products of compact pavings are compact

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Suppose that $(K_i, \mathcal{K}_i)$ is a paved space for each i in index set I. The http://planetmath.org/GeneralizedCartesianProductproduct $\prod_{i\in I} K_i$ is the set of all functions $x\colon I \to \bigcup_i K_i$ such that $x_i \in K_i$ for each i, and the product of subsets $S_i \subseteq K_i$ is

$$\prod_{i\in I} S_i = \left\{ x \in \prod_{i\in I} K_i \colon x_i \in S_i \text{ for each } i \in I \right\}.$$

Then, the product paving is defined by

$$\prod_{i\in I} \mathcal{K}_i = \left\{ \prod_{i\in I} S_i \colon S_i \in \mathcal{K}_i \text{ for each } i \in I \right\}.$$

Theorem 1. *Let $(K_i, \mathcal{K}_i)$ be compact paved spaces for $i \in I$. Then, $\prod_i \mathcal{K}_i$ is a compact paving on $\prod_i K_i$.*

Note that this result is a version of Tychonoff's theorem applying to paved spaces and, together with the fact that all compact pavings are closed subsets of a compact space, is easily seen to be equivalent to Tychonoff's theorem.

Theorem **??** is simple to prove directly. Suppose that $\{A_j \colon j \in J\}$ is a subset of $\prod_i \mathcal{K}_i$ satisfying the finite intersection property. Writing $A_j = \prod_{i\in I} S_{ij}$ for $S_{ij} \in \mathcal{K}_i$ gives

$$\bigcap_{j\in J'} A_j = \prod_{i\in I} \left(\bigcap_{j\in J'} S_{ij} \right) \tag{1}$$

for any $J' \subseteq J$. By the finite intersection property, this is nonempty whenever J' is finite, so $\bigcap_{j\in J'} S_{ij} \neq \emptyset$. Consequently, $\{S_{ij} \colon j \in J\} \subseteq \mathcal{K}_i$ satisfies the finite intersection property and, by compactness of $\mathcal{K}_i$, the intersection $\bigcap_{j\in J} S_{ij}$ is nonempty. So equation (**??**) shows that $\bigcap_{j\in J} A_j$ is nonempty.