proof of Möbius circle transformation theorem

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Case 1: $f(z) = az + b$.

Case 1a: The points on $|z - C| = R$ can be written as $z = C + Re^{i\theta}$. They are mapped to the points $w = aC + b + aRe^{i\theta}$ which all lie on the circle $|w - (aC + b)| = |a|R$.

Case 1b: The line $\mathrm{Re}(e^{i\theta}z) = k$ are mapped to the line $\mathrm{Re}\left(\frac{e^{i\theta}w}{a}\right) = k + \mathrm{Re}\left(\frac{b}{a}\right)$.

Case 2: $f(z) = \frac{1}{z}$.

Case 2a: Consider a circle passing through the origin. This can be written as $|z - C| = |C|$. This circle is mapped to the line $\mathrm{Re}(Cw) = \frac{1}{2}$ which does not pass through the origin. To show this, write $z = C + |C|e^{i\theta}$. $w = \frac{1}{z} = \frac{1}{C+|C|e^{i\theta}}$.

$$\mathrm{Re}(Cw) = \frac{1}{2}(Cw + \overline{Cw}) = \frac{1}{2}\left(\frac{C}{C + |C|e^{i\theta}} + \frac{\overline{C}}{\overline{C} + |C|e^{-i\theta}}\right)$$

$$= \frac{1}{2}\left(\frac{C}{C + |C|e^{i\theta}} + \frac{\overline{C}}{\overline{C} + |C|e^{-i\theta}}\frac{e^{i\theta}}{e^{i\theta}}\frac{C/|C|}{C/|C|}\right) = \frac{1}{2}\left(\frac{C}{C + |C|e^{i\theta}} + \frac{|C|e^{i\theta}}{|C|e^{i\theta} + C}\right) = \frac{1}{2}$$

Case 2b: Consider the line which does not pass through the origin. This can be written as $\mathrm{Re}(az) = 1$ for $a \neq 0$. Then $az + \overline{az} = 2$ which is mapped to $\frac{a}{w} + \frac{\overline{a}}{\overline{w}} = 2$. This is simplified as $a\overline{w} + \overline{a}w = 2w\overline{w}$ which becomes $(w - a/2)(\overline{w} - \overline{a}/2) = a\overline{a}/4$ or $\left|w - \frac{a}{2}\right| = \frac{|a|}{2}$ which is a circle passing through the origin.

Case 2c: Consider a circle which does not pass through the origin. This can be written as $|z - C| = R$ with $|C| \neq R$. This circle is mapped to the circle

$$\left|w - \frac{\overline{C}}{|C|^2 - R^2}\right| = \frac{R}{||C|^2 - R^2|}$$

which is another circle not passing through the origin. To show this, we will demonstrate that

$$\frac{\overline{C}}{|C|^2 - R^2} + \frac{C - z}{R}\frac{\overline{z}}{z}\frac{R}{|C|^2 - R^2} = \frac{1}{z}$$

Note:$\left|\frac{C-z}{R}\frac{\overline{z}}{z}\right| = 1$.

$$\frac{\overline{C}}{|C|^2 - R^2} + \frac{C - z}{R}\frac{\overline{z}}{z}\frac{R}{|C|^2 - R^2} = \frac{z\overline{C} - z\overline{z} + \overline{z}C}{z(|C|^2 - R^2)}$$

$$= \frac{C\overline{C} - (z - C)(\overline{z} - \overline{C})}{z(|C|^2 - R^2)} = \frac{|C|^2 - R^2}{z(|C|^2 - R^2)} = \frac{1}{z}$$

Case 2d: Consider a line passing through the origin. This can be written as $\mathrm{Re}(e^{i\theta}z) = 0$. This is mapped to the set $\mathrm{Re}\left(\frac{e^{i\theta}}{w}\right) = 0$, which can be rewritten as $\mathrm{Re}(e^{i\theta}\overline{w}) = 0$ or $\mathrm{Re}(we^{-i\theta}) = 0$ which is another line passing through the origin.

Case 3: An arbitrary Mobius transformation can be written as $f(z) = \frac{az+b}{cz+d}$. If $c = 0$, this falls into Case 1, so we will assume that $c \neq 0$. Let

$$f_1(z) = cz + d \qquad f_2(z) = \frac{1}{z} \qquad f_3(z) = \frac{bc - ad}{c}z + \frac{a}{c}$$

Then $f = f_3 \circ f_2 \circ f_1$. By Case 1, f_1 and f_3 map circles to circles and by Case 2, f_2 maps circles to circles.