a harmonic function on a graph which is bounded below and nonconstant

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There exists no harmonic function on all of the d-dimensional grid $\mathbb{Z}^d$ which is bounded below and nonconstant. This categorises a particular property of the grid; below we see that other graphs can admit such harmonic functions.

Let $\mathcal{T}_3 = (V_3, E_3)$ be a 3-regular tree. Assign "levels" to the vertices of $\mathcal{T}_3$ as follows: Fix a vertex $o \in V_3$, and let π be a branch of $\mathcal{T}_3$ (an infinite simple path) from o. For every vertex $v \in V_3$ of $\mathcal{T}_3$ there exists a *unique* shortest path from v to a vertex of π; let $\ell(v) = |\pi|$ be the length of this path.

Now define $f(v) = 2^{-\ell(v)} > 0$. Without loss of generality, note that the three neighbours u_1, u_2, u_3 of v satisfy $\ell(u_1) = \ell(v) - 1$ ("u_1 is the parent of v"), $\ell(u_2) = \ell(u_3) = \ell(v) + 1$ ("u_2, u_3 are the siblings of v"). And indeed, $\frac{1}{3}\left(2^{\ell(v)-1} + 2^{\ell(v)+1} + 2^{\ell(v)+1}\right) = 2^{\ell(v)}$.

So f is a positive nonconstant harmonic function on $\mathcal{T}_3$.