persistence of differential equations

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The persistence of analytic relations has important consequences for the theory of differential equations in the complex plane. Suppose that a function f satisfies a differential equation $F(x, f(x), f'(x), \ldots, f^{(n)}) = 0$ where F is a polynomial. This equation may be viewed as a polynomial relation between the $n+2$ functions $\mathrm{id}, f, f', \ldots, f^{(n)}$ hence, by the persistence of analytic relations, it will also hold for the analytic continuations of these functions. In other words, if an algebraic differential equation holds for a function in some region, it will still hold when that function is analytically continued to a larger region.

An interesting special case is that of the homogeneous linear differential equation with polynomial coefficients. In that case, we have the principle of superposition which guarantees that a linear combination of solutions is also a solution. Hence, if we start with a basis of solutions to our equation about some point and analytically continue them back to our starting point, we obtain linear combinations of those solutions. This observation plays a very important role in the theory of differential equations in the complex plane and is the foundation for the notion of monodromy group and Riemann's global characterization of the hypergeometric function.

For a less exalted illustrative example, we can consider the complex logarithm. The differential equation

$$xy'' + y' = 0$$

has as solutions $y = 1$ and $y = \log x$. While the former is as singly valued as functions get, the latter is multiply valued. Hence upon performong analytic continuation, we expect that the second solution will continue to a linear combination of the two solutions. This, of course is exactly what happens; upon analytic continuation, the second solution becomes the solution $y = \log x + n\pi i$ where n is an integer whose value depends on how we carry out the analytic continuation.