proof of open mapping theorem

Canonical name	ProofOfOpenMappingTheorem
Date of creation	2013-03-22 16:23:31
Last modified on	2013-03-22 16:23:31
Owner	Statusx (15142)
Last modified by	Statusx (15142)
Numerical id	9
Author	Statusx (15142)
Entry type	Proof
Classification	msc 30A99
Classification	msc 46A30

We prove that if $\Lambda : X \to Y$ is a continuous linear surjective map between Banach spaces, then Λ is an open map. It suffices to show that Λ maps the open unit ball in X to a neighborhood of the origin of Y.

Let U, V be the open unit balls in X, Y respectively. Then $X = \cup_{k \in \mathbb{N}} kU$, so, since Λ is surjective, $Y = \Lambda(X) = \Lambda(\cup_{k \in \mathbb{N}} kU) = \cup_{k \in \mathbb{N}} \Lambda(kU)$. By the Baire category theorem, Y is not the union of countably many nowhere dense sets, so there is some $k \in \mathbb{N}$ and some open set $W \subset Y$ such that W is contained in the closure of $\Lambda(kU)$.

Let $y_0 \in W$, and pick $\eta > 0$ so that $y_0 + y \in W$ for all y with $||y|| < \eta$. Then y_0 and $y_0 + y$ are limit points of $\Lambda(kU)$, so there are sequences x_i' and x_i'' in kU with $\Lambda x_i' \to y_0$ and $\Lambda x_i'' \to y_0 + y$. Letting $x_i = x_i'' - x_i'$, we have $||x_i|| < 2k$ and $\Lambda x_i \to y$. So for any $y \in \eta V$ there is a sequence x_i in $2kU$ with $\Lambda x_i \to y$. Then by the linearity of Λ, we have that for any $\epsilon > 0$ and any $y \in Y$, there is an $x \in X$ with:

$||x|| < \delta^{-1} ||y||$ and $||\Lambda x - y|| < \epsilon$ (1)

where $\delta = \eta / 2k$.

Now let $y \in \delta V$ and $\epsilon > 0$. Then there is some x_1 with $||x_1|| < 1$ and $||y - \Lambda x_1|| < \epsilon\delta$. Define a sequence x_n inductively as follows. Assume:

$||y - \Lambda(x_1 + x_2 + ... + x_n)|| < \epsilon\delta 2^{-n}$ (2)

Then by (1) we can pick x_{n+1} so that:

$||x_{n+1}|| < \epsilon 2^{-n}$ (3)

and $||y - \Lambda(x_1 + x_2 + ... + x_n) - \Lambda(x_{n+1})|| < \epsilon\delta 2^{-(n+1)}$, so (2) is satisfied for x_{n+1}.

Put $s_n = x_1 + x_2 + ... + x_n$. Then from (3), s_n is a Cauchy sequence, and so, since X is complete, it converges to some $x \in X$. By (2), $\Lambda s_n \to y$, and by the continuity of Λ, $\Lambda s_n \to \Lambda x$, so $\Lambda x = y$. Also, $||x|| = \lim_{n \to \infty} ||s_n|| \leq \sum_{n=1}^{\infty} ||x_n|| < 1 + \epsilon$. Thus $\Lambda((1 + \epsilon)U) \supset \eta V$, or $\Lambda(U) \supset (1 + \epsilon)^{-1} \delta V$. Since this is true for all $\epsilon > 0$, we have $\Lambda(U) \supset \cup_{\epsilon > 0} (1 + \epsilon)^{-1} \delta V = \delta V$.