potential of hollow ball

Canonical name	PotentialOfHollowBall
Date of creation	2013-03-22 17:16:46
Last modified on	2013-03-22 17:16:46
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	8
Author	pahio (2872)
Entry type	Example
Classification	msc 28A25
Classification	msc 26B10
Classification	msc 26B15
Related topic	JacobiDeterminant
Related topic	ChangeOfVariablesInIntegralOnMathbbRn
Related topic	SubstitutionNotation
Related topic	ModulusOfComplexNumber

Let $(\xi,\, \eta,\, \zeta)$ be a point bearing a mass m and $(x,\, y,\, z)$ a point. If the distance of these points is r, we can define the *potential* of $(\xi,\, \eta,\, \zeta)$ in $(x,\, y,\, z)$ as

$$\frac{m}{r} = \frac{m}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}.$$

The relevance of this concept appears from the fact that its partial derivatives

$$\frac{\partial}{\partial x}\Big(\frac{m}{r}\Big) = -\frac{m(x-\xi)}{r^3}, \quad \frac{\partial}{\partial y}\Big(\frac{m}{r}\Big) = -\frac{m(y-\eta)}{r^3}, \quad \frac{\partial}{\partial z}\Big(\frac{m}{r}\Big) = -\frac{m(z-\zeta)}{r^3}$$

are the components of the gravitational with which the material point $(\xi,\, \eta,\, \zeta)$ acts on one mass unit in the point $(x,\, y,\, z)$ (provided that the are chosen suitably).

The potential of a set of points $(\xi,\, \eta,\, \zeta)$ is the sum of the potentials of individual points, i.e. it may lead to an integral.

We determine the potential of all points $(\xi,\, \eta,\, \zeta)$ of a hollow ball, where the matter is located between two concentric spheres with radii R_0 and $R\,(> R_0)$. Here the of mass is assumed to be presented by a continuous function $\varrho = \varrho(r)$ at the distance r from the centre O. Let a be the distance from O of the point A, where the potential is to be determined. We chose O the origin and the ray OA the positive z-axis.

For obtaining the potential in A we must integrate over the ball shell where $R_0 \le r \le R$. We use the spherical coordinates r, φ and ψ which are tied to the Cartesian coordinates via

$$x = r\cos\varphi\cos\psi, \quad y = r\cos\varphi\sin\psi, \quad z = r\sin\varphi;$$

for attaining all points we set

$$R_0 \le r \le R, \quad -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}, \quad 0 \le \psi < 2\pi.$$

The cosines law implies that $PA = \sqrt{r^2 - 2ar\sin\varphi + a^2}$. Thus the potential is the triple integral

$$V(a) = \int_{R_0}^{R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} \frac{\varrho(r)\, r^2\cos\varphi}{\sqrt{r^2-2ar\sin\varphi+a^2}}\, dr\, d\varphi\, d\psi = 2\pi\int_{R_0}^{R}\varrho(r)\, r\, dr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{r\cos\varphi\, d\varphi}{\sqrt{r^2-2ar\sin\varphi+a^2}}, \tag{1}$$

where the factor $r^2 \cos\varphi$ is the coefficient for the coordinate changing

$$\left|\frac{\partial(x,\, y,\, z)}{\partial(r,\, \varphi,\, \psi)}\right| = \quad \operatorname{mod} \begin{vmatrix} \cos\varphi\cos\psi & \cos\varphi\sin\psi & \sin\varphi \\ -r\sin\varphi\cos\psi & -r\sin\varphi\sin\psi & r\cos\varphi \\ -r\cos\varphi\sin\psi & r\cos\varphi\cos\psi & 0 \end{vmatrix}.$$

We get from the latter integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r\cos\varphi\, d\varphi}{\sqrt{r^2 - 2ar\sin\varphi + a^2}} = -\frac{1}{a} \Big/_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - 2ar\sin\varphi + a^2} = \frac{1}{a}[(r+a) - |r-a|]. \tag{2}$$

Accordingly we have the two cases:

1°. The point A is outwards the hollow ball, i.e. $a > R$. Then we have $|r - a| = a - r$ for all $r \in [R_0,\, R]$. The value of the integral (2) is $\frac{2r}{a}$, and (1) gets the form

$$V(a) = \frac{4\pi}{a} \int_{R_0}^{R} \varrho(r)\, r^2\, dr = \frac{M}{a},$$

where M is the mass of the hollow ball. Thus *the potential outwards the hollow ball is exactly the same as in the case that all mass were concentrated to the centre.* A correspondent statement concerns the attractive

$$V'(a) = -\frac{M}{a^2}.$$

2°. The point A is in the cavity of the hollow ball, i.e. $a < R_0$. Then $|r - a| = r - a$ on the interval of integration of (2). The value of (2) is equal to 2, and (1) yields

$$V(a) = 4\pi \int_{R_0}^{R} \varrho(r)\, r\, dr,$$

which is on a. That is, *the potential of the hollow ball, when the of mass depends only on the distance from the centre, has in the cavity a constant value, and the hollow ball influences in no way on a mass inside it.*

References

[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset II.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1932).