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proof of converse of Möbius transformation cross-ratio preservation theorem

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Suppose that a, b, c, d are distinct. Consider the transform μ defined as

$$\mu(z) = \frac{(b-d)(c-d)}{(c-b)(z-d)} - \frac{b-d}{c-b}.$$

Simple calculation reveals that $\mu(b) = 1$, $\mu(c) = 0$, and $\mu(d) = \infty$. Furthermore, $\mu(a)$ equals the cross-ratio of a, b, c, d.

Suppose we have two tetrads with a common cross-ratio λ. Then, as above, we may construct a transform μ_1 which maps the first tetrad to $(\lambda, 1, 0, \infty)$ and a transform μ_2 which maps the first tetrad to $(\lambda, 1, 0, \infty)$. Then $\mu_2^{-1} \circ \mu_1$ maps the former tetrad to the latter and, by the group property, it is also a Möbius transformation.