trigonometric identity involving product of sines of roots of unity

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Let $n > 1$ be a positive integer, and $\zeta_n = e^{2i\pi/n}$, a primitive n^{th} root of unity.

The purpose of this article is to prove

Theorem 1. *Let* $m = \left\lfloor \frac{n}{2} \right\rfloor$. *Then*

$$\prod_{k=1}^{m} \sin^2\left(\frac{\pi k}{n}\right) = \prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = \frac{n}{2^{n-1}} \tag{1}$$

The theorem follows easily from the following simple lemma:

Lemma 2. *Let* $n > 1$ *be a positive integer. Then*

$$\prod_{k=1}^{n-1}(1 - \zeta_n^k) = n$$

Proof. We have $x^n - 1 = \prod_{k=1}^{n}(x - \zeta_n^k)$. Dividing both sides by $x - 1$ gives

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \ldots x^{n-1} = \prod_{k=1}^{n-1}(x - \zeta_n^k)$$

Substitute $x = 1$ to get the result. $\square$

Proof of Theorem **??**. Using the definition of ζ_n and the half-angle formulas, we have

$$\begin{aligned}
1 - \zeta_n^k &= 1 - \cos\left(\frac{2\pi k}{n}\right) - i\sin\left(\frac{2\pi k}{n}\right) \\
&= 2\sin^2\left(\frac{\pi k}{n}\right) - 2i\sin\left(\frac{\pi k}{n}\right)\cos\left(\frac{\pi k}{n}\right) \\
&= 2\sin\left(\frac{\pi k}{n}\right)\left(\sin\left(\frac{\pi k}{n}\right) - i\cos\left(\frac{\pi k}{n}\right)\right)
\end{aligned}$$

Note that $|\sin\theta - i\cos\theta| = \sin^2\theta + \cos^2\theta = 1$, so taking absolute values, we get

$$\left|1 - \zeta_n^k\right| = 2\left|\sin\left(\frac{\pi k}{n}\right)\right|$$

Now, for $1 \leq k \leq n-1$, $\sin\left(\frac{\pi k}{n}\right) > 0$ so is equal to its absolute value. Thus (using, for n even, the fact that $\sin\frac{\pi}{2} = 1$),

$$\begin{aligned}
\prod_{k=1}^{m} \sin^2\left(\frac{\pi k}{n}\right) &= \prod_{k=1}^{m} \sin\left(\frac{\pi k}{n}\right) \sin\left(\pi - \frac{\pi k}{n}\right) = \prod_{k=1}^{m} \sin\left(\frac{\pi k}{n}\right) \sin\left(\frac{\pi(n-k)}{n}\right) \\
&= \prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = \prod_{k=1}^{n-1} \left|\sin\left(\frac{\pi k}{n}\right)\right| \\
&= \frac{1}{2^{n-1}} \left|\prod_{k=1}^{n-1}(1 - \zeta_n^k)\right| = \frac{1}{2^{n-1}} \left|n\right| \\
&= \frac{n}{2^{n-1}}
\end{aligned}$$

$\square$

(Thanks to dh2718 for greatly simplifying the original proof.)