holomorphic function associated with continuous function

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Theorem. If $f(z)$ is continuous on a (finite) contour γ of the complex plane, then the contour integral

$$g(z) \ =: \ \int_\gamma \frac{f(t)}{t-z}\,dt, \tag{1}$$

defines a function $z \mapsto g(z)$ which is holomorphic in any domain D not containing points of γ. Moreover, the derivative has the expression

$$g'(z) \ = \ \int_\gamma \frac{f(t)}{(t-z)^2}\,dt. \tag{2}$$

Proof. The right hand side of (2) is defined since its integrand is continuous. On has to show that it equals

$$\lim_{\Delta z \to 0} \frac{g(z+\Delta z)-g(z)}{\Delta z}.$$

Let $z_1 =: z+\Delta z \notin \gamma$, $\Delta z \neq 0$. We may write first

$$\frac{g(z_1)-g(z)}{z_1-z} \ = \ \frac{1}{\Delta z}\int_\gamma f(t)\left[\frac{1}{t-z_1}-\frac{1}{t-z}\right]dt \ = \ \int_\gamma \frac{f(t)}{(t-z_1)(t-z)}\,dt,$$

whence

$$E \ =: \ \frac{g(z_1)-g(z)}{z_1-z} - \int_\gamma \frac{f(t)}{(t-z)^2} \ = \ \Delta z \cdot \int_\gamma \frac{f(t)}{(t-z_1)(t-z)^2}\,dt.$$

Because f is continuous in the compact set γ, there is a positive constant M such that

$$|f(t)| \ < \ M \quad \forall\, t \in \gamma.$$

As well, we have a positive constant d such that

$$|t-z| \ \geqq \ d \quad \forall\, t \in \gamma.$$

When we choose $|\Delta z| < \frac{d}{2}$, it follows that

$$|t-z_1| \ = \ |(t-z)-\Delta z| \ \geqq \ |t-z|-|\Delta z| \ > \ d-\frac{d}{2} \ = \ \frac{d}{2}.$$

Consequently,

$$\left|\frac{f(t)}{(t-z_1)(t-z)^2}\right| \ = \ \frac{|f(t)|}{|t-z_1||t-z|^2} \ < \ \frac{M}{\frac{d}{2}\cdot d^2} \ = \ \frac{2M}{d^3}$$

and, by the estimating theorem of contour integral,

$$|E| \ = \ |\Delta z| \cdot \left| \int_\gamma \frac{f(t)}{(t-z_1)(t-z)^2}\, dt \right| \ < \ |\Delta z| \cdot \frac{2M}{d^3} \cdot k,$$

where k is the length of the contour. The last expression tends to zero as $\Delta z \to 0$. This settles the proof.

Remark 1. By induction, one can prove the following generalisation of (2):

$$g^{(n)}(z) \ = \ n! \int_\gamma \frac{f(t)}{(t-z)^{n+1}}\, dt \qquad (n \ = \ 0,\, 1,\, 2,\, \ldots) \tag{3}$$

Remark 2. The contour γ may be . If it especially is a circle, then (1) defines a holomorphic function inside γ and another outside it.