residue at infinity

Canonical name	ResidueAtInfinity
Date of creation	2013-03-22 19:15:00
Last modified on	2013-03-22 19:15:00
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	9
Author	pahio (2872)
Entry type	Definition
Classification	msc 30D30
Related topic	Residue
Related topic	RegularAtInfinity

If in the Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k \tag{1}$$

of the function f, the coefficient c_n is distinct from zero ($n > 0$) and $c_{n+1} = c_{n+2} = \ldots = 0$, then there exists the numbers M and K such that

$$|z^{-n} f(z)| < M \quad \text{always when} \quad |z| > K.$$

In this case one says that ∞ is a *pole of order* n of the function f (cf. zeros and poles of rational function).

If there is no such positive integer n, (1) infinitely many positive powers of z, and one may say that ∞ is an *essential singularity* of f.

In both cases one can define for f the *residue at infinity* as

$$\frac{1}{2i\pi} \oint_C f(z)\, dz = c_{-1}, \tag{2}$$

where the integral is taken along a closed contour C which goes once anticlockwise around the origin, i.e. once clockwise around the point $z = \infty$ (see the Riemann sphere).

Then the usual form

$$\frac{1}{2i\pi} \oint_C f(z)\, dz = \sum_j \mathrm{Res}(f;\, a_j)$$

of the residue theorem may be expressed as follows:

The sum of all residues of an analytic function having only a finite number of points of singularity is equal to zero.

References

[1] Ernst Lindelöf: *Le calcul des résidus et ses applications à la théorie des fonctions.* Gauthier-Villars, Paris (1905).