fundamental theorem of calculus for Kurzweil-Henstock integral

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Let the $\int$ symbol denote the Kurzweil-Henstock integral. We can then give the most general version of the fundamental theorem of calculus.

Theorem. *Let $F\colon [a,b] \to \mathbb{R}$ and suppose the derivative $F'(x)$ exists for all $x \in [a,b]$. Then*

$$\int_a^b F'(x)dx = F(b) - F(a).$$

The reader should note the subtle difference from the standard version. Here we do not assume anything about F' except that it exists. For the standard version we usually assume that F' is continuous, and if we use the Lebesgue integral we must assume that F' is Lebesgue integrable. Part of this theorem is that F' is Kurzweil-Henstock integrable, hence no extra assumptions are necessary.

An example of a function where the standard version has problems is the function

$$F(x) := \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

F is differentiable everywhere, but

$$F'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Which is not continuous and in fact unbounded on any interval containing zero.