Lebesgue differentiation theorem

Canonical name	LebesgueDifferentiationTheorem
Date of creation	2013-03-22 13:27:36
Last modified on	2013-03-22 13:27:36
Owner	Koro (127)
Last modified by	Koro (127)
Numerical id	9
Author	Koro (127)
Entry type	Theorem
Classification	msc 28A15

Let f be a locally integrable function on $\mathbb{R}^n$ with Lebesgue measure m, i.e. $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. *Lebesgue's differentiation theorem* basically says that for almost every x, the averages

$$\frac{1}{m(Q)} \int_Q |f(y) - f(x)| dy$$

converge to 0 when Q is a cube containing x and $m(Q) \to 0$.

Formally, this means that there is a set $N \subset \mathbb{R}^n$ with $\mu(N) = 0$, such that for every $x \notin N$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for each cube Q with $x \in Q$ and $m(Q) < \delta$, we have

$$\frac{1}{m(Q)} \int_Q |f(y) - f(x)| dy < \varepsilon.$$

For $n = 1$, this can be restated as an analogue of the fundamental theorem of calculus for Lebesgue integrals. Given a $x_0 \in \mathbb{R}$,

$$\frac{d}{dx} \int_{x_0}^{x} f(t) dt = f(x)$$

for almost every x.