e is transcendental

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Theorem. Napier's constant e is transcendental.

This theorem was first proved by Hermite in 1873. The below proof is near the one given by Hurwitz. We at first derive a couple of auxiliary results.

Let $f(x)$ be any polynomial of μ and $F(x)$ the sum of its derivatives,

$$F(x) \; := \; f(x)+f'(x)+f''(x)+\ldots+f^{(\mu)}(x). \tag{1}$$

consider the product $\Phi(x) := e^{-x}F(x)$. The derivative of this is simply

$$\Phi'(x) \;\equiv\; e^{-x}(F'(x)-F(x)) \;\equiv\; -e^{-x}f(x).$$

Applying the http://planetmath.org/MeanValueTheoremmean value theorem to the function Φ on the interval with end points 0 and x gives

$$\Phi(x)-\Phi(0) \;=\; e^{-x}F(x)-F(0) \;=\; \Phi'(\xi)x \;=\; -e^{-\xi}f(\xi)x,$$

which implies that $F(0) = e^{-x}F(x)+e^{-\xi}f(\xi)x$. Thus we obtain the

Lemma 1. $F(0)e^x = F(x)+xe^{x-\xi}f(\xi)$ (ξ is between 0 and x)

When the polynomial $f(x)$ is expanded by the powers of $x-a$, one gets

$$f(x) \;\equiv\; f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2!}+\ldots+f^{(\mu)}(a)\frac{(x-a)^\mu}{\mu!};$$

comparing this with (1) one gets the

Lemma 2. The value $F(a)$ is obtained so that the polynomial $f(x)$ is expanded by the powers of $x-a$ and in this the powers $x-a$, $(x-a)^2$, $\ldots$, $(x-a)^\mu$ are replaced respectively by the numbers $1!$, $2!, \ldots, \mu!$.

Now we begin the proof of the theorem. We have to show that there cannot be any equation

$$c_0+c_1e+c_2e^2+\ldots+c_ne^n \;=\; 0 \tag{2}$$

with integer coefficients c_i and at least one of them distinct from zero. The proof is indirect. Let's assume the contrary. We can presume that $c_0 \neq 0$.

For any positive integer ν, lemma 1 gives

$$F(0)e^\nu \;=\; F(\nu)+\nu e^{\nu-\xi_\nu}f(\xi_\nu) \quad (0<\xi_\nu<\nu). \tag{3}$$

By virtue of this, one may write (2), multiplied by $F(0)$, as

$$c_0F(0)+c_1F(1)+c_2F(2)+\ldots+c_nF(n) \;=\; -[c_1e^{1-\xi_1}f(\xi_1)+2c_2e^{2-\xi_2}f(\xi_2)+\ldots+nc_ne^{n-\xi_n}f(\xi_n)]. \tag{4}$$

We shall show that the polynomial $f(x)$ can be chosen such that the left has absolute value less than 1.

We choose

$$f(x) \;:=\; \frac{x^{p-1}}{(p-1)!}[(x-1)(x-2)\cdots(x-n)]^p, \tag{5}$$

where p is a positive prime number on which we later shall set certain conditions. We must determine the corresponding values $F(0)$, $F(1)$, ..., $F(n)$.

For determining $F(0)$ we need, according to lemma 2, to expand $f(x)$ by the powers of x, getting

$$f(x) \;=\; \frac{1}{(p-1)!}[(-1)^{np}n!^px^{p-1}+A_1x^p+A_2x^p+1+\ldots]$$

where A_1, A_2, ... are integers, and to replace the powers x^{p-1}, x^p, x^{p+1}, ... with the numbers $(p-1)!$, $p!$, $(p+1)!$, ... We then get the expression

$$F(0) \;=\; \frac{1}{(p-1)!}[(-1)^{np}n!^p(p-1)!+A_1p!+A_2(p+1)!+\ldots] \;=\; (-1)^{np}n!^p+pK_0,$$

in which K_0 is an integer.

We now set for the prime p the condition $p > n$. Then, $n!$ is not divisible by p, neither is the former addend $(-1)^{np}n!^p$. On the other hand, the latter addend pK_0 is divisible by p. Therefore:

(α) $F(0)$ is a non-zero integer not divisible by p.

For determining $F(1)$, $F(2)$, ..., $F(n)$ we expand the polynomial $f(x)$ by the powers of $x-\nu$, putting $x := \nu+(x-\nu)$. Because $f(x)$ the factor $(x-\nu)^p$, we obtain an of the form

$$f(x) \;=\; \frac{1}{(p-1)!}[B_p(x-\nu)^p+B_{p+1}(x-\nu)^{p+1}+\ldots],$$

where the B_i's are integers. Using the lemma 2 then gives the result

$$F(\nu) \;=\; \frac{1}{(p-1)!}[p!B_p+(p+1)!B_{p+1}+\ldots] \;=\; pK_\nu,$$

with K_ν a certain integer. Thus:

(β) $F(1)$, $F(2)$, $\ldots$, $F(n)$ are integers all divisible by p.

So, the left hand of (4) is an integer having the form $c_0F(0)+pK$ with K an integer. The factor $F(0)$ of the first addend is by (α) indivisible by p. If we set for the prime p a new requirement $p > |c_0|$, then also the factor c_0 is indivisible by p, and thus likewise the whole addend $c_0F(0)$. We conclude that the sum is not divisible by p and therefore:

(γ) If p in (5) is a prime number greater than n and $|c_0|$, then the left of (4) is a nonzero integer.

We then examine the right hand of (4). Because the numbers ξ_1, ..., ξ_n all are positive (cf. (3)), so the $e^{1-\xi_1}$, ..., $e^{n-\xi_n}$ all are $< e^n$. If $0 < x < n$, then in the polynomial (5) the factors x, $x-1$, $\ldots$, $x-n$ all have the absolute value less than n and thus

$$|f(x)| \;<\; \frac{1}{(p-1)!}n^{p-1}(n^n)^p = n^n\cdot\frac{(n^{n+1})^{p-1}}{(p-1)!}.$$

Because ξ_1, $\ldots$, ξ_n all are between 0 and n (cf. (3)), we especially have

$$|f(\xi_\nu)| \;<\; n^n\cdot\frac{(n^{n+1})^{p-1}}{(p-1)!} \quad \forall\,\nu = 1,\,2,\,\ldots,\,n.$$

If we denote by c the greatest of the numbers $|c_0|$, $|c_1|$, $\ldots$, $|c_n|$, then the right hand of (4) has the absolute value less than

$$(1{+}2{+}\ldots{+}n)ce^nn^n\cdot\frac{(n^{n+1})^{p-1}}{(p-1)!} \;=\; \frac{n(n{+}1)}{2}c(en)^n\cdot\frac{(n^{n+1})^{p-1}}{(p-1)!}.$$

But the limit of $\frac{(n^{n+1})^{p-1}}{(p-1)!}$ is 0 as $p\to\infty$, and therefore the above expression is less than 1 as soon as p exeeds some number p_0.

If we determine the polynomial $f(x)$ from the equation (5) such that the prime p is greater than the greatest of the numbers n, $|c_0|$ and p_0 (which is possible since there are http://planetmath.org/ProofThatThereAreInfinitelyManyPrimesinfin many prime numbers), then the having the absolute value < 1. The contradiction proves that the theorem is right.

References

[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset I.* WSOY, Helsinki (1950).