bounded linear functionals on $L^p(\mu)$

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If μ is a positive measure on a set X, $1 \leq p \leq \infty$, and $g \in L^q(\mu)$, where q is the Hölder conjugate of p, then Hölder's inequality implies that the map $f \mapsto \int_X fgd\mu$ is a bounded linear functional on $L^p(\mu)$. It is therefore natural to ask whether or not all such functionals on $L^p(\mu)$ are of this form for some $g \in L^q(\mu)$. Under fairly mild hypotheses, and excepting the case $p = \infty$, the Radon-Nikodym Theorem answers this question affirmatively.

Theorem. *Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space, $1 \leq p < \infty$, and q the Hölder conjugate of p. If Φ is a bounded linear functional on $L^p(\mu)$, then there exists a unique $g \in L^q(\mu)$ such that*

$$\Phi(f) = \int_X fgd\mu \tag{1}$$

for all $f \in L^p(\mu)$. Furthermore, $\|\Phi\| = \|g\|_q$. Thus, under the stated hypotheses, $L^q(\mu)$ is isometrically isomorphic to the dual space of $L^p(\mu)$.

If $1 < p < \infty$, then the assertion of the theorem remains valid without the assumption that μ is σ-finite; however, even with this hypothesis, the result can fail in the case that $p = \infty$. In particular, the bounded linear functionals on $L^\infty(m)$, where m is Lebesgue measure on $[0, 1]$, are not all obtained in the above manner via members of $L^1(m)$. An explicit example illustrating this is constructed as follows: the assignment $f \mapsto f(0)$ defines a bounded linear functional on $C([0, 1])$, which, by the Hahn-Banach Theorem, may be extended to a bounded linear functional Φ on $L^\infty(m)$. Assume for the sake of contradiction that there exists $g \in L^1(m)$ such that $\Phi(f) = \int_{[0,1]} fgdm$ for every $f \in L^\infty(m)$, and for $n \in \mathbb{Z}^+$, define $f_n : [0, 1] \to \mathbb{C}$ by $f_n(x) = \max\{1 - nx, 0\}$. As each f_n is continuous, we have $\Phi(f_n) = \varphi(f_n) = 1$ for all n; however, because $f_n \to 0$ almost everywhere and $|f_n| \leq 1$, the Dominated Convergence Theorem, together with our hypothesis on g, gives

$$1 = \lim_{n\to\infty} \Phi(f_n) = \lim_{n\to\infty} \int_{[0,1]} f_n gdm = 0,$$

a contradiction. It follows that no such g can exist.