calculation of contour integral

Canonical name	CalculationOfContourIntegral
Date of creation	2013-03-22 19:14:16
Last modified on	2013-03-22 19:14:16
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	8
Author	pahio (2872)
Entry type	Example
Classification	msc 30E20
Classification	msc 30A99
Related topic	AntiderivativeOfComplexFunction
Related topic	SubstitutionNotation
Related topic	ProofOfCauchyIntegralFormula

We will determine the important complex integral

$$I := \oint_C (z-z_0)^n \, dz$$

where C is the circumference of the circle $|z-z_0| = \varrho$ taken anticlockwise and n an arbitrary integer.

Let's take the "direction angle" of the radius of C as the parametre t, i.e.

$$t := \arg|z-z_0|.$$

Then on C, we have

$$z-z_0 = \varrho e^{it}, \quad 0 \leqq t \leqq 2\pi$$

and

$$dz = i\varrho e^{it} \, dt, \quad (z-z_0)^n = \varrho^n e^{int},$$

whence

$$I = \int_0^{2\pi} \varrho^n e^{int} i\varrho e^{it} \, dt = i\varrho^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt.$$

In the case $n = -1$ one gets trivially $I = 2i\pi$. If $n \neq -1$, we obtain

$$I = i\varrho^{n+1} \Big/_{t=0}^{2\pi} \frac{e^{i(n+1)t}}{i(n+1)} = \frac{\varrho^{n+1}}{n+1}(1-1) = 0,$$

using the fact that $2i\pi$ is a `http://planetmath.org/PeriodicityOfExponentialFunction`period of the exponential function.

Hence we can write the result

$$\oint_C (z-z_0)^n \, dz = \begin{cases} 2i\pi & \text{if } n = -1, \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{-1\}. \end{cases}$$