analytic sets define a closure operator

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For a paving $\mathcal{F}$ on a set X, we denote the collection of all $\mathcal{F}$-http://planetmath.org/AnalyticS sets by $a(\mathcal{F})$. Then, $\mathcal{F}\mapsto a(\mathcal{F})$ is a closure operator on the subsets of X. That is,

1. $\mathcal{F}\subseteq a(\mathcal{F})$.
2. If $\mathcal{F}\subseteq\mathcal{G}$ then $a(\mathcal{F})\subseteq a(\mathcal{G})$.
3. $a(a(\mathcal{F}))=a(\mathcal{F})$.

For example, if $\mathcal{G}$ is a collection of $\mathcal{F}$-analytic sets then $\mathcal{G}\subseteq a(\mathcal{F})$ gives $a(\mathcal{G})\subseteq a(a(\mathcal{F}))=a(\mathcal{F})$ and so all $\mathcal{G}$-analytic sets are also $\mathcal{F}$-analytic. In particular, for a metric space, the analytic sets are the same regardless of whether they are defined with respect to the collection of open, closed or Borel sets.

Properties **??** and **??** follow directly from the definition of analytic sets. We just need to prove **??**. So, for any $A\in a(a(\mathcal{F}))$ we show that $A\in a(\mathcal{F})$. First, there is a http://planetmath.org/PavedSpacecompact paved space $(K,\mathcal{K})$ and $S\in(a(\mathcal{F})\times\mathcal{K})_{\sigma\delta}$ such that A is equal to the projection $\pi_X(S)$. Write

$$S=\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}A_{m,n}\times B_{m,n}$$

for $A_{m,n}\in a(\mathcal{F})$ and $B_{m,n}\in\mathcal{K}$. It is clear that $A_{m,n}\times B_{m,n}$ is $\mathcal{F}\times\mathcal{K}$-analytic and, as countable unions and intersections of analytic sets are analytic, S is also $\mathcal{F}\times\mathcal{K}$-analytic. Finally, since projections of analytic sets are analytic, $A=\pi_X(S)$ must be $\mathcal{F}$-analytic as required.