symmetric quartic equation

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0.1 Symmetric quartic

Besides the biquadratic equation, there are other of quartic equations

$$a_0z^4+a_1z^3+a_2z^2+a_3z+a_4 \;=\; 0 \qquad (a_0 \;\neq\; 0), \tag{1}$$

which can be reduced to quadratic equations. If the left hand side of (1) is $P(z)$, one may write the identity

$$\frac{P(z)}{z^2} \;=\; \left(a_0z^2+\frac{a_4}{z^2}\right)+\left(a_1z+\frac{a_3}{z}\right)+a_2. \tag{2}$$

If we assume first that $a_4 = a_0$ and $a_3 = a_1$, the identity is

$$\frac{P(z)}{z^2} \;=\; a_0\left(z^2+\frac{1}{z^2}\right)+a_1\left(z+\frac{1}{z}\right)+a_2.$$

We set $z+\dfrac{1}{z} := x$, whence $z^2+\dfrac{1}{z^2} = x^2-2$; hence the identity is simplified to

$$\frac{P(z)}{z^2} \;=\; a_0x^2+a_1x+a_2-2a_0.$$

Accordingly, we obtain the `http://planetmath.org/Equation`roots of the so-called *symmetric quartic equation*

$$a_0z^4+a_1z^3+a_2z^2+a_1z+a_0 \;=\; 0 \qquad (a_0 \;\neq\; 0), \tag{3}$$

if we first determine the roots x_1 and x_2 of the quadratic

$$a_0x^2+a_1x+a_2-2a_0 \;=\; 0$$

and then solve the equations $z+\frac{1}{z} = x_1$ and $z+\frac{1}{z} = x_2$ which can be written

$$z^2-x_1z+1 \;=\; 0, \quad z^2-x_2z+1 \;=\; 0. \tag{4}$$

Note, that the roots of either equations (4) are inverse numbers of each other (see properties of quadratic equations). Therefore, as well the inverse number of any root of the symmetric quartic (3) is a root of (3); this fact is, by the way, clear also because of the identity

$$z^4P\left(\frac{1}{z}\right) \;=\; P(z).$$

Example. The equation

$$2z^4 - 5z^3 + 4z^2 - 5z + 2 \;=\; 0$$

is symmetric. Thus we solve first

$$2x^2 - 5x + 4 - 2\cdot 2 \;=\; 0,$$

which yields $x_1 = 0$, $x_2 = \frac{5}{2}$. Secondly we solve

$$z^2 + 1 \;=\; 0, \quad z^2 - \frac{5}{2}z + 1 \;=\; 0$$

which yield all the four roots $z = \pm i$, $z = \frac{1}{2}$, $z = 2$ of the quartic.

0.2 Almost symmetric quartic

There is still the quartic equation

$$a_0z^4 + a_1z^3 + a_2z^2 - a_1z + a_0 \;=\; 0 \qquad (a_0 \neq 0), \tag{5}$$

which reduces to quadratics — the identity (2) for it reads

$$\frac{P(z)}{z^2} \;=\; a_0\left(z^2 + \frac{1}{z^2}\right) + a_1\left(z - \frac{1}{z}\right) + a_2.$$

The substitution $z - \dfrac{1}{z} := x$ converts it to

$$\frac{P(z)}{z^2} \;=\; a_0x^2 + a_1x + a_2 + 2a_0.$$

Thus, (5) can be solved by determining first the roots x_1 and x_2 of

$$a_0x^2 + a_1x + a_2 + 2a_0 \;=\; 0,$$

then the roots z of $z - \frac{1}{z} = x_1$ and $z - \frac{1}{z} = x_2$ which may written

$$z^2 - x_1z - 1 \;=\; 0, \quad z^2 - x_2z - 1 \;=\; 0.$$

Hence one infers, that if z is a root of (5), so is also its opposite inverse $-\frac{1}{z}$; this is apparent also due to the identity

$$z^4P\left(-\frac{1}{z}\right) \;=\; P(z).$$

References

[1] Ernst Lindelöf: *Johdatus korkeampaan analyysiin.* Fourth edition. Werner Söderström Osakeyhtiö, Porvoo ja Helsinki (1956).

[2] E. Lindelöf: *Einführung in die höhere Analysis.* Nach der ersten schwedischen und zweiten finnischen Auflage auf deutsch herausgegeben von E. Ullrich. Teubner, Leipzig (1934).