proof to Cauchy-Riemann equations (polar coordinates)

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Owner	Daume (40)
Last modified by	Daume (40)
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Author	Daume (40)
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If $f(z)$ is differentialble at z_0 then the following limit

$$f'(z_0) \quad = \quad \lim_{\xi \to 0} \frac{f(z_0 + \xi) - f(z_0)}{\xi}$$

will remain the same approaching from any direction. First we fix θ as θ_0 then we take the limit along the ray where the argument is equal to θ_0. Then

$$\begin{aligned}
f'(z_0) &= \lim_{h \to 0} \frac{f(r_0 e^{i\theta_0} + h e^{i\theta_0}) - f(r_0 e^{i\theta_0})}{h e^{i\theta_0}} \\
&= \lim_{h \to 0} \frac{f((r_0 + h) e^{i\theta_0}) - f(r_0 e^{i\theta_0})}{h e^{i\theta_0}} \\
&= \lim_{h \to 0} \frac{u(r_0 + h, \theta_0) + iv(r_0 + h, \theta_0) - u(r_0, \theta_0) - iv(r_0, \theta_0)}{h e^{i\theta_0}} \\
&= \frac{1}{e^{i\theta_0}} \left[\lim_{h \to 0} \frac{u(r_0 + h, \theta_0) - u(r_0, \theta_0)}{h} + i \lim_{h \to 0} \frac{v(r_0 + h, \theta_0) - v(r_0, \theta_0)}{h} \right] \\
&= \frac{1}{e^{i\theta_0}} \left[\frac{\partial u}{\partial r}(r_0, \theta_0) + i \frac{\partial v}{\partial r}(r_0, \theta_0) \right]
\end{aligned}$$

Similarly, if we take the limit along the circle with fixed r equals r_0. Then

$$\begin{aligned}
f'(z_0) &= \lim_{h \to 0} \frac{f(r_0 e^{i\theta_0} + r_0 e^{i(\theta_0 + h)}) - f(r_0 e^{i\theta_0})}{r_0 e^{i\theta_0}(e^{ih} - 1)} \\
&= \lim_{h \to 0} \frac{f(r_0 e^{i(\theta_0 + h)}) - f(r_0 e^{i\theta_0})}{h e^{i\theta_0}} \\
&= \lim_{h \to 0} \frac{u(r_0, \theta_0 + h) + iv(r_0, \theta_0 + h) - u(r_0, \theta_0) - iv(r_0, \theta_0)}{h e^{i\theta_0}} \\
&= \frac{1}{r_0 e^{i\theta_0}} \left[\lim_{h \to 0} \frac{u(r_0 + h, \theta_0) - u(r_0, \theta_0)}{h} \cdot \frac{h}{e^{ih} - 1} + i \lim_{h \to 0} \frac{v(r_0 + h, \theta_0) - v(r_0, \theta_0)}{h} \cdot \frac{h}{e^{ih} - 1} \right] \\
&= \frac{1}{r_0 e^{i\theta_0}} \left[\lim_{h \to 0} \frac{u(r_0 + h, \theta_0) - u(r_0, \theta_0)}{h} \cdot \lim_{h \to 0} \frac{h}{e^{ih} - 1} + i \lim_{h \to 0} \frac{v(r_0 + h, \theta_0) - v(r_0, \theta_0)}{h} \cdot \lim_{h \to 0} \frac{}{e^{ih}} \right. \\
&= \frac{1}{r_0 e^{i\theta_0}} \left[\frac{\partial u}{\partial \theta}(r_0, \theta_0) \frac{1}{i} + \frac{\partial v}{\partial \theta}(r_0, \theta_0) \right] \\
&= \frac{1}{r_0 e^{i\theta_0}} \left[\frac{\partial v}{\partial \theta}(r_0, \theta_0) - i \frac{\partial u}{\partial \theta}(r_0, \theta_0) \right]
\end{aligned}$$

Note: We use l'Hôpital's rule to obtain the following result used above $\lim_{h\to 0} \frac{h}{e^{ih}-1} = \frac{1}{i}$.

Now, since the limit is the same along the circle and the ray then they are equal:

$$\begin{aligned}
\frac{1}{e^{i\theta_0}}\left[\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right] &= \frac{1}{r_0 e^{i\theta_0}}\left[\frac{\partial v}{\partial \theta}(r_0,\theta_0) - i\frac{\partial u}{\partial \theta}(r_0,\theta_0)\right] \\
\left[\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right] &= \frac{1}{r_0}\left[\frac{\partial v}{\partial \theta}(r_0,\theta_0) - i\frac{\partial u}{\partial \theta}(r_0,\theta_0)\right]
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\
\frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta}
\end{aligned}$$

QED