subharmonic and superharmonic functions

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First let's look at the most general definition.

Definition. Let $G \subset \mathbb{R}^n$ and let $\varphi\colon G \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous function, then φ is *subharmonic* if for every $x \in G$ and $r > 0$ such that $\overline{B(x,r)} \subset G$ (the closure of the open ball of radius r around x is still in G) and every real valued continuous function h on $\overline{B(x,r)}$ that is harmonic in $B(x,r)$ and satisfies $\varphi(x) \leq h(x)$ for all $x \in \partial B(x,r)$ (boundary of $B(x,r)$) we have that $\varphi(x) \leq h(x)$ holds for all $x \in B(x,r)$.

Note that by the above, the function which is identically $-\infty$ is subharmonic, but some authors exclude this function by definition. We can define *superharmonic* functions in a similar fashion to get that φ is superharmonic if and only if $-\varphi$ is subharmonic.

If we restrict our domain to the complex plane we can get the following definition.

Definition. Let $G \subset \mathbb{C}$ be a region and let $\varphi\colon G \to \mathbb{R}$ be a continuous function. φ is said to be *subharmonic* if whenever $D(z,r) \subset G$ (where $D(z,r)$ is a closed disc around z of radius r) we have

$$\varphi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + re^{i\theta}) d\theta,$$

and φ is said to be *superharmonic* if whenever $D(z,r) \subset G$ we have

$$\varphi(z) \geq \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + re^{i\theta}) d\theta.$$

Intuitively what this means is that a subharmonic function is at any point no greater than the average of the values in a circle around that point. This implies that a non-constant subharmonic function does not achieve its maximum in a region G (it would achieve it at the boundary if it is continuous there). Similarly for a superharmonic function, but then a non-constant superharmonic function does not achieve its minumum in G. It is also easy to see that φ is subharmonic if and only if $-\varphi$ is superharmonic.

Note that when equality always holds in the above equation then φ would in fact be a harmonic function. That is, when φ is both subharmonic and superharmonic, then φ is harmonic.

It is possible to relax the continuity statement above to take φ only upper semi-continuous in the subharmonic case and lower semi-continuous in the superharmonic case. The integral will then however need to be the http://planetmath.org/Integral2Lebesgue integral rather than the Riemann integral which may not be defined for such a function. Another thing to note here is that we may take $\mathbb{R}^2$ instead of $\mathbb{C}$ since we never did use complex multiplication. In that case however we must rewrite the expression $z + re^{i\theta}$ in of the real and imaginary parts to get an expression in $\mathbb{R}^2$.

It is also possible generalize the range of the functions as well. A subharmonic function could have a range of $\mathbb{R} \cup \{-\infty\}$ and a superharmonic function could have a range of $\mathbb{R} \cup \{\infty\}$. With this generalization, if f is a holomorphic function then $\varphi(z) := \log|f(z)|$ is a subharmonic function if we define the value of $\varphi(z)$ at the zeros of f as $-\infty$. Again it is important to note that with this generalization we again must use the Lebesgue integral.

References

[1] John B. Conway. . Springer-Verlag, New York, New York, 1978.

[2] Steven G. Krantz. , AMS Chelsea Publishing, Providence, Rhode Island, 1992.