infinitely-differentiable function that is not analytic

Canonical name	InfinitelydifferentiableFunctionThatIsNotAnalytic
Date of creation	2013-03-22 12:46:15
Last modified on	2013-03-22 12:46:15
Owner	ariels (338)
Last modified by	ariels (338)
Numerical id	5
Author	ariels (338)
Entry type	Example
Classification	msc 30B10
Classification	msc 26A99

If $f \in \mathcal{C}^\infty$, then we can certainly *write* a Taylor series for f. However, analyticity requires that this Taylor series actually converge (at least across some radius of convergence) to f. It is not necessary that the power series for f converge to f, as the following example shows.

Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Then $f \in \mathcal{C}^\infty$, and for any $n \geq 0$, $f^{(n)}(0) = 0$ (see below). So the Taylor series for f around 0 is 0; since $f(x) > 0$ for all $x \neq 0$, clearly it does not converge to f.

Proof that $f^{(n)}(0) = 0$

Let $p(x), q(x) \in \mathbb{R}[x]$ be polynomials, and define

$$g(x) = \frac{p(x)}{q(x)} \cdot f(x).$$

Then, for $x \neq 0$,

$$g'(x) = \frac{(p'(x) + p(x)\frac{2}{x^3})q(x) - q'(x)p(x)}{q^2(x)} \cdot e^{-\frac{1}{x^2}}.$$

Computing (e.g. by applying `http://planetmath.org/LHpitalsRule`L'Hôpital's rule), we see that $g'(0) = \lim_{x\to 0} g'(x) = 0$.

Define $p_0(x) = q_0(x) = 1$. Applying the above inductively, we see that we may write $f^{(n)}(x) = \frac{p_n(x)}{q_n(x)} f(x)$. So $f^{(n)}(0) = 0$, as required.