self-similar fractals

Canonical name	SelfsimilarFractals
Date of creation	2013-03-22 16:05:12
Last modified on	2013-03-22 16:05:12
Owner	paolini (1187)
Last modified by	paolini (1187)
Numerical id	11
Author	paolini (1187)
Entry type	Definition
Classification	msc 28A80

Let (X,d) be a metric space and let $T_1,\dots,T_N$ be a finite number of contractions on X i.e. each $T_i\colon X\to X$ enjoys the property

$$d(T_i(x),T_i(y))\le \lambda_i d(x,y)$$

(T_i is λ_i-Lipschitz) for some $\lambda_i<1$.

Given a set $A\subset X$ we can define

$$T(A)=\bigcup_{i=1}^{N} T_i(A).$$

Definition 1. *A set K such that $T(K)=K$ (invariant set) is called a* self-similar fractal *with respect to the contractions* $\{T_1,\dots,T_N\}$.

The most famous example of self-similar fractal is the Cantor set. This is constructed in $X=\mathbb{R}$ with the usual Euclidean metric structure, by choosing $N=2$ contractions: $T_1(x)=x/3$, $T_2(x)=1-(1-x)/3$.

A more interesting example is the Koch curve in $X=\mathbb{R}^2$. In this case we choose $N=4$ similitudes with factor $1/3$.

By choosing other appropriate transformations one can obtain the beautiful example of the Barnsley Fern, which shows how the fractal geometry can successfully describe nature.

An important result is given by the following Theorem.

Theorem 1. *Let X be a complete metric space and let $T_1,\dots,T_N\colon X\to X$ be a given set of contractions. Then there exists one and only one non empty compact set $K\subset X$ such that $T(K)=K$.*

Notice that the empty set always satisfies the relation $T(\emptyset)=\emptyset$ and hence is not an interesting case. On the other hand, if at least one of the T_i is surjective (as happens in the examples above), then the whole set X satisfies $T(X)=X$.