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submanifold

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Defines	codimension of a manifold
Defines	local defining functions
Defines	real submanifold
Defines	smooth submanifold
Defines	real analytic submanifold
Defines	regular submanifold
Defines	imbedded submanifold
Defines	embedded submanifold
Defines	germ of a submanifold
Defines	open submanifold

There are several conflicting definitions of what a submanifold is, depending on which author you are reading. All that agrees is that a submanifold is a subset of a manifold which is itself a manifold, however how structure is inherited from the ambient space is not generally agreed upon. So let's start with differentiable submanifolds of $\mathbb{R}^n$ as that's the most useful case.

Definition. Let M be a subset of $\mathbb{R}^n$ such that for every point $p \in M$ there exists a neighbourhood U_p of p in $\mathbb{R}^n$ and m continuously differentiable functions $\rho_k \colon U \to \mathbb{R}$ where the differentials of ρ_k are linearly independent, such that

$$M \cap U = \{x \in U \mid \rho_k(x) = 0, 1 \leq k \leq m\}.$$

Then M is called a *submanifold* of $\mathbb{R}^n$ of dimension m and of codimension $n - m$.

If ρ_k are in fact smooth then M is a *smooth submanifold* and similarly if ρ is real analytic then M is a *real analytic submanifold.* If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and we have a submanifold there it is called a *real submanifold* in $\mathbb{C}^n$. ρ_k are usually called the *local defining functions.*

Let's now look at a more general definition. Let M be a manifold of dimension m. A subset $N \subset M$ is said to have the *submanifold property* if there exists an integer $n \leq m$, such that for each $p \in N$ there is a coordinate neighbourhood U and a coordinate function $\varphi \colon U \to \mathbb{R}^m$ of M such that $\varphi(p) = (0, 0, 0, \ldots, 0)$, $\varphi(U \cap N) = \{x \in \varphi(U) \mid x_{n+1} = x_{n+2} = \ldots = x_m = 0\}$ if $n < m$ or $N \cap U = U$ if $n = m$.

Definition. Let M be a manifold of dimension m. A subset $N \subset M$ with the submanifold property for some $n \leq m$ is called a *submanifold* of M of dimension n and of codimension $m - n$.

The ambiguity arises about what topology we require N to have. Some authors require N to have the relative topology inherited from M, others don't.

One could also mean that a subset is a submanifold if it is a disjoint union of submanifolds of different dimensions. It is not hard to see that if N is connected this is not an issue (whatever the topology on N is).

In case of differentiable manifolds, if we take N to be a subspace of M (the topology on N is the relative topology inherited from M) and the differentiable structure of N to be the one determined by the coordinate neighbourhoods above then we call N a *regular submanifold.*

If N is a submanifold and the inclusion map $i\colon N \to M$ is an imbedding, then we say that N is an *imbedded* (or *embedded*) *submanifold* of M.

Definition. Let $p \in M$ where M is a manifold. Then the equivalence class of all submanifolds $N \subset M$ such that $p \in N$ where we say N_1 is equivalent to N_2 if there is some open neighbourhood U of p such that $N_1 \cap U = N_2 \cap U$ is called the *germ of a submanifold* through the point p.

If $N \subset M$ is an open subset of M, then N is called the *open submanifold* of M. This is the easiest class of examples of submanifolds.

Example of a submanifold (a in fact) is the unit sphere in $\mathbb{R}^n$. This is in fact a hypersurface as it is of codimension 1.

References

[1] William M. Boothby. , Academic Press, San Diego, California, 2003.

[2] M. Salah Baouendi, Peter Ebenfelt, Linda Preiss Rothschild. , Princeton University Press, Princeton, New Jersey, 1999.