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# places of holomorphic function

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If $c$ is a complex constant and $f$ a holomorphic function in a domain $D$ of $\mathbb{C}$, then $f$ has in every compact (`http://planetmath.org/TopologyOfTheComplexPlane`closed and `http://planetmath.org/Bounded`bounded) subdomain of $D$ at most a finite set of `http://planetmath.org/node/9084`$c$-places, i.e. the points $z$ where $f(z) = c$, except when $f(z) \equiv c$ in the whole $D$.

*Proof.* Let $A$ be a subdomain of $D$. Suppose that there is an infinite amount of $c$-places of $f$ in $A$. By `http://planetmath.org/node/2125`Bolzano–Weierstrass theorem, these $c$-places have an accumulation point $z_0$, which belongs to the closed set $A$. Define the constant function $g$ such that

$$g(z) \;=\; c$$

for all $z$ in $D$. Then $g$ is holomorphic in the domain $D$ and $g(z) = c$ in an infinite subset of $D$ with the accumulation point $z_0$. Thus in the $c$-places of $f$ we have

$$g(z) \;=\; f(z).$$

Consequently, the identity theorem of holomorphic functions implies that

$$f(z) \;=\; g(z) \;=\; c$$

in the whole $D$. Q.E.D.