proof of limit rule of product

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Let f and g be http://planetmath.org/RealFunctionreal or complex functions having the limits

$$\lim_{x\to x_0} f(x) = F \quad \text{and} \quad \lim_{x\to x_0} g(x) = G.$$

Then also the limit $\lim\limits_{x\to x_0} f(x)g(x)$ exists and equals FG.

Proof. Let ε be any positive number. The assumptions imply the existence of the positive numbers δ_1, δ_2, δ_3 such that

$$|f(x)-F| < \frac{\varepsilon}{2(1+|G|)} \ \text{ when } \ 0 < |x-x_0| < \delta_1 \tag{1}$$

$$|g(x)-G| < \frac{\varepsilon}{2(1+|F|)} \ \text{ when } \ 0 < |x-x_0| < \delta_2, \tag{2}$$

$$|g(x)-G| < 1 \ \text{ when } \ 0 < |x-x_0| < \delta_3. \tag{3}$$

According to the condition (3) we see that

$$|g(x)| = |g(x)-G+G| \leqq |g(x)-G| + |G| < 1+|G| \ \text{ when } \ 0 < |x-x_0| < \delta_3.$$

Supposing then that $0 < |x-x_0| < \min\{\delta_1,\, \delta_2,\, \delta_3\}$ and using (1) and (2) we obtain

$$\begin{aligned}
|f(x)g(x)-FG| &= |f(x)g(x)-Fg(x)+Fg(x)-FG| \\
&\leqq |f(x)g(x)-Fg(x)| + |Fg(x)-FG| \\
&= |g(x)|\cdot|f(x)-F| + |F|\cdot|g(x)-G| \\
&< (1+|G|)\frac{\varepsilon}{2(1+|G|)} + (1+|F|)\frac{\varepsilon}{2(1+|F|)} \\
&= \varepsilon
\end{aligned}$$

This settles the proof.