proof of weak maximum principle for real domains

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Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
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Author	rspuzio (6075)
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First, we show that, if $\Delta f > 0$ (where Δ denotes the Laplacian on $\mathbb{R}^d$) on K, then f cannot attain a maximum on the interior of K. Assume, to the contrary, that f did attain a maximum at a point p located on the interior of K. By the second derivative test, the matrix of second partial derivatives of f at p would have to be negative semi-definite. This would imply that the trace of the matrix is negative. But the trace of this matrix is the Laplacian, which was assumed to be strictly positive on K, so it is impossible for f to attain a maximum on the interior of K.

Next, suppose that $\Delta f = 0$ on K but that f does not attain its maximum on the boundary of K. Since K is compact, f must attain its maximum somewhere, and hence there exists a point p located in the interior of K at which f does attain its maximum. Since K is compact, the boundary of K is also compact, and hence the image of the boundary of K under f is also compact. Since every element of this image is strictly smaller than $f(p)$, there must exist a constant C such that $f(x) < C < f(p)$ whenever x lies on the boundary of K. Furthermore Since K is a compact subset of $\mathbb{R}^d$, it is bounded. Hence, there exists a constant $R > 0$ so that $|x - p| < R$ for all $x \in K$.

Consider the function g defined as

$$g(x) = f(x) + (f(p) - C)\frac{|x - p|^2}{R^2}$$

At any point $x \in K$,

$$g(x) < f(x) + f(p) - C$$

In particular, if x lies on the boundary of K, this implies that

$$g(x) < f(p)$$

Since $g(p) = f(p)$ this inequality implies that g cannot attain a maximum on the boundary of K.

This leads to a contradiction. Note that, since $\Delta f = 0$ on K,

$$\Delta g = \frac{d(f(p) - C)}{R^2} > 0$$

which implies that g cannot attain a maximum on the interior of K. However, since K is compact, g must attain a maximum somewhere on K. Since we have ruled out both the possibility that this maximum occurs in the interior and the possibility that it occurs on the boundary, we have a contradiction. The only way out of this contradiction is to conclude that f does attain its maximum on the boundary of K.