example of changing variable

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If one performs in the improper integral

$$I \;:=\; \int_{-\infty}^{\infty}\frac{e^{kx}}{1\!+\!e^x}\,dx \qquad (0 < k < 1) \tag{1}$$

the http://planetmath.org/ChangeOfVariableInDefiniteIntegralchange of variable

$$x \;=\; -\ln t, \quad dx = -\frac{dt}{t},$$

the new lower limit becomes ∞ and the new upper limit 0; hence one obtains

$$I \;=\; -\int_{\infty}^{0}\frac{e^{-k\ln t}dt}{(1\!+\!e^{-\ln t})t} \;=\; \int_0^{\infty}\frac{t^{-k}}{t\!+\!1}\,dt.$$

Thus one has recurred I to the integral

$$\int_0^{\infty}\frac{x^{-k}}{x\!+\!1}\,dx, \tag{2}$$

the value of which has been determined in the entry using residue theorem near branch point. Accordingly, we may write the result

$$\int_{-\infty}^{\infty}\frac{e^{kx}}{1\!+\!e^x}\,dx \;=\; \frac{\pi}{\sin\pi k}.$$

Calculating the integral (1) directly is quite laborious: one has to use Cauchy residue theorem to the integral

$$\oint_c \frac{e^{kz}}{1\!+\!e^z}\,dz$$

about the perimetre c of the rectangle

$$-a \;\leqq\; \mathrm{Re}\, z \;\leqq\; a, \quad 0 \;\leqq\; \mathrm{Im}\, z \;\leqq\; 2\pi$$

and then to let $a \to \infty$ (one cannot use the same half-disk as in determining the integral (2)). As for using the http://planetmath.org/MethodsOfEvaluatingImproperInteg of differentiation under the integral sign or taking Laplace transform with respect to k yields a more complicated integral.