proof of equivalent definitions of analytic sets for Polish spaces

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Let A be a nonempty subset of a Polish space X. Then, letting $\mathcal{N}$ denote Baire space and Y be any uncountable Polish space, we show that the following are equivalent.

1. A is $\mathcal{F}$-`http://planetmath.org/AnalyticSet2`analytic.
2. A is the `http://planetmath.org/GeneralizedCartesianProduct`projection of a closed subset of $X \times \mathcal{N}$ onto X.
3. A is the `http://planetmath.org/DirectImage`image of a continuous function $f\colon Z \to X$ for some Polish space Z.
4. A is the image of a continuous function $f\colon \mathcal{N} \to X$.
5. A is the image of a Borel measurable function $f\colon Y \to X$.
6. A is the projection of a Borel subset of $X \times Y$ onto X.

(??) implies (??): Let $\mathcal{F}$ be the paving consisting of closed subsets of X. The collection of $\mathcal{K}$-analytic sets contains the Borel σ-algebra of X (see countable unions and intersections of analytic sets are analytic) and, as the `http://planetmath.org/AnalyticSetsDefineAClosureOperator`analytic sets are given by a closure operator it follows that it contains all analytic subsets of X. So, any analytic subset A of X is $\mathcal{F}$-analytic. Then, there is a closed $S \subseteq \mathcal{N}$ and a function $\theta\colon \mathbb{N} \to \mathcal{F}$ such that

$$A = \bigcup_{s \in S} \bigcap_{n=1}^{\infty} \theta(s_n)$$

(see proof of equivalent definitions of analytic sets for paved spaces). For each $m, n \in \mathbb{N}$ let $K_{m,n}$ denote the closed subset of $s \in \mathcal{N}$ with $s_n = m$. Then, we can rearrange the above expression to get $A = \pi_X(B)$ where $\pi_X\colon X \times \mathcal{N} \to X$ is the projection map and

$$B = (X \times S) \cap \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \theta(m) \times K_{m,n}.$$

It is easily seen that $\bigcup_m \theta(m) \times K_{m,n}$ is a closed subset of $X \times \mathcal{N}$ for each n, and therefore B is closed, as required.
(??) implies (??): Suppose that $A = \pi_X(S)$ for a closed subset S of $X \times \mathcal{N}$, where $\pi_X\colon X \times \mathcal{N} \to X$ is the projection map. As the product of Polish

spaces is Polish, and every closed subset of a Polish space is Polish, then S will be a Polish space under the subspace topology. So, we can take $Z = S$ and let $f\colon Z \to X$ be the restriction of π_X to Z.

(??) implies (??): Suppose that A is the image of a continuous function $g\colon Z \to X$, for a Polish space Z. As Baire space is universal for Polish spaces, there exists a continuous and `http://planetmath.org/Surjective`onto function $h\colon \mathcal{N} \to Z$. The result follows by taking $f = g \circ h$.

(??) implies (??): Suppose that A is the image of a continuous function $g\colon \mathcal{N} \to X$. Since uncountable Polish spaces are all Borel isomorphic (see Polish spaces up to Borel isomorphism), there is a Borel isomorphism $h\colon Y \to \mathcal{N}$. The result follows by taking $f = g \circ h$.

(??) implies (??): Suppose that A is the image of a Borel measurable function $f\colon Y \to X$, and let Γ be its `http://planetmath.org/Graph2`graph

$$\Gamma \equiv \{(f(y), y)\colon y \in Y\} \subseteq X \times Y.$$

The projection of Γ onto X is equal to $f(Y) = A$, so the result will follow once it is shown that Γ is a Borel set.

Choose a countable and dense subset $\{x_1, x_2, \dots\}$ of X, and let d be a metric generating the topology on X. Then, for integers $m, n \geq 1$, denote the open ball about x_m of radius $1/n$ by $B_{m,n}$. Since the x_m form a dense set, $\bigcup_m B_{m,n} = X$ for each n. Let us define

$$\Gamma_n \equiv \bigcup_{m=1}^{\infty} B_{m,n} \times f^{-1}(B_{m,n}) \subseteq X \times Y,$$

which contains Γ. Furthermore, since $f^{-1}(B_{m,n})$ are Borel, Γ_n are Borel sets. Suppose that $(x, y) \in \bigcap_n \Gamma_n$. Then, for each n, there is an m such that $x \in B_{m,n}$ and $y \in f^{-1}(B_{m,n})$. So,

$$d(x, f(y)) \leq d(x, x_m) + d(x_m, f(y)) \leq 2/n.$$

This holds for all n, showing that $y = f(x)$ and so $(x, y) \in \Gamma$. We have shown that $\Gamma = \bigcap_n \Gamma_n$ is Borel, as required.

(??) implies (??): This is an immediate consequence of the result that projections of analytic sets are analytic.