proof of $\frac{f(t)-f(s)}{t-s}\leq\frac{f(u)-f(s)}{u-s}\leq\frac{f(u)-f(t)}{u-t}$ for convex f

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We will prove

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s}. \tag{1}$$

The proof of the right-most inequality is similar.

Suppose (**??**) does not hold. Then for some s, t, u,

$$\frac{f(t)-f(s)}{t-s} > \frac{f(u)-f(s)}{u-s}. \tag{2}$$

This inequality is just the statement of the slope of the line segment $\overline{AB}, A = (t, f(t)), B = (s, f(s))$, being larger than the slope of the segment $\overline{CB}, C = (u, f(u))$. Since t is between s and u, and f is continuous, this implies

$$f(t) > h(x) = \frac{f(u)-f(s)}{u-s}(x-s) + f(s), \tag{3}$$

$s < x < u$. This contradicts convexity of f on (a, b). Hence, (**??**) is false and (**??**) follows.

Note that we have tacitly use the fact that $x = \lambda u + (1-\lambda)s$ and $h(x) = \lambda f(u) + (1-\lambda)f(s)$ for some λ.