Radon-Nikodym theorem

Canonical name	RadonNikodymTheorem
Date of creation	2013-03-22 13:26:15
Last modified on	2013-03-22 13:26:15
Owner	Koro (127)
Last modified by	Koro (127)
Numerical id	9
Author	Koro (127)
Entry type	Theorem
Classification	msc 28A15
Related topic	AbsolutelyContinuous
Related topic	BoundedLinearFunctionalsOnLpmu
Related topic	MartingaleProofOfTheRadonNikodymTheorem
Related topic	BoundedLinearFunctionalsOnLinftymu
Defines	Radon-Nikodym derivative

Let μ and ν be two σ-finite measures on the same measurable space $(\Omega, \mathscr{S})$, such that $\nu \ll \mu$ (i.e. ν is absolutely continuous with respect to μ.) Then there exists a measurable function f, which is nonnegative and finite, such that for each $A \in \mathscr{S}$,

$$\nu(A) = \int_A f d\mu.$$

This function is unique (any other function satisfying these conditions is equal to f μ-almost everywhere,) and it is called the *Radon-Nikodym derivative* of ν with respect to μ, denoted by $f = \frac{d\nu}{d\mu}$.

Remark. The theorem also holds if ν is a signed measure. Even if ν is not σ-finite the theorem holds, with the exception that f is not necessarely finite.

Some properties of the Radon-Nikodym derivative

Let ν, μ, and λ be σ-finite measures in $(\Omega, \mathscr{S})$.

1. If $\nu \ll \lambda$ and $\mu \ll \lambda$, then

$$\frac{d(\nu + \mu)}{d\lambda} = \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda} \quad \mu\text{-almost everywhere;}$$

2. If $\nu \ll \mu \ll \lambda$, then

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \mu\text{-almost everywhere;}$$

3. If $\mu \ll \lambda$ and g is a μ-integrable function, then

$$\int_\Omega g d\mu = \int_\Omega g \frac{d\mu}{d\lambda} d\lambda;$$

4. If $\mu \ll \nu$ and $\nu \ll \mu$, then

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu}\right)^{-1}.$$