equivalent conditions for uniform integrability

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Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and S be a bounded subset of $L^1(\Omega, \mathcal{F}, \mu)$. That is, $\int |f|\,d\mu$ is bounded over $f \in S$. Then, the following are equivalent.

1. For every $\epsilon > 0$ there is a $\delta > 0$ so that
$$\int_A |f|\,d\mu < \epsilon$$
for all $f \in S$ and $\mu(A) < \delta$.

2. For every $\epsilon > 0$ there is a $K > 0$ satisfying
$$\int_{|f|>K} |f|\,d\mu < \epsilon$$
for all $f \in S$.

3. There is a measurable function $\Phi\colon \mathbb{R} \to [0, \infty)$ such that $\Phi(x)/|x| \to \infty$ as $|x| \to \infty$ and
$$\int \Phi(f)\,d\mu$$
is bounded over all $f \in S$. Moreover, the function Φ can always be chosen to be symmetric and convex.

So, for bounded subsets of L^1, either of the above properties can be used to define uniform integrability. Conversely, when the measure space is finite, then conditions (**??**) and (**??**) are easily shown to imply that S is bounded in L^1.

To show the equivalence of these statements, let us suppose that $\int |f|\,d\mu < L$ for $f \in S$.

(??) implies (??)

For $\epsilon > 0$, property (**??**) gives a $\delta > 0$ so that $\int_A |f|\,d\mu < \epsilon$ whenever $f \in S$ and $\mu(A) < \delta$. Choosing $K > L/\delta$, Markov's inequality gives
$$\mu(|f| > K) \le K^{-1} \int |f|\,d\mu \le L/K < \delta$$
and, therefore, $\int_{|f|>K} |f|\,d\mu < \epsilon$.

(??) implies (??)

For each $n = 1, 2, \ldots$, property (**??**) gives a K_n satisfying

$$\int (|f| - K_n)_+ \, d\mu \leq \int_{|f|>K_n} |f| \, d\mu \leq 2^{-n}.$$

Without loss of generality, the K_n can be chosen to be increasing to infinity, so we can define $\Phi(x) = \sum_n (|x| - K_n)_+$. Then,

$$\int \Phi(f) \, d\mu = \sum_n \int (|f| - K_n)_+ \, d\mu \leq \sum_n 2^{-n} = 1.$$

(??) implies (??)

First, suppose that $\int \Phi(f) \, d\mu < M$ for $f \in S$. For $\epsilon > 0$, the condition that $\Phi(x)/|x| \to \infty$ as $|x| \to \infty$ gives a $K > 0$ such that $\Phi(x)/|x| \geq 2M/\epsilon$ whenever $|x| > K$. Setting $\delta = \epsilon/2K$,

$$\begin{aligned}
\int_A |f| \, d\mu &\leq \int_{|f|>K} |f| \, d\mu + K\mu(A) \\
&< (\epsilon/2M) \int_{|f|>K} \Phi(f) \, d\mu + K\delta \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

whenever $\mu(A) < \delta$ and $f \in S$.