proof of Gauss' digamma theorem

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Proof. The proof follows the argument given in [**?**], which in turn derives from that given in [**?**].

The first formula is the logarithmic derivative of

$$\Gamma(x+n) = (x+n-1)(x+n-2)\cdots x\Gamma(x)$$

By the partial fraction decomposition satisfied by the ψ function,

$$\psi\left(\frac{p}{q}\right) + \gamma = \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{q}{p+nq}\right) = \lim_{t\to 1^-}\sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{q}{p+nq}\right)t^{p+nq}$$

using Abel's limit theorem.

Now,

$$\sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{q}{p+nq}\right)t^{p+nq} = \sum_{n=0}^{\infty}\frac{t^{p+nq}}{n+1} - \sum_{n=0}^{\infty}\frac{qt^{p+nq}}{p+nq} = t^{p-q}\sum_{n=0}^{\infty}\frac{t^{(n+1)q}}{n+1} - q\sum_{n=0}^{\infty}\frac{t^{p+nq}}{p+nq}$$

Since

$$-\ln(1-t) = \sum_{n=1}^{\infty}\frac{t^n}{n}$$

the first term is

$$-t^{p-q}\ln(1-t^q)$$

Using the algorithm for `http://planetmath.org/ExtractingEveryNthTermOfASeries`extracting every q^{th} term of a series, the second term is

$$\sum_{n=0}^{q-1}\omega^{-np}\ln(1-\omega^n t)$$

and therefore

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{q}{p+nq}\right)t^{p+nq} &= -t^{p-q}\ln(1-t^q) + \sum_{n=0}^{q-1}\omega^{-np}\ln(1-\omega^n t)\\ &= -t^{p-q}\ln\frac{1-t^q}{1-t} - (t^{p-1}-1)\ln(1-t) + \sum_{n=1}^{q-1}\omega^{-np}\ln(1-\omega^n t)\end{aligned}$$

Let $t \to 1^-$ to get

$$\psi\left(\frac{p}{q}\right) = -\gamma - \ln q + \sum_{n=1}^{q-1}\omega^{-np}\ln(1-\omega^n)$$

Replace p by $q-p$ and add the two expressions to obtain

$$\psi\left(\frac{p}{q}\right)+\psi\left(\frac{q-p}{q}\right)=-2\gamma-2\ln q+2\sum_{n=1}^{q-1}\cos\left(\frac{2\pi np}{q}\right)\ln(1-\omega^n)$$

The left side is real, so it is equal to the real part of the right side. But

$$\Re(\ln(1-\omega^n))=\ln|1-\omega^n|^{1/2}=\ln\left|\left(1-\cos\frac{2\pi n}{q}\right)^2+\sin^2\frac{2\pi n}{q}\right|^{1/2}=\frac{1}{2}\ln\left(2-2\cos\frac{2\pi n}{q}\right)$$

and so

$$\psi\left(\frac{p}{q}\right)+\psi\left(\frac{q-p}{q}\right)=-2\gamma-2\ln q+\sum_{n=1}^{q-1}\cos\left(\frac{2\pi np}{q}\right)\ln(2-2\cos\frac{2\pi n}{q}) \quad (1)$$

But

$$\psi(x)-\psi(1-x)=\frac{d}{dx}\ln(\Gamma(x)\Gamma(1-x))=-\pi\cot\pi x$$

by the Euler reflection formula and thus

$$\psi\left(\frac{p}{q}\right)-\psi\left(\frac{q-p}{q}\right)=-\pi\cot\frac{\pi p}{q} \quad (2)$$

Add equations (**??**) and (**??**) to get

$$\begin{aligned}\psi\left(\frac{p}{q}\right)&=-\gamma-\frac{\pi}{2}\cot\frac{\pi p}{q}-\ln q+\frac{1}{2}\sum_{n=1}^{q-1}\cos\frac{2\pi np}{q}\ln\left(2-2\cos\frac{2\pi n}{q}\right)\\&=-\gamma-\frac{\pi}{2}\cot\frac{\pi p}{q}-\ln q+\sum_{n=1}^{q-1}\cos\frac{2\pi np}{q}\ln\left(2\sin\frac{\pi n}{q}\right)\end{aligned}$$

where the last equality holds since

$$\ln(2-2\cos(2\theta))=\ln(2-2(1-2\sin^2\theta)=\ln(4\sin^2\theta)=2\ln(2\sin\theta)$$

References

[1] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, 2001.

[2] J.L. Jensen [1915-1916], An elementary exposition of the theory of the gamma function, *Ann. Math.* **17**, 124-166.