Dynkin's lemma

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Dynkin's lemma is a result in measure theory showing that the http://planetmath.org/SigmaA
algebra generated by any given http://planetmath.org/PiSystemπ-system on a set X coincides with the Dynkin system generated the π-system. The result can be used to prove that measures are uniquely determined by their values on π-systems generating the required σ-algebra. For example, the Borel σ-algebra on $\mathbb{R}$ is generated by the π-system of open intervals (a, b) for $a < b$ and consequently the Lebesgue measure μ is uniquely determined by the property that $\mu((a, b)) = b - a$.

Note that this lemma generalizes the statement that a Dynkin system which is also a π-system is a σ-algebra.

Lemma (Dynkin)**.** *Let A be a π-system on a set X. Then $\mathcal{D}(A) = \sigma(A)$. That is, the smallest Dynkin system containing A coincides with the σ-algebra generated by A.*

Proof. As A is a π-system, the set $\mathcal{D}_1 \equiv \{S \subseteq X : S \cap T \in \mathcal{D}(A) \text{ for every } T \in A\}$ contains A. We show that $\mathcal{D}_1$ is also a Dynkin system.

First, for every $T \in A$, $X \cap T = T \in A$ so X is in $\mathcal{D}_1$. Second, if $S_1 \subseteq S_2$ are in $\mathcal{D}_1$ and $T \in A$ then $(S_2 \setminus S_1) \cap T = (S_2 \cap T) \setminus (S_1 \cap T)$ is in $\mathcal{D}(A)$ showing that $S_2 \setminus S_1 \in \mathcal{D}_1$. Finally. if $S_n \in \mathcal{D}_1$ is a sequence increasing to $S \subseteq X$ and $T \in A$ then $S_n \cap T$ is a sequence in $\mathcal{D}(A)$ increasing to $S \cap T$. As Dynkin systems are closed under limits of increasing sequences this shows that $S \cap T \in \mathcal{D}(A)$ and therefore $S \in \mathcal{D}_1$. So $\mathcal{D}_1$ is indeed a Dynkin system. In particular, $\mathcal{D}(A) \subseteq \mathcal{D}_1$ and $S \cap T \in \mathcal{D}(A)$ for all $S \in \mathcal{D}(A)$ and $T \in A$.

We now set $\mathcal{D}_2 \equiv \{S \subseteq X : S \cap T \in \mathcal{D}(A) \text{ for every } T \in \mathcal{D}(A)\}$ which, as shown above, contains A. Also, as in the argument above for $\mathcal{D}_1$, $\mathcal{D}_2$ is a Dynkin system. Therefore, $\mathcal{D}(A)$ is contained in $\mathcal{D}_2$ and it follows that $S \cap T \in \mathcal{D}(A)$ for any $S, T \in \mathcal{D}(A)$. So $\mathcal{D}(A)$ is both a π-system and a Dynkin system.

We can now show that $\mathcal{D}(A)$ is a σ-algebra. As it is a Dynkin system, $S^c = X \setminus S \in \mathcal{D}(A)$ for every $S \in \mathcal{D}(A)$ and, as it is also a π-system, this shows that $\mathcal{D}(A)$ is an algebra of sets on X. Finally, choose any sequence $A_n \in \mathcal{D}(A)$. Then, $\bigcup_{m=1}^n A_m$ is a sequence in $\mathcal{D}(A)$ increasing to $\bigcup_n A_n$ which, as $\mathcal{D}(A)$ is Dynkin system, must be in $\mathcal{D}(A)$. So, $\mathcal{D}(A)$ is a σ-algebra and must contain $\sigma(A)$. Conversely, as $\sigma(A)$ is a Dynkin system (as it is a σ-algebra) containing A, it must also contain $\mathcal{D}(A)$. $\square$

References

[1] David Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, 1991.