logarithmic integral

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Related topic	SineIntegral
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Related topic	ConvergenceOfIntegrals
Defines	logarithmic integral
Defines	logarithmus integralis
Defines	Eulerian logarithmic integral

The European or Eulerian version of *logarithmic integral* (in Latin *logarithmus integralis*) is defined as

$$\operatorname{Li} x := \int_2^x \frac{dt}{\ln t}, \tag{1}$$

and the American version is

$$\operatorname{li} x := \int_0^x \frac{dt}{\ln t}, \tag{2}$$

The integrand $\frac{1}{\ln t}$ has a singularity $t = 1$, and for $x > 1$ the latter definition is interpreted as the Cauchy principal value

$$\operatorname{li} x = \lim_{\varepsilon \to 0+} \left(\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t} \right).$$

The connection between (1) and (2) is

$$\operatorname{Li} x = \operatorname{li} x - \operatorname{li} 2.$$

The logarithmic integral appears in some physical problems and in a formulation of the prime number theorem ($\operatorname{Li} x$ gives a slightly better approximation for the prime counting function than $\operatorname{li} x$).

One has the asymptotic series expansion

$$\operatorname{Li} x = \frac{x}{\ln x} \sum_{n=0}^{\infty} \frac{n!}{(\ln x)^n}.$$

The definition of the logarithmic integral may be extended to the whole complex plane, and one gets the analytic function $\operatorname{Li} z$ having the branch point $z = 1$ and the derivative $\frac{1}{\log z}$.