functional monotone class theorem

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The monotone class theorem is a result in measure theory which allows statements about particularly simple classes of functions to be generalized to arbitrary measurable and bounded functions.

Theorem 1. *Let $(X, \mathcal{A})$ be a measurable space and $\mathcal{S}$ be a* `http://planetmath.org/PiSystem`*π-system generating the* `http://planetmath.org/SigmaAlgebra`*σ-algebra $\mathcal{A}$. Suppose that $\mathcal{H}$ be a vector space of real-valued functions on X containing the constant functions and satisfying the following,*

- *if $f\colon X \to \mathbb{R}$ is bounded and there is a sequence of nonnegative functions $f_n \in \mathcal{H}$ increasing pointwise to f, then $f \in \mathcal{H}$.*
- *for every set $A \in \mathcal{S}$ the characteristic function 1_A is in $\mathcal{H}$.*

Then, $\mathcal{H}$ contains every bounded and measurable function from X to $\mathbb{R}$.

That $\mathcal{H}$ is a vector space just means that it is closed under taking linear combinations, so $\lambda f + \mu g \in \mathcal{H}$ whenever $f, g \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{R}$.

As an example application, consider `http://planetmath.org/FubinisTheorem`Fubini's theorem, which states that for any two finite measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ then we may commute the order of integration

$$\int\int f(x,y)\,d\mu(x)\,d\nu(y) = \int\int f(x,y)\,d\nu(y)\,d\mu(x). \tag{1}$$

Here, $f\colon X \times Y \to \mathbb{R}$ is a bounded and $\mathcal{A}\otimes\mathcal{B}$-measurable function. The space $\mathcal{H}$ of functions for which this identity holds is easily shown to be linearly closed and, by the monotone convergence theorem, is closed under taking monotone limits of functions. Furthermore, the π-system of sets of the form $A \times B$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$ generates the σ-algebra $\mathcal{A} \otimes \mathcal{B}$ and

$$\int\int 1_{A\times B}(x,y)\,d\mu(x)\,d\nu(y) = \mu(A)\nu(B) = \int\int 1_{A\times B}(x,y)\,d\nu(y)\,d\mu(x).$$

So, $1_{A\times B} \in \mathcal{H}$ and the monotone class theorem allows us to conclude that all real valued and bounded $\mathcal{A}\otimes\mathcal{B}$-measurable functions are in $\mathcal{H}$, and equation (**??**) is satisfied.

An alternative, more general form of the monotone class theorem is as follows. The version of the monotone class theorem given above follows from this by letting $\mathcal{K}$ be the characteristic functions 1_A for $A \in \mathcal{S}$.

Theorem 2. *Let X be a set and $\mathcal{K}$ be a collection of bounded and real valued functions on X which is closed under multiplication, so that $fg \in \mathcal{K}$ for all $f, g \in \mathcal{K}$. Let $\mathcal{A}$ be the σ-algebra on X generated by $\mathcal{K}$.*

Suppose that $\mathcal{H}$ is a vector space of real valued functions on X containing $\mathcal{K}$ and the constant functions, and satisfying the following

- *if $f\colon X \to \mathbb{R}$ is bounded and there is a sequence of nonnegative functions $f_n \in \mathcal{H}$ increasing pointwise to f, then $f \in \mathcal{H}$.*

Then, $\mathcal{H}$ contains every bounded and real valued $\mathcal{A}$-measurable function on X.

Saying that $\mathcal{A}$ is the σ-algebra generated by $\mathcal{K}$ means that it is the smallest σ-algebra containing the sets $f^{-1}(B)$ for $f \in \mathcal{K}$ and Borel subset B of $\mathbb{R}$.

For example, letting $\mathcal{K}$ be as in theorem **??** and μ, ν be finite measures on $(X, \mathcal{A})$ such that $\mu(X) = \nu(X)$ and $\int f\,d\mu = \int f\,d\nu$ for all $f \in \mathcal{K}$, then $\int f\,d\mu = \int f\,d\nu$ for all bounded and measurable real valued functions f. Therefore, $\mu = \nu$. In particular, a finite measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is uniquely determined by its characteristic function $\chi(x) \equiv \int e^{ixy}\,d\mu(y)$ for $x \in \mathbb{R}$. Similarly, a finite measure μ on a bounded interval is uniquely determined by the integrals $\int x^n\,d\mu(x)$ for $n \in \mathbb{Z}_+$.