Cauchy integral formula

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The formulas. Let $D = \{z \in \mathbb{C} : |z - z_0| < R\}$ be an open disk in the complex plane, and let $f(z)$ be a holomorphic[1] function defined on some open domain that contains D and its boundary. Then, for every $z \in D$ we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ f'(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \\ &\vdots \\ f^{(n)}(z) &= \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \end{aligned}$$

Here $C = \partial D$ is the corresponding circular boundary contour, oriented counterclockwise, with the most obvious parameterization given by

$$\zeta = z_0 + Re^{it}, \quad 0 \leq t \leq 2\pi.$$

Discussion. The first of the above formulas underscores the "rigidity" of holomorphic functions. Indeed, the values of the holomorphic function inside a disk D are completely specified by its values on the boundary of the disk. The second formula is useful, because it gives the derivative in terms of an integral, rather than as the outcome of a limit process.

Generalization. The following technical generalization of the formula is needed for the treatment of removable singularities. Let S be a finite subset of D, and suppose that $f(z)$ is holomorphic for all $z \notin S$, but also that $f(z)$ is bounded near all $z \in S$. Then, the above formulas are valid for all $z \in D \setminus S$.

Using the Cauchy residue theorem, one can further generalize the integral formula to the situation where D is any domain and C is any closed rectifiable curve in D; in this case, the formula becomes

$$\eta(C, z) f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

where $\eta(C, z)$ denotes the winding number of C. It is valid for all points $z \in D \setminus S$ which are not on the curve C.

[1] It is necessary to draw a distinction between holomorphic functions (those having a complex derivative) and analytic functions (those representable by power series). The two concepts are, in fact, equivalent, but the standard proof of this fact uses the Cauchy Integral Formula with the (apparently) weaker holomorphicity hypothesis.