uniformly continuous

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Defines	uniformly continuous function

Let $f : A \to \mathbb{R}$ be a real function defined on a subset A of the real line. We say that f is *uniformly continuous* if, given an arbitrary small positive ε, there exists a positive δ such that whenever two points in A differ by less than δ, they are mapped by f into points which differ by less than ε. In symbols:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, y \in A \; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Every uniformly continuous function is also continuous, while the converse does not always hold. For instance, the function $f :]0, +\infty[\to \mathbb{R}$ defined by $f(x) = 1/x$ is continuous in its domain, but not uniformly.

A more general definition of uniform continuity applies to functions between metric spaces (there are even more general environments for uniformly continuous functions, i.e. uniform spaces). Given a function $f : X \to Y$, where X and Y are metric spaces with distances d_X and d_Y, we say that f is uniformly continuous if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, y \in X \; d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Uniformly continuous functions have the property that they map Cauchy sequences to Cauchy sequences and that they preserve uniform convergence of sequences of functions.

Any continuous function defined on a compact space is uniformly continuous (see Heine-Cantor theorem).