proof of Riemann's removable singularity theorem

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Suppose that f is holomorphic on $U \setminus \{a\}$ and $\lim_{z\to a}(z-a)f(z) = 0$. Let

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$$

be the Laurent series of f centered at a. We will show that $c_k = 0$ for $k < 0$, so that f can be holomorphically extended to all of U by defining $f(a) = c_0$.

For any non-negative integer n, the residue of $(z-a)^n f(z)$ at a is

$$\mathrm{Res}((z-a)^n f(z), a) = \frac{1}{2\pi i} \lim_{\delta\to 0^+} \oint_{|z-a|=\delta} (z-a)^n f(z) \mathrm{d}z.$$

This is equal to zero, because

$$\begin{aligned} \left| \oint_{|z-a|=\delta} (z-a)^n f(z) \mathrm{d}z \right| &\leq 2\pi\delta \max_{|z-a|=\delta} |(z-a)^n f(z)| \\ &= 2\pi\delta^n \max_{|z-a|=\delta} |(z-a)f(z)| \end{aligned}$$

which, by our assumption, goes to zero as $\delta \to 0$. Since the residue of $(z-a)^n f(z)$ at a is also equal to c_{-n-1}, the coefficients of all negative powers of z in the Laurent series vanish.

Conversely, if a is a removable singularity of f, then f can be expanded in a power series centered at a, so that

$$\lim_{z\to a} (z-a)f(z) = 0$$

because the constant term in the power series of $(z-a)f(z)$ is zero.

A corollary of this theorem is the following: if f is bounded near a, then

$$|(z-a)f(z)| \leq |z-a|M$$

for some $M > 0$. This implies that $(z-a)f(z) \to 0$ as $z \to a$, so a is a removable singularity of f.