Mergelyan's theorem

Canonical name	MergelyansTheorem
Date of creation	2013-03-22 14:23:59
Last modified on	2013-03-22 14:23:59
Owner	jirka (4157)
Last modified by	jirka (4157)
Numerical id	7
Author	jirka (4157)
Entry type	Theorem
Classification	msc 30E10
Related topic	RungesTheorem

Theorem (Mergelyan). *Let $K \subset \mathbb{C}$ be a compact subset of the complex plane such that $\mathbb{C} \backslash K$ (the complement of K) is connected, and let $f\colon K \to \mathbb{C}$ be a continuous function which is also holomorphic on the interior of K. Then f is the uniform limit on K of holomorphic polynomials (polynomials in one complex variable).*

So for any $\epsilon > 0$ one can find a polynomial $p(z) = \sum_{j=1}^{n} a_j z^j$ such that $|f(z) - p(z)| < \epsilon$ for all $z \in K$.

Do note that this theorem is not a weaker version of Runge's theorem. Here, we do not need f to be holomorphic on a neighbourhood of K, but just on the interior of K. For example, if the interior of K is empty, then f just needs to be continuous on K. Further, it could be that the closure of the interior of K might not be all of K. Consider $K = D \cup [-10, 10]$, where D is the closed unit disc. Then K has two lines coming out of either end of the disc and f needs to only be continuous there.

Also note that this theorem is distinct from the Stone-Weierstrass theorem. The point here is that the polynomials are holomorphic in Mergelyan's theorem.

References

[1] John B. Conway. . Springer-Verlag, New York, New York, 1978.

[2] Walter Rudin. . McGraw-Hill, Boston, Massachusetts, 1987.