proof of estimating theorem of contour integral

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WLOG consider $g(t) : \mathbb{R} \to \mathbb{C}$ a parameterization of the γ curve along which the integral is evaluated with $|g'(t)| = 1$. This amounts to a canonical parameterization and is always possible. Since the integral is independent of re-parameterization[1] the result will be completely general.

With this in mind, the contour integral can be explicitly written as

$$\int_\gamma f(z)dz = \int_0^L f(g(t))g'(t)dt \tag{1}$$

where L is the arc length of the curve γ.

Consider the set of all continuous functions $[0, L] \to \mathbb{C}$ as a vector space[2], we can define an inner product in it via

$$\langle f, g\rangle = \int_0^L f(t)\bar{g}(t)dt \tag{2}$$

The axioms are easy to verify:

- $\langle k_1a_1 + k_2a_2, a_3\rangle = \int_0^L (k_1a_1(t) + k_2a_2(t))\bar{a_3}(t)dt = k_1\langle a_1, a_3\rangle + k_2\langle a_2, a_3\rangle$
- $\langle a, b\rangle = \int_0^L a(t)\bar{b}(t)dt = \int_0^L \overline{b(t)\bar{a}(t)}dt = \overline{\int_0^L b(t)\bar{a}(t)dt} = \overline{\langle b, a\rangle}$
- $\langle a, a\rangle = \int_0^L a(t)\bar{a}(t)dt = \int_0^L |a(t)|^2dt \geq 0$ since the integrand is a non-negative (real) function, and 0 iff $|a|^2 = 0$ everywhere in the interval, that is: $\langle a, a\rangle = 0 \iff a = 0$

With all this in mind, equation **??** can be written as

$$\int_\gamma f(z)dz = \langle f \circ g, \bar{g}'\rangle \tag{3}$$

Where by definition $\|f\| = \sqrt{< f, f >}$ is the norm associated with the inner product defined previously.

Using Cauchy-Schwarz inequality we can write that

$$|\langle f \circ g, \bar{g}'\rangle| \leq \|f \circ g\|\|\bar{g}'\| \tag{4}$$

But since by assumption the parameterization g is canonic, $\|\bar{g}'\| = \|g'\| = \sqrt{\int_0^L 1dt} = \sqrt{L}$.

[1] apart from a possible sign change due to exchange of orientation of the path

[2] axioms are trivial to verify

On the other hand $\|f \circ g\| = \sqrt{\int_0^L f(g(t))\bar{f}(g(t))dt} \leq \sqrt{\int_0^L M^2 dt} = M\sqrt{L}$, where $|f(g(t))| \leq M$ for every point on γ.

The previous paragraphs imply that

$$\left| \int_\gamma f(z)dz \right| \leq ML \tag{5}$$

which is the result we aimed to prove.

Cauchy-Schwarz inequality says more, it also says that $|\langle a, b\rangle| = \|a\|\|b\| \iff a = \lambda b$ where λ is a constant.

So if $|\langle f \circ g, \bar{g}'\rangle = \|f \circ g\|\|\bar{g}'\|$ then $f \circ g = \lambda \bar{g}'$, where $\lambda \in \mathbb{C}$ is a constant. If g is a canonical parameterization $|g'| = 1$ and we get the absolute modulus $|\lambda| = |f \circ g|$ (which must be constant) and all that remains is to find the phase of λ which must also be constant.