proof of limit of nth root of n

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In this entry, we present a self-contained, elementary proof of the fact that $\lim_{n\to\infty} n^{1/n} = 1$. We begin by with inductive proofs of two integer inequalities — real numbers will not enter until the very end.

Lemma 1. *For all integers n greater than or equal to 5,*

$$2^n > n^2$$

Proof. We begin with a few easy observations. First, a bit of arithmetic:

$$2^5 = 32 > 25 = 5^2$$

Second, some algebraic manipulation of the inequality $n > 4$:

$$\begin{aligned} n-1 &> 3 \\ (n-1)^2 &> 9 \\ (n-1)^2 &> 2 \\ n^2 - 2n + 1 &> 2 \\ 2n^2 &> n^2 + 2n + 1 \\ 2n^2 &> (n+1)^2 \end{aligned}$$

These observations provide us with the makings of an inductive proof. Suppose that $2^n > n^2$ for some integer $n \geq 5$. Using the inequality we just showed,

$$2^{n+1} = 2 \cdot 2^n > 2n^2 > (n+1)^2.$$

Snce $2^5 > 5^2$ and $2^n > n^2$ implies that $2^{n+1} > (n+1)^2$ when $n \geq 5$ we conclude that $2^n > n^2$ dor all $n \geq 5$. $\square$

Lemma 2. *For all integers n greater than or equal to 3,*

$$n^{n+1} > (n+1)^n$$

Proof. We begin by noting that

$$3^4 = 81 > 64 = 4^3.$$

Next, we make assume that

$$(n-1)^n > n^{(n-1)}.$$

for some n. Multiplying both sides by n:

$$n(n-1)^n > n^n.$$

Multiplying both sides by $(n+1)^n$ and making use of the identity $(n+1)(n-1) = n^2 - 1$,

$$n(n^2-1)^n > n^n(n+1)^n.$$

Since $n^2 > n^2 - 1$, the left-hand side is less than n^{2n+1}, hence

$$n^{2n+1} > n^n(n+1)^n.$$

Canceling n^n from both sides,

$$n^{(n+1)} > (n+1)^n.$$

Hence, by induction, $n^{(n+1)} > (n+1)^n$ for all $n \geq 3$. $\square$

Theorem 1.

$$\lim_{n\to\infty} n^{1/n} = 1$$

Proof. Consider the subsequence where n is a power of 2. We then have

$$(2^m)^{(1/2^m)} = 2^{m/2^m}.$$

By lemma 1, $m/2^m < 1/m$ when $m \geq 5$. Hence, $(2^m)^{1/2^m} < 2^{1/m}$. Since $\lim_{m\to 0} 2^{1/m} = 1$, and $(2^m)^{1/2^m)} > 1$, we conclude by the squeeze rule that

$$\lim_{m\to 0}(2^m)^{1/2^m} = 1.$$

By lemma 2, the sequence $\{n^{1/n}\}$ is decreasing. It is clearly bounded from below by 1. Above, we exhibited a subsequence which tends towards 1. Thus it follows that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

$\square$