proof of argument principle

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Since f is meromorphic, f' is meromorphic, and hence f'/f is meromorphic. The singularities of f'/f can only occur at the zeros and the poles of f.

I claim that all singularities of f'/f are simple poles. Furthermore, if f has a zero at some point p, then the residue of the pole at p is positive and equals the multiplicity of the zero of f at p. If f has a pole at some point p, then the residue of the pole at p is negative and equals minus the multiplicity of the pole of f at p.

To prove these assertions, write $f(x) = (x-p)^n g(x)$ with $g(p) \neq 0$. Then

$$\frac{f'(x)}{f(x)} = \frac{n}{x-p} + \frac{g'(x)}{g(x)}$$

Since $g(p) \neq 0$, the only singularity of f'/f at p comes from the first summand. Since n is either the order of the zero of f at p if f has a zero at p or minus the order of the pole of f at p if f has a pole at p, the assertion is proven.

By the Cauchy residue theorem, the integral

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$$

equals the sum of the residues of f'/f. Combining this fact with the characterization of the poles of f'/f and their residues given above, one deduces that this integral equals the number of zeros of f minus the number of poles of f, counted with multiplicity.