planetmath.org

Math for the people, by the people.

proof of Vitali's Theorem

Canonical name	ProofOfVitalisTheorem
Date of creation	2013-03-22 13:45:50
Last modified on	2013-03-22 13:45:50
Owner	paolini (1187)
Last modified by	paolini (1187)
Numerical id	7
Author	paolini (1187)
Entry type	Proof
Classification	msc 28A05
Related topic	ProofOfPsuedoparadoxInMeasureTheory

Consider the equivalence relation in $[0,1)$ given by

$$x \sim y \quad \Leftrightarrow \quad x - y \in \mathbb{Q}$$

and let $\mathcal{F}$ be the family of all equivalence classes of $\sim$. Let V be a of $\mathcal{F}$ i.e. put in V an element for each equivalence class of $\sim$ (notice that we are using the axiom of choice).

Given $q \in \mathbb{Q} \cap [0,1)$ define

$$V_q = ((V+q) \cap [0,1)) \cup ((V+q-1) \cap [0,1))$$

that is V_q is obtained translating V by a quantity q to the right and then cutting the piece which goes beyond the point 1 and putting it on the left, starting from 0.

Now notice that given $x \in [0,1)$ there exists $y \in V$ such that $x \sim y$ (because V is a section of $\sim$) and hence there exists $q \in \mathbb{Q} \cap [0,1)$ such that $x \in V_q$. So

$$\bigcup_{q \in \mathbb{Q} \cap [0,1)} V_q = [0,1).$$

Moreover all the V_q are disjoint. In fact if $x \in V_q \cap V_p$ then $x - q$ (modulus $[0,1)$) and $x - p$ are both in V which is not possible since they differ by a rational quantity $q - p$ (or $q - p + 1$).

Now if V is Lebesgue measurable, clearly also V_q are measurable and $\mu(V_q) = \mu(V)$. Moreover by the countable additivity of μ we have

$$\mu([0,1)) = \sum_{q \in \mathbb{Q} \cap [0,1)} \mu(V_q) = \sum_q \mu(V).$$

So if $\mu(V) = 0$ we had $\mu([0,1)) = 0$ and if $\mu(V) > 0$ we had $\mu([0,1)) = +\infty$.

So the only possibility is that V is not Lebesgue measurable.