proof of identity theorem of holomorphic functions

Canonical name	ProofOfIdentityTheoremOfHolomorphicFunctions
Date of creation	2013-03-22 14:40:41
Last modified on	2013-03-22 14:40:41
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Numerical id	12
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Entry type	Proof
Classification	msc 30A99

Since $z_0 \in D$, there exists an $\epsilon_0 > 0$ the closed disk of radius ϵ about z_0 is contained in D. Furthermore, both f_1 and f_2 are analytic inside this disk. Since z_0 is a limit point, there must exist a sequence $x_k{}_{k=1}^{\infty}$ of distinct points of s which converges to z_0. We may further assume that $|x_k - z_0| < \epsilon_0$ for every k.

By the theorem on the radius of convergence of a complex function, the Taylor series of f_1 and f_2 about z_0 have radii of convergence greater than or equal to ϵ_0. Hence, if we can show that the Taylor series of the two functions at z_0 ar equal, we will have shown that $f_1(z) = g_1(z)$ whenever $z < \epsilon$.

The n-th coefficient of the Taylor series of a function is constructed from the n-th derivative of the function. The n-th derivative may be expressed as a limit of n-th divided differences

$$f^{(n)}(z_0) = \lim_{y_1, \dots y_n \to z_0} \frac{\Delta^n f(y_1, \dots, y_n)}{\Delta^n (y_1, \dots, y_n)}$$

Suppose we choose the points at which to compute the divided differences as points of the sequence x_i. Then we have

$$f^{(n)}(z_0) = \lim_{m \to \infty} \frac{\Delta^n f(x_{m+1}, \dots, x_{m+n})}{\Delta^n (x_{m+1}, \dots, x_{m+n})}$$

Since $f_1(x_i) = f_2(x_i)$, it follows that $f_1^{(n)}(z_0) = f_2^{(n)}(z_0)$ for all n and hence $f_1(z) = f_2(z)$ when $|z - z - 0| < \epsilon_0$.

If D happens to be a circle centred about z_0, we are done. If not, let z_1 be any point of D such that $|z_1 - z_0| \geq \epsilon$. Since every connected open subset of the plane is arcwise connected, there exists an arc C with endpoints z_1 and z_0.

Define the function $M \colon D \to \mathbb{R}$ as follows

$$M(z) = \sup\{r \mid |z - w| < r \Rightarrow w \in D\} \cap [0, 1]$$

Because D is open, it follows that $0 < M(z) \leq 1$ for all $z \in D$.

We will now show that M is continuous. Let w_1 and w_2 be any two distinct points of D. If $M(w_1) > |w_1 - w_2|$, then a disk of radius $M(w_1) - |w_1 - w_2|$ about w_2 will be contained in the disk of radius $M(w_1)$ about w_1. Hence, by the definition of M, it will follow that $M(w_2) \geq M(w_1) - |w_1 - w_2|$. Therefore, for any two points w_1 and w_2, it is the case that $|M(w_1) - M(w_2)| \leq |w_1 - w_2|$, which implies that M is continuous.

Since M is continuous and the arc C is compact, M attains a minimum value m on C. Let $\mu > 0$ be chosen smaller strictly less than both $m/2$ and ϵ_0. Consider the set of all open disks of radius μ centred about ponts of C. By the way μ was selected, each of these disks lies inside D. Since C is compact a finite subset of these disks will serve to cover D. In other words, there exsits a finite set of points $y_1, y_2, \ldots y_n$ such that, if $z \in C$, then $|z - y_j| < \mu$ for some $j \in \{1, 2, \ldots, n\}$. We may assume that the y_j's are ordered so that, as one traverses C from z_0 to z_1, one encounters y_j before one encounters y_{j+1}. This imples that $|y_j - y_{j+1}| < \mu$. Without loss of generality, we may assume that $y_1 = z_0$ and $y_n = z_1$.

We shall now show that $f_1(z) = f_2(z)$ when $|z - y_j| < \mu$ for all j by induction. From our definitions it follows that $f_1(z) = f_2(z)$ when $|z - y_1| < \mu$. Next, we shall now show that if $f_1(z) = f_2(z)$ when $|z - y_j| \leq m/2$, then $f_1(z) = f_2(z)$ when $|z - y_{j+i}| \leq m/2$. Since $|y_j - y_{j+1}| < \mu$, there exists a point $w \in C$ and a constant $\epsilon > 0$ such that $|w - z| < \epsilon$ implies $|z - y_j| \leq \mu$ and $|z - y_{j+i}| \leq \mu$. By the induction hypothesis, $f_1(z) = f_2(z)$ when $|z - y_j| < \mu$. Consider a disk of radius m about w. By the definition of m, this disk lies inside D and, by what we have already shown, $f_1(z) = f_2(z)$ when $|z - w| \leq m$. Since $|w - y_{j+1}| < \mu < m/2$, it follows from the triangle inequality that $f_1(z) = f_2(z)$ when $|z - y_{j+1}| < \mu$.

In particular, the proposition just proven implies that $f_1(z_1) = f_1(z_1)$ since $z_1 = y_n$. This means that we have shown that $f_1(z) = f_2(z)$ for all $z \in D$.