proof of Fubini's theorem for the Lebesgue integral

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Let μ_x and μ_y be measures on X and Y respectively, let μ be the product measure $\mu_x \otimes \mu_y$, and let $f(x,y)$ be μ-integrable on $A \subset X \times Y$. Then

$$\int_A f(x,y)d\mu = \int_X \left(\int_{A_x} f(x,y)d\mu_y \right) d\mu_x = \int_Y \left(\int_{A_y} f(x,y)d\mu_x \right) d\mu_y$$

where

$$A_x = \{y \mid (x,y) \in A\}, A_y = \{x \mid (x,y) \in A\}$$

Proof: Assume for now that $f(x,y) \geq 0$. Consider the set

$$U = X \times Y \times \mathbb{R}$$

equipped with the measure

$$\mu_u = \mu_x \otimes \mu_y \otimes \mu^1 = \mu \otimes \mu^1 = \mu_x \otimes \lambda$$

where μ^1 is ordinary Lebesgue measure and $\lambda = \mu_y \otimes \mu^1$. Also consider the set $W \subset U$ defined by

$$W = \{(x,y,z) \mid (x,y) \in A, 0 \leq z \leq f(x,y)\}$$

Then

$$\mu_u(W) = \int_A f(x,y)d\mu$$

And

$$\mu_u(W) = \int_X \lambda(W_x)\, d\mu_x$$

where

$$W_x = \{(y,z) \mid (x,y,z) \in W\}$$

However, we also have that

$$\lambda(W_x) = \int_{A_x} f(x,y)d\mu_y$$

Combining the last three equations gives us Fubini's theorem. To remove the restriction that $f(x,y)$ be nonnegative, write f as

$$f(x,y) = f^+(x,y) - f^-(x,y)$$

where

$$f^+(x,y) = \frac{|f(x,y)| + f(x,y)}{2}, f^-(x,y) = \frac{|f(x,y)| - f(x,y)}{2}$$

are both nonnegative.