(\sum_{k=1}^{n-1} \frac{\lambda_k}{1-\lambda_n} x_k\right) + \lambda_n f(x_n).
\end{aligned}$$

But, by inductive hypothesis, since $\sum_{k=1}^{n-1} \frac{\lambda_k}{1-\lambda_n} = 1$, we have:

$$f\left(\sum_{k=1}^{n-1} \frac{\lambda_k}{1-\lambda_n} x_k\right) \leq \sum_{k=1}^{n-1} \frac{\lambda_k}{1-\lambda_n} f(x_k),$$

so that

$$\begin{aligned}
f\left(\sum_{k=1}^{n} \lambda_k x_k\right) &\leq (1-\lambda_n) f\left(\sum_{k=1}^{n-1} \frac{\lambda_k}{1-\lambda_n} x_k\right) + \lambda_n f(x_n) \\
&\leq (1-\lambda_n) \sum_{k=1}^{n-1} \frac{\lambda_k}{1-\lambda_n} f(x_k) + \lambda_n f(x_n) \\
&= \sum_{k=1}^{n-1} \lambda_k f(x_k) + \lambda_n f(x_n) \\
&= \sum_{k=1}^{n} \lambda_k f(x_k)
\end{aligned}$$

which is the thesis.