Nevanlinna theory

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Nevanlinna theory deals with quantitative aspects of entire holomorphic mappings into complex varieties. Let $\omega = i\sum_{j,k}\omega_{jk}\,dz_j\wedge d\bar{z}_k$ be a smooth $(1,1)$-form on a complex manifold X. Suppose that ω is positive definite, i.e. the Hermitian matrix $(\omega_{jk}(z))$ is positive definite at every point. Thus ω can be viewed as a Hermitian metric on the tangent bundle T_X.

If $f\colon \mathbb{C}\to X$ is an entire curve, the *growth indicatrix* of f is the function

$$T_{f,\omega}(r) = \int_{r_0}^{r} t_{f,\omega}(\rho)\,\frac{d\varrho}{\varrho},$$

where

$$t_{f,\omega}(\varrho) = \int_{D(0,\varrho)} f^*\omega.$$

It is clear that $t_{f,\omega}(\varrho)$ is nothing more than the area with respect to ω of the image $f(D(0,\varrho))$ of the disk in the complex line centered at 0 and of ray ϱ. Now let L be an ample holomorphic line bundle over X (compact, connected). Then L carries an Hermitian metric h with positive curvature $\omega = \Theta_h(L)$ and so, in this case, one can suppose that our ω is in fact the curvature of this line bundle. Furthermore, it is clear that if one is merely interested in the order of growth $O(T_{f,\omega}(r))$ when r goes to infinity, then this order is independent of the choice of ω.

Let us consider an hypersurface $H = \{\sigma = 0\}\subset X$ defined by a global section of L: one would like to "measure"the intersections of the entire curve $f\colon \mathbb{C}\to X$ with H. For this purpose one looks at the holomorphic function $\sigma\circ f\colon \mathbb{C}\to L$ and introduces the *enumerating of zeros function*

$$N_{f,\sigma}(r) = \int_{r_0}^{r} n_{f,\sigma}(\varrho)\,\frac{d\varrho}{\varrho},$$

where $n_{f,\sigma}(\varrho)$ = number of zeros of $\sigma\circ f$ on $D(0,\varrho)$ counted with multiplicities. Finally one introduces the function $m_{f,\sigma}$, called *proximity function*, defined by

$$m_{f,\sigma}(r) = \frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{\|(\sigma\circ f)(re^{i\vartheta})\|_h}\,d\vartheta.$$

This function is non negative, once one has normalized σ with a constant in such a way that $\|\sigma\|_h\le 1$. Morally, $m_{f,\sigma}(r)$ is bigger and bigger when f often goes near $H = \{\sigma = 0\}$ on the circle of ray r.

The *first fundamental theorem of Nevanlinna* states the following:

Let (L, h) be an Hermitian line bundle with curvature form $\omega = \Theta_h(L) > 0$. For all section σ of L and all curve $f\colon \mathbb{C} \to X$ such that $f(\mathbb{C})$ is not entirely contained in the hypersurface $H = \{\sigma = 0\}$, one has

$$m_{f,\sigma}(r) + N_{f,\sigma}(r) = T_{f,\omega}(r) + O(1).$$

In particular, the order of growth of the left hand side when $r \to \infty$ does not depend on the choice of σ, but only on the growth indicatrix of f.

Classically, one introduces the *defect* of f with respect to $H = \{\sigma = 0\}$, defined by

$$\delta_\sigma(f) = \liminf_{r\to\infty} \frac{m_{f,\sigma}(r)}{T_{f,\omega}(r)} \in [0, 1].$$

In particular, the defect is equal to 1 if $\sigma \circ f$ is never zero, and equal to 0 if the enumerating of zeros function $N_{f,\sigma}(r)$ grows as much as possible. One of the most important results of Nevanlinna theory concerns the entire curves which map into the Riemann sphere $\mathbb{P}^1$ and states that the sum of defects $\sum_{a\in\mathbb{P}^1} \delta_a(f)$ is at most equal to 2. One of the essential steps for the proof of this statement is an estimate of the proximity function of the logarithmic derivative of a meromorphic function.

More precisely the following "logarithmic derivative lemma" holds:

Let $f\colon \mathbb{C} \to \mathbb{P}^1$ be a meromorphic function and $D^p \log f$ the p-th logarithmic derivative of f. Then, for all $\varepsilon > 0$, there exists a set E of finite Lebesgue measure in $\mathbb{R}_+$ such that

$$m_{D^p \log f,\infty}(r) \le (p + 1 + \varepsilon)(\log r + \log_+ T_{f,\omega}(r)) + O(1), \quad \forall r \in \mathbb{R}_+ \setminus E.$$

An important consequence of the logarithmic derivative lemma is what is called the *second fundamental theorem of Nevanlinna* from which it follows immediately the estimate for the sum of the defects introduced above:

Let $f\colon \mathbb{C} \to \mathbb{P}^1$ be a meromorphic function. Define the ramification divisor R_f of f as the sum $\sum e_j[w_j]$ where the w_j's are the points where f' is zero and the e_j's are the multiplicities of zero of f' at w_j (where $f(w_j) = \infty$ one looks at $1/f$ instead of f). Then, for all finite set $\{a_j\} \subset \mathbb{P}^1$, there exists a subset $E \subset \mathbb{R}_+$ of finite Lebesgue measure such that

$$N_{R_f}(r) + \sum_j m_{f,a_j}(r) \le 2T_{f,\omega}(r) + O(\log r + \log_+ T_{f,\omega}(r)), \quad \forall r \in \mathbb{R}_+ \setminus E,$$

where $N_{R_f}(r) = \int_{r_0}^{r} n_{R_f}(\varrho)\, d\varrho/\varrho$ is the enumerating function of the ramification divisor.

References

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