proof of the uniformization theorem

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Our proof relies on the well-known Newlander-Niremberg theorem which implies, in particular, that any Riemmanian metric on an oriented 2-dimensional real manifold defines a unique analytic structure.

We will merely use the fact that $H^1(X,\mathbb{R})=0$. If X is compact, then X is a complex curve of genus 0, so $X \simeq \mathbb{P}^1$. On the other hand, the elementary Riemann mapping theorem says that an open set $\Omega \subset \mathbb{C}$ with $H^1(\Omega,\mathbb{R})=0$ is either equal to $\mathbb{C}$ or biholomorphic to the unit disk. Thus, all we have to show is that a non compact Riemann surface X with $H^1(X,\mathbb{R})=0$ can be embedded in the complex plane $\mathbb{C}$.

Let Ω_ν be an exhausting sequence of relatively compact connected open sets with smooth boundary in X. We may assume that $X \setminus \Omega_\nu$ has no relatively compact connected components, otherwise we "fill the holes" of Ω_ν by taking the union with all such components. We let Y_ν be the double of the manifold with boundary $(\overline{\Omega}_\nu, \partial\Omega_\nu)$, i.e. the union of two copies of $\overline{\Omega}_\nu$ with opposite orientations and the boundaries identified. Then Y_ν is a compact oriented surface without boundary.

Fact: we have $H^1(Y_\nu,\mathbb{R})=0$. We postpone the proof of this fact to the end of the present paragraph and we continue with the proof of the uniformization theorem.

Extend the almost complex structure of $\overline{\Omega}_\nu$ in an arbitrary way to Y_ν, e.g. by an extension of a Riemmanian metric. Then Y_ν becomes a compact Riemann surface of genus 0, thus $Y_\nu \simeq \mathbb{P}^1$ and we obtain in particular a holomorphic embedding $\Phi_\nu\colon \Omega_\nu \to \mathbb{C}$. Fix a point $a \in \Omega_0$ and a non zero linear form $\xi^* \in T_aX$. We can take the composition of Φ_ν with an affine linear map $\mathbb{C} \to \mathbb{C}$ so that $\Phi_\nu(a)=0$ and $d\Phi_\nu(a)=\xi^*$. By the well-known properties of injective holomorphic maps, (Φ_ν) is then uniformly bounded on every small disk centered at a, thus also on every compact subset of X by a connectedness argument. Hence (Φ_ν) has a subsequence converging towards an injective holomorphic map $\Phi\colon X \to \mathbb{C}$.

Proof of the "fact": Let us first compute the cohomology with compact support $H^1_c(\Omega_\nu,\mathbb{R})$. Let u be a closed 1-form with compact support in Ω_ν. By Poincaré duality $H^1_c(X,\mathbb{R})=0$, so $u=df$ for some "test" function $f \in \mathcal{D}(X)$. As $df=0$ on a neighborhood of $X \setminus \Omega_\nu$ and as all connected components of this set are non compact, f must be equal to the constant zero near $X \setminus \Omega_\nu$. Hence $u=df$ is the zero class in $H^1_c(\Omega_\nu,\mathbb{R})$ and we get $H^1_c(\Omega_\nu,\mathbb{R}) = H^1(\Omega_\nu,\mathbb{R}) = 0$. The exact sequence of the pair $(\overline{\Omega}_\nu, \partial\Omega_n u)$

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$$\mathbb{R} = H^0(\overline{\Omega}_\nu, \mathbb{R}) \to H^0(\partial\Omega_\nu, \mathbb{R}) \to H^1(\overline{\Omega}_\nu, \partial\Omega_\nu; \mathbb{R}) \simeq H^1_c(\Omega_\nu, \mathbb{R}) = 0,$$

thus $H^0(\partial\Omega_\nu, \mathbb{R}) = \mathbb{R}$. Finally, the Mayer-Vietoris sequence applied to small neighborhoods of the two copies of $\overline{\Omega}_\nu$ in Y_ν gives an exact sequence

$$H^0(\overline{\Omega}_\nu, \mathbb{R})^{\oplus 2} \to H^0(\partial\Omega_\nu, \mathbb{R}) \to H^1(Y_\nu, \mathbb{R}) \to H^1(\overline{\Omega}_\nu, \mathbb{R})^{\oplus 2} = 0$$

where the first map is onto. Hence $H^1(Y_\nu, \mathbb{R}) = 0$.

References

J.-P. Demailly, *Complex Analytic and Algebraic Geometry.*