equivalent definitions of analytic sets

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For a paved space $(X,\mathcal{F})$ the $\mathcal{F}$-`http://planetmath.org/AnalyticSet2`analytic sets can be defined as the `http://planetmath.org/GeneralizedCartesianProduct`projections of sets in $(\mathcal{F}\times\mathcal{K})_{\sigma\delta}$ onto X, for compact paved spaces $(K,\mathcal{K})$. There are, however, many other equivalent definitions, some of which we list here.

In conditions **??** and **??** of the following theorem, Baire space $\mathcal{N}=\mathbb{N}^{\mathbb{N}}$ is the collection of sequences of natural numbers together with the product topology. In conditions **??** and **??**, Y can be any uncountable Polish space. For example, we may take $Y=\mathbb{R}$ with the standard topology.

Theorem. *Let $(X,\mathcal{F})$ be a paved space such that $\mathcal{F}$ contains the empty set, and A be a subset of X. The following are equivalent.*

1. *A is $\mathcal{F}$-analytic.*
2. *There is a closed subset S of $\mathcal{N}$ and $\theta\colon\mathbb{N}^2\to\mathcal{F}$ such that*
$$A=\bigcup_{s\in S}\bigcap_{n=1}^{\infty}\theta\left(n,s_n\right).$$
3. *There is a closed subset S of $\mathcal{N}$ and $\theta\colon\mathbb{N}\to\mathcal{F}$ such that*
$$A=\bigcup_{s\in S}\bigcap_{n=1}^{\infty}\theta\left(s_n\right).$$
4. *A is the result of a Souslin scheme on $\mathcal{F}$.*
5. *A is the projection of a set in $(\mathcal{F}\times\mathcal{G})_{\sigma\delta}$ onto X, where $\mathcal{G}$ is the collection of closed subsets of Y.*
6. *A is the projection of a set in $(\mathcal{F}\times\mathcal{K})_{\sigma\delta}$ onto X, where $\mathcal{K}$ is the collection of compact subsets of Y.*

For subsets of a measurable space, the following result gives a simple condition to be analytic. Again, the space Y can be any uncountable Polish space, and its Borel σ-algebra is denoted by $\mathcal{B}$. In particular, this result shows that a subset of the real numbers is analytic if and only if it is the projection of a Borel set from $\mathbb{R}^2$.

Theorem. *Let $(X,\mathcal{F})$ be a measurable space. For a subset A of X the following are equivalent.*

1. *A is $\mathcal{F}$-analytic.*

2. *A is the projection of an $\mathcal{F} \otimes \mathcal{B}$-measurable subset of $X \times Y$ onto X.*

We finally state some equivalent definitions of analytic subsets of a Polish space. Again, $\mathcal{N}$ denotes Baire space and Y is any uncountable Polish space.

Theorem. *For a nonempty subset A of a Polish space X the following are equivalent.*

1. *A is $\mathcal{F}$-`http://planetmath.org/AnalyticSet2`analytic.*

2. *A is the projection of a closed subset of $X \times \mathcal{N}$ onto X.*

3. *A is the projection of a Borel subset of $X \times Y$ onto X.*

4. *A is the `http://planetmath.org/DirectImage`image of a continuous function $f\colon Z \to X$ for some Polish space Z.*

5. *A is the image of a continuous function $f\colon \mathcal{N} \to X$.*

6. *A is the image of a Borel measurable function $f\colon Y \to X$.*