inner function

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Defines	singular inner function
Defines	outer function

If $f\colon \mathbb{D} \to \mathbb{C}$ is an analytic function on the unit disc, we denote by $f^*(e^{i\theta})$ the radial limit of f where it exists, that is

$$f^*(e^{i\theta}) := \lim_{r \to 1, r<1} f(re^{i\theta}).$$

A bounded analytic function on the disc will have radial limits almost everywhere (with respect to the Lebesgue measure on the $\partial\mathbb{D}$).

Definition. A bounded analytic function f is called an *inner function* if $|f^*(e^{i\theta})| = 1$ almost everywhere. If f has no zeros on the unit disc, then f is called a *singular inner function.*

Theorem. *Every inner function can be written as*

$$f(z) := \alpha B(z) \exp\left(- \int \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(e^{i\theta}) \right),$$

where μ is a positive singular measure on $\partial\mathbb{D}$, $B(z)$ is a Blaschke product and $|\alpha| = 1$ is a constant.

Note that all the zeros of the function come from the Blaschke product.

Definition. Let

$$f(z) := \exp\left(\int \frac{e^{i\theta}+z}{e^{i\theta}-z} h(e^{i\theta}) dm(e^{i\theta}) \right),$$

where h is a real valued Lebesgue integrable function on the unit circle and m is the Lebesgue measure. Then f is called an *outer function.*

The significance of these definitions is that every bounded holomorphic function can be written as an inner function times an outer function. See the `http://planetmath.org/FactorizationTheoremForHinftyFunctions`factorization theorem for H^∞ functions.

References

[1] John B. Conway. . Springer-Verlag, New York, New York, 1995.