closure properties of Cauchy-Riemann equations

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The set of solutions of the Cauchy-Riemann equations is closed under a surprisingly large number of operations. For convenience, let us introduce the notational conventions that f and g are complex functions with $f(x+iy) = u(x,y) + iv(x,y)$ and $g(x+iy) = p(x,y) + iq(x,y)$. Let D and D' denote open subsets of the complex plane.

Theorem 1. *If $f\colon D \to \mathbb{C}$ and $g\colon D \to \mathbb{C}$ satisfy the Cauchy-Riemann equations, so does $f+g$. Furthermore, if $z \in \mathbb{C}$, then zf satisfies the Cauchy-Riemann equations.*

Proof. This is an immediate consequence of the linearity of derivatives. $\square$

Theorem 2. *If $f\colon D \to \mathbb{C}$ and $g\colon D \to \mathbb{C}$ satisfy the Cauchy-Riemann equations, so does $f \cdot g$.*

Proof. Letting h and k denote the real and imaginary parts of $f \cdot g$ respectively, we have

$$\begin{aligned}
\frac{\partial h}{\partial x} - \frac{\partial k}{\partial y} &= \frac{\partial}{\partial x}(up - vq) - \frac{\partial}{\partial y}(uq + vp) \\
&= u\frac{\partial p}{\partial x} + p\frac{\partial u}{\partial x} - v\frac{\partial q}{\partial x} - q\frac{\partial v}{\partial x} - u\frac{\partial q}{\partial y} - q\frac{\partial u}{\partial y} - v\frac{\partial p}{\partial y} - p\frac{\partial v}{\partial y} \\
&= u\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right) - v\left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right) + p\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - q\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial h}{\partial y} + \frac{\partial k}{\partial x} &= \frac{\partial}{\partial y}(up - vq) + \frac{\partial}{\partial x}(uq + vp) \\
&= u\frac{\partial p}{\partial y} + p\frac{\partial u}{\partial y} - v\frac{\partial q}{\partial y} - q\frac{\partial v}{\partial y} + u\frac{\partial q}{\partial x} + q\frac{\partial u}{\partial x} + v\frac{\partial p}{\partial x} + p\frac{\partial v}{\partial x} \\
&= u\left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right) + v\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right) + p\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + q\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = 0.
\end{aligned}$$

$\square$

Theorem 3. *If $f\colon D \to D'$ and $g\colon D' \to \mathbb{C}$ satisfy the Cauchy-Riemann equations, so does $f \circ g$.*

Proof. Letting h and k denote the real and imaginary parts of $f \circ g$ respectively, we have

$$\begin{aligned}
\frac{\partial h}{\partial x} - \frac{\partial k}{\partial y} &= \frac{\partial}{\partial x} u(p(x,y), q(x,y)) - \frac{\partial}{\partial y} v(p(x,y), q(x,y)) \\
&= \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} - \frac{\partial v}{\partial p}\frac{\partial p}{\partial y} - \frac{\partial v}{\partial q}\frac{\partial q}{\partial y} \\
&= \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right) + \frac{\partial q}{\partial y}\left(\frac{\partial u}{\partial p} - \frac{\partial v}{\partial q}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right) - \frac{\partial p}{\partial y}\left(\frac{\partial u}{\partial q} + \frac{\partial v}{\partial p}\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial h}{\partial y} + \frac{\partial k}{\partial x} &= \frac{\partial}{\partial y} u(p(x,y), q(x,y)) + \frac{\partial}{\partial x} v(p(x,y), q(x,y)) \\
&= \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial v}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial x} \\
&= \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right) - \frac{\partial q}{\partial x}\left(\frac{\partial u}{\partial p} - \frac{\partial v}{\partial q}\right) - \frac{\partial u}{\partial q}\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right) + \frac{\partial p}{\partial x}\left(\frac{\partial u}{\partial q} + \frac{\partial v}{\partial p}\right) = 0
\end{aligned}$$

□

Theorem 4. *If $f : D \to \mathbb{C}$ satisfies the Cauchy-Riemann equations, and has non-vanishing Jacobian, then f^{-1} also satisfies the Cauchy-Riemann equations.*

Proof. Let us denote the real and imaginary parts of f^{-1} as h and k, respectively. Then, by definition of inverse function, we have

$$\begin{aligned}
u(h(x,y), k(x,y)) &= x \\
v(h(x,y), k(x,y)) &= y.
\end{aligned}$$

Taking derivatives,

$$\begin{aligned}
\frac{\partial u}{\partial h}\frac{\partial h}{\partial x} + \frac{\partial u}{\partial k}\frac{\partial k}{\partial x} &= 1 \\
\frac{\partial u}{\partial h}\frac{\partial h}{\partial y} + \frac{\partial u}{\partial k}\frac{\partial k}{\partial y} &= 0 \\
\frac{\partial v}{\partial h}\frac{\partial h}{\partial x} + \frac{\partial v}{\partial k}\frac{\partial k}{\partial x} &= 0 \\
\frac{\partial v}{\partial h}\frac{\partial h}{\partial y} + \frac{\partial v}{\partial k}\frac{\partial k}{\partial y} &= 1
\end{aligned}$$

By the Cauchy-Riemann equations, $\partial u/\partial h = \partial v/\partial k$ and $\partial u/\partial k = -\partial v/\partial h$. Using these relations to re-express the derivatives of u as derivatives of v, then subtracting the fourth equation form the first equation and adding the second and third equations, we obtain

$$\frac{\partial u}{\partial h}\left(\frac{\partial h}{\partial x} - \frac{\partial k}{\partial y}\right) + \frac{\partial u}{\partial k}\left(\frac{\partial h}{\partial y} + \frac{\partial k}{\partial x}\right) = 0$$
$$\frac{\partial u}{\partial h}\left(\frac{\partial h}{\partial y} + \frac{\partial k}{\partial x}\right) - \frac{\partial u}{\partial k}\left(\frac{\partial h}{\partial x} - \frac{\partial k}{\partial y}\right) = 0.$$

With a little algebraic manipulation, we may conclude

$$\left(\left(\frac{\partial u}{\partial h}\right)^2 + \left(\frac{\partial u}{\partial k}\right)^2\right)\left(\frac{\partial h}{\partial y} + \frac{\partial k}{\partial x}\right) = 0$$
$$\left(\left(\frac{\partial u}{\partial h}\right)^2 + \left(\frac{\partial u}{\partial k}\right)^2\right)\left(\frac{\partial h}{\partial x} - \frac{\partial k}{\partial y}\right) = 0.$$

Note that, by the Cauchy-Riemann equations, the Jacobian of f equals the common prefactor of these equations:

$$\frac{\partial(u,v)}{\partial(h,k)} = \frac{\partial u}{\partial h}\frac{\partial v}{\partial k} - \frac{\partial u}{\partial k}\frac{\partial v}{\partial h} = \left(\frac{\partial u}{\partial h}\right)^2 + \left(\frac{\partial u}{\partial k}\right)^2$$

Hence, by assumptions, this quantity differs from zero and we may cancel it to obtain the Cauchy-Riemann equations for f^{-1}. $\square$