Baouendi-Treves approximation theorem

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Suppose M is a real smooth manifold. Let $\mathcal{V}$ be a subbundle of the complexified tangent space $\mathbb{C}TM$ (that is $\mathbb{C} \otimes TM$). Let $n = \dim_{\mathbb{C}} \mathcal{V}$ and $d = \dim_{\mathbb{R}} M$. We will say that $\mathcal{V}$ is *integrable*, if it is integrable in the following sense. Suppose that for any point $p \in M$, there exist $m = d - n$ smooth complex valued functions $z_1, \ldots, z_m$ defined in a neighbourhood of p, such that the differentials $dz_1, \ldots, dz_m$ are $\mathbb{C}$-linearly independent and for all sections $L \in \Gamma(M, \mathcal{V})$ we have $Lz_k = 0$ for $k = 1, \ldots, m$. We say $z = (z_1, \ldots, z_m)$ are near p.

We say f is a if $Lf = 0$ for every $L \in \Gamma(M, \mathcal{V})$ in the sense of distributions (or classically if f is in fact smooth).

Theorem (Baouendi-Treves)**.** *Suppose M is a smooth manifold of real dimension d and $\mathcal{V}$ an integrable subbundle as above. Let $p \in M$ be fixed and let $z = (z_1, \ldots, z_m)$ be basic solutions near p. Then there exists a compact neighbourhood K of p, such that for any continuous solution $f \colon M \to \mathbb{C}$, there exists a sequence p_j of polynomials in m variables with complex coefficients such that*

$$p_j(z_1, \ldots, z_m) \to f \qquad \textit{uniformly in } K.$$

In particular we have the following corollary for CR submanifolds. A real smooth CR submanifold that is embedded in $\mathbb{C}^N$ has the CR vector fields as the integrable subbundle $\mathcal{V}$. Also the coordinate functions $z_1, \ldots, z_N$ can be taken as the basic solutions. We will require that M be a generic submanifold rather than just any CR submanifold to make sure that $\mathbb{C}^N$ is of the minimal dimension.

Corollary. *Let $M \subset \mathbb{C}^N$ be an embedded real smooth generic submanifold and $p \in M$. Then there exists a compact neighbourhood $K \subset M$ of p such that any continuous CR function f is uniformly approximated on K by polynomials in N variables.*

This result can be used to extend CR functions from CR submanifolds. For example, if we can fill a certain set with analytic discs attached to M, we can approximate f on $K \subset M$ and by the maximum principle we will be able to use the fact that uniform limits of holomorphic functions (in this case polynomials) are holomorphic. A key point is that while K is not arbitrary, it does not depend on f, it only depends on M and p.

Example. *Suppose $M \subset \mathbb{C}^2$ is given in coordinates (z, w) by $\operatorname{Im} w = |z|^2$. Note that for some $t > 0$, the map $\xi \mapsto (t\xi, t)$ is an attached analytic disc.*

By taking different $t > 0$, we can fill the set $\{(z, w) \mid \operatorname{Im} w \geq |z|^2\}$ by analytic discs attached to M. If f is a continuous CR function on M, then there exists some compact neighbourhood K of $(0, 0)$ such that f is uniformly approximated on K by holomorphic polynomials. By maximum principle we get that this sequence of holomorphic polynomials converges uniformly on all the discs for $t < \epsilon$ for some $\epsilon > 0$ (such that the boundary of the disc lies in K). Hence f extends to a holomorphic function on $\epsilon > \operatorname{Im} w > |z|^2$, and which is continuous on $\epsilon > \operatorname{Im} w \geq |z|^2$.

Using methods of the example it is possible (among many other results) to prove the following.

Corollary. *Suppose $M \subset \mathbb{C}^N$ be a smooth strongly pseudoconvex hypersurface and f a continuous CR function on M. Then f extends to a small neighbourhood on the pseudoconvex side of M as a holomorphic function.*

Using the above corollary we can prove the Hartogs phenomenon for hypersurfaces by reducing to the standard Hartogs phenomenon (although the theorem also holds without pseudoconvexity with a different proof).

Corollary. *Let $U \subset \mathbb{C}^N$ be a domain with smooth strongly pseudoconvex boundary. Suppose f is a continuous CR function on ∂U. Then there exists a function f holomorphic in U and continuous on $\bar{U}$, such that $F|_{\partial U} = f$.*

References

[1] M. Salah Baouendi, Peter Ebenfelt, Linda Preiss Rothschild. , Princeton University Press, Princeton, New Jersey, 1999.

[2] Albert Boggess. , CRC, 1991.