Cantor set

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The Cantor set C is the canonical example of an uncountable set of measure zero. We construct C as follows.

Begin with the unit interval $C_0 = [0, 1]$, and remove the open segment $R_1 := (\frac{1}{3}, \frac{2}{3})$ from the middle. We define C_1 as the two remaining pieces

$$C_1 := C_0 \setminus R_1 = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right] \tag{1}$$

Now repeat the process on each remaining segment, removing the open set

$$R_2 := \left(\frac{1}{9}, \frac{2}{9}\right) \bigcup \left(\frac{7}{9}, \frac{8}{9}\right) \tag{2}$$

to form the four-piece set

$$C_2 := C_1 \setminus R_2 = \left[0, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, 1\right] \tag{3}$$

Continue the process, forming $C_3, C_4, \ldots$ Note that C_k has 2^k pieces.

Also note that at each step, the endpoints of each closed segment will stay in the set forever—e.g., the point $\frac{2}{3}$ isn't touched as we remove sets.

The *Cantor set* is defined as

$$C := \bigcap_{k=1}^{\infty} C_k = C_0 \setminus \bigcup_{n=1}^{\infty} R_n \tag{4}$$

Cardinality of the Cantor set To establish cardinality, we want a bijection between the Cantor set and some set whose cardinality we know .

Start at C_1, which has two pieces. Mark the left-hand segment "0" and the right-hand segment "1". Then continue to C_2, and consider only the leftmost pair. Again, mark the segments "0" and "1", and do the same for the rightmost pair.

Keep doing this all the way down the C_k, starting at the left side and marking the segments 0, 1, 0, 1, 0, 1 as you encounter them, until you've labeled the entire Cantor set.

Now, pick a path through the tree starting at C_0 and going left-left-right-left. . . and so on. Mark a decimal point for C_0, and record the zeros and ones as you proceed. Each path has a unique number based on your decision at each step. Every point in the Cantor set will have a unique address dependent solely on the pattern of lefts and rights, 0's and 1's, required to reach it. The Cantor set therefore has the same cardinality as the set of sequences of 0's and 1's, which is $2^{\aleph_0}$, the cardinality of the continuum.

The Cantor set and ternary expansions Return, for a moment, to the earlier observation that numbers such as $\frac{1}{3}$, $\frac{2}{9}$, the endpoints of deleted intervals, are themselves never deleted. In particluar, consider the first deleted interval: the ternary expansions of its constituent numbers are precisely those that begin 0.1, and proceed thence with at least one non-zero "ternary" digit further along. Note also that the point $\frac{1}{3}$, with ternary expansion 0.1, may also be written $0.0\dot{2}$ (or $0.0\bar{2}$), which has no ternary digit 1. Similar descriptions apply to further deleted intervals. The result is that the Cantor set is precisely those numbers in the set $[0, 1]$ that have a ternary expansion contains no digits 1.

Measure of the Cantor set Let μ be Lebesgue measure. The measure of the sets R_k that we remove during the construction of the Cantor set are

$$\mu(R_1) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \tag{5}$$

$$\mu(R_2) = \left(\frac{2}{9} - \frac{1}{9}\right) + \left(\frac{8}{9} - \frac{7}{9}\right) = \frac{2}{9} \tag{6}$$

$$\vdots \tag{7}$$

$$\mu(R_k) = \sum_{n=1}^{k} \frac{2^{n-1}}{3^n} \tag{8}$$

Note that the R's are disjoint, which will allow us to sum their measures without worry. In the limit $k \to \infty$, this gives us

$$\mu\left(\bigcup_{n=1}^{\infty} R_n\right) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1. \tag{9}$$

But we have $\mu(C_0) = 1$ as well, so this means

$$\mu(C) = \mu\left(C_0 \setminus \bigcup_{n=1}^{\infty} R_n\right) = \mu(C_0) - \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 - 1 = 0. \tag{10}$$

Thus we have seen that the measure of C is zero (though see below for more on this topic). How many points are there in C? Lots, as we shall see.

So we have a set of measure zero (very tiny) with uncountably many points (very big). This non-intuitive result is what makes Cantor sets so interesting.

Cantor sets with positive measure Clearly, Cantor sets can be constructed for all sorts of "removals"—we can remove middle halves, or thirds, or any amount $\frac{1}{r}, r > 1$ we like. All of these Cantor sets have measure zero, since at each step n we end up with

$$L_n = \left(1 - \frac{1}{r}\right)^n \tag{11}$$

of what we started with, and $\lim_{n\to\infty} L_n = 0$ for any $r > 1$.

However, it is possible to construct Cantor sets with positive measure as well; the key is to remove less and less as we proceed. These Cantor sets have the same "shape" (topology) as the Cantor set we first constructed, and the same cardinality, but a different "size."

Again, start with the unit interval for C_0, and choose a number $0 < p < 1$. Let

$$R_1 := \left(\frac{2-p}{4}, \frac{2+p}{4}\right) \tag{12}$$

which has length (measure) $\frac{p}{2}$. Again, define $C_1 := C_0 \setminus R_1$. Now define

$$R_2 := \left(\frac{2-p}{16}, \frac{2+p}{16}\right) \bigcup \left(\frac{14-p}{16}, \frac{14+p}{16}\right) \tag{13}$$

which has measure $\frac{p}{4}$. Continue as before, such that each R_k has measure $\frac{p}{2^k}$; note again that all the R_k are disjoint. The resulting Cantor set has measure

$$\mu\left(C_0 \setminus \bigcup_{n=1}^{\infty} R_n\right) = 1 - \sum_{n=1}^{\infty} \mu(R_n) = 1 - \sum_{n=1}^{\infty} p\ 2^{-n} = 1 - p > 0.$$

Thus we have a whole family of Cantor sets of positive measure to accompany their vanishing brethren.