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identity theorem

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Lemma 1. *Let f be analytic on $\Omega \subseteq \mathbb{C}$ and let L be the set of accumulation points (also called limit points or cluster points) of $\{z \in \Omega \colon f(z) = 0\}$ in Ω. Then L is both open and closed in Ω.*

Proof. By definition of accumulation point, L is closed. To see that it is also open, let $z_0 \in L$, choose an open ball $B(z_0, r) \subseteq \Omega$ and write $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, z \in B(z_0, r)$. Now $f(z_0) = 0$, and hence either f has a zero of order m at z_0 (for some m), or else $a_n = 0$ for all n. In the former case, there is a function g analytic on Ω such that $f(z) = (z - z_0)^m g(z), z \in \Omega$, with $g(z_0) \neq 0$. By continuity of g, $g(z) \neq 0$ for all z sufficiently close to z_0, and consequently z_0 is an isolated point of$\{z \in \Omega \colon f(z) = 0\}$. But then $z_0 \notin L$, contradicting our assumption. Thus, it must be the case that $a_n = 0$ for all n, so that $f \equiv 0$ on $B(z_0, r)$. Consequently, $B(z_0, r) \in L$, proving that L is open in Ω. □

Theorem 1 (Identity theorem)**.** *Let Ω be a open connected subset of $\mathbb{C}$ (i.e., a domain). If f and g are analytic on Ω and $\{z \in \Omega \colon f(z) = g(z)\}$ has an accumulation point in Ω, then $f \equiv g$ on Ω.*

Proof. We have that $\{z \in \Omega \colon f(z) - g(z) = 0\}$ has an accumulation point, hence, according to the previous lemma, it is open and closed (also called "clopen"). But, as Ω is connected, the only closed and open subset at once is Ω itself, therefore $\{z \in \Omega \colon f(z) - g(z) = 0\} = \Omega$, i.e., $f \equiv g$ on Ω. □

Remark 1. This theorem provides a very powerful and useful tool to test whether two analytic functions, whose values coincide in some points, are indeed the same function. Namely, unless the points in which they are equal are isolated, they are the same function.