Kummer's acceleration method

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There are several methods for acceleration of the convergence of a given series

$$\sum_{n=1}^{\infty} a_n \;=\; S. \tag{1}$$

One of the simplest is the following one due to Kummer (1837).

We suppose that the terms a_n of (1) are nonzero. Let

$$\sum_{n=1}^{\infty} b_n \;=\; C$$

be a series with nonzero terms and the known sum C. We use the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} \;=\; \varrho \;\neq\; 0$$

and the identity

$$S \;=\; \varrho C + \sum_{n=1}^{\infty} \left(1 - \varrho \frac{b_n}{a_n}\right) a_n. \tag{2}$$

Thus the original series (1) has attained a new form (2) the convergence of which is faster because of

$$\lim_{n\to\infty} \left(1 - \varrho \frac{b_n}{a_n}\right) \;=\; 0.$$

Example. For replacing the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; S$$

by a faster converging series we may take

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \;=:\; C,$$

which, for its part, can be expressed as the telescoping series

$$C \;=\; \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) \;=\; 1.$$

Now we have $\varrho = 1,$ and using (2) we obtain

$$S \;=\; 1+\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)}.$$

The convergence of this series may accelerated similarly taking e.g.

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} \;=:\; C,$$

where now $C = \frac{1}{4};$ then we get

$$S \;=\; \frac{5}{4}+2\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)(n+2)}.$$

The procedure may be repeated N times in all, yielding the result

$$S \;=\; \sum_{n=1}^{N}\frac{1}{n^2}+N!\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)(n+2)\cdots(n+N)}.$$

As for the sum of this series, see `http://planetmath.org/valueoftheriemannzetafunctionat` zeta function at $s = 2$.

References

[1] Pascal Sebah & Xavier Gourdon: `http://numbers.computation.free.fr/Constants/constants.html` *Acceleration of the convergence of series* (2002).