dilogarithm function

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The *dilogarithm function*

$$\mathrm{Li}_2(x) \;=:\; \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \tag{1}$$

studied already by Leibniz, is a special case of the *polylogarithm function*

$$\mathrm{Li}_s(x) \;=:\; \sum_{n=1}^{\infty} \frac{x^n}{n^s}.$$

The radius of convergence of the series (1) is 1, whence the definition (1) is valid also in the unit disc of the complex plane. For $0 \leq x \leq 1$, the equation (1) is apparently equivalent to

$$\mathrm{Li}_2(x) \;=:\; -\int_0^x \frac{\ln(1-t)}{t}\,dt, \tag{2}$$

(cf. logarithm series of $\ln(1-x)$). The analytic continuation of Li_2 for $|z| \geq 1$ can be made by

$$\mathrm{Li}_2(z) \;=:\; -\int_0^z \frac{\log(1-t)}{t}\,dt. \tag{3}$$

Thus $\mathrm{Li}_2(z)$ is a multivalued analytic function of z. Its *principal branch* is single-valued and is got by taking the principal branch of the complex logarithm; then

$$z \;\in\; \mathbb{C}\smallsetminus[1,\,\infty[, \quad 0 < \arg(z-1) < 2\pi.$$

For real values of x we have

$$\mathrm{Im}(\mathrm{Li}_2(x)) \;=\; \begin{cases} 0 & \text{for } x \leq 1, \\ -\pi\ln x & \text{for } x > 1. \end{cases}$$

According to (2), the derivative of the dilogarithm is

$$\mathrm{Li}_2'(x) \;=\; \frac{-\ln(1-x)}{x}.$$

In terms of the Bernoulli numbers, the dilogarithm function has a series expansion more rapidly converging than (1):

$$\mathrm{Li}_2(x) \;=\; \sum_{n=1}^{\infty} B_{n-1} \frac{(-\ln(1-x))^n}{n!} \qquad (|\ln(1-x)| < 2\pi) \tag{4}$$

Some functional equations and values

$$\begin{aligned}
\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) \;&=\; \frac{1}{2}\mathrm{Li}_2(z^2), \\
\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) \;&=\; -\frac{1}{2}(\log(-z))^2 - \frac{\pi^2}{6}, \\
\mathrm{Li}_2(iz) - i\mathrm{Li}_2(z) \;&=\; \frac{1}{4}\mathrm{Li}_2(-z^2),
\end{aligned}$$

$$\mathrm{Li}_2(1) \;=\; \frac{\pi^2}{6}, \quad \mathrm{Li}_2(2) \;=\; \frac{\pi^2}{4} - i\pi\ln 2, \quad \mathrm{Li}_2(i) \;=\; -\frac{\pi^2}{48} - i\,G$$

Here, G is Catalan's constant.

References

[1] Anatol N. Kirillov: *Dilogarithm identities* (1994). Available `http://arxiv.org/pdf/hep-th/9408113v2.pdf`here.

[2] Leonard C. Maximon: The dilogarithm function for complex argument. – *Proc. R. Soc. Lond. A* **459** (2003) 2807–2819.