continuous almost everywhere versus equal to a continuous function almost everywhere

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The concept of almost everywhere can be somewhat tricky to people who are not familiar with it. Let m denote Lebesgue measure. Consider the following two statements about a function $f\colon \mathbb{R} \to \mathbb{R}$:

- f is `http://planetmath.org/Continuous`continuous almost everywhere with respect to m
- f is equal to a continuous function almost everywhere with respect to m

Although these two statements seem alike, they have quite different meanings. In fact, neither one of these statements implies the other.

Consider the function $\chi_{[0,\infty)}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$

This function is not continuous at 0, but it is continuous at all other $x \in \mathbb{R}$. Note that $m(\{0\}) = 0$. Thus, $\chi_{[0,\infty)}$ is continuous almost everywhere.

Suppose $\chi_{[0,\infty)}$ is equal to a continuous function almost everywhere. Let $A \subset \mathbb{R}$ be `http://planetmath.org/LebesgueMeasure`Lebesgue measurable with $m(A) = 0$ and $g\colon \mathbb{R} \to \mathbb{R}$ such that $\chi_{[0,\infty)}(x) = g(x)$ for all $x \in \mathbb{R} \setminus A$. Since $\chi_{[0,\infty)}(x) = 0$ for all $x < 0$ and $m(A \cap (-\infty, 0)) = 0$, there exists $a < 0$ such that $g(a) = 0$. Similarly, there exists $b \geq 0$ such that $g(b) = 1$. Since g is continuous, by the intermediate value theorem, there exists $c \in (a, b)$ with $g(c) = \frac{1}{2}$. Let $U = (0, 1)$. Since g is continuous, $g^{-1}(U)$ is open. Recall that $c \in g^{-1}(U)$. Thus, $g^{-1}(U) \neq \emptyset$. Since $g^{-1}(U)$ is a nonempty open set, $m(g^{-1}(U)) > 0$. On the other hand, $g^{-1}(U) \subseteq A$, yielding that $0 < m(g^{-1}(U)) \leq m(A) = 0$, a contradiction.

Now consider the function $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

Note that $m(\mathbb{Q}) = 0$. Thus, $\chi_{\mathbb{Q}} = 0$ almost everywhere. Since 0 is continuous, $\chi_{\mathbb{Q}}$ is equal to a continuous function almost everywhere. On the other hand, $\chi_{\mathbb{Q}}$ is not continuous almost everywhere. Actually, $\chi_{\mathbb{Q}}$ is not continuous at any $x \in \mathbb{R}$. Recall that $\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q}$ are both `http://planetmath.org/Dense`dense in $\mathbb{R}$. Therefore, for every $x \in \mathbb{R}$ and for every $\delta > 0$, there exist $x_1 \in (x - \delta, x + \delta) \cap \mathbb{Q}$ and $x_2 \in (x - \delta, x + \delta) \cap (\mathbb{R} \setminus \mathbb{Q})$. Since $\chi_{\mathbb{Q}}(x_1) = 1$ and $\chi_{\mathbb{Q}}(x_2) = 0$, it follows that $\chi_{\mathbb{Q}}$ is not continuous at x. (Choose any $\varepsilon \in (0, 1)$.)