uniqueness of measures extended from a π-system

Canonical name	UniquenessOfMeasuresExtendedFromApisystem
Date of creation	2013-03-22 18:33:08
Last modified on	2013-03-22 18:33:08
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	8
Author	gel (22282)
Entry type	Theorem
Classification	msc 28A12
Related topic	LebesgueMeasure
Related topic	DynkinsLemma

The following theorem allows measures to be uniquely defined by specifying their values on a `http://planetmath.org/PiSystem`π-system instead of having to specify the measure of every possible measurable set. For example, the collection of open intervals $(a,b)\subseteq\mathbb{R}$ forms a π-system generating the `http://planetmath.org/BorelSigmaAlgebra`Borel σ-algebra and consequently the Lebesgue measure μ is uniquely defined by the equality $\mu((a,b))=b-a$.

Theorem. *Let λ, μ be measures on a measurable space $(X,\mathcal{A})$. Suppose that A is a π-system on X generating $\mathcal{A}$ such that $\lambda=\mu$ on A and that there exists a sequence $S_n\in A$ with $\bigcup_{n=1}^{\infty}S_n=X$ and $\lambda(S_n)<\infty$. Then, $\lambda=\mu$.*

Proof. Choose any $T\in A$ such that $\lambda(T)<\infty$ and set $\mathcal{B}=\{S\in\mathcal{A}:\lambda(S\cap T)=\mu(S\cap T)\}$. For any $S\in A$, $S\cap T\in A$ and the requirement that λ,μ agree on A gives $S\in\mathcal{B}$, so $\mathcal{B}$ contains A. We show that $\mathcal{B}$ is a Dynkin system in order to apply Dynkin's lemma. It is clear that $X\in\mathcal{B}$. Suppose that $S_1\subseteq S_2$ are in $\mathcal{B}$. Then, the additivity of λ and μ gives

$$\lambda\left((S_2\setminus S_1)\cap T\right)=\lambda(S_2\cap T)-\lambda(S_1\cap T)=\mu(S_2\cap T)-\mu(S_1\cap T)=\mu\left((S_2\setminus S_1)\cap T\right)$$

and therefore $S_2\setminus S_1\in\mathcal{B}$. Now suppose that S_n is an increasing sequence of sets in $\mathcal{B}$ increasing to $S\subseteq X$. Then, monotone convergence of λ and μ gives

$$\lambda(S\cap T)=\lim_{n\to\infty}\lambda(S_n\cap T)=\lim_{n\to\infty}\mu(S_n\cap T)=\lambda(S\cap T),$$

so $S\in\mathcal{B}$ and $\mathcal{B}$ is a Dynkin system containing A. By Dynkin's lemma this shows that $\mathcal{B}$ contains $\sigma(A)=\mathcal{A}$.

We have shown that $\lambda(S\cap T)=\mu(S\cap T)$ for any $S\in\mathcal{A}$ and $T\in A$ with $\lambda(T)<\infty$. In the particular case where $X\in A$ and λ,μ are finite measures then it follows that $\lambda(S)=\mu(S)$ simply by taking $T=X$. More generally, choose a sequence of sets $T_n\in A$ satisfying $\lambda(T_n)<\infty$ and $\bigcup_n T_n=X$. For any $S\in\mathcal{A}$, $S_n\equiv(S\cap T_n)\setminus\bigcup_{m=1}^{n-1}T_m$ is a pairwise disjoint sequence of sets in $\mathcal{A}$ with $S_n\subseteq T_n$ and $\bigcup_n S_n=S$. So, $\lambda(S_n)=\mu(S_n)$ and the countable additivity of λ and μ gives

$$\lambda(S)=\sum_n\lambda(S_n)=\sum_n\mu(S_n)=\mu(S).$$

$\square$