measure-preserving

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Defines	invertible measure-preserving transformation
Defines	endomorphism of a measure space

1 Definition

Definition - Let $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$ be measure spaces, and $T : X_1 \to X_2$ be a measurable transformation. The transformation T is said to be *measure-preserving* if for all $A \in \mathfrak{B}_2$ we have that

$$\mu_1(T^{-1}(A)) = \mu_2(A),$$

where $T^{-1}(A)$ is, as usual, the set of points $x \in X_1$ such that $T(x) \in A$.

Additional Notation:

- If T is bijective, measure-preserving, and its inverse T^{-1} is also measure-preserving, then T is said to be an measure-preserving transformation.

- Measure-preserving transformations between the same measure space are sometimes called of the measure space.

Remarks:

- The fact that a map $T : X_1 \longrightarrow X_2$ is measure-preserving depends heavily on the sigma-algebras $\mathfrak{B}_i$ and measures μ_i involved. If other measures or sigma-algebras are also in consideration, one should make clear to which measure space is $T : X_1 \longrightarrow X_2$ measure-preserving.

- Measure-preserving maps are the morphisms on the category whose objects are measure spaces. This should be clear from the next results and examples.

2 Properties

- The composition of measure-preserving maps is again measure-preserving. Of course, we are supposing that the domains and codomains of the maps are such that the composition is possible.

- Let $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$ be measure spaces and $(X_1, \overline{\mathfrak{B}_1}, \overline{\mu_1})$ and $(X_2, \overline{\mathfrak{B}_2}, \overline{\mu_2})$ their completions. If $T : (X_1, \mathfrak{B}_1, \mu_1) \longrightarrow (X_2, \mathfrak{B}_2, \mu_2)$ is measure-preserving, then so is $T : (X_1, \overline{\mathfrak{B}_1}, \overline{\mu_1}) \longrightarrow (X_2, \overline{\mathfrak{B}_2}, \overline{\mu_2})$.

- Let $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$ be measure spaces and $T_1 : X_1 \longrightarrow X_1$, $T_2 : X_2 \longrightarrow X_2$ be measure-preserving maps. Then, the product map $T_1 \times T_2 : X_1 \times X_2 \longrightarrow X_1 \times X_2$, defined by

$$T_1 \times T_2\ (x_1, x_2) := (T_1(x_1), T_2(x_2))$$

is a measure-preserving transformation of $(T_1 \times T_2, \mathfrak{B}_1 \times \mathfrak{B}_1, \mu_1 \times \mu_2)$.

3 Examples

- The identity map of a measure space $(X, \mathfrak{B}, \mu)$ is always measure-preserving.

- Let G be a locally compact http://planetmath.org/TopologicalGroupgroup. For each $a \in G$, the transformation $T(g) := ag$ is measure-preserving relatively to any left Haar measure. Similarly, any right translation on G any right Haar measure.

- Every continuous surjective homomorphism between compact Hausdorff is measure-preserving relatively to the normalized Haar measure (see http://planetmath.org/ContinuousEpimorphismOfCompactGroupsPreservesHaarMe entry).