exponential integral

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The antiderivative of the function

$$x \mapsto \frac{e^{-x}}{x}$$

is not expressible in closed form. Thus such http://planetmath.org/ImproperIntegralintegrals as

$$\int_x^\infty \frac{e^{-t}}{t}\,dt \quad \text{and} \quad \int_\infty^{-x} \frac{e^{-t}}{t}\,dt,$$

define certain http://planetmath.org/ElementaryFunctionnon-elementary transcendental functions. They are called *exponential integrals* and denoted usually E_1 and Ei, respectively. Accordingly,

$$\mathrm{E}_1(x) \;:=\; \int_x^\infty \frac{e^{-t}}{t}\,dt$$

$$\mathrm{Ei}\,x \;:=\; \int_\infty^{-x} \frac{e^{-t}}{t}\,dt \;=\; -\int_{-x}^\infty \frac{e^{-t}}{t}\,dt \;:=\; \int_{-\infty}^{x} \frac{e^{-u}}{u}\,du.$$

Then one has the connection

$$\mathrm{E}_1(x) \;=\; -\mathrm{Ei}\,(-x).$$

For positive values of x the series expansion

$$\mathrm{Ei}\,x \;=\; \gamma + \ln x + \sum_{j=1}^{\infty} \frac{x^j}{j!j},$$

where γ is the http://planetmath.org/node/1883Euler–Mascheroni constant, is valid.

Note: Some authors use the convention $\mathrm{Ei}\,x \;:=\; \int_x^\infty \frac{e^{-t}}{t}\,dt$.

0.1 Laplace transform of $\frac{1}{t+a}$

By the definition of Laplace transform,

$$\mathcal{L}\{\frac{1}{t+a}\} \;=\; \int_0^\infty \frac{e^{-st}}{t+a}\,dt.$$

The http://planetmath.org/ChangeOfVariableInDefiniteIntegralsubstitution $t+a=u$ gives

$$\mathcal{L}\{\frac{1}{t+a}\} \;=\; \int_a^\infty \frac{e^{as-su}}{u}\,du \;=\; e^{as}\int_a^\infty \frac{e^{-su}}{u}\,du,$$

from which the substitution $su=t$ yields

$$\mathcal{L}\{\frac{1}{t+a}\} \;=\; e^{as}\int_{as}^\infty \frac{e^{-t}}{t}\,dt,$$

i.e.

$$\mathcal{L}\{\frac{1}{t+a}\} \;=\; e^{as}\mathrm{E}_1(as). \tag{1}$$

Using http://planetmath.org/LaplaceTransformOfDerivativethe rule $\mathcal{L}\{f'(t)\} = sF(s)-f(0)$, one easily derives from (1) the

$$\mathcal{L}\{\frac{1}{(t+a)^2}\} \;=\; \frac{1}{a}-se^{as}\mathrm{E}_1(as). \tag{2}$$