any σ-finite measure is equivalent to a probability measure

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The following theorem states that for any http://planetmath.org/SigmaFiniteσ-finite measure μ, there is an equivalent probability measure $\mathbb{P}$ — that is, the sets A satisfying $\mu(A) = 0$ are the same as those satisfying $\mathbb{P}(A) = 0$. This result allows statements about probability measures to be generalized to arbitrary σ-finite measures.

Theorem. *Any nonzero σ-finite measure μ on a measurable space $(X, \mathcal{A})$ is equivalent to a probability measure $\mathbb{P}$ on $(X, \mathcal{A})$. In particular, there is a positive measurable function $f\colon X \to (0, \infty)$ satisfying $\int f\, d\mu = 1$, and $\mathbb{P}(A) = \int_A f\, d\mu$ for all $A \in \mathcal{A}$.*

Proof. Let $A_1, A_2, \ldots$ be a sequence in $\mathcal{A}$ such that $\mu(A_k) < \infty$ and $\bigcup_k A_k = X$. Then it is easily verified that

$$g \equiv \sum_{k=1}^{\infty} 2^{-k} \frac{1_{A_k}}{1 + \mu(A_k)}$$

satisfies $1 \geq g > 0$ and $\int g\, d\mu < \infty$. So, setting $f = g / \int g\, d\mu$, we have $\int f\, d\mu = 1$ and therefore $\mathbb{P}(A) \equiv \int_A f\, d\mu$ is a probability measure equivalent to μ. $\square$