closure and interior of Cantor set

Canonical name	ClosureAndInteriorOfCantorSet
Date of creation	2013-03-22 15:43:21
Last modified on	2013-03-22 15:43:21
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	4
Author	rspuzio (6075)
Entry type	Theorem
Classification	msc 28A80

The Cantor set is closed and its interior is empty.

To prove the first assertion, note that each of the sets $C_0, C_1, C_2, \ldots$, being the union of a finite number of closed intervals is closed. Since the Cantor set is the intersection of all these sets and intersections of closed sets are closed, it follows that the Cantor set is closed.

To prove the second assertion, it suffices to show that given any open interval I, no matter how small, at least one point of that interval will not belong to the Cantor set. To accomplish this, the ternary characterization of the Cantor set is useful. Because rational numbers whose denominators are powers of 3 are dense, there exists a rational number $n/3^m$ contained in I. Expressed in base 3, this rational number has a finite expansion. If this expansion contains the digit "1", then our number does not belong to Cantor set, and we are done. If not, since I is open, there must exist a number $k > m$ such that $n/3^m + 1/3^k \in I$. Now, the last digit of the ternary expansion of this number is "1" by construction, so we also find a number not belonging to the interval in this case.