proof that the outer (Lebesgue) measure of an interval is its length

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We begin with the case in which we have a bounded interval, say $[a, b]$. Since the open interval $(a - \varepsilon, b + \varepsilon)$ contains $[a, b]$ for each positive number ε, we have $m^*[a, b] \leq b - a + 2\varepsilon$. But since this is true for each positive ε, we must have $m^*[a, b] \leq b - a$. Thus we only have to show that $m^*[a, b] \geq b - a$; for this it suffices to show that if $\{I_n\}$ is a countable open cover by intervals of $[a, b]$, then

$$\sum l(I_n) \geq b - a.$$

By the Heine-Borel theorem, any collection of open intervals $[a, b]$ contains a finite subcollection that also cover $[a, b]$ and since the sum of the lengths of the finite subcollection is no greater than the sum of the original one, it suffices to prove the inequality for finite collections $\{I_n\}$ that cover $[a, b]$. Since a is contained in $\bigcup I_n$, there must be one of the I_n's that contains a. Let this be the interval (a_1, b_1). We then have $a_1 < a < b_1$. If $b_1 \leq b$, then $b_1 \in [a, b]$, and since $b_1 \notin (a_1, b_1)$, there must be an interval (a_2, b_2) in the collection $\{I_n\}$ such that $b_1 \in (a_2, b_2)$, that is $a_2 < b_1 < b_2$. Continuing in this fashion, we obtain a sequence $(a_1, b_1), \ldots, (a_k, b_k)$ from the collection $\{I_n\}$ such that $a_i < b_{i-1} < b_i$. Since $\{I_n\}$ is a finite collection our process must terminate with some interval (a_k, b_k). But it terminates only if $b \in (a_k, b_k)$, that is if $a_k < b < b_k$. Thus

$$\begin{aligned}
\sum l(I_n) &\geq \sum l(a_i, b_i) \\
&= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1) \\
&= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \cdots - (a_2 - b_1) - a_1 \\
&> b_k - a_1,
\end{aligned}$$

since $a_i < b_{i-1}$. But $b_k > b$ and $a_1 < a$ and so we have $b_k - a_1 > b - a$, whence $\sum l(I_n) > b - a$. This shows that $m^*[a, b] = b - a$.

If I is any finite interval, then given $\varepsilon > 0$, there is a closed interval$J \subset I$ such that $l(J) > l(I) - \varepsilon$. Hence

$$l(I) - \varepsilon < l(J) = m^*J \leq m^*I \leq m^*\overline{I} = l(\overline{I}) = l(I),$$

where by $\overline{I}$ we the topological closure of I. Thus for each $\varepsilon > 0$, we have $l(I) - \varepsilon < m^*I \leq l(I)$, and so $m^*I = l(I)$.

If now I is an unbounded interval, then given any real number Δ, there is a closed interval $J \subset I$ with $l(J) = \Delta$. Hence $m^*I \geq m^*J = l(J) = \Delta$. Since $m^*I \geq \Delta$ for each Δ, it follows $m^*I = \infty = l(I)$.

References

Royden, H. L. *Real analysis. Third edition*. Macmillan Publishing Company, New York, 1988.