proof of Vitali convergence theorem

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Owner	stevecheng (10074)
Last modified by	stevecheng (10074)
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Author	stevecheng (10074)
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Theorem. *Let $f_1, f_2, \ldots$ be $\mathbf{L}^p$-integrable functions on a measure space (X, μ), for $1 \leq p < \infty$. The following conditions are necessary and sufficient for f_n to be a Cauchy sequence in the $\mathbf{L}^p(X, \mu)$ norm:*

(i) the sequence f_n is Cauchy in measure;

(ii) the functions $\{|f_n|^p\}$ are uniformly integrable; and

(iii) for each $\epsilon > 0$, there is a set A of finite measure, with $\|f_n \mathbf{1}(X \setminus A)\| < \epsilon$ for all n.

Proof. We abbreviate $|f_n - f_m|$ by f_{mn}.

Necessity of (i). Fix $t > 0$, and let $E_{mn} = \{f_{mn} \geq t\}$. Then

$$\mu(E_{mn})^{1/p} = \frac{1}{t}\|t\,\mathbf{1}(E_{mn})\| \leq \frac{1}{t}\|f_{mn}\| \to 0\,, \quad \text{as } m, n \to \infty.$$

Necessity of (ii). Select N such that $\|f_n - f_N\| < \epsilon$ when $n \geq N$. The family $\{|f_1|^p, \ldots, |f_{N-1}|^p, |f_N|^p\}$ is uniformly integrable because it consists of only *finitely* many integrable functions.

So for every $\epsilon > 0$, there is $\delta > 0$ such that $\mu(E) < \delta$ implies $\|f_n \mathbf{1}(E)\| < \epsilon$ for $n \leq N$. On the other hand, for $n > N$,

$$\|f_n \mathbf{1}(E)\| \leq \|(f_n - f_N)\mathbf{1}(E)\| + \|f_N \mathbf{1}(E)\| < 2\epsilon$$

for the same sets E, and thus the entire infinite sequence $\{|f_n|^p\}$ is uniformly integrable too.

Necessity of (iii). Select N such that $\|f_n - f_N\| < \epsilon$ for all $n \geq N$. Let φ be a simple function approximating f_N in $\mathbf{L}^p$ norm up to ϵ. Then $\|f_n - \varphi\| < 2\epsilon$ for all $n \geq N$. Let $A_N = \{\varphi \neq 0\}$ be the support of φ, which must have finite measure. It follows that

$$\begin{aligned}
\|f_n \mathbf{1}(X \setminus A_N)\| = \|f_n - f_n \mathbf{1}(A_N)\| &\leq \|f_n - \varphi\| + \|\varphi - f_n \mathbf{1}(A_N)\| \\
&= \|f_n - \varphi\| + \|(\varphi - f_n)\mathbf{1}(A_N)\| \\
&< 2\epsilon + 2\epsilon\,.
\end{aligned}$$

For each $n < N$, we can similarly construct sets A_n of finite measure, such that $\|f_n \mathbf{1}(X \setminus A_n)\| < 4\epsilon$. If we set $A = A_1 \cup \cdots \cup A_{N-1} \cup A_N$, a finite union, then A has finite measure, and clearly $\|f_n \mathbf{1}(X \setminus A)\| < 4\epsilon$ for any n.

Sufficiency. We show f_{mn} to be small for large m, n by a multi-step estimate:

$$\|f_{mn}\| \leq \|f_{mn}\mathbf{1}(A \setminus E_{mn})\| + \|f_{mn}\mathbf{1}(E_{mn})\| + \|f_{mn}\mathbf{1}(X \setminus A)\| .$$

Use condition (iii) to choose A of finite measure such that $\|f_n\mathbf{1}(X \setminus A)\| < \epsilon$ for every n. Then $\|f_{mn}\mathbf{1}(X \setminus A)\| < 2\epsilon$.

Let $t = \epsilon/\mu(A)^{1/p} > 0$, and $E_{mn} = \{f_{mn} \geq t\}$. By condition (ii) choose $\delta > 0$ so that $\|f_n\mathbf{1}(E)\| < \epsilon$ whenever $\mu(E) < \delta$. By condition (i), take N such that if $m, n \geq N$, then $\mu(E_{mn}) < \delta$; it follows immediately that $\|f_{mn}\mathbf{1}(E_{mn})\| < 2\epsilon$.

Finally, $\|f_{mn}\mathbf{1}(A \setminus E_{mn})\| \leq t\mu(A)^{1/p} = \epsilon$, since $f_{mn} < t$ on the complement of E_{mn}. Hence $\|f_{mn}\| < 5\epsilon$ for $m, n \geq N$. □

Remark. In the statement of the theorem, instead of dealing with Cauchy sequences, we can directly speak of convergence of f_n to f in $\mathbf{L}^p$ and in measure. This variation of the theorem is easily proved, for:

- a sequence converges in $\mathbf{L}^p$ if and only if it is Cauchy in $\mathbf{L}^p$;
- a sequence that converges in measure is automatically Cauchy in measure;
- a simple adaptation of the argument shows that $f_n \to f$ in $\mathbf{L}^p$ implies $f_n \to f$ in measure; and
- the limit in measure is unique.