proof of dominated convergence theorem

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It is not difficult to prove that f is measurable. In fact we can write

$$f(x) = \sup_n \inf_{k \geq n} f_k(x)$$

and we know that measurable functions are closed under the sup and inf operation.

Consider the sequence $g_n(x) = 2\Phi(x) - |f(x) - f_n(x)|$. Clearly g_n are nonnegative functions since $f - f_n \leq 2\Phi$. So, applying Fatou's Lemma, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_X |f - f_n|\, d\mu &\leq \limsup_{n\to\infty} \int_X |f - f_n|\, d\mu \\
&= -\liminf_{n\to\infty} \int_X -|f - f_n|\, d\mu \\
&= \int_X 2\Phi\, d\mu - \liminf_{n\to\infty} \int_X 2\Phi - |f - f_n|\, d\mu \\
&\leq \int_X 2\Phi\, d\mu - \int_X 2\Phi - \limsup_{n\to\infty} |f - f_n|\, d\mu \\
&= \int_X 2\Phi\, d\mu - \int_X 2\Phi\, d\mu = 0.
\end{aligned}$$