proof of Marty's theorem

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(i) Fix $K \subseteq \Omega$ compact. We have:

$$\frac{2|f'(z)|}{1+|f(z)|^2} \leq M_K \ \forall \ f \in \mathcal{F}, z \in K \tag{$*$}$$

Let V be a region with $K = \overline{V}$ and let $\gamma\colon [a,b] \to V$ be the C^1 curves connecting the points $P, Q \in \Omega$. Then we have:

$$\begin{aligned} d_\sigma(f(P), f(Q)) &= \inf_\gamma l_\sigma(f \circ \gamma) = \inf_\gamma \int_a^b \|(f \circ \gamma)'(t)\|_{\sigma, f\circ\gamma(t)} \, dt \\ &= \inf_\gamma \int_a^b \frac{2|f'(\gamma(t))|}{1+|f(\gamma(t))|^2} |\gamma'(t)| \, dt \\ &\underset{(??)}{\leq} M_K \inf_\gamma \int_a^b |\gamma'(t)| \, dt \\ &= M_K \inf_\gamma l(\gamma) = M_K |P-Q| \end{aligned}$$

Thus f is Lipschitz continuous and thus $\mathcal{F}$ is equicontinuous. By the Ascoli-Arzel Theorem we conclude that $\mathcal{F}$ is normal.

(ii) Now assume $\mathcal{F}$ to be normal. Define:

$$f^\sharp(z) := \frac{2|f'(z)|}{1+|f(z)|^2}$$

Let $K \subseteq \Omega$ be compact. To obtain contradiction assume $\{f^\sharp : f \in \mathcal{F}\}$ is not uniformly bounded on K. But then there exists a sequence $\{f_n\} \subset \mathcal{F}$ such that:

$$\max_{z \in K} f_n^\sharp(z) \to \infty \ (n \to \infty)$$

Since $\mathcal{F}$ is normal for each $P \in \Omega$ let there be a neighbourhood U_P such that $\{f_n\}$ converges normally to a meromorphic function f. But from $(1/f_n)^\sharp = f_n^\sharp$ we see that $\{f_n^\sharp\}$ converges normally on U_P. Since K is compact it can be covered by finitely many sets U_P. We conclude that $\{f_n^\sharp\}$ must be bounded on K and obtain a contradiction. $\square$