two series arising from the alternating zeta function

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The terms of the series defining the alternating zeta function

$$\eta(s) \;:=\; \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \qquad (\operatorname{Re}s > 0),$$

a.k.a. the Dirichlet eta function, may be split into their real and imaginary parts:

$$\frac{1}{n^s} \;=\; \frac{e^{-ib\ln n}}{n^a} \;=\; \frac{\cos(b\ln n)}{n^a} - \frac{i\sin(b\ln n)}{n^a}$$

Here, $s = a+ib$ with real a and b. It follows the equation

$$\eta(s) \;=\; -\sum_{n=1}^{\infty}\frac{(-1)^n}{n^a}\cos(b\ln n) + i\sum_{n=1}^{\infty}\frac{(-1)^n}{n^a}\sin(b\ln n) \tag{1}$$

containing two Dirichlet series.

The alternating zeta function and the Riemann zeta function are connected by the relation

$$\zeta(s) \;=\; \frac{\eta(s)}{1-2^{1-s}}$$

(see the `http://planetmath.org/AnalyticContinuationOfRiemannZetaToCriticalStrip`parent entry). The following conjecture concerning the above real part series and imaginary part series of (1) has been proved by Sondow [1] to be equivalent with the Riemann hypothesis.

Conjecture. If the equations

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^a}\cos(b\ln n) \;=\; 0 \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^a}\sin(b\ln n) \;=\; 0$$

are true for some pair of real numbers a and b, then

$$a \;=\; 1/2 \qquad\text{or}\qquad a \;=\; 1.$$

References

[1] Jonathan Sondow: A simple counterexample to Havil's "reformulation" of the Riemann hypothesis. – *Elemente der Mathematik* **67** (2012) 61–67. Also available `http://arxiv.org/pdf/0706.2840v3.pdf`here.