proof of Choquet's capacitability theorem

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Let $(X,\mathcal{F})$ be a paved space such that $\mathcal{F}$ is closed under finite unions and finite intersections, and let I be an $\mathcal{F}$-http://planetmath.org/ChoquetCapacitycapacity. We prove the capacitability theorem, which states that all $\mathcal{F}$-http://planetmath.org/AnalyticSe sets are $(\mathcal{F},I)$-http://planetmath.org/ChoquetCapacitycapacitable. The idea is to deduce it from the following special case.

Lemma. *With the above notation, every set in $\mathcal{F}_{\sigma\delta}$ is $(\mathcal{F},I)$-capacitable.*

Recall that $\mathcal{F}_{\sigma\delta}$ is the collection of countable intersections of countable unions in $\mathcal{F}$ and, since countable unions and intersections of analytic sets are analytic, all such sets are analytic. According to the capacitability theorem they should then be capacitable, and the lemma is indeed a special case.

Supposing that the lemma is true, then the capacitability theorem can be deduced as follows. For an $\mathcal{F}$-analytic set $A\subseteq X$, there is a http://planetmath.org/PavedSpacec paved space $(K,\mathcal{K})$ and $S\in(\mathcal{F}\times\mathcal{K})_{\sigma\delta}$ such that $A=\pi_X(S)$, where π_X is the projection map from $X\times K$ to X. Letting $\mathcal{G}$ be the closure under finite unions and finite intersections of the paving $\mathcal{F}\times\mathcal{K}$, then the composition $I\circ\pi_X$ is a $\mathcal{G}$-capacity, and the projection of any $(\mathcal{G},I\circ\pi_X)$-capacitable set onto X is itself $(\mathcal{F},I)$-capacitable (see http://planetmath.org/ExtendingACapacityToACartesianProductexte a capacity to a Cartesian product). In particular, $S\in\mathcal{G}_{\sigma\delta}$ so, by the lemma, is $(\mathcal{G},I\circ\pi_X)$-capacitable. Therefore, $A=\pi_X(S)$ is $(\mathcal{F},I)$-capacitable. It only remains to prove the lemma.

Proof of lemma. If $S\in\mathcal{F}_{\sigma\delta}$ then there exists $S_{m,n}\in\mathcal{F}$ such that

$$S=\bigcap_n\bigcup_m S_{m,n}.$$

For any positive integers $m_1,m_2,\dots,m_k$ let us write

$$S(m_1,m_2,\dots,m_k)\equiv\left(\bigcap_{n\le k}\bigcup_{m\le m_n}S_{m,n}\right)\bigcap\left(\bigcap_{n>k}\bigcup_m S_{m,n}\right).$$

In particular, $S()=S$ and, $I(S())=I(S)$. For any $\epsilon>0$ and $k\in\mathbb{N}$ suppose that we have chosen positive integers $m_1,\dots,m_{k-1}$ such that $I(S(m_1,\dots,m_{k-1}))>I(S)-\epsilon$. Since I is a capacity and $S(m_1,\dots,m_k)$ increases to $S(m_1,\dots,m_{k-1})$ as m_k increases to infinity,

$$I(S(m_1,\dots,m_k))\to I(S(m_1,\dots,m_{k-1}))$$

as m_k tends to infinity. So, by choosing m_k large enough, we have

$$I(S(m_1, \dots, m_k)) > I(S) - \epsilon.$$

Then, by induction, we can find an infinite sequence $m_1, m_2, \dots$ such that this inequality holds for every k. Setting

$$\begin{aligned} A_k &\equiv \bigcap_{n \le k} \bigcup_{m \le m_n} S_{m,n} \in \mathcal{F}, \\ A &\equiv \bigcap_{n} \bigcup_{m \le m_n} S_{m,n} = \bigcap_{k} A_k \in \mathcal{F}_\delta, \end{aligned}$$

then $A \subseteq S$. Furthermore, A_k contains $S(m_1, \dots, m_k)$ and decreases to A as k tends to infinity. As I is an $\mathcal{F}$-capacity this gives

$$I(A) = \lim_{k \to \infty} I(A_k) \ge \lim_{k \to \infty} I(S(m_1, \dots, m_k)) \ge I(S) - \epsilon.$$

So S is $(\mathcal{F}, I)$-capacitable, as required. $\square$