

Systems of First Order Linear Equations

7.1

1. Introduce the variables $x_1 = u$ and $x_2 = u'$. It follows that $x'_1 = x_2$ and

$$x'_2 = u'' = -2u - 0.5u'.$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= -2x_1 - 0.5x_2. \end{aligned}$$

3. First divide both sides of the equation by t^2 , and write

$$u'' = -\frac{1}{t}u' - \left(1 - \frac{1}{4t^2}\right)u.$$

Set $x_1 = u$ and $x_2 = u'$. It follows that $x'_1 = x_2$ and

$$x'_2 = u'' = -\frac{1}{t}u' - \left(1 - \frac{1}{4t^2}\right)u.$$

We obtain the system of equations

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= -\left(1 - \frac{1}{4t^2}\right)x_1 - \frac{1}{t}x_2. \end{aligned}$$

5. Let $x_1 = u$ and $x_2 = u'$; then $u'' = x_2'$. In terms of the new variables, we have

$$x_2' + 0.25x_2 + 4x_1 = 2 \cos 3t$$

with the initial conditions $x_1(0) = 1$ and $x_2(0) = -2$. The equivalent first order system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_1 - 0.25x_2 + 2 \cos 3t \end{aligned}$$

with the above initial conditions.

7.(a) Solving the first equation for x_2 , we have $x_2 = x_1' + 2x_1$. Substitution into the second equation results in

$$(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1).$$

That is, $x_1'' + 4x_1' + 3x_1 = 0$. The resulting equation is a second order differential equation with constant coefficients. The general solution is

$$x_1(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

With x_2 given in terms of x_1 , it follows that

$$x_2(t) = c_1 e^{-t} - c_2 e^{-3t}.$$

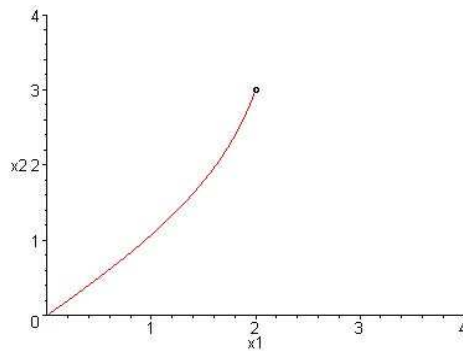
(b) Imposing the specified initial conditions, we obtain

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 - c_2 &= 3, \end{aligned}$$

with solution $c_1 = 5/2$ and $c_2 = -1/2$. Hence

$$x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \text{ and } x_2(t) = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

(c)



10.(a) Solving the first equation for x_2 , we obtain $x_2 = (x_1 - x_1')/2$. Substitution into the second equation results in

$$(x_1 - x_1')'/2 = 3x_1 - 2(x_1 - x_1').$$

Rearranging the terms, the single differential equation for x_1 is

$$x_1'' + 3x_1' + 2x_1 = 0.$$

(b) The general solution is

$$x_1(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

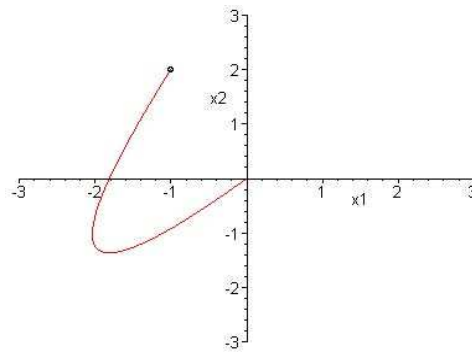
With x_2 given in terms of x_1 , it follows that

$$x_2(t) = c_1 e^{-t} + \frac{3}{2} c_2 e^{-2t}.$$

Invoking the specified initial conditions, $c_1 = -7$ and $c_2 = 6$. Hence

$$x_1(t) = -7e^{-t} + 6e^{-2t} \text{ and } x_2(t) = -7e^{-t} + 9e^{-3t}.$$

(c)



11.(a) Solving the first equation for x_2 , we have $x_2 = x_1'/2$. Substitution into the second equation results in

$$x_1''/2 = -2x_1.$$

The resulting equation is $x_1'' + 4x_1 = 0$.

(b) The general solution is

$$x_1(t) = c_1 \cos 2t + c_2 \sin 2t.$$

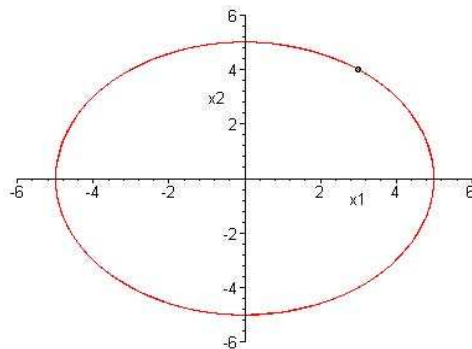
With x_2 given in terms of x_1 , it follows that

$$x_2(t) = -c_1 \sin 2t + c_2 \cos 2t.$$

Imposing the specified initial conditions, we obtain $c_1 = 3$ and $c_2 = 4$. Hence

$$x_1(t) = 3 \cos 2t + 4 \sin 2t \text{ and } x_2(t) = -3 \sin 2t + 4 \cos 2t.$$

(c)



12.(a) Solving the first equation for x_2 , we obtain $x_2 = x_1'/2 + x_1/4$. Substitution into the second equation results in

$$x_1''/2 + x_1'/4 = -2x_1 - (x_1'/2 + x_1/4)/2.$$

Rearranging the terms, the single differential equation for x_1 is

$$x_1'' + x_1' + \frac{17}{4}x_1 = 0.$$

(b) The general solution is

$$x_1(t) = e^{-t/2} [c_1 \cos 2t + c_2 \sin 2t].$$

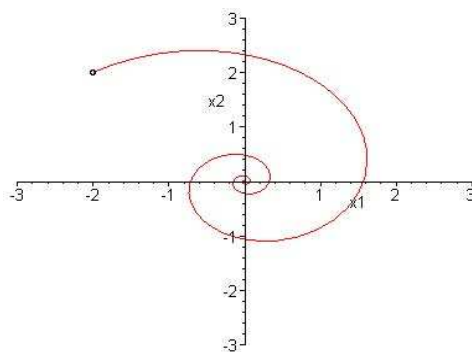
With x_2 given in terms of x_1 , it follows that

$$x_2(t) = e^{-t/2} [-c_1 \cos 2t + c_2 \sin 2t].$$

Imposing the specified initial conditions, we obtain $c_1 = -2$ and $c_2 = 2$. Hence

$$x_1(t) = e^{-t/2} [-2 \cos 2t + 2 \sin 2t] \quad \text{and} \quad x_2(t) = e^{-t/2} [2 \cos 2t + 2 \sin 2t].$$

(c)



13. Solving the first equation for V , we obtain $V = L \cdot I'$. Substitution into the second equation results in

$$L \cdot I'' = -\frac{I}{C} - \frac{L}{RC} I'.$$

Rearranging the terms, the single differential equation for I is

$$LRC \cdot I'' + L \cdot I' + R \cdot I = 0.$$

15. Direct substitution results in

$$\begin{aligned} (c_1 x_1(t) + c_2 x_2(t))' &= p_{11}(t) [c_1 x_1(t) + c_2 x_2(t)] + p_{12}(t) [c_1 y_1(t) + c_2 y_2(t)] \\ (c_1 y_1(t) + c_2 y_2(t))' &= p_{21}(t) [c_1 x_1(t) + c_2 x_2(t)] + p_{22}(t) [c_1 y_1(t) + c_2 y_2(t)]. \end{aligned}$$

Expanding the left-hand-side of the first equation,

$$\begin{aligned} c_1 x_1'(t) + c_2 x_2'(t) &= c_1 [p_{11}(t)x_1(t) + p_{12}(t)y_1(t)] + \\ &\quad + c_2 [p_{11}(t)x_2(t) + p_{12}(t)y_2(t)]. \end{aligned}$$

Repeat with the second equation to show that the system of ODEs is identically satisfied.

16. Based on the hypothesis,

$$\begin{aligned} x_1'(t) &= p_{11}(t)x_1(t) + p_{12}(t)y_1(t) + g_1(t) \\ x_2'(t) &= p_{11}(t)x_2(t) + p_{12}(t)y_2(t) + g_1(t). \end{aligned}$$

Subtracting the two equations,

$$x_1'(t) - x_2'(t) = p_{11}(t) [x_1'(t) - x_2'(t)] + p_{12}(t) [y_1'(t) - y_2'(t)].$$

Similarly,

$$y_1'(t) - y_2'(t) = p_{21}(t) [x_1'(t) - x_2'(t)] + p_{22}(t) [y_1'(t) - y_2'(t)].$$

Hence the difference of the two solutions satisfies the homogeneous ODE.

17. For rectilinear motion in one dimension, Newton's second law can be stated as

$$\sum F = m x''.$$

The resisting force exerted by a linear spring is given by $F_s = k \delta$, in which δ is the displacement of the end of a spring from its equilibrium configuration. Hence, with $0 < x_1 < x_2$, the first two springs are in tension, and the last spring is in compression. The sum of the spring forces on m_1 is

$$F_s^1 = -k_1 x_1 - k_2 (x_2 - x_1).$$

The total force on m_1 is

$$\sum F^1 = -k_1 x_1 + k_2 (x_2 - x_1) + F_1(t).$$

Similarly, the total force on m_2 is

$$\sum F^2 = -k_2(x_2 - x_1) - k_3x_2 + F_2(t).$$

18. One of the ways to transform the system is to assign the variables

$$y_1 = x_1, \quad y_2 = x_1', \quad y_3 = x_2, \quad y_4 = x_2'.$$

Before proceeding, note that

$$\begin{aligned} x_1'' &= \frac{1}{m_1} [-(k_1 + k_2)x_1 + k_2x_2 + F_1(t)] \\ x_2'' &= \frac{1}{m_2} [k_2x_1 - (k_2 + k_3)x_2 + F_2(t)] . \end{aligned}$$

Differentiating the new variables, we obtain the system of four first order equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \frac{1}{m_1} [-(k_1 + k_2)y_1 + k_2y_3 + F_1(t)] \\ y_3' &= y_4 \\ y_4' &= \frac{1}{m_2} [k_2y_1 - (k_2 + k_3)y_3 + F_2(t)] . \end{aligned}$$

19.(a) Taking a clockwise loop around each of the paths, it is easy to see that voltage drops are given by $V_1 - V_2 = 0$, and $V_2 - V_3 = 0$.

(b) Consider the right node. The current in is given by $I_1 + I_2$. The current leaving the node is $-I_3$. Hence the current passing through the node is $(I_1 + I_2) - (-I_3)$. Based on Kirchhoff's first law, $I_1 + I_2 + I_3 = 0$.

(c) In the capacitor,

$$C V_1' = I_1.$$

In the resistor,

$$V_2 = R I_2 .$$

In the inductor,

$$L I_3' = V_3 .$$

(d) Based on part (a), $V_3 = V_2 = V_1$. Based on part (b),

$$C V_1' + \frac{1}{R} V_2 + I_3 = 0 .$$

It follows that

$$C V_1' = -\frac{1}{R} V_1 - I_3 \quad \text{and} \quad L I_3' = V_1 .$$

21. Let I_1, I_2, I_3 , and I_4 be the current through the resistors, inductor, and capacitor, respectively. Assign V_1, V_2, V_3 , and V_4 as the respective voltage drops. Based on Kirchhoff's second law, the net voltage drops, around each loop, satisfy

$$V_1 + V_3 + V_4 = 0, \quad V_1 + V_3 + V_2 = 0 \quad \text{and} \quad V_4 - V_2 = 0.$$

Applying Kirchhoff's first law to the upper-right node,

$$I_3 - (I_2 + I_4) = 0.$$

Likewise, in the remaining nodes,

$$I_1 - I_3 = 0 \quad \text{and} \quad I_2 + I_4 - I_1 = 0.$$

That is,

$$V_4 - V_2 = 0, \quad V_1 + V_3 + V_4 = 0 \quad \text{and} \quad I_2 + I_4 - I_3 = 0.$$

Using the current-voltage relations,

$$V_1 = R_1 I_1, \quad V_2 = R_2 I_2, \quad L I_3' = V_3, \quad C V_4' = I_4.$$

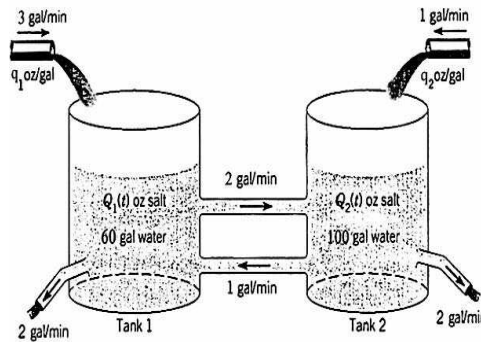
Combining these equations,

$$R_1 I_3 + L I_3' + V_4 = 0 \quad \text{and} \quad C V_4' = I_3 - \frac{V_4}{R_2}.$$

Now set $I_3 = I$ and $V_4 = V$, to obtain the system of equations

$$L I' = -R_1 I - V \quad \text{and} \quad C V' = I - \frac{V}{R_2}.$$

23.(a)



Let $Q_1(t)$ and $Q_2(t)$ be the amount of salt in the respective tanks at time t . Note that the volume of each tank remains constant. Based on conservation of mass, the rate of increase of salt, in any given tank, is given by

$$\text{rate of increase} = \text{rate in} - \text{rate out}.$$

For Tank 1, the rate of salt flowing into Tank 1 is

$$r_{in} = \left[q_1 \frac{\text{oz}}{\text{gal}} \right] \left[3 \frac{\text{gal}}{\text{min}} \right] + \left[\frac{Q_2}{100} \frac{\text{oz}}{\text{gal}} \right] \left[1 \frac{\text{gal}}{\text{min}} \right] = 3q_1 + \frac{Q_2}{100} \frac{\text{oz}}{\text{min}}.$$

The rate at which salt flow out of Tank 1 is

$$r_{out} = \left[\frac{Q_1}{60} \frac{\text{oz}}{\text{gal}} \right] \left[4 \frac{\text{gal}}{\text{min}} \right] = \frac{Q_1}{15} \frac{\text{oz}}{\text{min}}.$$

Hence

$$\frac{dQ_1}{dt} = 3q_1 + \frac{Q_2}{100} - \frac{Q_1}{15}.$$

Similarly, for Tank 2,

$$\frac{dQ_2}{dt} = q_2 + \frac{Q_1}{30} - \frac{3Q_2}{100}.$$

The process is modeled by the system of equations

$$\begin{aligned} Q_1' &= -\frac{Q_1}{15} + \frac{Q_2}{100} + 3q_1 \\ Q_2' &= \frac{Q_1}{30} - \frac{3Q_2}{100} + q_2. \end{aligned}$$

The initial conditions are $Q_1(0) = Q_1^0$ and $Q_2(0) = Q_2^0$.

(b) The equilibrium values are obtained by solving the system

$$\begin{aligned} -\frac{Q_1}{15} + \frac{Q_2}{100} + 3q_1 &= 0 \\ \frac{Q_1}{30} - \frac{3Q_2}{100} + q_2 &= 0. \end{aligned}$$

Its solution leads to $Q_1^E = 54q_1 + 6q_2$ and $Q_2^E = 60q_1 + 40q_2$.

(c) The question refers to a possible solution of the system

$$\begin{aligned} 54q_1 + 6q_2 &= 60 \\ 60q_1 + 40q_2 &= 50. \end{aligned}$$

It is possible to formally solve the system of equations, but the unique solution gives

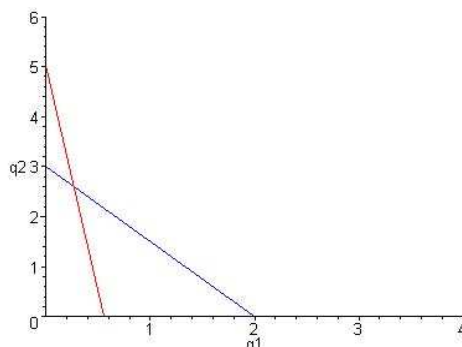
$$q_1 = \frac{7}{6} \frac{\text{oz}}{\text{gal}} \quad \text{and} \quad q_2 = -\frac{1}{2} \frac{\text{oz}}{\text{gal}},$$

which is not physically possible.

(d) We can write

$$\begin{aligned} q_2 &= -9q_1 + \frac{Q_1^E}{6} \\ q_2 &= -\frac{3}{2}q_1 + \frac{Q_2^E}{40}, \end{aligned}$$

which are the equations of two lines in the q_1 - q_2 -plane:



The intercepts of the first line are $Q_1^E/54$ and $Q_1^E/6$. The intercepts of the second line are $Q_2^E/60$ and $Q_2^E/40$. Therefore the system will have a unique solution, in the first quadrant, as long as $Q_1^E/54 \leq Q_2^E/60$ or $Q_2^E/40 \leq Q_1^E/6$. That is,

$$\frac{10}{9} \leq \frac{Q_2^E}{Q_1^E} \leq \frac{20}{3}.$$

7.2

2.(a)

$$\mathbf{A} - 2\mathbf{B} = \begin{pmatrix} 1+i-2i & -1+2i-6 \\ 3+2i-4 & 2-i+4i \end{pmatrix} = \begin{pmatrix} 1-i & -7+2i \\ -1+2i & 2+3i \end{pmatrix}.$$

(b)

$$3\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3+3i+i & -3+6i+3 \\ 9+6i+2 & 6-3i-2i \end{pmatrix} = \begin{pmatrix} 3+4i & 6i \\ 11+6i & 6-5i \end{pmatrix}.$$

(c)

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} (1+i)i+2(-1+2i) & 3(1+i)+(-1+2i)(-2i) \\ (3+2i)i+2(2-i) & 3(3+2i)+(2-i)(-2i) \end{pmatrix} \\ &= \begin{pmatrix} -3+5i & 7+5i \\ 2+i & 7+2i \end{pmatrix}. \end{aligned}$$

(d)

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} (1+i)i+3(3+2i) & (-1+2i)i+3(2-i) \\ 2(1+i)+(-2i)(3+2i) & 2(-1+2i)+(-2i)(2-i) \end{pmatrix} \\ &= \begin{pmatrix} 8+7i & 4-4i \\ 6-4i & -4 \end{pmatrix}. \end{aligned}$$

3.(c,d)

$$\begin{aligned}\mathbf{A}^T + \mathbf{B}^T &= \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 1 \\ 3 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 4 & 0 \\ 3 & -1 & 0 \\ 5 & -4 & 1 \end{pmatrix} = (\mathbf{A} + \mathbf{B})^T.\end{aligned}$$

4.(b)

$$\overline{\mathbf{A}} = \begin{pmatrix} 3+2i & 1-i \\ 2+i & -2-3i \end{pmatrix}.$$

(c) By definition, $\mathbf{A}^* = \overline{\mathbf{A}^T} = (\overline{\mathbf{A}})^T$.

5.

$$2(\mathbf{A} + \mathbf{B}) = 2 \begin{pmatrix} 5 & 3 & -2 \\ 0 & 2 & 5 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 6 & -4 \\ 0 & 4 & 10 \\ 4 & 4 & 6 \end{pmatrix}.$$

7. Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. The given operations in (a)-(d) are performed elementwise. That is,

$$(a) \ a_{ij} + b_{ij} = b_{ij} + a_{ij}.$$

$$(b) \ a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}.$$

$$(c) \ \alpha(a_{ij} + b_{ij}) = \alpha a_{ij} + \alpha b_{ij}.$$

$$(d) \ (\alpha + \beta) a_{ij} = \alpha a_{ij} + \beta a_{ij}.$$

In the following, let $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$ and $\mathbf{C} = (c_{ij})$.

(e) Calculating the generic element,

$$(\mathbf{BC})_{ij} = \sum_{k=1}^n b_{ik} c_{kj}.$$

Therefore

$$\begin{aligned}[\mathbf{A}(\mathbf{BC})]_{ij} &= \sum_{r=1}^n a_{ir} \left(\sum_{k=1}^n b_{rk} c_{kj} \right) = \\ &= \sum_{r=1}^n \sum_{k=1}^n a_{ir} b_{rk} c_{kj} = \sum_{k=1}^n \left[\left(\sum_{r=1}^n a_{ir} b_{rk} \right) c_{kj} \right].\end{aligned}$$

The last summation is recognized as

$$\sum_{r=1}^n a_{ir} b_{rk} = (\mathbf{AB})_{ik},$$

which is the ik -th element of the matrix \mathbf{AB} .

(f) Likewise,

$$\begin{aligned} [\mathbf{A}(\mathbf{B} + \mathbf{C})]_{ij} &= \sum_{k=1}^n a_{ik}(b_{kj} + c_{kj}) = \\ &= \sum_{k=1}^n a_{ik} b_{kj} + \sum_{k=1}^n a_{ik} c_{kj} = (\mathbf{AB})_{ij} + (\mathbf{AC})_{ij}. \end{aligned}$$

8.(a) $\mathbf{x}^T \mathbf{y} = 2(-1 + i) + 2(3i) + (1 - i)(3 - i) = 4i.$

(b) $\mathbf{y}^T \mathbf{y} = (-1 + i)^2 + 2^2 + (3 - i)^2 = 12 - 8i.$

(c) $(\mathbf{x}, \mathbf{y}) = 2(-1 - i) + 2(3i) + (1 - i)(3 + i) = 2 + 2i.$

(d) $(\mathbf{y}, \mathbf{y}) = (-1 + i)(-1 - i) + 2^2 + (3 - i)(3 + i) = 16.$

9. Indeed,

$$5 + 3i = \mathbf{x}^T \mathbf{y} = \sum_{j=1}^n x_j y_j = \mathbf{y}^T \mathbf{x},$$

and

$$3 - 5i = (\mathbf{x}, \mathbf{y}) = \sum_{j=1}^n x_j \bar{y}_j = \sum_{j=1}^n \bar{y}_j x_j = \overline{\sum_{j=1}^n y_j \bar{x}_j} = \overline{(\mathbf{y}, \mathbf{x})}.$$

11. First augment the given matrix by the identity matrix:

$$[\mathbf{A} | \mathbf{I}] = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Divide the first row by 3, to obtain

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Adding -6 times the first row to the second row results in

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 4 & -2 & 1 \end{pmatrix}.$$

Divide the second row by 4, to obtain

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 1 & -1/2 & 1/4 \end{pmatrix}.$$

Finally, adding $1/3$ times the second row to the first row results in

$$\begin{pmatrix} 1 & 0 & 1/6 & 1/12 \\ 0 & 1 & -1/2 & 1/4 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}^{-1} = \frac{1}{12} \begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}.$$

13. The augmented matrix is

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

Combining the elements of the first row with the elements of the second and third rows results in

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{pmatrix}.$$

Divide the elements of the second row by -3 , and the elements of the third row by 3 . Now subtracting the new second row from the first row yields

$$\begin{pmatrix} 1 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & -1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{pmatrix}.$$

Finally, combine the third row with the second row to obtain

$$\begin{pmatrix} 1 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

15. Elementary row operations yield

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix}.$$

Finally, combining the first and third rows results in

$$\begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix}, \text{ so } A^{-1} = \begin{pmatrix} 1/2 & -1/4 & 1/8 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

16. Elementary row operations yield

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 1 & 4 & -3 & 0 & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & -1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & 10/3 & -7/3 & -1/3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1/10 & 3/10 & 1/10 \\ 0 & 1 & 0 & -3/15 & 2/5 & -1/5 \\ 0 & 0 & 10/3 & -7/3 & -1/3 & 1 \end{pmatrix}.$$

Finally, normalizing the last row results in

$$\begin{pmatrix} 1 & 0 & 0 & 1/10 & 3/10 & 1/10 \\ 0 & 1 & 0 & -3/15 & 2/5 & -1/5 \\ 0 & 0 & 1 & -7/10 & -1/10 & 3/10 \end{pmatrix}, \text{ so } A^{-1} = \begin{pmatrix} 1/10 & 3/10 & 1/10 \\ -3/15 & 2/5 & -1/5 \\ -7/10 & -1/10 & 3/10 \end{pmatrix}.$$

17. Elementary row operations on the augmented matrix yield the row-reduced form of the augmented matrix

$$\begin{pmatrix} 1 & 0 & -1/7 & 0 & 1/7 & 2/7 \\ 0 & 1 & 3/7 & 0 & 4/7 & 1/7 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{pmatrix}.$$

The left submatrix cannot be converted to the identity matrix. Hence the given matrix is singular.

18. Elementary row operations on the augmented matrix yield

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

19. Elementary row operations on the augmented matrix yield

$$\begin{pmatrix} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 10 & 4 & -4 & 1 \end{pmatrix}.$$

Normalizing the last row and combining it with the others results in

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & -4/5 & 4/5 & -1/5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 6 & 13/5 & -8/5 & 2/5 \\ 0 & 1 & 0 & 0 & 5 & 11/5 & -6/5 & 4/5 \\ 0 & 0 & 1 & 0 & 0 & -1/5 & 1/5 & 1/5 \\ 0 & 0 & 0 & 1 & -2 & -4/5 & 4/5 & -1/5 \end{pmatrix}.$$

20. Suppose that \mathbf{A} is nonsingular, and that there exist matrices \mathbf{B} and \mathbf{C} , such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{I}$. Based on the properties of matrices, it follows that

$$\mathbf{A}(\mathbf{B} - \mathbf{C}) = \mathbf{AY} = \mathbf{0}_{n \times n}.$$

Write the difference of the two matrices, \mathbf{Y} , in terms of its columns as

$$\mathbf{Y} = [\mathbf{y}^{(1)} | \mathbf{y}^{(2)} | \dots | \mathbf{y}^{(n)}].$$

The j -th column of the product matrix, $\mathbf{A}\mathbf{Y}$, can be expressed as $\mathbf{A}\mathbf{y}^{(j)}$. Now since all columns of the product matrix consist only of zeros, we end up with n homogeneous systems of linear equations

$$\mathbf{A}\mathbf{y}^{(j)} = \mathbf{0}_{n \times 1}, j = 1, 2, \dots, n.$$

Since \mathbf{A} is nonsingular, each system must have a trivial solution. So $\mathbf{y}^{(j)} = \mathbf{0}_{n \times 1}$ for $j = 1, 2, \dots, n$. Hence $\mathbf{Y} = \mathbf{0}_{n \times n}$ and $\mathbf{B} = \mathbf{C}$.

21.(a)

$$\begin{aligned} \mathbf{A} + 3\mathbf{B} &= \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} + \begin{pmatrix} 6e^t & 3e^{-t} & 9e^{2t} \\ -3e^t & 6e^{-t} & 3e^{2t} \\ 9e^t & -3e^{-t} & -3e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 7e^t & 5e^{-t} & 10e^{2t} \\ -e^t & 7e^{-t} & 2e^{2t} \\ 8e^t & 0 & -e^{2t} \end{pmatrix}. \end{aligned}$$

(b) Based on the standard definition of matrix multiplication,

$$\mathbf{AB} = \begin{pmatrix} 2e^{2t} - 2 + 3e^{3t} & 1 + 4e^{-2t} - e^t & 3e^{3t} + 2e^t - e^{4t} \\ 4e^{2t} - 1 - 3e^{3t} & 2 + 2e^{-2t} + e^t & 6e^{3t} + e^t + e^{4t} \\ -2e^{2t} - 3 + 6e^{3t} & -1 + 6e^{-2t} - 2e^t & -3e^{3t} + 3e^t - 2e^{4t} \end{pmatrix}.$$

(c)

$$\frac{d\mathbf{A}}{dt} = \begin{pmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{pmatrix}.$$

(d) Note that

$$\int \mathbf{A}(t)dt = \begin{pmatrix} e^t & -2e^{-t} & e^{2t}/2 \\ 2e^t & -e^{-t} & -e^{2t}/2 \\ -e^t & -3e^{-t} & e^{2t} \end{pmatrix} + C.$$

Therefore

$$\begin{aligned} \int_0^1 \mathbf{A}(t)dt &= \begin{pmatrix} e & -2e^{-1} & e^2/2 \\ 2e & -e^{-1} & -e^2/2 \\ -e & -3e^{-1} & e^2 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 1/2 \\ 2 & -1 & -1/2 \\ -1 & -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e-1 & 2-2e^{-1} & e^2/2-1/2 \\ 2e-2 & 1-e^{-1} & 1/2-e^2/2 \\ 1-e & 3-3e^{-1} & e^2-1 \end{pmatrix}. \end{aligned}$$

The result can also be written as

$$(e-1) \begin{pmatrix} 1 & 2/e & \frac{1}{2}(e+1) \\ 2 & 1/e & -\frac{1}{2}(e+1) \\ -1 & 3/e & e+1 \end{pmatrix}.$$

23. First note that

$$\mathbf{x}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (e^t + t e^t) = \begin{pmatrix} 3e^t + 2t e^t \\ 2e^t + 2t e^t \end{pmatrix}.$$

We also have

$$\begin{aligned} \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} &= \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} (t e^t) \\ &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix} (t e^t) = \begin{pmatrix} 2e^t + 2t e^t \\ 3e^t + 2t e^t \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t = \begin{pmatrix} 3e^t + 2t e^t \\ 2e^t + 2t e^t \end{pmatrix}.$$

24. It is easy to see that

$$\mathbf{x}' = \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t} = \begin{pmatrix} -6e^{-t} \\ 8e^{-t} + 4e^{2t} \\ 4e^{-t} - 4e^{-2t} \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x} &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t}. \end{aligned}$$

26. Differentiation, elementwise, results in

$$\Psi' = \begin{pmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \Psi &= \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} e^t & e^{-2t} & e^{3t} \\ -4e^t & -e^{-2t} & 2e^{3t} \\ -e^t & -e^{-2t} & e^{3t} \end{pmatrix} \\ &= \begin{pmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{pmatrix}. \end{aligned}$$

7.3

4. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array} \right).$$

Adding -2 times the first row to the second row and subtracting the first row from the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right).$$

Adding the negative of the second row to the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We evidently end up with an equivalent system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0. \end{aligned}$$

Since there is no unique solution, let $x_3 = \alpha$, where α is arbitrary. It follows that $x_2 = \alpha$, and $x_1 = -\alpha$. Hence all solutions have the form

$$x = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

5. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right).$$

Adding -3 times the first row to the second row and adding the first row to the last row yields

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right).$$

Now add the negative of the second row to the third row to obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right).$$

We end up with an equivalent linear system

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \\ x_3 &= 0. \end{aligned}$$

Hence the unique solution of the given system of equations is $x_1 = x_2 = x_3 = 0$.

6. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ -2 & -4 & 2 & 4 \\ 2 & 4 & -2 & -4 \end{array}\right).$$

Adding 2 times the first row to the second row and subtracting 2 times the first row from the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

We evidently end up with an equivalent system of equations

$$x_1 + 2x_2 - x_3 = -2.$$

Since there is no unique solution, let $x_2 = \alpha$, and $x_3 = \beta$, where α, β are arbitrary. It follows that $x_1 = -2 - 2\alpha + \beta$. Hence all solutions have the form

$$\mathbf{x} = \begin{pmatrix} -2 - 2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix}.$$

8. Write the given vectors as columns of the matrix

$$\mathbf{X} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is evident that $\det(\mathbf{X}) = 0$. Hence the vectors are linearly dependent. In order to find a linear relationship between them, write $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

We obtain the system of equations

$$c_1 - c_3/2 = 0$$

$$c_2 + 5c_3/2 = 0.$$

Setting $c_3 = 2$, it follows that $c_1 = 1$ and $c_2 = -5$. Hence

$$\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + 2\mathbf{x}^{(3)} = \mathbf{0}.$$

10. The matrix containing the given vectors as columns is

$$\mathbf{X} = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix}.$$

We find that $\det(\mathbf{X}) = -70$. Hence the given vectors are linearly independent.

11. Write the given vectors as columns of the matrix

$$\mathbf{X} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix}.$$

The four vectors are necessarily linearly dependent. Hence there are nonzero scalars such that $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ -2 & 0 & 1 & -2 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right).$$

We end up with an equivalent linear system

$$\begin{aligned} c_1 + c_4 &= 0 \\ c_2 + c_4 &= 0 \\ c_3 &= 0. \end{aligned}$$

Let $c_4 = -1$. Then $c_1 = 1$ and $c_2 = 1$. Therefore we find that

$$\mathbf{x}^{(1)} + \mathbf{x}^{(2)} - \mathbf{x}^{(4)} = \mathbf{0}.$$

12. The matrix containing the given vectors as columns, \mathbf{X} , is of size $n \times m$. Since $n < m$, we can augment the matrix with $m - n$ rows of zeros. The resulting matrix, $\tilde{\mathbf{X}}$, is of size $m \times m$. Since $\tilde{\mathbf{X}}$ is a square matrix, with at least one row of zeros, it follows that $\det(\tilde{\mathbf{X}}) = 0$. Hence the column vectors of $\tilde{\mathbf{X}}$ are linearly dependent. That is, there is a nonzero vector, \mathbf{c} , such that $\tilde{\mathbf{X}}\mathbf{c} = \mathbf{0}_{m \times 1}$. If we write only the first n rows of the latter equation, we have $\mathbf{X}\mathbf{c} = \mathbf{0}_{n \times 1}$. Therefore the column vectors of \mathbf{X} are linearly dependent.

13. By inspection, we find that

$$\mathbf{x}^{(1)}(t) - 2\mathbf{x}^{(2)}(t) = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}.$$

Hence $3\mathbf{x}^{(1)}(t) - 6\mathbf{x}^{(2)}(t) + \mathbf{x}^{(3)}(t) = \mathbf{0}$, and the vectors are linearly dependent.

14. Two vectors are linearly dependent if and only if one is a nonzero scalar multiple of the other. However, there is no nonzero scalar, c , such that $2 \sin t = c \sin t$ and $\sin t = 2c \sin t$ for all $t \in (-\infty, \infty)$. Therefore the vectors are linearly independent.

17. The eigenvalues λ and eigenvectors \mathbf{x} satisfy the equation

$$\begin{pmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(3 - \lambda)(-1 - \lambda) + 8 = 0$, that is,

$$\lambda^2 - 2\lambda + 5 = 0.$$

The eigenvalues are $\lambda_1 = 1 - 2i$ and $\lambda_2 = 1 + 2i$. The components of the eigenvector $\mathbf{x}^{(1)}$ are solutions of the system

$$\begin{pmatrix} 2 + 2i & -2 \\ 4 & -2 + 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The two equations reduce to $(1 + i)x_1 = x_2$. Hence $\mathbf{x}^{(1)} = (1, 1 + i)^T$. Now setting $\lambda = \lambda_2 = 1 + 2i$, we have

$$\begin{pmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with solution given by $\mathbf{x}^{(2)} = (1, 1 - i)^T$.

18. The eigenvalues λ and eigenvectors \mathbf{x} satisfy the equation

$$\begin{pmatrix} -2 - \lambda & 1 \\ 1 & -2 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-2 - \lambda)(-2 - \lambda) - 1 = 0$, that is,

$$\lambda^2 + 4\lambda + 3 = 0.$$

The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. For $\lambda_1 = -3$, the system of equations becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1, -1)^T$. Substituting $\lambda = \lambda_2 = -1$, we have

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equations reduce to $x_1 = x_2$. Hence a solution vector is given by $\mathbf{x}^{(2)} = (1, 1)^T$.

19. The eigenvalues λ and eigenvectors \mathbf{x} satisfy the equation

$$\begin{pmatrix} 1 - \lambda & i \\ -i & 1 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(1 - \lambda)(1 - \lambda) - 1 = 0$, that is,

$$\lambda^2 - 2\lambda = 0.$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. For $\lambda_1 = 0$, the system of equations becomes

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 + ix_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1, i)^T$. Substituting $\lambda = \lambda_2 = 2$, we have

$$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equations reduce to $x_1 = ix_2$. So a solution vector is given by $\mathbf{x}^{(2)} = (1, -i)^T$.

20. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 1 - \lambda & \sqrt{3} \\ \sqrt{3} & -1 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, the determinant of the coefficient matrix must be zero. That is,

$$\lambda^2 - 4 = 0.$$

Hence the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$. Substituting the first eigenvalue, $\lambda = -2$, yields

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The system is equivalent to the equation $\sqrt{3}x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1, -\sqrt{3})^T$. Substitution of $\lambda = 2$ results in

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 = \sqrt{3}x_2$. A corresponding solution vector is $\mathbf{x}^{(2)} = (\sqrt{3}, 1)^T$.

21. The eigenvalues λ and eigenvectors \mathbf{x} satisfy the equation

$$\begin{pmatrix} -3 - \lambda & 3/4 \\ -5 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-3 - \lambda)(1 - \lambda) + 15/4 = 0$, that is,

$$\lambda^2 + 2\lambda + 3/4 = 0.$$

Hence the eigenvalues are $\lambda_1 = -3/2$ and $\lambda_2 = -1/2$. In order to determine the eigenvector corresponding to λ_1 , set $\lambda = -3/2$. The system of equations becomes

$$\begin{pmatrix} -3/2 & 3/4 \\ -5 & 5/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $-2x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1, 2)^T$. Substitution of $\lambda = \lambda_2 = -1/2$ results in

$$\begin{pmatrix} -5/2 & 3/4 \\ -5 & 3/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $10x_1 = 3x_2$. A corresponding solution vector is $\mathbf{x}^{(2)} = (3, 10)^T$.

23. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, with roots $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Setting $\lambda = \lambda_1 = 1$, we have

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system reduces to the equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1, 0, -1)^T$. Setting $\lambda = \lambda_2 = 2$, the reduced system of equations is

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (-2, 1, 0)^T$. Finally, setting $\lambda = \lambda_3 = 3$, the reduced system of equations is

$$\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (0, 1, -1)^T$.

24. For computational purposes, note that if λ is an eigenvalue of \mathbf{B} , then $c\lambda$ is an eigenvalue of the matrix $\mathbf{A} = c\mathbf{B}$. Eigenvectors are unaffected, since they are only determined up to a scalar multiple. So with

$$\mathbf{B} = \begin{pmatrix} 11 & -2 & 8 \\ -2 & 2 & 10 \\ 8 & 10 & 5 \end{pmatrix},$$

the associated characteristic equation is $\mu^3 - 18\mu^2 - 81\mu + 1458 = 0$, with roots $\mu_1 = -9$, $\mu_2 = 9$ and $\mu_3 = 18$. Hence the eigenvalues of the given matrix, \mathbf{A} , are $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$. Setting $\lambda = \lambda_1 = -1$, (which corresponds to using $\mu_1 = -9$ in the modified problem) the reduced system of equations is

$$\begin{aligned} 2x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1, 2, -2)^T$. Setting $\lambda = \lambda_2 = 1$, the reduced system of equations is

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - 2x_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (2, -2, -1)^T$. Finally, setting $\lambda = \lambda_2 = 1$, the reduced system of equations is

$$\begin{aligned}x_1 - x_3 &= 0 \\ 2x_2 - x_3 &= 0.\end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (2, 1, 2)^T$.

26. By definition,

$$(\mathbf{Ax}, \mathbf{y}) = \sum_{i=0}^n (\mathbf{Ax})_i \overline{y_i} = \sum_{i=0}^n \sum_{j=0}^n a_{ij} x_j \overline{y_i}.$$

Let $b_{ij} = \overline{a_{ji}}$, so that $a_{ij} = \overline{b_{ji}}$. Now interchanging the order of summation,

$$(\mathbf{Ax}, \mathbf{y}) = \sum_{j=0}^n x_j \sum_{i=0}^n a_{ij} \overline{y_i} = \sum_{j=0}^n x_j \sum_{i=0}^n \overline{b_{ji}} \overline{y_i}.$$

Now note that

$$\sum_{i=0}^n \overline{b_{ji}} \overline{y_i} = \overline{\sum_{i=0}^n b_{ji} y_i} = \overline{(\mathbf{A}^* \mathbf{y})_j}.$$

Therefore

$$(\mathbf{Ax}, \mathbf{y}) = \sum_{j=0}^n x_j \overline{(\mathbf{A}^* \mathbf{y})_j} = (\mathbf{x}, \mathbf{A}^* \mathbf{y}).$$

Now also $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^* \mathbf{y})$. By definition of a Hermitian matrix, $\mathbf{A} = \mathbf{A}^*$.

27. Suppose that $\mathbf{Ax} = \mathbf{0}$, but that $\mathbf{x} \neq \mathbf{0}$. Let $\mathbf{A} = (a_{ij})$. Using elementary row operations, it is possible to transform the matrix into one that is not upper triangular. If it were upper triangular, backsubstitution would imply that $\mathbf{x} = \mathbf{0}$. Hence a linear combination of all the rows results in a row containing only zeros. That is, there are n scalars, β_i , one for each row and not all zero, such that for each column j ,

$$\sum_{i=1}^n \beta_i a_{ij} = 0.$$

Now consider $\mathbf{A}^* = (b_{ij})$. By definition, $b_{ij} = \overline{a_{ji}}$, or $a_{ij} = \overline{b_{ji}}$. It follows that for each j ,

$$\sum_{i=1}^n \beta_i \overline{b_{ji}} = \sum_{k=1}^n \overline{b_{jk}} \beta_k = \sum_{k=1}^n b_{jk} \overline{\beta_k} = 0.$$

Let $\mathbf{y} = (\overline{\beta_1}, \overline{\beta_2}, \dots, \overline{\beta_n})^T$. Hence we have a nonzero vector, \mathbf{y} , such that $\mathbf{A}^* \mathbf{y} = \mathbf{0}$.

29. By linearity,

$$\mathbf{A}(\mathbf{x}^{(0)} + \alpha \boldsymbol{\xi}) = \mathbf{Ax}^{(0)} + \alpha \mathbf{A}\boldsymbol{\xi} = \mathbf{B} + \mathbf{0} = \mathbf{B}.$$

30. Let $c_{ij} = \overline{a_{ji}}$. By the hypothesis, there is a nonzero vector, \mathbf{y} , such that

$$\sum_{j=1}^n c_{ij} y_j = \sum_{j=1}^n \overline{a_{ji}} y_j = 0, \quad i = 1, 2, \dots, n.$$

Taking the conjugate of both sides, and interchanging the indices, we have

$$\sum_{i=1}^n a_{ij} \overline{y_i} = 0.$$

This implies that a linear combination of each row of \mathbf{A} is equal to zero. Now consider the augmented matrix $[\mathbf{A} | \mathbf{B}]$. Replace the last row by

$$\sum_{i=1}^n \overline{y_i} [a_{i1}, a_{i2}, \dots, a_{in}, b_i] = \left[0, 0, \dots, 0, \sum_{i=1}^n \overline{y_i} b_i \right].$$

We find that if $(\mathbf{B}, \mathbf{y}) = 0$, then the last row of the augmented matrix contains only zeros. Hence there are $n - 1$ remaining equations. We can now set $x_n = \alpha$, some parameter, and solve for the other variables in terms of α . Therefore the system of equations $\mathbf{Ax} = \mathbf{b}$ has a solution.

31. If $\lambda = 0$ is an eigenvalue of \mathbf{A} , then there is a nonzero vector, \mathbf{x} , such that

$$\mathbf{Ax} = \lambda \mathbf{x} = \mathbf{0}.$$

That is, $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution. This implies that the mapping defined by \mathbf{A} is not 1-to-1, and hence not invertible. On the other hand, if \mathbf{A} is singular, then $\det(\mathbf{A}) = 0$. Thus, $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution. The latter equation can be written as $\mathbf{Ax} = 0\mathbf{x}$.

32.(a) Based on Problem 26, $(\mathbf{Ax}, \mathbf{x}) = (\mathbf{x}, \mathbf{Ax})$.

(b) Let \mathbf{x} be an eigenvector corresponding to an eigenvalue λ . It then follows that $(\mathbf{Ax}, \mathbf{x}) = (\lambda \mathbf{x}, \mathbf{x})$ and $(\mathbf{x}, \mathbf{Ax}) = (\mathbf{x}, \lambda \mathbf{x})$. Based on the properties of the inner product, $(\lambda \mathbf{x}, \mathbf{x}) = \lambda(\mathbf{x}, \mathbf{x})$ and $(\mathbf{x}, \lambda \mathbf{x}) = \overline{\lambda}(\mathbf{x}, \mathbf{x})$. Then from part (a),

$$\lambda(\mathbf{x}, \mathbf{x}) = \overline{\lambda}(\mathbf{x}, \mathbf{x}).$$

(c) From part (b),

$$(\lambda - \overline{\lambda})(\mathbf{x}, \mathbf{x}) = 0.$$

Based on the definition of an eigenvector, $(\mathbf{x}, \mathbf{x}) = \|\mathbf{x}\|^2 > 0$. Hence we must have $\lambda - \overline{\lambda} = 0$, which implies that λ is real.

33. From Problem 32,

$$(\mathbf{Ax}^{(1)}, \mathbf{x}^{(2)}) = (\mathbf{x}^{(1)}, \mathbf{Ax}^{(2)}).$$

Hence

$$\lambda_1(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \overline{\lambda_2}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \lambda_2(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}),$$

since the eigenvalues are real. Therefore

$$(\lambda_1 - \lambda_2)(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0.$$

Given that $\lambda_1 \neq \lambda_2$, we must have $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$.

7.4

3. Equation (14) states that the Wronskian satisfies the first order linear ODE

$$\frac{dW}{dt} = (p_{11} + p_{22} + \cdots + p_{nn})W.$$

The general solution of this is given by Equation (15):

$$W(t) = C e^{\int (p_{11} + p_{22} + \cdots + p_{nn}) dt},$$

in which C is an arbitrary constant. Let \mathbf{X}_1 and \mathbf{X}_2 be matrices representing two sets of fundamental solutions. It follows that

$$\det(\mathbf{X}_1) = W_1(t) = C_1 e^{\int (p_{11} + p_{22} + \cdots + p_{nn}) dt}$$

$$\det(\mathbf{X}_2) = W_2(t) = C_2 e^{\int (p_{11} + p_{22} + \cdots + p_{nn}) dt}.$$

Hence $\det(\mathbf{X}_1)/\det(\mathbf{X}_2) = C_1/C_2$. Note that $C_2 \neq 0$.

4. First note that $p_{11} + p_{22} = -p(t)$. As shown in Problem 3,

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = c e^{-\int p(t) dt}.$$

For second order linear ODE, the Wronskian (as defined in Chapter 3) satisfies the first order differential equation $W' + p(t)W = 0$. It follows that

$$W[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}] = c_1 e^{-\int p(t) dt}.$$

Alternatively, based on the hypothesis,

$$\mathbf{y}^{(1)} = \alpha_{11} x_{11} + \alpha_{12} x_{12}$$

$$\mathbf{y}^{(2)} = \alpha_{21} x_{11} + \alpha_{22} x_{12}.$$

Direct calculation shows that

$$\begin{aligned} W[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}] &= \begin{vmatrix} \alpha_{11} x_{11} + \alpha_{12} x_{12} & \alpha_{21} x_{11} + \alpha_{22} x_{12} \\ \alpha_{11} x'_{11} + \alpha_{12} x'_{12} & \alpha_{21} x'_{11} + \alpha_{22} x'_{12} \end{vmatrix} \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x'_{12} - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x'_{11} \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{22} - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{21}. \end{aligned}$$

Here we used the fact that $\mathbf{x}'_1 = \mathbf{x}_2$. Hence

$$W[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}] = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}].$$

5. The particular solution satisfies the ODE $(\mathbf{x}^{(p)})' = \mathbf{P}(t)\mathbf{x}^{(p)} + \mathbf{g}(t)$. Now let \mathbf{x} be any solution of the homogeneous equation, $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. We know that $\mathbf{x} = \mathbf{x}^{(c)}$,

in which $\mathbf{x}^{(c)}$ is a linear combination of some fundamental solution. By linearity of the differential equation, it follows that $\mathbf{x} = \mathbf{x}^{(p)} + \mathbf{x}^{(c)}$ is a solution of the ODE. Based on the uniqueness theorem, all solutions must have this form.

7.(a) By definition,

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = \begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = (t^2 - 2t)e^t.$$

(b) The Wronskian vanishes at $t_0 = 0$ and $t_0 = 2$. Hence the vectors are linearly independent on $\mathcal{D} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$.

(c) It follows from Theorem 7.4.3 that one or more of the coefficients of the ODE must be discontinuous at $t_0 = 0$ and $t_0 = 2$. If not, the Wronskian would not vanish.

(d) Let

$$\mathbf{x} = c_1 \begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Then

$$\mathbf{x}' = c_1 \begin{pmatrix} 2t \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \mathbf{x} &= c_1 \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2 \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} e^t \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} c_1 [p_{11}t^2 + 2p_{12}t] + c_2 [p_{11} + p_{12}]e^t \\ c_1 [p_{21}t^2 + 2p_{22}t] + c_2 [p_{21} + p_{22}]e^t \end{pmatrix}. \end{aligned}$$

Comparing coefficients, we find that

$$\begin{aligned} p_{11}t^2 + 2p_{12}t &= 2t \\ p_{11} + p_{12} &= 1 \\ p_{21}t^2 + 2p_{22}t &= 2 \\ p_{21} + p_{22} &= 1. \end{aligned}$$

Solution of this system of equations results in

$$p_{11}(t) = 0, \quad p_{12}(t) = 1, \quad p_{21}(t) = \frac{2-2t}{t^2-2t}, \quad p_{22}(t) = \frac{t^2-2}{t^2-2t}.$$

Hence the vectors are solutions of the ODE

$$\mathbf{x}' = \frac{1}{t^2-2t} \begin{pmatrix} 0 & t^2-2t \\ 2-2t & t^2-2 \end{pmatrix} \mathbf{x}.$$

8. Suppose that the solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(m)}$ are linearly dependent at $t = t_0$. Then there are constants c_1, c_2, \dots, c_m (not all zero) such that

$$c_1 \mathbf{x}^{(1)}(t_0) + c_2 \mathbf{x}^{(2)}(t_0) + \dots + c_m \mathbf{x}^{(m)}(t_0) = \mathbf{0}.$$

Now let $\mathbf{z}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \cdots + c_m \mathbf{x}^{(m)}(t)$. Then clearly, $\mathbf{z}(t)$ is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, with $\mathbf{z}(t_0) = \mathbf{0}$. Furthermore, $\mathbf{y}(t) \equiv \mathbf{0}$ is also a solution, with $\mathbf{y}(t_0) = \mathbf{0}$. By the uniqueness theorem, $\mathbf{z}(t) = \mathbf{y}(t) = \mathbf{0}$. Hence

$$c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \cdots + c_m \mathbf{x}^{(m)}(t) = \mathbf{0}$$

on the entire interval $\alpha < t < \beta$. Going in the other direction is trivial.

9.(a) Let $\mathbf{y}(t)$ be any solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. It follows that

$$\mathbf{z}(t) + \mathbf{y}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \cdots + c_n \mathbf{x}^{(n)}(t) + \mathbf{y}(t)$$

is also a solution. Now let $t_0 \in (\alpha, \beta)$. Then the collection of vectors

$$\mathbf{x}^{(1)}(t_0), \mathbf{x}^{(2)}(t_0), \dots, \mathbf{x}^{(n)}(t_0), \mathbf{y}(t_0)$$

constitutes $n+1$ vectors, each with n components. Based on the assertion in Problem 12, Section 7.3, these vectors are necessarily linearly dependent. That is, there are $n+1$ constants $b_1, b_2, \dots, b_n, b_{n+1}$ (not all zero) such that

$$b_1 \mathbf{x}^{(1)}(t_0) + b_2 \mathbf{x}^{(2)}(t_0) + \cdots + b_n \mathbf{x}^{(n)}(t_0) + b_{n+1} \mathbf{y}(t_0) = \mathbf{0}.$$

From Problem 8, we have

$$b_1 \mathbf{x}^{(1)}(t) + b_2 \mathbf{x}^{(2)}(t) + \cdots + b_n \mathbf{x}^{(n)}(t) + b_{n+1} \mathbf{y}(t) = \mathbf{0}$$

for all $t \in (\alpha, \beta)$. Now $b_{n+1} \neq 0$, otherwise that would contradict the fact that the first n vectors are linearly independent. Hence

$$\mathbf{y}(t) = -\frac{1}{b_{n+1}}(b_1 \mathbf{x}^{(1)}(t) + b_2 \mathbf{x}^{(2)}(t) + \cdots + b_n \mathbf{x}^{(n)}(t)),$$

and the assertion is true.

(b) Consider $\mathbf{z}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \cdots + c_n \mathbf{x}^{(n)}(t)$, and suppose that we also have

$$\mathbf{z}(t) = k_1 \mathbf{x}^{(1)}(t) + k_2 \mathbf{x}^{(2)}(t) + \cdots + k_n \mathbf{x}^{(n)}(t).$$

Based on the assumption,

$$(k_1 - c_1) \mathbf{x}^{(1)}(t) + (k_2 - c_2) \mathbf{x}^{(2)}(t) + \cdots + (k_n - c_n) \mathbf{x}^{(n)}(t) = \mathbf{0}.$$

The collection of vectors

$$\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \dots, \mathbf{x}^{(n)}(t)$$

is linearly independent on $\alpha < t < \beta$. It follows that $k_i - c_i = 0$, for $i = 1, 2, \dots, n$.

7.5

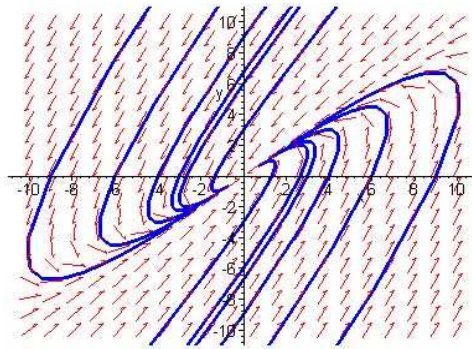
2.(a) Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, and substituting into the ODE, we obtain the algebraic equations

$$\begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 3r + 2 = 0$. The roots of the characteristic equation are $r_1 = -1$ and $r_2 = -2$. For $r = -1$, the two equations reduce to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = -2$ results in the single equation $3\xi_1 = 2\xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2, 3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}.$$

(b)



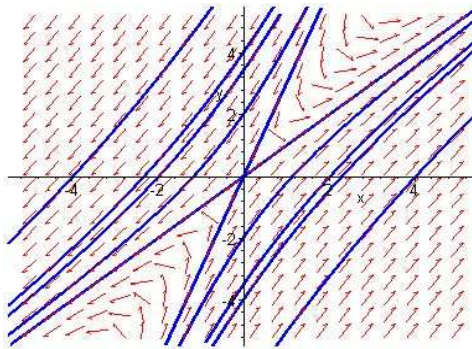
3.(a) Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = -1$ results in the single equation $3\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

(b)



The system has an unstable eigendirection along $\xi^{(1)} = (1, 1)^T$. Unless $c_1 = 0$, all solutions will diverge.

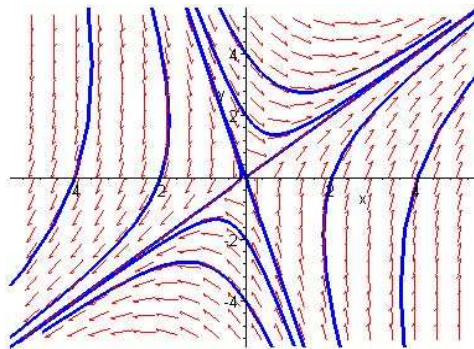
4.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + r - 6 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -3$. For $r = 2$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\xi^{(1)} = (1, 1)^T$. Substitution of $r = -3$ results in the single equation $4\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\xi^{(2)} = (1, -4)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

(b)



The system has an unstable eigendirection along $\xi^{(1)} = (1, 1)^T$. Unless $c_1 = 0$, all solutions will diverge.

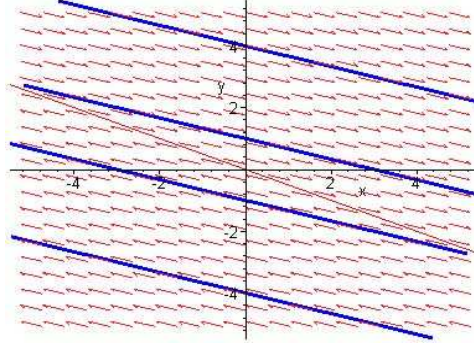
8.(a) Setting $\mathbf{x} = \xi e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 3-r & 6 \\ -1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - r = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = 0$. With $r = 1$, the system of equations reduces to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\xi^{(1)} = (3, -1)^T$. For the case $r = 0$, the system is equivalent to the equation $\xi_1 + 2\xi_2 = 0$. An eigenvector is $\xi^{(2)} = (2, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(b)



The entire line along the eigendirection $\xi^{(2)} = (2, -1)^T$ consists of equilibrium points. All other solutions diverge. The direction field changes across the line $x_1 + 2x_2 = 0$. Eliminating the exponential terms in the solution, the trajectories are given by

$$x_1 + 3x_2 = -c_2.$$

10. The characteristic equation is given by

$$\begin{vmatrix} 2-r & 2+i \\ -1 & -1-i-r \end{vmatrix} = r^2 - (1-i)r - i = 0.$$

The equation has complex roots $r_1 = 1$ and $r_2 = -i$. For $r = 1$, the components of the solution vector must satisfy $\xi_1 + (2+i)\xi_2 = 0$. Thus the corresponding eigenvector is $\xi^{(1)} = (2+i, -1)^T$. Substitution of $r = -i$ results in the single equation $\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\xi^{(2)} = (1, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2+i \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-it}.$$

11. Setting $\mathbf{x} = \xi e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 - 4r^2 - r + 4 = 0$. The roots of the characteristic equation are $r_1 = 4$, $r_2 = 1$ and $r_3 = -1$. Setting $r = 4$, we have

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system reduces to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\xi^{(1)} = (1, 1, 1)^T$. Setting $\lambda = 1$, the reduced system of equations is

$$\begin{aligned}\xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\xi_3 &= 0.\end{aligned}$$

A corresponding solution vector is given by $\xi^{(2)} = (1, -2, 1)^T$. Finally, setting $\lambda = -1$, the reduced system of equations is

$$\begin{aligned}\xi_1 + \xi_3 &= 0 \\ \xi_2 &= 0.\end{aligned}$$

A corresponding solution vector is given by $\xi^{(3)} = (1, 0, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^t + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}.$$

12. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 2 & 4 \\ 2 & -r & 2 \\ 4 & 2 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 6r^2 - 15r - 8 = 0$, with roots $r_1 = 8$, $r_2 = -1$ and $r_3 = -1$. Setting $r = r_1 = 8$, we have

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned}\xi_1 - \xi_3 &= 0 \\ 2\xi_2 - \xi_3 &= 0.\end{aligned}$$

A corresponding solution vector is given by $\xi^{(1)} = (2, 1, 2)^T$. Setting $r = -1$, the system of equations is reduced to the single equation

$$2\xi_1 + \xi_2 + 2\xi_3 = 0.$$

Two independent solutions are obtained as

$$\xi^{(2)} = (1, -2, 0)^T \text{ and } \xi^{(3)} = (0, -2, 1)^T.$$

Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-t}.$$

13. Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -8 & -5 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 + r^2 - 4r - 4 = 0$. The roots of the characteristic equation are $r_1 = 2$, $r_2 = -2$ and $r_3 = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system reduces to the equations

$$\begin{aligned} \xi_1 &= 0 \\ \xi_2 + \xi_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (0, 1, -1)^T$. Setting $\lambda = -1$, the reduced system of equations is

$$\begin{aligned} 2\xi_1 + 3\xi_3 &= 0 \\ \xi_2 - 2\xi_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (3, -4, -2)^T$. Finally, setting $\lambda = -2$, the reduced system of equations is

$$\begin{aligned} 7\xi_1 + 4\xi_3 &= 0 \\ 7\xi_2 - 5\xi_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (4, -5, -7)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} e^{-2t}.$$

15. Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned}c_1 + c_2 &= 2 \\c_1 + 3c_2 &= -1.\end{aligned}$$

Hence $c_1 = 7/2$ and $c_2 = -3/2$, and the solution of the IVP is

$$\mathbf{x} = \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

17. Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 0 & 2-r & 2 \\ -1 & 1 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 - 6r^2 + 11r - 6 = 0$. The roots of the characteristic equation are $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$. Setting $r = 1$, we have

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system reduces to the equations

$$\begin{aligned}\xi_1 &= 0 \\ \xi_2 + 2\xi_3 &= 0.\end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (0, -2, 1)^T$. Setting $\lambda = 2$, the reduced system of equations is

$$\begin{aligned}\xi_1 - \xi_2 &= 0 \\ \xi_3 &= 0.\end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (1, 1, 0)^T$. Finally, upon setting $\lambda = 3$, the reduced system of equations is

$$\begin{aligned}\xi_1 - 2\xi_3 &= 0 \\ \xi_2 - 2\xi_3 &= 0.\end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (2, 2, 1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{3t}.$$

Invoking the initial conditions, the coefficients must satisfy the equations

$$\begin{aligned}c_2 + 2c_3 &= 2 \\ -2c_1 + c_2 + 2c_3 &= 0 \\ c_1 + c_3 &= 1.\end{aligned}$$

It follows that $c_1 = 1$, $c_2 = 2$ and $c_3 = 0$. Hence the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

18. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 0 & -1 \\ 2 & -r & 0 \\ -1 & 2 & 4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 4r^2 - r + 4 = 0$, with roots $r_1 = -1$, $r_2 = 1$ and $r_3 = 4$. Setting $r = r_1 = -1$, we have

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\xi_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1, -2, 1)^T$. Setting $r = 1$, the system reduces to the equations

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 + 2\xi_3 &= 0. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 2, -1)^T$. Finally, upon setting $r = 4$, the system is equivalent to the equations

$$\begin{aligned} 4\xi_1 + \xi_3 &= 0 \\ 8\xi_2 + \xi_3 &= 0. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(3)} = (2, 1, -8)^T$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^t + c_3 \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} e^{4t}.$$

Invoking the initial conditions,

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 7 \\ -2c_1 + 2c_2 + c_3 &= 5 \\ c_1 - c_2 - 8c_3 &= 5. \end{aligned}$$

It follows that $c_1 = 3$, $c_2 = 6$ and $c_3 = -1$. Hence the solution of the IVP is

$$\mathbf{x} = 3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} + 6 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^t - \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} e^{4t}.$$

19. Set $\mathbf{x} = \boldsymbol{\xi} t^r$. Substitution into the system of differential equations results in

$$t \cdot r t^{r-1} \boldsymbol{\xi} = \mathbf{A} \boldsymbol{\xi} t^r,$$

which upon simplification yields is, $\mathbf{A} \boldsymbol{\xi} - r \boldsymbol{\xi} = \mathbf{0}$. Hence the vector $\boldsymbol{\xi}$ and constant r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$.

21. Setting $\mathbf{x} = \boldsymbol{\xi} t^r$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 3)^T$. It follows that

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

The Wronskian of this solution set is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = 2t^6$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

22. As shown in Problem 19, solution of the ODE requires analysis of the equations

$$\begin{pmatrix} 4-r & -3 \\ 8 & -6-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 2r = 0$. The roots of the characteristic equation are $r_1 = 0$ and $r_2 = -2$. For $r = 0$, the system of equations reduces to $4\xi_1 = 3\xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (3, 4)^T$. Setting $r = -2$ results in the single equation $2\xi_1 - \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 2)^T$. It follows that

$$\mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

The Wronskian of this solution set is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = 2t^{-2}$. These solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

23. Setting $\mathbf{x} = \boldsymbol{\xi} t^r$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - r - 2 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -1$. Setting $r = 2$, the system of equations reduces to $\xi_1 - 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2, 1)^T$.

With $r = -1$, the system is equivalent to the equation $2\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 2)^T$. It follows that

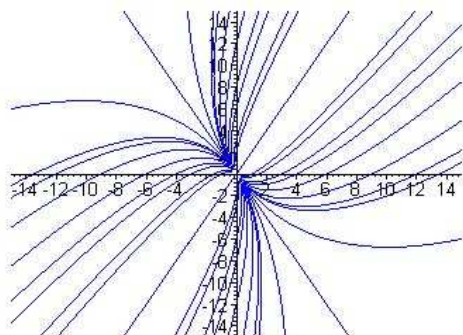
$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 \quad \text{and} \quad \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}.$$

The Wronskian of this solution set is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = 3t$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

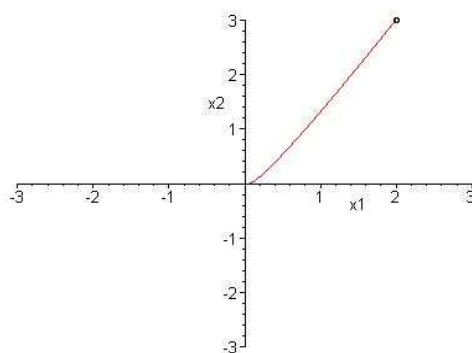
$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}.$$

24.(a) The general solution is

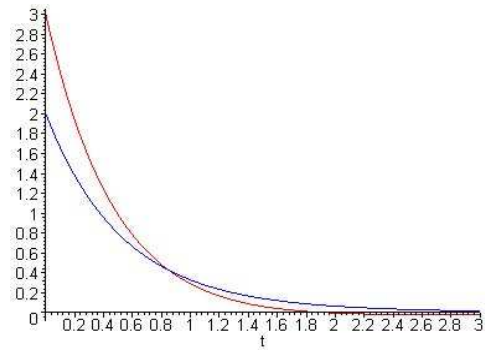
$$x = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}.$$



(b)

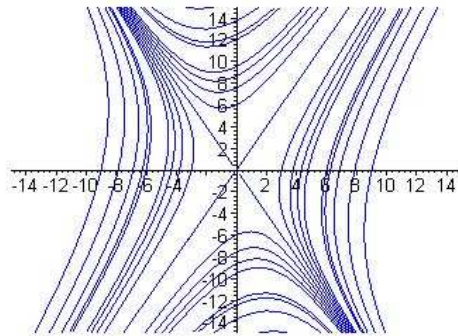


(c)

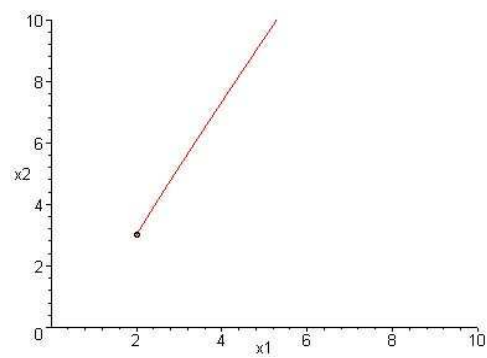


26.(a) The general solution is

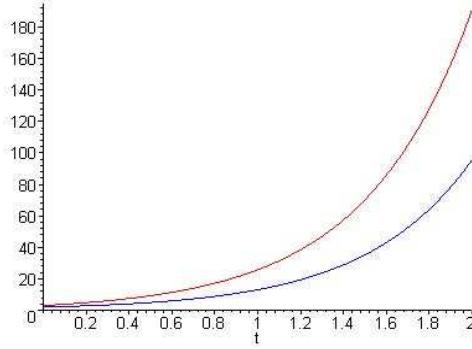
$$x = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}.$$



(b)



(c)



28.(a) We note that $(\mathbf{A} - r_i \mathbf{I})\boldsymbol{\xi}^{(i)} = \mathbf{0}$, for $i = 1, 2$.

(b) It follows that $(\mathbf{A} - r_2 \mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{A}\boldsymbol{\xi}^{(1)} - r_2 \boldsymbol{\xi}^{(1)} = r_1 \boldsymbol{\xi}^{(1)} - r_2 \boldsymbol{\xi}^{(1)}$.

(c) Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then there exist constants c_1 and c_2 , not both zero, such that $c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\xi}^{(2)} = \mathbf{0}$. Assume that $c_1 \neq 0$. It is clear that $(\mathbf{A} - r_2 \mathbf{I})(c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\xi}^{(2)}) = \mathbf{0}$. On the other hand,

$$(\mathbf{A} - r_2 \mathbf{I})(c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\xi}^{(2)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + \mathbf{0} = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}.$$

Since $r_1 \neq r_2$, we must have $c_1 = 0$, which leads to a contradiction.

(d) Note that $(\mathbf{A} - r_1 \mathbf{I})\boldsymbol{\xi}^{(2)} = (r_2 - r_1)\boldsymbol{\xi}^{(2)}$.

(e) Let $n = 3$, with $r_1 \neq r_2 \neq r_3$. Suppose that $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$ are indeed linearly dependent. Then there exist constants c_1 , c_2 and c_3 , not all zero, such that

$$c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\xi}^{(2)} + c_3 \boldsymbol{\xi}^{(3)} = \mathbf{0}.$$

Assume that $c_1 \neq 0$. It is clear that $(\mathbf{A} - r_2 \mathbf{I})(c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\xi}^{(2)} + c_3 \boldsymbol{\xi}^{(3)}) = \mathbf{0}$. On the other hand,

$$(\mathbf{A} - r_2 \mathbf{I})(c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\xi}^{(2)} + c_3 \boldsymbol{\xi}^{(3)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + c_3(r_3 - r_2)\boldsymbol{\xi}^{(3)}.$$

It follows that $c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + c_3(r_3 - r_2)\boldsymbol{\xi}^{(3)} = \mathbf{0}$. Based on the result of part (a), which is actually not dependent on the value of n , the vectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(3)}$ are linearly independent. Hence we must have $c_1(r_1 - r_2) = c_3(r_3 - r_2) = 0$, which leads to a contradiction.

29.(a) Let $x_1 = y$ and $x_2 = y'$. It follows that $x'_1 = x_2$ and

$$x'_2 = y'' = -\frac{1}{a}(cy + by').$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= -\frac{1}{a}(cx_1 + bx_2). \end{aligned}$$

(b) The coefficient matrix is given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}.$$

Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -r & 1 \\ -\frac{c}{a} & -\frac{b}{a} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have

$$\det(\mathbf{A} - r\mathbf{I}) = r^2 + \frac{b}{a}r + \frac{c}{a} = 0.$$

Multiplying both sides of the equation by a , we obtain $ar^2 + br + c = 0$.

30.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -1/10 - r & 3/40 \\ 1/10 & -1/5 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = 0$. The characteristic equation is $80r^2 + 24r + 1 = 0$, with roots $r_1 = -1/4$ and $r_2 = -1/20$. With $r = -1/4$, the system of equations reduces to $2\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, -2)^T$. Substitution of $r = -1/20$ results in the equation $2\xi_1 - 3\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (3, 2)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/4} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t/20}.$$

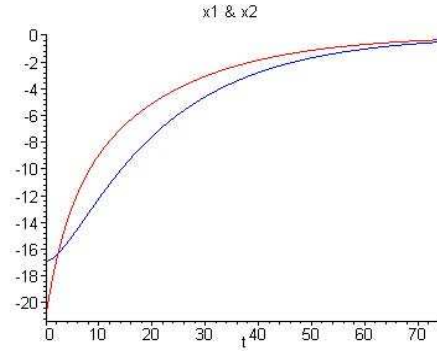
Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + 3c_2 &= -17 \\ -2c_1 + 2c_2 &= -21. \end{aligned}$$

Hence $c_1 = 29/8$ and $c_2 = -55/8$, and the solution of the IVP is

$$\mathbf{x} = \frac{29}{8} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/4} - \frac{55}{8} \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t/20}.$$

(b)



(c) Both functions are monotone increasing. It is easy to show that $-0.5 \leq x_1(t) < 0$ and $-0.5 \leq x_2(t) < 0$ provided that $t > T \approx 74.39$.

31.(a) For $\alpha = 1/2$, solution of the ODE requires that

$$\begin{pmatrix} -1-r & -1 \\ -1/2 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $2r^2 + 4r + 1 = 0$, with roots $r_1 = -1 + 1/\sqrt{2}$ and $r_2 = -1 - 1/\sqrt{2}$. With $r = -1 + 1/\sqrt{2}$, the system of equations is $\sqrt{2}\xi_1 + 2\xi_2 = 0$. The corresponding eigenvector is $\xi^{(1)} = (-\sqrt{2}, 1)^T$. Substitution of $r = -1 - 1/\sqrt{2}$ results in the equation $\sqrt{2}\xi_1 - 2\xi_2 = 0$. An eigenvector is $\xi^{(2)} = (\sqrt{2}, 1)^T$. The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{(-2+\sqrt{2})t/2} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{(-2-\sqrt{2})t/2}.$$

The eigenvalues are distinct and both negative. The equilibrium point is a stable node.

(b) For $\alpha = 2$, the characteristic equation is given by $r^2 + 2r - 1 = 0$, with roots $r_1 = -1 + \sqrt{2}$ and $r_2 = -1 - \sqrt{2}$. With $r = -1 + \sqrt{2}$, the system of equations reduces to $\sqrt{2}\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\xi^{(1)} = (1, -\sqrt{2})^T$. Substitution of $r = -1 - \sqrt{2}$ results in the equation $\sqrt{2}\xi_1 - \xi_2 = 0$. An eigenvector is $\xi^{(2)} = (1, \sqrt{2})^T$. The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{(-1+\sqrt{2})t} + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{(-1-\sqrt{2})t}.$$

The eigenvalues are of opposite sign, hence the equilibrium point is a saddle point.

32.(a) The system of differential equations is

$$\frac{d}{dt} \begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & -5/2 \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

Solution of the system requires analysis of the eigenvalue problem

$$\begin{pmatrix} -1/2 - r & -1/2 \\ 3/2 & -5/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 3r + 2 = 0$, with roots $r_1 = -1$ and $r_2 = -2$. With $r = -1$, the equations reduce to $\xi_1 - \xi_2 = 0$. A corresponding eigenvector is given by $\xi^{(1)} = (1, 1)^T$. Setting $r = -2$, the system reduces to the equation $3\xi_1 - \xi_2 = 0$. An eigenvector is $\xi^{(2)} = (1, 3)^T$. Hence the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t}.$$

(b) The eigenvalues are distinct and both negative. We find that the equilibrium point $(0, 0)$ is a stable node. Hence all solutions converge to $(0, 0)$.

33.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -\frac{R_1}{L} - r & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is

$$r^2 + \left(\frac{L + CR_1R_2}{LCR_2} \right) r + \frac{R_1 + R_2}{LCR_2} = 0.$$

The eigenvectors are real and distinct, provided that the discriminant is positive. That is,

$$\left(\frac{L + CR_1R_2}{LCR_2} \right)^2 - 4 \left(\frac{R_1 + R_2}{LCR_2} \right) > 0,$$

which simplifies to the condition

$$\left(\frac{1}{CR_2} - \frac{R_1}{L} \right)^2 - \frac{4}{LC} > 0.$$

(b) The parameters in the ODE are all positive. Observe that the sum of the roots is

$$-\frac{L + CR_1R_2}{LCR_2} < 0.$$

Also, the product of the roots is

$$\frac{R_1 + R_2}{LCR_2} > 0.$$

It follows that both roots are negative. Hence the equilibrium solution $I = 0, V = 0$ represents a stable node, which attracts all solutions.

(c) If the condition in part (a) is not satisfied, that is,

$$\left(\frac{1}{CR_2} - \frac{R_1}{L} \right)^2 - \frac{4}{LC} \leq 0,$$

then the real part of the eigenvalues is

$$\operatorname{Re}(r_{1,2}) = -\frac{L + CR_1R_2}{2LCR_2}.$$

As long as the parameters are all positive, then the solutions will still converge to the equilibrium point $(0, 0)$.

7.6

2.(a) Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -1-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

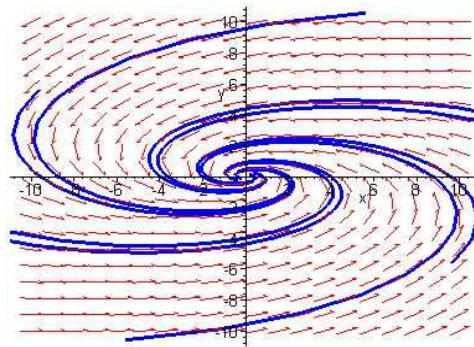
For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 2r + 5 = 0$. The roots of the characteristic equation are $r = -1 \pm 2i$. Substituting $r = -1 - 2i$, the two equations reduce to $\xi_1 + 2i\xi_2 = 0$. The two eigenvectors are $\boldsymbol{\xi}^{(1)} = (-2i, 1)^T$ and $\boldsymbol{\xi}^{(2)} = (2i, 1)^T$. Hence one of the complex-valued solutions is given by

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-(1+2i)t} = \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-t} (\cos 2t - i \sin 2t) = \\ &= e^{-t} \begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} + i e^{-t} \begin{pmatrix} -2 \cos 2t \\ -\sin 2t \end{pmatrix}. \end{aligned}$$

Based on the real and imaginary parts of this solution, the general solution is

$$\mathbf{x} = c_1 e^{-t} \begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix}.$$

(b)



3.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2 + i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2 + i, 1)^T$ and $\boldsymbol{\xi}^{(2)} = (2 - i, 1)^T$. Hence one of the complex-valued solutions is given by

$$\begin{aligned}\mathbf{x}^{(1)} &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos t + i \sin t) = \\ &= \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix}.\end{aligned}$$

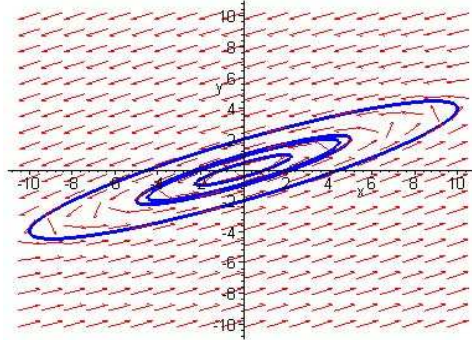
Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin t \\ -\cos t + 2 \sin t \end{pmatrix}.$$

(b)



4.(a) Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -5/2 \\ 9/5 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 - r + \frac{5}{2} = 0$. The roots of the characteristic equation are $r = (1 \pm 3i)/2$. With $r = (1 + 3i)/2$, the equations reduce to the single equation $(3 - 3i)\xi_1 - 5\xi_2 = 0$. The corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (5, 3 - 3i)^T$. Hence one of the complex-valued solutions is

$$\begin{aligned}\mathbf{x}^{(1)} &= \begin{pmatrix} 5 \\ 3-3i \end{pmatrix} e^{(1+3i)t/2} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{t/2} \left(\cos \frac{3}{2}t + i \sin \frac{3}{2}t \right) = \\ &= e^{t/2} \begin{pmatrix} 2 \cos \frac{3}{2}t - \sin \frac{3}{2}t \\ \cos \frac{3}{2}t \end{pmatrix} + ie^{t/2} \begin{pmatrix} \cos \frac{3}{2}t + 2 \sin \frac{3}{2}t \\ \sin \frac{3}{2}t \end{pmatrix}.\end{aligned}$$

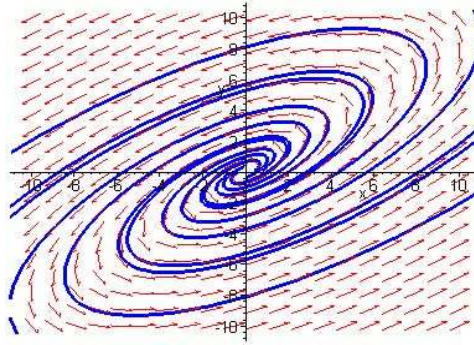
The general solution is

$$\mathbf{x} = c_1 e^{t/2} \begin{pmatrix} 2 \cos \frac{3}{2}t - \sin \frac{3}{2}t \\ \cos \frac{3}{2}t \end{pmatrix} + c_2 e^{t/2} \begin{pmatrix} \cos \frac{3}{2}t + 2 \sin \frac{3}{2}t \\ \sin \frac{3}{2}t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1 e^{t/2} \begin{pmatrix} 5 \cos \frac{3}{2}t \\ 3 \cos \frac{3}{2}t + 3 \sin \frac{3}{2}t \end{pmatrix} + c_2 e^{t/2} \begin{pmatrix} 5 \sin \frac{3}{2}t \\ -3 \cos \frac{3}{2}t + 3 \sin \frac{3}{2}t \end{pmatrix}.$$

(b)



5.(a) Setting $\mathbf{x} = \boldsymbol{\xi} t^r$ results in the algebraic equations

$$\begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

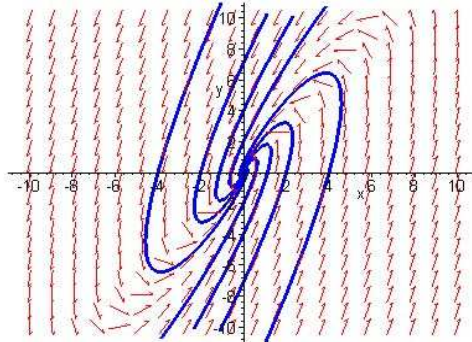
The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Substituting $r = -1 - i$ reduces the system of equations to $(2+i)\xi_1 - \xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (1, 2+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1, 2-i)^T$. Hence one of the complex-valued solutions is given by

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-(1+i)t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t} (\cos t - i \sin t) = \\ &= e^{-t} \begin{pmatrix} \cos t \\ 2 \cos t + \sin t \end{pmatrix} + i e^{-t} \begin{pmatrix} -\sin t \\ \cos t - 2 \sin t \end{pmatrix}. \end{aligned}$$

The general solution is

$$\mathbf{x} = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ -\cos t + 2 \sin t \end{pmatrix}.$$

(b)



6.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 1-r & 2 \\ -5 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

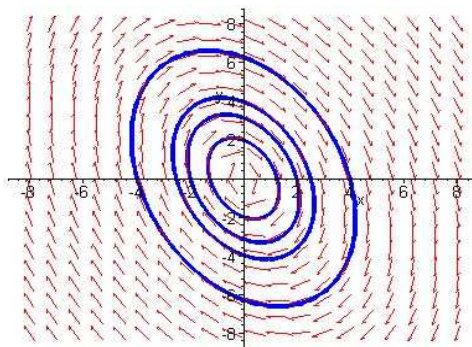
For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 9 = 0$. The roots of the characteristic equation are $r = \pm 3i$. Setting $r = 3i$, the two equations reduce to $(1 - 3i)\xi_1 + 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (-2, 1 - 3i)^T$. Hence one of the complex-valued solutions is given by

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} -2 \\ 1 - 3i \end{pmatrix} e^{3it} = \begin{pmatrix} -2 \\ 1 - 3i \end{pmatrix} (\cos 3t + i \sin 3t) = \\ &= \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} + i \begin{pmatrix} -2 \sin 3t \\ -3 \cos 3t + \sin 3t \end{pmatrix}. \end{aligned}$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 2 \sin 3t \\ 3 \cos 3t - \sin 3t \end{pmatrix}.$$

(b)



8. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -3-r & 0 & 2 \\ 1 & -1-r & 0 \\ -2 & -1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 + 4r^2 + 7r + 6 = 0$, with roots $r_1 = -2$, $r_2 = -1 - \sqrt{2}i$ and $r_3 = -1 + \sqrt{2}i$. Setting $r = -2$, the equations reduce to

$$\begin{aligned} -\xi_1 + 2\xi_3 &= 0 \\ \xi_1 + \xi_2 &= 0. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2, -2, 1)^T$. With $r = -1 - \sqrt{2}i$, the system of equations is equivalent to

$$\begin{aligned} (2 - i\sqrt{2})\xi_1 - 2\xi_3 &= 0 \\ \xi_1 + i\sqrt{2}\xi_2 &= 0. \end{aligned}$$

An eigenvector is given by $\boldsymbol{\xi}^{(2)} = (-i\sqrt{2}, 1, -1 - i\sqrt{2})^T$. Hence one of the complex-valued solutions is given by

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1 - i\sqrt{2} \end{pmatrix} e^{-(1+i\sqrt{2})it} = \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1 - i\sqrt{2} \end{pmatrix} e^{-t} (\cos \sqrt{2}t - i \sin \sqrt{2}t) = \\ &= e^{-t} \begin{pmatrix} -\sqrt{2} \sin \sqrt{2}t \\ \cos \sqrt{2}t \\ -\cos \sqrt{2}t - \sqrt{2} \sin \sqrt{2}t \end{pmatrix} + ie^{-t} \begin{pmatrix} -\sqrt{2} \cos \sqrt{2}t \\ -\sin \sqrt{2}t \\ -\sqrt{2} \cos \sqrt{2}t - \sin \sqrt{2}t \end{pmatrix}. \end{aligned}$$

The other complex-valued solution is $\mathbf{x}^{(3)} = \overline{\boldsymbol{\xi}^{(2)}} e^{r_3 t}$. The general solution is

$$\begin{aligned} \mathbf{x} &= c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} + \\ &+ c_2 e^{-t} \begin{pmatrix} \sqrt{2} \sin \sqrt{2}t \\ -\cos \sqrt{2}t \\ \cos \sqrt{2}t + \sqrt{2} \sin \sqrt{2}t \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} \sqrt{2} \cos \sqrt{2}t \\ \sin \sqrt{2}t \\ \sqrt{2} \cos \sqrt{2}t + \sin \sqrt{2}t \end{pmatrix}. \end{aligned}$$

It is easy to see that all solutions converge to the equilibrium point $(0, 0, 0)$.

10. Solution of the system of ODEs requires that

$$\begin{pmatrix} -3-r & 2 \\ -1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 4r + 5 = 0$, with roots $r = -2 \pm i$. Substituting $r = -2 + i$, the equations are equivalent to $\xi_1 - (1 - i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1 - i, 1)^T$. One of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(-2+i)t} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{-2t} (\cos t + i \sin t) =$$

$$= e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i e^{-2t} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

Hence the general solution is

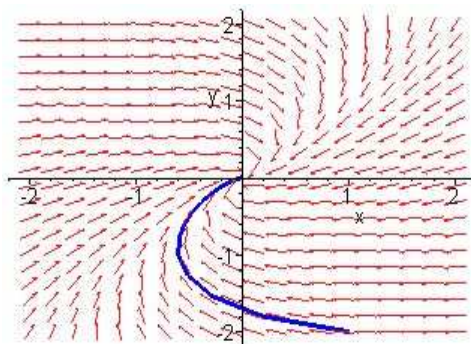
$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 - c_2 &= 1 \\ c_1 &= -2. \end{aligned}$$

Solving for the coefficients, the solution of the initial value problem is

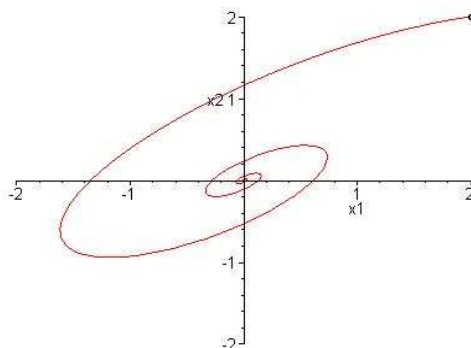
$$\begin{aligned} \mathbf{x} &= -2 e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} - 3 e^{-2t} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix} \\ &= e^{-2t} \begin{pmatrix} \cos t - 5 \sin t \\ -2 \cos t - 3 \sin t \end{pmatrix}. \end{aligned}$$



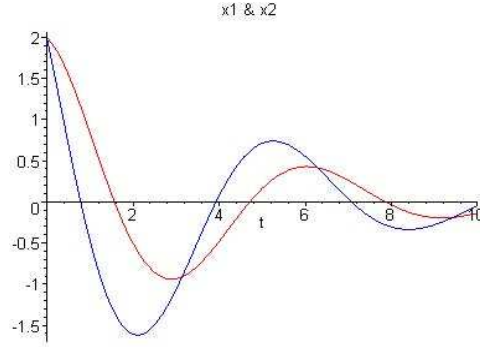
11.(a) With $\mathbf{x}(0) = (2, 2)^T$, the solution is

$$\mathbf{x} = e^{-t/4} \begin{pmatrix} 2 \cos t - 2 \sin t \\ 2 \cos t \end{pmatrix}.$$

(b)



(c)



12. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{4}{5} - r & 2 \\ -1 & \frac{6}{5} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $25r^2 - 10r + 26 = 0$, with roots $r = \frac{1}{5} \pm i$. Setting $r = 1/5 + i$, the two equations reduce to $\xi_1 - (1 - i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1 - i, 1)^T$. One of the complex-valued solutions is given by

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} e^{(\frac{1}{5} + i)t} = \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} e^{t/5} (\cos t + i \sin t) = \\ &= e^{t/5} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i e^{t/5} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}. \end{aligned}$$

Hence the general solution is

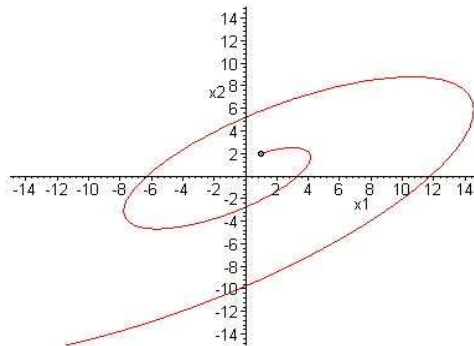
$$\mathbf{x} = c_1 e^{t/5} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{t/5} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

(b) Let $\mathbf{x}(0) = (x_1^0, x_2^0)^T$. The solution of the initial value problem is

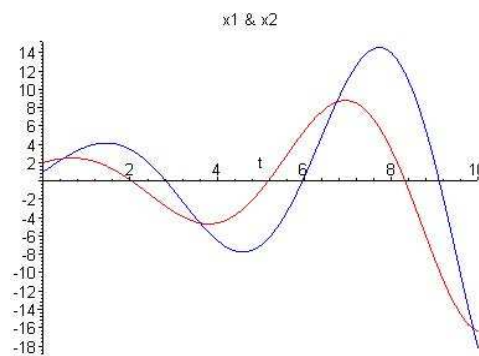
$$\begin{aligned} \mathbf{x} &= x_2^0 e^{t/5} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + (x_2^0 - x_1^0) e^{t/5} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix} \\ &= e^{t/5} \begin{pmatrix} x_1^0 \cos t + (2x_2^0 - x_1^0) \sin t \\ x_2^0 \cos t + (x_2^0 - x_1^0) \sin t \end{pmatrix}. \end{aligned}$$

With $\mathbf{x}(0) = (1, 2)^T$, the solution is

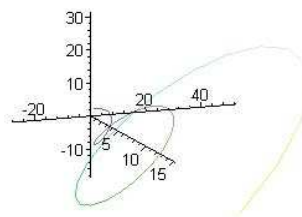
$$\mathbf{x} = e^{t/5} \begin{pmatrix} \cos t + 3 \sin t \\ 2 \cos t + \sin t \end{pmatrix}.$$



(c)



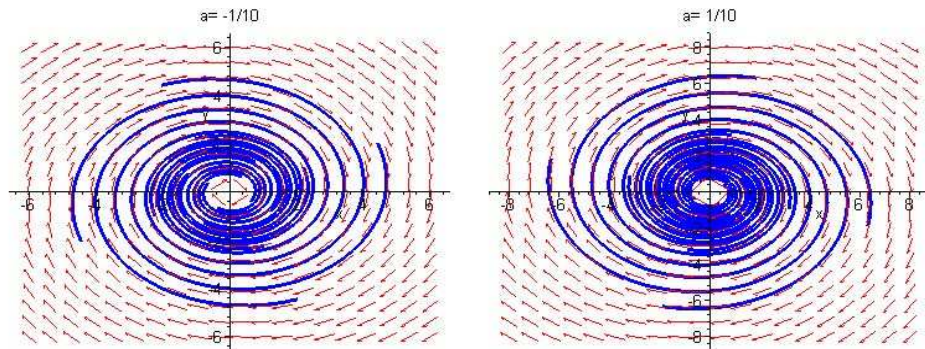
(d)



13.(a) The characteristic equation is $r^2 - 2\alpha r + 1 + \alpha^2 = 0$, with roots $r = \alpha \pm i$.

(b) When $\alpha < 0$ and $\alpha > 0$, the equilibrium point $(0, 0)$ is a stable spiral and an unstable spiral, respectively. The equilibrium point is a center when $\alpha = 0$.

(c)

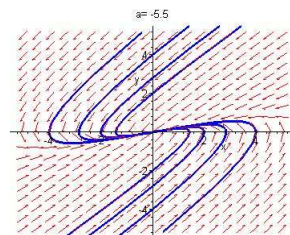
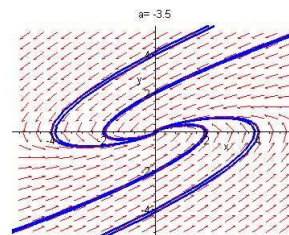
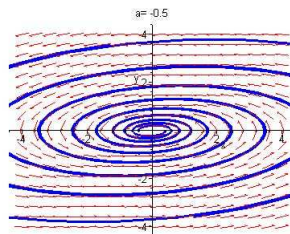
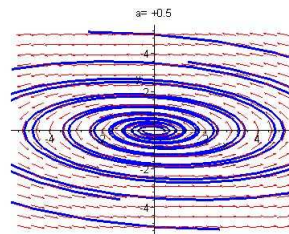


14.(a) The roots of the characteristic equation, $r^2 - \alpha r + 5 = 0$, are

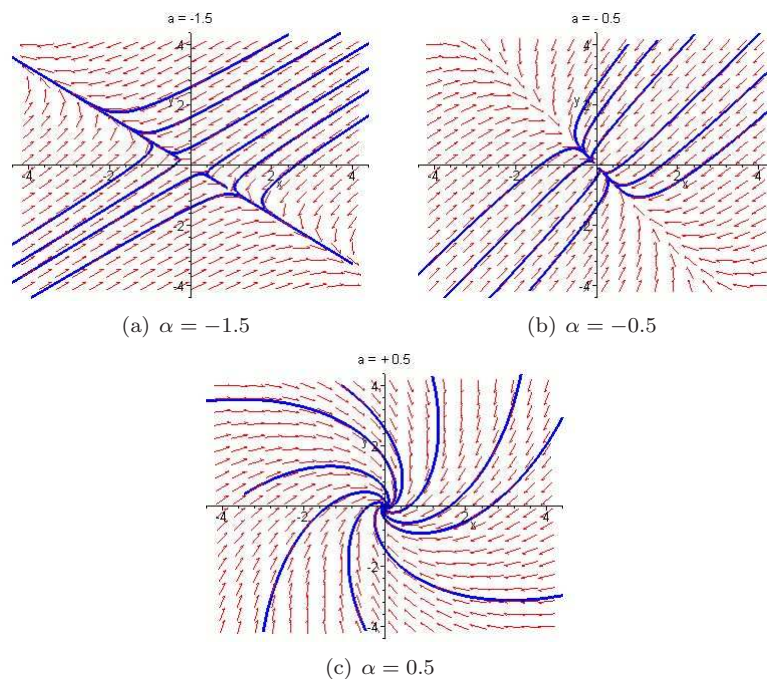
$$r_{1,2} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 20}.$$

(b) Note that the roots are complex when $-\sqrt{20} < \alpha < \sqrt{20}$. For the case when $\alpha \in (-\sqrt{20}, 0)$, the equilibrium point $(0, 0)$ is a stable spiral. On the other hand, when $\alpha \in (0, \sqrt{20})$, the equilibrium point is an unstable spiral. For the case $\alpha = 0$, the roots are purely imaginary, so the equilibrium point is a center. When $\alpha^2 > 20$, the roots are real and distinct. The equilibrium point becomes a node, with its stability dependent on the sign of α . Finally, the case $\alpha^2 = 20$ marks the transition from spirals to nodes.

(c)

(a) $\alpha = -5.5$ (b) $\alpha = -3.5$ (c) $\alpha = -0.5$ (d) $\alpha = 0.5$

17. The characteristic equation of the coefficient matrix is $r^2 + 2r + 1 + \alpha = 0$, with roots given formally as $r_{1,2} = -1 \pm \sqrt{-\alpha}$. The roots are real provided that $\alpha \leq 0$. First note that the sum of the roots is -2 and the product of the roots is $1 + \alpha$. For negative values of α , the roots are distinct, with one always negative. When $\alpha < -1$, the roots have opposite signs. Hence the equilibrium point is a saddle. For the case $-1 < \alpha < 0$, the roots are both negative, and the equilibrium point is a stable node. $\alpha = -1$ represents a transition from saddle to node. When $\alpha = 0$, both roots are equal. For the case $\alpha > 0$, the roots are complex conjugates, with negative real part. Hence the equilibrium point is a stable spiral.



19. The characteristic equation for the system is given by

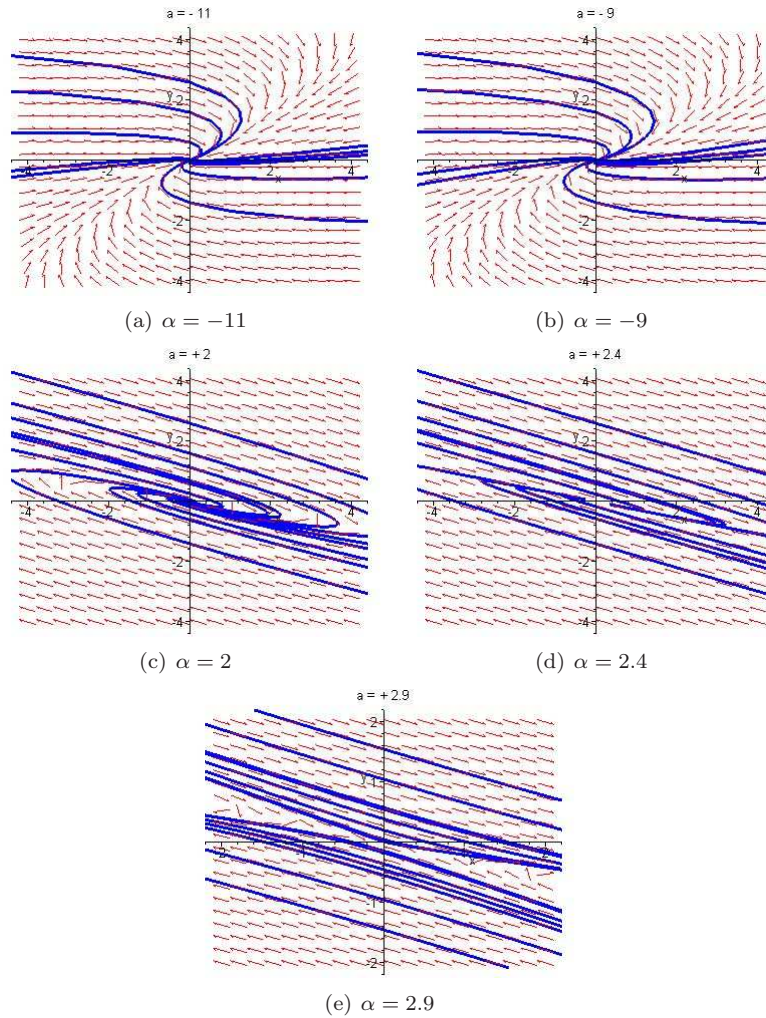
$$r^2 + (4 - \alpha)r + 10 - 4\alpha = 0.$$

The roots are

$$r_{1,2} = -2 + \frac{\alpha}{2} \pm \sqrt{\alpha^2 + 8\alpha - 24}.$$

First note that the roots are complex when $-4 - 2\sqrt{10} < \alpha < -4 + 2\sqrt{10}$. We also find that when $-4 - 2\sqrt{10} < \alpha < 2$, the equilibrium point is a stable spiral. For the case $\alpha = 2$, the equilibrium point is a center. When $2 < \alpha < -4 + 2\sqrt{10}$, the equilibrium point is an unstable spiral. For all other cases, the roots are real. When $\alpha > 2.5$, the roots have opposite signs, with the equilibrium point being a

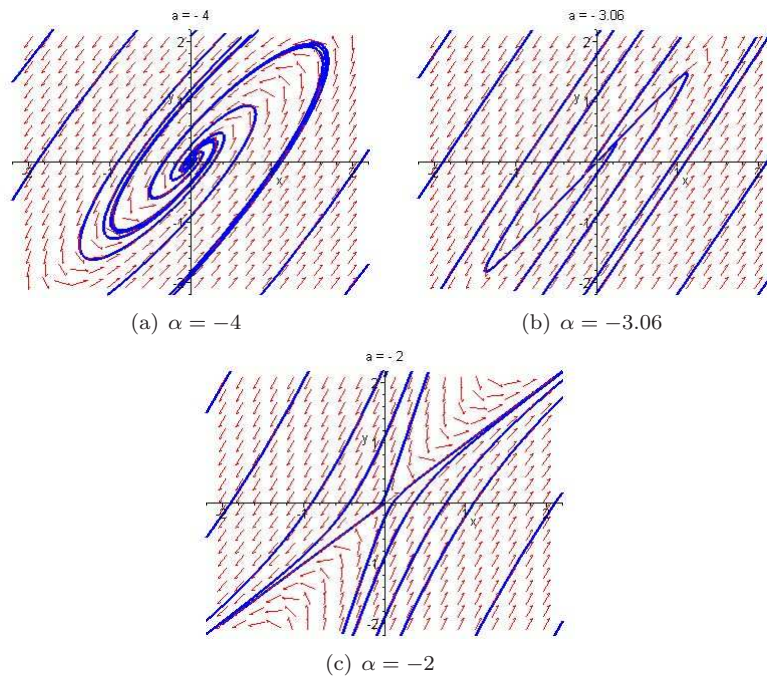
saddle. For the case $-4 + 2\sqrt{10} < \alpha < 2.5$, the roots are both positive, and the equilibrium point is an unstable node. Finally, when $\alpha < -4 - 2\sqrt{10}$, both roots are negative, with the equilibrium point being a stable node.



20. The characteristic equation is $r^2 + 2r - (24 + 8\alpha) = 0$, with roots

$$r_{1,2} = -1 \pm \sqrt{25 + 8\alpha}.$$

The roots are complex when $\alpha < -25/8$. Since the real part is negative, the origin is a stable spiral. Otherwise the roots are real. When $-25 < \alpha < -3$, both roots are negative, and hence the equilibrium point is a stable node. For $\alpha > -3$, the roots are of opposite sign and the origin is a saddle.



22. Based on the method in Problem 19 of Section 7.5, setting $\mathbf{x} = \boldsymbol{\xi} t^r$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation for the system is $r^2 + 1 = 0$, with roots $r_{1,2} = \pm i$. With $r = i$, the equations reduce to the single equation $\xi_1 - (2+i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i, 1)^T$. One complex-valued solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} t^i.$$

We can write $t^i = e^{i \ln t}$. Hence

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{i \ln t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} [\cos(\ln t) + i \sin(\ln t)] = \\ &= \begin{pmatrix} 2 \cos(\ln t) - \sin(\ln t) \\ \cos(\ln t) \end{pmatrix} + i \begin{pmatrix} \cos(\ln t) + 2 \sin(\ln t) \\ \sin(\ln t) \end{pmatrix}. \end{aligned}$$

Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \cos(\ln t) - \sin(\ln t) \\ \cos(\ln t) \end{pmatrix} + c_2 \begin{pmatrix} \cos(\ln t) + 2 \sin(\ln t) \\ \sin(\ln t) \end{pmatrix}.$$

Other combinations are also possible.

24.(a) The characteristic equation of the system is

$$r^3 + \frac{2}{5}r^2 + \frac{81}{80}r - \frac{17}{160} = 0,$$

with eigenvalues $r_1 = 1/10$, and $r_{2,3} = -1/4 \pm i$. For $r = 1/10$, simple calculations reveal that a corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (0, 0, 1)^T$. Setting $r = -1/4 - i$, we obtain the system of equations

$$\begin{aligned}\xi_1 - i\xi_2 &= 0 \\ \xi_3 &= 0.\end{aligned}$$

A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (i, 1, 0)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10}.$$

Another solution, which is complex-valued, is given by

$$\begin{aligned}\mathbf{x}^{(2)} &= \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-(\frac{1}{4}+i)t} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-t/4} (\cos t - i \sin t) = \\ &= e^{-t/4} \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + ie^{-t/4} \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}.\end{aligned}$$

Using the real and imaginary parts of $\mathbf{x}^{(2)}$, the general solution is constructed as

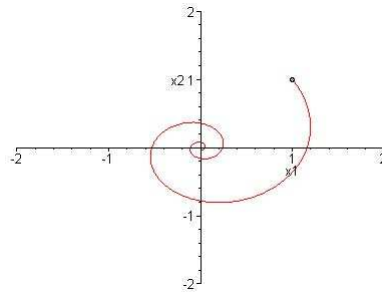
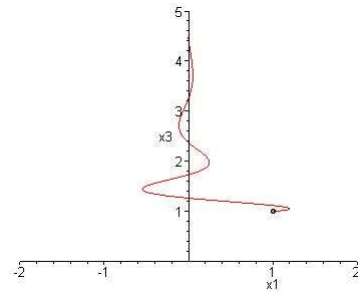
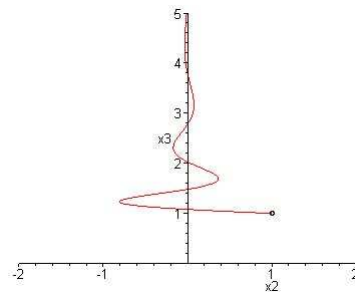
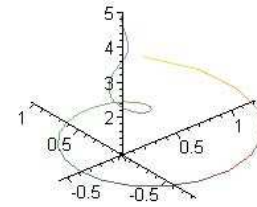
$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10} + c_2 e^{-t/4} \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + c_3 e^{-t/4} \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}.$$

(b) Let $\mathbf{x}(0) = (x_1^0, x_2^0, x_3^0)$. The solution can be written as

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ x_3^0 e^{t/10} \end{pmatrix} + e^{-t/4} \begin{pmatrix} x_2^0 \sin t + x_1^0 \cos t \\ x_2^0 \cos t - x_1^0 \sin t \\ 0 \end{pmatrix}.$$

With $\mathbf{x}(0) = (1, 1, 1)$, the solution of the initial value problem is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ e^{t/10} \end{pmatrix} + e^{-t/4} \begin{pmatrix} \sin t + \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix}.$$

(a) $x_1 - x_2$ (b) $x_1 - x_3$ (c) $x_2 - x_3$ (d) $x_1 x_2 x_3$

25.(a) Based on Problems 19-21 of Section 7.1, the system of differential equations is

$$\frac{d}{dt} \begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

With $R_1 = R_2 = 4$ ohms, $C = \frac{1}{2}$ farads and $L = 8$ henrys, the eigenvalue problem is

$$\begin{pmatrix} -\frac{1}{2} - r & -\frac{1}{8} \\ 2 & -\frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) The characteristic equation of the system is $r^2 + r + \frac{1}{2} = 0$, with eigenvalues

$$r_{1,2} = -\frac{1}{2} \pm \frac{1}{2}i.$$

Setting $r = -1/2 + i/2$, the algebraic equations reduce to $4i\xi_1 + \xi_2 = 0$. It follows that $\xi^{(1)} = (1, -4i)^T$. Hence one complex-valued solution is

$$\begin{aligned} \begin{pmatrix} I \\ V \end{pmatrix}^{(1)} &= \begin{pmatrix} 1 \\ -4i \end{pmatrix} e^{(-1+i)t/2} = \begin{pmatrix} 1 \\ -4i \end{pmatrix} e^{-t/2} [\cos(t/2) + i \sin(t/2)] = \\ &= e^{-t/2} \begin{pmatrix} \cos(t/2) \\ 4 \sin(t/2) \end{pmatrix} + i e^{-t/2} \begin{pmatrix} \sin(t/2) \\ -4 \cos(t/2) \end{pmatrix}. \end{aligned}$$

Therefore the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t/2} \begin{pmatrix} \cos(t/2) \\ 4 \sin(t/2) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \sin(t/2) \\ -4 \cos(t/2) \end{pmatrix}.$$

(c) Imposing the initial conditions, we arrive at the equations $c_1 = 2$ and $c_2 = -\frac{3}{4}$, and

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t/2} \begin{pmatrix} 2 \cos(t/2) - \frac{3}{4} \sin(t/2) \\ 8 \sin(t/2) + 3 \cos(t/2) \end{pmatrix}.$$

(d) Since the eigenvalues have negative real parts, all solutions converge to the origin.

26.(a) The characteristic equation of the system is

$$r^2 + \frac{1}{RC}r + \frac{1}{CL} = 0,$$

with eigenvalues

$$r_{1,2} = -\frac{1}{2RC} \pm \frac{1}{2RC} \sqrt{1 - \frac{4R^2C}{L}}.$$

The eigenvalues are real and different provided that

$$1 - \frac{4R^2C}{L} > 0.$$

The eigenvalues are complex conjugates as long as

$$1 - \frac{4R^2C}{L} < 0.$$

(b) With the specified values, the eigenvalues are $r_{1,2} = -1 \pm i$. The eigenvector corresponding to $r = -1 + i$ is $\xi^{(1)} = (1, -4i)^T$. Hence one complex-valued solution is

$$\begin{aligned} \begin{pmatrix} I \\ V \end{pmatrix}^{(1)} &= \begin{pmatrix} 1 \\ -1 + i \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 1 \\ -1 + i \end{pmatrix} e^{-t} (\cos t + i \sin t) = \\ &= e^{-t} \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + i e^{-t} \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}. \end{aligned}$$

Therefore the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}.$$

(c) Imposing the initial conditions, we arrive at the equations

$$\begin{aligned} c_1 &= 2 \\ -c_1 + c_2 &= 1, \end{aligned}$$

with $c_1 = 2$ and $c_2 = 3$. Therefore the solution of the IVP is

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t} \begin{pmatrix} 2 \cos t + 3 \sin t \\ \cos t - 5 \sin t \end{pmatrix}.$$

(d) Since $\operatorname{Re}(r_{1,2}) = -1$, all solutions converge to the origin.

27.(a) Suppose that $c_1 \mathbf{a} + c_2 \mathbf{b} = \mathbf{0}$. Since \mathbf{a} and \mathbf{b} are the real and imaginary parts of the vector $\boldsymbol{\xi}^{(1)}$, respectively, $\mathbf{a} = (\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}})/2$ and $\mathbf{b} = (\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}})/2i$. Hence

$$c_1(\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}}) - ic_2(\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}}) = \mathbf{0},$$

which leads to

$$(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\overline{\boldsymbol{\xi}^{(1)}} = \mathbf{0}.$$

Now since $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}^{(1)}}$ are linearly independent, we must have

$$\begin{aligned} c_1 - ic_2 &= 0 \\ c_1 + ic_2 &= 0. \end{aligned}$$

It follows that $c_1 = c_2 = 0$.

(c) Recall that

$$\begin{aligned} \mathbf{u}(t) &= e^{\lambda t}(\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) \\ \mathbf{v}(t) &= e^{\lambda t}(\mathbf{a} \cos \mu t + \mathbf{b} \sin \mu t). \end{aligned}$$

Consider the equation $c_1 \mathbf{u}(t_0) + c_2 \mathbf{v}(t_0) = \mathbf{0}$, for some t_0 . We can then write

$$c_1 e^{\lambda t_0}(\mathbf{a} \cos \mu t_0 - \mathbf{b} \sin \mu t_0) + c_2 e^{\lambda t_0}(\mathbf{a} \cos \mu t_0 + \mathbf{b} \sin \mu t_0) = \mathbf{0}. (*)$$

Rearranging the terms, and dividing by the exponential,

$$(c_1 + c_2) \cos \mu t_0 \mathbf{a} + (c_2 - c_1) \sin \mu t_0 \mathbf{b} = \mathbf{0}.$$

From part (b), since \mathbf{a} and \mathbf{b} are linearly independent, it follows that

$$(c_1 + c_2) \cos \mu t_0 = (c_2 - c_1) \sin \mu t_0 = 0.$$

Without loss of generality, assume that the trigonometric factors are nonzero. Otherwise proceed again from Equation (*), above. We then conclude that

$$c_1 + c_2 = 0 \text{ and } c_2 - c_1 = 0,$$

which leads to $c_1 = c_2 = 0$. Thus $\mathbf{u}(t_0)$ and $\mathbf{v}(t_0)$ are linearly independent for some t_0 , and hence the functions are linearly independent at every point.

28.(a) Let $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -\frac{k}{m} u.$$

In terms of the new variables, we obtain the system of two first order ODEs

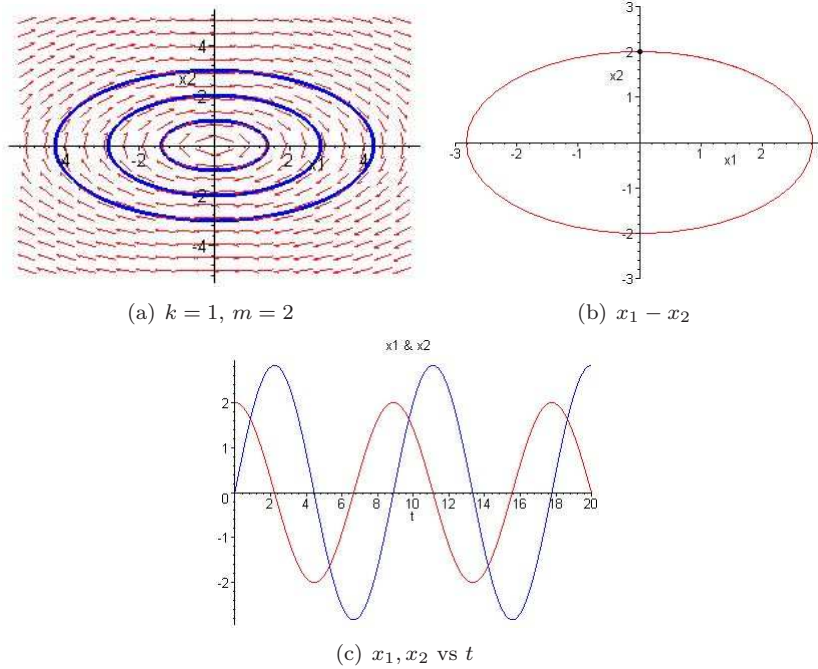
$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m} x_1. \end{aligned}$$

(b) The associated eigenvalue problem is

$$\begin{pmatrix} -r & 1 \\ -k/m & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + k/m = 0$, with roots $r_{1,2} = \pm i\sqrt{k/m}$.

(c) Since the eigenvalues are purely imaginary, the origin is a center. Hence the phase curves are ellipses, with a clockwise flow. For computational purposes, let $k = 1$ and $m = 2$.



(d) The general solution of the second order equation is

$$u(t) = c_1 \cos \sqrt{\frac{k}{m}} t + c_2 \sin \sqrt{\frac{k}{m}} t.$$

The general solution of the system of ODEs is given by

$$\mathbf{x} = c_1 \begin{pmatrix} \sqrt{\frac{m}{k}} \sin \sqrt{\frac{k}{m}} t \\ \cos \sqrt{\frac{k}{m}} t \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{\frac{m}{k}} \cos \sqrt{\frac{k}{m}} t \\ -\sin \sqrt{\frac{k}{m}} t \end{pmatrix}.$$

It is evident that the natural frequency of the system is equal to $\text{Im}(r_{1,2})$.

29.(a) Set $\mathbf{x} = (x_1, x_2)^T$. We can rewrite Equation (22) in the form

$$\begin{pmatrix} 2 & 0 \\ 0 & 9/4 \end{pmatrix} \begin{pmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -27/4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Multiplying both sides of this equation by the inverse of the diagonal matrix, we obtain

$$\begin{pmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{pmatrix} = \begin{pmatrix} -2 & 3/2 \\ 4/3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(b) Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$,

$$r^2 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{rt} = \begin{pmatrix} -2 & 3/2 \\ 4/3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{rt},$$

which can be written as

$$(\mathbf{A} - r^2 \mathbf{I}) \boldsymbol{\xi} = \mathbf{0}.$$

(c) The eigenvalues are $r_1^2 = -1$ and $r_2^2 = -4$, with corresponding eigenvectors

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

(d) The linearly independent solutions are

$$\mathbf{x}^{(1)} = \tilde{C}_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{it} \text{ and } \mathbf{x}^{(2)} = \tilde{C}_2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{2it}.$$

in which \tilde{C}_1 and \tilde{C}_2 are arbitrary complex coefficients. In scalar form,

$$\begin{aligned} x_1 &= 3c_1 \cos t + 3c_2 \sin t + 3c_3 \cos 2t + 3c_4 \sin 2t \\ x_2 &= 2c_1 \cos t + 2c_2 \sin t - 4c_3 \cos 2t - 4c_4 \sin 2t \end{aligned}$$

(e) Differentiating the above expressions,

$$\begin{aligned} x_1' &= -3c_1 \sin t + 3c_2 \cos t - 6c_3 \sin 2t + 6c_4 \cos 2t \\ x_2' &= -2c_1 \sin t + 2c_2 \cos t + 8c_3 \sin 2t - 8c_4 \cos 2t \end{aligned}$$

It is evident that $\mathbf{y} = (x_1, x_2, x_1', x_2')^T$ as in Equation (31).

31.(a) The second order system is given by

$$\begin{aligned} \frac{d^2 x_1}{dt^2} &= -2x_1 + x_2 \\ \frac{d^2 x_2}{dt^2} &= x_1 - 2x_2 \end{aligned}$$

Let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x'_1$ and $y_4 = x'_2$. In terms of the new variables, we have

$$\begin{aligned} y'_1 &= y_3 \\ y'_2 &= y_4 \\ y'_3 &= -2y_1 + y_2 \\ y'_4 &= y_1 - 2y_2 \end{aligned}$$

hence the coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{pmatrix}.$$

(b) The eigenvalues and corresponding eigenvectors of \mathbf{A} are:

$$\begin{aligned} r_1 &= i, \quad \boldsymbol{\xi}^{(1)} = (1, 1, i, i)^T \\ r_2 &= -i, \quad \boldsymbol{\xi}^{(2)} = (1, 1, -i, -i)^T \\ r_3 &= \sqrt{3}i, \quad \boldsymbol{\xi}^{(3)} = (1, -1, \sqrt{3}i, -\sqrt{3}i)^T \\ r_4 &= -\sqrt{3}i, \quad \boldsymbol{\xi}^{(4)} = (1, -1, -\sqrt{3}i, \sqrt{3}i)^T \end{aligned}$$

(c) Note that

$$\boldsymbol{\xi}^{(1)} e^{it} = \begin{pmatrix} 1 \\ 1 \\ i \\ i \end{pmatrix} (\cos t + i \sin t)$$

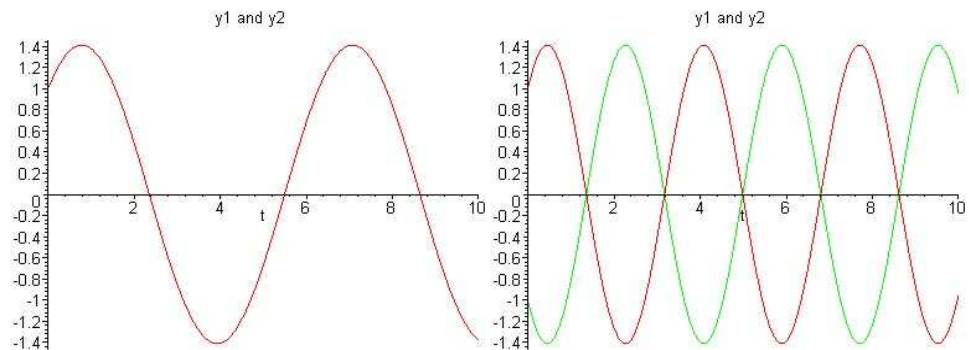
and

$$\boldsymbol{\xi}^{(3)} e^{\sqrt{3}it} = \begin{pmatrix} 1 \\ -1 \\ \sqrt{3}i \\ -\sqrt{3}i \end{pmatrix} (\cos \sqrt{3}t + i \sin \sqrt{3}t).$$

Hence the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} \cos t \\ \cos t \\ -\sin t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \sin t \\ \cos t \\ \cos t \end{pmatrix} + c_3 \begin{pmatrix} \cos \sqrt{3}t \\ -\cos \sqrt{3}t \\ -\sqrt{3} \sin \sqrt{3}t \\ \sqrt{3} \sin \sqrt{3}t \end{pmatrix} + c_4 \begin{pmatrix} \sin \sqrt{3}t \\ -\sin \sqrt{3}t \\ \sqrt{3} \cos \sqrt{3}t \\ -\sqrt{3} \cos \sqrt{3}t \end{pmatrix}.$$

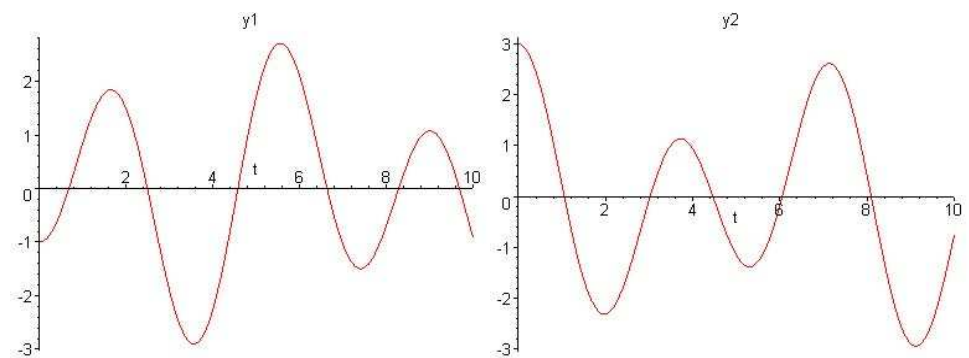
(d) The two modes have natural frequencies of $\omega_1 = 1$ rad/sec and $\omega_2 = \sqrt{3}$ rad/sec.



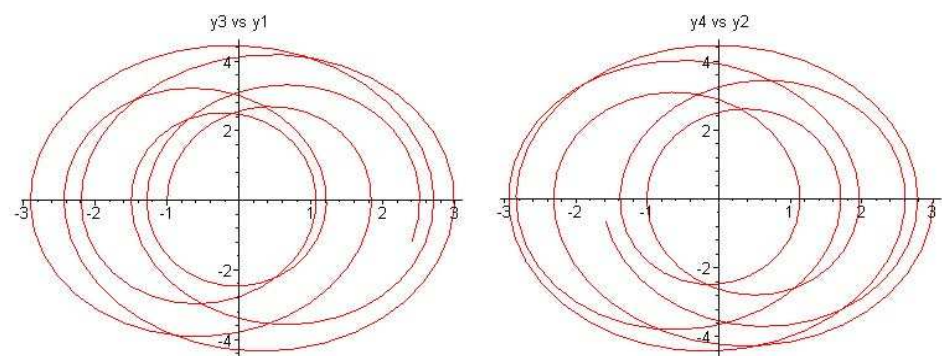
(e) For the initial condition $\mathbf{y}(0) = (-1, 3, 0, 0)^T$, it is necessary that

$$\begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \\ -\sqrt{3} \end{pmatrix},$$

resulting in the coefficients $c_1 = 1$, $c_2 = 0$, $c_3 = -2$ and $c_4 = 0$.



The solutions are not periodic, since the two natural frequencies are incommensurate.



7.7

1.(a) The eigenvalues and eigenvectors were found in Problem 1, Section 7.5.

$$r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix}.$$

(b) We now have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \boldsymbol{\Psi}^{-1}(0) = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

So that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{3} \begin{pmatrix} -e^{-t} + 4e^{2t} & 2e^{-t} - 2e^{2t} \\ -2e^{-t} + 2e^{2t} & 4e^{-t} - e^{2t} \end{pmatrix}.$$

3.(a,b) The eigenvalues and eigenvectors were found in Problem 3, Section 7.5. The general solution of the system is

$$\mathbf{x} = c_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 + 3c_2 &= 0, \end{aligned}$$

to obtain $c_1 = 3/2$, $c_2 = -1/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2}e^t - \frac{1}{2}e^{-t} \\ \frac{3}{2}e^t - \frac{3}{2}e^{-t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 3c_2 &= 1, \end{aligned}$$

to obtain $c_1 = -1/2$, $c_2 = 1/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -\frac{1}{2}e^t + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\boldsymbol{\Phi}(t) = \frac{1}{2} \begin{pmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{pmatrix}.$$

5.(a,b) The general solution, found in Problem 3, Section 7.6, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin t \\ -\cos t + 2 \sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 5c_1 &= 1 \\ 2c_1 - c_2 &= 0, \end{aligned}$$

resulting in $c_1 = 1/5$, $c_2 = 2/5$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} 5c_1 &= 0 \\ 2c_1 - c_2 &= 1, \end{aligned}$$

resulting in $c_1 = 0$, $c_2 = -1$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -5 \sin t \\ \cos t - 2 \sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\Phi(t) = \begin{pmatrix} \cos t + 2 \sin t & -5 \sin t \\ \sin t & \cos t - 2 \sin t \end{pmatrix}.$$

7.(a,b) The general solution, found in Problem 15, Section 7.5, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} e^{2t} \\ 3e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 + c_2 &= 0, \end{aligned}$$

resulting in $c_1 = -1/2$, $c_2 = 3/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2} \begin{pmatrix} -e^{2t} + 3e^{4t} \\ -3e^{2t} + 3e^{4t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$ require that

$$\begin{aligned} c_1 + c_2 &= 0 \\ 3c_1 + c_2 &= 1, \end{aligned}$$

resulting in $c_1 = 1/2$, $c_2 = -1/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2} \begin{pmatrix} e^{2t} - e^{4t} \\ 3e^{2t} - e^{4t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\Phi(t) = \frac{1}{2} \begin{pmatrix} -e^{2t} + 3e^{4t} & e^{2t} - e^{4t} \\ -3e^{2t} + 3e^{4t} & 3e^{2t} - e^{4t} \end{pmatrix}.$$

8.(a,b) The general solution, found in Problem 5, Section 7.6, is given by

$$\mathbf{x} = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ -\cos t + 2 \sin t \end{pmatrix}.$$

The specific solution corresponding to the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$ is

$$\mathbf{x} = e^{-t} \begin{pmatrix} \cos t + 2 \sin t \\ 5 \sin t \end{pmatrix}.$$

For the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, the solution is

$$\mathbf{x} = e^{-t} \begin{pmatrix} -\sin t \\ \cos t - 2 \sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\Phi(t) = e^{-t} \begin{pmatrix} \cos t + 2 \sin t & -\sin t \\ 5 \sin t & \cos t - 2 \sin t \end{pmatrix}.$$

9.(a,b) The general solution, found in Problem 13, Section 7.5, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} 4e^{-2t} \\ -5e^{-2t} \\ -7e^{-2t} \end{pmatrix} + c_2 \begin{pmatrix} 3e^{-t} \\ -4e^{-t} \\ -2e^{-t} \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ e^{2t} \\ -e^{2t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 1 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 0, \end{aligned}$$

resulting in $c_1 = -1/2$, $c_2 = 1$, $c_3 = 3/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -2e^{-2t} + 3e^{-t} \\ 5e^{-2t}/2 - 4e^{-t} + 3e^{2t}/2 \\ 7e^{-2t}/2 - 2e^{-t} - 3e^{2t}/2 \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 1 \\ -7c_1 - 2c_2 - c_3 &= 0, \end{aligned}$$

resulting in $c_1 = -1/4$, $c_2 = 1/3$, $c_3 = 13/12$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ 5e^{-2t}/4 - 4e^{-t}/3 + 13e^{2t}/12 \\ 7e^{-2t}/4 - 2e^{-t}/3 - 13e^{2t}/12 \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(3)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 1, \end{aligned}$$

resulting in $c_1 = -1/4$, $c_2 = 1/3$, $c_3 = 1/12$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ 5e^{-2t}/4 - 4e^{-t}/3 + e^{2t}/12 \\ 7e^{-2t}/4 - 2e^{-t}/3 - e^{2t}/12 \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\Phi(t) = \frac{1}{12} \begin{pmatrix} -24e^{-2t} + 36e^{-t} & -12e^{-2t} + 12e^{-t} & -12e^{-2t} + 12e^{-t} \\ 30e^{-2t} - 48e^{-t} + 18e^{2t} & 15e^{-2t} - 16e^{-t} + 13e^{2t} & 15e^{-2t} - 16e^{-t} + e^{2t} \\ 42e^{-2t} - 24e^{-t} - 18e^{2t} & 21e^{-2t} - 8e^{-t} - 13e^{2t} & 21e^{-2t} - 8e^{-t} - e^{2t} \end{pmatrix}.$$

12. The solution of the initial value problem is given by

$$\begin{aligned} \mathbf{x} &= \Phi(t)\mathbf{x}(0) = \begin{pmatrix} e^{-t} \cos 2t & -2e^{-t} \sin 2t \\ \frac{1}{2}e^{-t} \sin 2t & e^{-t} \cos 2t \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \\ &= e^{-t} \begin{pmatrix} 3 \cos 2t - 2 \sin 2t \\ \frac{3}{2} \sin 2t + \cos 2t \end{pmatrix}. \end{aligned}$$

13. Let

$$\Psi(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}.$$

It follows that

$$\Psi(t_0) = \begin{pmatrix} x_1^{(1)}(t_0) & \cdots & x_1^{(n)}(t_0) \\ \vdots & & \vdots \\ x_n^{(1)}(t_0) & \cdots & x_n^{(n)}(t_0) \end{pmatrix}$$

is a scalar matrix, which is invertible, since the solutions are linearly independent.

Let $\Psi^{-1}(t_0) = (c_{ij})$. Then

$$\Psi(t)\Psi^{-1}(t_0) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}.$$

The j -th column of the product matrix is

$$[\Psi(t)\Psi^{-1}(t_0)]^{(j)} = \sum_{k=1}^n c_{kj} \mathbf{x}^{(k)},$$

which is a solution vector, since it is a linear combination of solutions. Furthermore, the columns are all linearly independent, since the vectors $\mathbf{x}^{(k)}$ are. Hence the

product is a fundamental matrix. Finally, setting $t = t_0$, $\Psi(t_0)\Psi^{-1}(t_0) = \mathbf{I}$. This is precisely the definition of $\Phi(t)$.

14. The fundamental matrix $\Phi(t)$ for the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

is given by

$$\Phi(t) = \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}.$$

Direct multiplication results in

$$\begin{aligned} \Phi(t)\Phi(s) &= \frac{1}{16} \begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix} \begin{pmatrix} 2e^{3s} + 2e^{-s} & e^{3s} - e^{-s} \\ 4e^{3s} - 4e^{-s} & 2e^{3s} + 2e^{-s} \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 8(e^{3t+3s} + e^{-t-s}) & 4(e^{3t+3s} - e^{-t-s}) \\ 16(e^{3t+3s} - e^{-t-s}) & 8(e^{3t+3s} + e^{-t-s}) \end{pmatrix}. \end{aligned}$$

Hence

$$\Phi(t)\Phi(s) = \frac{1}{4} \begin{pmatrix} 2e^{3(t+s)} + 2e^{-(t+s)} & e^{3(t+s)} - e^{-(t+s)} \\ 4e^{3(t+s)} - 4e^{-(t+s)} & 2e^{3(t+s)} + 2e^{-(t+s)} \end{pmatrix}.$$

15.(a) Let s be arbitrary, but fixed, and t variable. Similar to the argument in Problem 13, the columns of the matrix $\Phi(t)\Phi(s)$ are linear combinations of fundamental solutions. Hence the columns of $\Phi(t)\Phi(s)$ are also solution of the system of equations. Further, setting $t = 0$, $\Phi(0)\Phi(s) = \mathbf{I}\Phi(s) = \Phi(s)$. That is, $\Phi(t)\Phi(s)$ is a solution of the initial value problem $\mathbf{Z}' = \mathbf{A}\mathbf{Z}$, with $\mathbf{Z}(0) = \Phi(s)$. Now consider the change of variable $\tau = t + s$. Let $\mathbf{W}(\tau) = \mathbf{Z}(\tau - s)$. The given initial value problem can be reformulated as

$$\frac{d}{d\tau} \mathbf{W} = \mathbf{A}\mathbf{W}, \text{ with } \mathbf{W}(s) = \Phi(s).$$

Since $\Phi(t)$ is a fundamental matrix satisfying $\Phi' = \mathbf{A}\Phi$, with $\Phi(0) = \mathbf{I}$, it follows that

$$\mathbf{W}(\tau) = [\Phi(\tau)\Phi^{-1}(s)]\Phi(s) = \Phi(\tau).$$

That is, $\Phi(t + s) = \Phi(\tau) = \mathbf{W}(\tau) = \mathbf{Z}(t) = \Phi(t)\Phi(s)$.

(b) Based on part (a), $\Phi(t)\Phi(-t) = \Phi(t + (-t)) = \Phi(0) = \mathbf{I}$. Hence $\Phi(-t) = \Phi^{-1}(t)$.

(c) It also follows that $\Phi(t - s) = \Phi(t + (-s)) = \Phi(t)\Phi(-s) = \Phi(t)\Phi^{-1}(s)$.

16. Let \mathbf{A} be a diagonal matrix, with $\mathbf{A} = [a_1\mathbf{e}^{(1)}, a_2\mathbf{e}^{(2)}, \dots, a_n\mathbf{e}^{(n)}]$. Note that for any positive integer k ,

$$\mathbf{A}^k = [a_1^k\mathbf{e}^{(1)}, a_2^k\mathbf{e}^{(2)}, \dots, a_n^k\mathbf{e}^{(n)}].$$

It follows, from basic matrix algebra, that

$$\mathbf{I} + \sum_{k=1}^m \mathbf{A}^k \frac{t^k}{k!} = \begin{pmatrix} \sum_{k=0}^m a_1^k \frac{t^k}{k!} & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^m a_2^k \frac{t^k}{k!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^m a_n^k \frac{t^k}{k!} \end{pmatrix}.$$

It can be shown that the partial sums on the left hand side converge for all t . Taking the limit as $m \rightarrow \infty$ on both sides of the equation, we obtain

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{a_1 t} & 0 & \cdots & 0 \\ 0 & e^{a_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_n t} \end{pmatrix}.$$

Alternatively, consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Since the ODEs are uncoupled, the vectors $\mathbf{x}^{(j)} = e^{a_j t} \mathbf{e}^{(j)}$, $j = 1, 2, \dots, n$, are a set of linearly independent solutions. Hence the matrix

$$\mathbf{x} = [e^{a_1 t} \mathbf{e}^{(1)}, e^{a_2 t} \mathbf{e}^{(2)}, \dots, e^{a_n t} \mathbf{e}^{(n)}]$$

is a fundamental matrix. Finally, since $\mathbf{X}(0) = \mathbf{I}$, it follows that

$$[e^{a_1 t} \mathbf{e}^{(1)}, e^{a_2 t} \mathbf{e}^{(2)}, \dots, e^{a_n t} \mathbf{e}^{(n)}] = \Phi(t) = e^{\mathbf{A}t}.$$

17.(a) Let $x_1 = u$ and $x_2 = u'$; then $u'' = x_2'$. In terms of the new variables, we have

$$x_2' + \omega^2 x_1 = 0$$

with the initial conditions $x_1(0) = u_0$ and $x_2(0) = v_0$. The equivalent first order system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\omega^2 x_1 \end{aligned}$$

which can be expressed in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

(b) Setting

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix},$$

it is easy to show that

$$\mathbf{A}^2 = -\omega^2 \mathbf{I}, \mathbf{A}^3 = -\omega^2 \mathbf{A} \text{ and } \mathbf{A}^4 = \omega^4 \mathbf{I}.$$

It follows inductively that

$$\mathbf{A}^{2k} = (-1)^k \omega^{2k} \mathbf{I}$$

and

$$\mathbf{A}^{2k+1} = (-1)^k \omega^{2k} \mathbf{A}.$$

Hence

$$\begin{aligned} e^{\mathbf{A}t} &= \sum_{k=0}^{\infty} \left[(-1)^k \frac{\omega^{2k} t^{2k}}{(2k)!} \mathbf{I} + (-1)^k \frac{\omega^{2k} t^{2k+1}}{(2k+1)!} \mathbf{A} \right] \\ &= \left[\sum_{k=0}^{\infty} (-1)^k \frac{\omega^{2k} t^{2k}}{(2k)!} \right] \mathbf{I} + \frac{1}{\omega} \left[\sum_{k=0}^{\infty} (-1)^k \frac{\omega^{2k+1} t^{2k+1}}{(2k+1)!} \right] \mathbf{A} \end{aligned}$$

and therefore

$$e^{\mathbf{A}t} = \cos \omega t \mathbf{I} + \frac{1}{\omega} \sin \omega t \mathbf{A}.$$

(c) From Equation (28),

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \left[\cos \omega t \mathbf{I} + \frac{1}{\omega} \sin \omega t \mathbf{A} \right] \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \\ &= \cos \omega t \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \frac{1}{\omega} \sin \omega t \begin{pmatrix} v_0 \\ -\omega^2 u_0 \end{pmatrix}. \end{aligned}$$

18.(a) Assuming that $\mathbf{x} = \phi(t)$ is a solution, then $\phi' = \mathbf{A}\phi$, with $\phi(0) = \mathbf{x}^0$. Integrate both sides of the equation to obtain

$$\phi(t) - \phi(0) = \int_0^t \mathbf{A}\phi(s) ds.$$

Hence

$$\phi(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\phi(s) ds.$$

(b) Proceed with the iteration

$$\phi^{(i+1)}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\phi^{(i)}(s) ds.$$

With $\phi^{(0)}(t) = \mathbf{x}^0$, and noting that \mathbf{A} is a constant matrix,

$$\begin{aligned} \phi^{(1)}(t) &= \mathbf{x}^0 + \int_0^t \mathbf{A}\mathbf{x}^0 ds \\ &= \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t. \end{aligned}$$

That is, $\phi^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0$.

(c) We then have

$$\begin{aligned} \phi^{(2)}(t) &= \mathbf{x}^0 + \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}s)\mathbf{x}^0 ds \\ &= \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t + \mathbf{A}^2 \mathbf{x}^0 \frac{t^2}{2} \\ &= \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} \right) \mathbf{x}^0. \end{aligned}$$

Now suppose that

$$\phi^{(n)}(t) = (\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \cdots + \mathbf{A}^n \frac{t^n}{n!})\mathbf{x}^0.$$

It follows that

$$\begin{aligned} \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}s + \mathbf{A}^2 \frac{s^2}{2} + \cdots + \mathbf{A}^n \frac{s^n}{n!})\mathbf{x}^0 ds &= \\ &= \mathbf{A}(\mathbf{I}t + \mathbf{A} \frac{t^2}{2} + \mathbf{A}^2 \frac{t^3}{3!} + \cdots + \mathbf{A}^n \frac{t^{n+1}}{(n+1)!})\mathbf{x}^0 \\ &= (\mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \mathbf{A}^3 \frac{t^3}{3!} + \cdots + \mathbf{A}^{n+1} \frac{t^{n+1}}{(n+1)!})\mathbf{x}^0. \end{aligned}$$

Therefore

$$\phi^{(n+1)}(t) = (\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \cdots + \mathbf{A}^{n+1} \frac{t^{n+1}}{(n+1)!})\mathbf{x}^0.$$

By induction, the asserted form of $\phi^{(n)}(t)$ is valid for all $n \geq 0$.

(d) Define $\phi^{(\infty)}(t) = \lim_{n \rightarrow \infty} \phi^{(n)}(t)$. It can be shown that the limit does exist. In fact,

$$\phi^{(\infty)}(t) = e^{\mathbf{A}t}\mathbf{x}^0.$$

Term-by-term differentiation results in

$$\begin{aligned} \frac{d}{dt}\phi^{(\infty)}(t) &= \frac{d}{dt}(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \cdots + \mathbf{A}^n \frac{t^n}{n!} + \cdots)\mathbf{x}^0 \\ &= (\mathbf{A} + \mathbf{A}^2 t + \cdots + \mathbf{A}^n \frac{t^{n-1}}{(n-1)!} + \cdots)\mathbf{x}^0 \\ &= \mathbf{A}(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \cdots + \mathbf{A}^{n-1} \frac{t^{n-1}}{(n-1)!} + \cdots)\mathbf{x}^0. \end{aligned}$$

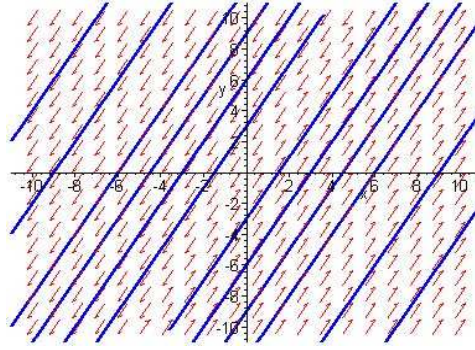
That is,

$$\frac{d}{dt}\phi^{(\infty)}(t) = \mathbf{A}\phi^{(\infty)}(t).$$

Furthermore, $\phi^{(\infty)}(0) = \mathbf{x}^0$. Based on uniqueness of solutions, $\phi(t) = \phi^{(\infty)}(t)$.

7.8

2.(a)



(b) All of the points on the line $x_2 = 2x_1$ are equilibrium points. Solutions starting at all other points become unbounded.

(c) Setting $\mathbf{x} = \boldsymbol{\xi} t^r$ results in the algebraic equations

$$\begin{pmatrix} 4-r & -2 \\ 8 & -4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with the single root $r = 0$. Substituting $r = 0$ reduces the system of equations to $2\xi_1 - \xi_2 = 0$. Therefore the only eigenvector is $\boldsymbol{\xi} = (1, 2)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which is a constant vector. In order to generate a second linearly independent solution, we must search for a generalized eigenvector. This leads to the system of equations

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This system also reduces to a single equation, $2\eta_1 - \eta_2 = 1/2$. Setting $\eta_1 = k$, some arbitrary constant, we obtain $\eta_2 = 2k - 1/2$. A second solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} k \\ 2k - 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + k \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Note that the last term is a multiple of $\mathbf{x}^{(1)}$ and may be dropped. Hence

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \right].$$

3.(c) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{3}{2} - r & 1 \\ -\frac{1}{4} & -\frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $\xi_1 - 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (2, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by finding a generalized eigenvector. We therefore analyze the system

$$\begin{pmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

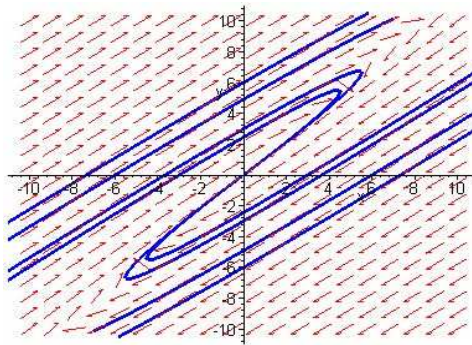
The equations reduce to the single equation $-\eta_1 + 2\eta_2 = 2$. Let $\eta_1 = 2k$. We obtain $\eta_2 = 1 + k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 2k \\ 1+k \end{pmatrix} e^{-t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + k \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \right].$$

4.(a)



(c) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -3 - r & \frac{5}{2} \\ -\frac{5}{2} & 2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + r + \frac{1}{4} = 0$. The only root is $r = -1/2$, which is an eigenvalue of multiplicity two. Setting $r = -1/2$ the coefficient matrix reduces the system to the single equation $-\xi_1 + \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}.$$

In order to obtain a second linearly independent solution, we find a solution of the system

$$\begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

These equations reduce to $-5\eta_1 + 5\eta_2 = 2$. Set $\eta_1 = k$, some arbitrary constant. Then $\eta_2 = k + 2/5$. A second solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t/2} + \begin{pmatrix} k \\ k + 2/5 \end{pmatrix} e^{-t/2} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}. \end{aligned}$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2} \right].$$

6. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r - 2 = 0$, with roots $r_1 = 2$ and $r_{2,3} = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0. \end{aligned}$$

A corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (1, 1, 1)^T$. Setting $r = -1$, the system of equations is reduced to the single equation

$$\xi_1 + \xi_2 + \xi_3 = 0.$$

An eigenvector is given by $\boldsymbol{\xi}^{(2)} = (1, 0, -1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r = -1$) is $\boldsymbol{\xi}^{(3)} = (0, 1, -1)^T$. Therefore the general solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}.$$

7.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 6r + 9 = 0$. The only root is $r = -3$, which is an eigenvalue of multiplicity two. Substituting $r = -3$ into the coefficient matrix, the system reduces to the single equation $\xi_1 - \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

For a second linearly independent solution, we search for a generalized eigenvector. Its components satisfy

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is, $4\eta_1 - 4\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = k + 1/4$. It follows that a second solution is given by

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} k + 1/4 \\ k \end{pmatrix} e^{-3t} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}. \end{aligned}$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} \right].$$

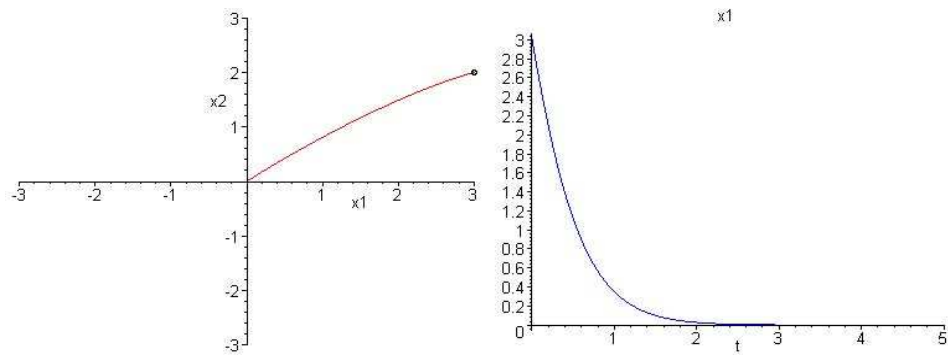
Imposing the initial conditions, we require that

$$\begin{aligned} c_1 + \frac{1}{4}c_2 &= 3 \\ c_1 &= 2, \end{aligned}$$

which results in $c_1 = 2$ and $c_2 = 4$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t}.$$

(b)



8.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{5}{2} - r & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $-\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (1, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by solving the system

$$\begin{pmatrix} -3/2 & 3/2 \\ -3/2 & 3/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $-3\eta_1 + 3\eta_2 = 2$. Let $\eta_1 = k$. We obtain $\eta_2 = 2/3 + k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} k \\ 2/3 + k \end{pmatrix} e^{-t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} \right].$$

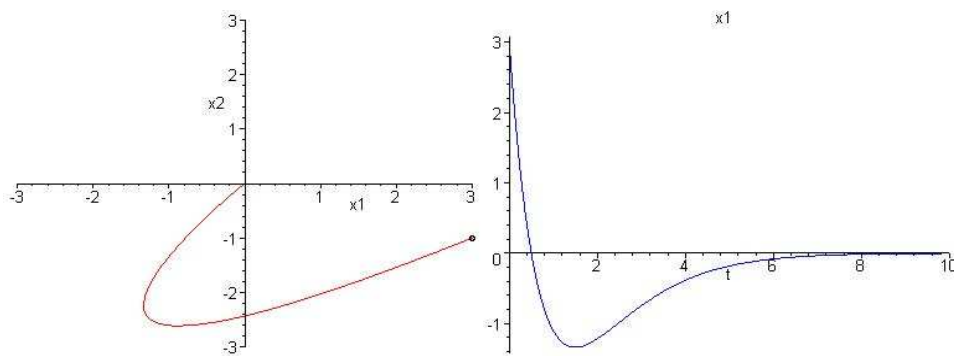
Imposing the initial conditions, find that

$$\begin{aligned} c_1 &= 3 \\ c_1 + \frac{2}{3}c_2 &= -1, \end{aligned}$$

so that $c_1 = 3$ and $c_2 = -6$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-t} - \begin{pmatrix} 6 \\ 6 \end{pmatrix} t e^{-t}.$$

(b)



10.(a) The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 9 \\ -1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with a single root $r = 0$. Setting $r = 0$, the two equations reduce to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\xi = (-3, 1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$

which is a constant vector. A second linearly independent solution is obtained from the system

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $\eta_1 + 3\eta_2 = -1$. Let $\eta_2 = k$. We obtain $\eta_1 = -1 - 3k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}t + \begin{pmatrix} -1 - 3k \\ k \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} + k \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} -3 \\ 1 \end{pmatrix}t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right].$$

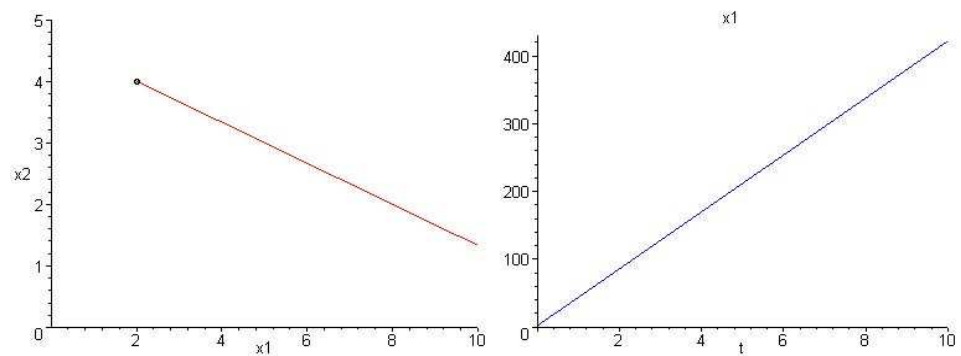
Imposing the initial conditions, we require that

$$\begin{aligned} -3c_1 - c_2 &= 2 \\ c_1 &= 4, \end{aligned}$$

which results in $c_1 = 4$ and $c_2 = -14$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 14 \begin{pmatrix} -3 \\ 1 \end{pmatrix}t.$$

(b)



12.(a) The characteristic equation of the system is $8r^3 + 60r^2 + 126r + 49 = 0$. The eigenvalues are $r_1 = -1/2$ and $r_{2,3} = -7/2$. The eigenvector associated with r_1 is $\xi^{(1)} = (1, 1, 1)^T$. Setting $r = -7/2$, the components of the eigenvectors must satisfy the relation

$$\xi_1 + \xi_2 + \xi_3 = 0.$$

An eigenvector vector is given by $\xi^{(2)} = (1, 0, -1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r = -7/2$) is $\xi^{(3)} = (0, 1, -1)^T$. Therefore the general solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-7t/2} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-7t/2}.$$

Invoking the initial conditions, we require that

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + c_3 &= 3 \\ c_1 - c_2 - c_3 &= -1. \end{aligned}$$

Hence the solution of the IVP is

$$\mathbf{x} = \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t/2} + \frac{2}{3} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-7t/2} + \frac{5}{3} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-7t/2}.$$

13. Setting $\mathbf{x} = \xi t^r$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 - 2r + 1 = 0$, with a single root of $r_{1,2} = 1$. With $r = 1$, the system reduces to a single equation $\xi_1 - 2\xi_2 = 0$. An eigenvector is given by $\xi = (2, 1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t.$$

In order to find a second linearly independent solution, we search for a generalized eigenvector whose components satisfy

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

These equations reduce to $\eta_1 - 2\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = 1 + 2k$. (Before proceeding, note that if we set $u = \ln t$, the original equation is transformed into a constant coefficient equation with independent variable u . Recall that a second solution is obtained by multiplication of the first

solution by the factor u . This implies that we must multiply first solution by a factor of $\ln t$.) Hence a second linearly independent solution is

$$\begin{aligned}\mathbf{x}^{(2)} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} t \ln t + \begin{pmatrix} 1+2k \\ k \end{pmatrix} t \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} t \ln t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + k \begin{pmatrix} 2 \\ 1 \end{pmatrix} t.\end{aligned}$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t \ln t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right].$$

15. The characteristic equation is

$$r^2 - (a+d)r + ad - bc = 0.$$

Hence the eigenvalues are

$$r_{1,2} = \frac{a+d}{2} \pm \frac{1}{2} \sqrt{(a+d)^2 - 4(ad-bc)}.$$

16.(a) Using the result in Problem 15, the eigenvalues are

$$r_{1,2} = -\frac{1}{2RC} \pm \frac{\sqrt{L^2 - 4R^2CL}}{2RCL}.$$

The discriminant vanishes when $L = 4R^2CL$.

(b) The system of differential equations is

$$\frac{d}{dt} \begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ -1 & -1 \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

The associated eigenvalue problem is

$$\begin{pmatrix} -r & \frac{1}{4} \\ -1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + r + 1/4 = 0$, with a single root of $r_{1,2} = -1/2$. Setting $r = -1/2$, the algebraic equations reduce to $2\xi_1 + \xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (1, -2)^T$. Hence one solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}.$$

A second solution is obtained from a generalized eigenvector whose components satisfy

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

It follows that $\eta_1 = k$ and $\eta_2 = 4 - 2k$. A second linearly independent solution is

$$\begin{aligned} \begin{pmatrix} I \\ V \end{pmatrix}^{(2)} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} t e^{-t/2} + \begin{pmatrix} k \\ 4 - 2k \end{pmatrix} e^{-t/2} \\ &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} t e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2} + k \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}. \end{aligned}$$

Dropping the last term, the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2} + c_2 \left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} t e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2} \right].$$

Imposing the initial conditions, we require that

$$\begin{aligned} c_1 &= 1 \\ -2c_1 + 4c_2 &= 2, \end{aligned}$$

which results in $c_1 = 1$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/2} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} t e^{-t/2}.$$

18.(a) The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r^2 + 3r - 1 = 0$, with a single root of multiplicity three, $r = 1$. Setting $r = 1$, we have

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system of algebraic equations reduce to a single equation

$$4\xi_1 - 3\xi_2 - 2\xi_3 = 0.$$

An eigenvector vector is given by $\xi^{(1)} = (1, 0, 2)^T$. Since the last equation has two free variables, a second linearly independent eigenvector (associated with $r = 1$) is $\xi^{(2)} = (0, 2, -3)^T$. Therefore two solutions are obtained as

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^t.$$

(b) It follows directly that $\mathbf{x}' = \xi t e^t + \xi e^t + \eta e^t$. Hence the coefficient vectors must satisfy $\xi t e^t + \xi e^t + \eta e^t = \mathbf{A} \xi t e^t + \mathbf{A} \eta e^t$. Rearranging the terms, we have

$$\xi e^t = (\mathbf{A} - \mathbf{I}) \xi t e^t + (\mathbf{A} - \mathbf{I}) \eta e^t.$$

Given an eigenvector ξ , it follows that $(\mathbf{A} - \mathbf{I})\eta = \xi$.

(c) Note that a linear combination of two eigenvectors, associated with the same eigenvalue, is also an eigenvector. Consider the equation

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}.$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 8 & -6 & -4 & 2c_2 \\ -4 & 3 & 2 & 2c_1 - 3c_2 \end{array} \right).$$

Using elementary row operations, we obtain

$$\left(\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 0 & 0 & 0 & -2c_1 + 2c_2 \\ 0 & 0 & 0 & 3c_1 - 3c_2 \end{array} \right).$$

It is evident that a solution exists provided $c_1 = c_2$.

(d) Let $c_1 = c_2 = 2$. The components of the generalized eigenvector must satisfy

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}.$$

Based on part (c), the equations reduce to the single equation $4\eta_1 - 3\eta_2 - 2\eta_3 = 2$. Let $\eta_1 = \alpha$ and $\eta_2 = 2\beta$, where α and β are arbitrary constants. We then have

$$\eta_3 = -1 + 2\alpha - 3\beta,$$

so that

$$\boldsymbol{\eta} = \begin{pmatrix} \alpha \\ 2\beta \\ -1 + 2\alpha - 3\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}.$$

Observe that $\boldsymbol{\eta} = \alpha\boldsymbol{\xi}^{(1)} + \beta\boldsymbol{\xi}^{(2)}$. Hence a third linearly independent solution is

$$\mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} e^t.$$

(e) Given the three linearly independent solutions, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^t & 0 & 2te^t \\ 0 & 2e^t & 4te^t \\ 2e^t & -3e^t & -2te^t - e^t \end{pmatrix}.$$

(f) We construct the transformation matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix},$$

with inverse

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 2 & -3/2 & -1 \end{pmatrix}.$$

The Jordan form of the matrix \mathbf{A} is

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

20.(a) Direct multiplication results in

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}, \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}, \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}.$$

(b) Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\mathbf{J}^{n+1} = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda \cdot \lambda^n & 0 & 0 \\ 0 & \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} \\ 0 & 0 & \lambda \cdot \lambda^n \end{pmatrix}.$$

Hence the result follows by mathematical induction.

(c) Note that \mathbf{J} is block diagonal. Hence each block may be exponentiated. Using the result in Problem 19,

$$e^{\mathbf{J}t} = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

(d) Setting $\lambda = 1$, and using the transformation matrix \mathbf{T} in Problem 18,

$$\mathbf{T}e^{\mathbf{J}t} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & 2e^t & 2te^t \\ 0 & 4e^t & 4te^t \\ 2e^t & -2e^t & -2te^t - e^t \end{pmatrix}.$$

Based on the form of \mathbf{J} , $e^{\mathbf{J}t}$ is the fundamental matrix associated with the solutions

$$\mathbf{y}^{(1)} = \boldsymbol{\xi}^{(1)}e^t, \mathbf{y}^{(2)} = (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})e^t \text{ and } \mathbf{y}^{(3)} = (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})te^t + \boldsymbol{\eta}e^t.$$

Hence the resulting matrix is the fundamental matrix associated with the solution set

$$\left\{ \boldsymbol{\xi}^{(1)}e^t, (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})e^t, (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})te^t + \boldsymbol{\eta}e^t \right\},$$

as opposed to the solution set in Problem 18, given by

$$\left\{ \boldsymbol{\xi}^{(1)}e^t, \boldsymbol{\xi}^{(2)}e^t, (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})te^t + \boldsymbol{\eta}e^t \right\}.$$

21.(a) Direct multiplication results in

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}, \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}, \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}.$$

(b) Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{J}^{n+1} &= \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} & n\lambda^{n-1} + \frac{n(n-1)}{2}\lambda \cdot \lambda^{n-2} \\ 0 & \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} \\ 0 & 0 & \lambda \cdot \lambda^n \end{pmatrix}. \end{aligned}$$

The result follows by noting that

$$n\lambda^{n-1} + \frac{n(n-1)}{2}\lambda \cdot \lambda^{n-2} = \left[n + \frac{n(n-1)}{2} \right] \lambda^{n-1} = \frac{n^2 + n}{2} \lambda^{n-1}.$$

(c) We first observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda^n \frac{t^n}{n!} &= e^{\lambda t} \\ \sum_{n=0}^{\infty} n\lambda^{n-1} \frac{t^n}{n!} &= t \sum_{n=1}^{\infty} \lambda^{n-1} \frac{t^{n-1}}{(n-1)!} = t e^{\lambda t} \\ \sum_{n=0}^{\infty} \frac{n(n-1)}{2} \lambda^{n-2} \frac{t^n}{n!} &= \frac{t^2}{2} \sum_{n=2}^{\infty} \lambda^{n-2} \frac{t^{n-2}}{(n-2)!} = \frac{t^2}{2} e^{\lambda t}. \end{aligned}$$

Therefore

$$e^{\mathbf{J}t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

(d) Setting $\lambda = 2$, and using the transformation matrix \mathbf{T} in Problem 17,

$$\mathbf{T}e^{\mathbf{J}t} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} 0 & e^{2t} & te^{2t} + 2e^{2t} \\ e^{2t} & te^{2t} + e^{2t} & \frac{t^2}{2}e^{2t} + te^{2t} \\ -e^{2t} & -te^{2t} & -\frac{t^2}{2}e^{2t} + 3e^{2t} \end{pmatrix}.$$

7.9

5. As shown in Problem 2, Section 7.8, the general solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 2t - \frac{1}{2} \end{pmatrix}.$$

An associated fundamental matrix is

$$\Psi(t) = \begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\Psi^{-1}(t) = \begin{pmatrix} 4t - 3 & -2t + 2 \\ 8t - 8 & -4t + 5 \end{pmatrix}.$$

We can now compute

$$\Psi^{-1}(t)\mathbf{g}(t) = -\frac{1}{t^3} \begin{pmatrix} 2t^2 + 4t - 1 \\ -2t - 4 \end{pmatrix},$$

and

$$\int \Psi^{-1}(t)\mathbf{g}(t) dt = \begin{pmatrix} -\frac{1}{2}t^{-2} + 4t^{-1} - 2 \ln t \\ -2t^{-2} - 2t^{-1} \end{pmatrix}.$$

Finally,

$$\mathbf{v}(t) = \Psi(t) \int \Psi^{-1}(t)\mathbf{g}(t) dt,$$

where

$$\begin{aligned} v_1(t) &= -\frac{1}{2}t^{-2} + 2t^{-1} - 2 \ln t - 2 \\ v_2(t) &= 5t^{-1} - 4 \ln t - 4. \end{aligned}$$

Note that the vector $(2, 4)^T$ is a multiple of one of the fundamental solutions. Hence we can write the general solution as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 2t - \frac{1}{2} \end{pmatrix} - \frac{1}{t^2} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} 2 \\ 5 \end{pmatrix} - 2 \ln t \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

6. The eigenvalues of the coefficient matrix are $r_1 = 0$ and $r_2 = -5$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix}.$$

The coefficient matrix is symmetric. Hence the system is diagonalizable. Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad \mathbf{T}^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Setting $\mathbf{x}=\mathbf{T}\mathbf{y}$, and $\mathbf{h}(t)=\mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$\begin{aligned}y_1' &= \frac{5+8t}{\sqrt{5}t} \\ y_2' &= -5y_2 + \frac{4}{\sqrt{5}}.\end{aligned}$$

The solutions are readily obtained as

$$y_1(t) = \sqrt{5} \ln t + \frac{4}{\sqrt{5}} t + c_1 \quad \text{and} \quad y_2(t) = c_2 e^{-5t} + \frac{4}{5\sqrt{5}}.$$

Transforming back to the original variables, we have $\mathbf{x}=\mathbf{T}\mathbf{y}$, with

$$\mathbf{x} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} y_1(t) + \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} y_2(t).$$

Hence the general solution is,

$$\mathbf{x} = k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ln t + \frac{4}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \frac{4}{25} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

7. The solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}.$$

Based on the simple form of the right hand side, we use the method of undetermined coefficients. Set $\mathbf{v}=\mathbf{a}e^t$. Substitution into the ODE yields

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$

In scalar form, after canceling the exponential, we have

$$\begin{aligned}a_1 &= a_1 + a_2 + 2 \\ a_2 &= 4a_1 + a_2 - 1,\end{aligned}$$

with $a_1 = 1/4$ and $a_2 = -2$. Hence the particular solution is

$$\mathbf{v} = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t,$$

so that the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + \frac{1}{4} \begin{pmatrix} e^t \\ -8e^t \end{pmatrix}.$$

8. The eigenvalues of the coefficient matrix are $r_1 = 1$ and $r_2 = -1$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

Use the method of undetermined coefficients. Since the right hand side is related to one of the fundamental solutions, set $\mathbf{v} = \mathbf{a}te^t + \mathbf{b}e^t$. Substitution into the ODE yields

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (e^t + te^t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} te^t + \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$$

In scalar form, we have

$$\begin{aligned} (a_1 + b_1)e^t + a_1te^t &= (2a_1 - a_2)te^t + (2b_1 - b_2)e^t + e^t \\ (a_2 + b_2)e^t + a_2te^t &= (3a_1 - 2a_2)te^t + (3b_1 - 2b_2)e^t - e^t. \end{aligned}$$

Equating the coefficients in these two equations, we find that

$$\begin{aligned} a_1 &= 2a_1 - a_2 \\ a_1 + b_1 &= 2b_1 - b_2 + 1 \\ a_2 &= 3a_1 - 2a_2 \\ a_2 + b_2 &= 3b_1 - 2b_2 - 1. \end{aligned}$$

It follows that $a_1 = a_2$. Setting $a_1 = a_2 = a$, the equations reduce to

$$\begin{aligned} b_1 - b_2 &= a - 1 \\ 3b_1 - 3b_2 &= 1 + a. \end{aligned}$$

Combining these equations, it is necessary that $a = 2$. As a result, $b_1 = b_2 + 1$. Choosing $a_1 = a_2 = 2$, and $b_2 = k$, some arbitrary constant, a particular solution is

$$\mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} k+1 \\ k \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t.$$

Since the second vector is a fundamental solution, the general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t.$$

9. Note that the coefficient matrix is symmetric. Hence the system is diagonalizable. The eigenvalues and eigenvectors are given by

$$r_1 = -\frac{1}{2}, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } r_2 = -2, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Setting $\mathbf{x} = \mathbf{T}\mathbf{y}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$\begin{aligned} y_1' &= -\frac{1}{2}y_1 + \sqrt{2}t + \frac{1}{\sqrt{2}}e^t \\ y_2' &= -2y_2 + \sqrt{2}t - \frac{1}{\sqrt{2}}e^t. \end{aligned}$$

Using any elementary method for first order linear equations, the solutions are

$$\begin{aligned}y_1(t) &= k_1 e^{-t/2} + \frac{\sqrt{2}}{3} e^t - 4\sqrt{2} + 2\sqrt{2}t \\y_2(t) &= k_2 e^{-2t} - \frac{1}{3\sqrt{2}} e^t - \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} t.\end{aligned}$$

Transforming back to the original variables, $\mathbf{x} = \mathbf{T}\mathbf{y}$, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} - \frac{1}{4} \begin{pmatrix} 17 \\ 15 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 5 \\ 3 \end{pmatrix} t + \frac{1}{6} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t.$$

10. Since the coefficient matrix is symmetric, the differential equations can be decoupled. The eigenvalues and eigenvectors are given by

$$r_1 = -4, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \text{ and } r_2 = -1, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} \quad \mathbf{T}^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

Setting $\mathbf{x} = \mathbf{T}\mathbf{y}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$\begin{aligned}y_1' &= -4y_1 + \frac{1}{\sqrt{3}}(1 + \sqrt{2})e^{-t} \\y_2' &= -y_2 + \frac{1}{\sqrt{3}}(1 - \sqrt{2})e^{-t}.\end{aligned}$$

The solutions are easily obtained as

$$\begin{aligned}y_1(t) &= k_1 e^{-4t} + \frac{1}{3\sqrt{3}}(1 + \sqrt{2})e^{-t} \\y_2(t) &= k_2 e^{-t} + \frac{1}{\sqrt{3}}(1 - \sqrt{2})te^{-t}.\end{aligned}$$

Transforming back to the original variables, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + \frac{1}{9} \begin{pmatrix} 2 + \sqrt{2} + 3\sqrt{3} \\ 3\sqrt{6} - \sqrt{2} - 1 \end{pmatrix} e^{-t} + \frac{1}{3} \begin{pmatrix} 1 - \sqrt{2} \\ \sqrt{2} - 2 \end{pmatrix} te^{-t}.$$

Note that

$$\begin{pmatrix} 2 + \sqrt{2} + 3\sqrt{3} \\ 3\sqrt{6} - \sqrt{2} - 1 \end{pmatrix} = \begin{pmatrix} 2 + \sqrt{2} \\ -\sqrt{2} - 1 \end{pmatrix} + 3\sqrt{3} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

The second vector is an eigenvector, hence the solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + \frac{1}{9} \begin{pmatrix} 2 + \sqrt{2} \\ -\sqrt{2} - 1 \end{pmatrix} e^{-t} + \frac{1}{3} \begin{pmatrix} 1 - \sqrt{2} \\ \sqrt{2} - 2 \end{pmatrix} te^{-t}.$$

11. Based on the solution of Problem 3 of Section 7.6, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} 5 \cos t & 5 \sin t \\ 2 \cos t + \sin t & -\cos t + 2 \sin t \end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\Psi^{-1}(t) = \frac{1}{5} \begin{pmatrix} \cos t - 2 \sin t & 5 \sin t \\ 2 \cos t + \sin t & -5 \cos t \end{pmatrix}.$$

It follows that

$$\Psi^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} \cos t \sin t \\ -\cos^2 t \end{pmatrix},$$

and

$$\int \Psi^{-1}(t)\mathbf{g}(t) dt = \begin{pmatrix} \frac{1}{2} \sin^2 t \\ -\frac{1}{2} \cos t \sin t - \frac{1}{2}t \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \Psi(t) \int \Psi^{-1}(t)\mathbf{g}(t) dt,$$

where

$$\begin{aligned} v_1(t) &= \frac{5}{2} \cos t \sin t - \cos^2 t + \frac{5}{2}t + 1 \\ v_2(t) &= \cos t \sin t - \frac{1}{2} \cos^2 t + t + \frac{1}{2}. \end{aligned}$$

Hence the general solution is

$$\begin{aligned} \mathbf{x} &= c_1 \begin{pmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin t \\ -\cos t + 2 \sin t \end{pmatrix} - \\ &\quad - t \sin t \begin{pmatrix} 5/2 \\ 1 \end{pmatrix} + t \cos t \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} - \cos t \begin{pmatrix} 5/2 \\ 1 \end{pmatrix}. \end{aligned}$$

13.(a) As shown in Problem 25 of Section 7.6, the solution of the homogeneous system is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1 e^{-t/2} \begin{pmatrix} \cos(t/2) \\ 4 \sin(t/2) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \sin(t/2) \\ -4 \cos(t/2) \end{pmatrix}.$$

Therefore the associated fundamental matrix is given by

$$\Psi(t) = e^{-t/2} \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ 4 \sin(t/2) & -4 \cos(t/2) \end{pmatrix}.$$

(b) The inverse of the fundamental matrix is

$$\Psi^{-1}(t) = \frac{e^{t/2}}{4} \begin{pmatrix} 4 \cos(t/2) & \sin(t/2) \\ 4 \sin(t/2) & -\cos(t/2) \end{pmatrix}.$$

It follows that

$$\Psi^{-1}(t)\mathbf{g}(t) = \frac{1}{2} \begin{pmatrix} \cos(t/2) \\ \sin(t/2) \end{pmatrix},$$

and

$$\int \Psi^{-1}(t)\mathbf{g}(t) dt = \begin{pmatrix} \sin(t/2) \\ -\cos(t/2) \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \Psi(t) \int \Psi^{-1}(t) \mathbf{g}(t) dt,$$

where

$$\begin{aligned} v_1(t) &= 0 \\ v_2(t) &= 4e^{-t/2}. \end{aligned}$$

Hence the general solution is

$$\mathbf{x} = c_1 e^{-t/2} \begin{pmatrix} \cos(t/2) \\ 4 \sin(t/2) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \sin(t/2) \\ -4 \cos(t/2) \end{pmatrix} + 4e^{-t/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Imposing the initial conditions, we require that

$$\begin{aligned} c_1 &= 0 \\ -4c_2 + 4 &= 0, \end{aligned}$$

which results in $c_1 = 0$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\mathbf{x} = e^{-t/2} \begin{pmatrix} \sin(t/2) \\ 4 - 4 \cos(t/2) \end{pmatrix}.$$

15. The general solution of the homogeneous problem is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2,$$

which can be verified by substitution into the system of ODEs. Since the vectors are linearly independent, a fundamental matrix is given by

$$\Psi(t) = \begin{pmatrix} t^{-1} & 2t^2 \\ 2t^{-1} & t^2 \end{pmatrix}.$$

The inverse of the fundamental matrix is

$$\Psi^{-1}(t) = \frac{1}{3} \begin{pmatrix} -t & 2t \\ 2t^{-2} & -t^{-2} \end{pmatrix}.$$

Dividing both equations by t , we obtain

$$\mathbf{g}(t) = \begin{pmatrix} -2 \\ t^3 - t^{-1} \end{pmatrix}.$$

Proceeding with the method of variation of parameters,

$$\Psi^{-1}(t) \mathbf{g}(t) = \begin{pmatrix} \frac{2}{3}t^4 + \frac{2}{3}t - \frac{2}{3} \\ -\frac{1}{3}t - \frac{4}{3}t^{-2} + \frac{1}{3}t^{-3} \end{pmatrix},$$

and

$$\int \Psi^{-1}(t) \mathbf{g}(t) dt = \begin{pmatrix} \frac{2}{15}t^5 + \frac{1}{3}t^2 - \frac{2}{3}t \\ -\frac{1}{6}t^2 + \frac{4}{3}t^{-1} - \frac{1}{6}t^{-2} \end{pmatrix}.$$

Hence a particular solution is obtained as

$$\mathbf{v} = \begin{pmatrix} -\frac{1}{5}t^4 + 3t - 1 \\ \frac{1}{10}t^4 + 2t - \frac{3}{2} \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 + \frac{1}{10} \begin{pmatrix} -2 \\ 1 \end{pmatrix} t^4 + \begin{pmatrix} 3 \\ 2 \end{pmatrix} t - \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}.$$

16. Based on the hypotheses,

$$\phi'(t) = \mathbf{P}(t)\phi(t) + \mathbf{g}(t) \text{ and } \mathbf{v}'(t) = \mathbf{P}(t)\mathbf{v}(t) + \mathbf{g}(t).$$

Subtracting the two equations results in

$$\phi'(t) - \mathbf{v}'(t) = \mathbf{P}(t)\phi(t) - \mathbf{P}(t)\mathbf{v}(t),$$

that is,

$$[\phi(t) - \mathbf{v}(t)]' = \mathbf{P}(t)[\phi(t) - \mathbf{v}(t)].$$

It follows that $\phi(t) - \mathbf{v}(t)$ is a solution of the homogeneous equation. According to Theorem 7.4.2,

$$\phi(t) - \mathbf{v}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \cdots + c_n \mathbf{x}^{(n)}(t).$$

Hence

$$\phi(t) = \mathbf{u}(t) + \mathbf{v}(t),$$

in which $\mathbf{u}(t)$ is the general solution of the homogeneous problem.

17.(a) Setting $t_0 = 0$ in Equation (34),

$$\begin{aligned} \mathbf{x} &= \Phi(t)\mathbf{x}^0 + \Phi(t) \int_0^t \Phi^{-1}(s)\mathbf{g}(s)ds \\ &= \Phi(t)\mathbf{x}^0 + \int_0^t \Phi(t)\Phi^{-1}(s)\mathbf{g}(s)ds. \end{aligned}$$

It was shown in Problem 15(c) in Section 7.7 that $\Phi(t)\Phi^{-1}(s) = \Phi(t-s)$. Therefore

$$\mathbf{x} = \Phi(t)\mathbf{x}^0 + \int_0^t \Phi(t-s)\mathbf{g}(s)ds.$$

(b) The principal fundamental matrix is identified as $\Phi(t) = e^{\mathbf{A}t}$. Hence

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}^0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)ds.$$

In Problem 27 of Section 3.6, the particular solution is given as

$$y(t) = \int_{t_0}^t K(t-s)g(s)ds,$$

in which the kernel $K(t)$ depends on the nature of the fundamental solutions.

