$C_0^\infty(U)$ is not empty

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Owner	matte (1858)
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Author	matte (1858)
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Theorem. If U is a non-empty open set in $\mathbb{R}^n$, then the set of smooth functions with compact support $C_0^\infty(U)$ is non-trivial (that is, it contains functions other than the zero function).

Remark. This theorem may seem to be obvious at first sight. A way to notice that it is not so obvious, is to formulate it for analytic functions with compact support: in that case, the result does not hold; in fact, there are no nonconstant analytic functions with compact support at all. One important consequence of this theorem is the existence of partitions of unity.

Proof of the theorem: Let us first prove this for $n = 1$: If $a < b$ be real numbers, then there exists a smooth non-negative function $f : \mathbb{R} \to \mathbb{R}$, whose `http://planetmath.org/SupportOfFunction`support is the compact set $[a, b]$.

To see this, let $\phi\colon \mathbb{R} \to \mathbb{R}$ be the function defined on `http://planetmath.org/InfinitelyDiffe` page, and let

$$f(x) = \phi(x - a)\phi(b - x).$$

Since ϕ is smooth, it follows that f is smooth. Also, from the definition of ϕ, we see that $\phi(x - a) = 0$ precisely when $x \leq a$, and $\phi(b - x) = 0$ precisely when $x \geq b$. Thus the support of f is indeed $[a, b]$.

Since U is non-empty and open there exists an $x \in U$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. Let f be smooth function such that $\operatorname{supp} f = [-\varepsilon/2, \varepsilon/2]$, and let

$$h(z) = f(\|x - z\|^2).$$

Since $\|\cdot\|^2$ (Euclidean norm) is smooth, the claim follows. $\square$