proof of l'Hôpital's rule for ∞/∞ form

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This is the proof of http://planetmath.org/LHpitalsRuleL'Hôpital's Rule in the case of the indeterminate form $\pm\infty/\infty$. Compared to http://planetmath.org/ProofO
proof for the $0/0$ case, more complicated estimates are needed.

Assume that

$$\lim_{x\to a} f(x) = \pm\infty\,, \quad \lim_{x\to a} g(x) = \pm\infty\,, \quad \lim_{x\to a} \frac{f'(x)}{g'(x)} = m\,,$$

where a and m are real numbers. The case when a or m is infinite only involves slight modifications to the arguments below.

Given $\epsilon > 0$. there is a $\delta > 0$ such that

$$\left|\frac{f'(\xi)}{g'(\xi)} - m\right| < \epsilon$$

whenever $0 < |\xi - a| < \delta$.

Let c and x be points such that $a - \delta < c < x < a$ or $a < x < c < a + \delta$. (That is, both c and x are within distance δ of a, but x is always closer.) By Cauchy's mean value theorem, there exists some ξ_x in between c and x (and hence $0 < |\xi_x - a| < \delta$) such that

$$\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(\xi_x)}{g'(\xi_x)}\,.$$

We can assume the values $f(x)$, $g(x)$, $f(x) - f(c)$, $g(x) - g(c)$ are *all non-zero* when x is close enough to a, say, when $0 < |x - a| < \delta'$ for some $0 < \delta' < \delta$. (So there is no division by zero in our equations.) This is because $f(x)$ and $g(x)$ were assumed to approach $\pm\infty$, so when x is close enough to a, they will exceed the fixed values $f(c)$, $g(c)$, and 0.

We write

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{f(x)}{f(x) - f(c)} \cdot \frac{g(x) - g(c)}{g(x)} \cdot \frac{f(x) - f(c)}{g(x) - g(c)} \\ &= \frac{1 - g(c)/g(x)}{1 - f(c)/f(x)} \cdot \frac{f'(\xi_x)}{g'(\xi_x)}\,. \end{aligned}$$

Note that

$$\lim_{x\to a} \frac{1 - g(c)/g(x)}{1 - f(c)/f(x)} = 1\,,$$

but ξ_x is not guaranteed to approach a as x approaches a, so we cannot just take the limit $x \to a$ directly. However: there exists $0 < \delta'' < \delta'$ so that

$$\left| \frac{1 - g(c)/g(x)}{1 - f(c)/f(x)} - 1 \right| < \frac{\epsilon}{|m| + \epsilon}$$

whenever $0 < |x - a| < \delta''$. Then

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - m \right| &= \left| \left(\frac{f'(\xi_x)}{g'(\xi_x)} - m \right) + \frac{f'(\xi_x)}{g'(\xi_x)} \left(\frac{1 - g(c)/g(x)}{1 - f(c)/f(x)} - 1 \right) \right| \\ &\leq \epsilon + (|m| + \epsilon) \frac{\epsilon}{|m| + \epsilon} = 2\epsilon \end{aligned}$$

for $0 < |x - a| < \delta''$.

This proves

$$\lim_{x \to a} \frac{f(x)}{g(x)} = m = \lim_{x \to a} \frac{f'(x)}{g'(x)}\,.$$

References

[1] Michael Spivak, *Calculus*, 3rd ed. Publish or Perish, 1994.