zeros of Dirichlet eta function

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As stated in the http://planetmath.org/AnalyticContinuationOfRiemannZetaToCriticalS entry, the definition of the Riemann zeta function may be http://planetmath.org/AnalyticConti continued from the half-plane $\Re s > 1$ to the half-plane $\Re s > 0$ by using the Dirichlet eta function $\eta(s)$ via the equation

$$\zeta(s) \;=\; \frac{\eta(s)}{1-\frac{2}{2^s}}. \tag{1}$$

Then only the status of the points

$$s_n \;:=\; 1+n\cdot\frac{2\pi i}{\ln 2} \qquad (n \in \mathbb{Z}) \tag{2}$$

which are the zeros of $1-\frac{2}{2^s}$, remains : are they poles of $\zeta(s)$ or not? E. Landau has 1909 signaled this problem, which has been elementarily solved not earlier than after 40 years, by D. V. Widder. He proved that those numbers, except $s=1$, are also zeros of $\eta(s)$. This means that they only are removable singularities of $\zeta(s)$ and that (1) in fact extends $\zeta(s)$ to every points of the half-plane $\Re s > 0$ except $s=1$.

A new direct proof by J. Sondow of the vanishing of the Dirichlet eta function at the points $s_n \neq 1$ was published in 2003. It is based on a relation between the partial sums $\eta_n(s)$ and $\zeta_n(s)$ of the series defining respectively the functions $\eta(s)$ and $\zeta(s)$ for $\Re s > 1$, which involves the approximation of an integral by a Riemann sum.

With some clever but not so complicated performed on finite sums, Sondow writes for any s the following:

$$\begin{aligned}
\eta_{2n}(s) &= 1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+-\ldots+\frac{(-1)^{2n-1}}{(2n)^s}\\
&= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots+\frac{(-1)^{2n-1}}{(2n)^s}-2\left(\frac{1}{2^s}+\frac{1}{4^s}+\ldots+\frac{1}{(2n)^s}\right)\\
&= \left(1-\frac{2}{2^s}\right)\zeta_{2n}(s)+\frac{2}{2^s}\left(\frac{1}{(n+1)^s}+\ldots+\frac{1}{(2n)^s}\right)\\
&= \left(1-\frac{2}{2^s}\right)\zeta_{2n}(s)+\frac{2n}{(2n)^s}\frac{1}{n}\left(\frac{1}{(1+1/n)^s}+\ldots+\frac{1}{(1+n/n)^s}\right)
\end{aligned}$$

Now if t is real, $s=1+it$, and $2^{1-s}=2^{-it}=1$, then the factor multiplying $\zeta_{2n}(s)$ is zero and consequently

$$\eta_{2n}(s) \;=\; \frac{1}{n^{it}}R_n(1/(1+x)^s,0,1)$$

where $R_n(f(x), a, b)$ denotes a special Riemann sum approximating the integral of $f(x)$ over $[a, b]$. For $s = 1$, i.e. $t = 0$, one gets

$$\eta(1) \;=\; \lim_{n\to\infty} \eta_{2n}(1) \;=\; \lim_{n\to\infty} R_n(1/(1+x), 0, 1) \;=\; \int_0^1 \frac{dx}{1+x} \;=\; \ln 2$$

and otherwise, when $t \neq 0$, one has $|n^{1-s}| = |n^{-it}| = 1$, giving

$$\begin{aligned} |\eta(s)| \;&=\; \lim_{n\to\infty} |\eta_{2n}(s)| \;=\; \lim_{n\to\infty} |R_n(1/(1+x)^s, 0, 1)| \\ &=\; \left|\int_0^1 \frac{dx}{(1+x)^s}\right| \;=\; \left|\frac{2^{1-s}-1}{1-s}\right| \;=\; \left|\frac{1-1}{-it}\right| \;=\; 0. \end{aligned}$$

Note. By (1) the Dirichlet eta function has as zeros also the zeros of the Riemann zeta function (see `http://planetmath.org/RiemannZetaFunction`Riemann hypothesis).

References

[1] E. Landau: *Handbuch der Lehre von der Verteilung der Primzahlen.* Erster Band. Berlin (1909); p. 161, 933.

[2] D. V. Widder: *The Laplace transform.* Princeton University Press (1946); p. 230.

[3] J. Sondow: "Zeros of the alternating zeta function on the line $\Re s = 1$". — *Amer. Math. Monthly* **110** (2003). Also available `http://arxiv.org/abs/math.NT/0209393`here.

[4] J. Sondow: "The Riemann hypothesis, simple zeros, and the asymptotic convergence degree of improper Riemann sums". — *Proc. Amer. Math. Soc.* **126** (1998). Also available `http://www.ams.org/journals/proc/1998-126-05/S0002-9939-98-04607-3/`here.