Fubini's theorem for the Lebesgue integral

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This is the version of the Fubini's Theorem for the Lebesgue integral. For the Riemann integral, see `http://planetmath.org/FubinisTheorem`the standard calculus version.

In the following suppose we will by convention define $\int_X f d\mu := 0$ in case f is not integrable. This simplifies notation and does not affect the results since it will turn out that such cases happen on a set of measure 0.

Also if we have a function $f\colon X \times Y \to \mathbb{F}$ then define $f_x(y) := f(x,y)$ and $f^y(x) := f(x,y)$.

Theorem (Fubini)**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are `http://planetmath.org/SigmaFinited` finite measure spaces. If $f \in L^1(X \times Y)$ then $f_x \in L^1(\nu)$ for μ-almost every x and $f^y \in L^1(\mu)$ for ν-almost every y. Further the functions $x \mapsto \int_Y f_x d\nu$ and $y \mapsto \int_X f^y d\nu$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively and*

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_X \left[\int_Y f(x,y)\, d\nu(y)\right] d\mu(x) = \int_Y \left[\int_X f(x,y)\, d\mu(x)\right] d\nu(y).$$

You can now see the reason for defining the integral even where f_x and f^y are not integrable since the functions $x \mapsto \int_Y f_x d\nu$ and $y \mapsto \int_X f^y d\nu$ are normally only almost everywhere defined, and we'd like to define them everywhere. Since we have changed the definition only on a set of measure zero, this does not change the final result and we can interchange the integrals freely without having to worry about where the functions are actually defined.

Note the of this theorem and Tonelli's theorem for non-negative functions. Here you actually need to check some integrability before switching the integral . A application of Tonelli's theorem actually shows that you can prove any one of these equations to show that $f \in L^1(X \times Y)$

$$\int_{X\times Y} |f|\, d(\mu\times\nu) < \infty,$$

$$\int_X \left[\int_Y |f(x,y)|\, d\nu(y)\right] d\mu(x) < \infty, \text{ or}$$

$$\int_Y \left[\int_X |f(x,y)|\, d\mu(x)\right] d\nu(y) < \infty.$$

If we take the counting measure on $\mathbb{N}$, then one can the Fubini theorem for sums.

Theorem (Fubini for sums)**.** *Suppose that f_{ij} is absolutely summable, that is $\sum_{i,j\in\mathbb{N}}|f_{ij}| < \infty$, then*

$$\sum_{i,j\in\mathbb{N}} f_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} f_{ij}.$$

In the above theorem we have used $\mathbb{N}$ as our set for simplicity and familiarity of notation. Any summable function f_{ij} will have only a countable number of non-zero elements and thus the theorem for arbitrary sets just reduces to the above case.

References

[1] Gerald B. Folland. . John Wiley & Sons, Inc., New York, New York, 1999