conditionally convergent real series

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Theorem. If the series

$$u_1 + u_2 + u_3 + \dots \tag{1}$$

with real terms u_i is conditionally convergent, i.e. converges but $|u_1| + |u_2| + |u_3| + \cdots$ diverges, then the both series

$$a_1 + a_2 + a_3 + \dots \quad \text{and} \quad -b_1 - b_2 - b_3 - \dots \tag{2}$$

consisting of the positive and negative terms of (1) are divergent — more accurately,

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_n \;=\; +\infty \quad \text{and} \quad \lim_{n\to\infty} \sum_{i=1}^{n} (-b_n) \;=\; -\infty.$$

Proof. If both of the series (2) were convergent, having the sums A and $-B$, then we had

$$0 \leqq |u_1| + |u_2| + \dots + |u_n| < A + B$$

for every n. This would however mean that (1) would converge absolutely, contrary to the conditional convergence. If, on the other hand, one of the series (2) were convergent and the other divergent, then we can see that (1) had to diverge, contrary to what is supposed in the theorem. In fact, if e.g. $a_1 + a_2 + a_3 + \dots$ were convergent, then the partial sum $a_1 + a_2 + \dots + a_n$ were below a finite bound for each n, whereas the n^{th} partial sum of the divergent one of (2) would tend to $-\infty$ as $n \to \infty$; then should also the n^{th} partial sum of (1) tend to $-\infty$.