counter-example to Tonelli's theorem

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The following observation demonstrates the necessity of the σ-finite assumption in Tonelli's and Fubini's theorem. Let X denote the closed unit interval $[0, 1]$ equipped with Lebesgue measure and Y the same set, but this time equipped with counting measure ν. Let

$$f(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y) = 0,$$

while

$$\int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) = 1.$$

The iterated integrals do not give the same value, this despite the fact that the integrand is a non-negative function.

Also observe that there does not exist a simple function on $X \times Y$ that is dominated by f. Hence,

$$\int_{X \times Y} f(x, y) d(\mu(x) \times \nu(y) = 0.$$

Therefore, the integrand is L^1 integrable relative to the product measure. However, as we observed above, the iterated integrals do not agree. This observation illustrates the need for the σ-finite assumption for Fubini's theorem.