conditional expectations are uniformly integrable

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The collection of all conditional expectations of an integrable random variable forms a uniformly integrable set. More generally, we have the following result.

Theorem. *Let S be a uniformly integrable set of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, the set*

$$\{\mathbb{E}[X \mid \mathcal{G}] : X \in S \textit{ and } \mathcal{G} \textit{ is a sub-}\sigma\textit{-algebra of } \mathcal{F}\}$$

is also uniformly integrable.

To prove the result, we first use the fact that uniform integrability implies that S is L^1-bounded. That is, there is a constant $L > 0$ such that $\mathbb{E}[|X|] \leq L$ for every $X \in S$. Also, choosing any $\epsilon > 0$, there is a $\delta > 0$ so that

$$\mathbb{E}[|X| 1_A] < \epsilon$$

for all $X \in S$ and $A \in \mathcal{F}$ with $\mathbb{P}(A) \leq \delta$.

Set $K = L/\delta$. Then, if $Y = \mathbb{E}[X \mid \mathcal{G}]$ for any $X \in S$ and $\mathcal{G} \subseteq \mathcal{F}$, Jensen's inequality gives

$$|Y| \leq \mathbb{E}[|X| \mid \mathcal{G}].$$

So, applying Markov's inequality,

$$\mathbb{P}(|Y| > K) \leq K^{-1}\mathbb{E}[|Y|] \leq K^{-1}\mathbb{E}[|X|] \leq L/K = \delta$$

and, therefore

$$\mathbb{E}[|Y| 1_{\{|Y|>K\}}] \leq \mathbb{E}[|X| 1_{\{|Y|>K\}}] < \epsilon.$$