proof of fundamental theorem of algebra (due to d'Alembert)

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This proof, due to d'Alembert, relies on the following three facts:

- Every polynomial with real coefficients which is of odd order has a real root. (This is a corollary of the intermediate value theorem.
- Every second order polynomial with complex coefficients has two complex roots.
- For every polynomial p with real coefficients, there exists a field E in which the polynomial may be factored into linear terms. (For more information, see the entry "splitting field".)

Note that it suffices to prove that every polynomial with real coefficients has a complex root. Given a polynomial with complex coefficients, one can construct a polynomial with real coefficients by multiplying the polynomial by its complex conjugate. Any root of the resulting polynomial will either be a root of the original polynomial or the complex conjugate of a root.

The proof proceeds by induction. Write the degree of the polynomial as $2^n(2m+1)$. If $n = 0$, then we know that it must have a real root. Next, assume that we already have shown that the fundamental theorem of algebra holds whenver $n < N$. We shall show that any polynomial of degree $2^N(2m+1)$ has a complex root if a certain other polynomial of order $2^{N-1}(2m'+1)$ has a root. By our hypothesis, the other polynomial does have a root, hence so does the original polynomial. Hence, by induction on n, every polynomial with real coefficients has a complex root.

Let p be a polynomial of order $d = 2^N(2m+1)$ with real coefficients. Let its factorization over the extension field E be

$$p(x) = (x - r_1)(x - r_2) \cdots (x - r_d)$$

Next construct the $d(d-1)/2 = 1$ polynomials

$$q_k(x) = \prod_{i<j} (x - r_i - r_j - k r_i r_j)$$

where k is an integer between 1 and $d(d-1)/2 = 1$. Upon expanding the product and collecting terms, the coefficient of each power of x is a symmetric function of the roots r_i. Hence it can be expressed in terms of the coefficients of p, so the coefficients of q_k will all be real.

Note that the order of each q_k is $d(d-1)/2 = 2^{N-1}(2m+1)(2^N(2m+1)-1)$. Hence, by the induction hypothesis, each q_k must have a complex root. By construction, each root of q_k can be expressed as $r_i+r_j+kr_ir_j$ for some choice of integers i and j. By the pigeonhole principle, there must exist integers i, j, k_1, k_2 such that both

$$u = r_i + r_j + k_1 r_i r_j$$

and

$$v = r_i + r_j + k_2 r_i r_j$$

are complex. But then r_i and r_j must be complex as well. because they are roots of the polynomial

$$x^2 + bx + c$$

where

$$b = -\frac{k_2 u + k_1 v}{(k_1 + k_2)}$$

and

$$c = \frac{u - v}{k_1 - k_2}$$

Note. D'Alembert was an avid supporter (in fact, the co-editor) of the famous French philosophical encyclopaedia. Therefore it is a fitting tribute to have his proof appear in the web pages of this encyclopaedia.

References

[1] Jean le Rond D'Alembert: "Recherches sur le calcul intégral". *Histoire de l'Acadḿie Royale des Sciences et Belles Lettres*, année MDCCXLVI, 182–224. Berlin (1746).

[2] R. Argand: "Réflexions sur la nouvelle théorie d'analyse". *Annales de mathématiques* **5**, 197–209 (1814).