Carathéodory's lemma

Canonical name	CaratheodorysLemma
Date of creation	2013-03-22 18:33:03
Last modified on	2013-03-22 18:33:03
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	19
Author	gel (22282)
Entry type	Theorem
Classification	msc 28A12
Related topic	CaratheodorysExtensionTheorem
Related topic	OuterMeasure2
Related topic	LebesgueOuterMeasure
Related topic	ConstructionOfOuterMeasures
Related topic	ProofOfCaratheodorysLemma
Related topic	ProofOfCaratheodorysExtensionTheorem
Defines	Carathéodory measurable

In measure theory, Carathéodory's lemma is used for constructing measures and, for example, can be applied to the construction of the Lebesgue measure and is used in the proof of Carathéodory's extension theorem. The idea is that to define a measure on a measurable space $(X, \mathcal{A})$ we would first construct an `http://planetmath.org/OuterMeasure2`outer measure, which is a set function defined on the power set of X. Then, this outer measure is restricted to $\mathcal{A}$ and Carathéodory's lemma is applied to show that this restriction does in fact result in a measure. For an example of this procedure, see the proof of Carathéodory's extension theorem.

Given an outer measure μ on a set X, the result first defines a collection of subsets of X — the μ-measurable sets. A subset $S \subseteq X$ is called μ-measurable (or Carathéodory measurable with respect to μ) if the equality

$$\mu(E) = \mu(E \cap S) + \mu(E \cap S^c)$$

holds for every $E \subseteq X$. Then, Caratheodory's lemma says that a measure is obtained by restricting μ to the μ-measurable sets.

Lemma (Carathéodory)**.** *Let μ be an outer measure on a set X, and $\mathcal{A}$ be the class of μ-measurable sets. Then $\mathcal{A}$ is a `http://planetmath.org/SigmaAlgebra`σ-algebra and the restriction of μ to $\mathcal{A}$ is a measure.*

It should be noted that for any outer measure μ and sets $S, E \subseteq X$, subadditivity of μ implies that the inequality $\mu(E) \leq \mu(E \cap S) + \mu(E \cap S^c)$ is always satisfied. So, only the reverse inequality is required and consequently S is μ-measurable if and only if

$$\mu(E) \geq \mu(E \cap S) + \mu(E \cap S^c)$$

for every $E \subseteq X$.

References

[1] David Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, 1991.

[2] Olav Kallenberg, *Foundations of modern probability*, Second edition. Probability and its Applications. Springer-Verlag, 2002.