proof of Casorati-Weierstrass theorem

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Assume that a is an essential singularity of f. Let $V \subset U$ be a punctured neighborhood of a, and let $\lambda \in \mathbb{C}$. We have to show that λ is a limit point of $f(V)$. Suppose it is not, then there is an $\epsilon > 0$ such that $|f(z) - \lambda| > \epsilon$ for all $z \in V$, and the function

$$g : V \to \mathbb{C}, z \mapsto \frac{1}{f(z) - \lambda}$$

is bounded, since $|g(z)| = \frac{1}{|f(z)-\lambda|} < \epsilon^{-1}$ for all $z \in V$. According to Riemann's removable singularity theorem, this implies that a is a removable singularity of g, so that g can be extended to a holomorphic function $\bar{g} : V \cup \{a\} \to \mathbb{C}$. Now

$$f(z) = \frac{1}{\bar{g}(z)} - \lambda$$

for $z \neq a$, and a is either a removable singularity of f (if $\bar{g}(z) \neq 0$) or a pole of order n (if $\bar{g}$ has a zero of order n at a). This contradicts our assumption that a is an essential singularity, which means that λ must be a limit point of $f(V)$. The argument holds for all $\lambda \in \mathbb{C}$, so $f(V)$ is dense in $\mathbb{C}$ for any punctured neighborhood V of a.

To prove the converse, assume that $f(V)$ is dense in $\mathbb{C}$ for any punctured neighborhood V of a. If a is a removable singularity, then f is bounded near a, and if a is a pole, $f(z) \to \infty$ as $z \to a$. Either of these possibilities contradicts the assumption that the image of any punctured neighborhood of a under f is dense in $\mathbb{C}$, so a must be an essential singularity of f.