L^p-space

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Entry type	Definition
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Synonym	L^p space
Synonym	essentially bounded function
Related topic	MeasureSpace
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Related topic	Measure
Related topic	FeynmannPathIntegral
Related topic	AmenableGroup
Related topic	VectorPnorm
Related topic	VectorNorm
Related topic	SobolevInequality
Related topic	L2SpacesAreHilbertSpaces
Related topic	RieszFischerTheorem
Related topic	BoundedLinearFunctionalsOnLpmu
Related topic	ConvolutionsOfComplexFunctionsOnLocallyCompactG
Defines	p-integrable function
Defines	L^∞
Defines	essentially bounded
Defines	L^p-norm

Definition Let $(X, \mathfrak{B}, \mu)$ be a measure space. Let $0 < p < \infty$. The L^p-*norm* of a function $f : X \to \mathbb{C}$ is defined as

$$||f||_p := \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \tag{1}$$

when the integral exists. The set of functions with finite L^p-norm forms a vector space V with the usual pointwise addition and scalar multiplication of functions. In particular, the set of functions with zero L^p-norm form a linear subspace of V, which for this article will be called K. We are then interested in the quotient space V/K, which consists of complex functions on X with finite L^p-norm, identified up to equivalence almost everywhere. This quotient space is the complex L^p-space on X.

Theorem If $1 \leq p < \infty$, the vector space V/K is complete with respect to the L^p norm.

The space L^∞. The space L^∞ is somewhat special, and may be defined without explicit reference to an integral. First, the L^∞-norm of f is defined to be the essential supremum of $|f|$:

$$||f||_\infty := \text{ess sup}\, |f| = \inf \{a \in \mathbb{R} : \mu(\{x : |f(x)| > a\}) = 0\} \tag{2}$$

However, if μ is the trivial measure, then essential supremum of every measurable function is defined to be 0.

The definitions of V, K, and L^∞ then proceed as above, and again we have that L^∞ is complete. Functions in L^∞ are also called *essentially bounded*.

Example Let $X = [0, 1]$ and $f(x) = \frac{1}{\sqrt{x}}$. Then $f \in L^1(X)$ but $f \notin L^2(X)$.