converse of Euler's homogeneous function theorem

Canonical name	ConverseOfEulersHomogeneousFunctionTheorem
Date of creation	2013-03-22 18:07:56
Last modified on	2013-03-22 18:07:56
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	7
Author	pahio (2872)
Entry type	Theorem
Classification	msc 26B12
Classification	msc 26A06
Classification	msc 15-00
Synonym	converse of Euler's theorem on homogeneous functions
Related topic	ConverseTheorem
Related topic	ChainRuleSeveralVariables
Related topic	Logarithm

Theorem. If the function f of the real variables $x_1, \ldots, x_k$ satisfies the identity

$$x_1 \frac{\partial f}{\partial x_1} + \ldots + x_k \frac{\partial f}{\partial x_k} = nf, \tag{1}$$

then f is a homogeneous function of degree n.

Proof. Let $f(tx_1, \ldots, tx_k) := \varphi(t)$. Differentiating with respect to t we obtain

$$\varphi'(t) = x_1 f'_{x_1}(tx_1, \ldots, tx_k) + \ldots + x_k f'_{x_k}(tx_1, \ldots, tx_k) = \frac{1}{t}[tx_1 f'_{x_1}(tx_1, \ldots, tx_k) + \ldots + tx_k f'_{x_k}(tx_1, \ldots$$

which by (1) may be written

$$\varphi'(t) = \frac{n}{t} f(tx_1, \ldots, tx_k) = \frac{n}{t}\varphi(t).$$

Accordingly,

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{n}{t},$$

which implies the integrated form

$$\ln|\varphi(t)| = \ln t^n + \ln C$$

for any positive t. Thus we have $\varphi(t) = Ct^n$, where C is independent on t. Choosing $t = 1$ we see that $C = \varphi(1)$, and therefore $\varphi(t) = t^n\varphi(1)$. This last equation means that

$$f(tx_1, \ldots, tx_k) = t^n f(x_1, \ldots, x_k)$$

saying that f is a (positively) homogeneous function of degree n.

References

[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset II.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1932).