locally bounded

Canonical name	LocallyBounded
Date of creation	2013-03-22 14:17:47
Last modified on	2013-03-22 14:17:47
Owner	jirka (4157)
Last modified by	jirka (4157)
Numerical id	9
Author	jirka (4157)
Entry type	Definition
Classification	msc 30A99
Classification	msc 54-00

Suppose that X is a topological space and Y a metric space.

Definition. A set $\mathcal{F}$ of functions $f\colon X \to Y$ is said to be *locally bounded* if for every $x \in X$, there exists a neighbourhood N of x such that $\mathcal{F}$ is uniformly bounded on N.

In the special case of functions on the complex plane where it is often used, the definition can be given as follows.

Definition. A set $\mathcal{F}$ of functions $f\colon G \subset \mathbb{C} \to \mathbb{C}$ is said to be *locally bounded* if for every $a \in G$ there exist constants $\delta > 0$ and $M > 0$ such that for all $z \in G$ such that $|z - a| < \delta$, $|f(z)| < M$ for all $f \in \mathcal{F}$.

As an example we can look at the set $\mathcal{F}$ of entire functions where $f(z) = z^2 + t$ for any $t \in [0, 1]$. Obviously each such f is unbounded itself, however if we take a small neighbourhood around any point we can bound all $f \in \mathcal{F}$. Say on an open ball $B(z_0, 1)$ we can show by triangle inequality that $|f(z)| \leq (|z_0| + 1)^2 + 1$ for all $z \in B(z_0, 1)$. So this set of functions is locally bounded.

Another example would be say the set of all analytic functions from some region G to the unit disc. All those functions are bounded by 1, and so we have a uniform bound even over all of G.

As a counterexample suppose the we take the constant functions $f_n(z) = n$ for all natural numbers n. While each of these functions is itself bounded, we can never find a uniform bound for all such functions.

References

[1] John B. Conway. . Springer-Verlag, New York, New York, 1978.