exponential function never vanishes

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In the entry http://planetmath.org/ExponentialFunctionexponential function one defines for real variable x the real exponential function $\exp x$, i.e. e^x, as the sum of power series:

$$e^x \;=\; \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

The series form implies immediately that the real exponential function attains only positive values when $x \geqq 0$. Also for $-1 \leqq x < 0$ the positiveness is easy to see by grouping the series terms pairwise.

In to study the sign of e^x for arbitrary real x, we may multiply the series of e^x and e^{-x} using http://planetmath.org/AbelsMultiplicationRuleForSeriesAbel's multiplication rule for series. We obtain

$$e^xe^{-x} \;=\; \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!} \;=\; \sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\frac{x^n}{j!(n-j)!} \;=\; \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{j=0}^{n}\binom{n}{j}(-1)^j \;=\; \sum_{n=0}^{\infty}\frac{x^n}{n!}\cdot 0^n.$$

The last sum equals 1. So, if $-x > 0$, then $e^{-x} > 0$, whence e^x must be positive.

Let us now consider arbitrary complex value $z = x+iy$ where x and y are real. Using the addition formula of complex exponential function and the Euler relation, we can write

$$e^z \;=\; e^{x+iy} \;=\; e^xe^{iy} \;=\; e^x(\cos y + i\sin y).$$

From this we see that the absolute value of e^z is e^x, which we above have proved to be positive. Accordingly, we may write the

Theorem. The complex exponential function never vanishes.