# comparison between Lebesgue and Riemann Integration

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The Riemann and Lebesgue integral are defined in different ways, with the latter generally perceived as the more general. The aim of this article is to clarify this claim by providing a number of, hopefully simple and convincing, examples and arguments. We restrict this article to a discussion of proper and improper integrals. For extensions and even more general definitions of the integral we refer to `http://www.math.vanderbilt.edu/~schectex/ccc/gauge/`:

# 1 Proper Integrals

## The Dirichlet Function

Our first example shows that functions exists that are Lebesgue integrable but not Riemann integrable.

Consider the characteristic function of the rational numbers in $[0, 1]$, i.e.,

$$1_{\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x \text{ rational,} \\ 0, & \text{elsewhere.} \end{cases}$$

This function, known as the Dirichlet function, is not Riemann integrable. To see this, take an arbitrary partition of the interval $[0, 1]$. The supremum of $1_{\mathbb{Q}}$ on any interval (with non-empty interior) is 1, whereas its infimum is 0. Hence, the upper Riemann sum of $1_{\mathbb{Q}}$ is 1 while the lower Riemann sum is 0. Clearly, the upper and lower Riemann sums converge to 1 and 0, respectively, in the limit of the size of largest interval in the partition going to zero. Obviously, these limits are not the same. As a bounded function is Riemann integrable if and only if the upper and lower Riemann sums converge to the same number, the Riemann integral of $1_{\mathbb{Q}}$ cannot exist.

On the other hand, $1_{\mathbb{Q}}$ turns out to be Lebesgue integrable, which we now show. Let us enumerate the rationals in $[0, 1]$ as $\{q_0, q_1, \ldots\}$. Now cover each $q_i$ with an open set $O_i$ of size $\epsilon/2^i$. Hence, the set $\{q_0, q_1, \ldots\}$ is contained in the set $\subset G_\epsilon := \cup_{i=0}^{\infty} O_i$. Now it is known that every countable union of open sets forms a Lebesgue measurable set. Therefore, $G_\epsilon$ is also a Lebesgue measurable set. Consequently, the Lebesgue integral of this set's characteristic function, i.e.,

$$1_{G_\epsilon}(x) = \begin{cases} 1, & \text{if } x \in O, \\ 0, & \text{elsewhere,} \end{cases}$$

exists.

In fact, the integral of $1_{G_\epsilon}$ is less than or equal to $\epsilon \sum_{i=0}^{\infty} 2^{-i} = \epsilon$ as the total length of the union of sets $O_i$ is less than or equal to $\epsilon$. (There might be overlaps between the $O_i$.) Now taking the limit $\epsilon \to 0$ it follows that for all $\epsilon$,

$$\int 1_{\mathbb{Q}}\, dx \leq \int 1_{G_\epsilon}\, dx \leq \epsilon$$

Since this holds for all $\epsilon > 0$, the left hand side must be 0.

## A 'Worse' Kind of Dirichlet Function

The above is, arguably, somewhat simple. The only reason that the Dirichlet function is Lebesgue, but not Riemann, integrable, is that its spikes occur on the rationals, a set of numbers which is, in comparison to the irrational numbers, a very small set. By modifying the Dirichlet function on *a set of measure zero*, that is, by removing its spikes, it becomes the zero function, which is evidently Riemann integrable. This reasoning might lead us to conjecture that it is possible to turn any Lebesgue integrable function into a Riemann integrable function by modifying it on a set of measure zero. This conjecture, however, is false as the next example shows.

Interestingly, the function we seek can be obtained in a more or less direct way from the previous example. First cover the rationals by open sets $O_i$ of length $y_i$, where the sequence of numbers of $\{y_i\}_{i=0,\ldots}$ is such that $y_i > 0$ for all $i$ but such that $\sum_i y_i = 1/2$. Observe that $G_y := \cup_i O_i$ is a measurable set with measure (length) less than or equal to $1/2$. But this implies that the complement $G_y^c$ of $G_y$, which contains only irrational numbers, has length at least $1/2$.

Observe that the characteristic function $1_{G_y}$ is nowhere continuous on $G_y^c$, as the rationals lie dense in the reals. Since, also, $G_y^c$ has measure at least equal to $1/2$ it is impossible to remove these discontinuity points by a mere modification on a countable number of measure zero sets.

Now there is a theorem by Lebesgue stating that a bounded function $f$ is Riemann integrable if and only if $f$ is continuous almost everywhere. Apparently, $1_{G_y}$ is bounded and discontinuous on a set with measure larger than 0. Thus, we may conclude that $1_{G_y}$ is not Riemann integrable.

To prove that $1_{G_y}$ is Lebesgue integrable follows easily if we approach the subject of integration from a somewhat more abstract point of view. This is the topic of the next section.

## Exchanging Limits and Integrals

Let us first present one further example: the sequence of characteristic functions of $\cup_{i=1}^{n} O_i$, where $O_i, i = 1 \ldots n \; and \;\; n < \infty$, are the open sets appearing in the definition of the 'worse Dirichlet function'. Clearly, any such function has a finite number of discontinuities, hence is Riemann integrable. However, these functions converge point-wise to $1_{G_y}$, which is *not* Riemann integrable. Apparently, sequences of Riemann integrable functions may converge to non-Riemann integrable functions. Interestingly, the sequence of integrals of $1_{\cup_{i=1}^{n} O_i}$, i.e. a sequence of reals, *has* a limit as it is increasing and bounded by $1/2$. This somewhat disturbing inconsistency is satisfactory resolved by Lebesgue's theory of integration.

As a matter of fact, the advantage of Lebesgue integration is perhaps best appreciated by interpreting this example from a more abstract (functional analysis) point of view. Stated a bit differently, we might approach the subject not from the bottom up (looking at individual functions) but from the top down (looking at classes of functions). In more detail, suppose we are allowed to apply an operator $T$ to any function that is an element of some function space. It would be nice if this space is closed under taking (point-wise) limits. In other words, besides being allowed to apply $T$ to some sequence of functions $f_n$, we are also allowed to apply $T$ to the function $f$ obtained as the limit of $f_n$. It would be even nicer if $\lim_n T f_n$ is the same as $T \lim_n f_n = T f$. (This is, for instance, useful when it is simple to compute $T f_n$ for each $n$, but difficult to compute $T f$, while the latter might be what really interests us.)

In the present case, i.e, integration, we perceive the integral of a function as a (continuous linear) operator. The class of Lebesgue integrable functions has the desired abstract properties (simple conditions to check whether the exchange of integral and limit is allowed), whereas the class of Riemann integrable functions does not.

Applying this to the above example, viz. the integration of $1_{G_y}$, we use Lebesgue Dominated Convergence Theorem, which states that when a sequence $\{f_n\}$ of Lebesgue measurable functions is bounded by a Lebesgue integrable function, the function $f$ obtained as the pointwise limit $f_n$ is also Lebesgue integrable, and $\int \lim_n f_n = \int f = \lim_n \int f_n$. Since, for all $n$, $1_{\cup_i^n O_i}$ is bounded and Lebesgue integrable, $1_{G_y}$ is also Lebesgue integrable, and reversing the (pointwise) limit and the integral is allowed.

## 1.1 Fubini's Theorem

Admittedly the function $1_{G_y}$ is rather artificial. A really powerful example of the consequences of being allowed to reverse integral and limit is provided by (the proof of) Fubini's theorem applied to the rectangle $Q = [a, b] \times [c, d]$. Compare the following two theorems. See, for instance, [**?**] or [**?**] for proofs of the first theorem, and [**?**] for the second theorem.

**Theorem 1.1.** *Riemann Case. Assume $f$ to be Riemann integrable on $Q$. Assume also that the one-dimensional function $x \mapsto f(x, y)$ is Riemann integrable for almost all $y \in [c, d]$. Then the function $y \mapsto \int_a^b f(x, y)\, dx$ is Riemann integrable and $\int_Q f(x, y)\, dx\, dy = \int_c^d \left( \int_a^b f(x, y)\, dx \right) dy$. Note: both conditions are satisfied if $f$ is continuous on $Q$.*

**Theorem 1.2.** *Lebesgue Case. Assume that $f$ is Lebesgue integrable on $Q$. Then the function $x \mapsto f(x, y)$ is Lebesgue integrable for almost all $y$ on $[c, d]$. As a consequence, the function $y \mapsto \int_a^b f(x, y)\, dx$ is Lebesgue integrable and $\int_Q f(x, y)\, dx\, dy = \int_c^d \left( \int_a^b f(x, y)\, dx \right) dy$.*

Observe that the second *assumption* in the Riemann case has turned into a *consequence* in the Lebesgue case. The main reason behind this difference is precisely that the class of Lebesgue measurable functions is closed under taking limits (under a bounded condition), whereas the class of Riemann integrable functions is not.

# 2 Improper Integrals

From the above the reader may conclude that whenever a function is Riemann integrable, it is Lebesgue integrable. This is true as long we only include *proper* integrals. If, on the other hand, we also consider *improper* integrals the statement is no longer valid. There exist functions whose improper Riemann integral exists, whereas the Lebesgue integral does not. Concentrating on functions defined on subsets of $\mathbb{R}^n$ the situation is as shown by the following Venn Diagram:

In the previous we already discussed the inclusion $R \subset L$.

Let us now integrate the function $1/\sqrt{x}$ over $[0, 1]$ to show that some functions exist $RI \cup L$ but are not in $R$. As $1/\sqrt{x}$ is not bounded on $[0, 1]$

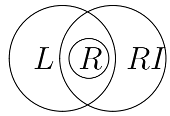


Figure 1: $R$, $RI$, and $L$, are the classes of Riemann, improper Riemann, and Lebesgue integrable functions, respectively.

every upper Riemann sum is infinite. On the other hand, both the improper Riemann integral and the Lebesgue integral exist, and give the same result.

Secondly, $L$ is not contained in $RI$ as follows from the fact that the Dirichlet function is not $RI$.

Finally, $RI$ is also not a subset of $L$. Define $f : [0, \infty) \mapsto [-1, 1]$ as 1 on $[0, 1)$, $-1/2$ on $[1, 2)$, $1/3$ on $[2, 3)$, $-1/4$ on $[3, 4)$, etc. The Riemann integral and the Lebesgue integral of $f$ over $[0, n)$ are both equal to $\sum_{k=1}^{n}(-1)^{k+1}1/k$. It is well known that this alternating sum converges, implying the existence of the improper Riemann integral. However, since a function is Lebesgue integrable if and only if its absolute value is also Lebesgue integrable, $f$ is not in $L$.

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