proof of Weierstrass M-test

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Consider the sequence of partial sums $s_n = \sum_{m=1}^{n} f_m$. Take any $p, q \in \mathbb{N}$ such that $p \leq q$,then, for every $x \in X$, we have

$$\begin{aligned} |s_q(x) - s_p(x)| &= \left| \sum_{m=p+1}^{q} f_m(x) \right| \\ &\leq \sum_{m=p+1}^{q} |f_m(x)| \\ &\leq \sum_{m=p+1}^{q} M_m \end{aligned}$$

But since $\sum_{n=1}^{\infty} M_n$ converges, for any $\epsilon > 0$ we can find an $N \in \mathbb{N}$ such that, for any $p, q > N$ and $x \in X$, we have $|s_q(x) - s_p(x)| \leq \sum_{m=p+1}^{q} M_m < \epsilon$. Hence the sequence s_n converges uniformly to $\sum_{n=1}^{\infty} f_n$.