proof of Egorov's theorem

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Let $E_{i,j} = \{x \in E : |f_j(x) - f(x)| < 1/i\}$. Since $f_n \to f$ almost everywhere, there is a set S with $\mu(S) = 0$ such that, given $i \in \mathbb{N}$ and $x \in E - S$, there is $m \in \mathbb{N}$ such that $j > m$ implies $|f_j(x) - f(x)| < 1/i$. This can be expressed by

$$E - S \subset \bigcup_{m\in\mathbb{N}} \bigcap_{j>m} E_{i,j},$$

or, in other words,

$$\bigcap_{m\in\mathbb{N}} \bigcup_{j>m} (E - E_{i,j}) \subset S.$$

Since $\{\bigcup_{j>m}(E - E_{i,j})\}_{m\in\mathbb{N}}$ is a decreasing nested sequence of sets, each of which has finite measure, and such that its intersection has measure 0, by `http://planetmath.org/PropertiesForMeasure`continuity from above we know that

$$\mu(\bigcup_{j>m} (E - E_{i,j})) \xrightarrow[m\to\infty]{} 0.$$

Therefore, for each $i \in \mathbb{N}$, we can choose m_i such that

$$\mu(\bigcup_{j>m_i} (E - E_{i,j})) < \frac{\delta}{2^i}.$$

Let

$$E_\delta = \bigcup_{i\in\mathbb{N}} \bigcup_{j>m_i} (E - E_{i,j}).$$

Then

$$\mu(E_\delta) \leq \sum_{i=1}^{\infty} \mu(\bigcup_{j>m_i} (E - E_{i,j})) < \sum_{i=1}^{\infty} \frac{\delta}{2^i} = \delta.$$

We claim that $f_n \to f$ uniformly on $E - E_\delta$. In fact, given $\varepsilon > 0$, choose n such that $1/n < \varepsilon$. If $x \in E - E_\delta$, we have

$$x \in \bigcap_{i\in\mathbb{N}} \bigcap_{j>m_i} E_{i,j},$$

which in particular implies that, if $j > m_n$, $x \in E_{n,j}$; that is, $|f_j(x) - f(x)| < 1/n < \varepsilon$. Hence, for each $\varepsilon > 0$ there is N (which is given by m_n above) such that $j > N$ implies $|f_j(x) - f(x)| < \varepsilon$ for each $x \in E - E_\delta$, as required. This the proof.