proof of existence and unicity of self-similar fractals

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We consider the space $\mathcal{K}(X) = \{K \subset X \colon K \text{ compact and non empty}\}$ endowed with the Hausdorff distance δ. Since Hausdorff metric inherits completeness, being X complete, $(\mathcal{K}(X), \delta)$ is complete too. We then consider the mapping $T \colon \mathcal{K}(X) \to \mathcal{K}(X)$ defined by

$$T(A) = \bigcup_{i=1}^{N} T_i(A).$$

We claim that T is a contraction. In fact, recalling that $\delta(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{\delta(A_1, B_1), \delta(A_2, B_2)\}$ while $\delta(T_i(A), T_i(B)) \leq \lambda_i \delta(A, B)$ if T_i is λ_i-Lipschitz, we have

$$\begin{aligned}\delta(T(A), T(B)) = \delta(\bigcup_i T_i(A), \bigcup_i T_i(B)) &\leq \max_i \delta(T_i(A), T_i(B)) \\ &\leq \max_i \lambda_i \delta(A, B) = \lambda \delta(A, B)\end{aligned}$$

with $\lambda = \max_i \lambda_i < 1$.

So T is a contraction on the complete metric space $\mathcal{K}(X)$ and hence, by Banach Fixed Point Theorem, there exists one and only one $K \in \mathcal{K}(X)$ such that $T(K) = K$.