invariance of formula for surface integration with respect to area under change of variables

Canonical name	InvarianceOfFormulaForSurfaceIntegrationWithRespectToAreaUnderChangeO
Date of creation	2013-03-22 15:07:32
Last modified on	2013-03-22 15:07:32
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	5
Author	rspuzio (6075)
Entry type	Proof
Classification	msc 28A75

First, we can use the chain rule for Jacobians to see how one of the terms in parentheses transforms:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(u',v')}\frac{\partial(u',v')}{\partial(u,v)}$$

A similar story holds for the other two factors. Combining them, we conclude that

$$\sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2} =$$

$$\sqrt{\left(\frac{\partial(x,y)}{\partial(u',v')}\frac{\partial(u',v')}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u',v')}\frac{\partial(u',v')}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u',v')}\frac{\partial(u',v')}{\partial(u,v)}\right)^2} =$$

$$\frac{\partial(u',v')}{\partial(u,v)}\sqrt{\left(\frac{\partial(x,y)}{\partial(u',v')}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u',v')}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u',v')}\right)^2}$$

Since the factor in parentheses in front of the square root is the Jacobi determinant, we can apply the rule change of variables in multidimensional integrals to conclude that

$$\int f(u,v)\sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2}\, du\, dv =$$

$$\int f(u',v')\sqrt{\left(\frac{\partial(x,y)}{\partial(u',v')}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u',v')}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u',v')}\right)^2}\, du'\, dv',$$

which shows that our formula gives the same answer for $\int_S f(u,v)\, d^2A$, no matter how we choose to parameterize S.