convexity of tangent function

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We will show that the tangent function is convex on the interval $[0, \pi/2)$. To do this, we will use the addition formula for the tangent and the fact that a continuous real function f is `http://planetmath.org/ConvexFunction`convex if and only if $f((x+y)/2) \leq (f(x)+f(y))/2$.

We start with the observation that, if $0 \leq x < 1$ and $0 \leq y < 1$, then by the `http://planetmath.org/ArithmeticGeometricMeansInequality`arithmetic-geometric mean inequality,

$$\begin{aligned} -2xy &\geq -x^2 - y^2 \\ 1 - 2xy + x^2y^2 &\geq 1 - x^2 - y^2 + x^2y^2 \\ (1-xy)^2 &\geq (1-x^2)(1-y^2), \end{aligned}$$

so

$$\frac{(1-xy)^2}{(1-x^2)(1-y^2)} \geq 1.$$

Let u and v be two numbers in the interval $[0, \pi/4)$. Set $x = \tan u$ and $y = \tan v$. Then $0 \leq x < 1$ and $0 \leq y < 1$. By the addition formula, we have

$$\begin{aligned} \tan(2u) &= \frac{2x}{1-x^2} \\ \tan(u+v) &= \frac{x+y}{1-xy} \\ \tan(2v) &= \frac{2y}{1-y^2}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{2}\left(\tan(2u) + \tan(2v)\right) &= \frac{x+y-x^2y-xy^2}{(1-x^2)(1-y^2)} \\ &= \frac{(x+y)(1-xy)}{(1-x^2)(1-y^2)} \\ &= \frac{x+y}{1-xy}\frac{(1-xy)^2}{(1-x^2)(1-y^2)} \\ &\geq \frac{x+y}{1-xy} = \tan(u+v), \end{aligned}$$

so the tangent function is convex.