Radon measure

Canonical name	RadonMeasure
Date of creation	2013-03-22 15:49:41
Last modified on	2013-03-22 15:49:41
Owner	ptr (5636)
Last modified by	ptr (5636)
Numerical id	16
Author	ptr (5636)
Entry type	Definition
Classification	msc 28C05
Classification	msc 28C15
Related topic	BorelMeasure
Related topic	SigmaFiniteBorelMeasureAndRelatedBorelConcepts
Defines	Radon space

Let X be a Hausdorff space. A Borel measure μ on X is said to be a *Radon measure* if it is:

1. finite on compact sets,
2. inner regular (tight), $\mu(A) = \sup\{\mu(V) \mid \text{compact } V \subset A\}$.

A finite Radon measure satisfies $\mu(A) = \inf\{\mu(G) \mid \text{open } G \supset A\}$.

Radon measures are not necessarily locally finite, although this is the case for locally compact and metric spaces. (Counterexample: spaces where only finite subsets are compact.)

A *Radon space* is a topological space on which every finite Borel measure is a Radon measure, this is the case, e.g. for Polish spaces or Hausdorff spaces that are continuous images of Polish spaces.

Radon measures are the "most important class of measures on arbitrary Hausdorff topological spaces" (König [?], p.xiv) and formed the base of the development of integration theory by Bourbaki and Schwartz. In particular for locally compact spaces one often *defines* Radon measures as linear functionals μ on the space $C^c(X)$ of continuous functions with compact support ('Riesz representation definition'). Berg *et al.* give the following summary [?], p. 62*f.*:

Given the finctional μ one defines *set functions*, in fact Borel measures, the *outer measure* μ^* and the *essential outer measure* $\mu^\bullet$ given by

$$
\begin{aligned}
{}^*(G) &= \sup\{\mu(f) | f \in C^c(X), 0 \leq f \leq 1_G\} \text{ for open } G \subset X \text{ and} && (1)\\
\mu^*(A) &= \inf\{\mu^*(A \cap K) | K \in \mathfrak{K}(X)\} \text{ for } A \subset X, && (2)\\
\mu^\bullet(A) &= \sup\{\mu^*(A \cap K) | K \in \mathfrak{K}(X)\} \text{ for } A \subset X. && (3)
\end{aligned}
$$

$\mu^\bullet$ is a Radon measure in our sense, while μ^* is not always Radon. For locally compact and σ-compact spaces, however, both coincide (on the Borel algebra) and are equivalent to our Radon measure. For general Hausdorff spaces, Bourbaki introduces $W^*(A) = \sup\{(W_K)(A \cap K) | K \in \mathfrak{K}(X)\}$, where W, called a *Radon premeasure*, associates a Radon measure W_K to each compact $K \subset X$, with $W_K | L = W_L, L \in \mathfrak{K}$. This is a Radon measure (on Borel sets), Bourbaki, however, calls it only so if it is in addition locally finite.

Consider now Borel measures $\nu : \mathfrak{B} \mapsto [0, \infty]$ which are

- finite on compact sets, $\nu | \mathfrak{K} < \infty$,

- inner regular on the open sets, $\nu(G) = \sup\{\nu(K)|K \subset G\}$ for G open, and K compact,

- outer regular, $\nu(B) = \inf\{\nu(G)|B \subset G$ for open G and Borel B.

then the measures ν correspond bijectively to locally finite Radon measures μ on X.

References

[1] Christian Berg, Jens Peter Reus, Paul Ressel: Harmonic analysis on semigroups. – Berlin, 1984 (Graduate Texts in Mathematics; 100)

[2] Heinz König: Measure and integration : an advanced course in basic procedures and applications.– Berlin, 1997.