Cauchy-Riemann equations (complex coordinates)

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Let $f\colon G \subset \mathbb{C} \to \mathbb{C}$ be a continuously differentiable function in the real sense, using $\mathbb{R}^2$ instead of $\mathbb{C}$, identifying $f(z)$ with $f(x,y)$ where $z = x+iy$ and we also write $\bar{z} = x - iy$ (the complex conjugate). Then we have the following partial derivatives:

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Sometimes these are written as f_z and $f_{\bar{z}}$ respectively.

The classical Cauchy-Riemann equations are equivalent to

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

This can be seen if we write $f = u + iv$ for real valued u and v and then the differentials become

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

In several complex dimensions, for a function $f\colon G \subset \mathbb{C}^n \to \mathbb{C}$ which maps $(z_1, \ldots, z_n) \mapsto f(z_1, \ldots, z_n)$ where $z_j = x_j + iy_j$ we generalize simply by

$$\frac{\partial f}{\partial z_j} := \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right),$$
$$\frac{\partial f}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right).$$

Then the Cauchy-Riemann equations are given by

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \qquad \text{for all } 1 \leq j \leq n.$$

That is, f is holomorphic if and only if it satisfies the above equations.

References

[1] Steven G. Krantz. , AMS Chelsea Publishing, Providence, Rhode Island, 1992.