construction of Riemann surface using paths

Note: All arcs and curves are assumed to be smooth in this entry.

Let f be a complex function defined in a disk D about a point $z_0 \in \mathbb{C}$. In this entry, we shall show how to construct a Riemann surface such that f may be analytically continued to a function on this surface by considering paths in the complex plane.

Let $\mathcal{P}$ denote the class of paths on the complex plane having z_0 as an endpoint along which f may be analytically continued. We may define an equivalence relation $\sim$ on this set — $C_1 \sim C_2$ if C_1 and C_2 have the same endpoint and there exists a one-parameter family of paths along which f can be analytically continued which includes C_1 and C_2.

Define $\mathcal{S}$ as the quotient of $\mathcal{P}$ modulo $\sim$. It is possible to extend f to a function on $\mathcal{S}$. If $C \in \mathcal{P}$, let $f(C)$ be the value of the analytic continuation of f at the endpoint of C (not z_0, of course, but the other endpoint). By the monodromy theorem, if $C_1 \sim C_2$, then $f(C_1) = f(C_2)$. Hence, f is well defined on the quotient $\mathcal{S}$.

Also, note that there is a natural projection map $\pi \colon \mathcal{S} \to \mathbb{C}$. If C is an equivalence class of paths in $\mathcal{S}$, define $\pi(C)$ to be the common endpoint of those paths (not z_0, of course, but the other endpoint).

Next, we shall define a class of subsets of $\mathcal{S}$. If f can be analytically continued from along a path C from z_0 to z_1 then there must exist an open disk D' centered about z_1 in which the continuation of f is analytic. Given any $z \in D'$, let $C(z)$ be the concatenation of the path C from z_0 to z_1 and the straight line segment from z_1 to z (which lies inside D'). Let $N(C, D') \subset \mathcal{S}$ be the set of all such paths.

We will define a topology of $\mathcal{S}$ by taking all these sets $N(C, D')$ as a basis. For this to be legitimate, it must be the case that, if C_3 lies in the intersection of two such sets, $N(C_1, D_1)$ and $N(C_2, D_2)$ there exists a basis element $N(C_3, D_3)$ contained in the intersection of $N(C_1, D_1)$ and $N(C_2, D_2)$. Since the endpoint of C_3 lies in the intersection of D_1 and D_2, there must exist a disk D_3 centered about this point which lies in the intersection of D_1 and D_2. It is easy to see that $N(C_3, D_3) \subset N(C_1, D_1) \cap N(C_2, D_2)$.

Note that this topology has the Hausdorff property. Suppose that C_1 and C_2 are distinct elements of $\mathcal{S}$. On the one hand, if $\pi(C_1) \neq \pi(C_2)$, then one can find disjoint open disks D_1 and D_2 centered about C_1 and C_2. Then $N(C_1, D_1) \cap N(C_2, D_2) = \emptyset$ because $\pi(N(C_1, D_1)) \cap \pi(N(C_2, D_2)) = D_1 \cap D_2 = \emptyset$. On the other hand, if $\pi(C_1) = \pi(C_2)$, then let D_3 be the smaller of the disks D_1 and D_2. Then $N(C_1, D_3) \cap N(C_2, D_3) = \emptyset$.

To complete the proof that $\mathcal{S}$ is a Riemann surface, we must exhibit co-

ordinate neighborhoods and homomorphisms. As coordinate neighborhoods, we shall take the neighborhoods $N(C, D)$ introduced above and as homomorphisms we shall take the restrictions of π to these neighborhoods. By the way that these neighborhoods have been defined, every element of $\mathcal{S}$ lies in at least one such neighborhood. When the domains of two of these homomorphisms overlap, the composition of one homomorphism with the inverse of the restriction of the other homomorphism to the overlap region is simply the identity map in the overlap region, which is analytic. Hence, $\mathcal{S}$ is a Riemann surface.