proof of equivalent definitions of analytic sets for paved spaces

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Let $(X,\mathcal{F})$ be a paved space with $\emptyset\in\mathcal{F}$, let $\mathcal{N}$ be Baire space, and let Y be any uncountable Polish space. For a subset A of X, we show that the following statements are equivalent.

1. A is $\mathcal{F}$-`http://planetmath.org/AnalyticSet2`analytic.
2. There is a closed subset S of $\mathcal{N}$ and $\theta\colon\mathbb{N}^2\to\mathcal{F}$ such that
$$A=\bigcup_{s\in S}\bigcap_{n=1}^{\infty}\theta\left(n,s_n\right).$$
3. There is a closed subset S of $\mathcal{N}$ and $\theta\colon\mathbb{N}\to\mathcal{F}$ such that
$$A=\bigcup_{s\in S}\bigcap_{n=1}^{\infty}\theta\left(s_n\right).$$
4. A is the result of a Souslin scheme on $\mathcal{F}$.
5. A is the projection of a set in $(\mathcal{F}\times\mathcal{K})_{\sigma\delta}$ onto X, where $\mathcal{K}$ is the collection of compact subsets of Y.
6. A is the projection of a set in $(\mathcal{F}\times\mathcal{G})_{\sigma\delta}$ onto X, where $\mathcal{G}$ is the collection of closed subsets of Y.

(??) implies (??): As A is analytic, there exists a compact paved space $(K,\mathcal{K})$ and a set $B\in(\mathcal{F}\times\mathcal{K})_{\sigma\delta}$ such that $A=\pi_X(B)$, where $\pi_X\colon X\times K\to X$ is the projection map. Write

$$B=\bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}A_{n,m}\times K_{n,m}$$

for $A_{n,m}\in\mathcal{F}$ and $K_{n,m}\in\mathcal{K}$. Rearranging this expression,

$$B=\bigcup_{s\in\mathcal{N}}\bigcap_{n=1}^{\infty}A_{n,s_n}\times K_{n,s_n}.$$

So, defining $S\subseteq\mathcal{N}$ by

$$S=\left\{s\in\mathcal{N}\colon\bigcap_{n=1}^{\infty}K_{n,s_n}\neq\emptyset\right\}.$$

gives

$$A = \pi_X(B) = \bigcup_{s\in S}\bigcap_{n=1}^{\infty} A_{n,s_n}.$$

Setting $\theta(n,m) = A_{n,m}$ gives the required expression, and it only remains to show that S is closed. So, let $s^1, s^2, \ldots$ be a sequence in S converging to a limit $s \in \mathcal{N}$. For any $k \geq 0$ then $s^r_n = s_n$ for all $n \leq k$ and large enough r. Hence,

$$\bigcap_{n\leq k} K_{n,s_n} = \bigcap_{n\leq k} K_{n,s^r_n} \supseteq \bigcap_{n=1}^{\infty} K_{n,s_n} \neq \emptyset.$$

So, the collection of sets K_{n,s_n} for $n = 1, 2, \ldots$ satisfies the finite intersection property, and `http://planetmath.org/PavedSpace`compactness of the paving $\mathcal{K}$ gives

$$\bigcap_{n=1}^{\infty} K_{n,s_n} \neq \emptyset,$$

showing that $s \in S$ and that S is indeed closed.

(??) implies (??): Supposing that A satisfies the required expression, choose any bijection $\phi\colon \mathbb{N} \to \mathbb{N}^2$. Then define $\tilde{\theta} \equiv \theta \circ \phi$ and $f\colon \mathcal{N} \to \mathcal{N}$ by $f(s) = t$ where $t_n = \phi^{-1}(n, s_n)$. As S is closed, it follows that $\tilde{S} = f(S)$ will also be closed and,

$$A = \bigcup_{s\in S}\bigcap_{n} \theta(n, s_n) = \bigcup_{s\in S}\bigcap_{n} \tilde{\theta}(\phi^{-1}(n, s_n)) = \bigcup_{s\in\tilde{S}}\bigcap_{n} \tilde{\theta}(s_n)$$

as required.

(??) implies (??): Suppose that A satisfies the required expression and define a Souslin scheme (A_s) as follows. For any $n \geq 1$ and $s \in \mathbb{N}^n$ let us set

$$A_s = \begin{cases} \theta(s_n), & \text{if } s = t|_n \text{ for some } t \in \mathcal{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, for $s \in \mathcal{N}$,

$$\bigcap_{n=1}^{\infty} A_{s|_n} = \begin{cases} \bigcap_{n=1}^{\infty} \theta(s_n), & \text{if } s \in S, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Here, if $s \notin S$, we have used the fact that S is closed to deduce that for large n, there is no $t \in S$ with $t|_n = s|_n$ and, therefore, $A_{s|_n} = \emptyset$. The result of the Souslin scheme (A_s) is then

$$\bigcup_{s \in \mathcal{N}} \bigcap_{n=1}^{\infty} A_{s|_n} = \bigcup_{s \in S} \bigcap_{n=1}^{\infty} \theta(s_n) = A$$

as required.

(??) implies (??): Suppose that A is the result of a Souslin scheme (A_s). Let us first consider the case where Y is Cantor space, $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, which is a compact Polish space. Then, for any $s \in \mathbb{N}^n$, let K_s be the set of $t \in \mathcal{C}$ such that $t_k = 1$ if $k = s_1 + \cdots + s_m$ for some $m \leq n$ and $t_k = 0$ for all other $k < s_1 + \cdots + s_n$. These are closed and, therefore, compact sets.

Given any sequence $s^1 \in \mathbb{N}^1, s^2 \in \mathbb{N}^2, \ldots$ it is easily seen that $\bigcap_n K_{s^n}$ is nonempty if and only if there is an $s \in \mathcal{N}$ such that $s|_n = s^n$ for all n. Define the set B in $(\mathcal{F} \times \mathcal{K})_{\sigma\delta}$ by

$$\begin{aligned} B &= \bigcap_{n=1}^{\infty} \bigcup_{s \in \mathbb{N}^n} A_s \times K_s \\ &= \bigcup_{s^1 \in \mathbb{N}^1, s^2 \in \mathbb{N}^2, \ldots} \bigcap_{n=1}^{\infty} A_{s^n} \times K_{s^n} \\ &= \bigcup_{s \in \mathcal{N}} \bigcap_{n=1}^{\infty} A_{s|_n} \times K_{s|_n}. \end{aligned}$$

The projection of B onto X is then

$$\pi_X(S) = \bigcup_{s \in \mathcal{N}} \bigcap_{n=1}^{\infty} A_{s|_n},$$

which is the result A of the scheme (A_s) as required. The result then generalizes to any uncountable Polish space Y, as all such spaces `http://planetmath.org/UncountablePo`
Cantor space.

(??) implies (??): This is trivial, since all compact sets are closed.

(??) implies (??): This is a consequence of the result that projections of analytic sets are analytic.