proof of PTAH inequality

Canonical name	ProofOfPTAHInequality
Date of creation	2013-03-22 16:55:00
Last modified on	2013-03-22 16:55:00
Owner	Mathprof (13753)
Last modified by	Mathprof (13753)
Numerical id	29
Author	Mathprof (13753)
Entry type	Proof
Classification	msc 26D15

In order to prove the PTAH inequality two lemmas are needed. The first lemma is quite general and does not depend on the specific P and Q that are defined for the PTAH inequality.

The setup for the first lemma is as follows:

We still have a measure space X with measure m. We have a subset $\Lambda \subseteq \mathbb{R}^n$. And we have a function $p : X \times \Lambda \to \mathbb{R}$ which is positive and is integrable in x for all $\lambda \in \Lambda$. Also, $p(x, \lambda) \log p(x, \lambda')$ is integrable in x for each pair $\lambda, \lambda' \in \Lambda$.

Define $P : \Lambda \to \mathbb{R}$ by

$$P(\lambda) = \int p(x, \lambda) dm(x)$$

and $Q : \Lambda \times \Lambda \to \mathbb{R}$
by

$$Q(\lambda, \lambda') = \int p(x, \lambda) \log p(x, \lambda') dm(x).$$

Lemma 1 (1) $P(\lambda) \log \frac{P(\lambda')}{P(\lambda)} \geq Q(\lambda, \lambda') - Q(\lambda, \lambda)$
(2) if $Q(\lambda, \lambda') \geq Q(\lambda, \lambda)$ then $P(\lambda') \geq P(\lambda)$. If equality holds then $p(x, \lambda) = p(x, \lambda')$ a.e [m].

Proof It is clear that (2) follows from (1), so we only need to prove (1). Define a measure $d\nu(x) = \frac{p(x,\lambda) dm(x)}{P(\lambda)}$. Then

$$\int d\nu(x) = 1$$

so we can use Jensen's inequality for the logarithm.

$$
\begin{aligned}
Q(\lambda,\lambda') - Q(\lambda,\lambda) &= \int p(x,\lambda)[\log p(x,\lambda') - \log p(x,\lambda)]dm(x) \\
&= \int p(x,\lambda) \log \frac{p(x,\lambda')}{p(x,\lambda)} dm(x) \\
&= P(\lambda) \int \log \frac{p(x,\lambda')}{p(x,\lambda)} d\nu(x) \\
&\leq P(\lambda) \log \int \frac{p(x,\lambda')}{p(x,\lambda)} d\nu(x) \\
&= P(\lambda) \log \int \frac{p(x,\lambda')}{P(\lambda)} dm(x) \\
&= P(\lambda) \log \frac{P(\lambda')}{P(\lambda)}.
\end{aligned}
$$

The next lemma uses the notation of the parent entry.

Lemma 2 Suppose $r_i \geq 0$ for $i = 1, \ldots, n$ and $\theta = (\theta_1, \ldots, \theta_n) \in \sigma$. If $\sum_j r_j > 0$ then

$$\prod_{i=1}^{n} {\theta_i}^{r_i} \leq \prod_i (\frac{r_i}{\sum_j r_j})^{r_i}.$$

Proof. Let $\lambda = (\lambda_i) \in \sigma$. By the concavity of the log function we have

$$\sum_i \lambda_i \log x_i \leq \log \sum_i \lambda_i x_i$$

where $x_i > 0$ for $\imath = 1, \ldots, n$.

so that

$$\prod_i {x_i}^{\lambda_i} \leq \sum_i \lambda_i x_i = \prod_i (\sum_j \lambda_j x_j)^{\lambda_i}. \tag{1}$$

It is enough to prove the lemma for the case where $r_i > 0$ for all i. We can also assume $\theta_i > 0$ for all i, otherwise the result is trivial.

Let $\rho = \sum_j r_j > 0$ and $\lambda_i = \frac{r_i}{\rho}$ so that $\rho\lambda_i = r_i$.

Raise each side of (1) to the ρ power:

$$\prod_i {x_i}^{r_i} \leq \prod_i (\sum_j \lambda_j x_j)^{r_i} \tag{2}$$

so that

$$\prod_i (\frac{x_i}{\sum_j \lambda_j x_j})^{r_i} \leq 1 \tag{3}$$

Multiply (3) by $\prod(\frac{r_i}{\rho})^{r_i}$ to get:

$$\prod_i (\frac{r_i x_i}{\sum_j r_j x_j})^{r_i} \leq \prod_i (r_i/\rho)^{r_i}. \tag{4}$$

Claim: There exist $x_i > 0$, $i = 1, \ldots, n$ such that

$$\theta_i = \frac{r_i x_i}{\sum_j r_j x_j}. \tag{5}$$

If so, then substituting into (4)

$$\prod_i \theta_i{}^{r_i} \leq \prod_i (\frac{r_i}{\rho})^{r_i} = \prod_i (\frac{r_i}{\sum_j r_j})^{r_i}$$

So it remains to prove the claim. We have to solve the system of equations $\theta_i \sum_j r_j x_j = r_i x_i$, $i = 1, \ldots, n$ for x_i. Rewriting this in matrix form, let $A = (a_{ij})$, $R = \mathrm{diag}(r_1, \ldots, r_n)$, and $x = \mathrm{diag}(x_1, \ldots, x_n)$, where $a_{ii} = \theta_i - 1$ and $a_{ij} = \theta_i$ if $i \neq j$, $i, j = 1, \ldots, n$. The columns sums of A are 0, since $\theta \in \sigma$. Hence A is singular and the homogenous system $ARx = 0$ has a nonzero solution, say x. Since R is nonsingular, it follows that $Rx \neq 0$. It follows that $r_i x_i \neq 0$ for some i and therefore $\sum_j r_j x_j \neq 0$. If necessary, we can replace x by $-x$ so that $\sum_j r_j x_j > 0$. From (5) it follows that $x_j > 0$ for all j.

Now we can prove the PTAH inequality. Let $r_i(\lambda) = \int a_i(x) \prod_j \lambda_j{}^{a_j(x)} dm(x)$.

We calculate $\frac{\partial P}{\partial \lambda_i}$ by differentiating under the integral sign. If $\lambda_i > 0$ then

$$\frac{\partial P}{\partial \lambda_i} = r_i(\lambda)/\lambda_i.$$

Thus

$$\lambda_i \frac{\partial P}{\partial \lambda_i} = r_i(\lambda). \tag{6}$$

If $\lambda_i = 0$ then by writing

$$r_i(\lambda) = \int_E a_i(x) \ldots dm(x) + \int_{E^c} \lambda_i{}^{a_i(x)} \ldots dm(x)$$

where $E = \{x \in X | a_i(x) = 0\}$ it is clear that each integral is 0, so that $r_i(\lambda) = 0$. So again, (6) holds. Therefore,

$$\frac{r_i(\lambda)}{\sum_j r_j(\lambda)} = \frac{\lambda_i \partial P/\partial\lambda_i}{\sum_j \lambda_j \partial P/\lambda_j} = \overline{\lambda_i}.$$

Then

$$\begin{aligned}
Q(\lambda, \lambda') &= \int \prod_j {\lambda_j}^{a_j(x)} \log \prod_i ({\lambda_i}')^{a_i(x)} dm(x) \\
&= \sum_i \log {\lambda_i}' \int a_i(x) \prod_j {\lambda_j}^{a_j(x)} dm(x) \\
&= \sum_i r_i(\lambda) \log {\lambda_i}' \\
&= \log \prod_i ({\lambda_i}')^{r_i(\lambda)} \\
&\leq \log \prod_i (\frac{r_i(\lambda)}{\sum_j r_j(\lambda)})^{r_i(\lambda)} \\
&= \log \prod_i (\overline{\lambda_i})^{r_i(\lambda)} \\
&= Q(\lambda, \overline{\lambda}).
\end{aligned}$$

Now by Lemma 1, with $\overline{\lambda} = \lambda'$ we get $P(\overline{\lambda}) \geq P(\lambda)$.