proof of growth of exponential function

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Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
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Author	rspuzio (6075)
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In this proof, we first restrict to when x and a are integers and only later lift this restricton.

Let $a > 0$ be an integer, let $b > 1$ be real, and let x be an integer.

Consider the following inequality

$$\left(1 + \frac{1}{x}\right)^a \leq 1 + \frac{a}{x}\left(1 + \frac{1}{x}\right)^{a-1}$$

If $x \geq 2$, then we have

$$\left(1 + \frac{1}{x}\right)^a \leq 1 + \frac{a}{x}\left(\frac{3}{2}\right)^{a-1}.$$

Define X to be the greater of 2 and $\lceil a(3/2)^{a-1}/(1 - \sqrt{b})\rceil$; when $x > X$, we have

$$\left(1 + \frac{1}{x}\right)^a \leq \sqrt{b}.$$

Rewrite x^a/b^x as follows when $x > X$:

$$\frac{x^a}{b^x} = \frac{X^a}{b^X} \prod_{n=X}^{x} \left(1 + \frac{1}{n}\right)^a \frac{1}{b}$$

By the inequality established above, each term in the product will be bounded by $1/\sqrt{b}$, hence

$$\frac{x^a}{b^x} \leq \frac{X^a}{b^X} \frac{1}{(\sqrt{b})^{x-X}}$$

Since $b > 1$, it is also the case that $\sqrt{b} > 1$, hence we have the inequality

$$(\sqrt{b})^n \geq 1 + n(\sqrt{b} - 1)$$

Combining the last two inequalities yields the following:

$$\frac{x^a}{b^x} \leq \frac{X^a}{b^X} \leq \frac{1}{1 + (x - X)(\sqrt{b} - 1)}$$

From this, it follows that $\lim_{x\to\infty} x^a/b^x = 0$ when a and x are integers.

Now we lift the restriction that a be an integer. Since the power function is increasing, $x^a/b^x \leq x^{\lceil a\rceil}/b^x$, so we have $\lim_{x\to\infty} x^a/b^x = 0$ for real values of a as well.

To lift the restriction on x, let us write $x = x_1 + x_2$ where x_1 is an integer and $0 \leq x_2 < 1$. Then we have

$$\frac{x^a}{b^x} = \frac{x_1^a}{b^{x_1}} \left(\frac{x_1 + x_2}{x_1} \right)^a b^{-x_2}$$

If $x > 2$, then $(x_1 + x_2)/x_2 < 1.5$. Since $x_2 \geq 0, b^{-x_2} \leq 1$. Hence, for all real $x > 2$, we have

$$\frac{x^a}{b^x} \leq 1.5^a \frac{x_1^a}{b^{x_1}}$$

From this inequality, it follows that $\lim_{x \to \infty} x^a / b^x = 0$ for real values of x as well.