proof of Fatou-Lebesgue theorem

Canonical name	ProofOfFatouLebesgueTheorem
Date of creation	2013-03-22 15:58:50
Last modified on	2013-03-22 15:58:50
Owner	Wkbj79 (1863)
Last modified by	Wkbj79 (1863)
Numerical id	13
Author	Wkbj79 (1863)
Entry type	Proof
Classification	msc 28A20

Since $\left|\int g\,d\mu\right| \le \int |g|\,d\mu \le \int \Phi\,d\mu < \infty$, we have that $\int g\,d\mu > -\infty$. Similarly, $\int h\,d\mu < \infty$.

The inequality $\liminf_{n\to\infty} \int f_n\,d\mu \le \limsup_{n\to\infty} \int f_n\,d\mu$ is obvious by definition of lim inf and lim sup.

Define a sequence of functions $k_n\colon X \to \mathbb{R}$ by $k_n(x) = f_n(x) + \Phi(x)$. Then each k_n is nonnegative (since $-f_n \le |f_n| \le \Phi$) and integrable (since $k_n \le |f_n| + \Phi \le 2\Phi$), as is $k := \liminf_{n\to\infty} k_n$. Fatou's lemma yields that $\int k\,d\mu \le \liminf_{n\to\infty} \int k_n\,d\mu$. Thus:

$$\begin{aligned}
\int g\,d\mu + \int \Phi\,d\mu &= \int (g+\Phi)\,d\mu \\
&= \int k\,d\mu \\
&\le \liminf_{n\to\infty} \int k_n\,d\mu \\
&= \liminf_{n\to\infty} \int (f_n + \Phi)\,d\mu \\
&= \liminf_{n\to\infty} \left(\int f_n\,d\mu + \int \Phi\,d\mu\right) \\
&= \liminf_{n\to\infty} \int f_n\,d\mu + \liminf_{n\to\infty} \int \Phi\,d\mu \\
&= \liminf_{n\to\infty} \int f_n\,d\mu + \int \Phi\,d\mu
\end{aligned}$$

Since $\int \Phi\,d\mu < \infty$, it follows that $\int g\,d\mu \le \liminf_{n\to\infty} \int f_n\,d\mu$.

Note that $|-f_n| = |f_n| \le \Phi$. Thus,

$$\begin{aligned}
-\int h\,d\mu &= \int -h\,d\mu \\
&= \int -\limsup_{n\to\infty} f_n\,d\mu \\
&= \int \liminf_{n\to\infty} (-f_n)\ d\mu \\
&\leq \liminf_{n\to\infty} \int -f_n\,d\mu \text{ by a previous },\\
&= \liminf_{n\to\infty} \left(-\int f_n\,d\mu\right) \\
&= -\limsup_{n\to\infty} \int f_n\,d\mu.
\end{aligned}$$

Hence, $\limsup_{n\to\infty} \int f_n\,d\mu \leq \int h\,d\mu$. It follows that $-\infty < \int g\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu \leq \limsup_{n\to\infty} \int f_n\,d\mu \leq \int h\,d\mu < \infty$. $\square$