argument of product and quotient

Canonical name	ArgumentOfProductAndQuotient
Date of creation	2013-03-22 17:45:20
Last modified on	2013-03-22 17:45:20
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	8
Author	pahio (2872)
Entry type	Theorem
Classification	msc 30-00
Classification	msc 26A09
Synonym	product and quotient of complex numbers
Related topic	Argument
Related topic	PolarCoordinates
Related topic	ModulusOfComplexNumber
Related topic	Complex
Related topic	EqualityOfComplexNumbers

Using the distributive law, we perform the multiplication

$$(\cos\varphi_1+i\sin\varphi_1)(\cos\varphi_2+i\sin\varphi_2) = (\cos\varphi_1\cos\varphi_2-\sin\varphi_1\sin\varphi_2)+i(\sin\varphi_1\cos\varphi_2+\cos\varphi_1\sin\varphi_2).$$

Using the addition formulas of `http://planetmath.org/GoniometricFormulae`cosine and `http://planetmath.org/GoniometricFormulae`sine we still obtain the formula

$$(\cos\varphi_1+i\sin\varphi_1)(\cos\varphi_2+i\sin\varphi_2) = \cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2). \qquad (1)$$

The inverse number of $\cos\varphi_2+i\sin\varphi_2$ is calculated as follows:

$$\frac{1}{\cos\varphi_2+i\sin\varphi_2} = \frac{\cos\varphi_2-i\sin\varphi_2}{(\cos\varphi_2-i\sin\varphi_2)(\cos\varphi_2+i\sin\varphi_2)} = \frac{\cos\varphi_2-i\sin\varphi_2}{\cos^2\varphi_2+\sin^2\varphi_2}$$

This equals $\cos\varphi_2-i\sin\varphi_2$, and since the cosine is an `http://planetmath.org/EvenFunction`even and the sine an odd function, we have

$$\frac{1}{\cos\varphi_2+i\sin\varphi_2} = \cos(-\varphi_2)+i\sin(-\varphi_2). \qquad (2)$$

The equations (1) and (2) imply

$$\frac{\cos\varphi_1+i\sin\varphi_1}{\cos\varphi_2+i\sin\varphi_2} = (\cos\varphi_1+i\sin\varphi_1)(\cos(-\varphi_2)+i\sin(-\varphi_2)) = \cos(\varphi_1+(-\varphi_2))+i\sin(\varphi_1+(-\varphi_2)),$$

i.e.

$$\frac{\cos\varphi_1+i\sin\varphi_1}{\cos\varphi_2+i\sin\varphi_2} = \cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2). \qquad (3)$$

According to the formulae (1) and (3), for the complex numbers

$$z_1 = r_1(\cos\varphi_1+i\sin\varphi_1) \quad\text{and}\quad z_2 = r_2(\cos\varphi_2+i\sin\varphi_2)$$

we have

$$z_1z_2 = r_1r_2(\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)),$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)).$$

Thus we have the

Theorem. The modulus of the product of two complex numbers equals the product of the moduli of the factors and the argument equals the sum of the arguments of the http://planetmath.org/Productfactors. The modulus of the quotient of two complex numbers equals the quotient of the moduli of the dividend and the divisor and the argument equals the difference of the arguments of the dividend and the divisor.

Remark. The equation (1) may be by induction generalised for more than two factors of the left hand ; then the special case where all factors are equal gives de Moivre identity.

Example. Since

$$(2+i)(3+i) = 5+5i \;=\; 5e^{\frac{\pi}{4}},$$

one has

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} \;=\; \frac{\pi}{4}.$$