proof of fundamental theorem of algebra

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If $f(x) \in \mathbb{C}[x]$ let a be a root of $f(x)$ in some extension of $\mathbb{C}$. Let K be a Galois closure of $\mathbb{C}(a)$ over $\mathbb{R}$ and set $G = \mathrm{Gal}(K/\mathbb{R})$. Let H be a Sylow 2-subgroup of G and let $L = K^H$ (the fixed field of H in K). By the Fundamental Theorem of Galois Theory we have $[L : \mathbb{R}] = [G : H]$, an odd number. We may write $L = \mathbb{R}(b)$ for some $b \in L$, so the minimal polynomial $m_{b,\mathbb{R}}(x)$ is irreducible over $\mathbb{R}$ and of odd degree. That degree must be 1, and hence $L = \mathbb{R}$, which means that $G = H$, a 2-group. Thus $G_1 = \mathrm{Gal}(K/\mathbb{C})$ is also a 2-group. If $G_1 \neq 1$ choose $G_2 \leq G_1$ such that $[G_1 : G_2] = 2$, and set $M = K^{G_2}$, so that $[M : \mathbb{C}] = [G_1 : G_2] = 2$. But any polynomial of degree 2 over $\mathbb{C}$ has roots in $\mathbb{C}$ by the quadratic formula, so such a field M cannot exist. This contradiction shows that $G_1 = 1$. Hence $K = \mathbb{C}$ and $a \in \mathbb{C}$, completing the proof.