Hausdorff measure

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Introduction

Given a real number $\alpha \geq 0$ we are going to define a Borel external measure $\mathcal{H}^\alpha$ on $\mathbb{R}^n$ with values in $[0, +\infty]$ which will comprehend and generalize the concepts of length (for $\alpha = 1$), area ($\alpha = 2$) and volume ($\alpha = 3$) of sets in $\mathbb{R}^n$. In particular if $M \subset \mathbb{R}^n$ is an m-dimensional regular surface then one will show that $\mathcal{H}^m(M)$ is the m-dimensional area of M. However, being an external measure, $\mathcal{H}^m$ is defined not only on regular surfaces but on every subset of $\mathbb{R}^n$ thus generalizing the concepts of length, area and volume. In particular, for $m = n$, it turns out that the Hausdorff measure $\mathcal{H}^n$ is nothing else than the Lebesgue measure of $\mathbb{R}^n$.

Given any fixed set $E \subset \mathbb{R}^n$ one can consider the measures $\mathcal{H}^\alpha(E)$ with α varying in $[0, +\infty)$. We will see that for a fixed set E there exists at most one value α such that $\mathcal{H}^\alpha(E)$ is finite and positive; while for every other value β one will have $\mathcal{H}^\beta(E) = 0$ if $\beta > \alpha$ and $\mathcal{H}^\beta(E) = +\infty$ if $\beta < \alpha$. For example, if E is a regular 2-dimensional surface then only $\mathcal{H}^2(E)$ (which is the area of the surface) may possibly be finite and different from 0 while, for example, the volume of E will be 0 and the length of E will be infinite.

This can be used to define the dimension of a set E (this is called the Hausdorff dimension). A very interesting fact is the existence of sets with dimension α which is not integer, as happens for most *fractals*.

Also, the measure $\mathcal{H}^\alpha$ is naturally defined on every metric space (X, d), not only on $\mathbb{R}^n$.

Definition

Let (X, d) be a metric space. Given $E \subset X$ we define the diameter of E as

$$\operatorname{diam}(E) := \sup_{x,y \in E} d(x, y).$$

Given a real number α we consider the conventional constant

$$\omega_\alpha = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2 + 1)}$$

where $\Gamma(x)$ is the gamma function.

For all $\delta > 0$, $\alpha \geq 0$ and $E \subset X$ let us define

$$\mathcal{H}^\alpha_\delta(E) := \inf \left\{ \sum_{j=0}^\infty \omega_\alpha \left(\frac{\operatorname{diam}(B_j)}{2} \right)^\alpha : B_j \subset X,\ \bigcup_{j=0}^\infty B_j \supset E,\ \operatorname{diam}(B_j) \leq \delta\ \forall j = 0, 1, \dots \right\}. \tag{1}$$

The *infimum* is taken over all possible enumerable families of sets $B_0, B_1, \dots, B_j, \dots$ which are sufficiently small ($\operatorname{diam} B_j \leq \delta$) and which cover E.

Notice that the function $\mathcal{H}^\alpha_\delta(E)$ is decreasing in δ. In fact given $\delta' > \delta$ the family of sequences B_j considered in the definition of $\mathcal{H}^\alpha_{\delta'}$ contains the family of sequences considered in the definition of $\mathcal{H}^\alpha_\delta$ and hence the infimum is smaller. So the limit in the following definition exists:

$$\mathcal{H}^\alpha(E) := \lim_{\delta \to 0^+} \mathcal{H}^\alpha_\delta(E). \tag{2}$$

The number $\mathcal{H}^\alpha(E) \in [0, +\infty]$ is called *α-dimensional Hausdorff measure* of the set $E \subset X$.