polynomial equation of odd degree

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Theorem. The equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \tag{1}$$

with odd degree n and real coefficients a_i ($a_0 \neq 0$) has at least one real root x.

Proof. Denote by $f(x)$ the left hand side of (1). We can write

$$f(x) = a_0x^n[1 + g(x)]$$

where $g(x) := \dfrac{a_1}{x} + \cdots + \dfrac{a_{n-1}}{x^{n-1}} + \dfrac{a_n}{x^n}$. But we have $\lim\limits_{|x|\to\infty} g(x) = 0$ because

$$\lim_{|x|\to\infty} \frac{a_i}{x^i} = 0$$

for all $i = 1, ..., n$. Thus there exists an $M > 0$ such that

$$|g(x)| < 1 \text{ for } |x| \geqq M.$$

Accordingly $1 + g(\pm M) > 0$ and

$$\mathrm{sign} f(\pm M) = (\mathrm{sign} a_0)(\mathrm{sign}(\pm M))^n \cdot 1 = (\mathrm{sign} a_0)(\pm 1)$$

since n is odd. Therefore the real polynomial function f has opposite signs in the end points of the interval $[-M, M]$. Thus the continuity of f guarantees, according to Bolzano's theorem, at least one zero x of f in that interval. So (1) has at least one real root x.