natural boundary

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It is not always possible to analytically continue a function given in a certain region. It might turn out that, as one approaches the boundary of the region (or a portion of the boundary), the function always blows up, so there is no way of extending it past that portion of the boundary to a larger region. When this happens, we say that our function has a *natural boundary.* More formally, we may make a definition as follows:

Definition 1 *Let* $\mathbf{D}$ *be an open subset of the complex plane and let* $f\colon \mathbf{D} \to \mathbb{C}$ *be analytic. Then the* natural boundary *of* f *is that subset* B *of* $\partial\mathbf{D}$ *such that, if* $z \in B$*, then there exists no open neighborhood* $\mathbf{N}$ *of* z *and no analytic function* $g\colon \mathbf{N} \to \mathbb{C}$ *such that* $f(w) = g(w)$ *for all* $w \in \mathbf{D} \cup \mathbf{N}$*.*

As an example of this phenomenon, consider the power series

$$\sum_{k=0}^{\infty} z^{k!}.$$

By comparison with the geometric series, it is seen that this series converges absolutely when $|z| < 1$:

$$\sum_{k=0}^{\infty} |z|^{k!} < \sum_{k=0}^{\infty} |z|^{k} = \frac{1}{1 - |z|}$$

However, when we try to take the limit $|z| \to 1$, we find that the series diverges. Namely, let p/q be a rational number and let r be a positive real variable. Then, if we set $z = r\exp(2i\pi p/q)$, then, when $k \geq q$, we have that q divides $k!$, so $z^{k!} = r^{k!}$. However,

$$\lim_{r \to 1} \sum_{k=q}^{\infty} r^{k!}$$

diverges, so our powers series diverges when we try to take the limit $z \to \exp(2i\pi p/q)$. Since numbers of the form $\exp(2i\pi p/q)$ are dense amongst complex numbers with norm 1, it follows that the limit $z \to z_0$ diverges whenver $|z_0| = 1$. Hence, the unit circle $|z| = 1$ forms a natural boundary for the function defined by our power series.

Natural boundaries are not so familiar to beginners because the functions which one encounters in the more elementary part of the subject, such as algebraic functions, exponential functions, and functions defined by linear differential equations, do not have natural boundaries. To be sure, one could technically call a singular point a natural boundary, but this is usually not done, the term "natural boundary" being reserved for cases where the set on which the function misbehaves consists of more than just isolated points, as in the example above.

However, when one gains some more experience and studies more advanced material, then natural boundaries arise rather frequently. For instance, theta functions, elliptic modular functions, and functions defined by non-linear differential equations have natural boundaries. Natural boundaries also play an important role in applications — for instance, in statistical mechanics, phase transitions (such as freezing and boiling) are associated with natural boundaries of the partition function.