Neumann problem

Canonical name	NeumannProblem
Date of creation	2013-03-22 15:19:59
Last modified on	2013-03-22 15:19:59
Owner	dczammit (9747)
Last modified by	dczammit (9747)
Numerical id	10
Author	dczammit (9747)
Entry type	Definition
Classification	msc 31B15
Classification	msc 31B05
Classification	msc 31A05
Related topic	HarmonicFunction

Suppose Ω is a region of $\mathbb{R}^n$ and $\partial\Omega$ is the boundary of Ω. Further suppose f is a function $f\colon \partial\Omega \to \mathbb{C}$, and suppose $\frac{\partial}{\partial n}$ corresponds to taking a derivative in a direction normal to the boundary $\partial\Omega$ at any point. Then the *Neumann problem* is to find a function $\phi\colon \Omega \cup \partial\Omega \to \mathbb{C}$ such that

$$\begin{aligned} \frac{\partial \phi}{\partial n} &= f, \quad \text{on } \partial\Omega, \\ \nabla^2 \phi &= 0, \quad \text{in } \Omega. \end{aligned}$$

Here ∇^2 represents the Laplacian operator and the second condition is that ϕ be a harmonic function on Ω. The condition for the existence of a solution ϕ of the Neumann problem is that integral of the normal derivative of the function ϕ, calculated over the entire boundary $\partial\Omega$, vanish. This follows from the identic equation

$$\int_{\partial\Omega} \frac{\partial \phi}{\partial n} d\sigma = \int_{\Omega} \nabla \cdot (\nabla \phi) d\tau = \int_{\Omega} \nabla^2 \phi \, d\tau$$

and from the fact that $\nabla^2 \phi = 0$.