space of analytic functions

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For what follows suppose that $G \subset \mathbb{C}$ is a region. We wish to take the set of all holomorphic functions on G, denoted by $\mathcal{O}(G)$, and make it into a metric space. We will define a metric such that convergence in this metric is the same as uniform convergence on compact subsets of G. We will call this the *space of analytic functions* on G.

It is known that there exists a sequence of compact subsets $K_n \subset G$ such that $K_n \subset K_{n+1}^\circ$ (interior of K_{n+1}), such that $\bigcup K_n^\circ = G$ and such that if K is any compact subset of G, then $K \subset K_n$ for some n. Now define the quantity $\rho_n(f,g)$ for $f, g \in \mathcal{O}(G)$ as

$$\rho_n(f,g) := \sup_{z \in K_n} \{|f(z) - g(z)|\}.$$

We define the metric on $\mathcal{O}(G)$ as

$$d(f,g) := \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1 + \rho_n(f,g)}.$$

This can be shown to be a metric. Furthermore, it can be shown that the topology generated by this metric is independent of the choice of K_n, even though the actual values of the metric do depend on the particular K_n we have chosen. Finally, it can be shown that convergence in d is the same as uniform convergence on compact subsets. It is known that if you have a sequence of analytic functions on G that converge uniformly on compact subsets, then the limit is in fact analytic in G, and thus $\mathcal{O}(G)$ is a complete space.

Similarly, we can treat the functions that are meromorphic on G, and define $M(G)$ to be the *space of meromorphic functions* on G. We assume that the functions take the value ∞ at their poles, so that they are defined at every point of G. That is, they take their values in the Riemann sphere, or the extended complex plane. We just need to replace the definition of $\rho_n(f,g)$ with

$$\rho_n(f,g) := \sup_{z \in K_n} \{\sigma(f(z), g(z))\},$$

where σ is either the spherical metric on the Riemann sphere, or alternatively the metric induced by embedding the Riemann sphere in $\mathbb{R}^3$. Both of those metrics produce the same topology, and that is all that we care about. The rest of the definition is the same as that of $\mathcal{O}(G)$. There is, however, one small glitch here. $M(G)$ is not a complete metric space. It is possible that functions in $M(G)$ go off to infinity pointwise, but this is the worst that can happen. For example, the sequence $f_n(z) = n$ is a sequence of meromorphic functions on G, and this sequence is Cauchy in $M(G)$, but the limit would be $f(z) = \infty$ and that is not a function in $M(G)$.

Remark. Note that $\mathcal{O}(G)$ is sometimes denoted by $H(G)$ in literature. Also note that $A(G)$ is usually reserved for functions which are analytic on G and continuous on $\bar{G}$ (closure of G).

Remark. We can similarly define the space of continuous functions, and treat $\mathcal{O}(G)$ and $M(G)$ as subspaces of that. That is, $\mathcal{O}(G)$ would be a subspace of $C(G, \mathbb{C})$ and $M(G)$ would be a subspace of $C(G, \hat{\mathbb{C}})$.

References

[1] John B. Conway. . Springer-Verlag, New York, New York, 1978.