biquadratic equation

Canonical name	BiquadraticEquation
Date of creation	2013-03-22 17:52:45
Last modified on	2013-03-22 17:52:45
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	11
Author	pahio (2872)
Entry type	Topic
Classification	msc 30-00
Classification	msc 12D99
Related topic	BiquadraticExtension
Related topic	BiquadraticField
Related topic	EulersDerivationOfTheQuarticFormula
Related topic	IrreduciblePolynomialsObtainedFromBiquadraticFields
Related topic	LogicalOr
Defines	biquadratic equation

A *biquadratic equation* (in a narrower sense) is the special case of the http://planetmath.org/QuarticFormulaquartic equation containing no odd degree terms:

$$ax^4 + bx^2 + c = 0 \tag{1}$$

Here, a, b, c are known real or complex numbers and $a \neq 0$.

For solving a biquadratic equation (1) one does not need the http://planetmath.org/QuarticF formula since the equation may be thought a quadratic equation with respect to x^2, i.e.

$$a(x^2)^2 + bx^2 + c = 0,$$

whence

$$x^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(see quadratic formula or http://planetmath.org/QuadraticEquationInMathbbCquadratic equation in $\mathbb{C}$). Taking square roots of the values of x^2 (see taking square root algebraically), one obtains the four http://planetmath.org/Equationroots of (1).

Example. Solve the biquadratic equation

$$x^4 + x^2 - 20 = 0. \tag{2}$$

We have

$$x^2 = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-20)}}{2 \cdot 1} = \frac{-1 \pm 9}{2}, \tag{3}$$

i.e. $x^2 = 4$ or $x^2 = -5$. The solution is

$$x = \pm 2 \quad \vee \quad x = \pm i\sqrt{5}. \tag{4}$$

Remark. In one wants to form of rational numbers a polynomial equation with rational coefficients and most possibly low degree by using two square root operations, then one gets always a biquadratic equation. A couple of examples:

1) $x = 1 + \sqrt{2} + \sqrt{3}$
$(x-1)^2 = 2 + 2\sqrt{6} + 3$
$y^2 - 5 = 2\sqrt{6}$

$y^4 - 10y^2 + 1 = 0$ (one has `http://planetmath.org/TchirnhausTransformations`substituted $x - 1 := y$)

2) $x = \sqrt{\sqrt{2} - 1}$
$x^2 = \sqrt{2} - 1$
$(x^2 + 1)^2 = 2$
$x^4 + 2x^2 - 1 = 0$