proof of equivalence of formulas for exp

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We present an elementary proof that:

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n .$$

There are of course other proofs, but this one has the advantage that it carries verbatim for the matrix exponential and the operator exponential.

At the outset, we observe that $\sum_{k=0}^{\infty} z^k/k!$ converges by the ratio test. For definiteness, the notation e^z below will refer to exactly this series.

Proof. We expand the right-hand in the straightforward manner:

$$\begin{aligned}
\left(1 + \frac{z}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{n}\right)^k \\
&= \sum_{k=0}^{n} \frac{n \cdot (n-1) \cdots (n-k+1)}{n^k} \frac{z^k}{k!} = \sum_{k=0}^{n} \pi(k,n)\, \frac{z^k}{k!} ,
\end{aligned}$$

where $\pi(k,n)$ denotes the coefficient

$$1 \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) .$$

Let $|z| \leq M$. Given $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that whenever $n \geq N$, then $\sum_{k=n+1}^{\infty} M^k/k! < \epsilon/2$, since the sum is the tail of the convergent series e^M.

Since $\lim_{n\to\infty} \pi(k,n) = 1$ for k, there is also a $N' \in \mathbb{N}$, with $N' \geq N$, so that whenever $n \geq N'$ and $0 \leq k \leq N$, then $|\pi(k,n) - 1| < \epsilon/(2e^M)$. (Note that k is chosen only from a *finite* set.)

Now, when $n \geq N'$, we have

$$\begin{aligned}
\left| \sum_{k=0}^{n} \pi(k,n) \frac{z^k}{k!} - \sum_{k=0}^{\infty} \frac{z^k}{k!} \right| &= \left| \sum_{k=0}^{n} (\pi(k,n) - 1) \frac{z^k}{k!} - \sum_{k=n+1}^{\infty} \frac{z^k}{k!} \right| \\
&\leq \sum_{k=0}^{n} |\pi(k,n) - 1|\, \frac{M^k}{k!} + \sum_{k=n+1}^{\infty} \frac{M^k}{k!} \\
&= \sum_{k=0}^{N} |\pi(k,n) - 1|\, \frac{M^k}{k!} + \sum_{k=N+1}^{n} |\pi(k,n) - 1|\, \frac{M^k}{k!} + \sum_{k=n+1}^{\infty} \frac{M^k}{k!} \\
&< \frac{\epsilon}{2e^M} \sum_{k=0}^{N} \frac{M^k}{k!} + \sum_{k=N+1}^{n} \frac{M^k}{k!} + \sum_{k=n+1}^{\infty} \frac{M^k}{k!}
\end{aligned}$$

(In the middle sum, we use the bound $|\pi(k,n)-1| = 1-\pi(k,n) \le 1$ for all k and n.)

$$< \frac{\epsilon}{2e^M} \cdot e^M + \frac{\epsilon}{2} = \epsilon\,. \qquad \square$$

In fact, we have proved uniform convergence of $\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$ over $|z| \le M$. Exploiting this fact we can also show:

$$\left(1+\frac{z}{n}+o\left(\frac{1}{n}\right)\right)^n = \left(1+\frac{z+o(1)}{n}\right)^n \to \sum_{k=0}^{\infty}\frac{z^k}{k!} \quad \text{(pointwise, as } n\to\infty)$$

Proof. $|z| < M$. Given $\epsilon > 0$, for large enough n, we have

$$\left|\left(1+\frac{w}{n}\right)^n - e^w\right| < \epsilon/2 \quad \text{uniformly for all } |w| \le M.$$

Since $o(1)\to 0$, for large enough n we can set $w = z+o(1)$ above. Since the exponential is continuous[1], for large enough n we also have $|e^{z+o(1)}-e^z| < \epsilon/2$. Thus

$$\left|\left(1+\frac{z+o(1)}{n}\right)^n - e^z\right| \le \left|\left(1+\frac{z+o(1)}{n}\right)^n - e^{z+o(1)}\right| + |e^{z+o(1)}-e^z| < \epsilon \qquad \square$$

[1]follows from uniform convergence on bounded subsets of either expression for e^z