Hausdorff dimension

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Defines	countable r-cover
Defines	diameter

Let Θ be a bounded subset of $\mathbb{R}^n$ let $N_\Theta(\epsilon)$ be the minimum number of balls of radius ϵ required to cover Θ. Then define the *Hausdorff dimension* d_H of Θ to be

$$d_H(\Theta) := -\lim_{\epsilon \to 0} \frac{\log N_\Theta(\epsilon)}{\log \epsilon}.$$

Hausdorff dimension is easy to calculate for simple objects like the Sierpinski gasket or a Koch curve. Each of these may be covered with a collection of scaled-down copies of itself. In fact, in the case of the Sierpinski gasket, one can take the individual triangles in each approximation as balls in the covering. At stage n, there are 3^n triangles of radius $\frac{1}{2^n}$, and so the Hausdorff dimension of the Sierpinski triangle is at most $-\frac{n \log 3}{n \log 1/2} = \frac{\log 3}{\log 2}$, and it can be shown that it is equal to $\frac{\log 3}{\log 2}$.

From some notes from Koro This definition can be extended to a general metric space X with distance function d.

Define the *diameter* $|C|$ of a bounded subset C of X to be $\sup_{x,y \in C} d(x,y)$.

Define a r-*cover* of X to be a collection of subsets C_i of X indexed by some countable set I, such that $|C_i| < r$ and $X = \cup_{i \in I} C_i$.

We also define the function

$$H_r^D(X) = \inf \sum_{i \in I} |C_i|^D$$

where the infimum is over all countable r-covers of X. The *Hausdorff dimension* of X may then be defined as

$$d_H(X) = \inf\{D \mid \lim_{r \to 0} H_r^D(X) = 0\}.$$

When X is a subset of $\mathbb{R}^n$ with any norm-induced metric, then this definition reduces to that given above.