# proof of Cauchy residue theorem



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| Canonical name | ProofOfCauchyResidueTheorem |
| Date of creation | 2013-03-22 13:42:04 |
| Last modified on | 2013-03-22 13:42:04 |
| Owner | paolini (1187) |
| Last modified by | paolini (1187) |
| Numerical id | 7 |
| Author | paolini (1187) |
| Entry type | Proof |
| Classification | msc 30E20 |

Being $f$ holomorphic by Cauchy-Riemann equations the differential form $f(z)\,dz$ is closed. So by the lemma about closed differential forms on a simple connected domain we know that the integral $\int_C f(z)\,dz$ is equal to $\int_{C'} f(z)\,dz$ if $C'$ is any curve which is homotopic to $C$. In particular we can consider a curve $C'$ which turns around the points $a_j$ along small circles and join these small circles with segments. Since the curve $C'$ follows each segment two times with opposite orientation it is enough to sum the integrals of $f$ around the small circles.

So letting $z = a_j + \rho e^{i\theta}$ be a parameterization of the curve around the point $a_j$, we have $dz = \rho i e^{i\theta}\,d\theta$ and hence

$$\int_C f(z)\,dz = \int_{C'} f(z)\,dz = \sum_j \eta(C, a_j) \int_{\partial B_\rho(a_j)} f(z)\,dz$$

$$= \sum_j \eta(C, a_j) \int_0^{2\pi} f(a_j + \rho e^{i\theta})\rho i e^{i\theta}\,d\theta$$

where $\rho > 0$ is chosen so small that the balls $B_\rho(a_j)$ are all disjoint and all contained in the domain $U$. So by linearity, it is enough to prove that for all $j$

$$i\int_0^{2\pi} f(a_j + e^{i\theta})\rho e^{i\theta}\,d\theta = 2\pi i \mathrm{Res}(f, a_j).$$

Let now $j$ be fixed and consider now the Laurent series for $f$ in $a_j$:

$$f(z) = \sum_{k\in\mathbb{Z}} c_k (z - a_j)^k$$

so that $\mathrm{Res}(f, a_j) = c_{-1}$. We have

$$\int_0^{2\pi} f(a_j+e^{i\theta})\rho e^{i\theta}\,d\theta = \sum_k \int_0^{2\pi} c_k(\rho e^{i\theta})^k \rho e^{i\theta}\,d\theta = \rho^{k+1}\sum_k c_k \int_0^{2\pi} e^{i(k+1)\theta}\,d\theta.$$

Notice now that if $k = -1$ we have

$$\rho^{k+1} c_k \int_0^{2\pi} e^{i(k+1)\theta}\,d\theta = c_{-1}\int_0^{2\pi} d\theta = 2\pi c_{-1} = 2\pi\,\mathrm{Res}(f, a_j)$$

while for $k \neq -1$ we have

$$\int_0^{2\pi} e^{i(k+1)\theta}\,d\theta = \left[\frac{e^{i(k+1)\theta}}{i(k+1)}\right]_0^{2\pi} = 0.$$

Hence the result follows.