harmonic conjugate function

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Two harmonic functions u and v from an http://planetmath.org/OpenSetopen subset A of $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$, which satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x, \tag{1}$$

are the *harmonic conjugate functions* of each other.

- The relationship between u and v has a geometric meaning: Let's determine the slopes of the constant-value curves $u(x,\,y) = a$ and $v(x,\,y) = b$ in any point $(x,\,y)$ by differentiating these equations. The first gives $u_x dx + u_y dy = 0$, or
$$\frac{dy}{dx}^{(u)} = -\frac{u_x}{u_y} = \tan\alpha,$$
and the second similarly
$$\frac{dy}{dx}^{(v)} = -\frac{v_x}{v_y}$$
but this is, by virtue of (1), equal to
$$\frac{u_y}{u_x} = -\frac{1}{\tan\alpha}.$$
Thus, by the condition of orthogonality, the curves intersect at right angles in every point.

- If one of u and v is known, then the other may be determined with (1): When e.g. the function u is known, we need only to the line integral
$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} (-u_y\,dx + u_x\,dy)$$
along any connecting $(x_0,\,y_0)$ and $(x,\,y)$ in A. The result is the harmonic conjugate v of u, unique up to a real addend if A is simply connected.

- It follows from the preceding, that every harmonic function has a harmonic conjugate function.

- The real part and the imaginary part of a holomorphic function are always the harmonic conjugate functions of each other.

Example. $\sin x\cosh y$ and $\cos x\sinh y$ are harmonic conjugates of each other.