change of variable in definite integral

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Theorem. Let the real function $x \mapsto f(x)$ be continuous on the interval $[a,\, b]$. We introduce via the the equation

$$x \;=\; \varphi(t)$$

a new variable t satisfying

- $\varphi(\alpha) \;=\; a, \quad \varphi(\beta) \;=\; b,$
- φ and φ' are continuous on the interval with endpoints α and β.

Then

$$\int_a^b f(x)\,dx \;=\; \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt.$$

Proof. As a continuous function, f has an antiderivative F. Then the composite function $F\circ\varphi$ is an antiderivative of $(f\circ\varphi)\cdot\varphi'$, since by the chain rule we have

$$\frac{d}{dt}F(\varphi(t)) \;=\; F'(\varphi(t))\,\varphi'(t) \;=\; f(\varphi(t))\,\varphi'(t).$$

Using the http://planetmath.org/node/40459Newton–Leibniz formula we obtain

$$\int_a^b f(x)\,dx \;=\; F(b)-F(a) \;=\; F(\varphi(\beta))-F(\varphi(\alpha)) \;=\; \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt,$$

Q.E.D.