taking square root algebraically

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For getting the square root of the complex number $a+ib$ $(a, b \in \mathbb{R})$ purely algebraically, one should solve the real part x and the imaginary part y of $\sqrt{a+ib}$ from the binomial equation

$$(x+iy)^2 = a+ib. \tag{1}$$

This gives

$$a+ib \;=\; x^2+2ixy-y^2 \;=\; (x^2-y^2)+i\cdot 2xy.$$

Comparing (see `http://planetmath.org/EqualityOfComplexNumbers`equality) the real parts and the imaginary parts yields the pair of real equations

$$x^2-y^2 = a, \qquad 2xy = b,$$

which may be written

$$x^2+(-y^2) = a, \qquad x^2\cdot(-y^2) = -\frac{b^2}{4}.$$

Note that the x and y must be chosen such that their product $(=\frac{b}{2})$ has the same sign as b. Using the properties of quadratic equation, one infers that x^2 and $-y^2$ are the roots of the equation

$$t^2-at-\frac{b^2}{4} = 0.$$

The quadratic formula gives

$$t = \frac{a \pm \sqrt{a^2+b^2}}{2},$$

and since $-y^2$ is the smaller root, $x^2 = \frac{a+\sqrt{a^2+b^2}}{2}$, $\quad -y^2 = \frac{a-\sqrt{a^2+b^2}}{2}$. So we obtain the result

$$x = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}, \qquad y = (\operatorname{sign} b)\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$$

(see the signum function). Because both may have also the

$$\sqrt{a+ib} \;=\; \pm\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + (\operatorname{sign} b)i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right). \tag{2}$$

The result shows that the real and imaginary parts of the square root of any complex number $a+ib$ can be obtained from the real part a and imaginary part b of the number by using only algebraic operations, i.e. the rational operations and the . Apparently, the same is true for all roots of a complex number with `http://planetmath.org/NthRoot`index an integer power of 2.

In practise, when determining the square root of a non-real complex number, one need not to remember the (2), but it's better to solve concretely the equation (1).

Exercise. Compute $\sqrt{i}$ and check it!