application of Cauchy–Schwarz inequality

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In determining the perimetre of ellipse one encounters the elliptic integral

$$\int_0^{\frac{\pi}{2}}\sqrt{1-\varepsilon^2\sin^2 t}\,dt,$$

where the parametre ε is the eccentricity of the ellipse ($0 \leqq \varepsilon < 1$). A good upper bound for the integral is obtained by utilising the `http://planetmath.org/node/1628`Cauchy-Schwarz inequality

$$\left|\int_a^b fg\right| \leqq \sqrt{\int_a^b f^2}\sqrt{\int_a^b g^2}$$

choosing in it $f(t) := 1$ and $g(t) := \sqrt{1-\varepsilon^2\sin^2 t}$. Then we get

$$\begin{aligned} 0 < \int_0^{\frac{\pi}{2}}\sqrt{1-\varepsilon^2\sin^2 t}\,dt &\leqq \sqrt{\int_0^{\frac{\pi}{2}} 1^2\,dt}\sqrt{\int_0^{\frac{\pi}{2}}\left(1-\varepsilon^2\sin^2 t\right)\,dt} \\ &= \sqrt{\frac{\pi}{2}}\sqrt{\int_0^{\frac{\pi}{2}}\left(1-\varepsilon^2\cdot\frac{1-\cos 2t}{2}\right)\,dt} \\ &= \frac{\pi}{2}\sqrt{1-\frac{\varepsilon^2}{2}}. \end{aligned}$$

Thus we have the estimation

$$\int_0^{\frac{\pi}{2}}\sqrt{1-\varepsilon^2\sin^2 t}\,dt \leqq \frac{\pi}{2}\sqrt{1-\frac{\varepsilon^2}{2}}.$$

It is the better approximation for the perimetre of ellipse the smaller is its eccentricity, i.e. the closer the ellipse is to circle. The accuracy is $O(\varepsilon^4)$