proof of existence of the Lebesgue measure

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Owner	gel (22282)
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First, let $\mathcal{C}$ be the collection of bounded open intervals of the real numbers. As this is a http://planetmath.org/PiSystemπ-system, http://planetmath.org/UniquenessOf of measures extended from a π-system shows that any measure defined on the σ-algebra $\sigma(\mathcal{C})$ is uniquely determined by its values restricted to $\mathcal{C}$. It remains to prove the existence of such a measure.

Define the length of an interval as $p((a,b)) = b - a$ for $a < b$. The Lebesgue outer measure $\mu^*\colon \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ is defined as

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} p(A_i) : A_i \in \mathcal{C},\ A \subseteq \bigcup_{i=1}^{\infty} A_i\right\}. \tag{1}$$

This is indeed an http://planetmath.org/OuterMeasure2outer measure (see construction of outer measures) and, furthermore, for any interval of the form (a,b) it agrees with the standard definition of length, $\mu^*((a,b)) = p((a,b)) = b-a$ (see http://planetmath.org/ProofThatTheOuterLebesgueMeasureOfAnInterva that the outer (Lebesgue) measure of an interval is its length).

We show that intervals $(-\infty,a)$ are http://planetmath.org/CaratheodorysLemmaμ^*-measurable. Choosing any $\epsilon > 0$ and interval $A \in \mathcal{C}$ the definition of p gives

$$p(A) = p(A \cap (-\infty,a)) + p(A \cap (a,\infty)).$$

So, choosing an arbitrary set $E \subseteq \mathbb{R}$ and a sequence $A_i \in \mathcal{C}$ covering E,

$$\begin{aligned}\sum_{i=1}^{\infty} p(A_i) &= \sum_{i=1}^{\infty} p(A_i \cap (-\infty,a)) + \sum_{i=1}^{\infty} p(A_i \cap (a,\infty))\\ &\geq \mu^*(E \cap (-\infty,a)) + \mu^*(E \cap (a,\infty)).\end{aligned}$$

So, from equation (**??**)

$$\mu^*(E) \geq \mu^*(E \cap (-\infty,a)) + \mu^*(E \cap (a,\infty)). \tag{2}$$

Also, choosing any $\epsilon > 0$ and using the subadditivity of μ^*,

$$\begin{aligned}\mu^*(E \cap (a,\infty)) &\geq \mu^*(E \cap (a-\epsilon,\infty)) - \mu^*(E \cap (a-\epsilon,a+\epsilon))\\ &\geq \mu^*(E \cap [a,\infty)) - \mu^*((a-\epsilon,a+\epsilon))\\ &= \mu^*(E \cap [a,\infty)) - 2\epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary, $\mu^*(E \cap (a,\infty)) \geq \mu^*(E \cap [a,\infty))$ and substituting into (**??**) shows that

$$\mu^*(E) \geq \mu^*(E \cap (-\infty,a)) + \mu^*(E \cap [a,\infty)).$$

Consequently, intervals of the form $(-\infty, a)$ are μ^*-measurable. As such intervals generate the Borel σ-algebra and, by Caratheodory's lemma, the μ^*-measurable sets form a σ-algebra on which μ^* is a measure, it follows that the restriction of μ^* to the Borel σ-algebra is itself a measure.