measurable function

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Let $\big(X, \mathcal{B}(X)\big)$ and $\big(Y, \mathcal{B}(Y)\big)$ be two measurable spaces. Then a function $f\colon X \to Y$ is called a *measurable function* if:

$$f^{-1}\big(\mathcal{B}(Y)\big) \subseteq \mathcal{B}(X)$$

where $f^{-1}\big(\mathcal{B}(Y)\big) = \{f^{-1}(E) \mid E \in \mathcal{B}(Y)\}$.

In other words, the inverse image of every $\mathcal{B}(Y)$-measurable set is $\mathcal{B}(X)$-measurable. The space of all measurable functions $f\colon X \to Y$ is denoted as

$$\mathcal{M}\Big(\big(X, \mathcal{B}(X)\big), \big(Y, \mathcal{B}(Y)\big)\Big).$$

Any measurable function into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel sigma algebra of the real numbers $\mathbb{R}$, is called a *Borel measurable function*.[1] The space of all Borel measurable functions from a measurable space $(X, \mathcal{B}(X))$ is denoted by $\mathcal{L}^0\big(X, \mathcal{B}(X)\big)$.

Similarly, we write $\bar{\mathcal{L}}^0\big(X, \mathcal{B}(X)\big)$ for $\mathcal{M}\Big(\big(X, \mathcal{B}(X)), (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}})\big)\Big)$, where $\mathcal{B}(\bar{\mathbb{R}})$ is the Borel sigma algebra of $\bar{\mathbb{R}}$, the set of extended real numbers.

Remark. If $f : X \to Y$ and $g : Y \to Z$ are measurable functions, then so is $g \circ f : X \to Z$, for if E is $\mathcal{B}(Z)$-measurable, then $g^{-1}(E)$ is $\mathcal{B}(Y)$-measurable, and $f^{-1}\big(g^{-1}(E)\big)$ is $\mathcal{B}(X)$-measurable. But $f^{-1}\big(g^{-1}(E)\big) = (g \circ f)^{-1}(E)$, which implies that $g \circ f$ is a measurable function.

Example:

- Let E be a subset of a measurable space X. Then the characteristic function χ_E is a measurable function if and only if E is measurable.

[1] More generally, a measurable function is called *Borel measurable* if the range space Y is a topological space with $\mathcal{B}(Y)$ the sigma algebra generated by all open sets of Y.