equality of complex numbers

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The equality relation "=" among the is determined as consequence of the definition of the complex numbers as elements of the quotient ring $\mathbb{R}/(X^2{+}1)$, which enables the of the complex numbers as the ordered pairs $(a,\, b)$ of real numbers and also as the sums $a{+}ib$ where $i^2 = -1$.

$$a_1 + ib_1 = a_2 + ib_2 \quad \Longleftrightarrow \quad a_1 = a_2 \,\wedge\, b_1 = b_2 \tag{1}$$

This condition may as well be derived by using the field properties of $\mathbb{C}$ and the properties of the real numbers:

$$\begin{aligned} a_1 + ib_1 = a_2 + ib_2 &\implies a_2 - a_1 = -i(b_2 - b_1) \\ &\implies (a_2 - a_1)^2 = -(b_2 - b_1)^2 \\ &\implies (a_2 - a_1)^2 + (b_2 - b_1)^2 = 0 \\ &\implies a_2 - a_1 = 0, \;\; b_2 - b_1 = 0 \\ &\implies a_1 = a_2, \quad b_1 = b_2 \end{aligned}$$

The implication in the reverse direction is apparent.

If $a + ib \neq 0$, then at least one of the real numbers a and b differs from 0. We can set

$$a = r\cos\varphi, \qquad b = r\sin\varphi, \tag{2}$$

where r is a uniquely determined positive number and φ is an angle which is uniquely determined up to an integer multiple of 2π. In fact, the equations (2) yield

$$a^2 + b^2 = r^2(\cos^2\varphi + \sin^2\varphi) = r^2,$$

whence

$$r = \sqrt{a^2 + b^2}. \tag{3}$$

Thus (2) implies

$$\cos\varphi = \frac{a}{\sqrt{a^2 + b^2}}, \qquad \sin\varphi = \frac{b}{\sqrt{a^2 + b^2}}. \tag{4}$$

The equations (4) are , since the sum of the squares of their is 1. So these equations determine the angle φ up to a multiple of 2π. We can write the

Theorem. Every complex number may be represented in the *polar form*

$$r(\cos \varphi + i \sin \varphi),$$

where r is the modulus and φ the argument of the number. Two complex numbers are equal if and only if they have equal moduli and, if the numbers do not vanish, their arguments differ by a multiple of 2π.