second derivative as simple limit

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If the real function f is twice differentiable in a neighbourhood of $x = x_0$, then

$$f''(x_0) = \lim_{h \to 0} \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2}. \tag{1}$$

Proof. The right hand side of the asserted equation is of the indeterminate form $\frac{0}{0}$. Using `http://planetmath.org/node/2657`l'Hôpital's rule, we obtain

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2} &= \lim_{h \to 0} \frac{f'(x_0+2h) \cdot 2 - 2f'(x_0+h)}{2h} - \frac{f'(x_0)}{h} + \frac{f'(x_0)}{h} \\
&= 2 \lim_{2h \to 0} \frac{f'(x_0+2h) - f'(x_0)}{2h} - \lim_{h \to 0} \frac{f'(x_0+h) - f'(x_0)}{h} \\
&= 2f''(x_0) - f''(x_0) \\
&= f''(x_0).
\end{aligned}$$