σ-algebra

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Introduction

When defining a measure for a set E we usually cannot hope to make every subset of E measurable. Instead we must usually restrict our attention to a specific collection of subsets of E, requiring that this collection be closed under operations that we would expect to preserve measurability. A σ-algebra is such a collection.

Definition

Given a set E, a *σ-algebra* in E is a collection $\mathcal{F}$ of subsets of E such that:

- $\varnothing \in \mathcal{F}$.
- Any union of countably many elements of $\mathcal{F}$ is an element of $\mathcal{F}$.
- The complement of any element of $\mathcal{F}$ in E is an element of $\mathcal{F}$.

Notes

It follows from the definition that any σ-algebra $\mathcal{F}$ in E also satisfies the properties:

- $E \in \mathcal{F}$.
- Any intersection of countably many elements of $\mathcal{F}$ is an element of $\mathcal{F}$.

Note that a σ-algebra is a field of sets that is closed under countable unions and countable intersections (rather than just finite unions and finite intersections).

Given any collection C of subsets of E, the σ-algebra $\sigma(C)$ *generated by* C is defined to be the smallest σ-algebra in E such that $C \subseteq \sigma(C)$. This is well-defined, as the intersection of any non-empty collection of σ-algebras in E is also a σ-algebra in E.

Examples

For any set E, the power set $\mathcal{P}(E)$ is a σ-algebra in E, as is the set $\{\varnothing, E\}$.

A more interesting example is the `http://planetmath.org/BorelSigmaAlgebra`Borel σ-algebra in $\mathbb{R}$, which is the σ-algebra generated by the open subsets of $\mathbb{R}$, or, equivalently, the σ-algebra generated by the compact subsets of $\mathbb{R}$.