criterion for interchanging summation and integration

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The following criterion for interchanging integration and summation is often useful in practise: Suppose one has a sequence of measurable functions $f_k \colon M \to \mathbb{R}$ (The index k runs over non-negative integers.) on some measure space M and can find another sequence of measurable functions $g_k \colon M \to \mathbb{R}$ such that $|f_k(x)| \leq g_k(x)$ for all k and almost all x and $\sum_{k=0}^{\infty} g_k(x)$ converges for almost all $x \in M$ and $\sum_{k=0}^{\infty} \int g_k(x)\,dx < \infty$. Then

$$\int_M \sum_{k=0}^{\infty} f_k(x)\,dx = \sum_{k=0}^{\infty} \int_M f_k(x)\,dx$$

This criterion is a corollary of the monotone and dominated convergence theorems. Since the g_k's are nonnegative, the sequence of partial sums is increasing, hence, by the monotone convergence theorem, $\int_M \sum_{k=0}^{\infty} g_k(x)\,dx < \infty$. Since $\sum_{k=0}^{\infty} g_k(x)$ converges for almost all x,

$$\left|\sum_{k=0}^{n} f_k(x)\right| \leq \sum_{k=0}^{n} |f_k(x)| \leq \sum_{k=0}^{n} g_k(x) \leq \sum_{k=0}^{\infty} g_k(x),$$

the dominated convergence theorem implies that we may integrate the sequence of partial sums term-by-term, which is tantamount to saying that we may switch integration and summation.

As an example of this method, consider the following:

$$\int_{-\infty}^{+\infty} \sum_{k=1}^{\infty} \frac{\cos(x/k)}{x^2 + k^4}\,dx$$

The idea behind the method is to pick our g's as simple as possible so that it is easy to integrate them and apply the criterion. A good choice here is $g_k(x) = 1/(x^2 + k^4)$. We then have $\int_{-\infty}^{+\infty} g_k(x)\,dx = \pi/k^2$ and, as $\sum_{k=1}^{\infty} k^{-2} < \infty$, we can interchange summation and integration:

$$\sum_{k=1}^{\infty} \int_{-\infty}^{+\infty} \frac{\cos(x/k)}{x^2 + k^4}\,dx.$$

Doing the integrals, we obtain the answer

$$\pi \sum_{k=1}^{\infty} \frac{e^{-k}}{k^2}$$