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# Haar integral

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| Canonical name | HaarIntegral |
| Date of creation | 2013-03-22 13:39:56 |
| Last modified on | 2013-03-22 13:39:56 |
| Owner | rspuzio (6075) |
| Last modified by | rspuzio (6075) |
| Numerical id | 9 |
| Author | rspuzio (6075) |
| Entry type | Definition |
| Classification | msc 28C05 |
| Defines | normalized Haar integral |

Let $\Gamma$ be a locally compact topological group and $\mathcal{C}$ be the algebra of all continuous real-valued functions on $\Gamma$ with compact support. In addition we define $\mathcal{C}^+$ to be the set of non-negative functions that belong to $\mathcal{C}$. The *Haar integral* is a real linear map $I$ of $\mathcal{C}$ into the field of the real number for $\Gamma$ if it satisfies:

- $I$ is not the zero map
- $I$ only takes non-negative values on $\mathcal{C}^+$
- $I$ has the following property $I(\gamma \cdot f) = I(f)$ for all elements $f$ of $\mathcal{C}$ and all element $\gamma$ of $\Gamma$.

The *Haar integral* may be denoted in the following way *(there are also other ways)*:

$$\int_{\gamma \in \Gamma} f(\gamma) \text{ or } \int_\Gamma f \text{ or } \int_\Gamma f d\gamma \text{ or } I(f)$$

The following are necessary and sufficient conditions for the existence of a unique Haar integral: There is a real-valued function $I^+$

1. *(Linearity).*$I^+(\lambda f + \mu g) = \lambda I^+(f) + \mu I^+(g)$ where $f, g \in \mathcal{C}^+$ and $\lambda, \mu \in \mathbb{R}_+$.
2. *(Positivity).* If $f(\gamma) \geq 0$ for all $\gamma \in \Gamma$ then $I^+(f(\gamma)) \geq 0$.
3. *(Translation-Invariance).* $I(f(\delta\gamma)) = I(f(\gamma))$ for any fixed $\delta \in \Gamma$ and every $f$ in $\mathcal{C}^+$.

An additional property is if $\Gamma$ is a compact group then the Haar integral has right translation-invariance: $\int_{\gamma \in \Gamma} f(\gamma\delta) = \int_{\gamma \in \Gamma} f(\gamma)$ for any fixed $\delta \in \Gamma$. In addition we can define *normalized Haar integral* to be $\int_\Gamma 1 = 1$ since $\Gamma$ is compact, it implies that $\int_\Gamma 1$ is finite.
*(The proof for existence and uniqueness of the Haar integral is presented in [**?**] on page 9.)*

*(the information of this entry is in part quoted and paraphrased from [**?**])*

# References

[GSS] Golubsitsky, Martin. Stewart, Ian. Schaeffer, G. David.: Singularities and Groups in Bifurcation Theory *(Volume II)*. Springer-Verlag, New York, 1988.

[HG] Gochschild, G.: The Structure of Lie Groups. Holden-Day, San Francisco, 1965.