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# proof of compact pavings are closed subsets of a compact space

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Let $(K, \mathcal{K})$ be a http://planetmath.org/paved spacecompact paved space. We use the http://planetmath.org/EveryFilterIsContainedInAnUltrafilterultrafilter lemma to show that there is a compact paving $\mathcal{K}'$ containing $\mathcal{K}$ that is closed under arbitrary intersections and finite unions.

We first show that the paving $\mathcal{K}_1$ consisting of all finite unions of elements of $\mathcal{K}$ is compact. Let $\mathcal{F} \subseteq \mathcal{K}_1$ satisfy the finite intersection property. It then follows that the collection of finite intersections of $\mathcal{F}$ is a http://planetmath.org/Filterfilter. The ultrafilter lemma says that $\mathcal{F}$ is contained in an ultrafilter $\mathcal{U}$.

By definition, the ultrafilter satisfies the finite intersection property. So, the compactness of $\mathcal{K}$ implies that $\mathcal{F}' \equiv \mathcal{U} \cap \mathcal{K}$ has nonempty intersection. Also, every element $S$ of $\mathcal{F}$ is a union of finitely many elements of $\mathcal{K}$, one of which must be in $\mathcal{U}$ (see http://planetmath.org/AlternativeCharacterizationOfUltrafilter characterization of ultrafilter). In particular, $S$ contains the intersection of $\mathcal{F}'$ and,

$$\bigcap \mathcal{F} \supseteq \bigcap \mathcal{F}' \neq \emptyset.$$

Consequently, $\mathcal{K}_1$ is compact.

Finally, we let $\mathcal{K}'$ be the set of arbitrary intersections of $\mathcal{K}_1$. This is closed under all arbitrary intersections and finite unions. Furthermore, if $\mathcal{F} \subseteq \mathcal{K}'$ satisfies the finite intersection property then so does

$$\mathcal{F}' \equiv \{A \in \mathcal{K}_1 \colon B \subseteq A \text{ for some } B \in \mathcal{F}\}.$$

The compactness of $\mathcal{K}_1$ gives

$$\bigcap \mathcal{F} = \bigcap \mathcal{F}' \neq \emptyset$$

as required.