proof of Radon-Nikodym theorem

Canonical name	ProofOfRadonNikodymTheorem
Date of creation	2013-03-22 18:58:03
Last modified on	2013-03-22 18:58:03
Owner	Ziosilvio (18733)
Last modified by	Ziosilvio (18733)
Numerical id	5
Author	Ziosilvio (18733)
Entry type	Proof
Classification	msc 28A15
Synonym	Hilbert spaces proof of Radon-Nikodym's theorem
Synonym	measure- theoretic proof of Radon-Nikodym theorem

The following proof of Radon-Nikodym theorem is based on the original argument by John von Neumann. We suppose that μ and ν are real, nonnegative, and finite. The extension to the σ-finite case is a standard exercise, as is μ-a.e. uniqueness of Radon-Nikodym derivative. Having done this, the thesis also holds for signed and complex-valued measures.

Let $(X, \mathcal{F})$ be a measurable space and let $\mu, \nu : \mathcal{F} \to [0, R]$ two finite measures on X such that $\nu(A) = 0$ for every $A \in \mathcal{F}$ such that $\mu(A) = 0$. Then $\sigma = \mu + \nu$ is a finite measure on X such that $\sigma(A) = 0$ if and only if $\mu(A) = 0$.

Consider the linear functional $T : L^2(X, \mathcal{F}, \sigma) \to \mathbb{R}$ defined by

$$Tu = \int_X u \, d\mu \;\; \forall u \in L^2(X, \mathcal{F}, \sigma) \, . \tag{1}$$

T is well-defined because μ is finite and dominated by σ, so that $L^2(X, \mathcal{F}, \sigma) \subseteq L^2(X, \mathcal{F}, \mu) \subseteq L^1(X, \mathcal{F}, \mu)$; it is also linear and bounded because $|Tu| \leq \|u\|_{L^2(X,\mathcal{F},\sigma)} \cdot \sqrt{\sigma(X)}$. By Riesz representation theorem, there exists $g \in L^2(X, \mathcal{F}, \sigma)$ such that

$$Tu = \int_X u \, d\mu = \int_X u \cdot g \, d\sigma \tag{2}$$

for every $u \in L^2(X, \mathcal{F}, \sigma)$. Then $\mu(A) = \int_A g \, d\sigma$ for every $A \in \mathcal{F}$, so that $0 < g \leq 1$ μ- and σ-a.e. (Consider the former with $A = \{x \mid g(x) \leq 0\}$ or $A = \{x \mid g(x) > 1\}$.) Moreover, the second equality in (**??**) holds when $u = \chi_A$ for $A \in \mathcal{F}$, thus also when u is a simple measurable function by linearity of integral, and finally when u is a (μ- and σ-a.e.) nonnegative $\mathcal{F}$-measurable function because of the monotone convergence theorem.

Now, $1/g$ is $\mathcal{F}$-measurable and nonnegative μ- and σ-a.e.; moreover, $\dfrac{1}{g} \cdot g = 1$ σ- and μ-a.e. Thus, for every $A \in \mathcal{F}$,

$$\int_A \frac{1}{g} \, d\mu = \int_A d\sigma = \sigma(A) \tag{3}$$

Since σ is finite, $1/g \in L^1(X, \mathcal{F}, \mu)$, and so is $f = \dfrac{1}{g} - 1$. Then for every $A \in \mathcal{F}$

$$\nu(A) = \sigma(A) - \mu(A) = \int_A \left(\frac{1}{g} - 1 \right) d\mu = \int_A f \, d\mu \, .$$