Euler relation

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Defines	Euler's identity

Euler's relation (also known as *Euler's formula*) is considered the first between the fields of algebra and geometry, as it relates the exponential function to the trigonometric sine and cosine functions.

Euler's relation states that

$$e^{ix} = \cos x + i \sin x$$

Start by noting that

$$i^k = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 4 \\ i & \text{if } k \equiv 1 \pmod 4 \\ -1 & \text{if } k \equiv 2 \pmod 4 \\ -i & \text{if } k \equiv 3 \pmod 4 \end{cases}$$

Using the Taylor series expansions of e^x, $\sin x$ and $\cos x$ (see the entries on the complex exponential function and the complex sine and cosine), it follows that

$$\begin{aligned} e^{ix} &= \textstyle\sum_{n=0}^{\infty} \frac{i^n x^n}{n!} \\ &= \textstyle\sum_{n=0}^{\infty} \left(\frac{x^{4n}}{(4n)!} + \frac{ix^{4n+1}}{(4n+1)!} - \frac{x^{4n+2}}{(4n+2)!} - \frac{ix^{4n+3}}{(4n+3)!} \right) \end{aligned}$$

Because the series expansion above is absolutely convergent for all x, we can rearrange the terms of the series as

$$\begin{aligned} e^{ix} &= \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= \cos x + i \sin x \end{aligned}$$

As a special case, we get the beautiful and well-known identity, often called *Euler's identity*:

$$e^{i\pi} = -1$$