proof of dominated convergence theorem

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Define the functions h_n^+ and h_n^- as follows:

$$h_n^+(x) = \sup\{f_m(x) \colon m \geq n\}$$

$$h_n^-(x) = \inf\{f_m(x) \colon m \geq n\}$$

These suprema and infima exist because, for every x, $|f_n(x)| \leq g(x)$. These functions enjoy the following properties:

For every n, $|h_n^\pm| \leq g$

The sequence h_n^+ is decreasing and the sequence h_n^- is increasing.

For every x, $\lim_{n\to\infty} h_n^\pm(x) = f(x)$

Each $h_n^\pm$ is measurable.

The first property follows from immediately from the definition of supremum. The second property follows from the fact that the supremum or infimum is being taken over a larger set to define $h_n^\pm(x)$ than to define $h_m^\pm(x)$ when $n > m$. The third property is a simple consequence of the fact that, for any sequence of real numbers, if the sequence converges, then the sequence has an upper limit and a lower limit which equal each other and equal the limit. As for the fourth statement, it means that, for every real number y and every integer n, the sets

$$\{x \mid h_n^-(x) \geq y\} \qquad \text{and} \qquad \{x \mid h_n^+(x) \leq y\}$$

are measurable. However, by the definition of $h_n^\pm$, these sets can be expressed as

$$\bigcup_{m\leq n} \{x \mid f_n(x) \leq y\} \qquad \text{and} \qquad \bigcup_{m\geq n} \{x \mid f_n(x) \leq y\}$$

respectively. Since each f_n is assumed to be measurable, each set in either union is measurable. Since the union of a countable number of measurable sets is itself measurable, these unions are measurable, and hence the functions $h_n^\pm$ are measurable.

Because of properties 1 and 4 above and the assumption that g is integrable, it follows that each $h_n^\pm$ is integrable. This conclusion and property 2 mean that the monotone convergence theorem is applicable so one can conclude that f is integrable and that

$$\lim_{n\to\infty} \int h_n^\pm(x)\, d\mu(x) = \int \lim_{n\to\infty} h_n^\pm(x)\, d\mu(x)$$

By property 3, the right hand side equals $\int f(x)\, d\mu(x)$.

By construction, $h_n^- \leq f_n \leq h_n^+$ and hence

$$\int h_n^- \leq \int f_n \leq \int h_n^+$$

Because the outer two terms in the above inequality tend towards the same limit as $n \to \infty$, the middle term is squeezed into converging to the same limit. Hence

$$\lim_{n\to\infty} \int f_n(x)\, d\mu(x) = \int f(x)\, d\mu(x)$$