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monotone class theorem

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Theorem. *Let $\mathcal{F}_0$ an algebra of subsets of Ω. Let $\mathcal{M}$ be the smallest monotone class such that $\mathcal{F}_0 \subset \mathcal{M}$ and $\sigma(\mathcal{F}_0)$ be the sigma algebra generated by $\mathcal{F}_0$. Then $\mathcal{M} = \sigma(\mathcal{F}_0)$.*

Proof. It is enough to prove that $\mathcal{M}$ is an algebra, because an algebra which is a monotone class is obviously a σ-algebra.

Let $\mathcal{M}_A = \{B \in \mathcal{M} | A \cap B, A \cap B^{\complement}$ and $A^{\complement} \cap B \in \mathcal{M}\}$. Then is clear that $\mathcal{M}_A$ is a monotone class and, in fact, $\mathcal{M}_A = \mathcal{M}$, for if $A \in \mathcal{F}_0$, then $\mathcal{F}_0 \subset \mathcal{M}_A$ since $\mathcal{F}_0$ is a field, hence $\mathcal{M} \subset \mathcal{M}_A$ by minimality of $\mathcal{M}$; consequently $\mathcal{M} = \mathcal{M}_A$ by definition of $\mathcal{M}_A$. But this shows that for any $B \in \mathcal{M}$ we have $A \cap B$, $A \cap B^{\complement}$ and $A^{\complement} \cap B \in \mathcal{M}$ for any $A \in \mathcal{F}_0$, so that $\mathcal{F}_0 \subset \mathcal{M}_B$ and again by minimality $\mathcal{M} = \mathcal{M}_B$. But what we have just proved is that $\mathcal{M}$ is an algebra, for if $A, B \in \mathcal{M} = \mathcal{M}_A$ we have showed that $A \cap B$, $A \cap B^{\complement}$ and $A^{\complement} \cap B \in \mathcal{M}$, and, of course, $\Omega \in \mathcal{M}$. $\square$

Remark 1. One of the main applications of the Monotone Class Theorem is that of showing that certain property is satisfied by all sets in an σ-algebra, generally starting by the fact that the field generating the σ-algebra satisfies such property and that the sets that satisfies it constitutes a monotone class.

Example 1. Consider an infinite sequence of independent random variables $\{X_n, n \in \mathbb{N}\}$. The definition of independence is

$$P(X_1 \in A_1, X_2 \in A_2, ..., X_n \in A_n) = P(X_1 \in A_1)P(X_2 \in A_2) \cdots P(X_n \in A_n)$$

for any Borel sets $A_1, A_2, .., A_n$ and any finite n. Using the Monotone Class Theorem one can show, for example, that any event in $\sigma(X_1, X_2, ..., X_n)$ is independent of any event in $\sigma(X_{n+1}, X_{n+2}, ...)$. For, by independence

$$P((X_1, X_2, ..., X_n) \in A, (X_{n+1}, X_{n+2}, ...) \in B) = P((X_1, X_2, ..., X_n) \in A)P((X_{n+1}, X_{n+2}, ...) \in B)$$

when A and B are measurable rectangles in $\mathcal{B}^n$ and $\mathcal{B}^\infty$ respectively. Now it is clear that the sets A which satisfies the above relation form a monotone class. So

$$P((X_1, X_2, ..., X_n) \in A, (X_{n+1}, X_{n+2}, ...) \in B) = P((X_1, X_2, ..., X_n) \in A)P((X_{n+1}, X_{n+2}, ...) \in B)$$

for every $A \in \sigma(X_1, X_2, ..., X_n)$ and any measurable rectangle $B \in \mathcal{B}^\infty$. A second application of the theorem shows finally that the above relation holds for any $A \in \sigma(X_1, X_2, ..., X_n)$ and $B \in \sigma(X_{n+1}, X_{n+2}, ...)$