Levi flat

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Suppose $M \subset \mathbb{C}^n$ is at least a C^2 hypersurface.

Definition. M is *Levi-flat* if it is pseudoconvex from both sides, or equivalently if and only if the Levi form of M vanishes identically.

Suppose M is locally defined by $\rho = 0$. The vanishing of the Levi form is equivalent to the complex Hessian of ρ vanishing on all holomorphic vectors tangent to the hypersurface. Hence M is Levi-flat if and only if the complex bordered Hessian of ρ is of rank two on the hypersurface. In other words, it is not hard to see that M is Levi-flat if and only if

$$\operatorname{rank} \begin{bmatrix} \rho & \rho_z \\ \rho_{\bar{z}} & \rho_{z\bar{z}} \end{bmatrix} = 2 \qquad \text{for all points on } \{\rho = 0\}.$$

Here ρ_z is the row vector $\left[\frac{\partial \rho}{\partial z_1}, \ldots, \frac{\partial \rho}{\partial z_n}\right]$, $\rho_{\bar{z}}$ is the column vector $\left[\frac{\partial \rho}{\partial z_1}, \ldots, \frac{\partial \rho}{\partial z_n}\right]^T$, and $\rho_{z\bar{z}}$ is the complex Hessian $\left[\frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}\right]_{ij}$.

Let T^cM be the complex tangent space of M, that is at each point $p \in M$, define $T^c_pM = J(T_pM) \cap T_pM$, where J is the complex structure. Since M is a hypersurface the dimension of T^c_pM is always $2n-2$, and so T^cM is a subbundle of TM. M is Levi-flat if and only if T^cM is involutive. Since the leaves are graphs of functions that satisfy the Cauchy-Riemann equations, the leaves are complex analytic. Hence, M is Levi-flat, if and only if it is foliated by complex hypersurfaces.

The cannonical example of a Levi-flat hypersurface is the hypersurface defined in $\mathbb{C}^n$ by the equation $\operatorname{Im} z_1 = 0$. In fact, locally, all real analytic Levi-flat hypersurfaces are biholomorphic to this example.

References

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[2] Steven G. Krantz. , AMS Chelsea Publishing, Providence, Rhode Island, 1992.