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general power

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Entry type	Definition
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Synonym	complex power
Related topic	Logarithm
Related topic	ExponentialOperation
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Related topic	UsingResidueTheoremNearBranchPoint
Defines	base of the power
Defines	base
Defines	exponent
Defines	branch

The *general power* z^μ, where $z\,(\neq 0)$ and μ are arbitrary complex numbers, is defined via the complex exponential function and complex logarithm (denoted here by "log") of the by setting

$$z^\mu := e^{\mu\log z} = e^{\mu(\ln|z|+i\arg z)}.$$

The number z is the *base of the power* z^μ and μ is its *exponent*.

Splitting the exponent $\mu = \alpha + i\beta$ in its real and imaginary parts one obtains

$$z^\mu = e^{\alpha\ln|z|-\beta\arg z}\cdot e^{i(\beta\ln|z|+\alpha\arg z)},$$

and thus

$$|z^\mu| = e^{\alpha\ln|z|-\beta\arg z}, \quad \arg z^\mu = \beta\ln|z|+\alpha\arg z.$$

This shows that both the modulus and the http://planetmath.org/Complexargument of the general power are in general multivalued. The modulus is unique only if $\beta = 0$, i.e. if the exponent $\mu = \alpha$ is real; in this case we have

$$|z^\mu| = |z|^\mu, \quad \arg z^\mu = \mu\cdot\arg z.$$

Let $\beta \neq 0$. If one lets the point z go round the origin anticlockwise, $\arg z$ gets an addition 2π and hence the z^μ has been multiplied by a having the modulus $e^{-2\pi\beta} \neq 1$, and we may say that z^μ has come to a new *branch*.

Examples

1. $z^{\frac{1}{m}}$, where m is a positive integer, coincides with the m^{th} http://planetmath.org/Calculati of z.

2. $3^2 = e^{2\log 3} = e^{2(\ln 3+2n\pi i)} = 9(e^{2\pi i})^{2n} = 9 \quad \forall n\in\mathbb{Z}$.

3. $i^i = e^{i\log i} = e^{i(\ln 1+\frac{\pi}{2}i-2n\pi i)} = e^{2n\pi-\frac{\pi}{2}}$ (with $n = 0,\,\pm1,\,\pm2,\,\ldots$); all these values are positive real numbers, the simplest of them is $\dfrac{1}{\sqrt{e^\pi}} \approx 0.20788$.

4. $(-1)^i = e^{(2n+1)\pi}$ (with $n = 0,\,\pm1,\,\pm2,\,\ldots$) also are situated on the positive real axis.

5. $(-1)^{\sqrt{2}} = e^{\sqrt{2}\log(-1)} = e^{\sqrt{2}i(\pi+2n\pi)} = e^{i(2n+1)\pi\sqrt{2}}$ (with $n = 0,\,\pm1,\,\pm2,\,\ldots$); all these are (meaning here that their imaginary parts are distinct from 0), situated on the circumference of the unit circle such that all points

of the circumference are accumulation points of the sequence of the $(-1)^{\sqrt{2}}$ (see http://planetmath.org/SequenceAccumulatingEverywhereIn11this entry).

6. $2^{1-i} = 2e^{2n\pi}(\cos\ln 2 + i\sin\ln 2)$ (with $n = 0, \pm1, \pm2, \ldots$), are situated on the half line beginning from the origin with the argument $\ln 2 \approx 0.69315$ radians.