proof of fundamental theorem of algebra (argument principle)

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The fundamental theorem of algebra can be proven using the argument principle. Not only is this proof interesting because it demonstrates an important result, it also serves as an example of how to use the argument principle. Since it is so simple, it can be thought of as a "toy model" (see toy theorem) for theorems on the zeros of analytic functions. For a variant of this proof using Rouché's theorem (which is a consequence of the argument principle) please see the proof of the fundamental theorem of algebra (Rouché's theorem).

Proof. Consider the rational function

$$g(z) = \frac{zf'(z)}{f(z)}.$$

Denote the degree of the polynomial f by n. Then we can write

$$g(z) = \frac{nz^n + \text{lower degree terms}}{z^n + \text{lower degree terms}}.$$

This makes it clear that $\lim_{z \to \infty} g(z) = n$. Hence there exists a real constant R such that $|g(z) - n| < 1/2$ whenever $|z| \geq R$.

Consider the integral

$$I = \oint_{|z|=R} \frac{f'(z)}{f(z)} dz.$$

This can be rewritten as

$$I = \oint_{|z|=R} \frac{g(z)}{z} dz.$$

Split the integral into two parts, writing $I = I_1 + I_2$ where

$$I_1 = \oint_{|z|=R} \frac{n}{z} dz, \qquad I_2 = \oint_{|z|=R} \frac{g(z) - n}{z} |dz|.$$

The integral I_1 is easy: $I_1 = 2\pi i n$. As for I_2, we shall bound it using our bound for $|g(z) - n|$.

$$|I_2| \leq \oint_{|z|=R} \frac{|g(z) - n|}{|z|} dz \leq \frac{1}{2} \oint \frac{|dz|}{|z|} = \pi.$$

Since polynomials are analytic functions in the whole complex plane, f is an analytic function of z when $|z| \leq R$, so the argument principle applies and we conclude that $I/2\pi i$ must equal the number of zeros of f, counted with multiplicity. Among other things, this means that $I/2\pi i$ must be an integer. By explicit computation, we already know that $I_1/2\pi i$ is also an integer. Hence $|I/(2\pi i) - I_1/(2\pi i)|$ is an integer. But

$$\left| \frac{I}{(2\pi i)} - \frac{I_1}{(2\pi i)} \right| = \left| \frac{I_2}{(2\pi i)} \right| \leq \frac{1}{2}.$$

Now, the only integer smaller that $1/2$ in absolute value is 0, so we must have $I = I_1$. This implies that f has n zeros (counting with multiplicity) when $|z| < R$. (By the way we chose R, $f(z) \neq 0$ whenever $z \geq R$, so f has exactly n zeros in the whole complex plane.)