proof of implicit function theorem

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We state the Theorem with a different notation:

Theorem 1. *Let Ω be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$ and let $f \in \mathcal{C}^1(\Omega, \mathbb{R}^m)$. Let $(x_0, y_0) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}^m$. If the matrix $D_y f(x_0, y_0)$ defined by*

$$D_y f(x_0, y_0) = \left(\frac{\partial f_j}{\partial y_k}(x_0, y_0) \right)_{j,k} \qquad j = 1, \dots, m \quad k = 1, \dots, m$$

is invertible, then there exists a neighbourhood $U \subset \mathbb{R}^n$ of x_0, a neighbourhood $V \subset \mathbb{R}^m$ of y_0 and a function $g \in \mathcal{C}^1(U, V)$ such that

$$f(x, y) = f(x_0, y_0) \Leftrightarrow y = g(x) \qquad \forall (x, y) \in U \times V.$$

Moreover

$$Dg(x) = -(D_y f(x, g(x)))^{-1} D_x f(x, g(x)).$$

Proof. Consider the function $F \in \mathcal{C}^1(\Omega, \mathbb{R}^n \times \mathbb{R}^m)$ defined by

$$F(x, y) = (x, f(x, y)).$$

One finds that

$$DF(x, y) = \left(\begin{array}{c|c} I_m & 0 \\ \hline D_x f & D_y f \end{array} \right).$$

Being $D_y f(x_0, y_0)$ invertible, $DF(x_0, y_0)$ is invertible too. Applying the inverse function Theorem to F we find that there exist a neighbourhood U of x_0 and V of y_0 and a function $G \in C^1(U \times V, \mathbb{R}^{n+m})$ such that $F(G(x, y)) = (x, y)$ for all $(x, y) \in U \times V$. Letting $G(x, y) = (G_1(x, y), G_2(x, y))$ (so that $G_1 \colon V \times W \to \mathbb{R}^n$, $G_2 \colon V \times W \to \mathbb{R}^m$) we hence have

$$(x, y) = F(G_1(x, y), G_2(x, y)) = (G_1(x, y), f(G_1(x, y), G_2(x, y)))$$

and hence $x = G_1(x, y)$ and $y = f(G_1(x, y), G_2(x, y)) = f(x, G_2(x, y))$. So we only have to set $g(x) = G_2(x, f(x_0, y_0))$ to obtain

$$f(x, g(x)) = f(x_0, y_0), \quad \forall x \in U.$$

Differentiating with respect to x we obtain

$$D_x f(x, g(x)) + D_y f(x, g(x)) Dg(x) = 0$$

which gives the desired formula for the computation of Dg. $\square$