generalized intermediate value theorem

Canonical name	GeneralizedIntermediateValueTheorem
Date of creation	2013-03-22 17:17:44
Last modified on	2013-03-22 17:17:44
Owner	azdbacks4234 (14155)
Last modified by	azdbacks4234 (14155)
Numerical id	8
Author	azdbacks4234 (14155)
Entry type	Theorem
Classification	msc 26A06
Related topic	OrderTopology
Related topic	TotalOrder
Related topic	Continuous
Related topic	ConnectedSpace
Related topic	ConnectednessIsPreservedUnderAContinuousMap

Theorem. *Let $f : X \to Y$ be a continuous function with X a connected space and Y a totally ordered set in the order topology. If $x_1, x_2 \in X$ and $y \in Y$ lies between $f(x_1)$ and $f(x_2)$, then there exists $x \in X$ such that $f(x) = y$.*

Proof. The sets $U = f(X) \cap (-\infty, y)$ and $V = f(X) \cap (y, \infty)$ are disjoint open subsets of $f(X)$ in the subspace topology, and they are both non-empty, as $f(x_1)$ is contained in one and $f(x_2)$ is contained in the other. If $y \notin f(X)$, then $U \cup V$ constitutes a of the space $f(X)$, contradicting the hypothesis that $f(X)$ is the continuous image of the connected space X. Thus there must exist $x \in X$ such that $f(x) = y$. $\square$

This version of the intermediate value theorem reduces to the familiar one of http://planetmath.org/node/7599real analysis when X is taken to be a closed interval in $\mathbb{R}$ and Y is taken to be $\mathbb{R}$.

References

[1] J. Munkres, *Topology*, 2nd ed. Prentice Hall, 1975.