proof of convergence condition of infinite product

Canonical name	ProofOfConvergenceConditionOfInfiniteProduct
Date of creation	2013-03-22 17:22:27
Last modified on	2013-03-22 17:22:27
Owner	fernsanz (8869)
Last modified by	fernsanz (8869)
Numerical id	5
Author	fernsanz (8869)
Entry type	Proof
Classification	msc 30E20

Proof. Let $p_n = \prod_{i=1}^{n} u_i$. We have to study the convergence of the sequence $\{p_n\}$. The sequence $\{p_n\}$ converges to a not null limit iff $\{\log p_n\}$ (log is restricted to its principal branch) converges to a finite limit. By the Cauchy criterion, this happens iff for every $\epsilon' > 0$ there exist N such that $|\log p_{n+k} - \log p_n| < \epsilon'$ for all $n > N$ and all $k = 1, 2, \ldots$, i.e, iff

$$\left|\log \frac{p_{n+k}}{p_n}\right| = |\log u_{n+1}u_{n+2}\cdots u_{n+k}| < \epsilon';$$

as $\log(z)$ is an injective function and continuous at $z = 1$ and $\log(1) = 0$ this will happen iff for every $\epsilon > 0$

$$|u_{n+1}u_{n+2}\cdots u_{n+k} - 1| < \epsilon$$

for n greater than N and $k = 1, 2, \ldots$ $\square$