D'Alembertian

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The D'Alembertian is the equivalent of the Laplacian in Minkowskian geometry. It is given by:

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Here we assume a Minkowskian metric of the form $(+,+,+,-)$ as typically seen in special relativity. The connection between the Laplacian in Euclidean space and the D'Alembertian is clearer if we write both operators and their corresponding metric.

0.1 Laplacian

$$\text{Metric: } ds^2 = dx^2 + dy^2 + dz^2$$

$$\text{Operator: } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

0.2 D'Alembertian

$$\text{Metric: } ds^2 = dx^2 + dy^2 + dz^2 - cdt^2$$

$$\text{Operator: } \Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

In both cases we simply differentiate twice with respect to each coordinate in the metric. The D'Alembertian is hence a special case of the generalised Laplacian.

1 Connection with the wave equation

The wave equation is given by:

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u$$

Factorising in terms of operators, we obtain:

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})u = 0$$

or

$$\Box u = 0$$

Hence the frequent appearance of the D'Alembertian in special relativity and electromagnetic theory.

2 Alternative notation

The symbols $\Box$ and $\Box^2$ are both used for the D'Alembertian. Since it is unheard of to square the D'Alembertian, this is not as confusing as it may appear. The symbol for the Laplacian, Δ or ∇^2, is often used when it is clear that a Minkowski space is being referred to.

3 Alternative definition

It is common to define Minkowski space to have the metric $(-,+,+,+)$, in which case the D'Alembertian is simply the negative of that defined above:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$