proof of the power rule

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The power rule can be derived by repeated application of the product rule.

Proof for all positive integers n

The power rule has been shown to hold for $n = 0$ and $n = 1$. If the power rule is known to hold for some $k > 0$, then we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} x^{k+1} &= \frac{\mathrm{d}}{\mathrm{d}x}(x \cdot x^k) \\
&= x \left(\frac{\mathrm{d}}{\mathrm{d}x} x^k \right) + x^k \\
&= x \cdot (k x^{k-1}) + x^k \\
&= k x^k + x^k \\
&= (k+1) x^k
\end{aligned}$$

Thus the power rule holds for all positive integers n.

Proof for all positive rationals n

Let $y = x^{p/q}$. We need to show

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x^{p/q}) = \frac{p}{q} x^{p/q-1} \tag{1}$$

The proof of this comes from implicit differentiation.

By definition, we have $y^q = x^p$. We now take the derivative with respect to x on both sides of the equality.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}y^q &= \frac{\mathrm{d}}{\mathrm{d}x}x^p \\
\frac{\mathrm{d}}{\mathrm{d}y}(y^q)\frac{\mathrm{d}y}{\mathrm{d}x} &= px^{p-1} \\
qy^{q-1}\frac{\mathrm{d}y}{\mathrm{d}x} &= px^{p-1} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} \\
&= \frac{p}{q}x^{p-1}y^{1-q} \\
&= \frac{p}{q}x^{p-1}x^{p(1-q)/q} \\
&= \frac{p}{q}x^{p-1+p/q-p} \\
&= \frac{p}{q}x^{p/q-1}
\end{aligned}$$

Proof for all positive irrationals n

For positive irrationals we claim continuity due to the fact that (**??**) holds for all positive rationals, and there are positive rationals that approach any positive irrational.

Proof for negative powers n

We again employ implicit differentiation. Let $u = x$, and differentiate u^n with respect to x for some non-negative n. We must show

$$\frac{\mathrm{d}u^{-n}}{\mathrm{d}x} = -nu^{-n-1} \tag{2}$$

By definition we have $u^n u^{-n} = 1$. We begin by taking the derivative with respect to x on both sides of the equality. By application of the product rule we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(u^n u^{-n}) &= 1 \\
u^n \frac{\mathrm{d}u^{-n}}{\mathrm{d}x} + u^{-n}\frac{\mathrm{d}u^n}{\mathrm{d}x} &= 0 \\
u^n \frac{\mathrm{d}u^{-n}}{\mathrm{d}x} + u^{-n}(nu^{n-1}) &= 0 \\
u^n \frac{\mathrm{d}u^{-n}}{\mathrm{d}x} &= -nu^{-1} \\
\frac{\mathrm{d}u^{-n}}{\mathrm{d}x} &= -nu^{-n-1}
\end{aligned}
$$