Euler reflection formula

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Theorem 1 *(Euler Reflection Formula)*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

Proof: We have

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)e^{-x/n}\right)$$

and thus

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(-x)} = -x^2e^{\gamma x}e^{-\gamma x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)e^{-x/n}\right)\left(\left(1-\frac{x}{n}\right)e^{x/n}\right) = -x^2\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$$

But $\Gamma(1-x) = -x\Gamma(-x)$ and thus

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(1-x)} = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$$

Now, using the `http://planetmath.org/ExamplesOfInfiniteProducts`formula for $\sin x/x$, we have

$$\sin(\pi x) = \pi x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$$

so that

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(1-x)} = \frac{\sin(\pi x)}{\pi}$$

and the result follows.