proof of conformal Möbius circle map theorem

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Let f be a conformal map from the unit disk Δ onto itself. Let $a = f(0)$. Let $g_a(z) = \frac{z-a}{1-\bar{a}z}$. Then $g_a \circ f$ is a conformal map from Δ onto itself, with $g_a \circ f(0) = 0$. Therefore, by Schwarz's Lemma for all $z \in \Delta$ $|g_a \circ f(z)| \leq |z|$.

Because f is a conformal map onto Δ, f^{-1} is also a conformal map of Δ onto itself. $(g_a \circ f)^{-1}(0) = 0$ so that by Schwarz's Lemma $|(g_a \circ f)^{-1}(w)| \leq |w|$ for all $w \in \Delta$. Writing $w = g_a \circ f(z)$ this becomes $|z| \leq |g_a \circ f(z)|$.

Therefore, for all $z \in \Delta$ $|g_a \circ f(z)| = |z|$. By Schwarz's Lemma, $g_a \circ f$ is a rotation. Write $g_a \circ f(z) = e^{i\theta} z$, or $f(z) = e^{i\theta} g_a^{-1}$.

Therefore, f is a Möbius Transformation.