proof of Carathéodory's extension theorem

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The first step is to extend the set function μ_0 to the power set $P(X)$. For any subset $S \subseteq X$ the value of $\mu^*(S)$ is defined by taking sequences S_i in A which cover S,

$$\mu^*(S) \equiv \inf \left\{ \sum_{i=1}^{\infty} \mu_0(S_i) : S_i \in A,\ S \subseteq \bigcup_{i=1}^{\infty} S_i \right\}. \tag{1}$$

We show that this is an `http://planetmath.org/OuterMeasure2`outer measure. First, it is clearly non-negative. Secondly, if $S = \emptyset$ then we can take $S_i = \emptyset$ in (**??**) to obtain $\mu^*(S) \leq \sum_i \mu_0(\emptyset) = 0$, giving $\mu^*(\emptyset)$=0. It is also clear that μ^* is increasing, so that if $S \subseteq T$ then $\mu^*(S) \leq \mu^*(T)$. The only remaining property to be proven is subadditivity. That is, if S_i is a sequence in $P(X)$ then

$$\mu^*\left(\bigcup_i S_i\right) \leq \sum_i \mu^*(S_i). \tag{2}$$

To prove this inequality, choose any $\epsilon > 0$ and, by the definition (**??**) of μ^*, for each i there exists a sequence $S_{i,j} \in A$ such that $S_i \subseteq \bigcup_j S_{i,j}$ and,

$$\sum_{j=1}^{\infty} \mu_0(S_{i,j}) \leq \mu^*(S_i) + 2^{-i}\epsilon.$$

As $\bigcup_i S_i \subseteq \bigcup_{i,j} S_{i,j}$, equation (**??**) defining μ^* gives

$$\mu^*\left(\bigcup_i S_i\right) \leq \sum_{i,j} \mu_0(S_{i,j}) = \sum_i \sum_j \mu_0(S_{i,j}) \leq \sum_i (\mu^*(S_i) + 2^{-i}\epsilon) = \sum_i \mu^*(S_i) + \epsilon.$$

As $\epsilon > 0$ is arbitrary, this proves subadditivity (**??**). So, μ^* is indeed an outer measure.

The next step is to show that μ^* agrees with μ_0 on A. So, choose any $S \in A$. The inequality $\mu^*(S) \leq \mu_0(S)$ follows from taking $S_1 = S$ and $S_i = \emptyset$ in (**??**), and it remains to prove the reverse inequality. So, let S_i be a sequence in A covering S, and set

$$S_i' = (S \cap S_i) \setminus \bigcup_{j=1}^{i-1} S_j \in A.$$

Then, S'_i are disjoint sets satisfying $\bigcup_{j=1}^{i} S'_j = S \cap \bigcup_{j=1}^{i} S_j$ and, therefore, $\bigcup_i S'_i = S$. By the countable additivity of μ_0,

$$\sum_i \mu_0(S_i) = \sum_i (\mu_0(S'_i) + \mu_0(S_i \setminus S'_i)) \geq \sum_i \mu_0(S'_i) = \mu_0(S).$$

As this inequality hold for any sequence $S_i \in A$ covering S, equation (**??**) gives $\mu^*(S) \geq \mu_0(S)$ and, by combining with the reverse inequality, shows that μ^* does indeed agree with μ_0 on A.

We have shown that μ_0 extends to an outer measure μ^* on the power set of X. The final step is to apply Carathéodory's lemma on the restriction of outer measures. A set $S \subseteq X$ is said to be μ^*-measurable if the inequality

$$\mu^*(E) \geq \mu^*(E \cap S) + \mu^*(E \cap S^c) \tag{3}$$

is satisfied for all subsets E of X. Carathéodory's lemma then states that the collection $\mathcal{F}$ of μ^*-measurable sets is a `http://planetmath.org/SigmaAlgebra`σ-algebra and that the restriction of μ^* to $\mathcal{F}$ is a measure. To complete the proof of the theorem it only remains to be shown that every set in A is μ^*-measurable, as it will then follow that $\mathcal{F}$ contains $\mathcal{A} = \sigma(A)$ and the restriction of μ^* to $\mathcal{A}$ is a measure.

So, choosing any $S \in A$ and $E \subseteq X$, the proof will be complete once it is shown that (**??**) is satisfied. Given any $\epsilon > 0$, equation (**??**) says that there is a sequence E_i in A such that $E \subseteq \bigcup_i E_i$ and

$$\sum_i \mu_0(E_i) \leq \mu^*(E) + \epsilon.$$

As $E \cap S \subseteq \bigcup_i (E_i \cap S)$ and $E \cap S^c \subseteq \bigcup_i (E_i \cap S^c)$,

$$\mu^*(E \cap S) + \mu^*(E \cap S^c) \leq \sum_i \mu_0(E_i \cap S) + \sum_i \mu_0(E_i \cap S^c) = \sum_i \mu_0(E_i) \leq \mu^*(E) + \epsilon.$$

Since ϵ is arbitrary, this shows that (**??**) is satisfied and S is μ^*-measurable.