argument principle

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Defines	argument principle

If a function f is meromorphic on the interior of a rectifiable simple closed curve C, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz \tag{1}$$

equals the difference between the number of zeros and the number of poles of f counted with multiplicity. (For example, a zero of order two counts as two zeros; a pole of order three counts as three poles.) This fact is known as the *argument principle*.

The principle may be stated in another form which makes the origin of the name apparent: If a function f is meromorphic on the interior of a rectifiable simple closed curve C and has m poles and n zeros on the interior of C, then the argument of f increases by $2\pi(n-m)$ upon traversing C. The relation of this statement to the previous statement is easy to see. Note that $f'/f = (\log f)'$ and that $\log(z) = \log|z| + i\arg z$. Substituting this into formula (**??**), we find

$$2\pi i(n-m) = \oint_C \frac{f'(z)}{f(z)}dz = \oint_C d\log|f(z)| + i\oint_C d\arg(f(z))\,.$$

The first integral on the rightmost side of this equation equals zero because $\log|f|$ is single-valued. The second integral on the rightmost side equals the change in the argument as one traverses C. Cancelling the i from both sides, we conclude that the change in the argument equals $2\pi(n-m)$.

Note also that the integral (**??**) is the winding number, about zero, of the image curve $f\circ C$.