complex tangent and cotangent

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The tangent and the cotangent function for complex values of the z are defined with the equations

$$\tan z := \frac{\sin z}{\cos z}, \quad \cot z := \frac{\cos z}{\sin z}.$$

Using the `http://planetmath.org/ComplexSineAndCosine`Euler's formulae, one also can define

$$\tan z := -i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}, \quad \cot z := i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}. \tag{1}$$

The subtraction formulae of `http://planetmath.org/ComplexSineAndCosine`cosine and sine yield an additional between the cotangent and tangent:

$$\cot\left(\frac{\pi}{2}-z\right) = \frac{\cos\left(\frac{\pi}{2}-z\right)}{\sin\left(\frac{\pi}{2}-z\right)} = \frac{\cos\frac{\pi}{2}\cos z+\sin\frac{\pi}{2}\sin z}{\sin\frac{\pi}{2}\cos z-\cos\frac{\pi}{2}\sin z} = \frac{\sin z}{\cos z} = \tan z.$$

Thus the properties of the tangent are easily derived from the corresponding properties of the cotangent.

Because of the identic equation $\cos^2 z+\sin^2 z = 1$ the cosine and sine do not vanish simultaneously, and so their quotient $\cot z$ is finite in all finite points z of the complex plane except in the zeros $z = n\pi$ $(n = 0, \pm1, \pm2, \ldots)$ of $\sin z$, where $\cot z$ becomes infinite. We shall see that these multiples of π are simple poles of $\cot z$.

If one moves from z to $z+\pi$, then both $\cos z$ and $\sin z$ change their signs (cf. antiperiodic function), and therefore their quotient remains unchanged. Accordingly, π is a period of $\cot z$. But if ω is an arbitrary period of $\cot z$, we have $\cot(z+\omega) = \cot z$, and especially $z = 0$ gives $\cot\omega = \infty$; then (1) says that $e^{i\omega} = e^{-i\omega}$, i.e. $e^{2i\omega} = 1$. Since the prime period of the complex exponential function is $2i\pi$, the last equation is valid only for the values $\omega = n\pi$ $(n = 0, \pm1, \pm2, \ldots)$. Thus we have shown that the prime period of $\cot z$ is π.

We know that

$$\frac{\sin z}{z} = \frac{\sin z-\sin 0}{z} \to \cos 0 = 1 \quad \text{as} \quad z \to 0;$$

therefore

$$z\cot z = \frac{z}{\sin z}\cdot\cos z \to 1\cdot\cos 0 = 1 \quad \text{as} \quad z \to 0.$$

This result, together with

$$\cot z \to \infty \quad \text{as} \quad z \to 0,$$

means that $z = 0$ is a simple pole of $\cot z$.

Because of the periodicity, $\cot z$ has the simple poles in the points $z = 0,\ \pm\pi,\ \pm 2\pi,\ \ldots$. Since one has the derivative

$$\frac{d \cot z}{dz} = -\frac{1}{\sin^2 z},$$

$\cot z$ is holomorphic in all finite points except those poles, which accumulate only to the point $z = \infty$. Thus the cotangent is a meromorphic function. The same concerns naturally the tangent function.

As all meromorphic functions, the cotangent may be expressed as a series with the http://planetmath.org/PartialFractionsOfExpressionspartial fraction terms of the form $\frac{a_{jk}}{(z-p_j)^k}$, where p_j's are the poles — see http://planetmath.org/Examples entry.

The http://planetmath.org/CmplexFunctionreal and imaginary parts of tangent and cotangent are seen from the formulae

$$\tan(x+iy) = \frac{\sin x \cos x + i \sinh y \cosh y}{\cos^2 x + \sinh^2 y},$$

$$\cot(x+iy) = \frac{\sin x \cos x - i \sinh y \cosh y}{\sin^2 x + \sinh^2 y},$$

which may be derived from (1) by substituting $z := x+iy$ $(x,\ y \in \mathbb{R})$.

References

[1] R. Nevanlinna & V. Paatero: *Funktioteoria.* Kustannusosakeyhtiö Otava. Helsinki (1963).