CR submanifold

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Defines	CR bundle
Defines	CR vector field
Defines	complex tangent space
Defines	complex bundle
Defines	space of antiholomorphic vectors
Defines	antiholomorphic vector
Defines	CR dimension

Suppose that $M \subset \mathbb{C}^N$ is a real submanifold of real dimension n. Take $p \in M$, then let $T_p(\mathbb{C}^N)$ be the tangent vectors of $\mathbb{C}^N$ at the point p. If we identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ by $z_j = x_j + iy_j$, we can take the following vectors as our basis

$$\frac{\partial}{\partial x_1}\bigg|_p, \frac{\partial}{\partial y_1}\bigg|_p, \ldots, \frac{\partial}{\partial x_N}\bigg|_p, \frac{\partial}{\partial y_N}\bigg|_p.$$

We define a real linear mapping $J\colon T_p(\mathbb{C}^N) \to T_p(\mathbb{C}^N)$ such that for any $1 \leq j \leq N$ we have

$$J\left(\frac{\partial}{\partial x_1}\bigg|_p\right) = \frac{\partial}{\partial y_1}\bigg|_p \qquad \text{and } J\left(\frac{\partial}{\partial y_1}\bigg|_p\right) = -\frac{\partial}{\partial x_1}\bigg|_p.$$

Where J is referred to as the complex structure on $T_p(\mathbb{C}^N)$. Note that $J^2 = -I$, that is applying J twice we just negate the vector.

Let $T_p(M)$ be the tangent space of M at the point p (that is, those vectors of $T_p(\mathbb{C}^N)$ which are tangent to M).

Definition. The subspace $T_p^c(M)$ defined as

$$T_p^c(M) := \{X \in T_p(M) \mid J(X) \in T_p(M)\}$$

is called the *complex tangent space* of M at the point p, and if the dimension of $T_p(M)$ is constant for all $p \in M$ then the corresponding vector bundle $T^c(M) := \bigcup_{p\in M} T_p^c(M)$ is called the *complex bundle* of M.

Do note that the complex tangent space is a real (not complex) vector space, despite its rather unfortunate name.

Let $\mathbb{C}T_p(M)$ and $\mathbb{C}T_p(\mathbb{C}^N)$ be the complexified vector spaces, by just allowing the coeffficents of the vectors to be complex numbers. That is for $X = \sum a_j \frac{\partial}{\partial x_1}\Big|_p + b_j \frac{\partial}{\partial x_1}\Big|_p$ we allow a_j and b_j to be complex numbers. Next we can extend the mapping J to be $\mathbb{C}$-linear on these new vector spaces and still get that $J^2 = -I$ as before. We notice that the operator J has two eigenvalues, i and $-i$.

Definition. Let $\mathcal{V}_p$ be the eigenspace of $\mathbb{C}T_p(M)$ corresponding to the eigenvalue $-i$. That is

$$\mathcal{V}_p := \{X \in \mathbb{C}T_p(M) \mid J(X) = -iX\}.$$

If the dimension of $\mathcal{V}_p$ is constant for all $p \in M$, then we get a corresponding vector bundle $\mathcal{V}$ which we call the *CR bundle* of M. A smooth section of the CR bundle is then called a *CR vector field*.

Definition. The submanifold M is called a *CR submanifold* (or just *CR manifold*) if the dimension of $\mathcal{V}_p$ is constant for all $p \in M$. The complex dimension of $\mathcal{V}_p$ will then be called the *CR dimension* of M.

An example of a CR submanifold is for example a hyperplane defined by $\operatorname{Im} z_N = 0$ where the CR dimension is $N - 1$. Another less trivial example is the Lewy hypersurface.

Note that sometimes $\mathcal{V}_p$ is written as $T_p^{0,1}(M)$ and referred to as the *space of antiholomorphic vectors*, where an *antiholomorphic vector* is a tangent vector which can be written in terms of the basis

$$\left.\frac{\partial}{\partial \bar{z}_j}\right|_p := \frac{1}{2}\left(\left.\frac{\partial}{\partial x_j}\right|_p + i\left.\frac{\partial}{\partial y_j}\right|_p\right).$$

The CR in the name refers to Cauchy-Riemann and that is because the vector space $\mathcal{V}_p$ corresponds to differentiating with respect to $\bar{z}_j$.

References

[1] M. Salah Baouendi, Peter Ebenfelt, Linda Preiss Rothschild. , Princeton University Press, Princeton, New Jersey, 1999.

[2] Albert Boggess. , CRC, 1991.