proof of Carathéodory's lemma

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A set $S \subseteq X$ is μ-measurable if and only if

$$\mu(E) \geq \mu(E \cap S) + \mu(E \cap S^c) \tag{1}$$

for every $E \subseteq X$. As this inequality is clearly satisfied if $S = \emptyset$ and is unchanged when S is replaced by S^c, then $\mathcal{A}$ contains the empty set and is closed under taking complements of sets. To show that $\mathcal{A}$ is a σ-algebra, it only remains to show that it is closed under taking countable unions of sets. Choose any sets $A, B \in \mathcal{A}$ and $E \subseteq X$. Then,

$$\begin{aligned}\mu(E) &\geq \mu(E \cap A) + \mu(E \cap A^c)\\ &\geq \mu(E \cap A) + \mu(E \cap A^c \cap B) + \mu(E \cap A^c \cap B^c)\\ &\geq \mu(E \cap (A \cup B)) + \mu(E \cap A^c \cap B^c)\end{aligned}$$

The first two inequalities here follow from applying (**??**) with A and then B in place of S, and the third uses the subadditivity of μ together with $A \cup (A^c \cap B) = A \cup B$. So (**??**) is satisfied with $A \cup B$ in place of S, showing that $\mathcal{A}$ is closed under taking pairwise unions and is therefore an algebra of sets on X. If A, B are disjoint sets in $\mathcal{A}$ then replacing E by $E \cap (A \cup B)$ and S by A in (**??**) gives $\mu(E \cap (A \cup B)) \geq \mu(E \cap A) + \mu(E \cap B)$. As the reverse inequality follows from subadditivity of μ, this implies that

$$\mu(E \cap (A \cup B)) = \mu(E \cap A) + \mu(E \cap B).$$

So, the map $A \mapsto \mu(E \cap A)$ is an additive set function on $\mathcal{A}$. In particular, taking $E = X$ shows that μ is additive on $\mathcal{A}$.

Now choose a sequence $A_i \in \mathcal{A}$, and set $B_i \equiv \bigcup_{j=1}^{i} A_j$ which are in the algebra $\mathcal{A}$. To prove that $\mathcal{A}$ is a σ-algebra it needs to be shown that $A \equiv \bigcup_i A_i = \bigcup_i B_i$ is itself in $\mathcal{A}$. First, as $B_i \in \mathcal{A}$ and $A^c \subseteq B_i^c$,

$$\mu(E) \geq \mu(E \cap B_i) + \mu(E \cap B_i^c) \geq \mu(E \cap B_i) + \mu(E \cap A^c).$$

As $C_i \equiv B_i \setminus B_{i-1}$ are pairwise disjoint sets in $\mathcal{A}$ satisfying $\bigcup_{j=1}^{i} C_j = B_i$ the additivity of $C \mapsto \mu(E \cap C)$ on $\mathcal{A}$ gives

$$\mu(E) \geq \sum_{j=1}^{i} \mu(E \cap C_j) + \mu(E \cap A^c).$$

So, letting i increase to infinity, the subadditivity of μ applied to $\bigcup_j (E \cap C_j) = E \cap A$ gives

$$\mu(E) \geq \sum_j \mu(E \cap C_j) + \mu(E \cap A^c) \geq \mu(E \cap A) + \mu(E \cap A^c).$$

This shows that A is μ-measurable and so $\mathcal{A}$ is a σ-algebra.

It only remains to show that the restriction of μ to $\mathcal{A}$ is a measure, for which it needs to be shown that μ is countably additive on $\mathcal{A}$. So, choose any pairwise disjoint sequence $A_i \in \mathcal{A}$ and set $A = \bigcup_i A_i$. The following inequality

$$\sum_{j=1}^{i} \mu(A_j) = \mu\left(\bigcup_{j=1}^{i} A_j\right) \leq \mu(A) \leq \sum_j \mu(A_j)$$

follows from the additivity of μ on $\mathcal{A}$, the requirement that μ is increasing and from the countable subadditivity of μ. Letting i increase to infinity gives $\mu(A) = \sum_j \mu(A_j)$ and μ is indeed countably additive on $\mathcal{A}$.