examples of Cauchy-Riemann equations

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To illustrate the Cauchy-Riemann equations, we may consider a few examples. Let f be the squaring function, i.e. for any complex number z, we have $f(z) = z^2$.

We now separate real from imaginary parts. Letting x and y be real variables we have

$$f(x+iy) = (x+iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + i(2xy).$$

Defining real functions u and v by $f(x+iy) = u(x,y) + iv(x,y)$ and taking derivatives, we have

$$\begin{aligned} u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy \\ \frac{\partial u}{\partial x} &= 2x \\ \frac{\partial u}{\partial y} &= -2y \\ \frac{\partial v}{\partial x} &= 2y \\ \frac{\partial v}{\partial y} &= 2x \end{aligned}$$

Since $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. the Cauchy-Riemann relations are seen to be satisfied.

Next, consider the complex conjugation function. In this case, $\overline{x+iy} = x - iy$, so we have $u(x,y) = x$ and $v(x,y) = y$. Taking derivatives,

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1 \\ \frac{\partial u}{\partial y} &= 0 \\ \frac{\partial v}{\partial x} &= 0 \\ \frac{\partial v}{\partial y} &= -1 \end{aligned}$$

Because $\partial u/\partial x \neq \partial v/\partial y$, the Cauchy-Riemann equations are *not* satisfied do the conjugation function is not holomorphic. Likewise, one can show that the functions $\Re, \Im, and |\cdot|$ which appear in complex analysis are not holomorphic.

For our next example, we try a polynomial. Let $f(Z) = z^3 + 2z + 5$. Writing $z = x + iy$ and $f = u + iv$, we find that $u(x, y) = x^3 - 3xy^2 + 2x + 5$ and $v(x, y) = 3x^2y - y^3 + 2y$. Taking partial derivatives, one can confirm that the Cauchy-Riemann equations are satisfied, so we have a holomorphic function.

More generally, we can show that *all* complex polynomials are holomorphic. Since the Cauchy-Riemann equations are linear, it suffices to check that integer powers are holomorphic. We can do this by an induction argument. That $f(z) = z$ satisfies the equations is trivial and we have shown that $f(z) = z^2$ also satisfies them. Let us assume that $f(z) = z^n$ happens to satisfy the Cauchy-Riemann equations for a particular value of n and write

$$\begin{aligned}(x + iy)^n &= u(x, y) + iv(x, y)\\ (x + iy)^{n+1} &= \tilde{u}(x, y) + i\tilde{v}(x, y)\end{aligned}$$

By elementary algebra, we have

$$\begin{aligned}\tilde{u}(x, y) &= xu(x, y) - yv(x, y)\\ \tilde{v}(x, y) &= yu(x, y) + xv(x, y).\end{aligned}$$

By elementary calculus, we have

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial x} &= u + x\frac{\partial u}{\partial x} - y\frac{\partial v}{\partial x}\\ \frac{\partial \tilde{u}}{\partial y} &= -v + x\frac{\partial u}{\partial y} - y\frac{\partial v}{\partial y}\\ \frac{\partial \tilde{v}}{\partial x} &= v + y\frac{\partial u}{\partial x} + x\frac{\partial v}{\partial x}\\ \frac{\partial \tilde{v}}{\partial y} &= u + y\frac{\partial u}{\partial y} + x\frac{\partial v}{\partial y}\end{aligned}$$

so

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial x} - \frac{\partial \tilde{v}}{\partial y} &= x\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\\ \frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x} &= x\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\end{aligned}$$

Since the terms in parentheses are zero on account of u and v satisfying the Cauchy-Riemann equations, it follows that $\tilde{u}$ and $\tilde{v}$ also satisfy the Cauchy-Riemann equations. By induction, $f(z) = z^n$ is holomorphic for all positive integers n.

As our next example, we consider the complex square root. As shown in the entry taking square root algebraically, we have the following equality:

$$\sqrt{x+iy} = \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + (\operatorname{sign} y) i \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}$$

Differentiating and simplifying,

$$\begin{aligned}
u(x,y) &= \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} \\
v(x,y) &= (\operatorname{sign} y)\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}} \\
\frac{\partial u}{\partial x} &= \frac{\sqrt{2}}{4}\frac{\frac{x}{\sqrt{x^2+y^2}}+1}{\sqrt{\sqrt{x^2+y^2}+x}} = \frac{\sqrt{2}}{4}\frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}} \\
\frac{\partial u}{\partial y} &= (\operatorname{sign} y)\frac{\sqrt{2}}{4}\frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}+x}} = (\operatorname{sign} y)\frac{\sqrt{2}}{4}\frac{y}{\sqrt{x^2+y^2}\sqrt{\sqrt{x^2+y^2}+x}} \\
\frac{\partial v}{\partial x} &= (\operatorname{sign} y)\frac{\sqrt{2}}{4}\frac{\frac{x}{\sqrt{x^2+y^2}}-1}{\sqrt{\sqrt{x^2+y^2}-x}} = -(\operatorname{sign} y)\frac{\sqrt{2}}{4}\frac{\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}} \\
\frac{\partial v}{\partial y} &= \frac{\sqrt{2}}{4}\frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}-x}} = \frac{\sqrt{2}}{4}\frac{y}{\sqrt{x^2+y^2}\sqrt{\sqrt{x^2+y^2}-x}}.
\end{aligned}$$

Pulling out a common factor and placing over a common denominator,

$$
\begin{aligned}
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= \frac{\sqrt{2}}{4\sqrt{x^2+y^2}} \left(\sqrt{\sqrt{x^2+y^2}+x} - \frac{y}{\sqrt{\sqrt{x^2+y^2}-x}} \right) \\
&= \frac{\sqrt{2}}{4\sqrt{x^2+y^2}} \frac{\sqrt{\sqrt{x^2+y^2}+x} \cdot \sqrt{\sqrt{x^2+y^2}-x} - y}{\sqrt{\sqrt{x^2+y^2}-x}} \\
&= \frac{\sqrt{2}}{4\sqrt{x^2+y^2}} \frac{\sqrt{(x^2+y^2)-y^2} - y}{\sqrt{\sqrt{x^2+y^2}-x}} = 0 \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= (\operatorname{sign} y) \frac{\sqrt{2}}{4\sqrt{x^2+y^2}} \left(\frac{y}{\sqrt{\sqrt{x^2+y^2}+x}} - \sqrt{\sqrt{x^2+y^2}-x} \right) \\
&= (\operatorname{sign} y) \frac{\sqrt{2}}{4\sqrt{x^2+y^2}} \frac{y - \sqrt{\sqrt{x^2+y^2}+x} \cdot \sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{\sqrt{x^2+y^2}+x}} \\
&= (\operatorname{sign} y) \frac{\sqrt{2}}{4\sqrt{x^2+y^2}} \frac{\sqrt{(x^2+y^2)-x^2} - y}{\sqrt{\sqrt{x^2+y^2}+x}} = 0,
\end{aligned}
$$

so the Cauchy-Riemann equations are satisfied. More generally, it can be shown that all complex algebraic functions and fractional powers satisfy the Cauchy-Riemann equations. However, as suggested by the above derivation, a direct verification could be tedious, so it is better to use an indirect approach.

Finally, we finish up with two examples of transcendental functions, the complex exponential and the complex logarithm, The complex exponential is defined as $\exp(x+iy) = \exp(x)(\cos y + i \sin y)$. Hence we have

$$\begin{aligned}
u(x,y) &= \exp(x)\cos(y) \\
v(x,y) &= \exp(x)\sin(y) \\
\frac{\partial u}{\partial x} &= \exp(x)\cos(y) \\
\frac{\partial u}{\partial y} &= -\exp(x)\sin(y) \\
\frac{\partial v}{\partial x} &= \exp(x)\sin(y) \\
\frac{\partial v}{\partial y} &= \exp(x)\cos(y)
\end{aligned}$$

Thus we see that the complex exponential function is holomorphic.

The complex logarithm may be defined as $\log(x+iy) = 1/2\log(x^2+y^2) + i\arctan(y/x)$. Hence we have

$$\begin{aligned}
u(x,y) &= 1/2\log(x^2+y^2) \\
v(x,y) &= \arctan(y/x) \\
\frac{\partial u}{\partial x} &= \frac{x}{x^2+y^2} \\
\frac{\partial u}{\partial y} &= \frac{y}{x^2+y^2} \\
\frac{\partial v}{\partial x} &= \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2} \\
\frac{\partial v}{\partial y} &= \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}
\end{aligned}$$

Hence the complex logarithm is holomorphic.