hypersurface

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Defines	real analytic hypersurface
Defines	real hypersurface
Defines	local defining function
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Defines	hypervariety

Definition. Let M be a subset of $\mathbb{R}^n$ such that for every point $p \in M$ there exists a neighbourhood U_p of p in $\mathbb{R}^n$ and a continuously differentiable function $\rho \colon U \to \mathbb{R}$ with $\operatorname{grad} \rho \neq 0$ on U, such that

$$M \cap U = \{x \in U \mid \rho(x) = 0\}.$$

Then M is called a *hypersurface*.

If ρ is in fact smooth then M is a *smooth hypersurface* and similarly if ρ is real analytic then M is a *real analytic hypersurface*. If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and we have a hypersurface there it is called a *real hypersurface* in $\mathbb{C}^n$. ρ is usually called the *local defining function*. Hypersurface is really special name for a submanifold of codimension 1. In fact if M is just a topological manifold of codimension 1, then it is often also called a hypersurface.

A `http://planetmath.org/RealAnalyticSubvariety`real or complex analytic subvariety of codimension 1 (the zero set of a real or complex analytic function) is called a *singular hypersurface*. That is the definition is the same as above, but we do not require $\operatorname{grad} \rho \neq 0$. Note that some authors leave out the word *singular* and then use *non-singular hypersurface* for a hypersurface which is also a manifold. Some authors use the word *hypervariety* to describe a singular hypersurface.

An example of a hypersurface is the hypersphere (of radius 1 for simplicity) which has the defining equation

$$x_1^2 + x_2^2 + \ldots + x_n^2 = 1.$$

Another example of a hypersurface would be the boundary of a domain in $\mathbb{C}^n$ with smooth boundary.

An example of a singular hypersurface in $\mathbb{R}^2$ is for example the zero set of $\rho(x_1, x_2) = x_1 x_2$ which is really just the two axis. Note that this hypersurface fails to be a manifold at the origin.

References

[1] M. Salah Baouendi, Peter Ebenfelt, Linda Preiss Rothschild. , Princeton University Press, Princeton, New Jersey, 1999.