converse of Darboux's theorem (analysis) is not true

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Darboux' theorem says that, if $f : \mathbb{R} \to \mathbb{R}$ has an antiderivative, than f has to satisfy the *intermediate value property*, namely, for any $a < b$, for any number C with $f(a) < C < f(b)$ or $f(b) < C < f(a)$, there exists a $c \in (a, b)$ such that $f(c) = C$. With this theorem, we understand that if f does not satisfy the intermediate value property, then no function F satisfies $F' = f$ on $\mathbb{R}$.

Now, we will give an example to show that the converse is not true, i.e., a function that satisfies the intermediate value property might still have no antiderivative.

Let

$$f(x) = \begin{cases} \dfrac{1}{x}\cos(\ln x) & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases} .$$

First let us see that f satisfies the intermediate value property. Let $a < b$. If $0 < a$ or $b \leq 0$, the property is satisfied, since f is continuous on $(-\infty, 0]$ and $(0, \infty)$. If $a \leq 0 < b$, we have $f(a) = 0$ and $f(b) = (1/b)\cos(\ln b)$. Let C be between $f(a)$ and (b). Let $a_0 = \exp(-2\pi k_0 + \pi)$ for some k_0 large enough such that $a_0 < b$. Then $f(a_0) = 0 = f(a)$, and since f is continuous on (a_0, b), we must have a $c \in (a_0, b)$ with $f(c) = C$.

Assume, for a contradiction that there exists a differentiable function F such that $F'(x) = f(x)$ on $\mathbb{R}$. Then consider the function $G(x) = \sin(\ln x)$ which is defined on $(0, \infty)$. We have $G'(x) = f(x)$ on $(0, \infty)$, and since it is a an open connected set, we must have $F(x) = G(x) + c$ on $(0, \infty)$ for some $c \in \mathbb{R}$. But then, we have

$$\limsup_{x \to 0^+} F(x) = \limsup_{x \to 0^+} G(x) + c = 1 + c$$

and

$$\liminf_{x \to 0^+} F(x) = \liminf_{x \to 0^+} G(x) + c = -1 + c$$

which contradicts the differentiability of F at 0.