measurable projection theorem

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The projection of a measurable set from the product $X \times Y$ of two measurable spaces need not itself be measurable. See a Lebesgue measurable but non-Borel set for an example. However, the following result can be shown. The notation $\mathcal{F}\times\mathcal{B}$ refers to the `http://planetmath.org/ProductSigmaAlgebra`product σ-algebra.

Theorem. *Let $(X,\mathcal{F})$ be a measurable space and Y be a Polish space with Borel σ-algebra $\mathcal{B}$. Then the `http://planetmath.org/ProjectionMap`projection of any $S \in \mathcal{F} \times \mathcal{B}$ onto X is universally measurable.*

In particular, if $\mathcal{F}$ is universally complete then the projection of S will be in $\mathcal{F}$, and this applies to all complete `http://planetmath.org/SigmaFinite`σ-finite measure spaces $(X,\mathcal{F},\mu)$. For example, the projection of any Borel set in $\mathbb{R}^n$ onto $\mathbb{R}$ is Lebesgue measurable.

The theorem is a direct consequence of the properties of `http://planetmath.org/AnalyticSet` sets, following from the result that projections of analytic sets are analytic and the fact that `http://planetmath.org/MeasurabilityOfAnalyticSets`analytic sets are universally measurable. Note, however, that the theorem itself does not refer at all to the concept of analytic sets.

The measurable projection theorem has important applications to the theory of continuous-time stochastic processes. For example, the début theorem, which says that the first time at which a progressively measurable stochastic process enters a given measurable set is a stopping time, follows easily. Also, if $(X_t)_{t\in\mathbb{R}_+}$ is a jointly measurable process defined on a measurable space $(\Omega,\mathcal{F})$, then the maximum process $X^*_t = \sup_{s\le t} X_s$ will be universally measurable since,

$$\{\omega\in\Omega\colon X^*_t > K\} = \pi_\Omega\left(\{(s,\omega)\colon s\le t,\ X_s > K\}\right)$$

is universally measurable, where $\pi_\Omega\colon \Omega\times\mathbb{R}_+\to\Omega$ is the projection map. Furthermore, this also shows that the topology of ucp convergence is well defined on the space of jointly measurable processes.