# proof of Clarkson inequality

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| Owner | CWoo (3771) |
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Suppose $2 \leq p < \infty$ and $f, g \in L^p$.

$$\begin{aligned} \left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p &= \int \left| \frac{f+g}{2} \right|^p \, d\mu + \int \left| \frac{f-g}{2} \right|^p \, d\mu \qquad (1) \\ &= \frac{1}{2^p} \left( \int |f+g|^p \, d\mu + \int |f-g|^p \, d\mu \right). \qquad (2) \end{aligned}$$

By the triangle inequality, we have the following two inequalities

$$|f+g|^p \leq |f|^p + |g|^p \qquad \text{and} \qquad |f-g|^p \leq |f|^p + |g|^p,$$

and summing the two inequalities we get

$$|f+g|^p + |f-g|^p \leq 2(|f|^p + |g|^p).$$

This means that expression (2) above is less than or equal to

$$\frac{1}{2^{p-1}} \int (|f|^p + |g|^p) \, d\mu. \qquad (3)$$

Hence it follows that

$$\begin{aligned} \left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p &\leq \frac{1}{2^{p-1}} \left( \int |f|^p \, d\mu + \int |g|^p \, d\mu \right) \\ &= \frac{1}{2^{p-1}} \left( \|f\|_p^p + \|g\|_p^p \right), \end{aligned}$$

which since $p \geq 2$ directly implies the desired inequality.