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proof of functional monotone class theorem

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We start by proving the following version of the monotone class theorem.

Theorem 1. *Let $(X, \mathcal{A})$ be a measurable space and $\mathcal{S}$ be a* `http://planetmath.org/PiSystem`*π-system generating the* `http://planetmath.org/SigmaAlgebra`*σ-algebra $\mathcal{A}$. Suppose that $\mathcal{H}$ be a vector space of real-valued functions on X containing the constant functions and satisfying the following,*

- *if $f\colon X \to \mathbb{R}_+$ is bounded and there is a sequence of nonnegative functions $f_n \in \mathcal{H}$ increasing pointwise to f, then $f \in \mathcal{H}$.*
- *for every set $A \in \mathcal{S}$ the characteristic function 1_A is in $\mathcal{H}$.*

Then, $\mathcal{H}$ contains every bounded and measurable function from X to $\mathbb{R}$.

Let $\mathcal{D}$ consist of the collection of subsets B of X such that the characteristic function 1_B is in $\mathcal{H}$. Then, by the conditions of the theorem, the constant function 1_X is in V so that $X \in \mathcal{D}$, and $\mathcal{S} \subseteq \mathcal{D}$. For any $A \subseteq B$ in $\mathcal{D}$ then $1_{B\setminus A} = 1_B - 1_A \in \mathcal{H}$, as $\mathcal{H}$ is closed under linear combinations, and therefore $B \setminus A$ is in $\mathcal{D}$. If $A_n \in \mathcal{D}$ is an increasing sequence, then $1_{A_n} \in \mathcal{H}$ increases pointwise to $1_{\bigcup_n A_n}$, which is therefore in $\mathcal{H}$, and $\bigcup_n A_n \in \mathcal{D}$. It follows that $\mathcal{D}$ is a Dynkin system, and Dynkin's lemma shows that it contains the σ-algebra $\mathcal{A}$.

We have shown that $1_A \in \mathcal{H}$ for every $A \in \mathcal{A}$. Now consider any bounded and measurable function $f\colon X \to \mathbb{R}$ taking values in a finite set $S \subseteq \mathbb{R}$. Then,

$$f = \sum_{s \in S} s 1_{f^{-1}(\{s\})}$$

is in $\mathcal{H}$.

We denote the floor function by $\lfloor \cdot \rfloor$. That is, $\lfloor a \rfloor$ is defined to be the largest integer less than or equal to the real number a. Then, for any nonnegative bounded and measurable $f\colon X \to \mathbb{R}$, the sequence of functions $f_n(x) = 2^{-n}\lfloor 2^n f(x) \rfloor$ each take values in a finite set, so are in $\mathcal{H}$, and increase pointwise to f. So, $f \in \mathcal{H}$.

Finally, as every measurable and bounded function $f\colon X \to \mathbb{R}$ can be written as the difference of its positive and negative parts $f = f_+ - f_-$, then $f \in \mathcal{H}$.

We now extend this result to prove the following more general form of the theorem.

Theorem 2. *Let X be a set and $\mathcal{K}$ be a collection of bounded and real valued functions on X which is closed under multiplication, so that $fg \in \mathcal{K}$ for all $f, g \in \mathcal{K}$. Let $\mathcal{A}$ be the σ-algebra on X generated by $\mathcal{K}$.*

Suppose that $\mathcal{H}$ is a vector space of bounded real valued functions on X containing $\mathcal{K}$ and the constant functions, and satisfying the following

- *if $f\colon X \to \mathbb{R}$ is bounded and there is a sequence of nonnegative functions $f_n \in \mathcal{H}$ increasing pointwise to f, then $f \in \mathcal{H}$.*

Then, $\mathcal{H}$ contains every bounded and real valued $\mathcal{A}$-measurable function on X.

Let us start by showing that $\mathcal{H}$ is closed under uniform convergence. That is, if f_n is a sequence in $\mathcal{H}$ and $\|f_n - f\| \equiv \sup_x |f_n(x) - f(x)|$ converges to zero, then $f \in \mathcal{H}$. By passing to a subsequence if necessary, we may assume that $\|f_n - f_m\| \leq 2^{-n}$ for all $m \geq n$. Define $g_n \equiv f_n - 2^{1-n} + 2 + \|f\|$. Then $g_n \in \mathcal{H}$ since $\mathcal{H}$ is a vector space containing the constant functions. Also, g_n are nonnegative functions increasing pointwise to $f + 2 + \|f\|$ which must therefore be in $\mathcal{H}$, showing that $f \in \mathcal{H}$ as required.

Now let $\mathcal{H}_0$ consist of linear combinations of constant functions and functions in $\mathcal{K}$ and $\bar{\mathcal{H}}_0$ be its `http://planetmath.org/Closure`closure under uniform convergence. Then $\bar{\mathcal{H}}_0 \subseteq \mathcal{H}$ since we have just shown that $\mathcal{H}$ is closed under uniform convergence. As $\mathcal{K}$ is already closed under products, $\mathcal{H}_0$ and $\bar{\mathcal{H}}_0$ will also be closed under products, so are `http://planetmath.org/Algebra`algebras. In particular, $p(f) \in \bar{\mathcal{H}}_0$ for every $f \in \bar{\mathcal{H}}_0$ and polynomial $p \in \mathbb{R}[X]$. Then, for any continuous function $p\colon \mathbb{R} \to \mathbb{R}$, the Weierstrass approximation theorem says that there is a sequence of polynomials p_n converging uniformly to p on bounded intervals, so $p_n(f) \to p(f)$ uniformly. It follows that $p(f) \in \bar{\mathcal{H}}_0$. In particular, the minimum of any two functions $f, g \in \bar{\mathcal{H}}_0$, $f \wedge g = f - |f - g|$ and the maximum $f \vee g = f + |g - f|$ will be in $\bar{\mathcal{H}}_0$.

We let $\mathcal{S}$ consist of the sets $A \subseteq X$ such that there is a sequence of nonnegative $f_n \in \bar{\mathcal{H}}_0$ increasing pointwise to 1_A. Once it is shown that this is a π-system generating the σ-algebra $\mathcal{A}$, then the result will follow from theorem **??**.

If $f_n, g_n \in \bar{\mathcal{H}}_0$ are nonnegative functions increasing pointwise to $1_A, 1_B$ then $f_n g_n$ increases pointwise to $1_{A\cap B}$, so $A \cap B \in \mathcal{S}$ and $\mathcal{S}$ is a π-system.

Finally, choose any $f \in \mathcal{K}$ and $a \in \mathbb{R}$. Then, $f_n = ((n(f - a)) \vee 0) \wedge 1$ is a sequence of functions in $\bar{\mathcal{H}}_0$ increasing pointwise to $1_{f^{-1}((a,\infty))}$. So, $f^{-1}((a, \infty)) \in \mathcal{S}$. As intervals of the form (a, ∞) generate the Borel σ-algebra on $\mathbb{R}$, it follows that $\mathcal{S}$ generates the σ-algebra $\mathcal{A}$, as required.