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second integral mean-value theorem

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If the real functions f and g are continuous and f monotonic on the interval $[a, b]$, then the equation

$$\int_a^b f(x)g(x) dx = f(a) \int_a^\xi g(x) dx + f(b) \int_\xi^b g(x) dx \quad (1)$$

is true for a value ξ in this interval.

Proof. We can suppose that $f(a) \neq f(b)$ since otherwise any value of ξ between a and b would do.

Let's first prove the auxiliary result, that if a function φ is continuous on an open interval I containing $[a, b]$ then

$$\lim_{h \rightarrow 0} \int_a^b \frac{\varphi(x+h) - \varphi(x)}{h} dx = \varphi(b) - \varphi(a). \quad (2)$$

In fact, when we take an antiderivative Φ of φ , then for every nonzero h the function

$$x \mapsto \frac{\Phi(x+h) - \Phi(x)}{h}$$

is an antiderivative of the integrand of (2) on the interval $[a, b]$. Thus we have

$$\int_a^b \frac{\varphi(x+h) - \varphi(x)}{h} dx = \frac{\Phi(b+h) - \Phi(x)}{h} - \frac{\Phi(a+h) - \Phi(x)}{h} \rightarrow \varphi(b) - \varphi(a) \quad \text{as } h \rightarrow 0.$$

The given functions f and g can be extended on an open interval I containing $[a, b]$ such that they remain continuous and f monotonic. We take an antiderivative G of g and a nonzero number h having small absolute value. Then we can write the identity

$$\int_a^b \frac{f(x+h)G(x+h) - f(x)G(x)}{h} dx = \int_a^b \frac{f(x+h)[G(x+h) - G(x)]}{h} dx - \int_a^b \frac{f(x+h) - f(x)}{h} G(x) dx \quad (3)$$

By (2), the left hand side of (3) may be written

$$\int_a^b \frac{f(x+h)G(x+h) - f(x)G(x)}{h} dx = f(b)G(b) - f(a)G(a) + \varepsilon_1(h) \quad (4)$$

where $\varepsilon_1(h) \rightarrow 0$ as $h \rightarrow 0$. Further, the function

$$x \mapsto \begin{cases} \frac{f(x+h)[G(x+h)-G(x)]}{h} & \text{for } h \neq 0 \\ f(x)g(x) & \text{for } h = 0 \end{cases}$$

is continuous in a rectangle $a \leq x \leq b$, $-\delta \leq h \leq \delta$, whence we have

$$\int_a^b \frac{f(x+h)[G(x+h)-G(x)]}{h} dx = \int_a^b f(x)g(x) dx + \varepsilon_2(h) \quad (5)$$

where $\varepsilon_2(h) \rightarrow 0$ as $h \rightarrow 0$. Because of the monotonicity of f , the expression $\frac{f(x+h)-f(x)}{h}$ does not change its sign when $a \leq x \leq b$. Then the usual integral mean value theorem guarantees for every h (sufficiently near 0) a number ξ_h of the interval $[a, b]$ such that

$$\int_a^b \frac{f(x+h)-f(x)}{h} G(x) dx = G(\xi_h) \int_a^b \frac{f(x+h)-f(x)}{h} dx,$$

and the auxiliary result (2) allows to write this as

$$\int_a^b \frac{f(x+h)-f(x)}{h} G(x) dx = G(\xi_h)[f(b)-f(a)+\varepsilon_3(h)] \quad (6)$$

with $\varepsilon_3(h) \rightarrow 0$ as $h \rightarrow 0$. Now the equations (4), (5) and (6) imply

$$f(b)G(b)-f(a)G(a) + \varepsilon_1(h) = \int_a^b f(x)g(x) dx + \varepsilon_2(h) + G(\xi_h)[f(b)-f(a)+\varepsilon_3(h)]. \quad (7)$$

Because $f(b)-f(a) \neq 0$, the expression $G(\xi_h)$ has a limit L for $h \rightarrow 0$. By the continuity of G there must be a number ξ between a and b such that $G(\xi) = L$. Letting then h tend to 0 we thus get the limiting equation

$$f(b)G(b)-f(a)G(a) = \int_a^b f(x)g(x) dx + G(\xi)[f(b)-f(a)],$$

which finally gives

$$\int_a^b f(x)g(x) dx = f(a)[G(\xi)-G(a)]+f(b)[G(b)-G(\xi)] = f(a) \int_a^\xi g(x) dx + f(b) \int_\xi^b g(x) dx$$

Q.E.D.