extending a capacity to a Cartesian product

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A capacity on a set X can be extended to a set function on a Cartesian product $X \times K$ simply by projecting any subset onto X, and then applying the original capacity.

Theorem. *Suppose that $(X, \mathcal{F})$ is a paved space such that $\mathcal{F}$ is closed under finite unions and finite intersections, and that $(K, \mathcal{K})$ is a compact paved space. Define $\mathcal{G}$ to be the closure under finite unions and finite intersections of the paving $\mathcal{F} \times \mathcal{K}$ on $X \times K$.*

If I is an $\mathcal{F}$-capacity and $\pi_X \colon X \times K \to X$ is the projection map, we can form the composition

$$
\begin{aligned}
&I \circ \pi_X \colon \mathcal{P}(X \times K) \to \mathbb{R}, \\
&I \circ \pi_X(S) = I(\pi_X(S)).
\end{aligned}
$$

Then $\pi_X(S) \in \mathcal{F}_\delta$ for any $S \in \mathcal{G}_\delta$, and $I \circ \pi_X$ is a $\mathcal{G}$-capacity.

This result justifies looking at capacities when considering projections from the Cartesian product $X \times K$ onto X. We see that the property of being a capacity is preserved under composing with such projections. However, additivity of set functions is not preserved, so the corresponding result would not be true if "capacity" was replaced by "measure" or "outer measure".

Recall that if $S \subseteq X \times K$ is $(\mathcal{G}, I \circ \pi_X)$-capacitable then, for any $\epsilon > 0$, there is an $A \in \mathcal{G}_\delta$ such that $A \subseteq S$ and $I \circ \pi_X(A) > I \circ \pi_X(S) - \epsilon$. However, $\pi_X(A) \subseteq \pi_X(S)$ and, by the above theorem, $\pi_X(A) \in \mathcal{F}_\delta$. This has the following consequence.

Lemma. *Let $S \subseteq X \times K$ be $(\mathcal{G}, I \circ \pi_X)$-capacitable. Then, $\pi_X(S)$ is $(\mathcal{F}, I)$-capacitable.*