monotonicity of the sequence $(1 + x/n)^n$

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Theorem 1. *Let x be a real number and let n be an integer such that $n > 0$ and $n + x > 0$. Then*

$$\left(\frac{n+x}{n}\right)^n < \left(\frac{n+1+x}{n+1}\right)^{n+1}.$$

Proof. We begin by dividing the two expressions to be compared:

$$\begin{aligned}\frac{\left(\frac{n+x+1}{n+1}\right)^{n+1}}{\left(\frac{n+x}{n}\right)^n} &= \frac{n+x+1}{n+1} \cdot \left(\frac{n(n+x+1)}{(n+x)(n+1)}\right)^n \\ &= \frac{n+x+1}{n+1} \cdot \left(\frac{n^2+nx+n}{n^2+nx+n+x}\right)^n \\ &= \frac{n+x+1}{n+1} \cdot \left(1 - \frac{x}{n^2+nx+n+x}\right)^n\end{aligned}$$

Now, when $x > 0$, we have

$$0 < \frac{x}{n^2+nx+n+x} < 1$$

whilst, when $x < 0$ and $n + x > 0$, we have,

$$\frac{x}{n^2+nx+n+x} < 0.$$

Therefore, we may apply an inequality for differences of powers to conclude

$$\begin{aligned}\left(1 - \frac{x}{n^2+nx+n+x}\right)^n &> 1 - \frac{nx}{n^2+nx+n+x} \\ &= \frac{n^2+n+x}{n^2+nx+n+x}\end{aligned}$$

Hence, we have

$$\begin{aligned}\frac{\left(\frac{n+x+1}{n+1}\right)^{n+1}}{\left(\frac{n+x}{n}\right)^n} &> \frac{(n+x+1)(n^2+n+x)}{(n+1)(n^2+nx+n+x)} \\ &= \frac{n^3+2n^2+n+n^2x+2nx+x+x^2}{n^3+2n^2+n+n^2x+2nx+x}\end{aligned}$$

Note that the numerator is greater than the denominator because it contains every term contained in the denominator and an extra term x^2. Hence this ratio is larger than 1; multiplying out, we obtain the inequality which was to be demonstrated. □