derivative of exponential function

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In this entry, we shall compute the derivative of the exponential function from its definition as a limit of powers.

Theorem 1. *If* $0 \leq x < 1$, *then*

$$1 + x \leq \exp x \leq \frac{1}{1-x}$$

Proof. By the inequalities for differences of powers, we have

$$x \leq \left(1 + \frac{x}{n}\right)^n - 1 \leq \frac{x}{1 - \left(\frac{n-1}{n}\right) x}.$$

Since $n - 1 < n$, and $x > 0$, we have $0 < (n - 1/n)x < x$. Because $x < 1$, this implies $1 - (n - 1/n)x > 1 - x$, so

$$\frac{x}{1 - \left(\frac{n-1}{n}\right) x} < \frac{x}{1-x}.$$

Hence

$$1 + x \leq \left(1 + \frac{x}{n}\right)^n \leq \frac{1}{1-x}.$$

Taking the limit as $n \to \infty$, we obtain our result. □

Theorem 2.

$$\lim_{x \to 0} \frac{\exp(x) - 1}{x} = 1$$

Proof. Assume $0 < x < 1$. By our bound, we have

$$1 \leq \frac{\exp(x) - 1}{x} \leq \frac{1}{1-x}.$$

Suppose that $-1 < x < 0$. Then, since $\exp(x) = 1/\exp(-x)$, we have

$$\frac{\exp(x) - 1}{x} = \frac{1}{\exp(-x)} \cdot \frac{1 - \exp(-x)}{x}.$$

From the inequality above, we have

$$1 \leq \frac{1 - \exp(-x)}{x} \leq \frac{1}{1+x}.$$

Hence

$$\frac{1}{\exp(-x)} \le \frac{\exp(x)-1}{x} \le \frac{1}{(1+x)\exp(-x)}.$$

By theorem 1, we have $1-x \le \exp(-x) \le 1/(1+x)$, so

$$1+x \le \frac{1-\exp(-x)}{x} \le \frac{1}{(1+x)(1-x)} = \frac{1}{1-x^2}.$$

By the squeeze rule, we conclude that

$$\lim_{x\to 0} \frac{1-\exp(-x)}{x} = 1$$

whether we approach the limit from the left or the right. □

Theorem 3.

$$\frac{d}{dx}\exp(x) = \exp(x)$$

Proof. By definition,

$$\frac{d}{dx}\exp(x) = \lim_{y\to x} \frac{\exp(y)-\exp(x)}{y-x}.$$

By the addition theorem for the exponential, we have

$$\frac{\exp(y)-\exp(x)}{y-x} = \exp(x)\cdot\frac{\exp(y-x)-1}{y-x},$$

so

$$\lim_{y\to x}\frac{\exp(y)-\exp(x)}{y-x} = \exp(x)\lim_{y\to x}\frac{\exp(y-x)-1}{y-x} = \exp(x)\lim_{y\to 0}\frac{\exp y - 1}{y}.$$

By theorem 2, the limit on the right-hand side equals 1, so we have

$$\lim_{y\to x}\frac{\exp(y)-\exp(x)}{y-x} = \exp(x).$$

□