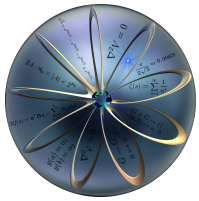

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## squeeze rule

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**Squeeze rule for sequences**

Let $f, g, h : \mathbb{N} \to \mathbb{R}$ be three sequences of real numbers such that

$$f(n) \leq g(n) \leq h(n)$$

for all $n$. If $\lim_{n\to\infty} f(n)$ and $\lim_{n\to\infty} h(n)$ exist and are equal, say to $a$, then $\lim_{n\to\infty} g(n)$ also exists and equals $a$.

The proof is fairly straightforward. Let $\epsilon$ be any real number $> 0$. By hypothesis there exist $M, N \in \mathbb{N}$ such that

$$|a - f(n)| < \epsilon \text{ for all } n \geq M$$

$$|a - h(n)| < \epsilon \text{ for all } n \geq N$$

Write $L = \max(M, N)$. For $n \geq L$ we have

- if $g(n) \geq a$:

$$|g(n) - a| = g(n) - a \leq h(n) - a < \epsilon$$

- else $g(n) < a$ and:

$$|g(n) - a| = a - g(n) \leq a - f(n) < \epsilon$$

So, for all $n \geq L$, we have $|g(n) - a| < \epsilon$, which is the desired conclusion.

**Squeeze rule for functions**

Let $f, g, h : S \to \mathbb{R}$ be three real-valued functions on a neighbourhood $S$ of a real number $b$, such that

$$f(x) \leq g(x) \leq h(x)$$

for all $x \in S - \{b\}$. If $\lim_{x\to b} f(x)$ and $\lim_{x\to b} h(x)$ exist and are equal, say to $a$, then $\lim_{x\to b} g(x)$ also exists and equals $a$.

Again let $\epsilon$ be an arbitrary positive real number. Find positive reals $\alpha$ and $\beta$ such that

$$|a - f(x)| < \epsilon \text{ whenever } 0 < |b - x| < \alpha$$

$$|a - h(x)| < \epsilon \text{ whenever } 0 < |b - x| < \beta$$

Write $\delta = \min(\alpha, \beta)$. Now, for any $x$ such that $|b - x| < \delta$, we have

- if $g(x) \geq a$:

$$|g(x) - a| = g(x) - a \leq h(x) - a < \epsilon$$

- else $g(x) < a$ and:

$$|g(x) - a| = a - g(x) \leq a - f(x) < \epsilon$$

and we are done.