computation of moment of spherical shell

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In using the formula for area integration over a sphere derived in the http://planetmath.org/node/6668last example, we need to keep in mind that to every point in xy plane, there correspond two points on the sphere, which are obtained by taking the two signs of the square root. The importance of this fact in obtaining a correct answer is illustrated by our next example, the calculation of the moment of inertia of a spherical shell.

The moment of a spherical shell is given by the integral

$$I = \int_S x^2 \, d^2 A.$$

While we could compute this by first converting to spherical coordinates and then using the result of http://planetmath.org/node/6664example 1, we can avoid the trouble of changing coordinates by treating the sphere as a graph. Using the result of the previous example, our integral becomes

$$\int_S x^2 \, d^2 A = 2 \int_{x^2+y^2<r^2} \frac{r x^2}{\sqrt{r^2 - x^2 - y^2}} \, dx \, dy,$$

where the factor of 2 takes into account the observation of the preceding paragraph that two points of the sphere correspond to each point of the xy plane. Computing this integral, we find

$$2 \int_{-r}^{+r} \int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} \frac{r x^2}{\sqrt{r^2 - x^2 - y^2}} \, dx \, dy =$$

$$2r \int_{-r}^{+r} \left(-\frac{1}{2} x \sqrt{r^2 - x^2 - y^2} + \frac{1}{2} (r^2 - y^2) \arcsin \frac{x}{\sqrt{r^2 - y^2}} \right) \Bigg|_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} dy =$$

$$2r \int_{-r}^{+r} \frac{\pi}{2} (r^2 - y^2) \, dy = \frac{4}{3} \pi r^4$$

Quick links:

- http://planetmath.org/node/6660main entry
- http://planetmath.org/node/6668previous example