$L^\infty(X,\mu)$

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Let X be a nonempty set and $\mathcal{A}$ be a σ-algebra on X. Also, let μ be a non-negative measure defined on $\mathcal{A}$. Two complex valued functions f and g are said to be equal almost everywhere on X (denoted as $f = g$ a.e. if $\mu\{x \in X : f(x) \neq g(x)\} = 0$. The relation of being equal almost everywhere on X defines an equivalence relation. It is a common practice in the integration theory to denote the equivalence class containing f by f itself. It is easy to see that if f_1, f_2 are equivalent and g_1, g_2 are equivalent, then $f_1 + g_1, f_2 + g_2$ are equivalent, and $f_1 g_1, f_2 g_2$ are equivalent. This naturally defines addition and multiplication among the equivalent classes of such functions. For a measureable $f\colon X \to \mathbb{C}$, we define

$$\|f\|_{\text{ess}} = \inf\{M > 0\colon \mu\{x : |f(x)| > M\} = 0\},$$

called the essential supremum of $|f|$ on X. Now we define,

$$L^\infty(X, \mu) = \{f : X \to \mathbb{C} : \|f\|_{\text{ess}} < \infty\}.$$

Here the elements of $L^\infty(X, \mu)$ are equivalence classes.

Properties of $L^\infty(X, \mu)$

1. The space $L^\infty(X, \mu)$ is a normed linear space with the norm $\|\cdot\|_{\text{ess}}$. Also, the metric defined by the norm is complete, making $L^\infty(X, \mu)$, a Banach space.

2. $L^\infty(X, \mu)$ is the dual of $L^1(X, \mu)$ if X is σ-finite.

3. $L^\infty(X, \mu)$ is closed under pointwise multiplication, and with this multiplication it becomes an algebra. Further, $L^\infty(X, \mu)$ is also a `http://planetmath.org/CAlgeb` algebra with the involution defined by $f^*(x) = \overline{f(x)}$. Since this C^*-algebra is also a dual of some Banach space, it is called von Neumann algebra.