nth root

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Related topic	SquareRoot
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Related topic	IrrationalNumber
Related topic	EvenEvenOddRule
Related topic	ExtensionOfValuationFromCompleteBaseField
Related topic	Radical5
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The phrase "*the n-th root of a number*" is a somewhat misleading concept that requires a fair amount of thought to make rigorous.

For n a positive integer, we define *an* n-th root of a number x to be a number y such that $y^n = x$. The number n is said to be the *index* of the root. Note that the term "number" here is ambiguous, as the discussion can apply in a variety of contexts (groups, rings, monoids, etc.) The purpose of this entry is specifically to deal with n-th roots of real and complex numbers.

In an effort to give meaning to the term *the* n-th root of a real number x, we define it to be the unique real number that y is *an* nth root of x and such that $\operatorname{sign}(x) = \operatorname{sign}(y)$, if such a number exists. We denote this number by $\sqrt[n]{x}$, or by $x^{\frac{1}{n}}$ if x is positive. This specific nth root is also called the *principal nth root*.

Example: $\sqrt[4]{81} = 3$ because $3^4 = 3 \times 3 \times 3 \times 3 = 81$, and 3 is the unique positive real number with this property.

Example: If $x+1$ is a positive real number, then we can write $\sqrt[5]{x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1}$ $x+1$ because $(x+1)^5 = (x^2+2x+1)^2(x+1) = x^5+5x^4+10x^3+10x^2+5x+1$. (See the Binomial Theorem and .)

The nth root operation is distributive for multiplication and division, but not for addition and subtraction. That is, $\sqrt[n]{x \times y} = \sqrt[n]{x} \times \sqrt[n]{y}$, and $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$. However, except in special cases, $\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y}$ and $\sqrt[n]{x-y} \neq \sqrt[n]{x} - \sqrt[n]{y}$.

Example: $\sqrt[4]{\frac{81}{625}} = \frac{3}{5}$ because $\left(\frac{3}{5}\right)^4 = \frac{3^4}{5^4} = \frac{81}{625}$.

Note that when we restrict our attention to real numbers, expressions like $\sqrt{-3}$ are undefined. Thus, for a more full definition of nth roots, we will have to incorporate the notion of complex numbers: *The nth roots of a complex number* $t = x + yi$ are all the complex numbers $z_1, z_2, \ldots, z_n \in \mathbb{C}$ that satisfy the condition $z_k^n = t$. Applying the fundamental theorem of algebra (complex version) to the function $x^n - t$ tells us that n such complex numbers always exist (counting multiplicity).

One of the more popular methods of finding these roots is through trigonometry and the geometry of complex numbers. For a complex number $z = x+iy$, recall that we can put z in polar form: $z = (r, \theta)$, where $r = \sqrt[2]{x^2 + y^2}$, and $\theta = \frac{\pi}{2}$ if $x = 0$, and $\theta = \arctan \frac{y}{x}$ if $x \neq 0$. (See the Pythagorean Theorem.) For the specific procedures involved, see calculating the nth roots of a complex number.