an integrable function that does not tend to zero

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Owner	silverfish (6603)
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Author	silverfish (6603)
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In this entry, we give an example of a function f such that f is Lebesgue integrable on $[0, \infty)$ but $f(x)$ does not tend to zero as $x \to \infty$.

First of all, let g_n be the function $\sin(2^n x)\chi_{[0, \frac{\pi}{2^n}]}$, where χ_I denotes the characteristic function of the interval I. In other words, χ takes the value 1 on I and 0 everywhere else.

Let μ denote Lebesgue measure. An easy computation shows

$$\int_{\mathbb{R}} g_n \, d\mu = 2^{1-n}, \tag{1}$$

and $g_n\left(\frac{\pi}{2^{n+1}}\right) = 1$. Let $h_n(x) = g_n(x - n\pi)$, so h_n is just a "shifted" version of g_n. Note that

$$h_n\left(n\pi + \frac{\pi}{2^{n+1}}\right) = 1. \tag{2}$$

We now construct our function f by defining $f = \sum_{r=0}^{\infty} h_r$. There are no convergence problems with this sum since for a given $x \in \mathbb{R}$, at most one h_r takes a non-zero value at x. Also $f(x)$ does not tend to 0 as $x \to \infty$ as there are arbitrarily large values of x for which f takes the value 1, by (**??**).

All that is left is to show that f is Lebesgue integrable. To do this rigorously, we apply the monotone convergence theorem (MCT) with $f_n = \sum_{r=0}^{n} h_r$. We must check the hypotheses of the MCT. Clearly $f_n \to f$ as $n \to \infty$, and the sequence (f_n) is monotone increasing, positive, and integrable. Furthermore, each f_n is continuous and zero except on a compact interval, so is integrable. Finally, from (**??**) we see that $\int_{\mathbb{R}} f_n \, d\mu \leq 4$ for all n. Therefore, the MCT applies and f is integrable.