bounded linear functionals on $L^{\infty}(\mu)$

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For any measure space $(X, \mathfrak{M}, \mu)$ and $g \in L^1(\mu)$, the following linear map can be defined

$$\begin{aligned}
&\Phi_g \colon L^\infty(\mu) \to \mathbb{R}, \\
&f \mapsto \Phi_g(f) \equiv \int fg \, d\mu.
\end{aligned}$$

It is easily shown that Φ_g is http://planetmath.org/OperatorNormbounded, so is a member of the dual space of Ł$^\infty(\mu)$. However, unless the measure space consists of a finite set of atoms, not every element of the dual of $L^\infty(\mu)$ can be written like this. Instead, it is necessary to restrict to linear maps satisfying a bounded convergence property.

Theorem. *Let $(X, \mathfrak{M}, \mu)$ be a* ***http://planetmath.org/SigmaFinite****σ-finite measure space and V be the space of bounded linear maps $\Phi \colon L^\infty(\mu) \to \mathbb{R}$ satisfying bounded convergence. That is, if $|f_n| \leq 1$ are in $L^\infty(\mu)$ and $f_n(x) \to 0$ for almost every $x \in X$, then $\Phi(f_n) \to 0$.*

Then $g \mapsto \Phi_g$ gives an isometric isomorphism from $L^1(\mu)$ to V.

Proof. First, the operator norm $\|\Phi_g\|$ is equal to the L^1-norm of g (see http://planetmath.org/LpNormIsDualToLqL^p-norm is dual to L^q), so the map $g \mapsto \Phi_g$ gives an isometric embedding from L^1 into the dual of L^∞. Furthermore, dominated convergence implies that Φ_g satisfies bounded convergence so $\Phi_g \in V$. We just need to show that $g \mapsto \Phi_g$ maps onto V.

So, suppose that $\Phi \in V$. It needs to be shows that $\Phi = \Phi_g$ for some $g \in L^1$. Defining an http://planetmath.org/Additiveadditive set function $\nu \colon \mathfrak{M} \to \mathbb{R}$ by

$$\nu(A) = \Phi(1_A)$$

for every set $A \in \mathfrak{M}$, the bounded convergence property for Φ implies that ν is countably additive and is therefore a finite signed measure. So, the Radon-Nikodym theorem gives a $g \in L^1$ such that $\nu(A) = \int_A g \, d\mu$ for every $A \in \mathfrak{M}$. Then, the equality

$$\Phi(fh) = \int fg \, d\mu$$

is satisfied for $f = 1_A$ with any $A \in \mathfrak{M}$ and the functional monotone class theorem extends this to any bounded and measurable $f \colon X \to \mathbb{C}$, giving $\Phi_g = \Phi$. $\square$