Laurent expansion of rational function

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The Laurent series expansion of a rational function may often be determined using the uniqueness of Laurent series coefficients in an annulus and applying geometric series. We will determine the expansion of

$$f(z) \;:=\; \frac{2z}{1+z^2}$$

by the powers of $z-i$.

We first have the partial fraction decomposition

$$f(z) \;=\; \frac{1}{z-i} + \frac{1}{z+i} \tag{1}$$

whence the principal part of the Laurent expansion contains $\dfrac{1}{z-i}$. Taking into account the poles $z = \pm i$ of f we see that there are two possible annuli for the Laurent expansion:

a) The annulus $0 < |z-i| < 2$. We can write

$$\frac{1}{z+i} \;=\; \frac{1}{2i+(z-i)} \;=\; \frac{1}{2i}\cdot\frac{1}{1-\left(-\frac{z-i}{2i}\right)} \;=\; \frac{1}{2i} - \frac{z-i}{(2i)^2} + \frac{(z-i)^2}{(2i)^3} - + \ldots$$

Thus

$$\frac{2z}{1+z^2} \;=\; \frac{1}{z-i} - \sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^{n+1}(z-i)^n \qquad (0 < |z-i| < 2).$$

b) The annulus $2 < |z-i| < \infty$. Now we write

$$\frac{1}{z+i} \;=\; \frac{1}{(z-i)+2i} \;=\; \frac{1}{z-i}\cdot\frac{1}{1-\left(-\frac{2i}{z-i}\right)} \;=\; \frac{1}{z-i} - \frac{2i}{(z-i)^2} + \frac{(2i)^2}{(z-i)^3} - + \ldots$$

Accordingly

$$\frac{2z}{1+z^2} \;=\; \frac{2}{z-i} + \sum_{n=2}^{\infty}\frac{(-2i)^{n-1}}{(z-i)^n} \qquad (2 < |z-i| < \infty).$$

This latter Laurent expansion consists of negative powers only, but $z = i$ isn't an essential singularity of f, though.