



derivation of geometric mean as the limit of the power mean

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Fix $x_1, x_2, \dots, x_n \in \mathbb{R}^+$. Then let

$$\mu(r) := \left(\frac{x_1^r + \dots + x_n^r}{n} \right)^{1/r}.$$

For $r \neq 0$, by definition $\mu(r)$ is the r th power mean of the x_i . It is also clear that $\mu(r)$ is a differentiable function for $r \neq 0$. What is $\lim_{r \rightarrow 0} \mu(r)$?

We will first calculate $\lim_{r \rightarrow 0} \log \mu(r)$ using <http://planetmath.org/LHopitalsRule>'Hôpital's rule.

$$\begin{aligned} \lim_{r \rightarrow 0} \log \mu(r) &= \lim_{r \rightarrow 0} \frac{\log \left(\frac{x_1^r + \dots + x_n^r}{n} \right)}{r} \\ &= \lim_{r \rightarrow 0} \frac{\left(\frac{x_1^r \log x_1 + \dots + x_n^r \log x_n}{n} \right)}{\left(\frac{x_1^r + \dots + x_n^r}{n} \right)} \\ &= \lim_{r \rightarrow 0} \frac{x_1^r \log x_1 + \dots + x_n^r \log x_n}{x_1^r + \dots + x_n^r} \\ &= \frac{\log x_1 + \dots + \log x_n}{n} \\ &= \log \sqrt[n]{x_1 \dots x_n}. \end{aligned}$$

It follows immediately that

$$\lim_{r \rightarrow 0} \left(\frac{x_1^r + \dots + x_n^r}{n} \right)^{1/r} = \sqrt[n]{x_1 \dots x_n}.$$