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compact pavings are closed subsets of a compact space

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Recall that a paving $\mathcal{K}$ is compact if every subcollection satisfying the finite intersection property has nonempty intersection. In particular, a topological space is `http://planetmath.org/Compact`compact if and only if its collection of closed subsets forms a compact paving. Compact paved spaces can therefore be constructed by taking closed subsets of a compact topological space. In fact, all compact pavings arise in this way, as we now show.

Given any compact paving $\mathcal{K}$ the following result says that the collection $\mathcal{K}'$ of all intersections of finite unions of sets in $\mathcal{K}$ is also compact.

Theorem 1. *Suppose that $(K,\mathcal{K})$ is a compact paved space. Let $\mathcal{K}'$ be the smallest collection of subsets of X such that $\mathcal{K} \subseteq \mathcal{K}'$ and which is closed under arbitrary intersections and finite unions. Then, $\mathcal{K}'$ is a compact paving.*

In particular,

$$\mathcal{T} \equiv \{K \setminus C \colon C \in \mathcal{K}'\} \cup \{\emptyset, K\}$$

is closed under arbitrary unions and finite intersections, and hence is a topology on K. The collection of closed sets defined with respect to this topology is $\mathcal{K}' \cup \{\emptyset, K\}$ which, by Theorem **??**, is a compact paving. So, the following is obtained.

Corollary. *A paving $(K,\mathcal{K})$ is compact if and only if there exists a topology on K with respect to which $\mathcal{K}$ are closed sets and K is compact.*