

Series Solutions of Second Order Linear Equations

5.1

1. Apply the ratio test:

$$\lim_{n \rightarrow \infty} \frac{|(x-3)^{n+1}|}{|(x-3)^n|} = \lim_{n \rightarrow \infty} |x-3| = |x-3|.$$

Hence the series converges absolutely for $|x-3| < 1$. The radius of convergence is $\rho = 1$. The series diverges for $x = 2$ and $x = 4$, since the n -th term does not approach zero.

3. Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|n! x^{2n+2}|}{|(n+1)! x^{2n}|} = \lim_{n \rightarrow \infty} \frac{x^2}{n+1} = 0.$$

The series converges absolutely for all values of x . Thus the radius of convergence is $\rho = \infty$.

4. Apply the ratio test:

$$\lim_{n \rightarrow \infty} \frac{|2^{n+1} x^{n+1}|}{|2^n x^n|} = \lim_{n \rightarrow \infty} 2|x| = 2|x|.$$

Hence the series converges absolutely for $2|x| < 1$, or $|x| < 1/2$. The radius of convergence is $\rho = 1/2$. The series diverges for $x = \pm 1/2$, since the n -th term does not approach zero.

6. Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|n(x-x_0)^{n+1}|}{|(n+1)(x-x_0)^n|} = \lim_{n \rightarrow \infty} \frac{n}{n+1} |(x-x_0)| = |(x-x_0)|.$$

Hence the series converges absolutely for $|(x-x_0)| < 1$. The radius of convergence is $\rho = 1$. At $x = x_0 + 1$, we obtain the harmonic series, which is divergent. At the other endpoint, $x = x_0 - 1$, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is conditionally convergent.

7. Apply the ratio test :

$$\lim_{n \rightarrow \infty} \frac{|3^n(n+1)^2(x+2)^{n+1}|}{|3^{n+1}n^2(x+2)^n|} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{3n^2} |(x+2)| = \frac{1}{3} |(x+2)|.$$

Hence the series converges absolutely for $\frac{1}{3} |x+2| < 1$, or $|x+2| < 3$. The radius of convergence is $\rho = 3$. At $x = -5$ and $x = +1$, the series diverges, since the n -th term does not approach zero.

8. Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|n^n(n+1)!x^{n+1}|}{|(n+1)^{n+1}n!x^n|} = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} |x| = \frac{1}{e} |x|,$$

since

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = e^{-1}.$$

Hence the series converges absolutely for $|x| < e$. The radius of convergence is $\rho = e$. At $x = \pm e$, the series diverges, since the n -th term does not approach zero. This follows from the fact that

$$\lim_{n \rightarrow \infty} \frac{n!e^n}{n^n\sqrt{2\pi n}} = 1.$$

10. We have $f(x) = e^x$, with $f^{(n)}(x) = e^x$, for $n = 1, 2, \dots$. Therefore $f^{(n)}(0) = 1$. Hence the Taylor expansion about $x_0 = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|n!x^{n+1}|}{|(n+1)!x^n|} = \lim_{n \rightarrow \infty} \frac{1}{n+1} |x| = 0.$$

The radius of convergence is $\rho = \infty$.

11. We have $f(x) = x$, with $f'(x) = 1$ and $f^{(n)}(x) = 0$, for $n = 2, \dots$. Clearly, $f(1) = 1$ and $f'(1) = 1$, with all other derivatives equal to zero. Hence the Taylor expansion about $x_0 = 1$ is

$$x = 1 + (x - 1).$$

Since the series has only a finite number of terms, it converges absolutely for all x .

14. We have $f(x) = 1/(1+x)$, $f'(x) = -1/(1+x)^2$, $f''(x) = 2/(1+x)^3, \dots$ with $f^{(n)}(x) = (-1)^n n!/(1+x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = (-1)^n n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \rightarrow \infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

15. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \dots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = n!$, for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \rightarrow \infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

16. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \dots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(2) = (-1)^{n+1} n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 2$ is

$$\frac{1}{1-x} = - \sum_{n=0}^{\infty} (-1)^n (x-2)^n.$$

Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|(x-2)^{n+1}|}{|(x-2)^n|} = \lim_{n \rightarrow \infty} |x-2| = |x-2|.$$

The series converges absolutely for $|x-2| < 1$, but diverges at $x = 1$ and $x = 3$.

17. Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|(n+1)x^{n+1}|}{|n x^n|} = \lim_{n \rightarrow \infty} \frac{n+1}{n} |x| = |x|.$$

The series converges absolutely for $|x| < 1$. Term-by-term differentiation results in

$$y' = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16x^3 + \dots$$

$$y'' = \sum_{n=2}^{\infty} n^2(n-1)x^{n-2} = 4 + 18x + 48x^2 + 100x^3 + \dots$$

Shifting the indices, we can also write

$$y' = \sum_{n=0}^{\infty} (n+1)^2 x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+2)^2(n+1)x^n.$$

20. Shifting the index in the second series, that is, setting $n = k + 1$,

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1} x^n.$$

Hence

$$\begin{aligned} \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} &= \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_1 + \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1}) x^k. \end{aligned}$$

21. Shifting the index by 2, that is, setting $m = n - 2$,

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n. \end{aligned}$$

22. Shift the index down by 2, that is, set $m = n + 2$. It follows that

$$\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{m=2}^{\infty} a_{m-2} x^m = \sum_{n=2}^{\infty} a_{n-2} x^n.$$

24. Clearly,

$$(1-x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

Shifting the index in the first series, that is, setting $k = n - 2$,

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n. \end{aligned}$$

Hence

$$(1-x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

Note that when $n=0$ and $n=1$, the coefficients in the second series are zero. So

$$(1-x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n] x^n.$$

26. Clearly,

$$\sum_{n=1}^{\infty} na_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}.$$

Shifting the index in the first series, that is, setting $k = n - 1$,

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k.$$

Shifting the index in the second series, that is, setting $k = n + 1$,

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{k=1}^{\infty} a_{k-1}x^k.$$

Combining the series, and starting the summation at $n = 1$,

$$\sum_{n=1}^{\infty} na_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} + a_{n-1}] x^n.$$

27. We note that

$$x \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n.$$

Shifting the index in the first series, that is, setting $k = n - 1$,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{k=1}^{\infty} k(k+1)a_{k+1}x^k = \sum_{k=0}^{\infty} k(k+1)a_{k+1}x^k,$$

since the coefficient of the term associated with $k = 0$ is zero. Combining the series,

$$x \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [n(n+1)a_{n+1} + a_n] x^n.$$

5.2

1.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n] x^n = 0.$$

Equating all the coefficients to zero,

$$(n+2)(n+1)a_{n+2} - a_n = 0, \quad n = 0, 1, 2, \dots$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, \dots$$

The subscripts differ by two, so for $k = 1, 2, \dots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \dots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \dots = \frac{a_1}{(2k+1)!}.$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

The linearly independent solutions are

$$y_1 = a_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \right) = a_0 \cosh x$$

$$y_2 = a_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \right) = a_1 \sinh x.$$

(c) The Wronskian is $a_0 a_1$.

4.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + k^2 x^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Rewriting the second summation,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=2}^{\infty} k^2 a_{n-2} x^n = 0,$$

that is,

$$2a_2 + 3 \cdot 2a_3x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + k^2a_{n-2}]x^n = 0.$$

Setting the coefficients equal to zero, we have $a_2 = 0$, $a_3 = 0$, and

$$(n+2)(n+1)a_{n+2} + k^2a_{n-2} = 0, \quad \text{for } n = 2, 3, 4, \dots$$

The recurrence relation can be written as

$$a_{n+2} = -\frac{k^2a_{n-2}}{(n+2)(n+1)}, \quad n = 2, 3, 4, \dots$$

The indices differ by four, so a_4, a_8, a_{12}, \dots are defined by

$$a_4 = -\frac{k^2a_0}{4 \cdot 3}, \quad a_8 = -\frac{k^2a_4}{8 \cdot 7}, \quad a_{12} = -\frac{k^2a_8}{12 \cdot 11}, \quad \dots$$

Similarly, a_5, a_9, a_{13}, \dots are defined by

$$a_5 = -\frac{k^2a_1}{5 \cdot 4}, \quad a_9 = -\frac{k^2a_5}{9 \cdot 8}, \quad a_{13} = -\frac{k^2a_9}{13 \cdot 12}, \quad \dots$$

The remaining coefficients are zero. Therefore the general solution is

$$y = a_0 \left[1 - \frac{k^2}{4 \cdot 3}x^4 + \frac{k^4}{8 \cdot 7 \cdot 4 \cdot 3}x^8 - \frac{k^6}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}x^{12} + \dots \right] + a_1 \left[x - \frac{k^2}{5 \cdot 4}x^5 + \frac{k^4}{9 \cdot 8 \cdot 5 \cdot 4}x^9 - \frac{k^6}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 4 \cdot 4}x^{13} + \dots \right].$$

Note that for the even coefficients,

$$a_{4m} = -\frac{k^2a_{4m-4}}{(4m-1)4m}, \quad m = 1, 2, 3, \dots$$

and for the odd coefficients,

$$a_{4m+1} = -\frac{k^2a_{4m-3}}{4m(4m+1)}, \quad m = 1, 2, 3, \dots$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(k^2x^4)^{m+1}}{3 \cdot 4 \cdot 7 \cdot 8 \dots (4m+3)(4m+4)}$$

$$y_2(x) = x \left[1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(k^2x^4)^{m+1}}{4 \cdot 5 \cdot 8 \cdot 9 \dots (4m+4)(4m+5)} \right].$$

(c) The Wronskian at 0 is 1.

6.(a,b) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Substitution into the ODE results in

$$(2+x^2) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

It follows that

$$4a_0 + 4a_2 + (3a_1 + 12a_3)x + \sum_{n=2}^{\infty} [2(n+2)(n+1)a_{n+2} + n(n-1)a_n - n a_n + 4a_n] x^n = 0.$$

Equating the coefficients to zero, we find that $a_2 = -a_0$, $a_3 = -a_1/4$, and

$$a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)} a_n, \quad n = 0, 1, 2, \dots$$

The indices differ by two, so for $k = 0, 1, 2, \dots$

$$a_{2k+2} = -\frac{(2k)^2 - 4k + 4}{2(2k+2)(2k+1)} a_{2k}$$

and

$$a_{2k+3} = -\frac{(2k+1)^2 - 4k + 2}{2(2k+3)(2k+2)} a_{2k+1}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \dots$$

$$y_2(x) = x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \dots$$

(c) The Wronskian at 0 is 1.

7.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

First write

$$x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

We then obtain

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + n a_n + 2a_n] x^n = 0.$$

It follows that $a_2 = -a_0$ and $a_{n+2} = -a_n/(n+1)$, $n = 0, 1, 2, \dots$. Note that the indices differ by two, so for $k = 1, 2, \dots$

$$a_{2k} = -\frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-3)(2k-1)} = \dots = \frac{(-1)^k a_0}{1 \cdot 3 \cdot 5 \dots (2k-1)}$$

and

$$a_{2k+1} = -\frac{a_{2k-1}}{2k} = \frac{a_{2k-3}}{(2k-2)2k} = \dots = \frac{(-1)^k a_1}{2 \cdot 4 \cdot 6 \dots (2k)}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} - \frac{x^6}{1 \cdot 3 \cdot 5} + \dots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \dots (2n-1)}$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \dots = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \dots (2n)}.$$

(c) The Wronskian at 0 is 1.

9.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Substitution into the ODE results in

$$(1+x^2) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - 4x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=2}^{\infty} n(n-1)a_nx^n$$

and

$$x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

It follows that

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4n a_n + 6a_n] x^n = 0.$$

Setting the coefficients equal to zero, we obtain $a_2 = -3a_0$, $a_3 = -a_1/3$, and

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)} a_n, \quad n = 0, 1, 2, \dots$$

Observe that for $n = 2$ and $n = 3$, we obtain $a_4 = a_5 = 0$. Since the indices differ by two, we also have $a_n = 0$ for $n \geq 4$. Therefore the general solution is a polynomial

$$y = a_0 + a_1x - 3a_0x^2 - a_1x^3/3.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - 3x^2 \quad \text{and} \quad y_2(x) = x - x^3/3.$$

(c) The Wronskian is $(x^2 + 1)^2$. At $x = 0$ it is 1.

10.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substitution into the ODE results in

$$(4 - x^2) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + 2 \sum_{n=0}^{\infty} a_nx^n = 0.$$

First write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=2}^{\infty} n(n-1)a_nx^n.$$

It follows that

$$2a_0 + 8a_2 + (2a_1 + 24a_3)x + \sum_{n=2}^{\infty} [4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n] x^n = 0.$$

We obtain $a_2 = -a_0/4$, $a_3 = -a_1/12$ and

$$4(n+2)a_{n+2} = (n-2)a_n, \quad n = 0, 1, 2, \dots$$

Note that for $n = 2$, $a_4 = 0$. Since the indices differ by two, we also have $a_{2k} = 0$ for $k = 2, 3, \dots$. On the other hand, for $k = 1, 2, \dots$,

$$a_{2k+1} = \frac{(2k-3)a_{2k-1}}{4(2k+1)} = \frac{(2k-5)(2k-3)a_{2k-3}}{4^2(2k-1)(2k+1)} = \dots = \frac{-a_1}{4^k(2k-1)(2k+1)}.$$

Therefore the general solution is

$$y = a_0 + a_1x - a_0 \frac{x^2}{4} - a_1 \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n(2n-1)(2n+1)}.$$

Hence the linearly independent solutions are $y_1(x) = 1 - x^2/4$ and

$$y_2(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \dots = x - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n(2n-1)(2n+1)}.$$

(c) The Wronskian at 0 is 1.

11.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substitution into the ODE results in

$$(3-x^2) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 3x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_nx^n = 0.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=2}^{\infty} n(n-1)a_nx^n$$

and

$$x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = \sum_{n=1}^{\infty} na_nx^n.$$

It follows that

$$6a_2 - a_0 + (-4a_1 + 18a_3)x + \sum_{n=2}^{\infty} [3(n+2)(n+1)a_{n+2} - n(n-1)a_n - 3na_n - a_n]x^n = 0.$$

We obtain $a_2 = a_0/6$, $2a_3 = a_1/9$, and

$$3(n+2)a_{n+2} = (n+1)a_n, \quad n = 0, 1, 2, \dots$$

The indices differ by two, so for $k = 1, 2, \dots$

$$a_{2k} = \frac{(2k-1)a_{2k-2}}{3(2k)} = \frac{(2k-3)(2k-1)a_{2k-4}}{3^2(2k-2)(2k)} = \dots = \frac{3 \cdot 5 \cdot \dots \cdot (2k-1)a_0}{3^k \cdot 2 \cdot 4 \cdot \dots \cdot (2k)}$$

and

$$a_{2k+1} = \frac{(2k)a_{2k-1}}{3(2k+1)} = \frac{(2k-2)(2k)a_{2k-3}}{3^2(2k-1)(2k+1)} = \cdots = \frac{2 \cdot 4 \cdot 6 \cdots (2k) a_1}{3^k \cdot 3 \cdot 5 \cdots (2k+1)}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdots (2n-1) x^{2n}}{3^n \cdot 2 \cdot 4 \cdots (2n)}$$

$$y_2(x) = x + \frac{2x^3}{9} + \frac{8x^5}{135} + \frac{16x^7}{945} + \cdots = x + \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n) x^{2n+1}}{3^n \cdot 3 \cdot 5 \cdots (2n+1)}.$$

(c) The Wronskian at 0 is 1.

12.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution into the ODE results in

$$(1-x) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Before proceeding, write

$$x \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

It follows that

$$2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)n a_{n+1} + n a_n - a_n] x^n = 0.$$

We obtain $a_2 = a_0/2$ and

$$(n+2)(n+1)a_{n+2} - (n+1)n a_{n+1} + (n-1)a_n = 0$$

for $n = 0, 1, 2, \dots$. Writing out the individual equations,

$$\begin{aligned} 3 \cdot 2 a_3 - 2 \cdot 1 a_2 &= 0 \\ 4 \cdot 3 a_4 - 3 \cdot 2 a_3 + a_2 &= 0 \\ 5 \cdot 4 a_5 - 4 \cdot 3 a_4 + 2 a_3 &= 0 \\ 6 \cdot 5 a_6 - 5 \cdot 4 a_5 + 3 a_4 &= 0 \\ &\vdots \end{aligned}$$

The coefficients are calculated successively as $a_3 = a_0/(2 \cdot 3)$, $a_4 = a_3/2 - a_2/12 = a_0/24$, $a_5 = 3a_4/5 - a_3/10 = a_0/120$, \dots . We can now see that for $n \geq 2$, a_n is proportional to a_0 . In fact, for $n \geq 2$, $a_n = a_0/(n!)$. Therefore the general solution is

$$y = a_0 + a_1 x + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!} + \frac{a_0 x^4}{4!} + \dots$$

Hence the linearly independent solutions are $y_2(x) = x$ and

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} = e^x - x.$$

(c) The Wronskian is $e^x(1-x)$. At $x = 0$ it is 1.

13.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution into the ODE results in

$$2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3 \sum_{n=0}^{\infty} a_n x^n = 0.$$

First write

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

We then obtain

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1)a_{n+2} + n a_n + 3a_n] x^n = 0.$$

It follows that $a_2 = -3a_0/4$ and

$$2(n+2)(n+1)a_{n+2} + (n+3)a_n = 0$$

for $n = 0, 1, 2, \dots$. The indices differ by two, so for $k = 1, 2, \dots$

$$\begin{aligned} a_{2k} &= -\frac{(2k+1)a_{2k-2}}{2(2k-1)(2k)} = \frac{(2k-1)(2k+1)a_{2k-4}}{2^2(2k-3)(2k-2)(2k-1)(2k)} = \dots \\ &= \frac{(-1)^k 3 \cdot 5 \dots (2k+1)}{2^k (2k)!} a_0. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k+2)a_{2k-1}}{2(2k)(2k+1)} = \frac{(2k)(2k+2)a_{2k-3}}{2^2(2k-2)(2k-1)(2k)(2k+1)} = \dots \\ &= \frac{(-1)^k 4 \cdot 6 \dots (2k)(2k+2)}{2^k (2k+1)!} a_1. \end{aligned}$$

Hence the linearly independent solutions are

$$\begin{aligned} y_1(x) &= 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 5 \dots (2n+1)}{2^n (2n)!} x^{2n} \\ y_2(x) &= x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \dots = x + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cdot 6 \dots (2n+2)}{2^n (2n+1)!} x^{2n+1}. \end{aligned}$$

(c) The Wronskian at 0 is 1.

15.(a) From Problem 2, we have

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n n! x^{2n+1}}{(2n+1)!}.$$

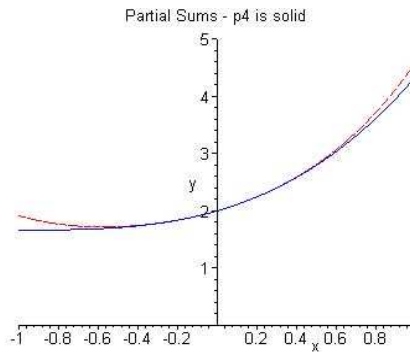
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 2y_1(x) + y_2(x)$. That is,

$$y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \dots$$

The four- and five-term polynomial approximations are

$$p_4 = 2 + x + x^2 + x^3/3, \text{ and } p_5 = 2 + x + x^2 + x^3/3 + x^4/4.$$

(b)



(c) The four-term approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.7$.

17.(a) From Problem 7, the linearly independent solutions are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \text{ and } y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

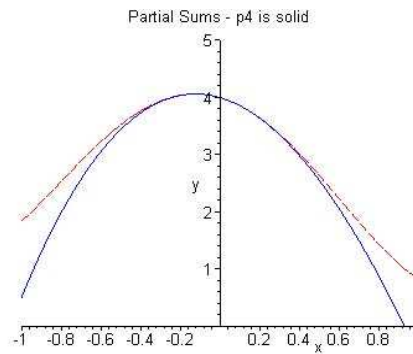
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 4y_1(x) - y_2(x)$. That is,

$$y(x) = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 - \frac{1}{8}x^5 - \frac{4}{15}x^6 + \cdots.$$

The four- and five-term polynomial approximations are

$$p_4 = 4 - x - 4x^2 + \frac{1}{2}x^3, \text{ and } p_5 = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4.$$

(b)



(c) The four-term approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.5$.

18.(a) From Problem 12, we have

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} \text{ and } y_2(x) = x.$$

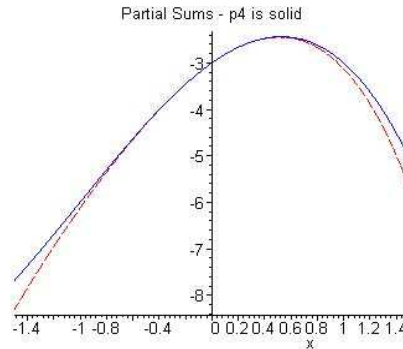
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = -3y_1(x) + 2y_2(x)$. That is,

$$y(x) = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 - \frac{1}{40}x^5 - \frac{1}{240}x^6 + \cdots.$$

The four- and five-term polynomial approximations are

$$p_4 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3, \text{ and } p_5 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4.$$

(b)



(c) The four-term approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.9$.

20. Two linearly independent solutions of Airy's equation (about $x_0 = 0$) are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)}.$$

Applying the ratio test to the terms of $y_1(x)$,

$$\lim_{n \rightarrow \infty} \frac{|2 \cdot 3 \cdots (3n-1)(3n) x^{3n+3}|}{|2 \cdot 3 \cdots (3n+2)(3n+3) x^{3n}|} = \lim_{n \rightarrow \infty} \frac{1}{(3n+1)(3n+2)(3n+3)} |x|^3 = 0.$$

Similarly, applying the ratio test to the terms of $y_2(x)$,

$$\lim_{n \rightarrow \infty} \frac{|3 \cdot 4 \cdots (3n)(3n+1) x^{3n+4}|}{|3 \cdot 4 \cdots (3n+3)(3n+4) x^{3n+1}|} = \lim_{n \rightarrow \infty} \frac{1}{(3n+2)(3n+3)(3n+4)} |x|^3 = 0.$$

Hence both series converge absolutely for all x .

21. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0.$$

First write

$$x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

We then obtain

$$2a_2 + \lambda a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - 2n a_n + \lambda a_n] x^n = 0.$$

Setting the coefficients equal to zero, it follows that

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)} a_n$$

for $n = 0, 1, 2, \dots$. Note that the indices differ by two, so for $k = 1, 2, \dots$

$$\begin{aligned} a_{2k} &= \frac{(4k-4-\lambda)a_{2k-2}}{(2k-1)2k} = \frac{(4k-8-\lambda)(4k-4-\lambda)a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \dots \\ &= (-1)^k \frac{\lambda \dots (\lambda-4k+8)(\lambda-4k+4)}{(2k)!} a_0. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= \frac{(4k-2-\lambda)a_{2k-1}}{2k(2k+1)} = \frac{(4k-6-\lambda)(4k-2-\lambda)a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \dots \\ &= (-1)^k \frac{(\lambda-2) \dots (\lambda-4k+6)(\lambda-4k+2)}{(2k+1)!} a_1. \end{aligned}$$

Hence the linearly independent solutions of the Hermite equation (about $x_0 = 0$) are

$$\begin{aligned} y_1(x) &= 1 - \frac{\lambda}{2!}x^2 + \frac{\lambda(\lambda-4)}{4!}x^4 - \frac{\lambda(\lambda-4)(\lambda-8)}{6!}x^6 + \dots \\ y_2(x) &= x - \frac{\lambda-2}{3!}x^3 + \frac{(\lambda-2)(\lambda-6)}{5!}x^5 - \frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7!}x^7 + \dots \end{aligned}$$

(b) Based on the recurrence relation

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)} a_n,$$

the series solution will terminate as long as λ is a nonnegative even integer. If $\lambda = 2m$, then one or the other of the solutions in part (b) will contain at most $m/2 + 1$ terms. In particular, we obtain the polynomial solutions corresponding to $\lambda = 0, 2, 4, 6, 8, 10$:

$\lambda = 0$	$y_1(x) = 1$
$\lambda = 2$	$y_2(x) = x$
$\lambda = 4$	$y_1(x) = 1 - 2x^2$
$\lambda = 6$	$y_2(x) = x - 2x^3/3$
$\lambda = 8$	$y_1(x) = 1 - 4x^2 + 4x^4/3$
$\lambda = 10$	$y_2(x) = x - 4x^3/3 + 4x^5/15$

(c) Observe that if $\lambda = 2n$, and $a_0 = a_1 = 1$, then

$$a_{2k} = (-1)^k \frac{2n \dots (2n - 4k + 8)(2n - 4k + 4)}{(2k)!}$$

and

$$a_{2k+1} = (-1)^k \frac{(2n - 2) \dots (2n - 4k + 6)(2n - 4k + 2)}{(2k + 1)!}.$$

for $k = 1, 2, \dots, [n/2]$. It follows that the coefficient of x^n , in y_1 and y_2 , is

$$a_n = \begin{cases} (-1)^k \frac{4^k k!}{(2k)!} & \text{for } n = 2k \\ (-1)^k \frac{4^k k!}{(2k+1)!} & \text{for } n = 2k + 1 \end{cases}$$

Then by definition,

$$H_n(x) = \begin{cases} (-1)^k 2^n \frac{(2k)!}{4^k k!} y_1(x) = (-1)^k \frac{(2k)!}{k!} y_1(x) & \text{for } n = 2k \\ (-1)^k 2^n \frac{(2k+1)!}{4^k k!} y_2(x) = (-1)^k \frac{2(2k+1)!}{k!} y_2(x) & \text{for } n = 2k + 1 \end{cases}$$

Therefore the first six Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

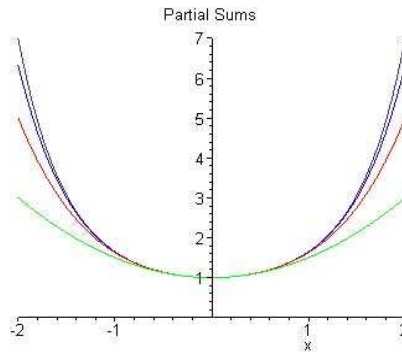
$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

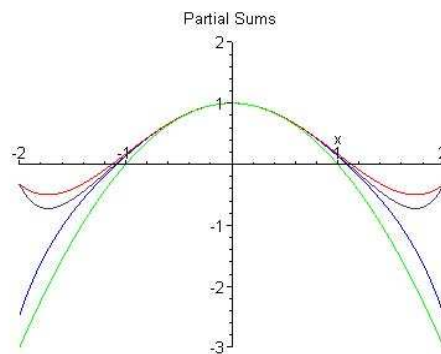
23. The series solution is given by

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{2^2 2!}x^4 + \frac{1}{2^3 3!}x^6 + \frac{1}{2^4 4!}x^8 + \dots$$



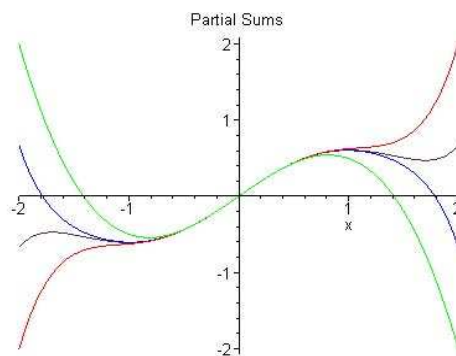
24. The series solution is given by

$$y(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \frac{x^8}{120} + \dots$$



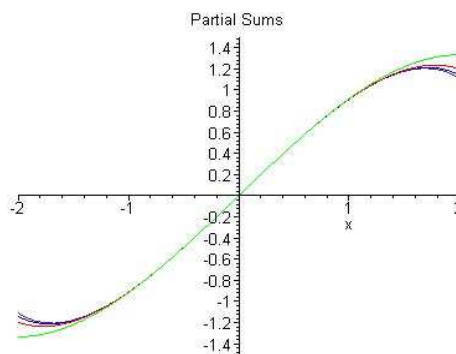
25. The series solution is given by

$$y(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \frac{x^9}{2 \cdot 4 \cdot 6 \cdot 8} - \dots$$



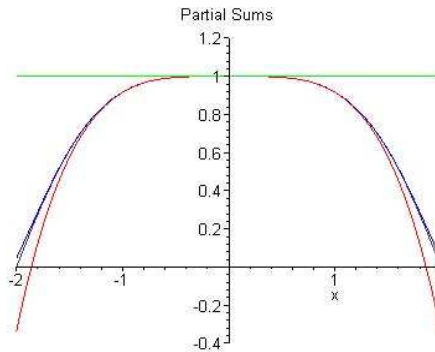
26. The series solution is given by

$$y(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \frac{x^9}{16128} - \dots$$



27. The series solution is given by

$$y(x) = 1 - \frac{x^4}{12} + \frac{x^8}{672} - \frac{x^{12}}{88704} + \dots$$



28. Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substitution into the ODE results in

$$(1-x) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - 2 \sum_{n=0}^{\infty} a_nx^n = 0.$$

After appropriately shifting the indices, it follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + na_n - 2a_n]x^n = 0.$$

We find that $a_2 = a_0$ and

$$(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + (n-2)a_n = 0$$

for $n = 1, 2, \dots$. Writing out the individual equations,

$$3 \cdot 2a_3 - 2 \cdot 1a_2 - a_1 = 0$$

$$4 \cdot 3a_4 - 3 \cdot 2a_3 = 0$$

$$5 \cdot 4a_5 - 4 \cdot 3a_4 + a_3 = 0$$

$$6 \cdot 5a_6 - 5 \cdot 4a_5 + 2a_4 = 0$$

\vdots

Since $a_0 = 0$ and $a_1 = 1$, the remaining coefficients satisfy the equations

$$3 \cdot 2a_3 - 1 = 0$$

$$4 \cdot 3a_4 - 3 \cdot 2a_3 = 0$$

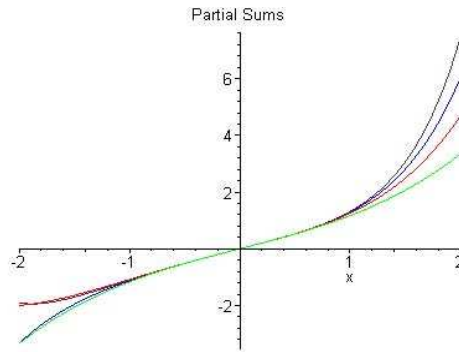
$$5 \cdot 4a_5 - 4 \cdot 3a_4 + a_3 = 0$$

$$6 \cdot 5a_6 - 5 \cdot 4a_5 + 2a_4 = 0$$

\vdots

That is, $a_3 = 1/6$, $a_4 = 1/12$, $a_5 = 1/24$, $a_6 = 1/45$, \dots . Hence the series solution of the initial value problem is

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6 + \frac{13}{1008}x^7 + \dots$$



5.3

2. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -(\sin x)y' - (\cos x)y.$$

Differentiating twice,

$$y''' = -(\sin x)y'' - 2(\cos x)y' + (\sin x)y$$

$$y^{(4)} = -(\sin x)y''' - 3(\cos x)y'' + 3(\sin x)y' + (\cos x)y.$$

Given that $\phi(0) = 0$ and $\phi'(0) = 1$, the first equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -2$ and $\phi^{(4)}(0) = 0$.

3. Let $y = \phi(x)$ be a solution of the initial value problem. First write

$$y'' = -\frac{1+x}{x^2}y' - \frac{3 \ln x}{x^2}y.$$

Differentiating twice,

$$y''' = \frac{-1}{x^3}[(x+x^2)y'' + (3x \ln x - x - 2)y' + (3 - 6 \ln x)y].$$

$$y^{(4)} = \frac{-1}{x^4}[(x^2+x^3)y''' + (3x^2 \ln x - 2x^2 - 4x)y'' + (6 + 8x - 12x \ln x)y' + (18 \ln x - 15)y].$$

Given that $\phi(1) = 2$ and $\phi'(1) = 0$, the first equation gives $\phi''(1) = 0$ and the last two equations give $\phi'''(1) = -6$ and $\phi^{(4)}(1) = 42$.

4. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -x^2 y' - (\sin x)y.$$

Differentiating twice,

$$\begin{aligned}y''' &= -x^2 y'' - (2x + \sin x)y' - (\cos x)y \\y^{(4)} &= -x^2 y''' - (4x + \sin x)y'' - (2 + 2\cos x)y' + (\sin x)y.\end{aligned}$$

Given that $\phi(0) = a_0$ and $\phi'(0) = a_1$, the first equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -a_0$ and $\phi^{(4)}(0) = -4a_1$.

5. Clearly, $p(x) = 4$ and $q(x) = 6x$ are analytic for all x . Hence the series solutions converge everywhere.

7. The zeroes of $P(x) = 1 + x^3$ are the three cube roots of -1 . They all lie on the unit circle in the complex plane. So for $x_0 = 0$, $\rho_{min} = 1$. For $x_0 = 2$, the nearest root is $e^{i\pi/3} = (1 + i\sqrt{3})/2$, hence $\rho_{min} = \sqrt{3}$.

8. The only root of $P(x) = x$ is zero. Hence $\rho_{min} = 1$.

9.(b) $p(x) = -x$ and $q(x) = -1$ are analytic for all x .

(c) $p(x) = -x$ and $q(x) = -1$ are analytic for all x .

(d) $p(x) = 0$ and $q(x) = kx^2$ are analytic for all x .

(e) The only root of $P(x) = 1 - x$ is 1. Hence $\rho_{min} = 1$.

(g) $p(x) = x$ and $q(x) = 2$ are analytic for all x .

(i) The zeroes of $P(x) = 1 + x^2$ are $\pm i$. Hence $\rho_{min} = 1$.

(j) The zeroes of $P(x) = 4 - x^2$ are ± 2 . Hence $\rho_{min} = 2$.

(k) The zeroes of $P(x) = 3 - x^2$ are $\pm\sqrt{3}$. Hence $\rho_{min} = \sqrt{3}$.

(l) The only root of $P(x) = 1 - x$ is 1. Hence $\rho_{min} = 1$.

(m) $p(x) = x/2$ and $q(x) = 3/2$ are analytic for all x .

(n) $p(x) = (1 + x)/2$ and $q(x) = 3/2$ are analytic for all x .

12. The Taylor series expansion of e^x , about $x_0 = 0$, is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{x^n}{n!} \right] \left[\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n \right] + x \sum_{n=0}^{\infty} a_nx^n = 0.$$

First note that

$$x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \dots + a_{n-1} x^n + \dots$$

The coefficient of x^n in the product of the two series is

$$c_n = 2a_2 \frac{1}{n!} + 6a_3 \frac{1}{(n-1)!} + 12a_4 \frac{1}{(n-2)!} + \dots \\ \dots + (n+1)n a_{n+1} + (n+2)(n+1)a_{n+2}.$$

Expanding the individual series, it follows that

$$2a_2 + (2a_2 + 6a_3)x + (a_2 + 6a_3 + 12a_4)x^2 + (a_2 + 6a_3 + 12a_4 + 20a_5)x^3 + \dots \\ \dots + a_0 x + a_1 x^2 + a_2 x^3 + \dots = 0.$$

Setting the coefficients equal to zero, we obtain the system $2a_2 = 0$, $2a_2 + 6a_3 + a_0 = 0$, $a_2 + 6a_3 + 12a_4 + a_1 = 0$, $a_2 + 6a_3 + 12a_4 + 20a_5 + a_2 = 0, \dots$. Hence the general solution is

$$y(x) = a_0 + a_1 x - a_0 \frac{x^3}{6} + (a_0 - a_1) \frac{x^4}{12} + (2a_1 - a_0) \frac{x^5}{40} + \left(\frac{4}{3}a_0 - 2a_1\right) \frac{x^6}{120} + \dots$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$y_1(x) = 1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \dots \\ y_2(x) = x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \dots$$

Since $p(x) = 0$ and $q(x) = xe^{-x}$ converge everywhere, $\rho = \infty$.

13. The Taylor series expansion of $\cos x$, about $x_0 = 0$, is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n \right] + \sum_{n=1}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

The coefficient of x^n in the product of the two series is

$$c_n = 2a_2 b_n + 6a_3 b_{n-1} + 12a_4 b_{n-2} + \dots + (n+1)n a_{n+1} b_1 + (n+2)(n+1)a_{n+2} b_0, \\ \text{in which } \cos x = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n + \dots. \text{ It follows that}$$

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty} (n-2)a_n x^n = 0.$$

Expanding the product of the series, it follows that

$$2a_2 - 2a_0 + 6a_3 x + (-a_2 + 12a_4)x^2 + (-3a_3 + 20a_5)x^3 + \dots \\ \dots - a_1 x + a_3 x^3 + 2a_4 x^4 + \dots = 0.$$

Setting the coefficients equal to zero, $a_2 - a_0 = 0$, $6a_3 - a_1 = 0$, $-a_2 + 12a_4 = 0$, $-3a_3 + 20a_5 + a_3 = 0$, \dots . Hence the general solution is

$$y(x) = a_0 + a_1x + a_0x^2 + a_1\frac{x^3}{6} + a_0\frac{x^4}{12} + a_1\frac{x^5}{60} + a_0\frac{x^6}{120} + a_1\frac{x^7}{560} + \dots$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$y_1(x) = 1 + x^2 + \frac{x^4}{12} + \frac{x^6}{120} + \dots$$

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \dots$$

The nearest zero of $P(x) = \cos x$ is at $x = \pm\pi/2$. Hence $\rho_{min} = \pi/2$.

14. The Taylor series expansion of $\ln(1+x)$, about $x_0 = 0$, is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substituting into the ODE,

$$\begin{aligned} & \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right] \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \\ & + \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} \right] \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - x \sum_{n=0}^{\infty} a_nx^n = 0. \end{aligned}$$

The first product is the series

$$2a_2 + (-2a_2 + 6a_3)x + (a_2 - 6a_3 + 12a_4)x^2 + (-a_2 + 6a_3 - 12a_4 + 20a_5)x^3 + \dots$$

The second product is the series

$$a_1x + (2a_2 - a_1/2)x^2 + (3a_3 - a_2 + a_1/3)x^3 + (4a_4 - 3a_3/2 + 2a_2/3 - a_1/4)x^4 + \dots$$

Combining the series and equating the coefficients to zero, we obtain

$$\begin{aligned} 2a_2 &= 0 \\ -2a_2 + 6a_3 + a_1 - a_0 &= 0 \\ 12a_4 - 6a_3 + 3a_2 - 3a_1/2 &= 0 \\ 20a_5 - 12a_4 + 9a_3 - 3a_2 + a_1/3 &= 0 \\ &\vdots \end{aligned}$$

Hence the general solution is

$$y(x) = a_0 + a_1x + (a_0 - a_1)\frac{x^3}{6} + (2a_0 + a_1)\frac{x^4}{24} + a_1\frac{7x^5}{120} + (\frac{5}{3}a_1 - a_0)\frac{x^6}{120} + \dots$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$y_1(x) = 1 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{120} + \dots$$

$$y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{24} + \frac{7x^5}{120} + \dots$$

The coefficient $p(x) = e^x \ln(1+x)$ is analytic at $x_0 = 0$, but its power series has a radius of convergence $\rho = 1$.

15. If $y_1 = x$ and $y_2 = x^2$ are solutions, then substituting y_2 into the ODE results in

$$2P(x) + 2xQ(x) + x^2R(x) = 0.$$

Setting $x = 0$, we find that $P(0) = 0$. Similarly, substituting y_1 into the ODE results in $Q(0) = 0$. Therefore $P(x)/Q(x)$ and $R(x)/P(x)$ may not be analytic. If they were, Theorem 3.2.1 would guarantee that y_1 and y_2 were the only two solutions. But note that an arbitrary value of $y(0)$ cannot be a linear combination of $y_1(0)$ and $y_2(0)$. Hence $x_0 = 0$ must be a singular point.

16. Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_nx^n = 0.$$

That is,

$$\sum_{n=0}^{\infty} [(n+1)a_{n+1} - a_n]x^n = 0.$$

Setting the coefficients equal to zero, we obtain

$$a_{n+1} = \frac{a_n}{n+1}$$

for $n = 0, 1, 2, \dots$. It is easy to see that $a_n = a_0/(n!)$. Therefore the general solution is

$$\begin{aligned} y(x) &= a_0 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right] \\ &= a_0 e^x. \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

17. Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - x \sum_{n=0}^{\infty} a_nx^n = 0.$$

That is,

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

Combining the series, we have

$$a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - a_{n-1}]x^n = 0.$$

Setting the coefficient equal to zero, $a_1 = 0$ and $a_{n+1} = a_{n-1}/(n+1)$ for $n = 1, 2, \dots$. Note that the indices differ by two, so for $k = 1, 2, \dots$

$$a_{2k} = \frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \dots = \frac{a_0}{2 \cdot 4 \dots (2k)}$$

and

$$a_{2k+1} = 0.$$

Hence the general solution is

$$y(x) = a_0 \left[1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \dots + \frac{x^{2n}}{2^n n!} + \dots \right] = a_0 e^{x^2/2}.$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

19. Let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$. Substituting into the ODE,

$$(1-x) \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Combining the series, we have

$$a_1 - a_0 + \sum_{n=1}^{\infty} [(n+1) a_{n+1} - n a_n - a_n] x^n = 0.$$

Setting the coefficients equal to zero, $a_1 = a_0$ and $a_{n+1} = a_n$ for $n = 0, 1, 2, \dots$. Hence the general solution is

$$y(x) = a_0 [1 + x + x^2 + x^3 + \dots + x^n + \dots] = a_0 \frac{1}{1-x}.$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + x \sum_{n=0}^{\infty} a_n x^n = 1 + x.$$

That is,

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 1 + x.$$

Combining the series, and the nonhomogeneous terms, we have

$$(a_1 - 1) + (2a_2 + a_0 - 1)x + \sum_{n=2}^{\infty} [(n+1) a_{n+1} + a_{n-1}] x^n = 0.$$

Setting the coefficients equal to zero, we obtain $a_1 = 1$, $2a_2 + a_0 - 1 = 0$, and

$$a_n = -\frac{a_{n-2}}{n}, \quad n = 3, 4, \dots$$

The indices differ by two, so for $k = 2, 3, \dots$

$$a_{2k} = -\frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \dots = \frac{(-1)^{k-1} a_2}{4 \cdot 6 \dots (2k)} = \frac{(-1)^k (a_0 - 1)}{2 \cdot 4 \cdot 6 \dots (2k)},$$

and for $k = 1, 2, \dots$

$$a_{2k+1} = -\frac{a_{2k-1}}{(2k+1)} = \frac{a_{2k-3}}{(2k-1)(2k+1)} = \dots = \frac{(-1)^k}{3 \cdot 5 \dots (2k+1)}.$$

Hence the general solution is

$$y(x) = a_0 + x + \frac{1-a_0}{2}x^2 - \frac{x^3}{3} + a_0 \frac{x^4}{2^2 2!} + \frac{x^5}{3 \cdot 5} - a_0 \frac{x^6}{2^3 3!} - \dots$$

Collecting the terms containing a_0 ,

$$y(x) = a_0 \left[1 - \frac{x^2}{2} + \frac{x^4}{2^2 2!} - \frac{x^6}{2^3 3!} + \dots \right] + \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2 2!} + \frac{x^5}{3 \cdot 5} + \frac{x^6}{2^3 3!} - \frac{x^7}{3 \cdot 5 \cdot 7} + \dots \right].$$

Upon inspection, we find that

$$y(x) = a_0 e^{-x^2/2} + \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2 2!} + \frac{x^5}{3 \cdot 5} + \frac{x^6}{2^3 3!} - \frac{x^7}{3 \cdot 5 \cdot 7} + \dots \right].$$

Note that the given ODE is first order linear, with integrating factor $\mu(x) = e^{x^2/2}$. The general solution is given by

$$y(x) = e^{-x^2/2} \int_0^x e^{u^2/2} du + (y(0) - 1)e^{-x^2/2} + 1.$$

23. If $\alpha = 0$, then $y_1(x) = 1$. If $\alpha = 2n$, then $a_{2m} = 0$ for $m \geq n+1$. As a result,

$$y_1(x) = 1 + \sum_{m=1}^n (-1)^m \frac{2^m n(n-1) \dots (n-m+1)(2n+1)(2n+3) \dots (2n+2m-1)}{(2m)!} x^{2m}.$$

$\alpha = 0$	1
$\alpha = 2$	$1 - 3x^2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$

If $\alpha = 2n+1$, then $a_{2m+1} = 0$ for $m \geq n+1$. As a result,

$$y_2(x) = x + \sum_{m=1}^n (-1)^m \frac{2^m n(n-1) \dots (n-m+1)(2n+3)(2n+5) \dots (2n+2m+1)}{(2m+1)!} x^{2m+1}.$$

$\alpha = 1$	x
$\alpha = 3$	$x - \frac{5}{3}x^3$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$

24.(a) Based on Problem 23,

$\alpha = 0$	1	$y_1(1) = 1$
$\alpha = 2$	$1 - 3x^2$	$y_1(1) = -2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$	$y_1(1) = \frac{8}{3}$

Normalizing the polynomials, we obtain

$$P_0(x) = 1$$

$$P_2(x) = -\frac{1}{2} + \frac{3}{2}x^2$$

$$P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$$

$\alpha = 1$	x	$y_2(1) = 1$
$\alpha = 3$	$x - \frac{5}{3}x^3$	$y_2(1) = -\frac{2}{3}$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$	$y_2(1) = \frac{8}{15}$

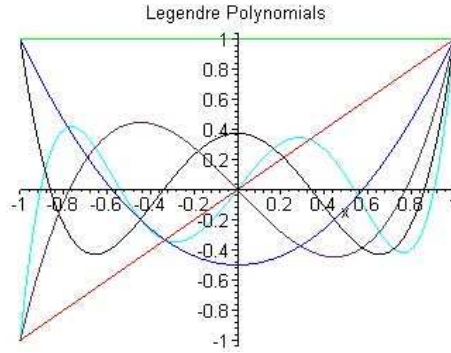
Similarly,

$$P_1(x) = x$$

$$P_3(x) = -\frac{3}{2}x + \frac{5}{2}x^3$$

$$P_5(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$$

(b)



(c) $P_0(x)$ has no roots. $P_1(x)$ has one root at $x = 0$. The zeros of $P_2(x)$ are at $x = \pm 1/\sqrt{3}$. The zeros of $P_3(x)$ are $x = 0, \pm\sqrt{3/5}$. The roots of $P_4(x)$ are given by $x^2 = (15 + 2\sqrt{30})/35, (15 - 2\sqrt{30})/35$. The roots of $P_5(x)$ are given by $x = 0$ and $x^2 = (35 + 2\sqrt{70})/63, (35 - 2\sqrt{70})/63$.

25. Observe that

$$P_n(-1) = \frac{(-1)^n}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} = (-1)^n P_n(1).$$

But $P_n(1) = 1$ for all nonnegative integers n .

27. We have

$$(x^2 - 1)^n = \sum_{k=0}^n \frac{(-1)^{n-k} n!}{k!(n-k)!} x^{2k},$$

which is a polynomial of degree $2n$. Differentiating n times,

$$\frac{d^n}{dx^n} (x^2 - 1)^n = \sum_{k=\mu}^n \frac{(-1)^{n-k} n!}{k!(n-k)!} (2k)(2k-1) \dots (2k-n+1) x^{2k-n},$$

in which the lower index is $\mu = [n/2] + 1$. Note that if $n = 2m + 1$, then $\mu = m + 1$. Now shift the index, by setting $k = n - j$. Hence

$$\begin{aligned} \frac{d^n}{dx^n} (x^2 - 1)^n &= \sum_{j=0}^{[n/2]} \frac{(-1)^j n!}{(n-j)!j!} (2n-2j)(2n-2j-1) \dots (n-2j+1) x^{n-2j} \\ &= n! \sum_{j=0}^{[n/2]} \frac{(-1)^j (2n-2j)!}{(n-j)!j!(n-2j)!} x^{n-2j}. \end{aligned}$$

Based on Problem 25,

$$\frac{d^n}{dx^n} (x^2 - 1)^n = n! 2^n P_n(x).$$

29. Since the $n + 1$ polynomials P_0, P_1, \dots, P_n are linearly independent, and the degree of P_k is k , any polynomial, f , of degree n can be expressed as a linear combination

$$f(x) = \sum_{k=0}^n a_k P_k(x).$$

Multiplying both sides by P_m and integrating,

$$\int_{-1}^1 f(x) P_m(x) dx = \sum_{k=0}^n a_k \int_{-1}^1 P_k(x) P_m(x) dx.$$

Based on Problem 28,

$$\int_{-1}^1 P_k(x) P_m(x) dx = \frac{2}{2m+1} \delta_{km}.$$

Hence

$$\int_{-1}^1 f(x) P_m(x) dx = \frac{2}{2m+1} a_m.$$

5.4

1. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r(r-1) + 4r + 2 = r^2 + 3r + 2.$$

The roots are $r = -2, -1$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x^{-2} + c_2 x^{-1}.$$

3. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r(r-1) - 3r + 4 = r^2 - 4r + 4.$$

The root is $r = 2$, with multiplicity two. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2 \ln |x|) x^2.$$

5. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r(r-1) - r + 1 = r^2 - 2r + 1.$$

The root is $r = 1$, with multiplicity two. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2 \ln |x|) x.$$

6. Substitution of $y = (x-1)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 7r + 12.$$

The roots are $r = -3, -4$. Hence the general solution, for $x \neq 1$, is

$$y = c_1 (x-1)^{-3} + c_2 (x-1)^{-4}.$$

7. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 5r - 1.$$

The roots are $r = -(5 \pm \sqrt{29})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{-(5+\sqrt{29})/2} + c_2 |x|^{-(5-\sqrt{29})/2}.$$

8. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 3r + 3.$$

The roots are complex, with $r = (3 \pm i\sqrt{3})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{3/2} \cos\left(\frac{\sqrt{3}}{2} \ln |x|\right) + c_2 |x|^{3/2} \sin\left(\frac{\sqrt{3}}{2} \ln |x|\right).$$

10. Substitution of $y = (x-2)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 4r + 8.$$

The roots are complex, with $r = -2 \pm 2i$. Hence the general solution, for $x \neq 2$, is

$$y = c_1 (x-2)^{-2} \cos(2 \ln |x-2|) + c_2 (x-2)^{-2} \sin(2 \ln |x-2|).$$

11. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + r + 4.$$

The roots are complex, with $r = -(1 \pm i\sqrt{15})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{-1/2} \cos\left(\frac{\sqrt{15}}{2} \ln |x|\right) + c_2 |x|^{-1/2} \sin\left(\frac{\sqrt{15}}{2} \ln |x|\right).$$

12. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 5r + 4.$$

The roots are $r = 1, 4$. Hence the general solution is

$$y = c_1 x + c_2 x^4.$$

14. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = 4r^2 + 4r + 17.$$

The roots are complex, with $r = -1/2 \pm 2i$. Hence the general solution, for $x > 0$, is

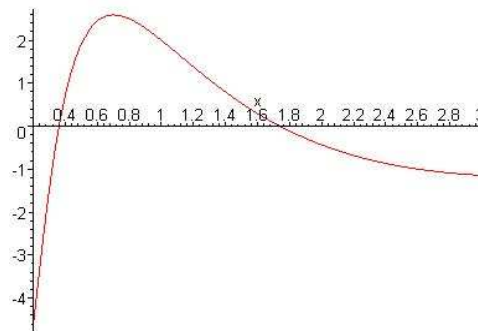
$$y = c_1 x^{-1/2} \cos(2 \ln x) + c_2 x^{-1/2} \sin(2 \ln x).$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 &= 2 \\ -\frac{1}{2}c_1 + 2c_2 &= -3 \end{aligned}$$

Hence the solution of the initial value problem is

$$y(x) = 2x^{-1/2} \cos(2 \ln x) - x^{-1/2} \sin(2 \ln x).$$



As $x \rightarrow 0^+$, the solution decreases without bound.

15. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 4r + 4.$$

The root is $r = 2$, with multiplicity two. Hence the general solution, for $x < 0$, is

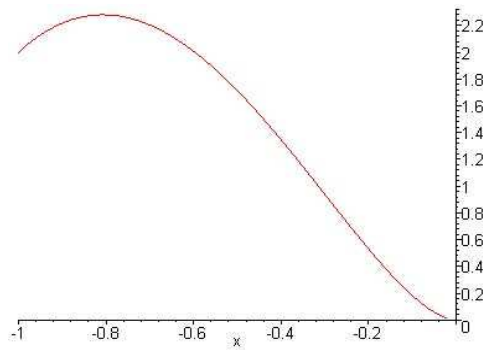
$$y = (c_1 + c_2 \ln |x|) x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 &= 2 \\ -2c_1 - c_2 &= 3 \end{aligned}$$

Hence the solution of the initial value problem is

$$y(x) = (2 - 7 \ln |x|) x^2.$$



We find that $y(x) \rightarrow 0$ as $x \rightarrow 0^-$.

18. We see that $P(x) = 0$ when $x = 0$ and 1 . Since the three coefficients have no factors in common, both of these points are singular points. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{2x}{x^2(1-x)^2} = 2.$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{4}{x^2(1-x)^2} = 4.$$

The singular point $x = 0$ is regular. Considering $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} (x-1) \frac{2x}{x^2(1-x)^2}.$$

The latter limit does not exist. Hence $x = 1$ is an irregular singular point.

19. $P(x) = 0$ when $x = 0$ and 1 . Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{x-2}{x^2(1-x)}.$$

The limit does not exist, and so $x = 0$ is an irregular singular point. Considering $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} (x-1) \frac{x-2}{x^2(1-x)} = 1.$$

$$\lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} (x-1)^2 \frac{-3x}{x^2(1-x)} = 0.$$

Hence $x = 1$ is a regular singular point.

20. $P(x) = 0$ when $x = 0$ and ± 1 . Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{2}{x^3(1-x^2)}.$$

The limit does not exist, and so $x = 0$ is an irregular singular point. Near $x = -1$,

$$\lim_{x \rightarrow -1} (x+1)p(x) = \lim_{x \rightarrow -1} (x+1) \frac{2}{x^3(1-x^2)} = -1.$$

$$\lim_{x \rightarrow -1} (x+1)^2 q(x) = \lim_{x \rightarrow -1} (x+1)^2 \frac{2}{x^3(1-x^2)} = 0.$$

Hence $x = -1$ is a regular singular point. At $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} (x-1) \frac{2}{x^3(1-x^2)} = -1.$$

$$\lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} (x-1)^2 \frac{2}{x^3(1-x^2)} = 0.$$

Hence $x = 1$ is a regular singular point.

22. The only singular point is at $x = 0$. We find that

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{x}{x^2} = 1.$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{x^2 - \nu^2}{x^2} = -\nu^2.$$

Hence $x = 0$ is a regular singular point.

23. The only singular point is at $x = -3$. We find that

$$\lim_{x \rightarrow -3} (x+3)p(x) = \lim_{x \rightarrow -3} (x+3) \frac{-2x}{x+3} = 6.$$

$$\lim_{x \rightarrow -3} (x+3)^2 q(x) = \lim_{x \rightarrow -3} (x+3)^2 \frac{1-x^2}{x+3} = 0.$$

Hence $x = -3$ is a regular singular point.

24. Dividing the ODE by $x(1-x^2)^3$, we find that

$$p(x) = \frac{1}{x(1-x^2)} \quad \text{and} \quad q(x) = \frac{2}{x(1+x)^2(1-x)^3}.$$

The singular points are at $x = 0$ and ± 1 . For $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{1}{x(1-x^2)} = 1.$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{2}{x(1+x)^2(1-x)^3} = 0.$$

Hence $x = 0$ is a regular singular point. For $x = -1$,

$$\lim_{x \rightarrow -1} (x+1)p(x) = \lim_{x \rightarrow -1} (x+1) \frac{1}{x(1-x^2)} = -\frac{1}{2}.$$

$$\lim_{x \rightarrow -1} (x+1)^2 q(x) = \lim_{x \rightarrow -1} (x+1)^2 \frac{2}{x(1+x)^2(1-x)^3} = -\frac{1}{4}.$$

Hence $x = -1$ is a regular singular point. For $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} (x-1) \frac{1}{x(1-x^2)} = -\frac{1}{2}.$$

$$\lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} (x-1)^2 \frac{2}{x(1+x)^2(1-x)^3}.$$

The latter limit does not exist. Hence $x = 1$ is an irregular singular point.

25. Dividing the ODE by $(x+2)^2(x-1)$, we find that

$$p(x) = \frac{3}{(x+2)^2} \quad \text{and} \quad q(x) = \frac{-2}{(x+2)(x-1)}.$$

The singular points are at $x = -2$ and 1 . For $x = -2$,

$$\lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} (x+2) \frac{3}{(x+2)^2}.$$

The limit does not exist. Hence $x = -2$ is an irregular singular point. For $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} (x-1) \frac{3}{(x+2)^2} = 0.$$

$$\lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} (x-1)^2 \frac{-2}{(x+2)(x-1)} = 0.$$

Hence $x = 1$ is a regular singular point.

26. $P(x) = 0$ when $x = 0$ and 3 . Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{x+1}{x(3-x)} = \frac{1}{3}.$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{-2}{x(3-x)} = 0.$$

Hence $x = 0$ is a regular singular point. For $x = 3$,

$$\lim_{x \rightarrow 3} (x-3)p(x) = \lim_{x \rightarrow 3} (x-3) \frac{x+1}{x(3-x)} = -\frac{4}{3}.$$

$$\lim_{x \rightarrow 3} (x-3)^2 q(x) = \lim_{x \rightarrow 3} (x-3)^2 \frac{-2}{x(3-x)} = 0.$$

Hence $x = 3$ is a regular singular point.

27. Dividing the ODE by $(x^2 + x - 2)$, we find that

$$p(x) = \frac{x+1}{(x+2)(x-1)} \quad \text{and} \quad q(x) = \frac{2}{(x+2)(x-1)}.$$

The singular points are at $x = -2$ and 1 . For $x = -2$,

$$\lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} \frac{x+1}{x-1} = \frac{1}{3}.$$

$$\lim_{x \rightarrow -2} (x+2)^2 q(x) = \lim_{x \rightarrow -2} \frac{2(x+2)}{x-1} = 0.$$

Hence $x = -2$ is a regular singular point. For $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} \frac{x+1}{x+2} = \frac{2}{3}.$$

$$\lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} \frac{2(x-1)}{(x+2)} = 0.$$

Hence $x = 1$ is a regular singular point.

29. Note that $p(x) = \ln|x|$ and $q(x) = 3x$. Evidently, $p(x)$ is not analytic at $x_0 = 0$. Furthermore, the function $x p(x) = x \ln|x|$ does not have a Taylor series about $x_0 = 0$. Hence $x = 0$ is an irregular singular point.

30. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. The Taylor series of $e^x - 1$, about $x = 0$, is

$$e^x - 1 = x + x^2/2 + x^3/6 + \dots$$

Hence the function $x p(x) = 2(e^x - 1)/x$ is analytic at $x = 0$. Similarly, the Taylor series of $e^{-x} \cos x$, about $x = 0$, is

$$e^{-x} \cos x = 1 - x + x^3/3 - x^4/6 + \dots$$

The function $x^2 q(x) = e^{-x} \cos x$ is also analytic at $x = 0$. Hence $x = 0$ is a regular singular point.

31. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. The Taylor series of $\sin x$, about $x = 0$, is

$$\sin x = x - x^3/3! + x^5/5! - \dots$$

Hence the function $x p(x) = -3 \sin x/x$ is analytic at $x = 0$. On the other hand, $q(x)$ is a rational function, with

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{1+x^2}{x^2} = 1.$$

Hence $x = 0$ is a regular singular point.

32. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. We find that

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{1}{x} = 1.$$

Although the function $R(x) = \cot x$ does not have a Taylor series about $x = 0$, note that $x^2 q(x) = x \cot x = 1 - x^2/3 - x^4/45 - 2x^6/945 - \dots$. Hence $x = 0$ is a regular singular point. Furthermore, $q(x) = \cot x/x^2$ is undefined at $x = \pm n\pi$. Therefore the points $x = \pm n\pi$ are also singular points. First note that

$$\lim_{x \rightarrow \pm n\pi} (x \mp n\pi)p(x) = \lim_{x \rightarrow \pm n\pi} (x \mp n\pi) \frac{1}{x} = 0.$$

Furthermore, since $\cot x$ has period π ,

$$q(x) = \cot x/x = \cot(x \mp n\pi)/x = \cot(x \mp n\pi) \frac{1}{(x \mp n\pi) \pm n\pi}.$$

Therefore

$$(x \mp n\pi)^2 q(x) = (x \mp n\pi) \cot(x \mp n\pi) \left[\frac{(x \mp n\pi)}{(x \mp n\pi) \pm n\pi} \right].$$

From above,

$$(x \mp n\pi) \cot(x \mp n\pi) = 1 - (x \mp n\pi)^2/3 - (x \mp n\pi)^4/45 - \dots$$

Note that the function in brackets is analytic near $x = \pm n\pi$. It follows that the function $(x \mp n\pi)^2 q(x)$ is also analytic near $x = \pm n\pi$. Hence all the singular points are regular.

34. The singular points are located at $x = \pm n\pi$, $n = 0, 1, \dots$. Dividing the ODE by $x \sin x$, we find that $x p(x) = 3 \csc x$ and $x^2 q(x) = x^2 \csc x$. Evidently, $x p(x)$ is not even defined at $x = 0$. Hence $x = 0$ is an irregular singular point. On the other hand, the Taylor series of $x \csc x$, about $x = 0$, is

$$x \csc x = 1 + x^2/6 + 7x^4/360 + \dots$$

Noting that $\csc(x \mp n\pi) = (-1)^n \csc x$,

$$\begin{aligned} (x \mp n\pi)p(x) &= 3(-1)^n (x \mp n\pi) \csc(x \mp n\pi)/x \\ &= 3(-1)^n (x \mp n\pi) \csc(x \mp n\pi) \left[\frac{1}{(x \mp n\pi) \pm n\pi} \right]. \end{aligned}$$

It is apparent that $(x \mp n\pi)p(x)$ is analytic at $x = \pm n\pi$. Similarly,

$$(x \mp n\pi)^2 q(x) = (x \mp n\pi)^2 \csc x = (-1)^n (x \mp n\pi)^2 \csc(x \mp n\pi),$$

which is also analytic at $x = \pm n\pi$. Hence all other singular points are regular.

36. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r + \beta = 0$. The roots are

$$r = \frac{1 \pm \sqrt{1 - 4\beta}}{2}.$$

If $\beta > 1/4$, the roots are complex, with $r_{1,2} = (1 \pm i\sqrt{4\beta - 1})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2} \cos\left(\frac{1}{2}\sqrt{4\beta - 1} \ln|x|\right) + c_2 |x|^{1/2} \sin\left(\frac{1}{2}\sqrt{4\beta - 1} \ln|x|\right).$$

Since the trigonometric factors are bounded, $y(x) \rightarrow 0$ as $x \rightarrow 0$. If $\beta = 1/4$, the roots are equal, and

$$y = c_1 |x|^{1/2} + c_2 |x|^{1/2} \ln |x|.$$

Since $\lim_{x \rightarrow 0} \sqrt{|x|} \ln |x| = 0$, $y(x) \rightarrow 0$ as $x \rightarrow 0$. If $\beta < 1/4$, the roots are real, with $r_{1,2} = (1 \pm \sqrt{1-4\beta})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2+\sqrt{1-4\beta}/2} + c_2 |x|^{1/2-\sqrt{1-4\beta}/2}.$$

Evidently, solutions approach zero as long as $1/2 - \sqrt{1-4\beta}/2 > 0$. That is,

$$0 < \beta < 1/4.$$

Hence all solutions approach zero for $\beta > 0$.

37. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r - 2 = 0$. The roots are $r = -1, 2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x^{-1} + c_2 x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 + 2c_2 &= \gamma \end{aligned}$$

Hence the solution of the initial value problem is

$$y(x) = \frac{2-\gamma}{3} x^{-1} + \frac{1+\gamma}{3} x^2.$$

The solution is bounded, as $x \rightarrow 0$, if $\gamma = 2$.

38. Substitution of $y = x^r$ results in the quadratic equation $r^2 + (\alpha - 1)r + 5/2 = 0$. Formally, the roots are given by

$$r = \frac{1 - \alpha \pm \sqrt{\alpha^2 - 2\alpha - 9}}{2} = \frac{1 - \alpha \pm \sqrt{(\alpha - 1 - \sqrt{10})(\alpha - 1 + \sqrt{10})}}{2}.$$

(i) The roots $r_{1,2}$ will be complex, if $|1 - \alpha| < \sqrt{10}$. For solutions to approach zero, as $x \rightarrow \infty$, we need $-\sqrt{10} < 1 - \alpha < 0$.

(ii) The roots will be equal, if $|1 - \alpha| = \sqrt{10}$. In this case, all solutions approach zero as long as $1 - \alpha = -\sqrt{10}$.

(iii) The roots will be real and distinct, if $|1 - \alpha| > \sqrt{10}$. It follows that

$$r_{max} = \frac{1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9}}{2}.$$

For solutions to approach zero, we need $1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9} < 0$. That is, $1 - \alpha < -\sqrt{10}$. Hence all solutions approach zero, as $x \rightarrow \infty$, as long as $\alpha > 1$.

42. $x = 0$ is the only singular point. Dividing the ODE by $2x^2$, we have $p(x) = 3/(2x)$ and $q(x) = -x^{-2}(1+x)/2$. It follows that

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{3}{2x} = \frac{3}{2},$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{-(1+x)}{2x^2} = -\frac{1}{2},$$

so $x = 0$ is a regular singular point. Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substitution into the ODE results in

$$2x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + 3x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - (1+x) \sum_{n=0}^{\infty} a_nx^n = 0.$$

That is,

$$2 \sum_{n=2}^{\infty} n(n-1)a_nx^n + 3 \sum_{n=1}^{\infty} na_nx^n - \sum_{n=0}^{\infty} a_nx^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

It follows that

$$-a_0 + (2a_1 - a_0)x + \sum_{n=2}^{\infty} [2n(n-1)a_n + 3na_n - a_n - a_{n-1}]x^n = 0.$$

Equating the coefficients to zero, we find that $a_0 = 0$, $2a_1 - a_0 = 0$, and

$$(2n-1)(n+1)a_n = a_{n-1}, \quad n = 2, 3, \dots$$

We conclude that all the a_n are equal to zero. Hence $y(x) = 0$ is the only solution that can be obtained.

44. Based on Problem 43, the change of variable, $x = 1/\xi$, transforms the ODE into the form

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + y = 0.$$

Evidently, $\xi = 0$ is a singular point. Now $p(\xi) = 2/\xi$ and $q(\xi) = 1/\xi^4$. Since the value of $\lim_{\xi \rightarrow 0} \xi^2 q(\xi)$ does not exist, $\xi = 0$, that is, $x = \infty$, is an irregular singular point.

46. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \left(1 - \frac{1}{\xi^2}\right) \frac{d^2y}{d\xi^2} + \left[2\xi^3 \left(1 - \frac{1}{\xi^2}\right) + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0,$$

that is,

$$(\xi^4 - \xi^2) \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2\xi}{\xi^2 - 1} \quad \text{and} \quad q(\xi) = \frac{\alpha(\alpha+1)}{\xi^2(\xi^2 - 1)}.$$

It follows that

$$\lim_{\xi \rightarrow 0} \xi p(\xi) = \lim_{\xi \rightarrow 0} \xi \frac{2\xi}{\xi^2 - 1} = 0,$$

$$\lim_{\xi \rightarrow 0} \xi^2 q(\xi) = \lim_{\xi \rightarrow 0} \xi^2 \frac{\alpha(\alpha + 1)}{\xi^2(\xi^2 - 1)} = -\alpha(\alpha + 1).$$

Hence $\xi = 0$ ($x = \infty$) is a regular singular point.

48. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2 y}{d\xi^2} + \left[2\xi^3 + 2\xi^2 \frac{1}{\xi} \right] \frac{dy}{d\xi} + \lambda y = 0,$$

that is,

$$\xi^4 \frac{d^2 y}{d\xi^2} + 2(\xi^3 + \xi) \frac{dy}{d\xi} + \lambda y = 0.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2(\xi^2 + 1)}{\xi^3} \text{ and } q(\xi) = \frac{\lambda}{\xi^4}.$$

It immediately follows that the limit $\lim_{\xi \rightarrow 0} \xi p(\xi)$ does not exist. Hence $\xi = 0$ ($x = \infty$) is an irregular singular point.

49. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2 y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} - \frac{1}{\xi} y = 0.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2}{\xi} \text{ and } q(\xi) = \frac{-1}{\xi^5}.$$

We find that

$$\lim_{\xi \rightarrow 0} \xi p(\xi) = \lim_{\xi \rightarrow 0} \xi \frac{2}{\xi} = 2,$$

but

$$\lim_{\xi \rightarrow 0} \xi^2 q(\xi) = \lim_{\xi \rightarrow 0} \xi^2 \frac{(-1)}{\xi^5}.$$

The latter limit does not exist. Hence $\xi = 0$ ($x = \infty$) is an irregular singular point.

5.5

1.(a) $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{1}{2x} = \frac{1}{2}.$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{1}{2} = 0.$$

Hence $x = 0$ is a regular singular point.

(b) Let

$$y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots) = \sum_{n=0}^{\infty} a_n x^{r+n}.$$

Then

$$y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2}.$$

Substitution into the ODE results in

$$\begin{aligned} 2 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0. \end{aligned}$$

That is,

$$2 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} = 0.$$

It follows that

$$\begin{aligned} a_0 [2r(r-1) + r] x^r + a_1 [2(r+1)r + r + 1] x^{r+1} + \\ + \sum_{n=2}^{\infty} [2(r+n)(r+n-1)a_n + (r+n)a_n + a_{n-2}] x^{r+n} = 0. \end{aligned}$$

Assuming that $a_0 \neq 0$, we obtain the indicial equation $2r^2 - r = 0$, with roots $r_1 = 1/2$ and $r_2 = 0$. It immediately follows that $a_1 = 0$. Setting the remaining coefficients equal to zero, we have

$$a_n = \frac{-a_{n-2}}{(r+n)[2(r+n)-1]}, \quad n = 2, 3, \dots$$

(c) For $r = 1/2$, the recurrence relation becomes

$$a_n = \frac{-a_{n-2}}{n(1+2n)}, \quad n = 2, 3, \dots$$

Since $a_1 = 0$, the odd coefficients are zero. Furthermore, for $k = 1, 2, \dots$,

$$a_{2k} = \frac{-a_{2k-2}}{2k(1+4k)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-3)(4k+1)} = \frac{(-1)^k a_0}{2^k k! 5 \cdot 9 \cdot 13 \dots (4k+1)}.$$

(d) For $r = 0$, the recurrence relation becomes

$$a_n = \frac{-a_{n-2}}{n(2n-1)}, \quad n = 2, 3, \dots$$

Since $a_1 = 0$, the odd coefficients are zero, and for $k = 1, 2, \dots$,

$$a_{2k} = \frac{-a_{2k-2}}{2k(4k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-5)(4k-1)} = \frac{(-1)^k a_0}{2^k k! 3 \cdot 7 \cdot 11 \dots (4k-1)}.$$

The two linearly independent solutions are

$$y_1(x) = \sqrt{x} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! 5 \cdot 9 \cdot 13 \dots (4k+1)} \right]$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! 3 \cdot 7 \cdot 11 \dots (4k-1)}.$$

3.(a) Note that $x p(x) = 0$ and $x^2 q(x) = x$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0,$$

and after multiplying both sides of the equation by x ,

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0.$$

It follows that

$$a_0 [r(r-1)] x^r + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + a_{n-1}] x^{r+n} = 0.$$

Setting the coefficients equal to zero, the indicial equation is $r(r-1) = 0$. The roots are $r_1 = 1$ and $r_2 = 0$. Here $r_1 - r_2 = 1$. The recurrence relation is

$$a_n = \frac{-a_{n-1}}{(r+n)(r+n-1)}, \quad n = 1, 2, \dots$$

(c) For $r = 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)}, \quad n = 1, 2, \dots$$

Hence for $n \geq 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \dots = \frac{(-1)^n a_0}{n!(n+1)!}.$$

Therefore one solution is

$$y_1(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+1)!}.$$

5.(a) Here $x p(x) = 2/3$ and $x^2 q(x) = x^2/3$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$. Substitution into the ODE results in

$$3 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + 2 \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0.$$

It follows that

$$\begin{aligned} & a_0 [3r(r-1) + 2r] x^r + a_1 [3(r+1)r + 2(r+1)] x^{r+1} + \\ & + \sum_{n=2}^{\infty} [3(r+n)(r+n-1)a_n + 2(r+n)a_n + a_{n-2}] x^{r+n} = 0. \end{aligned}$$

Assuming $a_0 \neq 0$, the indicial equation is $3r^2 - r = 0$, with roots $r_1 = 1/3$, $r_2 = 0$. Setting the remaining coefficients equal to zero, we have $a_1 = 0$, and

$$a_n = \frac{-a_{n-2}}{(r+n)[3(r+n)-1]}, \quad n = 2, 3, \dots$$

It immediately follows that the odd coefficients are equal to zero.

(c) For $r = 1/3$,

$$a_n = \frac{-a_{n-2}}{n(1+3n)}, \quad n = 2, 3, \dots$$

So for $k = 1, 2, \dots$,

$$a_{2k} = \frac{-a_{2k-2}}{2k(6k+1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-5)(6k+1)} = \frac{(-1)^k a_0}{2^k k! 7 \cdot 13 \dots (6k+1)}.$$

(d) For $r = 0$,

$$a_n = \frac{-a_{n-2}}{n(3n-1)}, \quad n = 2, 3, \dots$$

So for $k = 1, 2, \dots$,

$$a_{2k} = \frac{-a_{2k-2}}{2k(6k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-7)(6k-1)} = \frac{(-1)^k a_0}{2^k k! 5 \cdot 11 \dots (6k-1)}.$$

The two linearly independent solutions are

$$\begin{aligned} y_1(x) &= x^{1/3} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k! 7 \cdot 13 \dots (6k+1)} \left(\frac{x^2}{2}\right)^k \right] \\ y_2(x) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k! 5 \cdot 11 \dots (6k-1)} \left(\frac{x^2}{2}\right)^k. \end{aligned}$$

6.(a) Note that $x p(x) = 1$ and $x^2 q(x) = x - 2$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$. Substitution into the ODE results in

$$\begin{aligned} \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+1} - 2 \sum_{n=0}^{\infty} a_n x^{r+n} = 0. \end{aligned}$$

After adjusting the indices in the second-to-last series, we obtain

$$\begin{aligned} a_0 [r(r-1) + r - 2] x^r + \\ + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n - 2a_n + a_{n-1}] x^{r+n} = 0. \end{aligned}$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 - 2 = 0$, with roots $r = \pm\sqrt{2}$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{-a_{n-1}}{(r+n)^2 - 2}, \quad n = 1, 2, \dots$$

Note that $(r+n)^2 - 2 = (r+n+\sqrt{2})(r+n-\sqrt{2})$.

(c) For $r = \sqrt{2}$,

$$a_n = \frac{-a_{n-1}}{n(n+2\sqrt{2})}, \quad n = 1, 2, \dots$$

It follows that

$$a_n = \frac{(-1)^n a_0}{n!(1+2\sqrt{2})(2+2\sqrt{2}) \dots (n+2\sqrt{2})}, \quad n = 1, 2, \dots$$

(d) For $r = -\sqrt{2}$,

$$a_n = \frac{-a_{n-1}}{n(n-2\sqrt{2})}, \quad n = 1, 2, \dots,$$

and therefore

$$a_n = \frac{(-1)^n a_0}{n!(1-2\sqrt{2})(2-2\sqrt{2}) \dots (n-2\sqrt{2})}, \quad n = 1, 2, \dots$$

The two linearly independent solutions are

$$\begin{aligned} y_1(x) &= x^{\sqrt{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!(1+2\sqrt{2})(2+2\sqrt{2}) \dots (n+2\sqrt{2})} \right] \\ y_2(x) &= x^{-\sqrt{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!(1-2\sqrt{2})(2-2\sqrt{2}) \dots (n-2\sqrt{2})} \right]. \end{aligned}$$

7.(a) Here $x p(x) = 1 - x$ and $x^2 q(x) = -x$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$. Substitution into the ODE results in

$$\begin{aligned} \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} - \\ - \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0. \end{aligned}$$

After multiplying both sides by x ,

$$\begin{aligned} \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} - \\ - \sum_{n=0}^{\infty} (r+n)a_n x^{r+n+1} - \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0. \end{aligned}$$

After adjusting the indices in the last two series, we obtain

$$\begin{aligned} a_0 [r(r-1) + r] x^r + \\ + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n - (r+n)a_{n-1}] x^{r+n} = 0. \end{aligned}$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 = 0$, with roots $r_1 = r_2 = 0$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{a_{n-1}}{r+n}, \quad n = 1, 2, \dots$$

(c) With $r = 0$,

$$a_n = \frac{a_{n-1}}{n}, \quad n = 1, 2, \dots$$

Hence one solution is

$$y_1(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = e^x.$$

8.(a) Note that $xp(x) = 3/2$ and $x^2q(x) = x^2 - 1/2$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$. Substitution into the ODE results in

$$\begin{aligned} 2 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + 3 \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \\ + 2 \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0. \end{aligned}$$

After adjusting the indices in the second-to-last series, we obtain

$$a_0 [2r(r-1) + 3r - 1] x^r + a_1 [2(r+1)r + 3(r+1) - 1] + \\ + \sum_{n=2}^{\infty} [2(r+n)(r+n-1)a_n + 3(r+n)a_n - a_n + 2a_{n-2}] x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $2r^2 + r - 1 = 0$, with roots $r_1 = 1/2$ and $r_2 = -1$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{-2a_{n-2}}{(r+n+1)[2(r+n)-1]}, \quad n = 2, 3, \dots$$

Setting the remaining coefficients equal to zero, we have $a_1 = 0$, which implies that all of the odd coefficients are zero.

(c) With $r = 1/2$,

$$a_n = \frac{-2a_{n-2}}{n(2n+3)}, \quad n = 2, 3, \dots$$

So for $k = 1, 2, \dots$,

$$a_{2k} = \frac{-a_{2k-2}}{k(4k+3)} = \frac{a_{2k-4}}{(k-1)k(4k-5)(4k+3)} = \frac{(-1)^k a_0}{k! 7 \cdot 11 \dots (4k+3)}.$$

(d) With $r = -1$,

$$a_n = \frac{-2a_{n-2}}{n(2n-3)}, \quad n = 2, 3, \dots$$

So for $k = 1, 2, \dots$,

$$a_{2k} = \frac{-a_{2k-2}}{k(4k-3)} = \frac{a_{2k-4}}{(k-1)k(4k-11)(4k-3)} = \frac{(-1)^k a_0}{k! 5 \cdot 9 \dots (4k-3)}.$$

The two linearly independent solutions are

$$y_1(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n! 7 \cdot 11 \dots (4n+3)} \right] \\ y_2(x) = x^{-1} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n! 5 \cdot 9 \dots (4n-3)} \right].$$

9.(a) Note that $x p(x) = -x - 3$ and $x^2 q(x) = x + 3$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} - \sum_{n=0}^{\infty} (r+n)a_n x^{r+n+1} - 3 \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} \\ + \sum_{n=0}^{\infty} a_n x^{r+n+1} + 3 \sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

After adjusting the indices in the second-to-last series, we obtain

$$a_0 [r(r-1) - 3r + 3] x^r + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n - (r+n-2)a_{n-1} - 3(r+n-1)a_n] x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 - 4r + 3 = 0$, with roots $r_1 = 3$ and $r_2 = 1$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{(r+n-2)a_{n-1}}{(r+n-1)(r+n-3)}, \quad n = 1, 2, \dots$$

(c) With $r = 3$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)}, \quad n = 1, 2, \dots$$

It follows that for $n \geq 1$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)} = \frac{a_{n-2}}{(n-1)(n+2)} = \dots = \frac{2a_0}{n!(n+2)}.$$

Therefore one solution is

$$y_1(x) = x^3 \left[1 + \sum_{n=1}^{\infty} \frac{2x^n}{n!(n+2)} \right].$$

10.(a) Here $x p(x) = 0$ and $x^2 q(x) = x^2 + 1/4$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} + \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

After adjusting the indices in the second series, we obtain

$$a_0 \left[r(r-1) + \frac{1}{4} \right] x^r + a_1 \left[(r+1)r + \frac{1}{4} \right] x^{r+1} + \sum_{n=2}^{\infty} \left[(r+n)(r+n-1)a_n + \frac{1}{4}a_n + a_{n-2} \right] x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 - r + \frac{1}{4} = 0$, with roots $r_1 = r_2 = 1/2$. Setting the remaining coefficients equal to zero, we find that $a_1 = 0$. The recurrence relation is

$$a_n = \frac{-4a_{n-2}}{(2r+2n-1)^2}, \quad n = 2, 3, \dots$$

(c) With $r = 1/2$,

$$a_n = \frac{-a_{n-2}}{n^2}, \quad n = 2, 3, \dots$$

Since $a_1 = 0$, the odd coefficients are zero. So for $k \geq 1$,

$$a_{2k} = \frac{-a_{2k-2}}{4k^2} = \frac{a_{2k-4}}{4^2(k-1)^2k^2} = \cdots = \frac{(-1)^k a_0}{4^k(k!)^2}.$$

Therefore one solution is

$$y_1(x) = \sqrt{x} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \right].$$

12.(a) Dividing through by the leading coefficient, the ODE can be written as

$$y'' - \frac{x}{1-x^2} y' + \frac{\alpha^2}{1-x^2} y = 0.$$

For $x = 1$,

$$p_0 = \lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} \frac{x}{x+1} = \frac{1}{2}.$$

$$q_0 = \lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} \frac{\alpha^2(1-x)}{x+1} = 0.$$

For $x = -1$,

$$p_0 = \lim_{x \rightarrow -1} (x+1)p(x) = \lim_{x \rightarrow -1} \frac{x}{x-1} = \frac{1}{2}.$$

$$q_0 = \lim_{x \rightarrow -1} (x+1)^2 q(x) = \lim_{x \rightarrow -1} \frac{\alpha^2(x+1)}{(1-x)} = 0.$$

Hence $x = -1$ and $x = 1$ are regular singular points. As shown in Example 1, the indicial equation is given by

$$r(r-1) + p_0 r + q_0 = 0.$$

In this case, both sets of roots are $r_1 = 1/2$ and $r_2 = 0$.

(b) Let $t = x - 1$, and $u(t) = y(t+1)$. Under this change of variable, the differential equation becomes

$$(t^2 + 2t)u'' + (t+1)u' - \alpha^2 u = 0.$$

Based on part (a), $t = 0$ is a regular singular point. Set $u = \sum_{n=0}^{\infty} a_n t^{r+n}$. Substitution into the ODE results in

$$\begin{aligned} & \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n t^{r+n} + 2 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n t^{r+n-1} + \\ & + \sum_{n=0}^{\infty} (r+n)a_n t^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n t^{r+n-1} - \alpha^2 \sum_{n=0}^{\infty} a_n t^{r+n} = 0. \end{aligned}$$

Upon inspection, we can also write

$$\sum_{n=0}^{\infty} (r+n)^2 a_n t^{r+n} + 2 \sum_{n=0}^{\infty} (r+n)(r+n-\frac{1}{2})a_n t^{r+n-1} - \alpha^2 \sum_{n=0}^{\infty} a_n t^{r+n} = 0.$$

After adjusting the indices in the second series, it follows that

$$a_0 \left[2r(r - \frac{1}{2}) \right] t^{r-1} + \sum_{n=0}^{\infty} \left[(r+n)^2 a_n + 2(r+n+1)(r+n+\frac{1}{2}) a_{n+1} - \alpha^2 a_n \right] t^{r+n} = 0.$$

Assuming that $a_0 \neq 0$, the indicial equation is $2r^2 - r = 0$, with roots $r = 0, 1/2$. The recurrence relation is

$$(r+n)^2 a_n + 2(r+n+1)(r+n+\frac{1}{2}) a_{n+1} - \alpha^2 a_n = 0, \quad n = 0, 1, 2, \dots$$

With $r_1 = 1/2$, we find that for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{4\alpha^2 - (2n-1)^2}{4n(2n+1)} a_{n-1} \\ &= (-1)^n \frac{[1-4\alpha^2][9-4\alpha^2] \dots [(2n-1)^2-4\alpha^2]}{2^n(2n+1)!} a_0. \end{aligned}$$

With $r_2 = 0$, we find that for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{\alpha^2 - (n-1)^2}{n(2n-1)} a_{n-1} \\ &= (-1)^n \frac{\alpha(-\alpha)[1-\alpha^2][4-\alpha^2] \dots [(n-1)^2-\alpha^2]}{n! \cdot 3 \cdot 5 \dots (2n-1)} a_0. \end{aligned}$$

The two linearly independent solutions of the Chebyshev equation are

$$\begin{aligned} y_1(x) &= |x-1|^{1/2} \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{(1-4\alpha^2)(9-4\alpha^2) \dots ((2n-1)^2-4\alpha^2)}{2^n(2n+1)!} (x-1)^n \right) \\ y_2(x) &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\alpha(-\alpha)(1-\alpha^2)(4-\alpha^2) \dots ((n-1)^2-\alpha^2)}{n! \cdot 3 \cdot 5 \dots (2n-1)} (x-1)^n. \end{aligned}$$

13.(a) Here $xp(x) = 1-x$ and $x^2q(x) = \lambda x$, which are both analytic at $x = 0$. In fact,

$$p_0 = \lim_{x \rightarrow 0} xp(x) = 1 \text{ and } q_0 = \lim_{x \rightarrow 0} x^2q(x) = 0.$$

(b) The indicial equation is $r(r-1) + r = 0$, with roots $r_{1,2} = 0$.

(c) Set

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

Substitution into the ODE results in

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^{n-1} - \\ - \sum_{n=0}^{\infty} na_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0. \end{aligned}$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} na_nx^n + \lambda \sum_{n=0}^{\infty} a_nx^n = 0.$$

It follows that

$$a_1 + \lambda a_0 + \sum_{n=1}^{\infty} [(n+1)^2 a_{n+1} - (n-\lambda)a_n] x^n = 0.$$

Setting the coefficients equal to zero, we find that $a_1 = -\lambda a_0$, and

$$a_n = \frac{(n-1-\lambda)}{n^2} a_{n-1}, \quad n = 2, 3, \dots$$

That is, for $n \geq 2$,

$$a_n = \frac{(n-1-\lambda)}{n^2} a_{n-1} = \dots = \frac{(-\lambda)(1-\lambda)\dots(n-1-\lambda)}{(n!)^2} a_0.$$

Therefore one solution of the Laguerre equation is

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)(1-\lambda)\dots(n-1-\lambda)}{(n!)^2} x^n.$$

Note that if $\lambda = m$, a positive integer, then $a_n = 0$ for $n \geq m+1$. In that case, the solution is a polynomial

$$y_1(x) = 1 + \sum_{n=1}^m \frac{(-\lambda)(1-\lambda)\dots(n-1-\lambda)}{(n!)^2} x^n.$$

5.6

2.(a) $P(x) = 0$ only for $x = 0$. Furthermore, $xp(x) = -2 - x$ and $x^2q(x) = 2 + x^2$. It follows that

$$p_0 = \lim_{x \rightarrow 0} (-2 - x) = -2$$

$$q_0 = \lim_{x \rightarrow 0} (2 + x^2) = 2$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) - 2r + 2 = 0,$$

that is, $r^2 - 3r + 2 = 0$, with roots $r_1 = 2$ and $r_2 = 1$.

4. The coefficients $P(x)$, $Q(x)$, and $R(x)$ are analytic for all $x \in \mathbb{R}$. Hence there are no singular points.

5.(a) $P(x) = 0$ only for $x = 0$. Furthermore, $xp(x) = 3\frac{\sin x}{x}$ and $x^2q(x) = -2$. It follows that

$$p_0 = \lim_{x \rightarrow 0} 3 \frac{\sin x}{x} = 3$$

$$q_0 = \lim_{x \rightarrow 0} -2 = -2$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) + 3r - 2 = 0,$$

that is, $r^2 + 2r - 2 = 0$, with roots $r_1 = -1 + \sqrt{3}$ and $r_2 = -1 - \sqrt{3}$.

6.(a) $P(x) = 0$ for $x = 0$ and $x = -2$. We note that $p(x) = x^{-1}(x+2)^{-1}/2$, and $q(x) = -(x+2)^{-1}/2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x \rightarrow 0} \frac{1}{2(x+2)} = \frac{1}{4}$$

$$q_0 = \lim_{x \rightarrow 0} \frac{-x^2}{2(x+2)} = 0$$

and therefore $x = 0$ is a regular singular point.

For the singularity at $x = -2$,

$$p_0 = \lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} \frac{1}{2x} = -\frac{1}{4}$$

$$q_0 = \lim_{x \rightarrow -2} (x+2)^2q(x) = \lim_{x \rightarrow -2} \frac{-(x+2)}{2} = 0$$

and therefore $x = -2$ is a regular singular point.

(b) For $x = 0$: the indicial equation is given by

$$r(r-1) + \frac{1}{4}r = 0,$$

that is, $r^2 - \frac{3}{4}r = 0$, with roots $r_1 = \frac{3}{4}$ and $r_2 = 0$.

For $x = -2$: the indicial equation is given by

$$r(r-1) - \frac{1}{4}r = 0,$$

that is, $r^2 - \frac{5}{4}r = 0$, with roots $r_1 = \frac{5}{4}$ and $r_2 = 0$.

7.(a) $P(x) = 0$ only for $x = 0$. Furthermore, $xp(x) = \frac{1}{2} + \frac{\sin x}{2x}$ and $x^2q(x) = 1$. It follows that

$$p_0 = \lim_{x \rightarrow 0} xp(x) = 1$$

$$q_0 = \lim_{x \rightarrow 0} x^2q(x) = 1$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) + r + 1 = 0,$$

that is, $r^2 + 1 = 0$, with complex conjugate roots $r = \pm i$.

8.(a) Note that $P(x) = 0$ only for $x = -1$. We find that $p(x) = 3(x-1)/(x+1)$, and $q(x) = 3/(x+1)^2$. It follows that

$$\begin{aligned} p_0 &= \lim_{x \rightarrow -1} (x+1)p(x) = \lim_{x \rightarrow -1} 3(x-1) = -6 \\ q_0 &= \lim_{x \rightarrow -1} (x+1)^2 q(x) = \lim_{x \rightarrow -1} 3 = 3 \end{aligned}$$

and therefore $x = -1$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) - 6r + 3 = 0,$$

that is, $r^2 - 7r + 3 = 0$, with roots $r_1 = (7 + \sqrt{37})/2$ and $r_2 = (7 - \sqrt{37})/2$.

10.(a) $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x(x-2)^{-2}(x+2)^{-1}$, and $q(x) = 3(x-2)^{-1}(x+2)^{-1}$. For the singularity at $x = 2$,

$$\lim_{x \rightarrow 2} (x-2)p(x) = \lim_{x \rightarrow 2} \frac{2x}{x^2 - 4},$$

which is undefined. Therefore $x = 0$ is an irregular singular point. For the singularity at $x = -2$,

$$\begin{aligned} p_0 &= \lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} \frac{2x}{(x-2)^2} = -\frac{1}{4} \\ q_0 &= \lim_{x \rightarrow -2} (x+2)^2 q(x) = \lim_{x \rightarrow -2} \frac{3(x+2)}{x-2} = 0 \end{aligned}$$

and therefore $x = -2$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) - \frac{1}{4}r = 0,$$

that is, $r^2 - \frac{5}{4}r = 0$, with roots $r_1 = \frac{5}{4}$ and $r_2 = 0$.

11.(a) $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x/(4-x^2)$, and $q(x) = 3/(4-x^2)$. For the singularity at $x = 2$,

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 2} (x-2)p(x) = \lim_{x \rightarrow 2} \frac{-2x}{x+2} = -1 \\ q_0 &= \lim_{x \rightarrow 2} (x-2)^2 q(x) = \lim_{x \rightarrow 2} \frac{3(2-x)}{x+2} = 0 \end{aligned}$$

and therefore $x = 2$ is a regular singular point.

For the singularity at $x = -2$,

$$p_0 = \lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} \frac{2x}{2-x} = -1$$

$$q_0 = \lim_{x \rightarrow -2} (x+2)^2 q(x) = \lim_{x \rightarrow -2} \frac{3(x+2)}{2-x} = 0$$

and therefore $x = -2$ is a regular singular point.

(b) For $x = 2$: the indicial equation is given by

$$r(r-1) - r = 0,$$

that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$.

For $x = -2$: the indicial equation is given by

$$r(r-1) - r = 0,$$

that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$.

12.(a) $P(x) = 0$ for $x = 0$ and $x = -3$. We note that $p(x) = -2x^{-1}(x+3)^{-1}$, and $q(x) = -1/(x+3)^2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} \frac{-2}{x+3} = -\frac{2}{3}$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} \frac{-x^2}{(x+3)^2} = 0$$

and therefore $x = 0$ is a regular singular point.

For the singularity at $x = -3$,

$$p_0 = \lim_{x \rightarrow -3} (x+3)p(x) = \lim_{x \rightarrow -3} \frac{-2}{x} = \frac{2}{3}$$

$$q_0 = \lim_{x \rightarrow -3} (x+3)^2 q(x) = \lim_{x \rightarrow -3} (-1) = -1$$

and therefore $x = -3$ is a regular singular point.

(b) For $x = 0$: the indicial equation is given by

$$r(r-1) - \frac{2}{3}r = 0,$$

that is, $r^2 - \frac{5}{3}r = 0$, with roots $r_1 = \frac{5}{3}$ and $r_2 = 0$.

For $x = -3$: the indicial equation is given by

$$r(r-1) + \frac{2}{3}r - 1 = 0,$$

that is, $r^2 - \frac{1}{3}r - 1 = 0$, with roots $r_1 = (1 + \sqrt{37})/6$ and $r_2 = (1 - \sqrt{37})/6$.

13.(a) Note that $p(x) = 1/x$ and $q(x) = -1/x$. Furthermore, $xp(x) = 1$ and $x^2q(x) = -x$. It follows that

$$p_0 = \lim_{x \rightarrow 0} (1) = 1$$

$$q_0 = \lim_{x \rightarrow 0} (-x) = 0$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) + r = 0,$$

that is, $r^2 = 0$, with roots $r_1 = r_2 = 0$.

(c) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_nx^n = 0.$$

After adjusting the indices in the first series, we obtain

$$a_1 - a_0 + \sum_{n=1}^{\infty} [n(n+1)a_{n+1} + (n+1)a_{n+1} - a_n]x^n = 0.$$

Setting the coefficients equal to zero, it follows that for $n \geq 0$,

$$a_{n+1} = \frac{a_n}{(n+1)^2}.$$

So for $n \geq 1$,

$$a_n = \frac{a_{n-1}}{n^2} = \frac{a_{n-2}}{n^2(n-1)^2} = \dots = \frac{1}{(n!)^2} a_0.$$

With $a_0 = 1$, one solution is

$$y_1(x) = 1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \dots + \frac{1}{(n!)^2}x^n + \dots$$

For a second solution, set $y_2(x) = y_1(x) \ln x + b_1x + b_2x^2 + \dots + b_nx^n + \dots$. Substituting into the ODE, we obtain

$$L[y_1(x)] \cdot \ln x + 2y_1'(x) + L\left[\sum_{n=1}^{\infty} b_n x^n\right] = 0.$$

Since $L[y_1(x)] = 0$, it follows that

$$L\left[\sum_{n=1}^{\infty} b_n x^n\right] = -2y_1'(x).$$

More specifically,

$$\begin{aligned} b_1 + \sum_{n=1}^{\infty} [n(n+1)b_{n+1} + (n+1)b_{n+1} - b_n]x^n &= \\ &= -2 - x - \frac{1}{6}x^2 - \frac{1}{72}x^3 - \frac{1}{1440}x^4 - \dots \end{aligned}$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} b_1 &= -2 \\ 4b_2 - b_1 &= -1 \\ 9b_3 - b_2 &= -1/6 \\ 16b_4 - b_3 &= -1/72 \\ &\vdots \end{aligned}$$

Solving these equations for the coefficients, $b_1 = -2$, $b_2 = -3/4$, $b_3 = -11/108$, $b_4 = -25/3456$, \dots . Therefore a second solution is

$$y_2(x) = y_1(x) \ln x + \left[-2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 - \dots \right].$$

14.(a) Here $x p(x) = 2x$ and $x^2 q(x) = 6xe^x$. Both of these functions are analytic at $x = 0$, therefore $x = 0$ is a regular singular point. Note that $p_0 = q_0 = 0$.

(b) The indicial equation is given by

$$r(r-1) = 0,$$

that is, $r^2 - r = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c) In order to find the solution corresponding to $r_1 = 1$, set $y = x \sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+1} x^{n+1} + 2 \sum_{n=0}^{\infty} (n+1)a_n x^{n+1} + 6e^x \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

After adjusting the indices in the first two series, and expanding the exponential function,

$$\begin{aligned} &\sum_{n=1}^{\infty} n(n+1)a_n x^n + 2 \sum_{n=1}^{\infty} n a_{n-1} x^n + 6a_0 x + (6a_0 + 6a_1)x^2 + \\ &+ (6a_2 + 6a_1 + 3a_0)x^3 + (6a_3 + 6a_2 + 3a_1 + a_0)x^4 + \dots = 0. \end{aligned}$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 2a_1 + 2a_0 + 6a_0 &= 0 \\ 6a_2 + 4a_1 + 6a_0 + 6a_1 &= 0 \\ 12a_3 + 6a_2 + 6a_2 + 6a_1 + 3a_0 &= 0 \\ 20a_4 + 8a_3 + 6a_3 + 6a_2 + 3a_1 + a_0 &= 0 \\ &\vdots \end{aligned}$$

Setting $a_0 = 1$, solution of the system results in $a_1 = -4$, $a_2 = 17/3$, $a_3 = -47/12$, $a_4 = 191/120$, \dots . Therefore one solution is

$$y_1(x) = x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \dots$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x) = a y_1(x) \ln x + 1 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

Substituting into the ODE, we obtain

$$a L[y_1(x)] \cdot \ln x + 2a y_1'(x) + 2a y_1(x) - a \frac{y_1(x)}{x} + L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 0.$$

Since $L[y_1(x)] = 0$, it follows that

$$L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = -2a y_1'(x) - 2a y_1(x) + a \frac{y_1(x)}{x}.$$

More specifically,

$$\begin{aligned} & \sum_{n=1}^{\infty} n(n+1)c_{n+1}x^n + 2 \sum_{n=1}^{\infty} n c_n x^n + 6 + (6 + 6c_1)x + \\ & + (6c_2 + 6c_1 + 3)x^2 + \dots = -a + 10ax - \frac{61}{3}ax^2 + \frac{193}{12}ax^3 + \dots \end{aligned}$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 6 &= -a \\ 2c_2 + 8c_1 + 6 &= 10a \\ 6c_3 + 10c_2 + 6c_1 + 3 &= -\frac{61}{3}a \\ 12c_4 + 12c_3 + 6c_2 + 3c_1 + 1 &= \frac{193}{12}a \\ &\vdots \end{aligned}$$

Solving these equations for the coefficients, $a = -6$. In order to solve the remaining equations, set $c_1 = 0$. Then $c_2 = -33$, $c_3 = 449/6$, $c_4 = -1595/24$, \dots . Therefore a second solution is

$$y_2(x) = -6 y_1(x) \ln x + \left[1 - 33x^2 + \frac{449}{6}x^3 - \frac{1595}{24}x^4 + \dots \right].$$

15.(a) Note the $p(x) = 6x/(x-1)$ and $q(x) = 3x^{-1}(x-1)^{-1}$. Furthermore, $x p(x) = 6x^2/(x-1)$ and $x^2 q(x) = 3x/(x-1)$. It follows that

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} \frac{6x^2}{x-1} = 0 \\ q_0 &= \lim_{x \rightarrow 0} \frac{3x}{x-1} = 0 \end{aligned}$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) = 0,$$

that is, $r^2 - r = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c) In order to find the solution corresponding to $r_1 = 1$, set $y = x \sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\begin{aligned} \sum_{n=1}^{\infty} n(n+1)a_n x^{n+1} - \sum_{n=1}^{\infty} n(n+1)a_n x^n + \\ + 6 \sum_{n=0}^{\infty} (n+1)a_n x^{n+2} + 3 \sum_{n=0}^{\infty} a_n x^{n+1} = 0. \end{aligned}$$

After adjusting the indices, it follows that

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)a_{n-1} x^n - \sum_{n=1}^{\infty} n(n+1)a_n x^n + \\ + 6 \sum_{n=2}^{\infty} (n-1)a_{n-2} x^n + 3 \sum_{n=1}^{\infty} a_{n-1} x^n = 0. \end{aligned}$$

That is,

$$-2a_1 + 3a_0 + \sum_{n=2}^{\infty} [-n(n+1)a_n + (n^2 - n + 3)a_{n-1} + 6(n-1)a_{n-2}] x^n = 0.$$

Setting the coefficients equal to zero, we have $a_1 = 3a_0/2$, and for $n \geq 2$,

$$n(n+1)a_n = (n^2 - n + 3)a_{n-1} + 6(n-1)a_{n-2}.$$

If we assign $a_0 = 1$, then we obtain $a_1 = 3/2$, $a_2 = 9/4$, $a_3 = 51/16$, \dots . Hence one solution is

$$y_1(x) = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \frac{51}{16}x^4 + \frac{111}{40}x^5 + \dots$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x) = a y_1(x) \ln x + 1 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

Substituting into the ODE, we obtain

$$2ax y_1'(x) - 2a y_1'(x) + 6ax y_1(x) - a y_1(x) + a \frac{y_1(x)}{x} + L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 0,$$

since $L[y_1(x)] = 0$. It follows that

$$L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 2a y_1'(x) - 2ax y_1'(x) + a y_1(x) - 6ax y_1(x) - a \frac{y_1(x)}{x}.$$

Now

$$\begin{aligned} L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 3 + (-2c_2 + 3c_1)x + (-6c_3 + 5c_2 + 6c_1)x^2 + \\ + (-12c_4 + 9c_3 + 12c_2)x^3 + (-20c_5 + 15c_4 + 18c_3)x^4 + \dots \end{aligned}$$

Substituting for $y_1(x)$, the right hand side of the ODE is

$$a + \frac{7}{2}ax + \frac{3}{4}ax^2 + \frac{33}{16}ax^3 - \frac{867}{80}ax^4 - \frac{441}{10}ax^5 + \dots$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 3 &= a \\ -2c_2 + 3c_1 &= \frac{7}{2}a \\ -6c_3 + 5c_2 + 6c_1 &= \frac{3}{4}a \\ -12c_4 + 9c_3 + 12c_2 &= \frac{33}{16}a \\ &\vdots \end{aligned}$$

We find that $a = 3$. In order to solve the second equation, set $c_1 = 0$. Solution of the remaining equations results in $c_2 = -21/4$, $c_3 = -19/4$, $c_4 = -597/64, \dots$. Hence a second solution is

$$y_2(x) = 3y_1(x) \ln x + \left[1 - \frac{21}{4}x^2 - \frac{19}{4}x^3 - \frac{597}{64}x^4 + \dots \right].$$

16.(a) After multiplying both sides of the ODE by x , we find that $x p(x) = 0$ and $x^2 q(x) = x$. Both of these functions are analytic at $x = 0$, hence $x = 0$ is a regular singular point.

(b) Furthermore, $p_0 = q_0 = 0$. So the indicial equation is $r(r-1) = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c) In order to find the solution corresponding to $r_1 = 1$, set $y = x \sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty} n(n+1)a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

That is,

$$\sum_{n=1}^{\infty} [n(n+1)a_n + a_{n-1}] x^n = 0.$$

Setting the coefficients equal to zero, we find that for $n \geq 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)}.$$

It follows that

$$a_n = \frac{-a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \dots = \frac{(-1)^n a_0}{(n!)^2(n+1)}.$$

Hence one solution is

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \frac{1}{2880}x^5 + \dots$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x) = a y_1(x) \ln x + 1 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

Substituting into the ODE, we obtain

$$a L[y_1(x)] \cdot \ln x + 2a y_1'(x) - a \frac{y_1(x)}{x} + L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 0.$$

Since $L[y_1(x)] = 0$, it follows that

$$L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = -2a y_1'(x) + a \frac{y_1(x)}{x}.$$

Now

$$L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 1 + (2c_2 + c_1)x + (6c_3 + c_2)x^2 + (12c_4 + c_3)x^3 + (20c_5 + c_4)x^4 + (30c_6 + c_5)x^5 + \dots$$

Substituting for $y_1(x)$, the right hand side of the ODE is

$$-a + \frac{3}{2}ax - \frac{5}{12}ax^2 + \frac{7}{144}ax^3 - \frac{1}{320}ax^4 + \dots$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 1 &= -a \\ 2c_2 + c_1 &= \frac{3}{2}a \\ 6c_3 + c_2 &= -\frac{5}{12}a \\ 12c_4 + c_3 &= \frac{7}{144}a \\ &\vdots \end{aligned}$$

Evidently, $a = -1$. In order to solve the second equation, set $c_1 = 0$. We then find that $c_2 = -3/4$, $c_3 = 7/36$, $c_4 = -35/1728$, \dots . Therefore a second solution is

$$y_2(x) = -y_1(x) \ln x + \left[1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \dots \right].$$

19.(a) After dividing by the leading coefficient, we find that

$$p_0 = \lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} \frac{\gamma - (1 + \alpha + \beta)x}{1 - x} = \gamma.$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} \frac{-\alpha\beta x}{1 - x} = 0.$$

Hence $x = 0$ is a regular singular point. The indicial equation is $r(r - 1) + \gamma r = 0$, with roots $r_1 = 1 - \gamma$ and $r_2 = 0$.

(b) For $x = 1$,

$$p_0 = \lim_{x \rightarrow 1} (x - 1)p(x) = \lim_{x \rightarrow 1} \frac{-\gamma + (1 + \alpha + \beta)x}{x} = 1 - \gamma + \alpha + \beta.$$

$$q_0 = \lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} \frac{\alpha\beta(x-1)}{x} = 0.$$

Hence $x = 1$ is a regular singular point. The indicial equation is

$$r^2 - (\gamma - \alpha - \beta)r = 0,$$

with roots $r_1 = \gamma - \alpha - \beta$ and $r_2 = 0$.

(c) Given that $r_1 - r_2$ is not a positive integer, we can set $y = \sum_{n=0}^{\infty} a_n x^n$. Substitution into the ODE results in

$$x(1-x) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + [\gamma - (1+\alpha+\beta)x] \sum_{n=1}^{\infty} n a_n x^{n-1} - \alpha\beta \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\begin{aligned} \sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n + \gamma \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \\ - (1+\alpha+\beta) \sum_{n=1}^{\infty} n a_n x^n - \alpha\beta \sum_{n=0}^{\infty} a_n x^n = 0. \end{aligned}$$

Combining the series, we obtain

$$\gamma a_1 - \alpha\beta a_0 + [(2+2\gamma)a_2 - (1+\alpha+\beta+\alpha\beta)a_1]x + \sum_{n=2}^{\infty} A_n x^n = 0,$$

in which

$$A_n = (n+1)(n+\gamma)a_{n+1} - [n(n-1) + (1+\alpha+\beta)n + \alpha\beta]a_n.$$

Note that $n(n-1) + (1+\alpha+\beta)n + \alpha\beta = (n+\alpha)(n+\beta)$. Setting the coefficients equal to zero, we have $\gamma a_1 - \alpha\beta a_0 = 0$, and

$$a_{n+1} = \frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)} a_n$$

for $n \geq 1$. Hence one solution is

$$\begin{aligned} y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 2!}x^2 + \\ + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2) \cdot 3!}x^3 + \dots \end{aligned}$$

Since the nearest other singularity is at $x = 1$, the radius of convergence of $y_1(x)$ will be at least $\rho = 1$.

(d) Given that $r_1 - r_2$ is not a positive integer, we can set $y = x^{1-\gamma} \sum_{n=0}^{\infty} b_n x^n$. Then substitution into the ODE results in

$$\begin{aligned} x(1-x) \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma)a_n x^{n-\gamma-1} + \\ + [\gamma - (1+\alpha+\beta)x] \sum_{n=0}^{\infty} (n+1-\gamma)a_n x^{n-\gamma} - \alpha\beta \sum_{n=0}^{\infty} a_n x^{n+1-\gamma} = 0. \end{aligned}$$

That is,

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma)a_n x^{n+1-\gamma} + \\ & + \gamma \sum_{n=0}^{\infty} (n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta) \sum_{n=0}^{\infty} (n+1-\gamma)a_n x^{n+1-\gamma} - \\ & - \alpha\beta \sum_{n=0}^{\infty} a_n x^{n+1-\gamma} = 0. \end{aligned}$$

After adjusting the indices,

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=1}^{\infty} (n-\gamma)(n-1-\gamma)a_{n-1} x^{n-\gamma} + \\ & + \gamma \sum_{n=0}^{\infty} (n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta) \sum_{n=1}^{\infty} (n-\gamma)a_{n-1} x^{n-\gamma} - \\ & - \alpha\beta \sum_{n=1}^{\infty} a_{n-1} x^{n-\gamma} = 0. \end{aligned}$$

Combining the series, we obtain

$$\sum_{n=1}^{\infty} B_n x^{n-\gamma} = 0,$$

in which

$$B_n = n(n+1-\gamma)b_n - [(n-\gamma)(n-\gamma+\alpha+\beta) + \alpha\beta] b_{n-1}.$$

Note that $(n-\gamma)(n-\gamma+\alpha+\beta) + \alpha\beta = (n+\alpha-\gamma)(n+\beta-\gamma)$. Setting $B_n = 0$, it follows that for $n \geq 1$,

$$b_n = \frac{(n+\alpha-\gamma)(n+\beta-\gamma)}{n(n+1-\gamma)} b_{n-1}.$$

Therefore a second solution is

$$\begin{aligned} y_2(x) = x^{1-\gamma} & \left[1 + \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{(2-\gamma)1!} x + \right. \\ & \left. + \frac{(1+\alpha-\gamma)(2+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma)}{(2-\gamma)(3-\gamma)2!} x^2 + \dots \right]. \end{aligned}$$

(e) Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{1}{\xi} \left(1 - \frac{1}{\xi}\right) \frac{d^2 y}{d\xi^2} + \left\{ 2\xi^3 \frac{1}{\xi} \left(1 - \frac{1}{\xi}\right) - \xi^2 \left[\gamma - (1+\alpha+\beta) \frac{1}{\xi} \right] \right\} \frac{dy}{d\xi} - \alpha\beta y = 0.$$

That is,

$$(\xi^3 - \xi^2) \frac{d^2 y}{d\xi^2} + [2\xi^2 - \gamma\xi^2 + (-1+\alpha+\beta)\xi] \frac{dy}{d\xi} - \alpha\beta y = 0.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{(2-\gamma)\xi + (-1 + \alpha + \beta)}{\xi^2 - \xi} \text{ and } q(\xi) = \frac{-\alpha\beta}{\xi^3 - \xi^2}.$$

It follows that

$$p_0 = \lim_{\xi \rightarrow 0} \xi p(\xi) = \lim_{\xi \rightarrow 0} \frac{(2-\gamma)\xi + (-1 + \alpha + \beta)}{\xi - 1} = 1 - \alpha - \beta,$$

$$q_0 = \lim_{\xi \rightarrow 0} \xi^2 q(\xi) = \lim_{\xi \rightarrow 0} \frac{-\alpha\beta}{\xi - 1} = \alpha\beta.$$

Hence $\xi = 0$ ($x = \infty$) is a regular singular point. The indicial equation is

$$r(r-1) + (1 - \alpha - \beta)r + \alpha\beta = 0,$$

or $r^2 - (\alpha + \beta)r + \alpha\beta = 0$. Evidently, the roots are $r = \alpha$ and $r = \beta$.

21.(a) Note that

$$p(x) = \frac{\alpha}{x^s} \text{ and } q(x) = \frac{\beta}{x^t}.$$

It follows that

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} \alpha x^{1-s},$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} \beta x^{2-t}.$$

Hence if $s > 1$ or $t > 2$, one or both of the limits does not exist. Therefore $x = 0$ is an irregular singular point.

(c) Let $y = a_0 x^r + a_1 x^{r+1} + \dots + a_n x^{r+n} + \dots$. Write the ODE as

$$x^3 y'' + \alpha x^2 y' + \beta y = 0.$$

Substitution of the assumed solution results in

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r+1} + \alpha \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} + \beta \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Adjusting the indices, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} (n-1+r)(n+r-2) a_{n-1} x^{n+r} + \alpha \sum_{n=1}^{\infty} (n-1+r) a_{n-1} x^{n+r} + \\ + \beta \sum_{n=0}^{\infty} a_n x^{n+r} = 0. \end{aligned}$$

Combining the series,

$$\beta a_0 + \sum_{n=1}^{\infty} A_n x^{n+r} = 0,$$

in which $A_n = \beta a_n + (n-1+r)(n+r+\alpha-2)a_{n-1}$. Setting the coefficients equal to zero, we have $a_0 = 0$. But for $n \geq 1$,

$$a_n = \frac{(n-1+r)(n+r+\alpha-2)}{\beta} a_{n-1}.$$

Therefore, regardless of the value of r , it follows that $a_n = 0$, for $n = 1, 2, \dots$

5.7

3. Here $x p(x) = 1$ and $x^2 q(x) = 2x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + 2 \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0.$$

After adjusting the indices in the last series, we obtain

$$a_0 [r(r-1) + r] x^r + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n + 2a_{n-1}] x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 = 0$, with double root $r = 0$. Setting the remaining coefficients equal to zero, we have for $n \geq 1$,

$$a_n(r) = -\frac{2}{(n+r)^2} a_{n-1}(r).$$

It follows that

$$a_n(r) = \frac{(-1)^n 2^n}{[(n+r)(n+r-1)\dots(1+r)]^2} a_0, \quad n \geq 1.$$

Since $r = 0$, one solution is given by

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} x^n.$$

For a second linearly independent solution, we follow the discussion in Section 5.6. First note that

$$\frac{a'_n(r)}{a_n(r)} = -2 \left[\frac{1}{n+r} + \frac{1}{n+r-1} + \dots + \frac{1}{1+r} \right].$$

Setting $r = 0$,

$$a'_n(0) = -2 H_n a_n(0) = -2 H_n \frac{(-1)^n 2^n}{(n!)^2}.$$

Therefore,

$$y_2(x) = y_1(x) \ln x - 2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n H_n}{(n!)^2} x^n.$$

4. Here $x p(x) = 4$ and $x^2 q(x) = 2 + x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$. Substitution into the ODE results in

$$\begin{aligned} \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + 4 \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+1} + 2 \sum_{n=0}^{\infty} a_n x^{r+n} = 0. \end{aligned}$$

After adjusting the indices in the second-to-last series, we obtain

$$\begin{aligned} a_0 [r(r-1) + 4r + 2] x^r + \\ + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + 4(r+n)a_n + 2a_n + a_{n-1}] x^{r+n} = 0. \end{aligned}$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 + 3r + 2 = 0$, with roots $r_1 = -1$ and $r_2 = -2$. Setting the remaining coefficients equal to zero, we have for $n \geq 1$,

$$a_n(r) = -\frac{1}{(n+r+1)(n+r+2)} a_{n-1}(r).$$

It follows that

$$a_n(r) = \frac{(-1)^n}{[(n+r+1)(n+r)\dots(2+r)][(n+r+2)(n+r)\dots(3+r)]} a_0, \quad n \geq 1.$$

Since $r_1 = -1$, one solution is given by

$$y_1(x) = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!(n+1)!} x^n.$$

For a second linearly independent solution, we follow the discussion in Section 5.6. Since $r_1 - r_2 = N = 1$, we find that

$$a_1(r) = -\frac{1}{(r+2)(r+3)},$$

with $a_0 = 1$. Hence the leading coefficient in the solution is

$$a = \lim_{r \rightarrow -2} (r+2) a_1(r) = -1.$$

Further,

$$(r+2) a_n(r) = \frac{(-1)^n}{(n+r+2)[(n+r+1)(n+r)\dots(3+r)]^2}.$$

Let $A_n(r) = (r+2) a_n(r)$. It follows that

$$\frac{A'_n(r)}{A_n(r)} = -\frac{1}{n+r+2} - 2 \left[\frac{1}{n+r+1} + \frac{1}{n+r} + \dots + \frac{1}{3+r} \right].$$

Setting $r = r_2 = -2$,

$$\frac{A'_n(-2)}{A_n(-2)} = -\frac{1}{n} - 2 \left[\frac{1}{n-1} + \frac{1}{n-2} + \dots + 1 \right] = -H_n - H_{n-1}.$$

Hence

$$c_n(-2) = -(H_n + H_{n-1}) A_n(-2) = -(H_n + H_{n-1}) \frac{(-1)^n}{n!(n-1)!}.$$

Therefore,

$$y_2(x) = -y_1(x) \ln x + x^{-2} \left[1 - \sum_{n=1}^{\infty} \frac{(-1)^n (H_n + H_{n-1})}{n!(n-1)!} x^n \right].$$

6. Let $y(x) = v(x)/\sqrt{x}$. Then $y' = x^{-1/2} v' - x^{-3/2} v/2$ and $y'' = x^{-1/2} v'' - x^{-3/2} v' + 3x^{-5/2} v/4$. Substitution into the ODE results in

$$\left[x^{3/2} v'' - x^{1/2} v' + 3x^{-1/2} v/4 \right] + \left[x^{1/2} v' - x^{-1/2} v/2 \right] + \left(x^2 - \frac{1}{4} \right) x^{-1/2} v = 0.$$

Simplifying, we find that

$$v'' + v = 0,$$

with general solution $v(x) = c_1 \cos x + c_2 \sin x$. Hence

$$y(x) = c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x.$$

8. The absolute value of the ratio of consecutive terms is

$$\left| \frac{a_{2m+2} x^{2m+2}}{a_{2m} x^{2m}} \right| = \frac{|x|^{2m+2} 2^{2m} (m+1)! m!}{|x|^{2m} 2^{2m+2} (m+2)!(m+1)!} = \frac{|x|^2}{4(m+2)(m+1)}.$$

Applying the ratio test,

$$\lim_{m \rightarrow \infty} \left| \frac{a_{2m+2} x^{2m+2}}{a_{2m} x^{2m}} \right| = \lim_{m \rightarrow \infty} \frac{|x|^2}{4(m+2)(m+1)} = 0.$$

Hence the series for $J_1(x)$ converges absolutely for all values of x . Furthermore, since the series for $J_0(x)$ also converges absolutely for all x , term-by-term differentiation results in

$$\begin{aligned} J_0'(x) &= \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!} = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{2m+1}}{2^{2m+1} (m+1)! m!} = \\ &= -\frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m+1)! m!}. \end{aligned}$$

Therefore, $J_0'(x) = -J_1(x)$.

9.(a) Note that $x p(x) = 1$ and $x^2 q(x) = x^2 - \nu^2$, which are both analytic at $x = 0$. Thus $x = 0$ is a regular singular point. Furthermore, $p_0 = 1$ and $q_0 = -\nu^2$. Hence the indicial equation is $r^2 - \nu^2 = 0$, with roots $r_1 = \nu$ and $r_2 = -\nu$.

(b) Set $y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$. Substitution into the ODE results in

$$\begin{aligned} \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{r+n} = 0. \end{aligned}$$

After adjusting the indices in the second-to-last series, we obtain

$$\begin{aligned} a_0 [r(r-1) + r - \nu^2] x^r + a_1 [(r+1)r + (r+1) - \nu^2] + \\ + \sum_{n=2}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n - \nu^2 a_n + a_{n-2}] x^{r+n} = 0. \end{aligned}$$

Setting the coefficients equal to zero, we find that $a_1 = 0$, and

$$a_n = \frac{-1}{(r+n)^2 - \nu^2} a_{n-2},$$

for $n \geq 2$. It follows that $a_3 = a_5 = \dots = a_{2m+1} = \dots = 0$. Furthermore, with $r = \nu$,

$$a_n = \frac{-1}{n(n+2\nu)} a_{n-2}.$$

So for $m = 1, 2, \dots$,

$$a_{2m} = \frac{-1}{2m(2m+2\nu)} a_{2m-2} = \frac{(-1)^m}{2^{2m} m!(1+\nu)(2+\nu)\dots(m-1+\nu)(m+\nu)} a_0.$$

Hence one solution is

$$y_1(x) = x^\nu \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\nu)(2+\nu)\dots(m-1+\nu)(m+\nu)} \left(\frac{x}{2}\right)^{2m} \right].$$

(c) Assuming that $r_1 - r_2 = 2\nu$ is not an integer, simply setting $r = -\nu$ in the above results in a second linearly independent solution

$$y_2(x) = x^{-\nu} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\nu)(2-\nu)\dots(m-1-\nu)(m-\nu)} \left(\frac{x}{2}\right)^{2m} \right].$$

(d) The absolute value of the ratio of consecutive terms in $y_1(x)$ is

$$\begin{aligned} \left| \frac{a_{2m+2} x^{2m+2}}{a_{2m} x^{2m}} \right| &= \frac{|x|^{2m+2} 2^{2m} m!(1+\nu)\dots(m+\nu)}{|x|^{2m} 2^{2m+2} (m+1)!(1+\nu)\dots(m+1+\nu)} \\ &= \frac{|x|^2}{4(m+1)(m+1+\nu)}. \end{aligned}$$

Applying the ratio test,

$$\lim_{m \rightarrow \infty} \left| \frac{a_{2m+2} x^{2m+2}}{a_{2m} x^{2m}} \right| = \lim_{m \rightarrow \infty} \frac{|x|^2}{4(m+1)(m+1+\nu)} = 0.$$

Hence the series for $y_1(x)$ converges absolutely for all values of x . The same can be shown for $y_2(x)$. Note also, that if ν is a positive integer, then the coefficients in the series for $y_2(x)$ are undefined.

10.(a) It suffices to calculate $L[J_0(x) \ln x]$. Indeed,

$$[J_0(x) \ln x]' = J_0'(x) \ln x + \frac{J_0(x)}{x}$$

and

$$[J_0(x) \ln x]'' = J_0''(x) \ln x + 2\frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2}.$$

Hence

$$\begin{aligned} L[J_0(x) \ln x] &= x^2 J_0''(x) \ln x + 2x J_0'(x) - J_0(x) + \\ &\quad + x J_0'(x) \ln x + J_0(x) + x^2 J_0(x) \ln x. \end{aligned}$$

Since $x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$,

$$L[J_0(x) \ln x] = 2x J_0'(x).$$

(b) Given that $L[y_2(x)] = 0$, after adjusting the indices in part (a), we have

$$b_1 x + 2^2 b_2 x^2 + \sum_{n=3}^{\infty} (n^2 b_n + b_{n-2}) x^n = -2x J_0'(x).$$

Using the series representation of $J_0'(x)$ in Problem 8,

$$b_1 x + 2^2 b_2 x^2 + \sum_{n=3}^{\infty} (n^2 b_n + b_{n-2}) x^n = -2 \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n}}{2^{2n} (n!)^2}.$$

(c) Equating the coefficients on both sides of the equation, we find that

$$b_1 = b_3 = \dots = b_{2m+1} = \dots = 0.$$

Also, with $n = 1$, $2^2 b_2 = 1/(1!)^2$, that is, $b_2 = 1/[2^2(1!)^2]$. Furthermore, for $m \geq 2$,

$$(2m)^2 b_{2m} + b_{2m-2} = -2 \frac{(-1)^m (2m)}{2^{2m} (m!)^2}.$$

More explicitly,

$$\begin{aligned} b_4 &= -\frac{1}{2^2 4^2} \left(1 + \frac{1}{2}\right) \\ b_6 &= \frac{1}{2^2 4^2 6^2} \left(1 + \frac{1}{2} + \frac{1}{3}\right) \\ &\vdots \end{aligned}$$

It can be shown, in general, that

$$b_{2m} = (-1)^{m+1} \frac{H_m}{2^{2m} (m!)^2}.$$

11. Bessel's equation of order one is

$$x^2 y'' + x y' + (x^2 - 1)y = 0.$$

Based on Problem 9, the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. Set $y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$. Substitution into the ODE results in

$$\begin{aligned} \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0. \end{aligned}$$

After adjusting the indices in the second-to-last series, we obtain

$$\begin{aligned} a_0 [r(r-1) + r - 1] x^r + a_1 [(r+1)r + (r+1) - 1] x^{r+1} + \\ + \sum_{n=2}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n - a_n + a_{n-2}] x^{r+n} = 0. \end{aligned}$$

Setting the coefficients equal to zero, we find that $a_1 = 0$, and

$$a_n(r) = \frac{-1}{(r+n)^2 - 1} a_{n-2}(r) = \frac{-1}{(n+r+1)(n+r-1)} a_{n-2}(r),$$

for $n \geq 2$. It follows that $a_3 = a_5 = \dots = a_{2m+1} = \dots = 0$. Solving the recurrence relation,

$$a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)(2m+r-1)^2 \dots (r+3)^2(r+1)} a_0.$$

With $r = r_1 = 1$,

$$a_{2m}(1) = \frac{(-1)^m}{2^{2m}(m+1)! m!} a_0.$$

For a second linearly independent solution, we follow the discussion in Section 5.6. Since $r_1 - r_2 = N = 2$, we find that

$$a_2(r) = -\frac{1}{(r+3)(r+1)},$$

with $a_0 = 1$. Hence the leading coefficient in the solution is

$$a = \lim_{r \rightarrow -1} (r+1) a_2(r) = -\frac{1}{2}.$$

Further,

$$(r+1) a_{2m}(r) = \frac{(-1)^m}{(2m+r+1) [(2m+r-1) \dots (3+r)]^2}.$$

Let $A_n(r) = (r+1) a_n(r)$. It follows that

$$\frac{A'_{2m}(r)}{A_{2m}(r)} = -\frac{1}{2m+r+1} - 2 \left[\frac{1}{2m+r-1} + \dots + \frac{1}{3+r} \right].$$

Setting $r = r_2 = -1$, we calculate

$$\begin{aligned} c_{2m}(-1) &= -\frac{1}{2}(H_m + H_{m-1})A_{2m}(-1) = \\ &= -\frac{1}{2}(H_m + H_{m-1})\frac{(-1)^m}{2m[(2m-2)\dots 2]^2} = -\frac{1}{2}(H_m + H_{m-1})\frac{(-1)^m}{2^{2m-1}m!(m-1)!}. \end{aligned}$$

Note that $a_{2m+1}(r) = 0$ implies that $A_{2m+1}(r) = 0$, so

$$c_{2m+1}(-1) = \left[\frac{d}{dr} A_{2m+1}(r) \right]_{r=r_2} = 0.$$

Therefore,

$$\begin{aligned} y_2(x) &= -\frac{1}{2} \left[x \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!m!} \left(\frac{x}{2}\right)^{2m} \right] \ln x + \\ &+ \frac{1}{x} \left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m(H_m + H_{m-1})}{m!(m-1)!} \left(\frac{x}{2}\right)^{2m} \right]. \end{aligned}$$

Based on the definition of $J_1(x)$,

$$y_2(x) = -J_1(x) \ln x + \frac{1}{x} \left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m(H_m + H_{m-1})}{m!(m-1)!} \left(\frac{x}{2}\right)^{2m} \right].$$

12. Consider a solution of the form $y(x) = \sqrt{x} f(\alpha x^\beta)$. Then

$$y' = \frac{df}{d\xi} \cdot \frac{\alpha\beta x^\beta}{\sqrt{x}} + \frac{f(\xi)}{2\sqrt{x}}$$

in which $\xi = \alpha x^\beta$. Hence

$$y'' = \frac{d^2 f}{d\xi^2} \cdot \frac{\alpha^2 \beta^2 x^{2\beta}}{x\sqrt{x}} + \frac{df}{d\xi} \cdot \frac{\alpha\beta^2 x^\beta}{x\sqrt{x}} - \frac{f(\xi)}{4x\sqrt{x}},$$

and

$$x^2 y'' = \alpha^2 \beta^2 x^{2\beta} \sqrt{x} \frac{d^2 f}{d\xi^2} + \alpha\beta^2 x^\beta \sqrt{x} \frac{df}{d\xi} - \frac{1}{4} \sqrt{x} f(\xi).$$

Substitution into the ODE results in

$$\alpha^2 \beta^2 x^{2\beta} \frac{d^2 f}{d\xi^2} + \alpha\beta^2 x^\beta \frac{df}{d\xi} - \frac{1}{4} f(\xi) + (\alpha^2 \beta^2 x^{2\beta} + \frac{1}{4} - \nu^2 \beta^2) f(\xi) = 0.$$

Simplifying, and setting $\xi = \alpha x^\beta$, we find that

$$\xi^2 \frac{d^2 f}{d\xi^2} + \xi \frac{df}{d\xi} + (\xi^2 - \nu^2) f(\xi) = 0, \quad (*)$$

which is a Bessel equation of order ν . Therefore, the general solution of the given ODE is

$$y(x) = \sqrt{x} [c_1 f_1(\alpha x^\beta) + c_2 f_2(\alpha x^\beta)],$$

in which $f_1(\xi)$ and $f_2(\xi)$ are the linearly independent solutions of (*).