reduction formulas for integration of powers

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The following reduction formulas, with integer n and via integration by parts, may be used for lowing ($n > 0$) or raising ($n < 0$) the the powers:

- $\displaystyle\int \sin^n x\,dx \;=\; -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \qquad (n \gtrless 0)$
- $\displaystyle\int \cos^n x\,dx \;=\; \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \qquad (n \gtrless 0)$
- $\displaystyle\int (\ln x)^n\,dx \;=\; x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \qquad (n \gtrless 0)$
- $\displaystyle\int \frac{1}{(1+x^2)^n}\,dx \;=\; \frac{1}{2n-2}\cdot\frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{1}{(1+x^2)^{n-1}}\,dx \quad (n > 1)$

Example. For finding $\displaystyle\int \frac{dx}{\sin^3 x}$, we apply the first formula with $n := -1$, getting first

$$\int \frac{dx}{\sin x} \;=\; -\frac{1}{-1}\cdot\frac{\cos x}{\sin^2 x} + \frac{-2}{-1}\int \frac{dx}{\sin^3 x}.$$

From this we solve

$$\int \frac{dx}{\sin^3 x} \;=\; -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \int \frac{dx}{\sin x} \;=\; -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \ln\left|\tan\frac{x}{2}\right| + C$$

(see integration of rational function of sine and cosine).

Note 1. Instead of the two first formulae, it is simpler in the cases when n is a positive odd or a negative even number to use the following

$$\int \sin^{2m+1} x\,dx \;=\; \int \sin^{2m} x\sin x\,dx \;=\; -\int (1-\cos^2 x)^m(-\sin x)\,dx,$$

$$\int \cos^{2m+1} x\,dx \;=\; \int \cos^{2m} x\cos x\,dx \;=\; \int (1-\sin^2 x)^m\cos x\,dx,$$

$$\int \frac{1}{\sin^{2m} x}\,dx \;=\; \int \frac{1}{\sin^{2m-2} x}\cdot\frac{1}{\sin^2 x}\,dx \;=\; -\int (1+\cot^2 x)^{m-1}\,d\cot x,$$

$$\int \frac{1}{\cos^{2m} x}\,dx \;=\; \int \frac{1}{\cos^{2m-2} x}\cdot\frac{1}{\cos^2 x}\,dx \;=\; \int (1+\tan^2 x)^{m-1}\,d\tan x,$$

which may be found after making the powers on the right hand sides to polynomials.

Note 2. $\int \tan^n x\,dx$ $(n \in \mathbb{Z}_+)$ is obtained easily by the `http://planetmath.org/Integration`
$\tan x := t,\;\; dx = \frac{dt}{t^2+1}$ and a division; e.g.

$$\begin{aligned}
\int \tan^5 x\,dx \;&=\; \int \frac{t^5}{t^2+1}\,dt \;=\; \int \left(t^3 - t + \frac{t}{t^2+1}\right) dt \\
&= \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2}\ln(t^2+1) + C \\
&= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln\sqrt{\tan^2 x + 1} + C.
\end{aligned}$$