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proof of Hartogs' theorem

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Lemma. *Suppose that g is a smooth differential $(0,1)$-form with compact support in $\mathbb{C}^n$. Then there exists a smooth function ψ such that*

$$\bar{\partial}\psi = g, \tag{1}$$

and ψ has compact support if $n \geq 2$.

Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ be our coordinates. We note that $(0,1)$-form means a differential form given by

$$\sum_{k=1}^{n} g_k(z) d\bar{z}_k.$$

The operator $\bar{\partial}$ is the so-called d-bar operator and we are looking for a smooth function ϕ solving the equation inhomogeneous $\bar{\partial}$ equation. It is important that g has compact support, otherwise solutions to (**??**) are much harder to obtain.

Proof. Written out in detail we can think of g as n different functions $g_1, \ldots, g_n$, where are date g_k satisfy the compatibility condition

$$\frac{\partial g_k}{\partial \bar{z}_l} = \frac{\partial g_l}{\partial \bar{z}_k} \text{ for all } k,\ l.$$

Then we write equation (**??**) as

$$\frac{\partial \psi}{\partial \bar{z}_k} = g_k \text{ for all } k.$$

We assume also that g_k have compact support.

This system of equations has a solution (many equations in fact). We can obtain an explicit solution as follows.

$$\psi(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}.$$

ψ is smooth by differentiating under the integral. When $n \geq 2$, this solution will also have compact support since g_1 has compact support and as z tends to infinity $(\zeta, z_2, \ldots, z_n)$ also tends to infinity no matter what ζ is. The reader should notice that there is one direction which does not work. But if ψ has bounded support except for the line defined by $z_2 = z_3 = \cdots = z_n = 0$, then the support must be compact by continuity of ψ. It should also be clear why ψ does not have compact support if $n = 1$.

One might be wondering why we picked z_1 and g_1 in the construction of ψ. It does not matter, we will get different solutions we we use z_k and g_k, but it will still have compact support. Further one might wonder why we only use one part of the data, and still get an actual solution. The answer here is that the compatibility condition relates all the data, so we only need to look at one.

We still must check that this really is a solution. We apply the compatibility condition. Let $k \geq 2$.

$$\begin{aligned}\frac{\partial \psi}{\partial \bar{z}_k}(z_1, z_2, \ldots, z_n) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_1}{\partial \bar{z}_k}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_k}{\partial \bar{z}_1}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}.\end{aligned}$$

Note that the integral can be taken over a large ball B that contains the support of g_k. We apply the generalized Cauchy formula, where the boundary part of the integral is obviously zero since it is over a set where g_k is zero.

$$\begin{aligned}\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_k}{\partial \bar{z}_1}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} &= \frac{1}{2\pi i} \int_{B} \frac{\frac{\partial g_k}{\partial \bar{z}_1}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} \\ &= g_k(z_1, \ldots, z_n).\end{aligned}$$

Hence $\frac{\partial \psi}{\partial \bar{z}_k} = g_k$.

When $k = 1$, change coordinates to see that

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta}.$$

Next differentiate in $\bar{z}_k$ and change coordinates back and apply the generalized Cauchy formula as before to get that $\frac{\partial \psi}{\partial \bar{z}_1} = g_1$. □

Proof of Theorem. Let $U \subset \mathbb{C}^n$, K a compact subset of U and f be a holomorphic function defined on $U \setminus K$ and $U \setminus K$ to be connected. By the smooth version of Urysohn's lemma we can find a smooth function φ which is 1 in a neighbourhood of K and is compactly supported in U. Let $f_0 := (1 - \varphi)f$, which is identically zero on K and holomorphic near the boundary of U (since there φ is 0). We let $g = \bar{\partial} f_0$, that is $g_k = \frac{\partial f_0}{\partial \bar{z}_k}$. Let us see why g_k is compactly supported. The only place to check is on $U \setminus K$ as elsewhere we have 0 automatically,

$$\frac{\partial f_0}{\partial \bar{z}_k} = \frac{\partial}{\partial \bar{z}_k}((1-\varphi)f) = -f\frac{\partial \varphi}{\partial \bar{z}_k}.$$

By Lemma **??** we find a compactly supported solution ψ to $\bar{\partial}\psi = g$.

Set $\tilde{f} := f_0 - \psi$. Let us check that this is the desired extention. Firstly let us check it is holomorphic,

$$\frac{\partial \tilde{f}}{\partial \bar{z}_k} = \frac{\partial f_0}{\partial \bar{z}_k} - \frac{\partial \psi}{\partial \bar{z}_k} = g_k - g_k = 0.$$

It is not hard to see that ψ is compactly supported in U. This follows by the fact that $U \setminus K$ is connected and the fact that ψ is holomorphic on the set where g is identically zero. By unique continuation of holomorphic functions, support of ψ is no larger than that of g. $\square$