testing for continuity via closure operation

Canonical name	TestingForContinuityViaClosureOperation
Date of creation	2013-03-22 19:09:11
Last modified on	2013-03-22 19:09:11
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Last modified by	CWoo (3771)
Numerical id	8
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Entry type	Result
Classification	msc 26A15
Classification	msc 54C05

Proposition 1. *Let X, Y be topological spaces, and $f : X \to Y$ a function. Then the following are equivalent:*

1. *f is continuous,*
2. *for any closed set $D \subseteq Y$, the set $f^{-1}(D)$ is closed in X,*
3. *$f(\overline{A}) \subseteq \overline{f(A)}$, where $\overline{A}$ is the closure of A,*
4. *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$,*
5. *$f^{-1}(C^\circ) \subseteq f^{-1}(C)^\circ$, where C° is the interior of C.*

Proof.

- (1) $\Leftrightarrow$ (2). Use the identity $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$ for any function f. Then $f^{-1}(Y - D) = X - f^{-1}(D)$. So if D is closed (or open), $f^{-1}(Y - D)$ is open (or closed), whence $f^{-1}(D)$ is closed (or open).

- (2) $\Leftrightarrow$ (3). Suppose first that $f : X \to Y$ is continuous. Since

$$\overline{f(A)} = \bigcap \{C \mid C \text{ closed in } Y, \text{ and } f(A) \subseteq C\},$$

  $A \subseteq f^{-1}f(A) \subseteq f^{-1}(C)$, which is closed in X. So $\overline{A} \subseteq f^{-1}(C)$, and therefore $f(\overline{A}) \subseteq ff^{-1}(C) \subseteq C$. As a result,

$$f(\overline{A}) \subseteq \bigcap \{C \mid C \text{ closed in } Y, \text{ and } f(A) \subseteq C\} = \overline{f(A)}.$$

  Conversely, let V be closed in Y. Then $\overline{V} = V$. Let $U = f^{-1}(V)$. So $f(U) = V$. Let $W = \overline{U}$. Then $f(W) = f(\overline{U}) \subseteq \overline{f(U)} = \overline{V} = V$. So $W \subseteq f^{-1}f(W) \subseteq f^{-1}(V) = U \subseteq \overline{U} = W$. As a result, $U = W$ is closed.

- (3) $\Leftrightarrow$ (4). First, assume (2). Let $B \subseteq Y$ and $A = f^{-1}(B)$. So $f(A) \subseteq B$. Then $f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B}$. As a result, $\overline{f^{-1}(B)} = \overline{A} \subseteq f^{-1}f(\overline{A}) \subseteq f^{-1}(\overline{B})$.

  Conversely, assume (3). Let $A \subseteq X$ and $B = f(A)$. So $A \subseteq f^{-1}(B)$. Then

$$f(\overline{A}) \subseteq f(\overline{f^{-1}(B)}) \subseteq ff^{-1}(\overline{B}) \subseteq \overline{B} = \overline{f(A)}.$$

- (4) $\Leftrightarrow$ (5). First, assume (3). We use the identity: $C^\circ = Y - \overline{Y - C}$. Then

$$\begin{aligned} f^{-1}(C^\circ) &= f^{-1}(Y - \overline{Y - C}) = f^{-1}(Y) - f^{-1}(\overline{Y - C}) \subseteq X - \overline{f^{-1}(Y - C)} \\ &= X - \overline{f^{-1}(Y) - f^{-1}(C)} = X - \overline{X - f^{-1}(C)} = f^{-1}(C)^\circ. \end{aligned}$$

Conversely, assume (4). We use the identity $\overline{B} = Y - (Y - B)^\circ$. Then

$$\begin{aligned} \overline{f^{-1}(B)} &= X - (X - f^{-1}(B))^\circ = X - f^{-1}(Y - B)^\circ \\ &\subseteq X - f^{-1}((Y - B)^\circ) = f^{-1}(Y - (Y - B)^\circ) = f^{-1}(\overline{B}). \end{aligned}$$

$\square$