counter-example of Fubini's theorem for the Lebesgue integral

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The following observation demonstrates the necessity of the integrability assumption in Fubini's theorem. Let

$$Q = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$$

denote the upper, right quadrant. Let $R \subset Q$ be the region in the quadrant bounded by the lines $y = x, y = x - 1$, and let let $S \subset Q$ be a similar region, but this time bounded by the lines $y = x - 1,\ y = x - 2$. Let

$$f = \chi_S - \chi_R,$$

where χ denotes a characteristic function.

Observe that the Lebesgue measure of R and of S is infinite. Hence, f is not a Lebesgue-integrable function. However for every $x \geq 0$ the function

$$g(x) = \int_0^\infty f(x, y)\, dy$$

is integrable. Indeed,

$$g(x) = \begin{cases} -x & \text{for } 0 \leq x \leq 1, \\ x - 2 & \text{for } 1 \leq x \leq 2, \\ 0 & \text{for } x \geq 2. \end{cases}$$

Similarly, for $y \geq 0$, the function

$$h(y) = \int_0^\infty f(x, y)\, dx$$

is integrable. Indeed,

$$h(y) = 0, \quad y \geq 0.$$

Hence, the values of the iterated integrals

$$\int_0^\infty g(x)\, dx = -1,$$

$$\int_0^\infty h(y)\, dy = 0,$$

are finite, but do not agree. This does not contradict Fubini's theorem because the value of the planar Lebesgue integral

$$\int_Q f(x, y)\, d\mu(x, y),$$

where $\mu(x, y)$ is the planar Lebesgue measure, is not defined.