variant of Cauchy integral formula

Canonical name	VariantOfCauchyIntegralFormula
Date of creation	2013-03-22 18:54:15
Last modified on	2013-03-22 18:54:15
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	6
Author	pahio (2872)
Entry type	Theorem
Classification	msc 30E20
Synonym	Cauchy integral formula
Related topic	CauchyIntegralFormula
Related topic	CorollaryOfCauchyIntegralTheorem
Related topic	ExampleOfFindingTheGeneratingFunction
Related topic	GeneratingFunctionOfLaguerrePolynomials
Related topic	GeneratingFunctionOfHermitePolynomials

Theorem. Let $f(z)$ be holomorphic in a domain A of $\mathbb{C}$. If C is a closed contour not intersecting itself which with its domain is contained in A and if z is an arbitrary point inside C, then

$$f(z) \;=\; \frac{1}{2i\pi}\oint_C \frac{f(t)}{t-z}\,dt. \tag{1}$$

Proof. Let ε be any positive number. Denote by C_r the circles with radius r and centered in z. We have

$$\oint_C \frac{f(t)}{t-z}\,dt \;=\; \oint_C \frac{f(z)+(f(t)-f(z))}{t-z}\,dt \;=\; \underbrace{\oint_C \frac{f(z)}{t-z}\,dt}_{I} + \underbrace{\oint_C \frac{f(t)-f(z)}{t-z}\,dt}_{J}.$$

According to the corollary of Cauchy integral theorem and its example, we may write

$$I \;=\; f(z)\oint_C \frac{dt}{t-z} \;=\; 2i\pi f(z).$$

If $0 < r <$ some r_0, we have

$$J \;=\; \oint_{C_r} \frac{f(t)-f(z)}{t-z}\,dt.$$

The continuity of f in the point z implies, that

$$|f(t)-f(z)| < \varepsilon$$

when $0 < |t-z| <$ some δ_ε i.e. when

$$t \in C_r \text{ and } 0 < r < \text{ some } r_1. \tag{2}$$

If (2) is in , we obtain first

$$\left|\frac{f(t)-f(z)}{t-z}\right| \;=\; \frac{|f(t)-f(z)|}{|t-z|} \;=\; \frac{|f(t)-f(z)|}{r} \;<\; \frac{\varepsilon}{r},$$

whence, by the estimation theorem of integral,

$$|J| \;\leqq\; \frac{\varepsilon}{r}\cdot 2\pi r \;=\; 2\pi\varepsilon \quad \text{for} \quad 0 < r < \min\{r_0,\, r_1\},$$

and lastly

$$\left|\frac{1}{2i\pi}\oint_C \frac{f(t)}{t-z}\,dt - f(z)\right| \;=\; \left|\frac{1}{2i\pi}J\right| \;\leqq\; \frac{1}{2\pi}\cdot 2\pi\varepsilon \;=\; \varepsilon \quad \text{when } 0 < r < \min\{r_0,\, r_1\}. \tag{3}$$

This result implies (1).