proof of extending a capacity to a Cartesian product

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Let $(X, \mathcal{F})$ be a paved space such that $\mathcal{F}$ is closed under finite unions and finite intersections, and $(K, \mathcal{K})$ be a compact paved space. Define $\mathcal{G}$ to be the closure under finite unions and finite intersections of the paving $\mathcal{F} \times \mathcal{K}$ on $X \times K$. For an $\mathcal{F}$-capacity I, define

$$\begin{aligned}
&\tilde{I} \colon \mathcal{P}(X \times K) \to \mathbb{R}, \\
&\tilde{I}(S) = I(\pi_X(S)),
\end{aligned}$$

where π_X is the projection map onto X. We show that $\tilde{I}$ is a $\mathcal{G}$-capacity and that $\pi_X(S) \in \mathcal{F}_\delta$ whenever $S \in \mathcal{G}_\delta$.

Clearly, the property that $\tilde{I}$ is an increasing set function follows from the fact that I satisfies this property. Furthermore, if $S_n \subseteq X \times K$ is an increasing sequence of sets with $S = \bigcup_n S_n$ then $\pi_X(S_n)$ is an increasing sequence and

$$\tilde{I}(S) = I(\pi_X(S)) = I\left(\bigcup_n \pi_X(S_n)\right) = \lim_{n\to\infty} I(\pi_X(S_n)) = \lim_{n\to\infty} \tilde{I}(S_n).$$

To prove that $\tilde{I}$ is a $\mathcal{G}$-capacity, it only remains to show that if S_n is a sequence in $\mathcal{G}$ decreasing to $S \subseteq X \times K$ then $\tilde{I}(S_n) \to \tilde{I}(S)$. Note that any S in $\mathcal{G}$ can be written as $S = \bigcap_{j=1}^m \bigcup_{k=1}^{n_j} A_{j,k} \times K_{j,k}$ for sets $A_{j,k} \in \mathcal{F}$ and $K_{j,k} \in \mathcal{K}$. The projection onto X is then

$$\pi_X(S) = \bigcup \left\{ \bigcap_{j=1}^m A_{j,k_j} \colon k_j \le n_j,\ \bigcap_{j=1}^m K_{j,k_j} \ne \emptyset \right\}$$

which, as $\mathcal{F}$ is closed under finite unions and finite intersections, must be in $\mathcal{F}$. Furthermore, for any $x \in X$,

$$S_x \equiv \{y \in K \colon (x,y) \in S\} = \bigcap_{j=1}^m \bigcup \{K_{j,k} \colon k \le n_j, x \in A_{j,k}\}.$$

This shows that S_x is in the closure $\mathcal{K}^*$ of $\mathcal{K}$ under finite unions and finite intersections. Furthermore, since `http://planetmath.org/CompactPavingsAreClosedSubsetsOfAC` pavings are closed subsets of a compact topological space, $\mathcal{K}^*$ is itself a compact paving.

Now let S_n be a decreasing sequence of sets in $\mathcal{G}$ and set $S = \bigcap_n S_n$. Then $\pi_X(S) \subseteq \pi_X(S_n)$ for each n, giving $\pi_X(S) \subseteq \bigcap_n \pi_X(S_n)$. To prove the reverse inequality, consider $x \in \bigcap_n \pi_X(S_n)$. Then, $(S_n)_x$ is a nonempty set in $\mathcal{K}^*$ for all n. By compactness, $S_x = \bigcap_n (S_n)_x$ must also be nonempty and therefore $x \in \pi_X(S)$. This shows that

$$\bigcap_n \pi_X(S_n) = \pi_X(S).$$

Furthermore, as we have shown that $\pi_X(S_n) \in \mathcal{F}$ and, as I is an $\mathcal{F}$-capacity,

$$\tilde{I}(S_n) = I(\pi_X(S_n)) \to I(\pi_X(S)) = \tilde{I}(S).$$

So $\tilde{I}$ is a $\mathcal{G}$-capacity.

We finally show that if $S \in \mathcal{G}_\delta$ then $\pi_X(S) \in \mathcal{F}_\delta$. By definition, there is a sequence $S_n \in \mathcal{G}$ such that $S = \bigcap_n S_n$. Setting $S'_n = \bigcap_{m \leq n} S_m$ then, since $\mathcal{G}$ is closed under finite unions and finite intersections, $S'_n \in \mathcal{G}$. Furthermore, S'_n decreases to S so, as shown above, $\pi_X(S'_n) \in \mathcal{F}$ and

$$\pi_X(S) = \bigcap_n \pi_X(S'_n) \in \mathcal{F}_\delta$$

as required.