continuous derivative implies bounded variation

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Theorem. If the real function f has continuous derivative on the interval $[a,\, b]$, then on this interval,

- f is of bounded variation,
- f can be expressed as difference of two continuously differentiable monotonic functions.

Proof. 1º. The continuous function $|f'|$ has its greatest value M on the closed interval $[a,\, b]$, i.e.

$$|f'(x)| \leqq M \quad \forall x \in [a,\, b].$$

Let D be an arbitrary partition of $[a,\, b]$, with the points

$$x_0 = a < x_1 < x_2 < \ldots < x_{n-1} < b = x_n.$$

Consider f on a subinterval $[x_{i-1},\, x_i]$. By the mean-value theorem, there exists on this subinterval a point ξ_i such that $f(x_i) - f(x_{i-1}) = f'(\xi_i)(x_i - x_{i-1})$. Then we get

$$S_D := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} |f'(\xi_i)|(x_i - x_{i-1}) \leqq M \sum_{i=1}^{n}(x_i - x_{i-1}) = M(b-a).$$

Thus the total variation satisfies

$$\sup_D \{\text{all } S_D\text{'s}\} \leqq M(b-a) < \infty,$$

whence f is of bounded variation on the interval $[a,\, b]$.

2º. Define the functions G and H by setting

$$G := \frac{|f'| + f'}{2}, \quad H := \frac{|f'| - f'}{2}.$$

We see that these are non-negative and that $f' = G - H$. Define then the functions g and h on $[a,\, b]$ by

$$g(x) := f(a) + \int_a^x G(t)\, dt, \quad h(x) := \int_a^x H(t)\, dt.$$

Because G and H are non-negative, the functions g and h are monotonically nondecreasing. We have also

$$(g-h)(x) = f(a) + \int_a^x (G(t) - H(t))\,dt = f(a) + \int_a^x f'(t)\,dt = f(x),$$

whence $f = g - h$. Since G and H are by their definitions continuous, the monotonic functions g and h have continuous derivatives $g' = G$, $h' = H$. So g and h fulfil the requirements of the theorem.

Remark. It may be proved that each function of bounded variation is difference of two bounded monotonically increasing functions.