proof of Rouché's theorem

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Owner	rspuzio (6075)
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Author	rspuzio (6075)
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Consider the integral

$$N(\lambda) = \frac{1}{2\pi i} \oint_C \frac{f'(z) + \lambda g'(z)}{f(z) + \lambda g(z)} dz$$

where $0 \leq \lambda \leq 1$. By the hypotheses, the function $f + \lambda g$ is non-singular on C or on the interior of C and has no zeros on C. Hence, by the argument principle, $N(\lambda)$ equals the number of zeros (counted with multiplicity) of $f + \lambda g$ contained inside C. Note that this means that $N(\lambda)$ must be an integer.

Since C is compact, both $|f|$ and $|g|$ attain minima and maxima on C. Hence there exist positive real constants a and b such that

$$|f(z)| > a > b > |g(z)|$$

for all z on C. By the triangle inequality, this implies that $|f(z) + \lambda g(z)| > a - b$ on C. Hence $1/(f + \lambda g)$ is a continuous function of λ when $0 \leq \lambda \leq 1$ and $z \in C$. Therefore, the integrand is a continuous function of C and λ. Since C is compact, it follows that $N(\lambda)$ is a continuous function of λ.

Now there is only one way for a continuous function of a real variable to assume only integer values – that function must be constant. In particular, this means that the number of zeros of $f + \lambda g$ inside C is the same for all λ. Taking the extreme cases $\lambda = 0$ and $\lambda = 1$, this means that f and $f + g$ have the same number of zeros inside C.