completion of a measure space

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If the measure space $(X, \mathscr{S}, \mu)$ is not complete, then it can be completed in the following way. Let

$$\mathscr{Z} = \bigcup_{E \in \mathscr{S}, \mu(E)=0} \mathscr{P}(E),$$

i.e. the family of all subsets of sets whose μ-measure is zero. Define

$$\overline{\mathscr{S}} = \{A \cup B : A \in \mathscr{S},\, B \in \mathscr{Z}\}.$$

We assert that $\overline{\mathscr{S}}$ is a σ-algebra. In fact, it clearly contains the emptyset, and it is closed under countable unions because both $\mathscr{S}$ and $\mathscr{Z}$ are. We thus need to show that it is closed under complements. Let $A \in \mathscr{S}$, $B \in \mathscr{Z}$ and suppose $E \in \mathscr{S}$ is such that $B \subset E$ and $\mu(E) = 0$. Then we have

$$(A \cup B)^c = A^c \cap B^c = A^c \cap (E-(E-B))^c = A^c \cap (E^c \cup (E-B)) = (A^c \cap E^c) \cup (A^c \cap (E-B)),$$

where $A^c \cap E^c \in \mathscr{S}$ and $A^c \cap (E-B) \in \mathscr{Z}$. Hence $(A \cup B)^c \in \overline{\mathscr{S}}$.

Now we define $\overline{\mu}$ on $\overline{\mathscr{S}}$ by $\overline{\mu}(A \cup B) = \mu(A)$, whenever $A \in \mathscr{S}$ and $B \in \mathscr{Z}$. It is easily verified that this defines in fact a measure, and that $(X, \overline{\mathscr{S}}, \overline{\mu})$ is the completion of $(X, \mathscr{S}, \mu)$.