proof of existence of the essential supremum

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Suppose that $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space and $\mathcal{S}$ is a collection of measurable functions $f\colon \Omega \to \bar{\mathbb{R}}$. We show that the essential supremum of $\mathcal{S}$ exists and furthermore, if it is nonempty then there is a sequence $f_n \in \mathcal{S}$ such that

$$\operatorname{ess\,sup} \mathcal{S} = \sup_n f_n.$$

As http://planetmath.org/AnySigmaFiniteMeasureIsEquivalentToAProbabilityMeasure σ-finite measure is equivalent to a probability measure, we may suppose without loss of generality that μ is a probability measure. Also, without loss of generality, suppose that $\mathcal{S}$ is nonempty, and let $\mathcal{S}'$ consist of the collection of maximums of finite sequences of functions in $\mathcal{S}$. Then choose any continuous and strictly increasing $\theta\colon \bar{\mathbb{R}} \to \mathbb{R}$. For example, we can take

$$\theta(x) = \begin{cases} x/(1+|x|), & \text{if } |x| < \infty, \\ 1, & \text{if } x = \infty, \\ -1, & \text{if } x = -\infty. \end{cases}$$

As $\theta(f)$ is a bounded and measurable function for all $f \in \mathcal{S}'$, we can set

$$\alpha = \sup \left\{ \int \theta(f)\, d\mu : f \in \mathcal{S}' \right\}.$$

Then choose a sequence g_n in $\mathcal{S}'$ such that $\int \theta(g_n)\, d\mu \to \alpha$. By replacing g_n by the maximum of $g_1, \ldots, g_n$ if necessary, we may assume that $g_{n+1} \geq g_n$ for each n. Set

$$f = \sup_n g_n.$$

Also, every g_n is the maximum of a finite sequence of functions $g_{n,1}, \ldots, g_{n,m_n}$ in $\mathcal{S}$. Therefore, there exists a sequence $f_n \in \mathcal{S}$ such that

$$\{f_1, f_2, \ldots\} = \{g_{n,m} : n \geq 1, 1 \leq m \leq m_n\}.$$

Then,

$$f = \sup_n f_n.$$

It only remains to be shown that f is indeed the essential supremum of $\mathcal{S}$. First, by continuity of θ and the dominated convergence theorem,

$$\int \theta(f)\, d\mu = \lim_{n\to\infty} \int \theta(g_n)\, d\mu = \alpha.$$

Similarly, for any $g \in \mathcal{S}$,

$$\int \theta(f \vee g) = \lim_{n\to\infty} \int \theta(g_n \vee g)\, d\mu \leq \alpha.$$

It follows that $\theta(f \vee g) - \theta(f)$ is a nonnegative function with nonpositive integral, and so is equal to zero μ-almost everywhere. As θ is strictly increasing, $f \vee g = f$ and therefore $f \geq g$ μ-almost everywhere.

Finally, suppose that $g\colon \Omega \to \bar{\mathbb{R}}$ satisfies $g \geq h$ (μ-a.e.) for all $h \in \mathcal{S}$. Then, $g \geq f_n$ and,

$$g \geq \sup_n f_n = f$$

μ-a.e., as required.