Riesz representation theorem (of linear functionals on function spaces)

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Defines	Riesz-Markov theorem

This entry should not be mistaken with the entry on the Riesz representation theorem of http://planetmath.org/BoundedOperatorbounded linear functionals on an Hilbert space.

The Riesz provided here basically that linear functionals on certain spaces of functions can be seen as integration against measures. In other , for some spaces of functions all linear functionals have the form

$$f \longmapsto \int f \, d\mu$$

for some measure μ.

There are many versions of these Riesz , and which version is used depends upon the generality wishes to achieve, the difficulty of proof, the of space of functions involved, the of linear functionals involved, the of the "" space involved, and also the of measures involved.

We present here some possible Riesz of general use.

Notation - In the following we adopt the following conventions:

- X is a locally compact Hausdorff space.
- $C_c(X)$ denotes the space of real valued continuous functions on X with compact support.
- $C_0(X)$ denotes the space of real valued continuous functions on X that vanish at infinity.
- all function spaces are endowed with the sup-norm $\|.\|_\infty$
- a linear functional L is said to be if $0 \leq L(f)$ whenever $0 \leq f$.

Theorem 1 (Riesz-Markov) - Let L be a positive linear functional on $C_c(X)$. There exists a unique Radon measure μ on X, whose underlying http://planetmath.org/SigmaAlgebraσ-algebra is the σ-algebra generated by all compact sets, such that

$$L(f) = \int_X f \, d\mu$$

Moreover, μ is finite if and only if L is bounded.

Notice that when X is http://planetmath.org/SigmaCompactσ-compact the underlying σ-algebra for these measures is precisely the http://planetmath.org/BorelSigmaA
σ-algebra of X.

Theorem 2 - Let L be a positive linear functional on $C_0(X)$. There exists a unique finite Radon measure μ on X such that

$$L(f) = \int_X f \, d\mu$$

Theorem 3 (Dual of $C_0(X)$) - Let L be a linear functional on $C_0(X)$. There exists a unique finite `http://planetmath.org/SignedMeasure`signed Borel measure on X such that

$$L(f) = \int_X f \, d\mu$$

0.0.1 Complex version:

Here $C_0(X)$ denotes the space of complex valued continuous functions on X that vanish at infinity.

Theorem 4 - Let L be a linear functional on $C_0(X)$. There exists a unique finite complex Borel measure μ on X such that

$$L(f) = \int_X f \, d\mu$$