persistence of analytic relations

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The principle of persistence of analytic relations states that any algebraic relation between several analytic functions which holds on a sufficiently large set also holds wherever the functions are defined.

A more explicit statement of this principle is as follows: Let $f_1, f_2, \dots f_n$ be complex analytic functions. Suppose that there exists an open set D on which all these functions are defined and that there exists a polynomial p of n variables such that $p(f_1(z), f_2(z), \dots, f_n(z)) = 0$ whenever z lies in a subset X of D which has a limit point in D. Then $p(f_1(z), f_2(z), \dots f_n(z)) = 0$ for all $z \in D$.

This fact is a simple consequence of the rigidity theorem for analytic functions. If $f_1, f_2, \dots f_n$ are all analytic in D, then $p(f_1(z), f_2(z), \dots f_n(z))$ is also analytic in D. Hence, if $p(f_1(z), f_2(z), \dots, f_n(z)) = 0$ when z in X, then $p(f_1(z), f_2(z), \dots, f_n(z)) = 0$ for all $z \in D$.

This principle is very useful in establishing identites involving analytic functions because it means that it suffices to show that the identity holds on a small subset. For instance, from the fact that the familiar identity $\sin^2 x + \cos^2 x = 1$ holds for all real x, it automatically holds for all complex values of x. This principle also means that it is unnecessary to specify for which values of the variable an algebraic relation between analytic functions holds since, if such a relation holds, it will hold for all values for which the functions appearing in the relation are defined.