proof of maximal modulus principle

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$f : U \to \mathbb{C}$ is holomorphic and therefore continuous, so $|f|$ will also be continuous on U. $K \subset U$ is compact and since $|f|$ is continuous on K it must attain a maximum and a minimum value there.

Suppose the maximum of $|f|$ is attained at z_0 in the interior of K.

By definition there will exist $r > 0$ such that the set $S_r = \{z \in \mathbb{C} : |z - z_0|^2 \leq r^2\} \subset K$.

Consider C_r the boundary of the previous set parameterized counterclockwise. Since f is holomorphic by hypothesis, Cauchy integral formula says that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \tag{1}$$

A canonical parameterization of C_r is $z = z_0 + re^{i\frac{\theta}{r}}$, for $\theta \in [0, 2\pi r]$.

$$f(z_0) = \frac{1}{2\pi r} \int_0^{2\pi r} f(z_0 + re^{i\frac{\theta}{r}}) d\theta \tag{2}$$

Taking modulus on both sides and using the estimating theorem of contour integral

$$|f(z_0)| \leq \max_{z \in C_r} |f(z)|$$

Since $|f(z_0)|$ is a maximum, the last inequality must be verified by having the equality in the $\leq$ verified.

The `http://planetmath.org/ProofOfEstimatingTheoremOfContourIntegral`proof of the estimating theorem of contour integral implies that equality is only verified when

$$\frac{f(z_o + re^{i\frac{\theta}{r}})}{re^{i\frac{\theta}{r}}} = \lambda \overline{ie^{i\frac{\theta}{r}}}$$

where $\lambda \in \mathbb{C}$ is a constant. Therefore, $f(z_o + re^{i\frac{\theta}{r}})$ is constant and to verify equation **??** its value must be $f(z_0)$.

So f is holomorphic and constant on a circumference. It's a well known result that if 2 holomorphic functions are equal on a curve, then they are equal on their entire domain, so f is constant.

to see this in this particular circumstance is using equation **??** to calculate the value of f on a point $\xi \in$ interior S_r different than z_0. Bearing in mind that $f(z) = f(z_0)$ is constant in C_r the formula reads $f(\xi) = \frac{f(z_0)}{2\pi i} \oint_{C_r} \frac{1}{z - \xi} dz =$

$f(z_0)$. So f is really constant in the interior of S_r and the only holomorphic function defined in K that is constant in the interior of S_r is the constant function on all K.

Thus if the maximum of $|f|$ is attained in the interior of K, then f is constant. If f isn't constant, the maximum must be attained somewhere in K, but not in its interior. Since K is compact, by definition it must be attained at ∂K.