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subanalytic set

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Related topic	SemialgebraicSet
Defines	subanalytic
Defines	semianalytic set
Defines	semianalytic
Defines	semianalytic function
Defines	subanalytic function
Defines	semianalytic mapping
Defines	subanalytic mapping
Defines	dimension of a subanalytic set

Let $U \subset \mathbb{R}^n$. Suppose $\mathcal{A}(U)$ is any ring of real valued functions on U. Define $\mathcal{S}(\mathcal{A}(U))$ to be the smallest set of subsets of U, which contain the sets $\{x \in U \mid f(x) > 0\}$ for all $f \in \mathcal{A}(U)$, and is closed under finite union, finite intersection and complement.

Definition. A set $V \subset \mathbb{R}^n$ is *semianalytic* if and only if for each $x \in \mathbb{R}^n$, there exists a neighbourhood U of x, such that $V \cap U \in \mathcal{S}(\mathcal{O}(U))$, where $\mathcal{O}(U)$ denotes the real-analytic real valued functions.

Unlike for semialgebraic sets, there is no Tarski-Seidenberg theorem for semianalytic sets, and projections of semianalytic sets are in general not semianalytic.

Definition. We say $V \subset \mathbb{R}^n$ is a *subanalytic* set if for each $x \in \mathbb{R}^n$, there exists a relatively compact semianalytic set $X \subset \mathbb{R}^{n+m}$ and a neighbourhood U of x, such that $V \cap U$ is the projection of X onto the first n coordinates.

In particular all semianalytic sets are subanalytic. On an open dense set subanalytic sets are submanifolds and hence we can define dimension. Hence at a point p, where a set A is a submanifold, the dimension $\dim_p A$ is the dimension of the submanifold. The *dimension* of the subanalytic set is the maximum $\dim_p A$ for all p where A is a submanifold. Semianalytic sets are contained in a real-analytic subvariety of the same dimension. However, subanalytic sets are not in general contained in any subvariety of the same dimension. We do have however the following.

Theorem. *A subanalytic set A can be written as a locally finite union of submanifolds.*

The set of subanalytic sets is still not completely closed under projections however. Note that a real-analytic subvariety that is not relatively compact can have a projection which is not a locally finite union of submanifolds, and hence is not subanalytic.

Definition. Let $U \subset \mathbb{R}^n$. A mapping $f\colon U \to \mathbb{R}^m$ is said to be subanalytic (resp. semianalytic) if the graph of f (i.e. the set $\{(x, y) \in U \times \mathbb{R}^m \mid x, y = f(x)\}$) is subanalytic (resp. semianalytic)

References

[1] Edward Bierstone and Pierre D. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 5–42. http://www.ams.org/mathscinet-getitem?mr=89k:32011MR 89k:32011