signum function

Canonical name	SignumFunction
Date of creation	2013-03-22 13:36:41
Last modified on	2013-03-22 13:36:41
Owner	yark (2760)
Last modified by	yark (2760)
Numerical id	11
Author	yark (2760)
Entry type	Definition
Classification	msc 30-00
Classification	msc 26A06
Related topic	ModulusOfComplexNumber
Related topic	HeavisideStepFunction
Related topic	PlusSign
Related topic	SineIntegralInInfinity
Related topic	ListOfImproperIntegrals
Defines	Heavyside step function
Defines	step function

The *signum function* is the function $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{when } x < 0, \\ 0 & \text{when } x = 0, \\ 1 & \text{when } x > 0. \end{cases}$$

The following properties hold:

1. For all $x \in \mathbb{R}$, $\operatorname{sgn}(-x) = -\operatorname{sgn}(x)$.

2. For all $x \in \mathbb{R}$, $|x| = \operatorname{sgn}(x)x$.

3. For all $x \neq 0$, $\frac{d}{dx}|x| = \operatorname{sgn}(x)$.

Here, we should point out that the signum function is often defined simply as 1 for $x > 0$ and -1 for $x < 0$. Thus, at $x = 0$, it is left undefined. See for example [**?**]. In applications such as the Laplace transform this definition is adequate, since the value of a function at a single point does not change the analysis. One could then, in fact, set $\operatorname{sgn}(0)$ to any value. However, setting $\operatorname{sgn}(0) = 0$ is motivated by the above relations. On a related note, we can extend the definition to the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ by defining $\operatorname{sgn}(\infty) = 1$ and $\operatorname{sgn}(-\infty) = -1$.

A related function is the *Heaviside step function* defined as

$$H(x) = \begin{cases} 0 & \text{when } x < 0, \\ 1/2 & \text{when } x = 0, \\ 1 & \text{when } x > 0. \end{cases}$$

Again, this function is sometimes left undefined at $x = 0$. The motivation for setting $H(0) = 1/2$ is that for all $x \in \mathbb{R}$, we then have the relations

$$\begin{aligned} H(x) &= \frac{1}{2}(\operatorname{sgn}(x) + 1), \\ H(-x) &= 1 - H(x). \end{aligned}$$

This first relation is clear. For the second, we have

$$\begin{aligned} 1 - H(x) &= 1 - \frac{1}{2}(\operatorname{sgn}(x) + 1) \\ &= \frac{1}{2}(1 - \operatorname{sgn}(x)) \\ &= \frac{1}{2}(1 + \operatorname{sgn}(-x)) \\ &= H(-x). \end{aligned}$$

Example Let $a < b$ be real numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be the piecewise defined function

$$f(x) = \begin{cases} 4 & \text{when } x \in (a, b), \\ 0 & \text{otherwise.} \end{cases}$$

Using the Heaviside step function, we can write

$$f(x) = 4\big(H(x-a) - H(x-b)\big) \tag{1}$$

almost everywhere. Indeed, if we calculate f using equation **??** we obtain $f(x) = 4$ for $x \in (a, b)$, $f(x) = 0$ for $x \notin [a, b]$, and $f(a) = f(b) = 2$. Therefore, equation **??** holds at all points except a and b. $\square$

1 Signum function for complex arguments

For a complex number z, the signum function is defined as [**?**]

$$\operatorname{sgn}(z) = \begin{cases} 0 & \text{when } z = 0, \\ z/|z| & \text{when } z \neq 0. \end{cases}$$

In other words, if z is non-zero, then $\operatorname{sgn} z$ is the projection of z onto the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Clearly, the complex signum function reduces to the real signum function for real arguments. For all $z \in \mathbb{C}$, we have

$$z \operatorname{sgn} \overline{z} = |z|,$$

where $\overline{z}$ is the complex conjugate of z.

References

[1] E. Kreyszig, *Advanced Engineering Mathematics*, John Wiley & Sons, 1993, 7th ed.

[2] G. Bachman, L. Narici, *Functional analysis*, Academic Press, 1966.