essential supremum

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Essential supremum of a function

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let f be a Borel measurable function from Ω to the extended real numbers $\bar{\mathbb{R}}$. The *essential supremum* of f is the smallest number $a \in \bar{\mathbb{R}}$ for which f only exceeds a on a set of measure zero. This allows us to generalize the maximum of a function in a useful way.

More formally, we define $\operatorname{ess\,sup} f$ as follows. Let $a \in \mathbb{R}$, and define

$$M_a = \{x : f(x) > a\},$$

the subset of X where $f(x)$ is greater than a. Then let

$$A_0 = \{a \in \mathbb{R} : \mu(M_a) = 0\},$$

the set of real numbers for which M_a has measure zero. The essential supremum of f is

$$\operatorname{ess\,sup} f := \inf A_0.$$

The supremum is taken in the set of extended real numbers so, $\operatorname{ess\,sup} f = \infty$ if $A_0 = \emptyset$ and $\operatorname{ess\,sup} f = -\infty$ if $A_0 = \mathbb{R}$.

Essential supremum of a collection of functions

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $\mathcal{S}$ be a collection of measurable functions $f \colon \Omega \to \bar{\mathbb{R}}$. The Borel σ-algebra on $\bar{\mathbb{R}}$ is used.

If $\mathcal{S}$ is countable then we can define the pointwise supremum of the functions in $\mathcal{S}$, which will itself be measurable. However, if $\mathcal{S}$ is uncountable then this is often not useful, and does not even have to be measurable. Instead, the *essential supremum* can be used.

The essential supremum of $\mathcal{S}$, written as $\operatorname{ess\,sup} \mathcal{S}$, if it exists, is a measurable function $f \colon \Omega \to \bar{\mathbb{R}}$ satisfying the following.

- $f \geq g$, μ-`http://planetmath.org/AlmostSurely`almost everywhere, for any $g \in \mathcal{S}$.
- if $g \colon \Omega \to \bar{\mathbb{R}}$ is measurable and $g \geq h$ (μ-a.e.) for every $h \in \mathcal{S}$, then $g \geq f$ (μ-a.e.).

Similarly, the *essential infimum*, $\operatorname{ess\,inf} \mathcal{S}$ is defined by replacing the inequalities '$\geq$' by '$\leq$' in the above definition.

Note that if f is the essential supremum and $g\colon \Omega \to \bar{\mathbb{R}}$ is equal to f μ-almost everywhere, then g is also an essential supremum. Conversely, if f, g are both essential supremums then, from the above definition, $f \leq g$ and $g \leq f$, so $f = g$ (μ-a.e.). So, the essential supremum (and the essential infimum), if it exists, is only defined almost everywhere.

It can be shown that, for a σ-finite measure μ, the `http://planetmath.org/ExistenceOfTheEs` supremum and essential infimum always exist. Furthermore, they are always equal to the supremum or infimum of some countable subset of $\mathcal{S}$.