vanishing of gradient in domain

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Theorem. If the function f is defined in a `http://planetmath.org/Domain2`domain D of $\mathbb{R}^n$ and all the partial derivatives of a f vanish identically in D, i.e.

$$\nabla f \;\equiv\; \vec{0} \quad \text{in } D,$$

then the function has a constant value in the whole domain.

Proof. For the sake of simpler notations, think that $n = 3$; thus we have

$$f'_x(x,\,y,\,z) \;=\; f'_y(x,\,y,\,z) \;=\; f'_z(x,\,y,\,z) \;=\; 0 \quad \text{for all } (x,\,y,\,z) \in D. \quad (1)$$

Make the antithesis that there are the points $P_0 = (x_0,\,y_0,\,z_0)$ and $P_1 = (x_1,\,y_1,\,z_1)$ of D such that $f(x_0,\,y_0,\,z_0) \neq f(x_1,\,y_1,\,z_1)$. Since D is connected, one can form the broken line $P_0Q_1Q_2\ldots Q_kP_1$ contained in D. When one now goes along this broken line from P_0 to P_1, one mets the first corner where the value of f does not equal $f(x_0,\,y_0,\,z_0)$. Thus D contains a line segment, the end points of which give unequal values to f. When necessary, we change the notations such that this line segment is P_0P_1. Now, f'_x, f'_y, f'_z are continuous in D because they vanish. The mean-value theorem for several variables guarantees an interior point $(a,\,b,\,c)$ of the segment such that

$$0 \;\neq\; f(x_1,\,y_1,\,z_1)-f(x_0,\,y_0,\,z_0) \;=\; f'_x(a,\,b,\,c)(x_1-x_0)+f'_y(a,\,b,\,c)(y_1-y_0)+f'_z(a,\,b,\,c)(z_1-z_0).$$

But by (1), the last sum must vanish. This contradictory result shows that the antithesis is wrong, which settles the proof.