finite fields of sets

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If S is a finite set then any field of subsets of S (see "field of sets" in the entry on rings of sets) can be described as the set of unions of subsets of a partition of S.

Note that, if P is a partition of S and $A, B \subset P$, we have

$$\begin{aligned}
\overline{\bigcup A} &= \bigcup (P \setminus A) \\
(\bigcup A) \cup (\bigcup B) &= \bigcup (A \cup B) \\
(\bigcup A) \cap (\bigcup B) &= \bigcup (A \cap B)
\end{aligned}$$

so $\{\bigcup X \mid X \subset P\}$ is a field of sets.

Now assume that $\mathcal{F}$ is a field of subsets of a finite set S. Let us define the set of "prime elements" of $\mathcal{F}$ as follows:

$$P = \{X \in (\mathcal{F} \setminus \varnothing) \mid (Y \subset X) \wedge (Y \in \mathcal{F}) \Rightarrow (Y = \varnothing \vee Y = X)\}$$

The choice of terminology "prime element" is meant to be a suggestive mnemonic of how the only divisors of a prime number are 1 and the number itself.

We claim that P is a partition. To justify this claim, we need to show that elements of P are pairwise disjoint and that $\bigcup P = S$.

Suppose that A and B are prime elements. Since, by definition, $A \in \mathcal{F}$ and $B \in \mathcal{F}$ and $\mathcal{F}$ is a field of sets, $A \cap B \in \mathcal{F}$. Since $A \cap B \subset A$, we must either have $A \cap B = \varnothing$ or $A \cap B = A$. In the former case, A and B are disjoint, whilst in the latter case $A = B$.

Suppose that x is any element of S. Then we claim that the set X defined as

$$X = \bigcup \{Y \in \mathcal{F} \mid x \in Y\}$$

is a prime element of $\mathcal{F}$. To begin, note that, since $\mathcal{F}$ is finite, a forteriori any subset of $\mathcal{F}$ is finite and, since fields of sets are assumed to be closed under intersection, it follows that the intersection of a susbet of $\mathcal{F}$ is an element of $\mathcal{F}$, in particular $X \in \mathcal{F}$.

Suppose that $Z \subset X$ and $Z \in \mathcal{F}$. If $x \notin Z$, then $x \in \overline{Z}$. Since $\mathcal{F}$ is a field of sets, $\overline{Z} \in \mathcal{F}$. Hence, by the construction of X, it is the case that $X \subset \overline{Z}$, hence $X \cap Z = \varnothing$. Together with $Z \subset X$, this implies $Z = \varnothing$. If $x \in Z$, then, by construction, $X \subset Z$, which implies $X = Z$.

Thus, we see that X is a prime set. Since x was arbitrarily chosen, this means that every element of S is contained in a prime element of $\mathcal{F}$, so the union of all prime elements is S itself. Together with the previously shown fact that prime elements are pairwise disjoint, this shows that the prime elements for a partition of S.

Let A be an arbitrary element of $\mathcal{F}$. Since $P \subset \mathcal{F}$, it is the case that $(\forall X \in P) A \cap X \in \mathcal{F}$. Since P is a partition of S,

$$A = \bigcup \{A \cap X \mid X \in P\}$$

so every element of $\mathcal{F}$ can be expressed as a union of elements of P.