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proof of Minkowski inequality

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For $p=1$ the result follows immediately from the triangle inequality, so we may assume $p>1$.

We have

$$|a_k+b_k|^p = |a_k+b_k||a_k+b_k|^{p-1} \leq (|a_k|+|b_k|)|a_k+b_k|^{p-1}$$

by the triangle inequality. Therefore we have

$$|a_k+b_k|^p \leq |a_k||a_k+b_k|^{p-1} + |b_k||a_k+b_k|^{p-1}$$

Set $q=\frac{p}{p-1}$. Then $\frac{1}{p}+\frac{1}{q}=1$, so by the Hölder inequality we have

$$\sum_{k=0}^{n} |a_k||a_k+b_k|^{p-1} \leq \left(\sum_{k=0}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=0}^{n} |a_k+b_k|^{(p-1)q}\right)^{\frac{1}{q}}$$

$$\sum_{k=0}^{n} |b_k||a_k+b_k|^{p-1} \leq \left(\sum_{k=0}^{n} |b_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=0}^{n} |a_k+b_k|^{(p-1)q}\right)^{\frac{1}{q}}$$

Adding these two inequalities, dividing by the factor common to the right sides of both, and observing that $(p-1)q=p$ by definition, we have

$$\left(\sum_{k=0}^{n} |a_k+b_k|^p\right)^{1-\frac{1}{q}} \leq \frac{\sum_{k=0}^{n}(|a_k|+|b_k|)|a_k+b_k|^{p-1}}{\left(\sum_{k=0}^{n} |a_k+b_k|^p\right)^{\frac{1}{q}}} \leq \left(\sum_{k=0}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=0}^{n} |b_k|^p\right)^{\frac{1}{p}}$$

Finally, observe that $1-\frac{1}{q}=\frac{1}{p}$, and the result follows as required. The proof for the integral version is analogous.