existence of power series

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In this entry we shall demonstrate the logical equivalence of the holomorphic and analytic concepts. As is the case with so many basic results in complex analysis, the proof of these facts hinges on the Cauchy integral theorem, and the Cauchy integral formula.

Holomorphic implies analytic.

Theorem 1 *Let $U \subset \mathbb{C}$ be an open domain that contains the origin, and let $f : U \to \mathbb{C}$, be a function such that the complex derivative*

$$f'(z) = \lim_{\zeta \to 0} \frac{f(z+\zeta) - f(z)}{\zeta}$$

exists for all $z \in U$. Then, there exists a power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \|z\| < R, \quad a_k \in \mathbb{C}$$

for a sufficiently small radius of convergence $R > 0$.

Note: it is just as easy to show the existence of a power series representation around every basepoint in $z_0 \in U$; one need only consider the holomorphic function $f(z - z_0)$.

Proof. Choose an $R > 0$ sufficiently small so that the disk $\|z\| \leq R$ is contained in U. By the Cauchy integral formula we have that

$$f(z) = \frac{1}{2\pi i} \oint_{\|\zeta\|=R} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad \|z\| < R,$$

where, as usual, the integration contour is oriented counterclockwise. For every ζ of modulus R, we can expand the integrand as a geometric power series in z, namely

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)/\zeta}{1 - z/\zeta} = \sum_{k=0}^{\infty} \frac{f(\zeta)}{\zeta^{k+1}} \, z^k, \quad \|z\| < R.$$

The circle of radius R is a compact set; hence $f(\zeta)$ is bounded on it; and hence, the power series above converges uniformly with respect to ζ. Consequently, the order of the infinite summation and the integration operations can be interchanged. Hence,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \|z\| < R,$$

where

$$a_k = \frac{1}{2\pi i} \oint_{\|\zeta\|=R} \frac{f(\zeta)}{\zeta^{k+1}},$$

as desired. QED

Analytic implies holomorphic.

Theorem 2 *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \quad \|z\| < \epsilon$$

be a power series, converging in $D = D_\epsilon(0)$, the open disk of radius $\epsilon > 0$ about the origin. Then the complex derivative

$$f'(z) = \lim_{\zeta \to 0} \frac{f(z+\zeta) - f(z)}{\zeta}$$

exists for all $z \in D$, i.e. the function $f : D \to \mathbb{C}$ is holomorphic.

Note: this theorem generalizes immediately to shifted power series in $z - z_0,\ z_0 \in \mathbb{C}$.

Proof. For every $z_0 \in D$, the function $f(z)$ can be recast as a power series centered at z_0. Hence, without loss of generality it suffices to prove the theorem for $z = 0$. The power series

$$\sum_{n=0}^{\infty} a_{n+1} \zeta^n, \quad \zeta \in D$$

converges, and equals $(f(\zeta) - f(0))/\zeta$ for $\zeta \neq 0$. Consequently, the complex derivative $f'(0)$ exists; indeed it is equal to a_1. QED