periodicity of exponential function

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Theorem. The only periods of the complex exponential function $z \mapsto e^z$ are the multiples of $2\pi i$. Thus the function is *one-periodic.*

Proof. Let ω be any period of the exponential function, i.e. $e^{z+\omega} = e^z e^\omega = e^z$ for all $z \in \mathbb{C}$. Because e^z is always $\neq 0$, we have

$$e^\omega \;=\; 1. \tag{1}$$

If we set $\omega =: a+ib$ with a and b reals, (1) gets the form

$$e^a \cos b + ie^a \sin b \;=\; 1, \tag{2}$$

which implies (see equality of complex numbers)

$$e^a \cos b \;=\; 1, \qquad e^a \sin b \;=\; 0.$$

As these equations are squared and added, we obtain $e^{2a} = 1$ which , since a is real, that $a = 0$. Thus the preceding equations get the form

$$\cos b \;=\; 1, \qquad \sin b \;=\; 0.$$

These result that $b = n\cdot 2\pi$ and therefore

$$\omega \;=\; n\cdot 2\pi i \qquad (n \;=\; 0,\, \pm 1,\, \pm 2,\, \pm 3,\, \ldots)$$

Q.E.D.

References

[1] Ernst Lindelöf: *Johdatus funktioteoriaan* (‘Introduction to function theory’). Mercatorin kirjapaino, Helsinki (1936).