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convergence of complex term series

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A series

$$\sum_{\nu=1}^{\infty} c_\nu \ = \ c_1+c_2+c_3+\ldots \tag{1}$$

with complex terms

$$c_\nu \ = \ a_\nu+ib_\nu \qquad (a_\nu,\, b_\nu \in \mathbb{R} \ \ \forall\, \nu)$$

is convergent iff the sequence of its partial sums converges to a complex number.

Theorem 1. The series (1) converges iff the series

$$\sum_{\nu=1}^{\infty} a_\nu \quad \text{and} \quad \sum_{\nu=1}^{\infty} b_\nu \tag{2}$$

formed by real parts and the imaginary parts of its terms both are convergent.

Proof. Let $\varepsilon > 0$. Denote

$$\sum_{\nu=1}^{n} a_\nu \ := \ s_n, \quad \sum_{\nu=1}^{n} b_\nu \ := \ t_n, \quad \sum_{\nu=1}^{n} c_\nu \ := \ u_n.$$

If the series (2) are convergent with sums S and T, then there is a number N such that

$$|s_n - S| < \frac{\varepsilon}{2}, \quad |t_n - T| < \frac{\varepsilon}{2} \quad \text{when} \quad n \geqq N.$$

Accordingly,

$$|u_n-(S+iT)| = \sqrt{(s_n - S)^2 + (t_n - T)^2} \leqq |s_n-S|+|t_n-T| < \varepsilon \quad \text{when} \quad n \geqq N,$$

i.e. the series (1) converges to $S+iT$. If, conversely, (1) converges to a complex number

$$u \ = \ s+it \quad (s,\, t \in \mathbb{R}),$$

then

$$|s_n-s| \ \leqq \ |(s_n-s)+i(t_n-t)| \ = \ |u_n-u|, \quad |t_n-t| \ \leqq \ |(s_n-s)+i(t_n-t)| \ = \ |u_n-u|,$$

and consequently, $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, i.e. the series (2) are convergent with sums the real numbers s and t.

Theorem 2. The series (1) converges absolutely iff the series (2) both converge absolutely.

Proof. The absolute convergence of (1) means that the series

$$\sum_{\nu=1}^{\infty} |c_\nu|$$

converges. But since $|c_\nu|^2 = |a_\nu|^2 + |b_\nu|^2$, we have

$$|a_\nu| \leqq |c_\nu|; \quad |b_\nu| \leqq |c_\nu| \leqq |a_\nu| + |c_\nu|.$$

From these inequalities we can infer the assertion of the theorem 2.

References

[1] R. Nevanlinna & V. Paatero: *Funktioteoria*. Kustannusosakeyhtiö Otava. Helsinki (1963).