proof of countable unions and intersections of analytic sets are analytic

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Owner	gel (22282)
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Author	gel (22282)
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Let $(X,\mathcal{F})$ be a paved space and $(A_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}$-http://planetmath.org/Analytic sets. We show that their union and intersection is also analytic.

From the definition of $\mathcal{F}$-analytic sets, there exist compact paved spaces $(K_n,\mathcal{K}_n)$ and $S_n\in(\mathcal{F}\times\mathcal{K}_n)_{\sigma\delta}$ such that

$$A_n=\{x\in X\colon (x,y)\in S_n \text{ for some } y\in K_n\}\,.$$

We start by showing that $\bigcap_n A_n$ is analytic. Let $K=\prod_n K_n$ and $\mathcal{K}=\prod_n\mathcal{K}_n$ be the product paving, and $\pi_n\colon K\to K_n$ be the projection map. Then $x\in\bigcap_n A_n$ if and only if for each n there is a $y_n\in K_n$ with $(x,y_n)\in S_n$. Equivalently, setting $y=(y_1,y_2,\cdots)$, then $(x,y)\in\bigcap_n\pi_n^{-1}(S_n)$. However, this is in $(\mathcal{F}\times\mathcal{K}_n)_{\sigma\delta}$ and we can write,

$$\bigcap_n A_n=\pi_X\left(\bigcap_n \pi_n^{-1}(S_n)\right),$$

where $\pi_X\colon X\times K\to X$ is the projection map. As products of compact pavings are compact, $(K,\mathcal{K})$ is compact and it follows from the definition that $\bigcap_n A_n$ is $\mathcal{F}$-analytic.

We now show that $\bigcup_n A_n$ is analytic. Let $K=\sum_n K_n$ and $\mathcal{K}=\sum_n\mathcal{K}_n$ be the direct sum paving, http://planetmath.org/SumsOfCompactPavingsAreCompactwhich is compact. Also, write $S_n=\bigcap_{m=1}^\infty T_{m,n}$ for $T_{m,n}\in(\mathcal{F}\times\mathcal{K}_n)_\sigma$. We identify K_n with a subset of K, so that K is the union of the disjoint sets K_n. Then $x\in\bigcup_n A_n$ if and only if $(x,y)\in S_n$ for some n and some $y\in K$,

$$\bigcup_n A_n=\pi_X\left(\bigcup_n S_n\right).$$

However, the fact that K_{n_1},K_{n_2} are disjoint for $n_1\neq n_2$ says that T_{m,n_1},T_{m,n_2} are disjoint and, therefore,

$$\bigcup_n S_n=\bigcup_n\bigcap_m T_{m,n}=\bigcap_m\bigcup_n T_{m,n}\in(\mathcal{F}\times\mathcal{K})_{\sigma\delta}\,.$$

So $\bigcup_n A_n$ is $\mathcal{F}$-analytic.