projections of analytic sets are analytic

Canonical name	ProjectionsOfAnalyticSetsAreAnalytic
Date of creation	2013-03-22 18:46:21
Last modified on	2013-03-22 18:46:21
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	8
Author	gel (22282)
Entry type	Theorem
Classification	msc 28A05
Related topic	MeasurableProjectionTheorem

Projections along compact paved spaces

Given sets X and K, the projection map $\pi_X\colon X\times K\to X$ is defined by $\pi_X(x,y)=x$. An important property of http://planetmath.org/AnalyticSet2analytic sets is that they are stable under projections.

Theorem 1. *Let $(X,\mathcal{F})$ be a http://planetmath.org/PavedSpacepaved space, $(K,\mathcal{K})$ be a http://planetmath.org/PavedSpacecompact paved space and $\pi_X\colon X\times K\to X$ be the projection map.*

If $S\subseteq X\times K$ is $\mathcal{F}\times\mathcal{K}$-analytic then $\pi_X(S)$ is $\mathcal{F}$-analytic.

The proof of this follows easily from the definition of analytic sets. First, there is a compact paved space $(K',\mathcal{K}')$ and a set $T\in(\mathcal{F}\times\mathcal{K}\times\mathcal{K}')_{\sigma\delta}$ such that $S=\pi_{X\times K}(T)$. Then,

$$\pi_X(S)=\pi_X\left(\pi_{X\times K}(T)\right)=\pi_X(T).$$

However, $(K\times K',\mathcal{K}\times\mathcal{K}')$ is a compact paved space (see http://planetmath.org/ProductsOfComp of compact pavings are compact), which shows that $\pi_X(S)$ satisfies the definition of $\mathcal{F}$-analytic sets.

Projections along Polish spaces

Theorem **??** above can be used to prove the following result for projections from the product of a measurable space and a Polish space. For http://planetmath.org/SigmaAlgebraσ-algebras $\mathcal{F}$ and $\mathcal{B}$, we use the notation $\mathcal{F}\otimes\mathcal{B}$ for the http://planetmath.org/ProductSigmaAlgebraproduct σ-algebra, in order to distinguish it from the product paving $\mathcal{F}\times\mathcal{B}$.

Theorem 2. *Let $(X,\mathcal{F})$ be a measurable space and Y be a Polish space with http://planetmath.org/BorelSigmaAlgebraBorel σ-algebra $\mathcal{B}$.*

If $S\subseteq X\times Y$ is $\mathcal{F}\otimes\mathcal{B}$-analytic, then its projection onto X is $\mathcal{F}$-analytic.

An immediate consequence of this is the measurable projection theorem.

Although Theorem **??** applies to arbitrary Polish spaces, it is enough to just consider the case where Y is the space of real numbers $\mathbb{R}$ with the standard topology. Indeed, all Polish spaces are Borel isomorphic to either the real numbers or a discrete subset of the reals (see Polish spaces up to Borel isomorphism), so the general case follows from this.

If $Y=\mathbb{R}$, then the Borel σ-algebra is generated by the compact paving $\mathcal{K}$ of closed and bounded intervals. The collection $a(\mathcal{F}\times\mathcal{K})$ of analytic sets is closed under countable unions and countable intersections so, by the monotone class theorem, includes the product σ-algebra $\mathcal{F} \otimes \mathcal{B}$. Then, as the analytic sets define a closure operator,

$$a(\mathcal{F} \otimes \mathcal{B}) \subseteq a(a(\mathcal{F} \times \mathcal{K})) = a(\mathcal{F} \times \mathcal{K}).$$

Thus every $\mathcal{F} \otimes \mathcal{B}$-analytic set is $\mathcal{F} \times \mathcal{K}$-analytic, and the result follows from Theorem **??**.