finite limit implying uniform continuity

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Theorem. If the real function f is continuous on the interval $[0, \infty)$ and the limit $\lim_{x\to\infty} f(x)$ exists as a finite number a, then f is uniformly continuous on that interval.

Proof. Let $\varepsilon > 0$. According to the limit condition, there is a positive number M such that

$$|f(x)-a| \;<\; \frac{\varepsilon}{2} \quad \forall x > M. \tag{1}$$

The function is continuous on the finite interval $[0,\, M+1]$; hence f is also uniformly continuous on this compact interval. Consequently, there is a positive number $\delta < 1$ such that

$$|f(x_1)-f(x_2)| \;<\; \varepsilon \quad \forall\, x_1,\, x_2 \in [0,\, M+1] \text{ with } |x_1-x_2| < \delta. \tag{2}$$

Let x_1, x_2 be nonnegative numbers and $|x_1-x_2| < \delta$. Then $|x_1-x_2| < 1$ and thus both numbers either belong to $[0,\, M+1]$ or are greater than M. In the latter case, by (1) we have

$$|f(x_1)-f(x_2)| \;=\; |f(x_1)-a+a-f(x_2)| \;\leqq\; |f(x_1)-a| + |f(x_2)-a| \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon. \tag{3}$$

So, one of the conditions (2) and (3) is always in , whence the assertion is true.