multifunction

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Defines	multiple valued function

It is common practice among complex analysts to speak of *multiple valued functions* in contexts of "functions" such as $\sqrt{z}$. This somewhat informal notion can be made very precise when the "function" has finitely many values (as the $\sqrt{z}$ does).

Let X and Y be sets and denote by Y^m_{sym} the m^{th} symmetric power of Y.

Definition. A function $f\colon X \to Y^m_{sym}$ is called a *multifunction*, or an m-function from X to Y, where m is the multiplicity.

We can think of the value of f at any point as a set of m (or fewer) elements. Let Y be a topological space (resp. $\mathbb{C}$) A multifunction is said to be continuous (resp. holomorphic) if all the elementary symmetric polynomials of the elements of f are continuous (resp. holomorphic). Equivalently, f is continuous (resp. holomorphic) if it is continuous (resp. holomorphic) as functions to $Y^m_{sym} \cong Y^m$ (resp. $\mathbb{C}^m_{sym} \cong \mathbb{C}^m$).

With this definition $\sqrt{z}$ is a holomorphic multifunction (or a 2-function), into $\mathbb{C}^2_{sym}$.

Define the *multigraph* of f to be the set:

$$\{(x, y) \mid X \times Y \mid y \in f(x)\}.$$

The multigraph of $\sqrt{z}$ is the corresponding Riemann surface imbedded in $\mathbb{C}^2$. In general, with the aid of the Weierstrass preparation theorem we can realize any codimension 1 analytic set in $\mathbb{C}^n$ as a multigraph over $\mathbb{C}^{n-1}$. The roots of any Weierstrass polynomial (or in general of any monic polynomial with holomorphic coefficients) are a holomorphic multifunction.

References

[1] Hassler Whitney. . Addison-Wesley, Philippines, 1972.