proof of fundamental theorem of algebra (due to Cauchy)

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We will prove that any equation

$$f(z) \;:=\; z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n \;=\; 0,$$

where the coefficients a_j are complex numbers and $n \geqq 1$, has at least one `http://planetmath.org/Equation`root in $\mathbb{C}$.

Proof. We can suppose that $a_n \neq 0$. Denote $z := x+iy$ where $x,\, y$ are real. Then the function

$$g(x,\, y) \;:=\; |f(z)| \;=\; |f(x+iy)|$$

is defined and continuous in the whole $\mathbb{R}^2$. Let $c := \sum_{j=1}^{n} |a_j|$; it is positive. Using the triangle inequality we make the estimation

$$\begin{aligned}
|f(z)| \;&=\; |z|^n \left|1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_n}{z^n}\right| \\
&\geqq\; \left(1 - \frac{|a_1|}{|z|} - \frac{|a_2|}{|z|^2} + \ldots - \frac{|a_n|}{|z|^n}\right) \\
&\geqq\; \left(1 - \frac{|a_1|}{|z|} - \frac{|a_2|}{|z|} + \ldots - \frac{|a_n|}{|z|}\right) \\
&=\; |z|^n \left(1 - \frac{c}{|z|}\right) \;\geqq\; \frac{1}{2}|z|^n,
\end{aligned}$$

being true for $|z| > \max\{1,\, 2c\}$. Denote $r := \max\{1,\, 2c,\, \sqrt[n]{2|a_n|}\}$. Consider the disk $x^2+y^2 \leqq r^2$. Because it is compact, the function $g(x,\, y)$ attains at a point $(x_0,\, y_0)$ of the disk its absolute minimum value (infimum) in the disk. If $|z| > r$, we have

$$g(x,\, y) \;=\; |f(z)| \;\geqq\; \frac{1}{2}|z|^n \;>\; \frac{1}{2}r^n \;\geqq\; \frac{1}{2}\left(\sqrt[n]{2|a_n|}\right)^n \;=\; |a_n| \;>\; 0.$$

Thus

$$g(x_0,\, y_0) \;\leqq\; g(0,\, 0) \;=\; |a_n| \;<\; |f(z)| \quad \text{for } |z| \;>\; r.$$

Hence $g(x_0,\, y_0)$ is the absolute minimum of $g(x,\, y)$ in the whole complex plane. We show that $g(x_0,\, y_0) = 0$. Therefore we make the antithesis that $g(x_0,\, y_0) > 0$.

Denote $z_0 := x_0+iy_0, \quad z := z_0+u$ and

$$f(z) \;=\; f(z_0+u) \;:=\; b_n + b_{n-1}u + b_{n-2}u^2 + \ldots + b_1 u^{n-1} + u^n.$$

Then $b_n = f(z_0) \neq 0$ by the antithesis. Moreover, denote

$$c_j \;:=\; \frac{b_j}{b_n} \quad (j \;=\; 1,\, 2,\, \ldots,\, n), \quad c_0 \;:=\; \frac{1}{b_n}.$$

and assume that $c_{n-1} = c_{n-2} = \ldots = c_{n-k+1} = 0$ but $c_{n-k} \neq 0$. Thus we may write

$$f(z) \;=\; b_n(1 + c_{n-k}u^k + c_{n-k-1}u^{k+1} + \ldots + c_0 u^n).$$

If $c_{n-k} = p(\cos\alpha + i\sin\alpha)$ and $u = \varrho(\cos\varphi + i\sin\varphi)$, then

$$c_{n-k}u^k \;=\; p\varrho^k[\cos(\alpha + k\varphi) + i\sin(\alpha + k\varphi)]$$

by de Moivre identity. Choosing $\varrho \leqq 1$ and $\varphi = \frac{\pi-\alpha}{k}$ we get $c_{n-k}u^k = -p\varrho^k$ and can make the estimation

$$|\underbrace{c_{n-k-1}u^{k+1} + \ldots + c_0 u^n}_{h(u)}| \;\leqq\; |c_{n-k-1}|\varrho^{k+1} + \ldots + |c_0|\varrho^n \;\leqq\; (|c_{n-k-1}| + \ldots + |c_0|)\varrho^{k+1} \;:=\; R\varrho^{k+1}$$

where R is a constant. Let now $\varrho = \min\{1,\, \sqrt[k]{\frac{1}{p}},\, \frac{p}{2R}\}$. We obtain

$$\begin{aligned}
|f(z)| \;&=\; |b_n|\cdot|1 - p\varrho^k + h(u)| \\
&\leqq\; |b_n|\left[|1 - p\varrho^k| + |h(u)|\right] \\
&\leqq\; |b_n|\left[1 - p\varrho^k + R\varrho^{k+1}\right] \\
&\leqq\; |b_n|\left[1 - \varrho^k(p - R\varrho)\right] \\
&\leqq\; |b_n|\left[1 - \varrho^k(p - R\cdot\frac{p}{2R})\right] \\
&\leqq\; |b_n|\left[1 - \frac{1}{p}\cdot\frac{p}{2}\right] \\
&\leqq\; \frac{|b_n|}{2} \;<\; |b_n| \;=\; |f(z_0)|,
\end{aligned}$$

which result is impossible since $|f(z_0|$ was the absolute minimum. Consequently, the antithesis is wrong, and the proof is settled.