proof of pseudoparadox in measure theory

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Since this paradox depends crucially on the axiom of choice, we will place the application of this controversial axiom at the head of the proof rather than bury it deep within the bowels of the argument.

One can define an equivalence relation $\sim$ on $\mathbb{R}$ by the condition that $x \sim y$ if and only if $x - y$ is rational. By the Archimedean property of the real line, for every $x \in \mathbb{R}$ there will exist a number $y \in [0, 1)$ such that $y \sim x$. Therefore, by the axiom of choice, there will exist a choice function $f \colon \mathbb{R} \to [0, 1)$ such that $f(x) = f(y)$ if and only if $x \sim y$.

We shall use our choice function f to exhibit a bijection between $[0, 1)$ and $[0, 2)$. Let w be the "wrap-around function" which is defined as $w(x) = x$ when $x \geq 0$ and $w(x) = x + 2$ when $x < 0$. Define $g \colon [0, 1) \to \mathbb{R}$ by

$$g(x) = w(2x - f(x))$$

From the definition, it is clear that, since $x \sim f(x)$ and $w(x) \sim x$, $g(x) \sim x$. Also, it is easy to see that g maps $[0, 1)$ into $[0, 2)$. If $f(x) \leq 2x$, then $g(x) = 2x - f(x) \leq 2x < 2$. On the other hand, if $f(x) > 2x$, then $g(x) = 2x + 2 - f(x)$. Since $2x - f(x)$ is strictly negative, $g(x) < 2$. Since $f(x) < 1$, $g(x) > 0$.

Next, we will show that g is injective. Suppose that $g(x) = g(y)$ and $x < y$. By what we already observed, $x \sim y$, so $y - x$ is a non-negative rational number and $f(x) = f(y)$. There are 3 possible cases: 1) $f(x) \leq 2x \leq 2y$ In this case, $g(x) = g(y)$ implies that $2x - f(x) = 2y - f(x)$, which would imply that $x = y$. 2) $2x < f(x) < 2y$ In this case, $g(x) = g(y)$ implies $2 + 2x - f(x) = 2y - f(x)$ which, in turn, implies that $y = x + 1$, which is impossible if both x and y belong to $[0, 1)$. 3) $2x < 2y < f(x)$ In this case, $g(x) = g(y)$ implies that $2x + 2 - f(x) = 2y + 2 - f(x)$, which would imply that $x = y$. The only remaining possibility is that $x = y$, so $g(x) = g(y)$ implies that $x = y$.

Next, we show that g is surjective. Pick a number y in $[0, 2)$. We need to find a number $x \in [0, 1)$ such that $w(2x - f(y)) = y$. If $f(y) + y < 2$, we can choose $x = (f(y) + y)/2$. If $2 \leq f(y) + y$, we can choose $x = (f(y) + y)/2 - 1$.

Having shown that g is a bijection between $[0, 1)$ and $[0, 2)$, we shall now complete the proof by examining the action of g. As we already noted, $g(x) - x$ is a rational number. Since the rational numbers are countable, we can arrange them in a series $r_0, r_1, r_2 \ldots$ such that no number is counted twice. Define $A_i \subset C_1$ as

$$A_i = \{x \in [0, 1) \mid g(x) = r_i\}$$

It is obvious from this definition that the A_i are mutually disjoint. Furthermore, $\bigcup_{i=1}^{\infty} A_i = [0,1)$ and $\bigcup_{i=1}^{\infty} B_i = [0,2)$ where B_i is the translate of A_i by r_i.