coefficients of Laurent series

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Suppose that f is analytic in the annulus $\{z \in \mathbb{C} \colon R_1 < |z-a| < R_2\}$, where R_1 may be 0 and R_2 may be ∞. Then the coefficients of the http://planetmath.org/LaurentSeriesLaurent series

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

of f can be obtained from

$$c_n \;=\; \frac{1}{2\pi i}\oint_\gamma \frac{f(t)}{(t-a)^{n+1}}\,dt \quad (n = 0,\,\pm 1,\,\pm 2,\,\ldots), \tag{1}$$

where the http://planetmath.org/ContourIntegralpath γ goes anticlockwise once around the point $z = a$ within the annulus. Especially, the residue of f in the point a is

$$c_{-1} \;=\; \frac{1}{2\pi i}\oint_\gamma f(t)\,dt. \tag{2}$$

Remark. Usually, the Laurent series of a function, i.e. the coefficients c_n, are not determined by using the integral formula (1), but directly from known series . Often it is sufficient to know the value of c_{-1} or the residue, which is used to compute integrals (see the Cauchy residue theorem — cf. (2)). There is also the usable

Rule. In the case that the limit $\lim\limits_{z\to a}(z-a)f(z)$ exists and has a non-zero value r, the point $z = a$ is a pole of the 1 for the function f and

$$\mathrm{Res}(f;\,a) \;=\; r.$$

Examples

1. Let $f(z) := \frac{1}{\sin z}$, and $a = 0$. Using the Taylor series of the complex sine we obtain

$$\lim_{z\to 0} z\frac{1}{\sin z} \;=\; \lim_{z\to 0}\frac{1}{1-\frac{z^2}{3!}+-\ldots} \;=\; 1,$$

whence $\mathrm{Res}(\frac{1}{\sin z};\,0) = 1$. Thus we can write

$$\oint_\gamma \frac{dz}{\sin z} \;=\; 2\pi i,$$

where the must be chosen such that it encloses only the pole 0 of $\frac{1}{\sin z}$.

2. The Taylor series of the complex exponential function gives the Laurent series

$$e^{\frac{1}{z}} \equiv 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

which shows that $\operatorname{Res}(e^{\frac{1}{z}};\, 0) = 1$.