domain of holomorphy

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Definition. An open set $\Omega \subset \mathbb{C}^n$ is called a *domain of holomorphy* if there do not exist non-empty open sets $U \subset \Omega$ and $V \subset \mathbb{C}^n$ where V is connected, $V \not\subset \Omega$ and $U \subset \Omega \cap V$ such that for every holomorphic function f on Ω there exists a holomorphic function g on V such that $f = g$ on U.

When $n = 1$, then every open set is a domain of holomorphy. For an example, assume that the boundary of $\Omega \subset \mathbb{C}$ is a Jordan curve for simplicity. We can define a holomorphic function which has zeros which accumulate on the boundary of the domain and thus the function cannot be continued past any point in the boundary. If you could extend the function, it would be identically zero.

Alternatively given any open set $\Omega \subset \mathbb{C}$ and any point $p \in \partial\Omega$, the function $z \mapsto \frac{1}{z-p}$ is holomorphic in Ω, but cannot be continued past p.

For $n \geq 2$ many domains are not domains of holomorphy. For example if you take $\mathbb{C}^2 \setminus \{0\}$, this is no longer a domain of holomorphy by `http://planetmath.org/HartogsTheorem`Hartogs's theorem. It turns out that a domain is a domain of holomorphy if and only if the boundary is pseudoconvex. In particular, every convex (in the classical sense) domain is a domain of holomorphy. examples of domains of holomorphy are $\mathbb{C}^n$, an open ball, or a polydisc.

References

[1] Lars Hörmander. , North-Holland Publishing Company, New York, New York, 1973.

[2] Steven G. Krantz. , AMS Chelsea Publishing, Providence, Rhode Island, 1992.