proof of capacity generated by a measure

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Owner	gel (22282)
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For a finite measure space $(X, \mathcal{F}, \mu)$, define

$$\mu^*: \mathcal{P}(X) \to \mathbb{R}_+,$$
$$\mu^*(S) = \inf \{\mu(A) : A \in \mathcal{F},\ A \supseteq S\}.$$

We show that μ^* is an $\mathcal{F}$-capacity and that a subset $S \subseteq X$ is $(\mathcal{F}, \mu^*)$-http://planetmath.org/ChoquetCapacitycapacitable if and only if it is in the http://planetmath.org/CompleteMeasurecompletion of $\mathcal{F}$ with respect to μ.

Note, first of all, that $\mu^*(S) = \mu(S)$ for any $S \in \mathcal{F}$. That μ^* is increasing follows directly from the definition. If $A_n \in \mathcal{F}$ is a decreasing sequence of sets then $A = \bigcap_n A_n$ is also in $\mathcal{F}$ and, by http://planetmath.org/PropertiesForMeasurecontinuity from above for measures,

$$\mu^*(A_n) = \mu(A_n) \to \mu(A) = \mu^*(A)$$

as $n \to \infty$.

Now suppose that S_n is an increasing sequence of subsets of X and set $S = \bigcup_n S_n$. Then, $\mu^*(S_n) \leq \mu^*(S)$ for each n and, hence, $\lim_{n\to\infty} \mu^*(S_n) \leq \mu^*(S)$.

To prove the reverse inequality, choose any $\epsilon > 0$ and sequence $A_n \in \mathcal{F}$ with $S_n \subseteq A_n$ and $\mu(A_n) \leq \mu^*(S_n) + 2^{-n}\epsilon$. Then, $A_m \cap A_n \supseteq S_n$ whenever $m \geq n$ and, therefore,

$$\mu(A_n \setminus A_m) = \mu(A_n) - \mu(A_m \cap A_n) \leq \mu(A_n) - \mu^*(S_n) \leq 2^{-n}\epsilon.$$

Additivity of μ then gives

$$\mu\left(\bigcup_{m\leq n} A_m\right) \leq \mu(A_n) + \sum_{m=1}^{n-1} \mu(A_m \setminus A_n) \leq \mu^*(S_n) + \epsilon.$$

So, by continuity from below for measures,

$$\mu^*(S) \leq \mu\left(\bigcup_n A_n\right) = \lim_{n\to\infty} \mu\left(\bigcup_{m\leq n} A_m\right) \leq \lim_{n\to\infty} \mu^*(S_n) + \epsilon.$$

Choosing ϵ arbitrarily small shows that $\mu^*(S_n) \to \mu^*(S)$ and, therefore, μ^* is indeed an $\mathcal{F}$-capacity.

Now suppose that S is in the completion of $\mathcal{F}$ with respect to μ, so that there exists $A, B \in \mathcal{F}$ with $A \subseteq S \subseteq B$ and $\mu(B \setminus A) = 0$. Then,

$$\mu^*(A) = \mu(A) = \mu(B) \geq \mu^*(S)$$

and S is indeed $(\mathcal{F}, \mu^*)$-capacitable. Conversely, let S be $(\mathcal{F}, \mu^*)$-capacitable. Then, there exists $A_n, B_n \in \mathcal{F}$ such that $A_n \subseteq S \subseteq B_n$ and

$$\mu(A_n) \geq \mu^*(S) - 1/n,\ \ \mu(B_n) \leq \mu^*(S) + 1/n.$$

Setting $A = \bigcup_n A_n$ and $B = \bigcap_n B_n$ gives $A \subseteq S \subseteq B$ and

$$\mu(A) \geq \mu^*(S) \geq \mu(B).$$

So $\mu(B \setminus A) = \mu(B) - \mu(A) = 0$, as required.