construction of outer measures

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The following theorem is used in measure theory to construct http://planetmath.org/OuterMe measures on a set X, starting with a non-negative function on a collection of subsets of X. For example, if we take X to be the real numbers, $\mathcal{C}$ to be the collection of bounded open intervals of $\mathbb{R}$ and define p by $p((a,b)) = b - a$ for real numbers $a < b$, then the Lebesgue outer measure is obtained.

Theorem. *Let X be a set, $\mathcal{C}$ be a family of subsets of X containing the empty set and $p\colon \mathcal{C} \to \mathbb{R} \cup \{\infty\}$ be a function satisfying $p(\emptyset) = 0$. Then the function $\mu^*\colon \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ defined by*

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} p(A_i) : A_i \in \mathcal{C},\ A \subseteq \bigcup_{i=1}^{\infty} A_i\right\} \tag{1}$$

is an outer measure.

Proof. The definition of μ^* immediately gives $\mu^*(A) \le \mu^*(B)$ for sets $A \subseteq B$, and if $A = \emptyset$ then we can take $A_i = \emptyset$ in (**??**) to obtain $\mu^*(\emptyset) \le \sum_i p(\emptyset) = 0$, giving $\mu^*(\emptyset) = 0$. Only the countable subadditivity of μ^* remains to be shown. That is, if A_i is a sequence in $\mathcal{P}(X)$ then

$$\mu^*\left(\bigcup_i A_i\right) \le \sum_i \mu^*(A_i). \tag{2}$$

To prove this inequality, we may restrict to the case where $\mu^*(A_i) < \infty$ for each i so that, choosing any $\epsilon > 0$, equation (**??**) says that there exists a sequence $A_{i,j} \in \mathcal{C}$ such that $A_i \subseteq \bigcup_j A_{i,j}$ and,

$$\sum_{j=1}^{\infty} p(A_{i,j}) \le \mu^*(A_i) + 2^{-i}\epsilon.$$

As $\bigcup_i A_i \subseteq \bigcup_{i,j} A_{i,j}$, equation (**??**) defining μ^* gives

$$\mu^*\left(\bigcup_i A_i\right) \le \sum_{i,j} p(A_{i,j}) = \sum_i \sum_j p(A_{i,j}) \le \sum_i (\mu^*(A_i) + 2^{-i}\epsilon) = \sum_i \mu^*(A_i) + \epsilon.$$

As $\epsilon > 0$ is arbitrary, this proves subadditivity (**??**). $\square$

Although this result is rather general, placing few restrictions on the function p, there is no guarantee that the outer measure μ^* will agree with p for the sets in $\mathcal{C}$ nor that $\mathcal{C}$ will consist of `http://planetmath.org/CaratheodorysLemma`μ^*-measurable sets.

For example, if $X = \mathbb{R}$, $\mathcal{C}$ consists of the bounded open intervals, and $p((a,b)) = (b-a)^2$ for real numbers $a < b$, then $\mu^*((a,b)) = 0 \neq p((a,b))$.

Alternatively if $p((a,b)) = \sqrt{b-a}$ for all $a < b$ then it follows that $\mu^*((a,b)) = \sqrt{b-a}$ so

$$\mu^*((0,1)) + \mu^*([1,2)) = 1 + 1 \neq \mu^*((0,2)) = \sqrt{2},$$

and $(0,1)$ is not μ^*-measurable.