proof of monotone convergence theorem

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It is enough to prove the following

Theorem 1 *Let (X, μ) be a measurable space and let $f_k \colon X \to \mathbb{R} \cup \{+\infty\}$ be a monotone increasing sequence of positive measurable functions (i.e. $0 \leq f_1 \leq f_2 \leq \dots$). Then $f(x) = \lim_{k\to\infty} f_k(x)$ is measurable and*

$$\lim_{n\to\infty} \int_X f_k \, d\mu = \int_X f(x) \, d\mu.$$

First of all by the monotonicity of the sequence we have

$$f(x) = \sup_k f_k(x)$$

hence we know that f is measurable. Moreover being $f_k \leq f$ for all k, by the monotonicity of the integral, we immediately get

$$\sup_k \int_X f_k \, d\mu \leq \int_X f(x) \, d\mu.$$

So take any simple measurable function s such that $0 \leq s \leq f$. Given also $\alpha < 1$ define

$$E_k = \{x \in X \colon f_k(x) \geq \alpha s(x)\}.$$

The sequence E_k is an increasing sequence of measurable sets. Moreover the union of all E_k is the whole space X since $\lim_{k\to\infty} f_k(x) = f(x) \geq s(x) > \alpha s(x)$. Moreover it holds

$$\int_X f_k \, d\mu \geq \int_{E_k} f_k \, d\mu \geq \alpha \int_{E_k} s \, d\mu.$$

Since s is a simple measurable function it is easy to check that $E \mapsto \int_E s \, d\mu$ is a measure and hence

$$\sup_k \int_X f_k \, d\mu \geq \alpha \int_X s \, d\mu.$$

But this last inequality holds for every $\alpha < 1$ and for all simple measurable functions s with $s \leq f$. Hence by the definition of Lebesgue integral

$$\sup_k \int_X f_k \, d\mu \geq \int_X f \, d\mu$$

which completes the proof.