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modulus of complex number

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Synonym	absolute value
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Related topic	AbsoluteValue
Related topic	Subadditive
Related topic	SignumFunction
Related topic	ComplexConjugate
Related topic	PotentialOfHollowBall
Related topic	ConvergenceOfRiemannZetaSeries
Related topic	RealPartSeriesAndImaginaryPartSeries
Related topic	ArgumentOfProductAndSum
Related topic	ArgumentOfProductAndQuotient
Related topic	EqualityOfComplexNumbers

Definition Let z be a complex number, and let $\overline{z}$ be the complex conjugate of z. Then the *modulus*, or *absolute value*, of z is defined as

$$|z| := \sqrt{z\overline{z}}.$$

There is also the notation

$$\operatorname{mod} z$$

for the modulus of z.

If we write z in polar form as $z = re^{i\phi}$ with $r \geq 0,\ \phi \in [0,\, 2\pi),$ then $|z| = r$. It follows that the modulus is a positive real number or zero. Alternatively, if a is the real part of z, and b the imaginary part, then

$$|z| \quad = \quad \sqrt{a^2 + b^2}, \tag{1}$$

which is simply the Euclidean norm of the point $(a,\, b) \in \mathbb{R}^2$. It follows that the modulus satisfies the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|,$$

also

$$|\Re z| \leq |z|, \quad |\Im z| \leq |z|, \quad |z| \leq |\Re z| + |\Im z|.$$

Modulus is :

$$|z_1 z_2| = |z_1| \cdot |z_2|, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Since $\mathbb{R} \subset \mathbb{C}$, the definition of modulus includes the real numbers. Explicitly, if we write $x \in \mathbb{R}$ in polar form, $x = re^{i\phi},\ r > 0,\ \phi \in [0, 2\pi),$ then $\phi = 0$ or $\phi = \pi$, so $e^{i\phi} = \pm 1$. Thus,

$$|x| = \sqrt{x^2} = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}.$$