analytic continuation of Riemann zeta to critical strip

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Defines	alternating zeta function

The $\frac{1}{n^s} = e^{-s\log n}$ (see the general power) of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^s} \;=\; 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots, \tag{1}$$

defining the Riemann zeta function $\zeta(s)$ for $\Re s > 1$, are holomorphic in the whole s-plane and the series converges uniformly in any closed disc of the half-plane $\Re s > 1$ ((let $s = \sigma+it$ with σ, $t \in \mathbb{R}$ and $\sigma > 1$; then $|\frac{1}{n^s}| = \frac{1}{n^\sigma} \leq \frac{1}{n^{1+d}}$ for a positive d for all $n = 1, 2, \ldots$; the series $\sum_{n=1}^{\infty}\frac{1}{n^{1+d}}$ converges since $1+d > 1$; thus the series (1) converges uniformly in the closed half-plane $\Re s \geq 1+d$, by the http://planetmath.org/WeierstrassCriterionOfUniformConvergenceWeierstrass criterion)). Therefore we can infer (see http://planetmath.org/TheoremsOnComplexFunctionSe on complex function series) that the sum $\zeta(s)$ of (1) is holomorphic in the domain $\Re s > 1$.

We use also the fact that the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \;=\; 1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+-\ldots \tag{2}$$

defining the Dirichlet eta function $\eta(s)$, a.k.a. the *alternating zeta function*, is convergent for $\Re s > 0$ and its sum is holomorphic in this half-plane.

If we multiply the series (1) by the difference $1-\dfrac{2}{2^s}$, every other of the series changes its sign and we get the series (2). So we can write

$$\zeta(s) \;=\; \frac{\eta(s)}{1-\frac{2}{2^s}}, \tag{3}$$

which is valid when the denominator does not vanish and $\Re s > 1$. The zeros of the denominator are obtained from $2^s = 2$, i.e. from

$$e^{s\log 2} \;=\; e^{\log 2}.$$

This gives $s\log 2 - \log 2 = n\cdot 2i\pi$ (see the periodicity of exponential function), i.e.

$$s \;=\; 1+n\cdot\frac{2\pi i}{\ln 2} \quad (n \in \mathbb{Z}). \tag{4}$$

Thus the zeros of the denominator of (3) are on the line $\Re s = 1$.

Now the function on the right hand side of (3) is holomorphic in the set

$$D \;:=\; \{s\in\mathbb{C} \vdots\; \Re s>0\}\smallsetminus\{1+n\cdot\frac{2\pi i}{\ln 2} \vdots\; n\in\mathbb{Z}\}$$

and the values of this function coincide with the values of zeta function in the half-plane $\Re s>1$.

This result means that, via the equation (3), the zeta function has been `http://planetmath.org/AnalyticContinuation`analytically continued to the domain D, as far as to the imaginary axis.

Remark. In reality, all points (4) except $s=1$ are removable singularities of $\zeta(s)$ given by (3), due to the fact that they are also zeros of $\eta(s)$. The fact is considered in the entry zeros of Dirichlet eta function.

Charles Hermite has shown that the zeta function may be analytically continued to the whole s-plane except for a simple pole at $s=1$, by using the equation

$$\zeta(s) \;=\; \frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}}{e^x-1}\,dx. \tag{5}$$

See `http://planetmath.org/AnalyticContinuationOfRiemannZetaUsingIntegral`this article.