infinite product measure

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Let $(E_i, \mathcal{B}_i, \mu_i)$ be measure spaces, where $i \in I$ an index set, possibly infinite. We define the *product* of $(E_i, \mathcal{B}_i, \mu_i)$ as follows:

1. let $E = \prod E_i$, the Cartesian product of E_i,
2. let $\mathcal{B} = \sigma((\mathcal{B}_i)_{i \in I})$, the smallest sigma algebra containing subsets of E of the form $\prod B_i$ where $B_i = E_i$ for all but a finite number of $i \in I$.

Then $(E, \mathcal{B})$ is a measurable space. The next task is to define a measure μ on $(E, \mathcal{B})$ so that $(E, \mathcal{B}, \mu)$ becomes in addition a measure space. Before proceeding to define μ, we make the assumption that

each μ_i is a *totally finite measure*, that is, $\mu_i(E_i) < \infty$.

In fact, we can now turn each $(E_i, \mathcal{B}_i, \mu_i)$ into a probability space by introducing for each $i \in I$ a new measure:

$$\overline{\mu}_i = \frac{\mu_i}{\mu_i(E_i)}.$$

With the assumption that each $(E_i, \mathcal{B}_i, \mu_i)$ is a probability space, it can be shown that there is a *unique* measure μ defined on $\mathcal{B}$ such that, for any $B \in \mathcal{B}$ expressible as a product of $B_i \in \mathcal{B}_i$ with $B_i = E_i$ for all $i \in I$ except on a finite subset J of I:

$$\mu(B) = \prod_{j \in J} \mu_j(B_j).$$

Then $(E, \mathcal{B}, \mu)$ becomes a measure space, and in particular, a probability space. μ is sometimes written $\prod \mu_i$.

Remarks.

- If I is infinite, one sees that the total finiteness of μ_i can not be dropped. For example, if I is the set of positive integers, assume $\mu_1(E_1) < \infty$ and $\mu_2(E_2) = \infty$. Then $\mu(B)$ for

$$B := B_1 \times \prod_{i>1} E_i = B_1 \times E_2 \times \prod_{i>2} E_i, \text{ where } B_1 \in \mathcal{B}_1$$

  would not be well-defined (on the one hand, it is $\mu_1(B_1) < \infty$, but on the other it is $\mu_1(B_1)\mu_2(E_2) = \infty$).

- The above construction agrees with the result when I is finite (see `http://planetmath.org/ProductMeasure`finite product measure).