uniformly integrable

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Let μ be a positive measure on a measurable space. A collection of functions $\{f_\alpha\} \subset \mathbf{L}^1(\mu)$ is *uniformly integrable*, if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \int_E f_\alpha \, d\mu \right| < \epsilon \quad \text{whenever } \mu(E) < \delta, \text{ for any } \alpha.$$

(The absolute value sign outside of the integral above may appear under the integral sign instead without affecting the definition.)

The usefulness of this definition comes from the Vitali convergence theorem, which uses it to characterize the convergence of functions in $\mathbf{L}^1(\mu)$.

Definition in probability theory

In probability , a different, and slightly stronger, definition of "uniform integrability", is more commonly used:

A collection of functions $\{f_\alpha\} \subset \mathbf{L}^1(\mu)$ is *uniformly integrable*, if for every $\epsilon > 0$, there exists $t \geq 0$ such that

$$\int_{[|f_\alpha| \geq t]} |f_\alpha| \, d\mu < \epsilon \quad \text{for every } \alpha.$$

Assuming μ is a probability measure, this definition is equivalent to the previous one together with the condition that $\int |f_\alpha| \, d\mu$ is uniformly bounded for all α.

Properties

1. If a finite number of collections are uniformly integrable, then so is their finite union.

2. A single $f \in \mathbf{L}^1(\mu)$ is always uniformly integrable.

   To see this, observe that f must be almost everywhere non-infinite. Thus $f \cdot 1_{[|f|>k]}$ goes to zero a.e. as $k \to \infty$, and it is bounded by $|f|$. Then $\int_{[|f|>k]} |f| d\mu \to 0$ by the dominated convergence theorem. Choosing k big enough so that $\int_{[|f|>k]} |f| d\mu < \epsilon$, and letting $\delta = \epsilon/k$, we have, when $\mu(E) < \delta$,

   $$\int_E |f| d\mu = \int_{E \cap [|f| \leq k]} |f| d\mu + \int_{E \cap [|f| > k]} |f| d\mu \leq k\mu(E) + \epsilon = 2\epsilon \, .$$

Examples

1. If g is an integrable function, then the collection consisting of all measurable functions f dominated by g — that is, $|f| \leq g$ — is uniformly integrable.

2. If X is a $\mathbf{L}^1$ random variable on a probability space Ω, then the set of all of its conditional expectations,

$$\{\mathbb{E}[X \mid \mathcal{G}] \colon \mathcal{G} \text{ is a } \sigma\text{-algebra of } \Omega\},$$

is always uniformly integrable.

3. If there is an unbounded increasing function $\phi\colon [0,\infty) \to [0,\infty)$ such that

$$\int |f_\alpha| \phi(|f_\alpha|)\, d\mu$$

is uniformly bounded for all α, then the collection $\{f_\alpha\}$ is uniformly integrable.

References

[1] Kai Lai Chung. *A Course in Probability Theory*, third ed. Academic Press, 2001.

[2] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications*, second ed. Wiley-Interscience, 1999.

[3] Jeffrey S. Rosenthal. *A First Look at Rigorous Probability Theory*. World Scientific, 2003.