digamma and polygamma function

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Defines	digamma function
Defines	polygamma function

The *digamma function* is defined as the logarithmic derivative of the gamma function:

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Likewise the *polygamma functions* are defined as higher order logarithmic derivatives of the gamma function:

$$\psi^{(n)}(z) = \frac{d^n}{dz^n}\log\Gamma(z).$$

These equations enjoy functional equations which are closely related to those of the gamma function:

$$\begin{aligned}
\psi(z+1) &= \psi(z) + \frac{1}{z} \\
\psi(1-z) &= \psi(z) + \pi\cot\pi z \\
\psi(2z) &= \frac{1}{2}\psi(z) + \frac{1}{2}\psi\left(z+\frac{1}{2}\right) + \log 2 \\
\psi^{(n)}(z+1) &= \psi^{(n)}(z) + (-1)^n\frac{n!}{z^{n+1}}
\end{aligned}$$

These functions have poles at the negative integers and can be expressed as partial fraction series:

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{z+k}\right), \qquad (1)$$

$$\psi^{(n)}(z) = (-1)^n n! \sum_{k=0}^{\infty}\frac{1}{(z+k)^n} \qquad (2)$$

Here, γ is `http://planetmath.org/EulerMascheroniConstant`Euler–Mascheroni constant. Substituting $z = 1$ to (1), one gets the value

$$\Gamma'(1) = -\gamma.$$