evaluating the gamma function at 1/2

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In the entry on the gamma function it is mentioned that $\Gamma(1/2) = \sqrt{\pi}$. In this entry we reduce the proof of this claim to the problem of computing the area under the bell curve. First note that by definition of the gamma function,

$$\begin{aligned} \Gamma(1/2) &= \int_0^\infty e^{-x} x^{-1/2}\, dx \\ &= 2\int_0^\infty e^{-x} \frac{1}{2\sqrt{x}}\, dx. \end{aligned}$$

Performing the substitution $u = \sqrt{x}$, we find that $du = \frac{1}{2\sqrt{x}}\, dx$, so

$$\Gamma(1/2) = 2\int_0^\infty e^{-u^2}\, du = \int_{\mathbb{R}} e^{-u^2}\, du,$$

where the last equality holds because e^{-u^2} is an even function. Since the area under the bell curve is $\sqrt{\pi}$, it follows that $\Gamma(1/2) = \sqrt{\pi}$.