proof of arithmetic-geometric-harmonic means inequality

Canonical name	ProofOfArithmeticgeometricharmonicMeansInequality
Date of creation	2013-03-22 15:09:37
Last modified on	2013-03-22 15:09:37
Owner	Mathprof (13753)
Last modified by	Mathprof (13753)
Numerical id	10
Author	Mathprof (13753)
Entry type	Proof
Classification	msc 26D15

For the Arithmetic Geometric Inequality, I claim it is enough to prove that if $\prod_{i=1}^n x_i = 1$ with $x_i \geq 0$ then $\sum_{i=1}^n x_i \geq n$. The arithmetic geometric inequality for $y_1, \ldots, y_n$ will follow by taking $x_i = \frac{y_i}{\sqrt[n]{\prod_{k=1}^n y_k}}$. The geometric harmonic inequality follows from the arithmetic geometric by taking $x_i = \frac{1}{y_i}$.

So, we show that if $\prod_{i=1}^n x_i = 1$ with $x_i \geq 0$ then $\sum_{i=1}^n x_i \geq n$ by induction on n.

Clear for $n = 1$.

Induction Step: By reordering indices we may assume the x_i are increasing, so $x_n \geq 1 \geq x_1$. Assuming the statement is true for $n-1$, we have $x_2 + \cdots + x_{n-1} + x_1 x_n \geq n - 1$. Then,

$$\sum_{i=1}^n x_i \geq n - 1 + x_n + x_1 - x_1 x_n$$

by adding $x_1 + x_n$ to both sides and subtracting $x_1 x_n$. And so,

$$\begin{aligned}
\sum_{i=1}^n x_i &\geq n + (x_n - 1) + (x_1 - x_1 x_n) \\
&= n + (x_n - 1) - x_1(x_n - 1) \\
&= n + (x_n - 1)(1 - x_1) \\
&\geq n
\end{aligned}$$

The last line follows since $x_n \geq 1 \geq x_1$.