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proof of Hermite-Hadamard integral inequality

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First of all, let's recall that a convex function on a open interval (a,b) is continuous on (a,b) and admits left and right derivative $f^{+}(x)$ and $f^{-}(x)$ for any $x\in(a,b)$. For this reason, it's always possible to construct at least one `http://planetmath.org/ConvexFunctionsLieAboveTheirSupportingLines`supporting line for $f(x)$ at any $x_0\in(a,b):$ if $f(x_0)$ is differentiable in x_0, one has $r(x)=f(x_0)+f'(x_0)(x-x_0)$; if not, it's obvious that all $r(x)=f(x_0)+c(x-x_0)$ are supporting lines for any $c\in[f^{-}(x_0),f^{+}(x_0)]$.
Let now $r(x)=f\left(\frac{a+b}{2}\right)+c\left(x-\frac{a+b}{2}\right)$ be a supporting line of $f(x)$ in $x=\frac{a+b}{2}\in(a,b)$. Then, $r(x)\leq f(x)$. On the other side, by convexity definition, having defined $s(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$ the line connecting the points $(a,f(a))$ and $(b,f(b))$, one has $f(x)\leq s(x)$. Shortly,

$$r(x)\leq f(x)\leq s(x)$$

Integrating both inequalities between a and b

$$\int_a^b r(x)\,dx\leq\int_a^b f(x)\,dx\leq\int_a^b s(x)dx$$

$$\begin{aligned}
&\int_a^b r(x)\,dx\\
=\;&\int_a^b\left[f\left(\frac{a+b}{2}\right)+c\left(x-\frac{a+b}{2}\right)\right]dx\\
=\;&f\left(\frac{a+b}{2}\right)(b-a)+c\int_a^b\left(x-\frac{a+b}{2}\right)dx\\
=\;&f\left(\frac{a+b}{2}\right)(b-a)
\end{aligned}$$

$$\begin{aligned}
&\int_a^b s(x)dx\\
=\;&\int_a^b\left[f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\right]dx\\
=\;&f(a)(b-a)+\frac{f(b)-f(a)}{b-a}\int_a^b(x-a)dx\\
=\;&\frac{f(a)+f(b)}{2}(b-a)
\end{aligned}$$

and so

$$f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b f\left(x\right) dx \leq \frac{f(a)+f(b)}{2}(b-a)$$

which is the thesis.