regularity theorem for the Laplace equation

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Warning: This entry is still in the process of being written, hence is not yet .

Let D be an open subset of $\mathbb{R}^n$. Suppose that $f\colon D \to \mathbb{R}$ is twice differentiable and satisfies Laplace's equation. Then f has derivatives of all orders and is, in fact analytic.

Proof: Let $\mathbf{p}$ be any point of D. We shall show that f is analytic at $\mathbf{p}$. Since D is an open set, there must exist a real number $r > 0$ such that the closed ball of radius r about $\mathbf{p}$ lies inside of D.

Since f satisfies Laplace's equation, we can express the value of f inside this ball in terms of its values on the boundary of the ball by using Poisson's formula:

$$f(\mathbf{x}) = \frac{1}{r^{n-1}A(n-1)} \int_{|\mathbf{y}-p|=r} f(\mathbf{y}) \frac{r^2 - |\mathbf{x}-\mathbf{p}|^2}{|\mathbf{x}-\mathbf{y}|^n} \, d\Omega(\mathbf{y})$$

Here, $A(k)$ denotes the `http://planetmath.org/node/4495`area of the k-dimensional sphere and $d\Omega$ denotes the measure on the sphere of radius r about $\mathbf{p}$.

We shall show that f is analytic by deriving a convergent power series for f. From this, it will automatically follow that f has derivatives of all orders, so a separate proof of this fact will not be necessary.

Since this involves manipulating power series in several variables, we shall make use of multi-index notation to keep the equations from becoming unnecessarily complicated and drowning in a plethora of indices.

First, note that since f is assumed to be twice differentiable in D, it is continuous in D and, hence, since the sphere of radius r about s is compact, it attains a maximum on this sphere. Let us denote this maxmum by M. Next, let us consider the quantity

$$\frac{1}{|\mathbf{x}-\mathbf{y}|^n}$$

which appears in the integral. We may write this quantity more explicitly as

$$\left(|\mathbf{y}-\mathbf{p}|^2 - 2(\mathbf{x}-\mathbf{p}) \cdot (\mathbf{y}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|^2\right)^{-\frac{n}{2}}.$$

Since the values of the variable y has been restricted by the condition $|\mathbf{y}-\mathbf{p}| = r$, we may rewrite this as

$$\frac{1}{r^n} \left(1 + \frac{-2(\mathbf{x}-\mathbf{p}) \cdot (\mathbf{y}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|^2}{r^2}\right)^{-\frac{n}{2}}.$$

Assume that $|\mathbf{x}-\mathbf{p}| < r/4$. Then we have

$$\left|\frac{-2(\mathbf{x}-\mathbf{p})\cdot(\mathbf{y}-\mathbf{p})+|\mathbf{x}-\mathbf{p}|^2}{r^2}\right| \le \frac{2|(\mathbf{x}-\mathbf{p})\cdot(\mathbf{y}-\mathbf{p})|}{r^2} + \frac{|\mathbf{x}-\mathbf{p}|^2}{r^2} \le$$

$$\frac{2|\mathbf{x}-\mathbf{p}|\,|\mathbf{y}-\mathbf{p}|}{r^2} + \left(\frac{|\mathbf{x}-\mathbf{p}|}{r}\right)^2 \le 2\cdot\frac{1}{4} + \left(\frac{1}{4}\right)^2 = \frac{9}{16} < 1.$$

Since this absolute value is less than one, we may apply the binomial theorem to obtain the series

$$\frac{1}{|\mathbf{x}-\mathbf{y}|^n} = \frac{1}{r^n}\left(1+\frac{-2(\mathbf{x}-\mathbf{p})\cdot(\mathbf{y}-\mathbf{p})+|\mathbf{x}-\mathbf{p}|^2}{r^2}\right)^{\frac{n}{2}} =$$

$$\sum_{m=0}^{\infty}\frac{\left(\frac{n}{2}\right)^m}{m!}\left(\frac{-2(\mathbf{x}-\mathbf{p})\cdot(\mathbf{y}-\mathbf{p})+|\mathbf{x}-\mathbf{p}|^2}{r^2}\right)^m$$

Note that each term in this sum is a polynomial in $x-p$. The powers of the various components of $x-p$ that appear in the m-th term range between m and $2m$. Moreover, let us note that we can strengthen the assertion used to show that the binomial series converged by inserting absolute value bars. If we write

$$\frac{-2(x-p)\cdot(y-p)+|x-p|^2}{r^2} = \sum_{k=0}^{n} c_k(y)(x-p)_k + \sum_{k_1,k_2=0}^{n} c_{k_1k_2}(y)(x-p)_{k_1}(x-p)_{k_2},$$

(actually, the coefficients $c_{k_1k_2}$ depend on y trivially, but the dependence on y has been indicated for the sake of uniformity) then

$$\sum_{k=0}^{n}|c_k(y)|\,|(x-p)_k| + \sum_{k_1,k_2=0}^{n}|c_{k_1k_2}(y)|\,|(x-p)_{k_1}|\,|(x-p)_{k_2}| \le \frac{9}{16}.$$

Raising this to the m-th power, we see that, if we define

$$\left(\frac{-2(x-p)\cdot(y-p)+|x-p|^2}{r^2}\right)^m = \sum_{k_1,k_2,\ldots,k_m=0}^{n} c_{k_1k_2,\cdots k_m}(y)(x-p)_{k_1}(x-p)_{k_2}\cdots(x-p)_{k_m},$$

then we have

$$\sum_{k_1,k_2,\ldots,k_m=0}^{n}|c_{k_1k_2,\cdots k_m}(y)|\,|(x-p)_{k_1}|\,|(x-p)_{k_2}|\cdots|(x-p)_{k_m}| \le \left(\frac{9}{16}\right)^m$$

Because of the fact that one may freely rearrange and regroup the terms in an absolutely convegent series, we may conclude that the expansion of $|x-y|^{-n}$ in powers of $x-p$ converges absolutely. Furthermore, there exist constants $b_{k_1 k_2, \cdots k_m}$ such that the term involving $|(x-p)_{k_1}|\,|(x-p)_{k_2}| \cdots |(x-p)_{k_m}|$ in the power series is bounded by $b_{k_1 k_2, \cdots k_m}$.