area formula

Canonical name	AreaFormula
Date of creation	2013-03-22 14:27:36
Last modified on	2013-03-22 14:27:36
Owner	paolini (1187)
Last modified by	paolini (1187)
Numerical id	12
Author	paolini (1187)
Entry type	Theorem
Classification	msc 28A78
Related topic	ChangeOfVariablesInIntegralOnMathbbRn

Let $\mathcal{H}^m$ denote the Hausdorff measure. Let $m \leq n$ and consider a Lipschitz function $f\colon \mathbb{R}^m \to \mathbb{R}^n$. If $A \subset \mathbb{R}^m$ is a Lebesgue measurable set, the equality

$$\int_A J_f(x)\,dx = \int_{\mathbb{R}^n} \mathcal{H}^0(f^{-1}(\{y\}) \cap A)\,d\mathcal{H}^m y$$

holds, where

$$J_f(x) = \sqrt{\det(Df(x) \cdot Df(x)^*)}$$

is the Jacobian determinant of f in the point x and represent the m-volume of the image of the unit cube under the linear map $Df(x)$.

If $u \in L^1(\mathbb{R}^m)$ then one has

$$\int_{\mathbb{R}^m} u(x) J_f(x)\,dx = \int_{\mathbb{R}^n} \sum_{x \in f^{-1}(\{y\})} u(x)\,d\mathcal{H}^m y.$$

Notice that this formula is a generalization of the change of variables in integrals on $\mathbb{R}^n$.