# triangle inequality of complex numbers

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**Theorem.** All complex numbers $z_1$ and $z_2$ satisfy the triangle inequality

$$|z_1+z_z| \leqq |z_1| + |z_2|. \tag{1}$$

*Proof.*

$$\begin{aligned}
|z_1+z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} \\
&= (z_1+z_2)(\overline{z_1}+\overline{z_2}) \\
&= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2 \\
&= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} \\
&= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\overline{z_2}) \\
&\leqq |z_1|^2 + |z_2|^2 + 2|z_1\overline{z_2}| \\
&= |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |\overline{z_2}| \\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

Taking then the nonnegative square root, one obtains the asserted inequality.

**Remark.** Since the real numbers are complex numbers, the inequality (1) and its proof are valid also for all real numbers; however the inequality may be simplified to

$$|x+y|^2 \leqq (x+y)^2 = x^2+2xy+y^2 \leqq x^2+2|x||y|+y^2 = |x|^2+2|x||y|+|y|^2 = (|x|+|y|)^2.$$