proof of criterion for conformal mapping of Riemannian spaces

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In this attachment, we prove that the a mapping f of Riemannian (or pseudo-Riemannian) spaces (M, g) and (N, h) is conformal if and only if $f^*h = sg$ for some scalar field s (on M).

The key observation is that the angle A between curves S and T which intersect at a point P is determined by the tangent vectors to these two curves (which we shall term s and t) and the metric at that point, like so:

$$\cos A = \frac{g(s,t)}{\sqrt{g(s,s)}\sqrt{g(t,t)}}$$

Moreover, given any tangent vector at a point, there will exist at least one curve to which it is the tangent. Also, the tangent vector to the image of a curve under a map is the pushforward of the tangent to the original curve under the map; for instance, the tangent to $f(S)$ at $f(P)$ is f^*s. Hence, the mapping f is conformal if and only if

$$\frac{g(u,v)}{\sqrt{g(u,u)}\sqrt{g(v,v)}} = \frac{h(f^*u, f^*v)}{\sqrt{h(f^*u, f^*u)}\sqrt{h(f^*v, f^*v)}}$$

for all tangent vectors u and v to the manifold M. By the way pushforwards and pullbacks work, this is equivalent to the condition that

$$\frac{g(u,v)}{\sqrt{g(u,u)}\sqrt{g(v,v)}} = \frac{(f^*h)(u,v)}{\sqrt{(f^*h)(u,u)}\sqrt{(f^*h)(v,v)}}$$

for all tangent vectors u and v to the manifold N. Now, by elementary algebra, the above equation is equivalent to the requirement that there exist a scalar s such that, for all u and v, it is the case that $g(u,v) = sh^*(u,v)$ or, in other words, $f^*h = sg$ for some scalar field s.