improper limits

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Entry type	Definition
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Synonym	infinite limits
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Related topic	LHpitalsRule
Related topic	ExtendedRealNumbers
Related topic	LimitRulesOfFunctions
Related topic	IntegratingTanXOver0fracpi2
Related topic	IndeterminateForm
Related topic	ExampleOfJumpDiscontinuity
Related topic	ListOfCommonLimits
Related topic	LimitsOfNaturalLogarithm
Related topic	SecondDerivativeAsSimpleLimit
Related topic	AngleBetweenTwoLines
Defines	limit at infinity
Defines	mnemonic of infinite

In calculus there is often used such expressions as "the limit of a function is infinite", and one may write for instance that

$$\lim_{x\to 0} \frac{1}{x^2} \;=\; \infty.$$

Such "limits" are actually of the limit notion, and can be defined exactly. They are called *improper limits.*

Definition. Let the real function f be defined in a neighbourhood of the point x_0.

$$\lim_{x\to x_0} f(x) \;=\; \infty$$

iff for every real number M there exists a number δ_M such that

$$f(x) \;>\; M$$

as soon as

$$0 \;<\; |x - x_0| \;<\; \delta_M.$$

In a similar way we can define the improper limit $-\infty$ of a real function. The definition may be extended also to the cases $x \to \pm\infty$, when one speaks of *limits at infinity.*

Note 1. If $\lim_{x\to x_0} f(x) \;=\; \infty$ and $\lim_{x\to x_0} g(x) \;=\; a > 0$, then we have

$$\lim_{x\to x_0} f(x)g(x) \;=\; \infty.$$

Hence we can say that $\infty \cdot a = \infty$ when $a > 0$. There are some other "mnemonics of infinite" (cf. the extended real numbers):

$$\infty \cdot a \;=\; -\infty \qquad (a \;<\; 0)$$

$$\pm\infty + a \;=\; \pm\infty$$

$$\frac{a}{\pm\infty} \;=\; 0$$

$$\infty + \infty \;=\; \infty$$

$$\infty \cdot \infty \;=\; \infty$$

$$-\infty \cdot \infty \;=\; -\infty$$

On the contrary, there exist no mnemonics for the cases

$$\infty \cdot 0,\ \ \infty - \infty,\ \ \frac{\infty}{\infty},\ \ \frac{0}{0},\ \ 0^0,\ \ \infty^0,\ \ 1^\infty;$$

they are and depend on the instance (cf. the indeterminate form).

Note 2. In the complex plane, the expression

$$\lim_{z \to z_0} f(z) \ = \ \infty$$

means that $\lim\limits_{z \to z_0} |f(z)| \ = \ \infty.$