L^p-norm is dual to L^q

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If $(X, \mathfrak{M}, \mu)$ is any measure space and $1 \leq p, q \leq \infty$ are http://planetmath.org/ConjugateInd
conjugates then, for $f \in L^p$, the following linear function can be defined

$$\begin{aligned} \Phi_f \colon L^q &\to \mathbb{C}, \\ g &\mapsto \Phi_f(g) \equiv \int fg \, d\mu. \end{aligned}$$

The http://planetmath.org/HolderInequalityHölder inequality shows that this gives a well defined and bounded linear map. Its operator norm is given by

$$\|\Phi_f\| = \{\|fg\|_1 : g \in L^q, \|g\|_q = 1\}.$$

The following theorem shows that the operator norm of Φ_f is equal to the L^p-norm of f.

Theorem. *Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and p, q be Hölder conjugates. Then, any measurable function $f \colon X \to \mathbb{C}$ has L^p-norm*

$$\|f\|_p = \sup\{\|fg\|_1 : g \in L^q, \|g\|_q = 1\}. \tag{1}$$

Furthermore, if either $p < \infty$ and $\|f\|_p < \infty$ or $p = 1$ then μ is not required to be σ-finite.

Note that the σ-finite condition is required, except in the cases mentioned. For example, if μ is the measure satisfying $\mu(A) = \infty$ for every nonempty set A, then $L^p(\mu) = \{0\}$ for $p < \infty$ and it is easily checked that equality (**??**) fails whenever $f = 1$ and $p > 1$.