contour integral

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Defines	contour

Let f be a complex-valued function defined on the image of a http://planetmath.org/Curvecur
α: $[a,b] \to \mathbb{C}$, let $P = \{a_0, ..., a_n\}$ be a http://planetmath.org/Partition3partition of $[a,b]$. We will restrict our attention to *contours*, i.e. curves for which the parametric equations consist of a finite number of continuously differentiable arcs. If the sum

$$\sum_{i=1}^{n} f(z_i)(\alpha(a_i) - \alpha(a_{i-1})),$$

where z_i is some point $\alpha(t_i)$ such that $a_{i-1} \leqslant t_i \leqslant a_i$, converges as n tends to infinity and the greatest of the numbers $a_i - a_{i-1}$ tends to zero, then we define the *contour integral* of f along α to be the integral

$$\int_\alpha f(z)dz := \int_a^b f(\alpha(t))d\alpha(t)$$

Notes

(i) If $\mathrm{Im}(\alpha)$ is a segment of the real axis, then this definition reduces to that of the Riemann integral of $f(x)$ between $\alpha(a)$ and $\alpha(b)$.

(ii) An alternative definition, making use of the Riemann-Stieltjes integral, is based on the fact that the definition of this can be extended without any other changes in the wording to cover the cases where f and α are complex-valued functions.

Now let α be any curve $[a,b] \to \mathbb{R}^2$. Then α can be expressed in terms of the components (α_1, α_2) and can be associated with the complex-valued function

$$z(t) = \alpha_1(t) + i\alpha_2(t).$$

Given any complex-valued function of a complex variable, f say, defined on $\mathrm{Im}(\alpha)$ we define the **contour integral** of f along α, denoted by

$$\int_\alpha f(z)dz$$

by

$$\int_\alpha f(z)dz = \int_a^b f(z(t))dz(t)$$

whenever the complex Riemann-Stieltjes integral on the right exists.

(iii) Reversing the direction of the curve changes the sign of the integral.

(iv) The contour integral always exists if α is rectifiable and f is continuous.

(v) If α is piecewise smooth and the contour integral of f along α exists, then

$$\int_\alpha f dz = \int_a^b f(z(t))z'(t)dt.$$