Lewy extension theorem

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Let $M \subset \mathbf{C}^n$ be a smooth real hypersurface. Let ρ be a defining function for M near p. That is, for some neighbourhood of p, the submanifold M is defined by $\rho = 0$. For a neighbourhood $U \subset \mathbb{C}^n$, define the set U_+ to be the set $U \cap \{\rho > 0\}$. We will say that M has at least one negative eigenvalue if the Levi form defined by ρ has at least one negative eigenvalue. That is, if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho(p)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k < 0 \quad \text{for some} \quad w \in \mathbb{C}^n \quad \text{such that} \quad \sum_{j=1}^{n} w_j \frac{\partial \rho(p)}{\partial z_j} = 0.$$

Theorem. *Let f be a smooth CR function on M. Suppose that near $p \in M$ the Levi form of M has at least one positive eigenvalue at p. Then there exists a neighbourhood U of p, such that for every smooth CR function f on M, there exists a function F holomorphic in U_+ and C^1 up to M, such that $F|_{U \cap M} = f|_{U \cap M}$.*

By considering $-\rho$ instead of ρ as a defining function, we get the corresponding result for at least one negative eigenvalue. If the Levi form of M has both positive and negative eigenvalues at a point, then f extends to both sides of M and is then a restriction of a holomorphic function.

A point is the fact that U is fixed and does not depend on f. To see why this is necessary, imagine a Levi flat example. Let M be defined in $\mathbb{C}^2$ in coordinates (z, w) by $\operatorname{Im} w = 0$. The domains $U_\epsilon := \{|\operatorname{Im} w| < \epsilon\}$, for $\epsilon > 0$, are pseudoconvex and hence there exist functions holomorphic on Ω_ϵ (and hence CR on M) that do not extend past any point of the boundary. No neighbourhood of a point on M fits in all U_ϵ. So at least one nonzero eigenvalue of the Levi form is needed.

The statement of this theorem is not exactly the theorem that Lewy formulated[**?**], but this is generally called the Lewy extension. There have been many results in this direction since Lewy's original paper, but this is the most result.

References

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[4] Hans Lewy. *Ann. of Math.* (2) **64** (1956), 514–522.