sums of compact pavings are compact

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Suppose that $(K_i, \mathcal{K}_i)$ is a paved space for each i in an index set I. The direct sum, or http://planetmath.org/DisjointUniondisjoint union, $\sum_{i\in I} K_i$ is the union of the disjoint sets $K_i \times \{i\}$. The direct sum of the paving $\mathcal{K}_i$ is defined as

$$\sum_{i\in I} \mathcal{K}_i = \left\{ \sum_{i\in I} S_i \colon S_i \in \mathcal{K}_i \cup \{\emptyset\} \text{ is empty for all but finitely many } i \right\}.$$

Theorem. *Let $(K_i, \mathcal{K}_i)$ be compact paved spaces for $i \in I$. Then, $\sum_i \mathcal{K}_i$ is a compact paving on $\sum_i K_i$.*

The paving $\mathcal{K}'$ consisting of subsets of $\sum_i \mathcal{K}_i$ of the form $\sum_i S_i$ where $S_i = \emptyset$ for all but a single $i \in I$ is easily shown to be compact. Indeed, if $\mathcal{K}'' \subseteq \mathcal{K}'$ satisfies the finite intersection property then there is an $i \in I$ such that $S \subseteq K_i \times \{i\}$ for every $S \in \mathcal{K}''$. Compactness of $\mathcal{K}_i$ gives $\bigcap \mathcal{K}'' \neq \emptyset$.

Then, as $\sum_i \mathcal{K}_i$ consists of finite unions of sets in $\mathcal{K}'$, it is a compact paving (see compact pavings are closed subsets of a compact space).