a Lebesgue measurable but non-Borel set

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We give an example of a subset of the real numbers which is Lebesgue measurable, but not Borel measurable.

Let S consist of the set of all irrational real numbers with continued fraction of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

such that there exists an infinite sequence $0 < i_1 < i_2 < \cdots$ where each a_{i_k} divides $a_{i_{k+1}}$. It can be shown that this set is Lebesgue measurable, but not Borel measurable.

In fact, it can be shown that S is an `http://planetmath.org/AnalyticSet2`analytic set, meaning that it is the image of a continuous function $f\colon X \to \mathbb{R}$ for some Polish space X and, consequently, S is a universally measurable set.

This example is due to Lusin (1927).