Levi pseudoconvex

Canonical name	LeviPseudoconvex
Date of creation	2013-03-22 14:30:37
Last modified on	2013-03-22 14:30:37
Owner	jirka (4157)
Last modified by	jirka (4157)
Numerical id	10
Author	jirka (4157)
Entry type	Definition
Classification	msc 32T15
Classification	msc 32T05
Related topic	DomainOfHolomorphy
Related topic	Pseudoconvex
Related topic	BiholomorphismsOfStronglyPseudoconvexDomainsExtendToTheBoundary
Defines	Levi form
Defines	strongly Levi pseudoconvex
Defines	strongly pseudoconvex
Defines	strictly pseudoconvex
Defines	weakly pseudoconvex
Defines	weakly Levi pseudoconvex
Defines	holomorphic tangent vector

Let $G \subset \mathbb{C}^n$ be a http://planetmath.org/Domain2domain (open connected subset) with C^2 boundary, that is the boundary is locally the graph of a twice continuously differentiable function. Let $\rho \colon \mathbb{C}^n \to \mathbb{R}$ be a defining function of G, that is ρ is a twice continuously differentiable function such that $\operatorname{grad} \rho(z) \neq 0$ for $z \in \partial G$ and $G = \{z \in \mathbb{C}^n \mid \rho(z) < 0\}$ (such a function always exists).

Definition. Let $p \in \partial G$ (boundary of G). We call the space of vectors $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ such that

$$\sum_{k=1}^{n} \frac{\partial \rho}{\partial z_k}(p) w_k = 0,$$

the space of *holomorphic tangent vectors* at p and denote it $T_p^{1,0}(\partial G)$.

$T_p^{1,0}(\partial G)$ is an $n-1$ dimensional complex vector space and is a subspace of the complexified http://planetmath.org/TangentSpacereal tangent space, that is $\mathbb{C} \otimes_{\mathbb{R}} T_p(\partial G)$.

Note that when $n = 1$ then the complex tangent space contains just the zero vector.

Definition. The point $p \in \partial G$ is called *Levi pseudoconvex* (or just *pseudoconvex*) if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq 0,$$

for all $w \in T_p^{1,0}(\partial G)$. The point is called *strongly Levi pseudoconvex* (or just *strongly pseudoconvex* or also *strictly pseudoconvex*) if the inequality above is strict. The expression on the left is called the *Levi form*.

Note that if a point is not strongly Levi pseudoconvex then it is sometimes called a *weakly Levi pseudoconvex* point.

The Levi form really acts on an $n-1$ dimensional space, so the expression above may be confusing as it only acts on $T_p^{1,0}(\partial G)$ and not on all of $\mathbb{C}^n$.

Definition. The domain G is called Levi pseudoconvex if every boundary point is Levi pseudoconvex. Similarly G is called strongly Levi pseudoconvex if every boundary point is strongly Levi pseudoconvex.

Note that in particular all convex domains are pseudoconvex.

It turns out that G with C^2 boundary is a domain of holomorphy if and only if G is Levi pseudoconvex.

References

[1] M. Salah Baouendi, Peter Ebenfelt, Linda Preiss Rothschild. , Princeton University Press, Princeton, New Jersey, 1999.

[2] Steven G. Krantz. , AMS Chelsea Publishing, Providence, Rhode Island, 1992.