Physics

ACT

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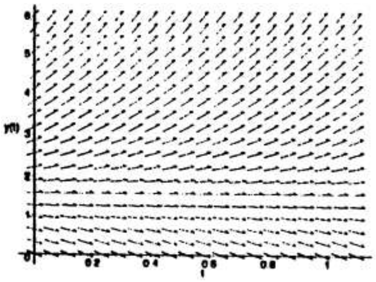
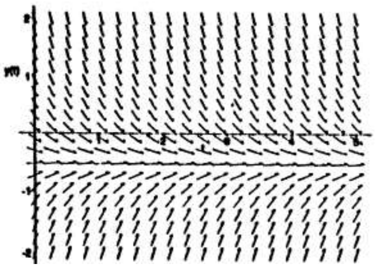
Capítulo 1

Section 1.1, Page 8



2. For $y > 3/2$ we see that $y' > 0$ and thus $y(t)$ is increasing there. For $y < 3/2$ we have $y' < 0$ and thus $y(t)$ is decreasing there. Hence $y(t)$ diverges from $3/2$ as $t\to\infty$.



4. Observing the direction field, we see that for $y>-1/2$ we have $y'<0$, so the solution is decreasing here. Likewise, for $y<-1/2$ we have $y'>0$ and thus $y(t)$ is increasing here. Since the slopes get closer to zero as y gets closer to $-1/2$, we conclude that $y\to-1/2$ as $t\to\infty$.

7. If all solutions approach 3, then 3 is the equilibrium solution and we want $\frac{dy}{dt} < 0$ for $y > 3$ and $\frac{dy}{dt} > 0$ for $y < 3$. Thus $\frac{dy}{dt} = 3-y$.

11. For $y = 0$ and $y = 4$ we have $y' = 0$ and thus $y = 0$ and $y = 4$ are equilibrium solutions. For $y > 4$, $y' < 0$ so if $y(0) > 4$ the solution approaches $y = 4$ from above. If $0 < y(0) < 4$, then $y' > 0$ and the solutions "grow" to $y = 4$ as $t\to\infty$. For $y(0) < 0$ we see that $y' < 0$ and the solutions diverge from 0.

13. Since $y' = y^2$, $y = 0$ is the equilibrium solution and $y' > 0$ for all y. Thus if $y(0) > 0$, solutions will diverge from 0 and if $y(0) < 0$, solutions will aproach $y = 0$ as $t\to\infty$.

15a. Let $q(t)$ be the number of grams of the substance in the water at any time. Then $\frac{dq}{dt} = 300(.01) - \frac{300q}{1{,}000{,}000} = 300(10^{-2} - 10^{-6}q)$.

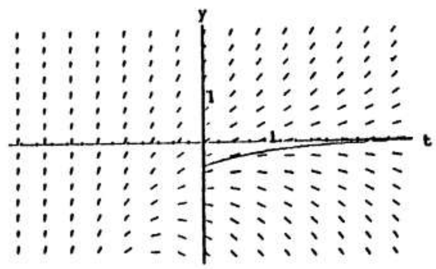
15b. The equilibrium solution occurs when $q' = 0$, or $c = 10^4$gm, independent of the amount present at $t = 0$ (all solutions approach the equilibrium solution).

16. The D.E. expressing the evaporation is $\frac{dV}{dt} = -aS$, $a > 0$.

Now $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, so $S = 4\pi\left(\frac{3}{4\pi}\right)^{2/3} V^{2/3}$.

Thus $\frac{dv}{dt} = -kV^{2/3}$, for $k > 0$.

21.



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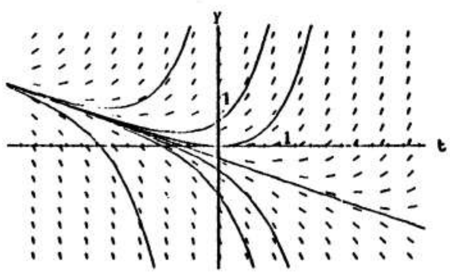

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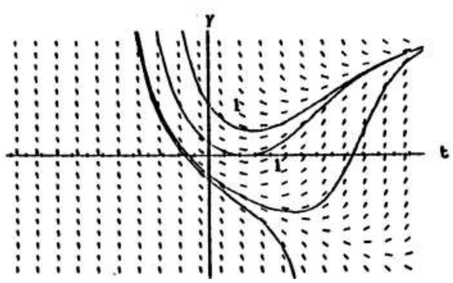

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1b. $dy/dt = -2y+5$ can be rewritten as $\frac{dy}{y-5/2} = -2dt$. Thus $\ln|y-5/2| = -2t+c_1$, or $y-5/2 = ce^{-2t}$. $y(0) = y_0$ yields $c = y_0 - 5/2$, so $y = 5/2 + (y_0-5/2)e^{-2t}$.
If $y_0 > 5/2$, the solution starts above the equilibrium solution and decreases exponentially and approaches 5/2 as $t\to\infty$. Conversely, if $y < 5/2$, the solution starts below 5/2 and grows exponentially and approaches 5/2 from

below as $t \to \infty$.

4a. Rewrite Eq.(ii) as $\dfrac{dy/dt}{y} = a$ and thus $\ln|y| = at+c_1$; or $y = ce^{at}$.

4b. If $y = y_1(t) + k$, then $\dfrac{dy}{dt} = \dfrac{dy_1}{dt}$. Substituting both these into Eq.(i) we get $\dfrac{dy_1}{dt} = a(y_1+k) - b$. Since $\dfrac{dy_1}{dt} = ay_1$, this leaves $ak - b = 0$ and thus $k = b/a$. Hence $y = y_1(t) + b/a$ is the solution to Eq(i).

6b. From Eq.(11) we have $p = 900 + ce^{t/2}$. If $p(0) = p_0$, then $c = p_0 - 900$ and thus $p = 900 + (p_0 - 900)e^{t/2}$. If $p_0 < 900$, this decreases, so if we set $p = 0$ and solve for T (the time of extinction) we obtain $e^{T/2} = 900/(900-p_0)$, or $T = 2\ln[900/(900-p_0)]$.

8a. Use Eq.(26).

8b. Use Eq.(29).

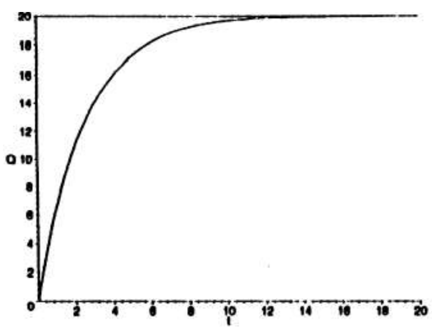
10a. $\dfrac{dQ}{dt} = -rQ$ yields $\dfrac{dQ/dt}{Q} = -r$, or $\ln|Q| = -rt + c_1$. Thus $Q = ce^{-rt}$ and $Q(0) = 100$ yields $c = 100$. Hence $Q = 100e^{-rt}$. Setting $t = 1$, we have $82.04 = 100e^{-r}$, which yields $r = .1980$/wk or $r = .02828$/day.

13a. Rewrite the D.E. as $\dfrac{dQ/dt}{Q-CV} = \dfrac{-1}{CR}$, thus upon integrating and simplifying we get $Q = De^{-t/CR} + CV$. $Q(0) = 0 \Rightarrow D = -CV$ and thus $Q(t) = CV(1 - e^{-t/CR})$.

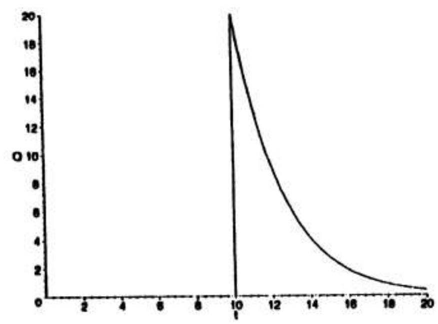
13b. $\lim\limits_{t\to\infty} Q(t) = CV$ since $\lim\limits_{t\to\infty} e^{-t/CR} = 0$.

13c. In this case $R\dfrac{dQ}{dt} + \dfrac{Q}{C} = 0$, $Q(t_1) = CV$. The solution of this D.E. is $Q(t) = Ee^{-t/CR}$, so $Q(t_1) = Ee^{-t_1/CR} = CV$, or $E = CVe^{t_1/CR}$. Thus $Q(t) = CVe^{t_1/CR}e^{-t/CR} = CVe^{-(t-t_1)/CR}$.

13a. CV = 20, CR = 2.5



13c. CV = 20, CR = 2.5 $t_1 = 10$



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2. The D.E. is second order since there is a second derivative of y appearing in the equation. The equation is nonlinear due to the y^2 term (as well as due to the y^2 term multiplying the y'' term).

6. This is a third order D.E. since the highest derivative is y''' and it is linear since y and all its derivatives appear to the first power only. The terms t^2 and $\cos^2 t$ do not affect the linearity of the D.E.

8. For $y_1(t) = e^{-3t}$ we have $y_1'(t) = -3e^{-3t}$ and $y_1''(t) = 9e^{-3t}$. Substitution of these into the D.E. yields
$9e^{-3t} + 2(-3e^{-3t}) - 3(e^{-3t}) = (9-6-3)e^{-3t} = 0$.

14. Recall that if $u(t) = \int_0^t f(s)ds$, then $u'(t) = f(t)$.

16. Differentiating e^{rt} twice and substituting into the D.E. yields $r^2e^{rt} - e^{rt} = (r^2-1)e^{rt}$. If $y = e^{rt}$ is to be a solution of the D.E. then the last quantity must be zero for all t. Thus $r^2-1 = 0$ since e^{rt} is never zero.

19. Differentiating t^r twice and substituting into the D.E. yields $t^2[r(r-1)t^{r-2}] + 4t[rt^{r-1}] + 2t^r = [r^2+3r+2]t^r$. If $y = t^r$ is to be a solution of the D.E., then the last term must be zero for all t and thus $r^2 + 3r + 2 = 0$.

22. The D.E. is second order since there are second partial derivatives of $u(x,y)$ appearing. The D.E. is nonlinear due to the product of $u(x,y)$ times u_x (or u_y).

26. Since $\frac{\partial u_1}{\partial t} = -\alpha^2 e^{-\alpha^2 t}\sin x$ and $\frac{\partial^2 u_1}{\partial x^2} = -e^{-\alpha^2 t}\sin x$ we have $\alpha^2[-e^{-\alpha^2 t}\sin x] = -\alpha^2 e^{-\alpha^2 t}\sin x$, which is true for all t and x.

CHAPTER 2

Section 2.1, Page 38

1. $\mu(t) = \exp(\int 3dt) = e^{3t}$. Thus $e^{3t}(y'+3y) = e^{3t}(t+e^{-2t})$ or $\frac{d}{dt}(ye^{3t}) = te^{3t} + e^t$. Integration of both sides yields $ye^{3t} = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + c$, and division by e^{3t} gives the general solution. Note that $\int te^{3t}dt$ is evaluated by integration by parts, with $u = t$ and $dv = e^{3t}dt$.

2. $\mu(t) = e^{-2t}$.

3. $\mu(t) = e^t$.

4. $\mu(t) = \exp(\int \frac{dt}{t}) = e^{\ln t} = t$, so $(ty)' = 3t\cos 2t$, and integration by parts yields the general solution.

6. The equation must be divided by t so that it is in the form of Eq.(3): $y' + (2/t)y = (\sin t)/t$. Thus $\mu(t) = \exp(\int \frac{2dt}{t} = t^2$, and $(t^2y)' = t\sin t$. Integration then yields $t^2y = -t\cos t + \sin t + c$.

7. $\mu(t) = e^{t^2}$.

8. $\mu(t) = \exp(\int \frac{4tdt}{1+t^2}) = (1+t^2)^2$.

11. $\mu(t) = e^t$ so $(e^ty)' = 5e^t\sin 2t$. To integrate the right side you can integrate by parts (twice), use an integral table, or use a symbolic computational software program to find $e^ty = e^t(\sin 2t - 2\cos 2t) + c$.

13. $\mu(t) = e^{-t}$ and $y = 2(t-1)e^{2t} + ce^t$. To find the value for c, set $t = 0$ in y and equate to 1, the initial value of y. Thus $-2+c = 1$ and $c = 3$, which yields the solution of the given initial value problem.

15. $\mu(t) = \exp(\int \frac{2dt}{t}) = t^2$ and $y = t^2/4 - t/3 + 1/2 + c/t^2$. Setting $t = 1$ and $y = 1/2$ we have $c = 1/12$.

18. $\mu(t) = t^2$. Thus $(t^2y)' = t\sin t$ and $t^2y = -t\cos t + \sin t + c$. Setting $t = \pi/2$ and $y = 1$ yields $c = \pi^2/4 - 1$.

20. $\mu(t) = te^t$.

Capítulo 2

CHAPTER 2

Section 2.1, Page 38

1. $\mu(t) = \exp(\int 3dt) = e^{3t}$. Thus $e^{3t}(y'+3y) = e^{3t}(t+e^{-2t})$ or $\frac{d}{dt}(ye^{3t}) = te^{3t} + e^t$. Integration of both sides yields $ye^{3t} = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + c$, and division by e^{3t} gives the general solution. Note that $\int te^{3t}dt$ is evaluated by integration by parts, with $u = t$ and $dv = e^{3t}dt$.

2. $\mu(t) = e^{-2t}$.

3. $\mu(t) = e^t$.

4. $\mu(t) = \exp(\int \frac{dt}{t}) = e^{\ln t} = t$, so $(ty)' = 3t\cos 2t$, and integration by parts yields the general solution.

6. The equation must be divided by t so that it is in the form of Eq.(3): $y' + (2/t)y = (\sin t)/t$. Thus $\mu(t) = \exp(\int \frac{2dt}{t} = t^2$, and $(t^2y)' = t\sin t$. Integration then yields $t^2y = -t\cos t + \sin t + c$.

7. $\mu(t) = e^{t^2}$.

8. $\mu(t) = \exp(\int \frac{4tdt}{1+t^2}) = (1+t^2)^2$.

11. $\mu(t) = e^t$ so $(e^ty)' = 5e^t\sin 2t$. To integrate the right side you can integrate by parts (twice), use an integral table, or use a symbolic computational software program to find $e^ty = e^t(\sin 2t - 2\cos 2t) + c$.

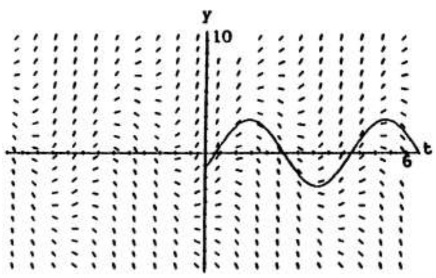
13. $\mu(t) = e^{-t}$ and $y = 2(t-1)e^{2t} + ce^t$. To find the value for c, set $t = 0$ in y and equate to 1, the initial value of y. Thus $-2+c = 1$ and $c = 3$, which yields the solution of the given initial value problem.

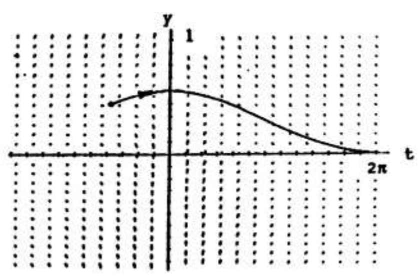
15. $\mu(t) = \exp(\int \frac{2dt}{t}) = t^2$ and $y = t^2/4 - t/3 + 1/2 + c/t^2$. Setting $t = 1$ and $y = 1/2$ we have $c = 1/12$.

18. $\mu(t) = t^2$. Thus $(t^2y)' = t\sin t$ and $t^2y = -t\cos t + \sin t + c$. Setting $t = \pi/2$ and $y = 1$ yields $c = \pi^2/4 - 1$.

20. $\mu(t) = te^t$.

21b. $\mu(t) = e^{-t/2}$ so $(e^{-t/2}y)' = 2e^{-t/2}\cos t$. Integrating (see comments in #11) and dividing by $e^{-t/2}$ yields
$y(t) = -\frac{4}{5}\cos t + \frac{8}{5}\sin t + ce^{t/2}$. Thus $y(0) = -\frac{4}{5} + c = a$,
or $c = a + \frac{4}{5}$ and $y(t) = -\frac{4}{5}\cos t + \frac{8}{5}\sin t + (a + \frac{4}{5})e^{t/2}$.
If $(a + \frac{4}{5}) = 0$, then the solution is oscillatory for all t, while if $(a + \frac{4}{5}) \neq 0$, the solution is unbounded as $t \to \infty$. Thus $a_0 = -\frac{4}{5}$.

21a. 

24a. 

24b. $\mu(t) = \exp\int \frac{2dt}{t} = t^2$, so $(t^2y)' = \sin t$ and
$y(t) = \frac{-\cos t}{t^2} + \frac{c}{t^2}$. Setting $t = -\frac{\pi}{2}$ yields
$\frac{4c}{\pi^2} = a$ or $c = \frac{a\pi^2}{4}$ and hence $y(t) = \frac{a\pi^2/4 - \cos t}{t^2}$, which is unbounded as $t \to 0$ unless $a\pi^2/4 = 1$ or $a_0 = 4/\pi^2$.

24c. For $a = 4/\pi^2$ $y(t) = \frac{1 - \cos t}{t^2}$. To find the limit as $t \to 0$ L'Hopital's Rule must be used:
$\lim\limits_{t\to 0} y(t) = \lim\limits_{t\to 0}\frac{\sin t}{2t} = \lim\limits_{t\to 0}\frac{\cos t}{2} = \frac{1}{2}$.

28. $(e^{-t}y)' = e^{-t} + 3e^{-t}\sin t$ so
$e^{-t}y = -e^{-t} - 3e^{-t}(\frac{\sin t+\cos t}{2}) + c$ or
$y(t) = -1 - (\frac{3}{2})e^{-t}(\sin t+\cos t) + ce^{t}$.
If $y(t)$ is to remain bounded, we must have $c = 0$. Thus
$y(0) = -1 - \frac{3}{2} + c = y_0$ or $c = y_0 + \frac{5}{2} = 0$ and $y_0 = -\frac{5}{2}$.

30. $\mu(t) = e^{at}$ so the D.E. can be written as
$(e^{at}y)' = be^{at}e^{-\lambda t} = be^{(a-\lambda)t}$. If $a \neq \lambda$, then integration and solution for y yields $y = [b/(a-\lambda)]e^{-\lambda t} + ce^{-at}$. Then $\lim_{x\to\infty} y$ is zero since both λ and a are positive numbers.
If $a = \lambda$, then the D.E. becomes $(e^{at}y)' = b$, which yields $y = (bt+c)/e^{\lambda t}$ as the solution. L'Hopital's Rule gives
$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{(bt+c)}{e^{\lambda t}} = \lim_{t\to\infty} \frac{b}{\lambda e^{\lambda t}} = 0.$$

32. There is no unique answer for this situation. One possible response is to assume $y(t) = ce^{-2t} + 3 - t$, then $y'(t) = -2ce^{-2t} - 1$ and thus $y' + 2y = 5 - 2t$.

35. This problem demonstrates the central idea of the method of variation of parameters for the simplest case. The solution (ii) of the homogeneous D.E. is extended to the corresponding nonhomogeneous D.E. by replacing the constant A by a function A(t), as shown in (iii).

36. Assume $y(t) = A(t)\exp(-\int(-2)dt) = A(t)e^{2t}$. Differentiating y(t) and substituting into the D.E. yields $A'(t) = t^2$ since the terms involving A(t) add to zero. Thus $A(t) = t^3/3 + c$, which substituted into y(t) yields the solution.

37. $y(t) = A(t)\exp(-\int\frac{dt}{t}) = A(t)/t$.

<u>Section 2.2, Page 45</u>

Problems 1 through 20 follow the pattern of the examples worked in this section. The first eight problems, however, do not have I.C. so the integration constant, c, cannot be found.

1. Write the equation in the form $ydy = x^2dx$. Integrating the left side with respect to y and the right side with respect to x yields
$$\frac{y^2}{2} = \frac{x^3}{3} + C, \text{ or } 3y^2 - 2x^3 = c.$$

4. For $y \neq -3/2$ multiply both sides of the equation by $3 + 2y$ to get the separated equation
$(3+2y)dy = (3x^2-1)dx$. Integration then yields

$3y + y^2 = x^3 - x + c$.

6. We need $x \neq 0$ and $|y| < 1$ for this problem to be defined. Separating the variables we get $(1-y^2)^{-1/2}dy = x^{-1}dx$. Integrating each side yields $\arcsin y = \ln|x|+c$, so $y = \sin[\ln|x|+c]$, $x \neq 0$ (note that $|y| < 1$). Also, $y = \pm 1$ satisfy the D.E., since both sides are zero.

10a. Separating the variables we get $ydy = (1-2x)dx$, so $\frac{y^2}{2} = x - x^2 + c$. Setting $x = 0$ and $y = -2$ we have $2 = c$ and thus $y^2 = 2x - 2x^2 +$ or $y = -\sqrt{2x - 2x^2 + 4}$. The negative square root must be used since $y(0) = -2$.

10c. Rewriting $y(x)$ as $-\sqrt{2(2-x)(x+1)}$, we see that y is defined for $-1 \le x \le 2$, However, since y' does not exist for $x = -1$ or $x = 2$, the solution is valid only for the open interval $-1 < x < 2$.

13. Separate variables by factoring the denominator of the right side to get $ydy = \frac{2x}{1+x^2}dx$. Integration yields $y^2/2 = \ln(1+x^2)+c$ and use of the I.C. gives $c = 2$. Thus $y = \pm [2\ln(1+x^2)+4]^{1/2}$, but we must discard the plus square root because of the I.C. Since $1 + x^2 > 0$, the solution is valid for all x.

15. Separating variables and integrating yields $y + y^2 = x^2 + c$. Setting $y = 0$ when $x = 2$ yields $c = -4$ or $y^2 + y = x^2-4$. To solve for y complete the square on the left side by adding 1/4 to both sides. This yields $y^2 + y + \frac{1}{4} = x^2 - 4 + \frac{1}{4}$ or $(y + \frac{1}{2})^2 = x^2 - 15/4$. Taking the square root of both sides yields $y + \frac{1}{2} = \pm\sqrt{x^2 - 15/4}$, where the positive square root must be taken in order to satisfy the I.C. Thus $y = -\frac{1}{2} + \sqrt{x^2 - 15/4}$, which is defined for $x^2 \ge 15/4$ or $x \ge \sqrt{15}/2$. The possibility that $x < -\sqrt{15}/2$ is discarded due to the I.C.

17a. Separating variables gives $(2y-5)dy = (3x^2-e^x)dx$ and integration then gives $y^2 - 5y = x^3 - e^x + c$. Setting $x = 0$ and $y = 1$ we have $1 - 5 = 0 - 1 + c$, or $c = -3$. Thus $y^2 - 5y - (x^3-e^x-3) = 0$ and using the quadratic formula then gives

$$y(x) = \frac{5 \pm \sqrt{25+4(x^3-e^x-3)}}{2} = \frac{5}{2} - \sqrt{\frac{13}{4} + x^3 - e^x}.$$ The negative square root is chosen due to the I.C.

17c. The interval of definition for y must be found numerically. Approximate values can be found by plotting $y_1(x) = \frac{13}{4} + x^3$ and $y_2(x) = e^x$ and noting the values of x where the two curves cross.

19a. As above we start with cos3ydy = -sin2xdx and integrate to get $\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + c$. Setting $y = \pi/3$ when $x = \pi/2$ (from the I.C.) we find that $0 = -\frac{1}{2} + c$ or $c = \frac{1}{2}$, so that $\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + \frac{1}{2} = \cos^2 x$ (using the appropriate trigonometric identity). To solve for y we must choose the branch that passes through the point $(\pi/2,\pi/3)$ and thus $3y = \pi - \arcsin(3\cos^2 x)$, or $y = \frac{\pi}{3} - \frac{1}{3}\arcsin(3\cos^2 x)$.

19c. The solution in part a is defined only for $0 \le 3\cos^2 x \le 1$, or $-\sqrt{1/3} \le \cos x \le \sqrt{1/3}$. Taking the indicated square roots and then finding the inverse cosine of each side yields $.9553 \le x \le 2.1863$, or $|x-\pi/2| \le 0.6155$, as the approximate interval.

21. We have $(3y^2-6y)dy = (1+3x^2)dx$ so that $y^3-3y^2 = x + x^3 - 2$, once the I.C. are used. From the D.E., the integral curve will have a vertical tangent when $3y^2 - 6y = 0$, or $y = 0,2$. For $y = 0$ we have $x^3 + x - 2 = 0$, which is satisfied for $x = 1$, which is the only zero of the function $w = x^3 + x - 2$. Likewise, for $y = 2$, $x = -1$.

23. Separating variables gives $y^{-2}dy = (2+x)dx$, so

$-y^{-1} = 2x + \frac{x^2}{2} + c$. $y(0) = 1$ yields $c = -1$ and thus

$y = \dfrac{-1}{\frac{x^2}{2} + 2x - 1} = \dfrac{2}{2 - 4x - x^2}$. This gives

$\dfrac{dy}{dx} = \dfrac{8 + 4x}{(2-4x-x^2)^2}$, so the minimum value is attained at $x = -2$. Note that the solution is defined for $-2 - \sqrt{6} < x < -2 + \sqrt{6}$ (by finding the zeros of the denominator) and has vertical asymptotes at the end points of the interval.

25. Separating variables and integrating yields $3y + y^2 = \sin 2x + c$. $y(0) = -1$ gives $c = -2$ so that $y^2 + 3y + (2-\sin 2x) = 0$. The quadratic formula then gives $y = -\frac{3}{2} + \sqrt{\sin 2x + 1/4}$, which is defined for $-.126 < x < 1.697$ (found by solving $\sin 2x = -.25$ for x and noting $x = 0$ is the initial point). Thus we have $\dfrac{dy}{dx} = \dfrac{\cos 2x}{(\sin 2x + \frac{1}{4})}$, which yields $x = \pi/4$ as the only critical point in the above interval. Using the second derivative test or graphing the solution clearly indicates the critical point is a maximum.

27a. By sketching the direction field and using the D.E.we note that $y' < 0$ for $y > 4$ and y' approaches zero as y approaches 4. For $0 < y < 4$, $y' > 0$ and again approaches zero as y approaches 4. Thus $\lim_{t\to\infty} y = 4$ if $y_0 > 0$. For $y_0 < 0$, $y' < 0$ for all y and hence y becomes negatively unbounded $(-\infty)$ as t increases. If $y_0 = 0$, then $y' = 0$ for all t, so $y = 0$ for all t.

27b. Separating variables and using a partial fraction expansion we have $(\frac{1}{y} - \frac{1}{y-4})dy = \frac{4}{3}t\,dt$. Hence $\ln\left|\frac{y}{y-4}\right| = \frac{2}{3}t^2 + c_1$ and thus $\left|\frac{y}{y-4}\right| = e^{c_1}e^{2t^2/3} = ce^{2t^2/3}$, where c is positive. For $y_0 = .5$ this becomes $\dfrac{y}{4-y} = ce^{2t^2/3}$ and thus $c = \dfrac{.5}{3.5} = \dfrac{1}{7}$. Using this value for c and solving for y yields $y(t) = \dfrac{4}{1 + 7e^{-2t^2/3}}$.

Setting this equal to 3.98 and solving for t yields $t = 3.29527$.

29. Separating variables yields $\dfrac{cy+d}{ay+b}\,dy = dx$. If $a \neq 0$ and $ay+b \neq 0$ then $dx = \left(\dfrac{c}{a} + \dfrac{ad-bc}{a(ay+b)}\right)dy$. Integration then yields the desired answer.

30c. If $v = y/x$ then $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ and thus the D.E. becomes $v + x\dfrac{dv}{dx} = \dfrac{v-4}{1-v}$. Subtracting v from both sides yields $x\dfrac{dv}{dx} = \dfrac{v^2-4}{1-v}$.

30d. The last equation in (c) separates into $\dfrac{1-v}{v^2-4}dv = \dfrac{1}{x}dx$. To integrate the left side use partial fractions to write $\dfrac{1-v}{v-4} = \dfrac{A}{v-2} + \dfrac{B}{v+2}$, which yields $A = -1/4$ and $B = -3/4$. Integration then gives $-\dfrac{1}{4}\ln|v-2| - \dfrac{3}{4}\ln|v+2| = \ln|x| - k$, or $\ln|x^4||v-2||v+2|^3 = 4k$ after manipulations using properties of the ln function.

31a. Simplifying the right side of the D.E. gives $dy/dx = 1 + (y/x) + (y/x)^2$ so the equation is homogeneous.

31b. The substitution $y = vx$ leads to $v + x\dfrac{dv}{dx} = 1 + v + v^2$ or $\dfrac{dv}{1 + v^2} = \dfrac{dx}{x}$. Solving, we get $\arctan v = \ln|x| + c$. Substituting for v we obtain $\arctan(y/x) - \ln|x| = c$.

33b. Dividing the numerator and denominator of the right side by x and substituting $y = vx$ we get $v + x\dfrac{dv}{dx} = \dfrac{4v - 3}{2-v}$ which can be rewritten as $x\dfrac{dv}{dx} = \dfrac{v^2 + 2v - 3}{2 - v}$. Note that $v = -3$ and $v = 1$ are solutions of this equation. For $v \neq 1, -3$ separating variables gives

$\frac{2 - v}{(v+3)(v-1)}\,dv = \frac{1}{x}dx$. Applying a partial fraction decomposition to the left side we obtain $[\frac{1}{4}\frac{1}{v-1} - \frac{5}{4}\frac{1}{v+3}]dv = \frac{dx}{x}$, and upon integrating both sides we find that $\frac{1}{4}\ln|v-1| - \frac{5}{4}\ln|v+3| = \ln|x| + c$. Substituting for v and performing some algebraic manipulations we get the solution in the implicit form $|y-x| = c|y+3x|^5$. v = 1 and v = −3 yield y = x and y = −3x, respectively, as solutions also.

35b. As in Prob.33, substituting y = vx into the D.E. we get $v + x\frac{dv}{dx} = \frac{1+3v}{1-v}$, or $x\frac{dv}{dx} = \frac{(v+1)^2}{1-v}$. Note that v = -1 (or y = -x) satisfies this D.E. Separating variables yields $\frac{1-v}{(v+1)^2}dv = \frac{dx}{x}$. Integrating the left side by parts we obtain $\frac{v-1}{v+1} - \ln|v+1| = \ln|x| + c$. Letting $v = \frac{y}{x}$ then yields $\frac{y-x}{y+x} - \ln\left|\frac{y+x}{x}\right| = \ln|x| + c$, or $\frac{y-x}{y+x} - \ln|y+x| = c$. This answer differs from that in the text. The answer in the text can be obtained by integrating the left side, above, using partial fractions. By differentiating both answers, it can be verified that indeed both forms satisfy the D.E.

<u>Section 2.3, Page 57</u>

1. Note that q(0) = 200 gm, where q(t) represents the amount of dye present at any time.

2. Let S(t) be the amount of salt that is present at any time t, then S(0) = 0 is the original amount of salt in the tank, 2 is the amount of salt entering per minute, and 2(S/120) is the amount of salt leaving per minute (all amounts measured in grams). Thus
dS/dt = 2 - 2S/120, S(0) = 0.

3. We must first find the amount of salt that is present after 10 minutes. For the first 10 minutes (if we let Q(t) be the amount of salt in the tank):
 = (2) - 2, Q(0) = 0. This I.V.P. has the solution: Q(10) = 50(1-) 9.063 lbs. of salt in the tank after the first 10

minutes. At this point no more salt is allowed to enter, so the new I.V.P. (letting P(t) be the amount of salt in the tank) is:
= (0)(2) - 2, P(0) = Q(10) = 9.063. The solution of this problem is P(t) = 9.063, which yields P(10) = 7.42 lbs.

4. Salt flows into the tank at the rate of (1)(3) lb/min. and it flows out of the tank at the rate of (2) lb/min. since the volume of water in the tank at any time t is 200 + (1)(t) gallons (due to the fact that water flows into the tank faster than it flows out). Thus the I.V.P. is dQ/dt = 3 - Q(t), Q(0) = 100.

7a. Set = 0 in Eq.(16) (or solve Eq.(15) with S(0) = 0).

7b. Set r = .075, t = 40 and S(t) = $1,000,000 in the answer to (a) and then solve for k.

7c. Set k = $2,000, t = 40 and S(t) = $1,000,000 in the answer to (a) and then solve numerically for r.

9. The rate of accumulation due to interest is .1S and the rate of decrease is k dollars per year and thus dS/dt = .1S - k, S(0) = $8,000. Solving this for S(t) yields S(t) = 8000 - 10k(-1). Setting S = 0 and substitution of t = 3 gives k = $3,086.64 per year. For 3 years this totals $9,259.92, so $1,259.92 has been paid in interest.

10. Since we are assuming continuity, either convert the monthly payment into an annual payment or convert the yearly interest rate into a monthly interest rate for 240 months. Then proceed as in Prob. 9.

12a. Using Eq. (15) we have - S = 800(1) or
S (800t), S(0) 100,000. Using an integrating factor and integration by parts (or using a D.E. solver) we get S(t) t c. Using the I.C. yields c . Substituting this value into S, setting S(t) 0, and solving numerically for t yields t 135.363 months.

16a. This problem can be done numerically using appropriate D.E. solver software. Analytically we have
(.1.2sint)dt by separating variables and thus
y(t) cexp(.1t.2cost). y(0) 1 gives c , so
y(t) exp(.2.1t.2cost). Setting y 2 yields
ln2 .2 .1 .2cos, which can be solved numerically to give
2.9632. If y(0) , then as above,
y(t) exp(.2.1t.2cost). Thus if we set y 2 we get the

same numerical equation for and hence the doubling time has not changed.

18. From Eq.(26) we have 19 = 7 + (20-7) or k = -ln = ln(13/12). Hence if (T) = 15 we get: 15 = 7 + 13. Solving for T yields T = ln(8/13)/-ln(13/12) = ln(13/8)/ln(13/12) min.

19. Hint: let Q(t) be the quantity of carbon monoxide in the room at any time t. Then the concentration is given by x(t) = Q(t)/1200.

20a. The required I.V.P. is dQ/dt = kr + P - r, Q(0) = V. Since c = Q(t)/V, the I.V.P. may be rewritten Vc(t) = kr + P - rc, c(0) = , which has the solution c(t) = k + + (- k -).

20b. Set k = 0, P = 0, t = T and c(T) = .5 in the solution found in (a).

21a. If we measure x positively upward from the ground, then Eq.(4) of Section 1.1 becomes m = -mg, since there is no air resistance. Thus the I.V.P. for v(t) is dv/dt = -g, v(0) = 20. Hence = v(t) = 20 - gt and x(t) = 20t - (g/2) + c. Since x(0) = 30, c = 30 and $x(t) = 20t - (g/2)t^2 + 30$. At the maximum height $v(t_m) = 0$ and thus $t_m = 20/9.8 = 2.04$ sec., which when substituted in the equation for x(t) yields the maximum height.

21b. At the ground $x(t_g) = 0$ and thus $20t_g - 4.9t_g^2 + 30 = 0$.

22. The I.V.P. in this case is $m\frac{dv}{dt} = -\frac{1}{30}v - mg$, v(0) = 20, where the positive direction is measured upward.

24a. The I.V.P. is $m\frac{dv}{dt} = mg - .75v$, v(0) = 0 and v is measured positively downward. Since m = 180/32, the D.E. becomes $\frac{dv}{dt} = 32 - \frac{2}{15}v$ and thus $v(t) = 240(1-e^{-2t/15})$ so that v(10) = 176.7 ft/sec.

24b. Integration of v(t) as found in (a) yields $x(t) = 240t + 1800(e^{-2t/15}-1)$ where x is measured positively down from the altitude of 5000 feet. Set t = 10 to find the distance traveled when the parachute opens.

24c. After the parachute opens the I.V.P. is $m\frac{dv}{dt} = mg-12v$, $v(0) = 176.7$, which has the solution $v(t) = 161.7e^{-32t/15} + 15$ and where t = 0 now represents the time the parachute opens. Letting $t\to\infty$ yields the limiting velocity.

24d. Integrate v(t) as found in (c) to find $x(t) = 15t - 75.8e^{-32t/15} + C_2$. $C_2 = 75.8$ since $x(0) = 0$, x now being measured from the point where the parachute opens. Setting x = 3925.5 will then yield the length of time the skydiver is in the air after the parachute opens.

26a. Again, if x is measured positively upward, then Eq.(4) of Sect.1.1 becomes $m\frac{dv}{dt} = -mg - kv$.

26b. From part (a) $v(t) = -\frac{mg}{k} + [v_0 + \frac{mg}{k}]e^{-kt/m}$. As $k \to 0$ this has the indeterminant form of $-\infty + \infty$. Thus rewrite v(t) as $v(t) = [-mg + (v_0k + mg)e^{-kt/m}]/k$ which has the indeterminant form of 0/0, as $k \to 0$ and hence L'Hopital's Rule may be applied with k as the variable.

27a. The equation of motion is $m(dv/dt) = w-R-B$ which, in this problem, is $\frac{4}{3}\pi a^3\rho(dv/dt) = \frac{4}{3}\pi a^3\rho g - 6\pi\mu av - \frac{4}{3}\pi a^3\rho' g$. The limiting velocity occurs when $dv/dt = 0$.

27b. Since the droplet is motionless, $v = dv/dt = 0$, we have the equation of motion $0 = (\frac{4}{3})\pi a^3\rho g - Ee - (\frac{4}{3})\pi a^3\rho' g$, where ρ is the density of the oil and ρ' is the density of air. Solving for e yields the answer.

28. All three parts can be answered from one solution if k represents the resistance and if the method of solution of Example 4 is used. Thus we have $m\frac{dv}{dt} = mv\frac{dv}{dx} = mg - kv$, $v(0) = 0$, where we have assumed the velocity is a function of x. The solution of this

I.V.P. involves a logarithmic term, and thus the answers to parts (a) and (c) must be found using a numerical procedure.

29b. Note that 32 ft/sec^2 = 78,545 m/hr^2.

30. This problem is the same as Example 4 through Eq.(29). In this case the I.C. is $v(\xi R) = v_o$, so $c = \dfrac{v_0{}^2}{2} - \dfrac{gR}{1+\xi}$. The escape velocity, v_e, is found by noting that $v_o^2 \geq \dfrac{2gR}{1+\xi}$ in order for v^2 to always be positive. From Example 4, the escape velocity for a surface launch is $v_e(0) = \sqrt{2gR}$. We want the escape velocity of $x_o = \xi R$ to have the relation $v_e(\xi R) = .85v_e(0)$, which yields $\xi = (0.85)^{-2} - 1 \cong 0.384$. If R = 4000 miles then $x_o = \xi R = 1536$ miles.

31b. From part a) $\dfrac{dx}{dt} = v = u\cos A$ and hence $x(t) = (u\cos A)t + d_1$. Since $x(0) = 0$, we have $d_1 = 0$ and $x(t) = (u\cos A)t$. Likewise $\dfrac{dy}{dt} = -gt + u\sin A$ and therefore $y(t) = -gt^2/2 + (u\sin A)t + d_2$. Since $y(0) = h$ we have $d_2 = h$ and $y(t) = -gt^2/2 + (u\sin A)t + h$.

31d. Let t_w be the time the ball reaches the wall. Then $x(t_w) = L = (u\cos A)t_w$ and thus $t_w = \dfrac{L}{u\cos A}$. For the ball to clear the wall $y(t_w) \geq H$ and thus (setting $t_w = \dfrac{L}{u\cos A}$, g = 32 and h = 3 in y) we get

$$\frac{-16L^2}{u^2\cos^2 A} + L\tan A + 3 \geq H.$$

31e. Setting L = 350 and H = 10 we get $\dfrac{-161.98}{\cos^2 A} + 350\dfrac{\sin A}{\cos A} \geq 7$ or $7\cos^2 A - 350\cos A\sin A + 161.98 \leq 0$. This can be solved numerically or by plotting the left side as a function of A and finding where the zero crossings are.

31f. Setting L = 350, and H = 10 in the answer to part d yields $\frac{-16(350)^2}{u^2\cos^2 A} + 350\tan A = 7$, where we have chosen the equality sign since we want to just clear the wall. Solving for u^2 we get $u^2 = \frac{1,960,000}{175\sin 2A - 7\cos^2 A}$. Now u will have a minimum when the denominator has a maximum. Thus $350\cos 2A + 7\sin 2A = 0$, or $\tan 2A = -50$, which yields $A = .7954$ rad. and $u = 106.89$ ft./sec.

Section 2.4, Page 72

1. If the equation is written in the form of Eq.(1), then $p(t) = (\ln t)/(t-3)$ and $g(t) = 2t/(t-3)$. These are defined and continuous on the intervals $(0,3)$ and $(3,\infty)$, but since the initial point is $t = 1$, the solution will be continuous on $0 < t < 3$.

4. $p(t) = 2t/(2-t)(2+t)$ and $g(t) = 3t^2/(2-t)(2+t)$.

8. Theorem 2.4.2 guarantees a unique solution to the D.E. through any point (t_0,y_0) such that $t_0^2 + y_0^2 < 1$ since $\frac{\partial f}{\partial y} = -y(1-t^2-y^2)^{1/2}$ is defined and continuous only for $1-t^2-y^2 > 0$. Note also that $f = (1-t^2-y^2)^{1/2}$ is defined and continuous in this region as well as on the boundary $t^2+y^2 = 1$. The boundary can't be included in the final region due to the discontinuity of $\frac{\partial f}{\partial y}$ there.

11. In this case $f = \frac{1+t^2}{y(3-y)}$ and $\frac{\partial f}{\partial y} = \frac{1+t^2}{y(3-y)^2} - \frac{1+t^2}{y^2(3-y)}$, which are both continuous everywhere except for $y = 0$ and $y = 3$.

13. The D.E. may be written as $ydy = -4tdt$ so that $\frac{y^2}{2} = -2t^2+c$, or $y^2 = c-4t^2$. The I.C. then yields $y_0^2 = c$, so that $y^2 = y_0^2 - 4t^2$ or $y = \pm\sqrt{y_0^2-4t^2}$, which is defined for $4t^2 < y_0^2$ or $|t| < |y_0|/2$. Note that $y_0 \neq 0$ since Theorem 2.4.2 does not hold there.

17. From the direction field and the given D.E. it is noted that for $t > 0$ and $y < 0$ that $y' < 0$, so $y \to -\infty$ for $y_0 < 0$. Likewise, for $0 < y_0 < 3$, $y' > 0$ and $y' \to 0$ as

$y \to 3$, so $y \to 3$ for $0 < y_0 < 3$ and for $y_0 > 3$, $y' < 0$ and again $y' \to 0$ as $y \to 3$, so $y \to 3$ for $y_0 > 3$. For $y_0 = 3$, $y' = 0$ and $y = 3$ for all t and for $y_0 = 0$, $y' = 0$ and $y = 0$ for all t.

22a. For $y_1 = 1-t$, $y_1' = -1 = \dfrac{-t+[t^2+4(1-t)]^{1/2}}{2}$

$$= \frac{-t+[(t-2)^2]^{1/2}}{2}$$

$$= \frac{-t+|t-2|}{2} = -1 \text{ if}$$

$(t-2) \geq 0$, by the definition of absolute value. Setting $t = 2$ in y_1 we get $y_1(2) = -1$, as required.

22b. By Theorem 2.4.2 we are guaranteed a unique solution only where $f(t,y) = \dfrac{-t+(t^2+4y)^{1/2}}{2}$ and $f_y(t,y) = (t^2+4y)^{-1/2}$ are continuous. In this case the initial point $(2,-1)$ lies in the region $t^2 + 4y \leq 0$, in which case $\dfrac{\partial f}{\partial y}$ is not continuous and hence the theorem is not applicable and there is no contradiction.

22c. If $y = y_2(t)$ then we must have $ct + c^2 = -t^2/4$, which is not possible since c must be a constant.

23a. To show that $\phi(t) = e^{2t}$ is a solution of the D.E., take its derivative and substitute into the D.E.

24. $[c\,\phi(t)]' + p(t)[c\phi(t)] = c[\phi'(t) + p(t)\phi(t)] = 0$ since $\phi(t)$ satisfies the given D.E.

25. $[y_1(t) + y_2(t)]' + p(t)[y_1(t) + y_2(t)] =$
$y_1'(t) + p(t)y_1(t) + y_2'(t) + p(t)y_2(t) = 0 + g(t)$.

27a. For $n = 0,1$, the D.E. is linear and Eqs.(3) and (4) apply.

27b. Let $v = y^{1-n}$ then $\dfrac{dv}{dt} = (1-n)y^{-n}\dfrac{dy}{dt}$ so $\dfrac{dy}{dt} = \dfrac{1}{1-n}y^n\dfrac{dv}{dt}$, which makes sense when $n \neq 0,1$. Substituting into the D.E. yields $\dfrac{y^n}{1-n}\,\dfrac{dv}{dt} + p(t)y = q(t)y^n$ or

$v' + (1-n)p(t)y^{1-n} = (1-n)q(t)$. Setting $v = y^{1-n}$ then yields a linear D.E. for v.

28. $n = 3$ so $v = y^{-2}$ and $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$ or $\frac{dy}{dt} = -\frac{1}{2}\,y^3\frac{dv}{dt}$.
Substituting this into the D.E. gives
$-\frac{1}{2}y^3\frac{dv}{dt} + \frac{2}{t}y = \frac{1}{t^2}y^3$. Simplifying and setting
$y^{-2} = v$ then gives the linear D.E.
$v' - \frac{4}{t}v = -\frac{2}{t^2}$, where $\mu(t) = \frac{1}{t^4}$ and
$v(t) = ct^4 + \frac{2}{5t} = \frac{2+5ct^5}{5t}$. Thus $y = \pm[5t/(2+5ct^5)]^{1/2}$.

29. $n = 2$ so $v = y^{-1}$ and $\frac{dv}{dt} = -y^2\frac{dv}{dt}$. Thus the D.E. becomes $-y^2\frac{dv}{dt} - ry = -ky^2$ or $\frac{dv}{dt} + rv = k$. Hence $\mu(t) = e^{rt}$ and $v = k/r + ce^{-rt}$. Setting $v = 1/y$ then yields the solution.

32. Since g(t) is continuous on the interval $0 \le t \le 1$ we may solve the I.V.P. $y_1' + 2y_1 = 1$, $y_1(0) = 0$ on that interval to obtain $y_1 = 1/2 - (1/2)e^{-2t}$, $0 \le t \le 1$. g(t) is also continuous for $1 < t$; and hence we may solve $y_2' + 2y_2 = 0$ to obtain $y_2 = ce^{-2t}$, $1 < t$. The solution y of the original I.V.P. must be continuous (since its derivative must exist) and hence we need c in y_2 so that y_2 at 1 has the same value as y_1 at 1. Thus
$ce^{-2} = 1/2 - e^{-2}/2$ or $c = (1/2)(e^2-1)$ and we obtain

$$y = \begin{cases} 1/2 - (1/2)e^{-2t} & 0 \le t \le 1 \\ 1/2(e^2-1)e^{-2t} & 1 \le t \end{cases} \qquad \text{and}$$

$$y' = \begin{cases} e^{-2t} & 0 \le t \le 1 \\ (1-e^2)e^{-2t} & 1 < t. \end{cases}$$

Evaluating the two parts of y′ at $t_0 = 1$ we see that they are different, and hence y′ is not continuous at $t_0 = 1$.

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Problems 1 through 13 follow the pattern illustrated in Fig.2.5.3 and the discussion following Eq.(11).

3. The critical points are found by setting $\frac{dy}{dt}$ equal to zero. Thus y = 0,1,2 are the critical points. The graph of y(y-1)(y-2) is positive for 0 < y < 1 and 2 < y and negative for 1 < y < 2. Thus y(t) is increasing ($\frac{dy}{dt} > 0$) for 0 < y < 1 and 2 < y and decreasing ($\frac{dy}{dt} < 0$) for 1 < y < 2. Therefore 0 and 2 are unstable critical points while 1 is an asymptotically stable critical point.

6. $\frac{dy}{dt}$ is zero only when arctany is zero (ie, y = 0). $\frac{dy}{dt} > 0$ for y < 0 and $\frac{dy}{dt} < 0$ for y > 0. Thus y = 0 is an asymptotically stable critical point.

7c. Separate variables to get $\frac{dy}{(1-y)^2} = kt$. Integration yields $\frac{1}{1-y} = kt + c$, or $y = 1 - \frac{1}{kt + c} = \frac{kt + c - 1}{kt + c}$. Setting t = 0 and $y(0) = y_0$ yields $y_0 = \frac{c-1}{c}$ or $c = \frac{1}{1-y_0}$. Hence $y(t) = \frac{(1-y_0)kt + y_0}{(1-y_0)kt + 1}$. Note that for $y_0 < 1$ $y \to (1-y_0)k/(1-y_0)k = 1$ as $t \to \infty$. For $y_0 > 1$ notice that the denominator will have a zero for some value of t, depending on the values chosen for y_0 and k. Thus the solution has a discontinuity at that point.

9. Setting $\frac{dy}{dt} = 0$ we find y = 0, $\pm$ 1 are the critical points. Since $\frac{dy}{dt} > 0$ for y < -1 and y > 1 while $\frac{dy}{dt} < 0$ for -1 < y < 1 we may conclude that y = -1 is asymptotically stable, y = 0 is semistable, and y = 1 is unstable.

11. $y = b^2/a^2$ and $y = 0$ are the only critical points. For $0 < y < b^2/a^2$, $\frac{dy}{dt} < 0$ and thus $y = 0$ is asymptotically stable. For $y > b^2/a^2$, $dy/dt > 0$ and thus $y = b^2/a^2$ is unstable.

14. If $f'(y_1) < 0$ then the slope of f is negative at y_1 and thus $f(y) > 0$ for $y < y_1$ and $f(y) < 0$ for $y > y_1$ since $f(y_1) = 0$. Hence y_1 is an asysmtotically stable critical point. A similar argument will yield the result for $f'(y_1) > 0$.

16b. By taking the derivative of $y \ln(K/y)$ it can be shown that the graph of $\frac{dy}{dt}$ vs y has a maximum point at $y = K/e$. Thus $\frac{dy}{dt}$ is positive and increasing for $0 < y < K/e$ and thus $y(t)$ is concave up for that interval. Similarly $\frac{dy}{dt}$ is positive and decreasing for $K/e < y < K$ and thus $y(t)$ is concave down for that interval.

16c. $\ln(K/y)$ is very large for small values of y and thus $(ry)\ln(K/y) > ry(1 - y/K)$ for small y. Since $\ln(K/y)$ and $(1 - y/K)$ are both strictly decreasing functions of y and since $\ln(K/y) = (1 - y/K)$ only for $y = K$, we may conclude that $\frac{dy}{dt} = (ry)\ln(K/y)$ is never less than $\frac{dy}{dt} = ry(1 - y/K)$.

17a. If $u = \ln(y/K)$ then $y = Ke^u$ and $\frac{dy}{dt} = Ke^u\frac{du}{dt}$ so that the D.E. becomes $du/dt = -ru$.

18a. The D.E. is $dV/dt = k - \alpha\pi r^2$. The volume of a cone of height L and radius r is given by $V = \pi r^2 L/3$ where $L = hr/a$ from symmetry. Solving for r yields the desired solution.

18b. Equilibrium is given by $k - \alpha\pi r^2 = 0$.

18c. The equilibrium height must be less than h.

20b. Use the results of Problem 14.

20d. Differentiate Y with respect to E.

21a. Set $\frac{dy}{dt} = 0$ and solve for y using the quadratic formula.

21b. Use the results of Problem 14.

21d. If $h > rK/4$ there are no critical points (see part a) and $\frac{dy}{dt} < 0$ for all t.

24a. If $z = x/n$ then $dz/dt = \frac{1}{n}\frac{dx}{dt} - \frac{x}{n^2}\frac{dn}{dt}$. Use of Equations (i) and (ii) then gives the I.V.P. (iii).

24b. Separate variables to get $\frac{\nu dz}{z(1-\nu z)} = -\beta dt$. Using partial fractions this becomes $\frac{dz}{z} + \frac{dz}{1-z} = -\beta dt$. Integration and solving for z yields the answer.

24c. Find z(20).

26a. Plot dx/dt vs x and observe that x = p and x = q are critical points. Also note that $dx/dt > 0$ for $x < \min(p,q)$ and $x > \max(p,q)$ while $dx/dt < 0$ for x between $\min(p,q)$ and $\max(p,q)$. Thus $x = \min(p,q)$ is an asymptotically stable point while $x = \max(p,q)$ is unstable. To solve the D.E., separate variables and use partial fractions to obtain $\frac{1}{q-p}\left[\frac{dx}{q-x} - \frac{dx}{p-x}\right] = \alpha dt$. Integration and solving for x yields the solution.

26b. x = p is a semistable critical point and since $\frac{dx}{dt} > 0$, x(t) is an increasing function. Thus for x(0) = 0, x(t) approaches p as $t \to \infty$. To solve the D.E., separate variables and integrate.

<u>Section 2.6, Page 95</u>

3. $M(x,y) = 3x^2-2xy+2$ and $N(x,y) = 6y^2-x^2+3$, so $M_y = -2x = N_x$ and thus the D.E. is exact. Integrating M(x,y) with

respect to x we get $\psi(x,y) = x^3 - x^2y + 2x + H(y)$. Taking the partial derivative of this with respect to y and setting it equal to N(x,y) yields $-x^2+h'(y) = 6y^2-x^2+3$, so that $h'(y) = 6y^2 + 3$ and $h(y) = 2y^3 + 3y$. Substitute this h(y) into $\psi(x,y)$ and recall that the equation which defines y(x) implicitly is $\psi(x,y) = c$. Thus $x^3 - x^2y + 2x + 2y^3 + 3y = c$ is the equation that yields the solution.

5. Writing the equation in the form $M(x,y)dx + N(x,y)dy = 0$ gives $M(x,y) = ax + by$ and $N(x,y) = bx + cy$. Now $M_y = b = N_x$ and the equation is exact. Integrating M(x,y) with respect to x yields $\psi(x,y) = (a/2)x^2 + bxy + h(y)$. Differentiating ψ with respect to y (x constant) and setting $\psi_y(x,y) = N(x,y)$ we find that $h'(y) = cy$ and thus $h(y) = (c/2)y^2$. Hence the solution is given by $(a/2)x^2 + bxy + (c/2)y^2 = k$.

7. $M_y(x,y) = e^x\cos y - 2\sin x = N_x(x,y)$ and thus the D.E. is exact. Integrating M(x,y) with respect to x gives $\psi(x,y) = e^x\sin y + 2y\cos x + h(y)$. Finding $\psi_y(x,y)$ from this and setting that equal to N(x,y) yields $h'(y) = 0$ and thus h(y) is a constant. Hence an implicit solution of the D.E. is $e^x\sin y + 2y\cos x = c$. The solution $y = 0$ is also valid since it satisfies the D.E. for all x.

9. If you try to find $\psi(x,y)$ by integrating M(x,y) with respect to x you must integrate by parts. Instead find $\psi(x,y)$ by integrating N(x,y) with respect to y to obtain $\psi(x,y) = e^{xy}\cos 2x - 3y + g(x)$. Now find g(x) by differentiating $\psi(x,y)$ with respect to x and set that equal to M(x,y), which yields $g'(x) = 2x$ or $g(x) = x^2$.

12. As long as $x^2 + y^2 \neq 0$, we can simplify the equation by multiplying both sides by $(x^2 + y^2)^{3/2}$. This gives the exact equation $xdx + ydy = 0$. The solution to this equation is given implicitly by $x^2 + y^2 = c$. If you apply Theorem 2.6.1 and its construction without the simplification, you get $(x^2 + y^2)^{-1/2} = C$ which can be written as $x^2 + y^2 = c$ under the same assumption required for the simplification.

14. $M_y = 1$ and $N_x = 1$, so the D.E. is exact. Integrating

M(x,y) with respect to x yields
$\psi(x,y) = 3x^3 + xy - x + h(y)$. Differentiating this with respect to y and setting $\psi_y(x,y) = N(x,y)$ yields
$h'(y) = -4y$ or $h(y) = -2y^2$. Thus the implicit solution is $3x^3 + xy - x - 2y^2 = c$. Setting x = 1 and y = 0 gives c = 2 so that $2y^2 - xy + (2+x-3x^3) = 0$ is the implicit solution satisfying the given I.C. Use the quadratic formula to find y(x), where the negative square root is used in order to satisfy the I.C. The solution will be valid for $24x^3 + x^2 - 8x - 16 > 0$.

15. We want $M_y(x,y) = 2xy + bx^2$ to be equal to $N_x(x,y) = 3x^2 + 2xy$. Thus we must have b = 3. This gives $\psi(x,y) = \frac{1}{2}x^2y^2 + x^3y + h(y)$ and consequently $h'(y) = 0$. After multiplying through by 2, the solution is given implicitly by $x^2y^2 + 2x^3y = c$.

19. $M_y(x,y) = 3x^2y^2$ and $N_x(x,y) = 1 + y^2$ so the equation is not exact by Theorem 2.6.1. Multiplying by the integrating factor $\mu(x,y) = 1/xy^3$ we get
$x + \frac{(1+y^2)}{y^3}y' = 0$, which is an exact equation since $M_y = N_x = 0$ (it is also separable). In this case
$\psi = \frac{1}{2}x^2 + h(y)$ and $h'(y) = y^{-3} + y^{-1}$ so that
$x^2 - y^{-2} + 2\ln|y| = c$ gives the solution implicitly.

22. Multiplication of the given D.E. (which is not exact) by $\mu(x,y) = xe^x$ yields $(x^2 + 2x)e^x\sin y\, dx + x^2e^x\cos y\, dy$, which is exact since $M_y(x,y) = N_x(x,y) = (x^2+2x)e^x\cos y$. To solve this exact equation it's easiest to integrate $N(x,y) = x^2e^x\cos y$ with respect to y to get $\psi(x,y) = x^2e^x\sin y + g(x)$. Solving for g(x) yields the implicit solution.

23. This problem is similar to the derivation leading up to Eq.(26). Assuming that μ depends only on y, we find from Eq.(25) that $\mu' = Q\mu$, where $Q = (N_x - M_y)/M$ must depend on y alone. Solving this last D.E. yields $\mu(y)$ as given. This method provides an alternative approach to Problems 27 through 30.

25. The equation is not exact so we must attempt to find an integrating factor. Since $\frac{1}{N}(M_y - N_x) = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = 3$ is a function of x alone there is an integrating factor depending only on x, as shown in Eq.(26). Then $d\mu/dx = 3\mu$, and the integrating factor is $\mu(x) = e^{3x}$. Hence the equation can be solved as in Example 4.

26. An integrating factor can be found which is a function of x only, yielding $\mu(x) = e^{-x}$. Alternatively, you might recognize that $y' - y = e^{2x} - 1$ is a linear first order equation which can be solved as in Section 2.1.

27. Using the results of Problem 23, it can be shown that $\mu(y) = y$ is an integrating factor. Thus multiplying the D.E. by y gives $ydx + (x - y\sin y)dy = 0$, which can be identified as an exact equation. Alternatively, one can rewrite the last equation as $(ydx + xdy) - y\sin y\, dy = 0$. The first term is $d(xy)$ and the last can be integrated by parts. Thus we have $xy + y\cos y - \sin y = c$.

29. Multiplying by siny we obtain $e^x\sin y\, dx + e^x\cos y\, dy + 2y\, dy = 0$, and the first two terms are just $d(e^x\sin y)$. Thus, $e^x\sin y + y^2 = c$.

31. Using the results of Problem 24, it can be shown that $\mu(xy) = xy$ is an integrating factor. Thus, multiplying by xy we have $(3x^2y + 6x)dx + (x^3 + 3y^2)dy = 0$, which can be identified as an exact equation. Alternatively, we can observe that the above equation can be written as $d(x^3y) + d(3x^2) + d(y^3) = 0$, so that $x^3y + 3x^2 + y^3 = c$.

<u>Section 2.7, Page 103</u>

1d. The exact solution to this I.V.P. is $y = \phi(t) = t + 2 - e^{-t}$.

3a. The Euler formula is $y_{n+1} = y_n + h(2y_n - t_n + 1/2)$ for $n = 0,1,2,3$ and with $t_0 = 0$ and $y_0 = 1$. Thus
$y_1 = y_0 + .1(2y_0 - t_0 + 1/2) = 1.25$,
$y_2 = 1.25 + .1[2(1.25) - (.1) + 1/2] = 1.54$,
$y_3 = 1.54 + .1[2(1.54) - (.2) + 1/2] = 1.878$, and
$y_4 = 1.878 + .1[2(1.878) - (.3) + 1/2] = 2.2736$.

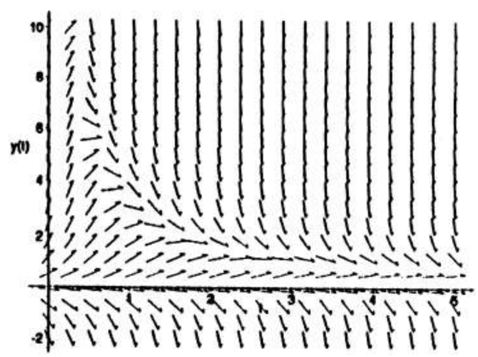
3b. Use the same formula as in Problem 3a, except now $h = .05$ and $n = 0,1...7$. Notice that only results for $n = 1,3,5$ and 7 are needed to compare with part a.

3c. Again, use the same formula as above with $h = .025$ and $n = 0,1...15$. Notice that only results for $n = 3,7,11$ and 15 are needed to compare with parts a and b.

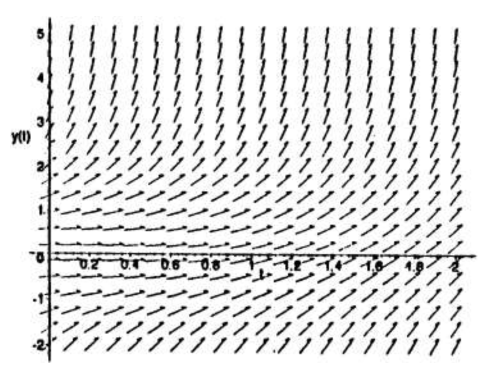
3d. $y' = 1/2 - t + 2y$ is a first order linear D.E. Rewrite the equation in the form $y' - 2y = 1/2 - t$ and multiply both sides by the integrating factor e^{-2t} to obtain $(e^{-2t}y)' = (1/2 - t)e^{-2t}$. Integrating the right side by parts and multiplying by e^{2t} we obtain $y = ce^{-2t} + t/2$. The I.C. $y(0) = 1 \rightarrow c = 1$ and hence the solution of the I.V.P. is $y = \phi(x) = e^{2t} + t/2$. Thus $\phi(0.1) = 1.2714$, $\phi(0.2) = 1.59182$, $\phi(0.3) = 1.97212$, and $\phi(0.4) = 2.42554$.

4d. The exact solution to this I.V.P. is $y = \phi(t) = (6\cos t + 3\sin t - 6e^{-2t})/5$.

6. For $y(0) > 0$ the solutions appear to converge. For $y(0)<0$ the solutions diverge.



9. All solutions seem to diverge.



13a. The Euler formula is

$$y_{n+1} = y_n + h\left(\frac{4-t_n y_n}{1+y_n^2}\right), \text{ where } t_0 = 0 \text{ and}$$

$y_0 = y(0) = -2$. Thus, for $h = .1$, we get

$y_1 = -2 + .1(4/5) = -1.92$

$$y_2 = -1.92 + .1\left(\frac{4-.1(-1.92)}{1+(1.92)^2}\right) = -1.83055$$

$$y_3 = -1.83055 + .1\left(\frac{4-.2(-1.83055)}{1+(1.83055)^2}\right) = -1.7302$$

$$y_4 = -1.7302 + .1\left(\frac{4-.3(-1.7302)}{1+(1.7302)^2}\right) = -1.617043$$

$$y_5 = -1.617043 + .1\left(\frac{4-.4(-1.617043)}{1+(1.617043)^2}\right) = -1.488494.$$

Thus, $y(.5) \cong -1.488494$.

15a. The Euler formula is

$$y_{n+1} = y_n + .1\left(\frac{3t_n^2}{3y_n^2-4}\right), \text{ where } t_0 = 1 \text{ and } y_0 = 0. \text{ Thus}$$

$$y_1 = 0 + .1\left(\frac{3}{-4}\right) = -.075 \text{ and}$$

$$y_2 = -.075 + .1\left(\frac{3(1.1)^2}{3(.075)^2-4}\right) = .166134.$$

15c. There are two factors that explain the large differences. From the limit, the slope of y, y′, becomes very “large” for values of y near -1.155. Also, the slope changes sign at y = -1.155. Thus for part a, $y(1.7) \cong y_7 = -1.178$, which is close to -1.155 and the slope y′ here is large and positive, creating the large change in $y_8 \cong y(1.8)$. For part b, $y(1.65) \cong -1.125$, resulting in a large negative slope, which yields $y(1.70) \cong -3.133$. The slope at this point is now positive and the remainder of the solutions “grow” to -3.098 for the approcimation to y(1.8).

16. For the four step sizes given, the approximte values for y(.8) are 3.5078, 4.2013, 4.8004 and 5.3428. Thus, since these changes are still rather “large”, it is hard to give an estimate other than y(.8) is at least 5.3428. By using h = .005, .0025 and .001, we find further approximate values of y(.8) to be 5.576, 5.707 and 5.790. Thus a better estimate now is for y(.8) to be between 5.8 and 6. No reliable estimate is obtainable for y(1), which is consistent with the direction field of Prob.9.

18. It is helpful, in understanding this problem, to also calculate $y'(t_n) = y_n(.1\,y_n^2 - t_n)$. For $\alpha = 2.38$ this term remains positive and grows very large for $t_n > 2$. On the other hand, for $\alpha = 2.37$ this term decreases and eventually becomes negative for $t_n \cong 1.6$ (for h = .01). For $\alpha = 2.37$ and h = .1, .05 and .01, y(2.00) has the approximations of 4.48, 4.01 and 3.50 respectively. A small step size must be used, due to the sensitivety of the slope field, given by $y_n\,(.1y_n^2 - t_n)$.

22. Using Eq.(8) we have $y_{n+1} = y_n + h(2y_n - 1) = (1+2h)y_n - h$. Setting n + 1 = k (and hence n = k-1) this becomes $y_k = (1 + 2h)y_{k-1} - h$, for k = 1,2,... . Since $y_0 = 1$, we have $y_1 = 1 + 2h - h = 1 + h = (1 + 2h)/2 + 1/2$, and hence $y_2 = (1 + 2h)y_1 - h = (1 + 2h)^2/2 + (1 + 2h)/2 - h$
$= (1 + 2h)^2/2 + 1/2$;
$y_3 = (1 + 2h)y_2 - h = (1 + 2h)^3/2 + (1 + 2h)/2 - h$
$= (1 + 2h)^3/2 + 1/2$. Continuing in this fashion (or using induction) we obtain $y_k = (1 + 2h)^k/2 + 1/2$. For fixed x > 0 choose h = x/k. Then substitute for h in the last formula to obtain $y_k = (1 + 2x/k)^k/2 + 1/2$. Letting $k \to \infty$ we find (See hint for Problem 20d.)
$y(x) = y_k \to e^{2x}/2 + 1/2$, which is the exact solution.

<u>Section 2.8, Page 113</u>

1. Let s = t-1 and w(s) = y(t(s)) - 2, then when t = 1 and y = 2 we have s = 0 and w(0) = 0. Also,
$$\frac{dw}{ds} = \frac{dw}{dt}\cdot\frac{dt}{ds} = \frac{d}{dt}(y-2)\frac{dt}{ds} = \frac{dy}{dt} \text{ and hence}$$
$$\frac{dw}{ds} = (s+1)^2 + (w+2)^2,$$ upon substitution into the given D.E.

4a. Following Ex. 1 of the text, from Eq.(7) we have $\phi_{n+1}(t) = \int_0^t f(s,\phi(s))ds$, where $f(t,\phi) = -1 - \phi$ for this problem. Thus if $\phi_0(t) = 0$, then $\phi_1(t) = -\int_0^t ds = -t$;
$$\phi_2(t) = -\int_0^t (1-s)ds = -t + \frac{t^2}{2};$$

$$\phi_3(t) = -\int_0^t (1-s+\frac{s^2}{2})ds = -t+\frac{t^2}{2}-\frac{t^3}{2\cdot 3};$$

$$\phi_4(t) = -\int_0^t (1-s+\frac{s^2}{2}-\frac{s^3}{3!})ds = -t+\frac{t^2}{2}-\frac{t^3}{3!}+\frac{t^4}{4!}.$$

Based upon these we hypothesize that: $\phi_n(t) = \sum_{k=1}^{n} \frac{(-1)^k t^k}{k!}$ and use mathematical induction to verify this form for $\phi_n(t)$. Using Eq.(7) again we have:

$$\phi_{n+1}(t) = -\int_0^t [1+\phi_n(s)]\, ds = -t - \sum_{k=1}^{n} \frac{(-1)^k t^{k+1}}{(k+1)!}$$

$= \sum_{k=0}^{n} \frac{(-1)^{k+1} t^{k+1}}{(k+1)!} = \sum_{i=1}^{n+1} \frac{(-1)^i t^i}{i!}$, where $i = k+1$. Since this is the same form for $\phi_{n+1}(t)$ as derived from $\phi_n(t)$ above, we have verified by mathematical induction that $\phi_n(t)$ is as given.

4c. From part a, let $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{(-1)^k t^k}{k!}$

$$= -t + \frac{t^2}{2} - \frac{t^3}{3!} + \ldots .$$

Since this is a power series, recall from calculus that:

$e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} = 1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{3!} + \ldots$. If we let $a = -1$, then we have $e^{-t} = 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \ldots = 1 + \phi(t)$.

Hence $\phi(t) = e^{-t} - 1$.

7. As in Prob.4,

$$\phi_1(t) = \int_0^t (s\phi_0(s)+1)ds = s\Big|_0^t = t$$

$$\phi_2(t) = \int_0^t (s^2+1)ds = (\frac{s^3}{3}+s)\Big|_0^t = t+\frac{t^3}{3}$$

$$\phi_3(t) = \int_0^t (s^2+\frac{s^4}{3}+1)ds = (\frac{s^3}{3}+\frac{s^5}{3\cdot 5}+s)\Big|_0^t = t+\frac{t^3}{3}+\frac{t^5}{3\cdot 5}.$$

Based upon these we hypothesize that:

$\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k-1}}{1\cdot3\cdot5\cdots(2k-1)}$ and use mathematical induction to verify this form for $\phi_n(t)$. Using Eq.(7) again we have:

$$\phi_{n+1}(t) = \int_0^t \left(\sum_{k=1}^{n} \frac{s^{2k}}{1\cdot3\cdot5\cdots(2k-1)} + 1\right)ds$$

$$= \sum_{k=1}^{n} \frac{t^{2k+1}}{1\cdot3\cdot5\cdots(2k+1)} + t$$

$$= \sum_{k=0}^{n} \frac{t^{2k+1}}{1\cdot3\cdot5\cdots(2k+1)}$$

$$= \sum_{i=1}^{n+1} \frac{t^{2i-1}}{1\cdot3\cdot5\cdots(2i-1)},$$ where $i = k+1$. Since this is the same form for $\phi_{n+1}(t)$ as derived from $\phi_n(t)$ above, we have verified by mathematical induction that $\phi_n(t)$ is as given.

11. Recall that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)$. Thus, for

$$\phi_2(t) = t - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + O(t^7)$$ we have

$$\sin[\phi_2(t)] = \left(t - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right) - \frac{\left(t - \frac{t^2}{2!}\right)^3}{3!} + \frac{t^5}{5!} + O(t^7).$$

<u>Section 2.9, Page 124</u>

2. Using the given difference equation we have for $n=0$, $y_1 = y_0/2$; for $n=1$, $y_2 = 2y_1/3 = y_0/3$; and for $n=2$, $y_3 = 3y_2/4 = y_0/4$. Thus we guess that $y_n = y_0/(n+1)$, and the given equation then gives $y_{n+1} = \frac{n+1}{n+2}y_n = y_0/(n+2)$, which, by mathematical induction, verifies $y_n = y_0/(n+1)$ as the solution for all n.

5. From the given equation we have $y_1 = .5y_0+6$.

$y_2 = .5y_1 + 6 = (.5)^2y_0 + 6(1 + \frac{1}{2})$ and

$y_3 = .5y_2 + 6 = (.5)^3y_0 + 6(1 + \frac{1}{2} + \frac{1}{4})$. In general, then

$$y_n = (.5)^ny_0 + 6(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}})$$

$$= (.5)^ny_0 + 6(\frac{1 - (1/2)^n}{1 - 1/2})$$

$$= (.5)^ny_0 + 12 - (.5)^n12$$

$= (.5)^n(y_0-12) + 12$. Mathematical induction can now be used to prove that this is the correct solution.

10. The governing equation is $y_{n+1} = \rho y_n-b$, which has the solution $y_n = \rho^ny_0 - \frac{1-\rho^n}{1-\rho}b$ (Eq.(14) with a negative b). Setting $y_{360} = 0$ and solving for b we obtain

$b = \frac{(1-\rho)\rho^{360}y_0}{1-\rho^{360}}$, where $\rho = 1.0075$ for part a.

13. You must solve Eq.(14) numerically for ρ when $n = 240$, $y_{240} = 0$, $b = -\$900$ and $y_0 = \$95,000$.

14. Substituting $u_n = \frac{\rho-1}{\rho} + v_n$ into Eq.(21) we get

$$\frac{\rho-1}{\rho} + v_{n+1} = \rho(\frac{\rho-1}{\rho} + v_n)(1 - \frac{\rho-1}{\rho} - v_n) \text{ or}$$

$$v_{n+1} = -\frac{\rho-1}{\rho} + (\rho-1 + \rho v_n)(\frac{1}{\rho} - v_n)$$

$$= \frac{1-\rho}{\rho} + \frac{\rho-1}{\rho} - (\rho-1)v_n + v_n - \rho v_n^2 = (2-\rho)v_n - \rho v_n^2.$$

15a. For $u_0 = .2$ we have $u_1 = 3.2u_0(1-u_0) = .512$ and $u_2 = 3.2u_1(1-u_1) = .7995392$. Likewise $u_3 = .51288406$, $u_4 = .7994688$, $u_5 = .51301899$, $u_6 = .7994576$ and $u_7 = .5130404$. Continuing in this fashion, $u_{14} = u_{16} = .79945549$ and $u_{15} = u_{17} = .51304451$.

17. For both parts of this problem a computer spreadsheet was used and an initial value of $u_0 = .2$ was chosen. Different initial values or different computer programs may need a slightly different number of iterations to reach the limiting value.

17a. The limiting value of .65517 (to 5 decimal places) is reached after approximately 100 iterations for $\rho = 2.9$. The limiting value of .66102 (to 5 decimal places) is reached after approximately 200 iterations for $\rho = 2.95$. The limiting value of .66555 (to 5 decimal places) is reached after approximately 910 iterations for $\rho = 2.99$.

17b. The solution oscillates between .63285 and .69938 after approximately 400 iterations for $\rho = 3.01$. The solution oscillates between.59016 and .73770 after approximately 130 iterations for $\rho = 3.05$. The solution oscillates between .55801 and .76457 after approximately 30 iterations for $\rho = 3.1$. For each of these cases additional iterations verified the oscillations were correct to five decimal places.

18. For an initial value of .2 and $\rho = 3.448$ we have the solution oscillating between .4403086 and .8497146. After approximately 3570 iterations the eighth decimal place is still not fixed, though. For the same initial value and $\rho = 3.45$ the solution oscillates between the four values: .43399155, .84746795, .44596778 and .85242779 after 3700 iterations.. For $\rho = 3.449$, the solution is still varying in the fourth decimal place after 3570 iterations, but there appear to be four values.

<u>Miscellaneous Problems, Page 126</u>

Before trying to find the solution of a D.E. it is necessary to know its type. The student should first classify the D.E. before reading this section, which indentifies the type of each equation in Problems 1 through 32.

1. Linear
2. Homogeneous
3. Exact
4. Linear equation in $x(y)$
5. Exact
6. Linear
7. Letting $u = x^2$ yields $\frac{dy}{dx} = 2x\frac{dy}{du}$ and thus $\frac{du}{dy} - 2yu = 2y^3$ which is linear in $u(y)$.

8. Linear
9. Exact
10. Integrating factor depends on x only
11. Exact
12. Linear
13. Homogeneous
14. Exact or homogeneous
15. Separable
16. Homogeneous
17. Linear
18. Linear or homogeneous
19. Integrating factor depends on x only
20. Separable
21. Homogeneous
22. Separable
23. Bernoulli equation
24. Separable
25. Exact
26. Integrating factor depends on x only
27. Integrating factor depends on x only
28. Exact
29. Homogeneous
30. Linear equation in x(y)
31. Separable
32. Integrating factor depends on y only.

Capítulo 3

CHAPTER 3

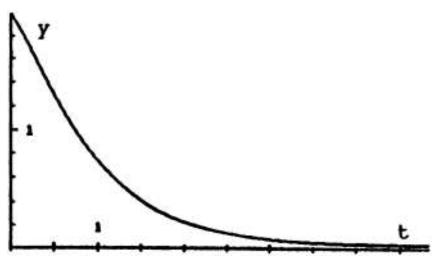
Section 3.1, Page 136

3. Assume $y = e^{rt}$, which gives $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substitution into the D.E. yields $(6r^2-r-1)e^{rt} = 0$. Since $e^{rt}\neq 0$, we have the characteristic equation $6r^2-r-1 = 0$, or $(3r+1)(2r-1) = 0$. Thus $r = -1/3,\ 1/2$ and $y = c_1e^{t/2} + c_2e^{-t/3}$.

5. The characteristic equation is $r^2 + 5r = 0$, so the roots are $r_1 = 0$, and $r_2 = -5$. Thus $y = c_1e^{0t} + c_2e^{-5t} = c_1 + c_2e^{-5t}$.

7. The characteristic equation is $r^2 - 9r + 9 = 0$ so that $r = (9\pm\sqrt{81-36})/2 = (9\pm 3\sqrt{5})/2$ using the quadratic formula. Hence $y = c_1\exp[(9+3\sqrt{5})t/2] + c_2\exp[(9-3\sqrt{5})t/2]$.

10. Substituting $y = e^{rt}$ in the D.E. we obtain the characteristic equation $r^2 + 4r + 3 = 0$, which has the roots $r_1 = -1$, $r_2 = -3$. Thus $y = c_1e^{-t} + c_2e^{-3t}$ and $y' = -c_1e^{-t} - 3c_2e^{-3t}$. Substituting $t = 0$ we then have $c_1 + c_2 = 2$ and $-c_1 - 3c_2 = -1$, yielding $c_1 = 5/2$ and $c_2 = -1/2$. Thus $y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$ and hence $y \to 0$ as $t \to \infty$.



15. The characteristic equation is $r^2 + 8r - 9 = 0$, so that $r_1 = 1$ and $r_2 = -9$ and the general solution is $y = c_1e^{t} + c_2e^{-9t}$. Since the I.C. are given at $t = 1$, it is convenient to write the general solution in the form $y = k_1e^{(t-1)} + k_2e^{-9(t-1)}$. Note that $c_1 = k_1e^{-1}$ and $c_2 = k_2e^{9}$. The advantage of the latter form of the general solution becomes clear when we apply

the I.C. $y(1) = 1$ and $y'(1) = 0$. This latter form of y gives $y' = k_1e^{(t-1)} - 9k_2e^{-9(t-1)}$ and thus setting $t = 1$ in y and y' yields the equations $k_1 + k_2 = 1$ and $k_1 - 9k_2 = 0$. Solving for k_1 and k_2 we find that $y = (9e^{(t-1)} + e^{-9(t-1)})/10$. Since $e^{(t-1)}$ has a positive exponent for $t > 1$, $y \to \infty$ as $t \to \infty$.

17. Comparing the given solution to Eq(17), we see that $r_1 = 2$ and $r_2 = -3$ are the two roots of the characteristic equation. Thus we have $(r-2)(r+3) = 0$, or $r^2 + r - 6 = 0$ as the characteristic equation. Hence the given solution is for the D.E. $y'' + y' - 6y = 0$.

19. The roots of the characteristic equation are $r = 1, -1$ and thus the general solution is $y(t) = c_1e^t + c_2e^{-t}$. $y(0) = c_1 + c_2 = \frac{5}{4}$ and $y'(0) = c_1 - c_2 = -\frac{3}{4}$, yielding $y(t) = \frac{1}{4}e^t + e^{-t}$. From this $y'(t) = \frac{1}{4}e^t - e^{-t} = 0$ or $e^{2t} = 4$ or $t = \ln 2$. The second derivative test or a graph of the solution indicates this is a minimum point.

21. The general solution is $y = c_1e^{-t} + c_2e^{2t}$. Using the I.C. we obtain $c_1 + c_2 = \alpha$ and $-c_1 + 2c_2 = 2$, so adding the two equations we find $3c_2 = \alpha + 2$. If y is to approach zero as $t \to \infty$, c_2 must be zero. Thus $\alpha = -2$.

24. The roots of the characteristic equation are given by $r = -2, \alpha - 1$ and thus $y(t) = c_1e^{-2t} + c_2e^{(\alpha-1)t}$. Hence, for $\alpha < 1$, all solutions tend to zero as $t \to \infty$. For $\alpha > 1$, the second term becomes unbounded, but not the first, so there are no values of α for which all solutions become unbounded.

25a. The characteristic equation is $2r^2 + 3r - 2 = 0$, so $r_1 = -2$ and $r_2 = 1/2$ and $y = c_1e^{-2t} + c_2e^{t/2}$. The I.C. yield $c_1 + c_2 = 2$ and $-2c_1 + \frac{1}{2}c_2 = -\beta$ so that $c_1 = (1 + 2\beta)/5$ and $c_2 = (4-2\beta)/5$.

25c. From part (a), if $\beta = 2$ then $y(t) = e^{-2t}$ and the solution simply decays to zero. For $\beta > 2$, the solution becomes unbounded negatively, and again there is no minimum point.

27. The second solution must decay faster than e^{-t}, so choose e^{-2t} or e^{-3t} etc. as the second solution. Then proceed as in Problem 17.

28. Let $v = y'$, then $v' = y''$ and thus the D.E. becomes $t^2v' + 2tv - 1 = 0$ or $t^2v' + 2tv = 1$. The left side is recognized as $(t^2v)'$ and thus we may integrate to obtain $t^2v = t + c$ (otherwise, divide both sides of the D.E. by t^2 and find the integrating factor, which is just t^2 in this case). Solving for $v = dy/dt$ we find $dy/dt = 1/t + c/t^2$ so that $y = \ln t + c_1/t + c_2$.

30. Set $v = y'$, then $v' = y''$ and thus the D.E. becomes $v' + tv^2 = 0$. This equation is separable and has the solution $-v^{-1} + t^2/2 = c$ or $v = y' = -2/(c_1 - t^2)$ where $c_1 = 2c$. We must consider separately the cases $c_1 = 0$, $c_1 > 0$ and $c_1 < 0$. If $c_1 = 0$, then $y' = 2/t^2$ or $y = -2/t + c_2$. If $c_1 > 0$, let $c_1 = k^2$. Then $y' = -2/(k^2-t^2) = -(1/k)[1/(k-t) + 1/(k+t)]$, so that $y = (1/k)\ln|(k-t)/(k+t)|+c_2$. If $c_1 < 0$, let $c_1 = -k^2$. Then $y' = 2/(k^2+ t^2)$ so that $y = (2/k)\tan^{-1}(t/k) + c_2$. Finally, we note that y = constant is also a solution of the D.E.

34. Following the procedure outlined, let $v = dy/dt$ and $y'' = dv/dt = v\,dv/dy$. Thus the D.E. becomes $yv\,dv/dy + v^2 = 0$, which is a separable equation with the solution $v = c_1/y$. Next let $v = dy/dt = c/y$, which again separates to give the solution $y^2 = c_1t + c_2$.

37. Again let $v = y'$ and $v' = v\,dv/dy$ to obtain $2y^2v\,dv/dy + 2yv^2 = 1$. This is an exact equation with solution $v = \pm\, y^{-1}(y + c_1)^{1/2}$. To solve this equation,

we write it in the form $\pm\, ydy/(y+c_1)^{1/2} = dt$. On observing that the left side of the equation can be written as $\pm[(y+c_1) - c_1]dy/(y+c_1)^{1/2}$ we integrate and find $\pm\,(2/3)(y-2c_1)(y+c_1)^{1/2} = t + c_2$.

39. If $v = y'$, then $v' = vdv/dy$ and the D.E. becomes $v\,dv/dy + v^2 = 2e^{-y}$. Dividing by v we obtain $dv/dy + v = 2v^{-1}e^{-y}$, which is a Bernoulli equation (see Prob.27, Section 2.4). Let $w(y) = v^2$, then $dw/dy = 2v\,dv/dy$ and the D.E. then becomes $dw/dy + 2w = 4e^{-y}$, which is linear in w. Its solution is $w = v^2 = ce^{-2y} + 4e^{-y}$. Setting $v = dy/dt$, we obtain a separable equation in y and t, which is solved to yield the solution.

40. Since both t and y are missing, either approach used above will work. In this case it's easier to use the approach of Problems 28-33, so let $v = y'$ and thus $v' = y''$ and the D.E. becomes $vdv/dt = 2$.

43. The variable y is missing. Let $v = y'$, then $v' = y''$ and the D.E. becomes $vv' - t = 0$. The solution of the separable equation is $v^2 = t^2 + c_1$. Substituting $v = y'$ and applying the I.C. $y'(1) = 1$, we obtain $y' = t$. The positive square root was chosen because $y' > 0$ at $t = 1$. Solving this last equation and applying the I.C. $y(1) = 2$, we obtain $y = t^2/2 + 3/2$.

<u>Section 3.2, Page 145</u>

2. $W(\cos t,\sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$.

4. $W(x,xe^x) = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^x$.

8. Dividing by (t-1) we have $p(t) = -3t/(t-1)$, $q(t) = 4/(t-1)$ and $g(t) = \sin t/(t-1)$, so the only point of discontinuity is $t = 1$. By Theorem 3.2.1, the largest interval is $-\infty < t < 1$, since the initial point is $t_0 = -2$.

12. $p(x) = 1/(x-2)$ and $q(x) = \tan x$, so $x = \pi/2, 2, 3\pi/2...$ are points of discontinuity. Since $t_0 = 3$, the interval specified by Theorem 3.2.1 is $2 < x < 3\pi/2$.

14. For $y = t^{1/2}$, $y' = \frac{1}{2}t^{-1/2}$ and $y'' = -\frac{1}{4}t^{-3/2}$. Thus $yy'' + (y')^2 = -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0$. Similarly $y = 1$ is also a solution. If $y = c_1(1) + c_2t^{1/2}$ is substituted in the D.E. you will get $-c_1c_2/4t^{3/2}$, which is zero only if $c_1 = 0$ or $c_2 = 0$. Thus the linear combination of two solutions is not, in general, a solution. Theorem 3.2.2 is not contradicted however, since the D.E. is not linear.

15. $y = \phi(t)$ is a solution of the D.E. so $L[\phi](t) = g(t)$. Since L is a linear operator, $L[c\phi](t) = cL[\phi](t) = cg(t)$. But, since $g(t) \neq 0$, $cg(t) = g(t)$ if and only if $c = 1$. This is not a contradiction of Theorem 3.2.2 since the linear D.E. is not homogeneous.

18. $W(f,g) = \begin{vmatrix} t & g \\ 1 & g' \end{vmatrix} = tg' - g = t^2e^t$, or $g' - \frac{1}{t}g = te^t$. This has an integrating factor of $\frac{1}{t}$ and thus $\frac{1}{t}g' - \frac{1}{t^2}g = e^t$ or $(\frac{1}{t}g)' = e^t$. Integrating and multiplying by t we obtain $g(t) = te^t + ct$.

21. From Section 3.1, e^t and e^{-2t} are two solutions, and since $W(e^t,e^{-2t}) \neq 0$ they form a fundamental set of solutions. To find the fundamental set specified by Theorem 3.2.5, let $y(t) = c_1e^t + c_2e^{-2t}$, where c_1 and c_2 satisfy
$c_1 + c_2 = 1$ and $c_1 - 2c_2 = 0$ for y_1. Solving, we find $y_1 = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}$. Likewise, c_1 and c_2 satisfy $c_1 + c_2 = 0$ and $c_1 - 2c_2 = 1$ for y_2, so that $y_2 = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$.

25. For $y_1 = x$, we have $x^2(0) - x(x+2)(1) + (x+2)(x) = 0$ and for $y_2 = xe^x$ we have $x^2(x+2)e^x - x(x+2)(x+1)e^x + (x+2)xe^x = 0$. From Problem 4, $W(x, xe^x) = x^2e^x \neq 0$ for $x > 0$, so y_1 and y_2 form a fundamental set of solutions.

27. Suppose that $P(x)y'' + Q(x)y' + R(x)y = [P(x)y']' + [f(x)y]'$. On expanding the right side and equating coefficients, we find $f'(x) = R(x)$ and $P'(x) + f(x) = Q(x)$. These two conditions on f can be satisfied if $R(x) = Q'(x) - P''(x)$ which gives the necessary condition $P''(x) - Q'(x) + R(x) = 0$.

30. We have $P(x) = x$, $Q(x) = -\cos x$, and $R(x) = \sin x$ and the condition for exactness is satisfied. Also, from Problem 27, $f(x) = Q(x) - P'(x) = -\cos x - 1$, so the D.E. becomes $(xy')' - [(1 + \cos x)y]' = 0$. Hence $xy' - (1 + \cos x)y = c_1$. This is a first order linear D.E. and the integrating factor (after dividing by x) is $\mu(x) = \exp[-\int x^{-1}(1 + \cos x)dx]$. The general solution is $y = [\mu(x)]^{-1}[c_1\int_{x_0}^{x} t^{-1}\mu(t)dt + c_2]$.

32. We want to choose $\mu(x)$ and $f(x)$ so that $\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = [\mu(x)P(x)y']' + [f(x)y]'$. Expand the right side and equate coefficients of y'', y' and y. This gives $\mu'(x)P(x) + \mu(x)P'(x) + f(x) = \mu(x)Q(x)$ and $f'(x) = \mu(x)R(x)$. Differentiate the first equation and then eliminate $f'(x)$ to obtain the adjoint equation $P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$.

34. $P = 1-x^2$, $Q = -2x$ and $R = \alpha(\alpha+1)$. Thus $2P' - Q = -4x + 2x = -2x = Q$ and $P'' - Q' + R = -2 + 2 + \alpha(\alpha+1) = \alpha(\alpha+1) = R$.

36. Write the adjoint D.E. given in Problem 32 as $\hat{P}\mu'' + \hat{Q}\mu' + \hat{R}\mu = 0$ where $\hat{P} = P$, $\hat{Q} = 2P' - Q$, and $\hat{R} = P'' - Q' + R$. The adjoint of this equation, namely the adjoint of the adjoint, is $\hat{P}y'' + (2\hat{P}' - \hat{Q})y' + (\hat{P}'' - \hat{Q}' + \hat{R})y = 0$. After substituting for $\hat{P}$, $\hat{Q}$, and $\hat{R}$ and simplifying, we obtain $Py'' + Qy' + Ry = 0$. This is the same as the original equation.

37. From Problem 32 the adjoint of $Py'' + Qy' + Ry = 0$ is $P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$. The two equations are the same if $2P' - Q = Q$ and $P'' - Q' + R = R$. This will be true if $P' = Q$. Hence the original D.E. is self-adjoint if $P' = Q$. For Problem 33, $P(x) = x^2$ so $P'(x) = 2x$ and $Q(x) = x$. Hence the Bessel equation of order ν is not self-adjoint. In a similar manner we find that Problems 34 and 35 are self-adjoint.

Section 3.3, Page 152

2. Since $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ we have $\cos 3\theta - (4\cos^3\theta - 3\cos\theta) = 0$ for all θ. From Eq.(1) we have $k_1 = 1$ and $k_2 = -1$ and thus $\cos 3\theta$ and $4\cos^3\theta - 3\cos\theta$ are linearly dependent.

6. $W(t,t^{-1}) = \begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix} = -2/t \neq 0.$

7. For $t>0$ $g(t) = t$ and hence $f(t) - 3g(t) = 0$ for all t. Therefore f and g are linearly dependent on $0<t$. For $t<0$ $g(t) = -t$ and $f(t) + 3g(t) = 0$, so again f and g are linearly dependent on $t<0$. For any interval that includes the origin, such as $-1<t<2$, there is no c for which $f(t) + cg(t) = 0$ for all t, and hence f and g are linearly independent on this interval.

12. The D.E. is linear and homogeneous. Hence, if y_1 and y_2 are solutions, then $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ are solutions. $W(y_3,y_4) = y_3y_4' - y_3'y_4 = (y_1 + y_2)(y_1' - y_2') - (y_1' + y_2')(y_1 - y_2) = -2(y_1y_2' - y_1'y_2) = -2W(y_1,y_2)$, is not zero since y_1 and y_2 are linearly independent solutions. Hence y_3 and y_4 form a fundamental set of solutions. Conversely, solving the first two equations for y_1 and y_2, we have $y_1 = (y_3+y_4)/2$ and $y_2 = (y_3-y_4)/2$, so y_1 and y_2 are solutions. Finally, from above we have $W(y_1y_2) = -W(y_3,y_4)/2$.

15. Writing the D.E. in the form of Eq.(7), we have $p(t) = -(t+2)/t$. Thus Eq.(8) yields

$$W(t) = c\exp[-\int \frac{-(t+2)}{t}\,dt] = ct^2e^t.$$

20. From Eq.(8) we have $W(y_1,y_2) = c\exp[-\int p(t)dt]$, where $p(t) = 2/t$ from the D.E. Thus $W(y_1,y_2) = c/t^2$. Since $W(y_1,y_2)(1) = 2$ we find $c = 2$ and thus $W(y_1,y_2)(5) = 2/25$.

24. Let c be the point in I at which both y_1 and y_2 vanish. Then $W(y_1,y_2)(c) = y_1(c)y_2'(c) - y_1'(c)y_2(c) = 0$. Hence, by Theorem 3.3.3 the functions y_1 and y_2 cannot form a fundamental set.

26. Suppose that y_1 and y_2 have a point of inflection at t_0 and either $p(t_0) \neq 0$ or $q(t_0) \neq 0$. Since $y_1''(t_0) = 0$ and $y_2''(t_0) = 0$ it follows from the D.E. that $p(t_0)y_1'(t_0) + q(t_0)y_1(t_0) = 0$ and $p(t_0)y_2'(t_0) + q(t_0)y_2(t_0) = 0$. If $p(t_0) = 0$ and $q(t_0) \neq 0$ then $y_1(t_0) = y_2(t_0) = 0$, and $W(y_1,y_2)(t_0) = 0$ so the solutions cannot form a fundamental set. If $p(t_0) \neq 0$ and $q(t_0) = 0$ then $y_1'(t_0) = y_2'(t_0) = 0$ and $W(y_1,y_2)(t_0) = 0$, so again the solutions cannot form a fundamental set. If $p(t_0) \neq 0$ and $q(t_0) = 0$ then $y_1'(t_0) = q(t_0)y_1(t_0)/p(t_0)$ and $y_2'(t_0) = q(t_0)y_2(t_0)/p(t_0)$ and thus

$$\begin{aligned} W(y_1,y_2)(t_0) &= y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \\ &= y_1(t_0)[q(t_0)y_2(t_0)/p(t_0)] - \\ &\quad [q(t_0)y_1(t_0)/p(t_0)]y_2(t_0) \\ &= 0. \end{aligned}$$

27. Let $-1 < t_0, t_1 < 1$ and $t_0 \neq t_1$. If $y_1 = t$ and $y_2 = t^2$ are linearly dependent then $c_1t_1 + c_2t_1^2 = 0$ and $c_1t_0 + c_2t_0^2 = 0$ have a solution for c_1 and c_2 such that c_1 and c_2 are not both zero. But this system of equations has a non-zero solution only if $t_1 = 0$ or $t_0 = 0$ or $t_1 = t_0$. Hence, the only set c_1 and c_2 that satisfies the system for every choice of t_0 and t_1 in $-1 < t < 1$ is $c_1 = c_2 = 0$. Therefore t and t^2 are linearly independent

on $-1 < t < 1$. Next, $W(t,t^2) = t^2$ clearly vanishes at $t = 0$. Since $W(t,t^2)$ vanishes at $t = 0$, but t and t^2 are linearly independent on $-1 < t < 1$, it follows that t and t^2 cannot be solutions of Eq.(7) on $-1 < t < 1$. To show that the functions $y_1 = t$ and $y_2 = t^2$ are solutions of $t^2y'' - 2ty' + 2y = 0$, substitute each of them in the equation. Clearly, they are solutions. There is no contradiction to Theorem 3.3.3 since $p(t) = -2/t$ and $q(t) = 2/t^2$ are discontinuous at $t = 0$, and hence the theorem does not apply on the interval $-1 < t < 1$.

28. On $0 < t < 1$, $f(t) = t^3$ and $g(t) = t^3$. Hence there are nonzero constants, $c_1 = 1$ and $c_2 = -1$, such that $c_1f(t) + c_2g(t) = 0$ for each t in $(0,1)$. On $-1 < t < 0$, $f(t) = -t^3$ and $g(t) = t^3$; thus $c_1 = c_2 = 1$ defines constants such that $c_1f(t) + c_2g(t) = 0$ for each t in $(-1,0)$. Thus f and g are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$. We will show that $f(t)$ and $g(t)$ are linearly independent on $-1 < t < 1$ by demonstrating that it is impossible to find constants c_1 and c_2, not both zero, such that $c_1f(t) + c_2g(t) = 0$ for all t in $(-1,1)$. Assume that there are two such nonzero constants and choose two points t_0 and t_1 in $-1 < t < 1$ such that $t_0 < 0$ and $t_1 > 0$. Then $-c_1t_0^3 + c_2t_0^3 = 0$ and $c_1t_1^3 + c_2t_1^3 = 0$. These equations have a nontrivial solution for c_1 and c_2 only if the determinant of coefficients is zero. But the determinant of coefficients is $-2t_0^3t_1^3 \neq 0$ for t_0 and t_1 as specified. Hence $f(t)$ and $g(t)$ are linearly independent on $-1 < t < 1$.

Section 3.4, Page 158

1. $\exp(1+2i) = e^{1+2i} = ee^{2i} = e(\cos 2 + i\sin 2)$.

5. Recall that $2^{1-i} = e^{\ln(2^{1-i})} = e^{(1-i)\ln 2}$

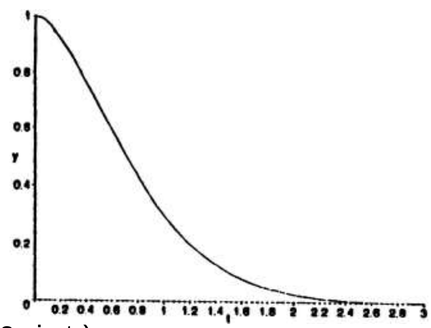
7. As in Section 3.1, we seek solutions of the form $y = e^{rt}$. Substituting this into the D.E. yields the characteristic equation $r^2 - 2r + 2 = 0$, which has the roots $r_1 = 1 + i$

and $r_2 = 1 - i$, using the quadratic formula. Thus $\lambda = 1$ and $\mu = 1$ and from Eq.(17) the general solution is $y = c_1e^t\cos t + c_2e^t\sin t$.

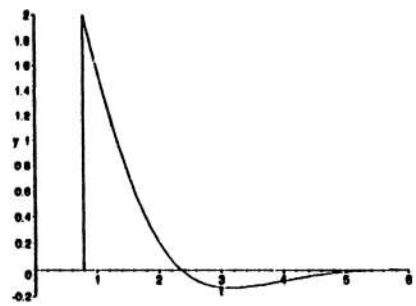
11. The characteristic equation is $r^2 + 6r + 13 = 0$, which has the roots $r = \dfrac{-6\pm\sqrt{-16}}{2} = -3\pm2i$. Thus $\lambda = -3$ and $\mu = 2$, so Eq.(17) becomes $y = c_1e^{-3t}\cos 2t + c_2e^{-3t}\sin 2t$.

14. The characteristic equation is $9r^2 + 9r - 4$, which has the real roots $-4/3$ and $1/3$. Thus the solution has the same form as in Section 3.1, $y(t) = c_1e^{t/3} + c_2e^{-4t/3}$.

18. The characteristic equation is $r^2 + 4r + 5 = 0$, which has the roots $r_1,r_2 = -2 \pm i$. Thus $y = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$ and $y' = (-2c_1+c_2)e^{-2t}\cos t + (-c_1-2c_2)e^{-2t}\sin t$, so that $y(0)= c_1= 1$ and $y'(0)= -2c_1+c_2= 0$, or $c_2 = 2$. Hence $y = e^{-2t}(\cos t + 2\sin t)$.



22. The characteristic equation is $r^2 + 2r + 2 = 0$, so $r_1,r_2 = -1 \pm i$. Since the I.C. are given at $\pi/4$ we want to alter Eq.(17) by letting $c_1 = e^{\pi/4}d_1$ and $c_2 = e^{\pi/4}d_2$. Thus, for $\lambda = -1$ and $\mu = 1$ we have $y = e^{-(t-\pi/4)}(d_1\cos t + d_2\sin t)$; so $y' = -e^{-(t-\pi/4)}(d_1\cos t + d_2\sin t) + e^{-(t-\pi/4)}(-d_1\sin t + d_2\cos t)$. Thus $\sqrt{2}\,d_1/2 + \sqrt{2}\,d_2/2 = 2$ and $-\sqrt{2}\,d_1 = -2$ and hence $y = \sqrt{2}\,e^{-(t-\pi/4)}(\cos t + \sin t)$.



23a. The characteristic equation is $3r^2 - r + 2 = 0$, which has the roots $r_1,r_2 = \dfrac{1}{6} \pm \dfrac{\sqrt{23}}{6}i$. Thus $u(t) = e^{t/6}(c_1\cos\dfrac{\sqrt{23}}{6}t + c_2\sin\dfrac{\sqrt{23}}{6}t)$ and we obtain $u(0) = c_1 = 2$

and $u'(0) = \frac{1}{6}c_1 + \frac{\sqrt{23}}{6}c_2 = 0$. Solving for c_2 we find $u(t) = e^{t/6}(2\cos\frac{\sqrt{23}}{6}t - \frac{2}{\sqrt{23}}\sin\frac{\sqrt{23}}{6}t)$.

23b. To estimate the first time that $|u(t)| = 10$ plot the graph of $u(t)$ as found in part (a). Use this estimate in an appropriate computer software program to find $t = 10.7598$.

25a. The characteristic equation is $r^2 + 2r + 6 = 0$, so $r_1, r_2 = -1 \pm \sqrt{5}\,i$ and $y(t) = e^{-t}(c_1\cos\sqrt{5}\,t + c_2\sin\sqrt{5}\,t)$. Thus $y(0) = c_1 = 2$ and $y'(0) = -c_1 + \sqrt{5}\,c_2 = \alpha$ and hence $y(t) = e^{-t}(2\cos\sqrt{5}\,t + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5}\,t)$.

25b. $y(1) = e^{-1}(2\cos\sqrt{5} + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5}) = 0$ and hence $\alpha = -2 - \frac{2\sqrt{5}}{\tan\sqrt{5}} = 1.50878$.

25c. For $y(t) = 0$ we must have $2\cos\sqrt{5}\,t + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5}\,t = 0$ or $\tan\sqrt{5}\,t = \frac{-2\sqrt{5}}{\alpha+2}$. For the given α (actually, for $\alpha > -2$) this yields $\sqrt{5}\,t = \pi - \arctan\frac{2\sqrt{5}}{\alpha+2}$ since $\arctan x < 0$ when $x < 0$.

25d. From part (c) $\arctan\frac{2\sqrt{5}}{\alpha+2} \to 0$ as $\alpha \to \infty$, so $t \to \pi/\sqrt{5}$.

31. $\frac{d}{dt}[e^{\lambda t}(\cos\mu t + i\sin\mu t)] = \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t)$ $+ e^{\lambda t}(-\mu\sin\mu t + i\mu\cos\mu t) = \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t)$ $+ i\mu e^{\lambda t}(i\sin\mu t + \cos\mu t) = e^{\lambda t}(\lambda+i\mu)(\cos\mu t + i\sin\mu t)$. Setting $r = \lambda+i\mu$ we then have $\frac{d}{dt}e^{rt} = re^{rt}$.

33. Suppose that $t = a$ and $t = b$ $(b>a)$ are consecutive zeros of y_1. We must show that y_2 vanishes once and only once in the interval $a < t < b$. Assume that it does not

vanish. Then we can form the quotient y_1/y_2 on the interval $a \le t \le b$. Note $y_2(a) \ne 0$ and $y_2(b) \ne 0$, otherwise y_1 and y_2 would not be linearly independent solutions. Next, y_1/y_2 vanishes at $t = a$ and $t = b$ and has a derivative in $a < t < b$. By Rolles theorem, the derivative must vanish at an interior point. But

$$\left(\frac{y_1}{y_2}\right)' = \frac{y_1'y_2 - y_2'y_1}{y_2^2} = \frac{-W(y_1,y_2)}{y_2^2},$$ which cannot be zero

since y_1 and y_2 are linearly independent solutions. Hence we have a contradiction, and we conclude that y_2 must vanish at a point between a and b. Finally, we show that it can vanish at only one point between a and b. Suppose that it vanishes at two points c and d between a and b. By the argument we have just given we can show that y_1 must vanish between c and d. But this contradicts the hypothesis that a and b are consecutive zeros of y_1.

35. We use the result of Problem 34. Note that $q(t) = e^{-t^2} > 0$ for $-\infty < t < \infty$. Next, we find that $(q' + 2pq)/q^{3/2} = 0$. Hence the D.E. can be transformed into an equation with constant coefficients by letting $x = u(t) = \int e^{-t^2/2}dt$. Substituting $x = u(t)$ in the differential equation found in part (b) of Problem 34 we obtain, after dividing by the coefficient of d^2y/dx^2, the D.E. $d^2y/dx^2 - y = 0$. Hence the general solution of the original D.E. is
$y = c_1\cos x + c_2\sin x$, $x = \int e^{-t^2/2}dt$.

38. Rewrite the D.E. as $y'' + (\alpha/t)y' + (\beta/t^2)y = 0$ so that $p = \alpha/t$ and $q = \beta/t^2$, which satisfy the conditions of parts (c) and (d) of Problem 34. Thus
$x = \int(1/t^2)^{1/2}dt = \ln t$ will transform the D.E. into
$dy^2/dx^2 + (\alpha-1)dy/dx + \beta y = 0$. Note that since β is constant, it can be neglected in defining x.

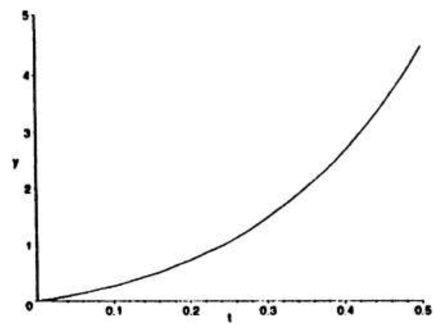
39. By direct substitution, or from Problem 38, $x = \ln t$ will transform the D.E. into $d^2y/dx^2 + y = 0$, since $\alpha = 1$ and $\beta = 1$. Thus $y = c_1\cos x + c_2\sin x$, with $x = \ln t$, $t > 0$.

Section 3.5, Page 166

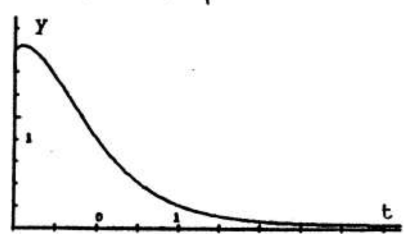
1. Substituting $y = e^{rt}$ into the D.E., we find that $r^2 - 2r + 1 = 0$, which gives $r_1 = 1$ and $r_2 = 1$. Since the roots are equal, the second linearly independent solution is te^t and thus the general solution is $y = c_1e^t + c_2te^t$.

9. The characteristic equation is $25r^2 - 20r + 4 = 0$, which may be written as $(5r-2)^2 = 0$ and hence the roots are $r_1, r_2 = 2/5$. Thus $y = c_1e^{2t/5} + c_2\,te^{2t/5}$.

12. The characteristic equation is $r^2 - 6r + 9 = (r-3)^2$, which has the repeated root $r = 3$. Thus $y = c_1e^{3t} + c_2te^{3t}$, which gives $y(0) = c_1 = 0$, $y'(t) = c_2(e^{3t}+3te^{3t})$ and $y'(0) = c_2 = 2$. Hence $y(t) = 2te^{3t}$.



14. The characteristic equation is $r^2 + 4r + 4 = (r+2)^2 = 0$, which has the repeated root $r = -2$. Since the I.C. are given at $t = -1$, write the general solution as $y = c_1e^{-2(t+1)} + c_2te^{-2(t+1)}$. Then $y' = -2c_1e^{-2(t+1)} + c_2e^{-2(t+1)} - 2c_2te^{-2(t+1)}$ and hence $c_1-c_2 = 2$ and $-2c_1+3c_2 = 1$ which yield $c_1 = 7$ and $c_2 = 5$. Thus $y = 7e^{-2(t+1)} + 5te^{-2(t+1)}$, a decaying exponential as shown in the graph.



17a. The characteristic equation is $4r^2 + 4r + 1 = (2r+1)^2 = 0$, so we have $y(t) = (c_1+c_2t)e^{-t/2}$. Thus $y(0) = c_1 = 1$ and $y'(0) = -c_1/2 + c_2 = 2$ and hence $c_2 = 5/2$ and $y(t) = (1 + 5t/2)e^{-t/2}$.

17b. From part (a), $y'(t) = -\frac{1}{2}(1 + 5t/2)e^{-t/2} + \frac{5}{2}e^{-t/2} = 0$, when $-\frac{1}{2} - \frac{5t}{4} + \frac{5}{2} = 0$, or $t_0 = \frac{8}{5}$ and $y_0 = 5e^{-4/5}$.

17c. From part (a), $-\frac{1}{2} + c_2 = b$ or $c_2 = b + \frac{1}{2}$ and $y(t) = [1 + (b + \frac{1}{2})t]e^{-t/2}$.

17d. From part (c), $y'(t) = -\frac{1}{2}[1 + (b+\frac{1}{2})t]e^{-t/2} + (b+\frac{1}{2})e^{-t/2} = 0$ which yields $t_M = \frac{4b}{2b+1}$ and $y_M = (1 + \frac{2b+1}{2} \cdot \frac{4b}{2b+1})e^{-2b/(2b+1)} = (1 + 2b)e^{-2b/(2b+1)}$.

19. If $r_1 = r_2$ then $y(t) = (c_1 + c_2t)e^{r_1t}$. Since the exponential is never zero, $y(t)$ can be zero only if $c_1 + c_2t = 0$, which yields at most one positive value of t if c_1 and c_2 differ in sign. If $r_2 > r_1$ then $y(t) = c_1e^{r_1t} + c_2e^{r_2t} = e^{r_1t}(c_1 + c_2e^{(r_2-r_1)t})$. Again, this is zero only if c_1 and c_2 differ in sign, in which case $t = \frac{\ln(-c_1/c_2)}{(r_2-r_1)}$.

21. If $r_2 \neq r_1$ then $\phi(t;r_1,r_2) = (e^{r_2t} - e^{r_1t})/(r_2 - r_1)$ is defined for all t. Note that ϕ is a liner combination of two solutions, e^{r_1t} and e^{r_2t}, of the D.E. Hence, ϕ is a solution of the differential equation. Think of r_1 as fixed and let $r_2 \to r_1$. The limit of ϕ as $r_2 \to r_1$ is indeterminate. If we use L'Hopital's rule, we find $\lim_{r_2 \to r_1} \frac{e^{r_2t} - e^{r_1t}}{r_2 - r_1} \cdot = \lim_{r_2 \to r_1} \frac{te^{r_2t}}{1} = te^{r_1t}$. Hence, the solution $\phi(t;r_1,r_2) \to te^{r_1t}$ as $r_2 \to r_1$.

25. Let $y_2 = v/t$. Then $y_2' = v'/t - v/t^2$ and $y_2'' = v''/t - 2v'/t^2 + 2v/t^3$. Substituting in the D.E. we obtain
$t^2(v''/t - 2v'/t^2 + 2v/t^3) + 3t(v'/t - v/t^2) + v/t = 0$.
Simplifying the left side we get $tv'' + v' = 0$, which yields $v' = c_1/t$. Thus $v = c_1\ln t + c_2$. Hence a second solution is $y_2(t) = (c_1\ln t + c_2)/t$. However, we may set $c_2 = 0$ and $c_1 = 1$ without loss of generality and thus we have $y_2(t) = (\ln t)/t$ as a second solution. Note that in

the form we actually calculated, $y_2(t)$ is a linear combination of $1/t$ and $\ln t/t$, and hence is the general solution.

27. In this case the calculations are somewhat easier if we do not use the explicit form for $y_1(x) = \sin x^2$ at the beginning but simply set $y_2(x) = y_1 v$. Substituting this form for y_2 in the D.E. gives $x(y_1 v)'' - (y_1 v)' + 4x^3(y_1 v) = 0$. On carrying out the differentiations and making use of the fact that y_1 is a solution, we obtain $xy_1 v'' + (2xy_1' - y_1)v' = 0$. This is a first order linear equation for v', which has the solution $v' = cx/(\sin x^2)^2$. Setting $u = x^2$ allows integration of this to get $v = c_1 \cot x^2 + c_2$. Setting $c_1 = 1$, $c_2 = 0$ and multiplying by $y_1 = \sin x^2$ we obtain $y_2(x) = \cos x^2$ as the second solution of the D.E.

30. Substituting $y_2(x) = y_1(x)v(x)$ in the D.E. gives $x^2(y_1 v)'' + x(y_1 v)' + (x^2 - \frac{1}{4})y_1 v = 0$. On carrying out the differentiations and making use of the fact that y_1 is a solution, we obtain $x^2 y_1 v'' + (2x^2 y_1' + xy_1)v' = 0$. This is a first order linear equation for v', $v'' + (2y_1'/y_1 + 1/x)v' = 0$, with solution

$$v'(x) = c\exp[-\int(2\frac{y_1'}{y_1} + \frac{1}{x})dx] = c\exp[-2\ln y_1 - \ln x]$$

$$= c\frac{1}{xy_1^{\ 2}} = \frac{c}{x(x^{-1}\sin^2 x)} = c\csc^2 x,$$

where c is an arbitrary constant, which we will take to be one. Then $v(x) = \int \csc^2 x\, dx = -\cot x + k$ where again k is an arbitrary constant which can be taken equal to zero. Thus $y_2(x) = y_1(x)v(x) = (x^{-1/2}\sin x)(-\cot x) = -x^{-1/2}\cos x$. The second solution is usually taken to be $x^{-1/2}\cos x$. Note that $c = -1$ would have given this solution.

31b. Let $y_2(x) = e^x v(x)$, then $y_2' = e^x v' + e^x v$, and $y_2'' = e^x v'' + 2e^x v' + e^x v$. Substituting in the D.E. we obtain $xe^x v'' + (xe^x - Ne^x)v' = 0$, or $v'' + (1-N/x)v' = 0$. This is a first order linear D.E. for v' with integrating factor $\mu(x) = \exp[\int(1-N/x)dx] = x^{-N}e^{x}$. Hence $(x^{-N}e^{x}v')' = 0$, and $v' = cx^{N}e^{-x}$ which gives $v(x) = c\int x^{N}e^{-x}dx + k$. On taking $k = 0$ we obtain as the second solution $y_2(x) = ce^{x}\int x^{N}e^{-x}dx$. The integral can be evaluated by using the method of integration by parts. At each stage let $u = x^N$ or x^{N-1}, or whatever the power of x that remains, and let $dv = e^{-x}$. Note that this dv is not related to the v(x) in $y_2(x)$. For N = 2 we have

$$y_2(x) = ce^{x}\int x^2e^{-x}dx = ce^{x}[x^2\frac{e^{-x}}{-1} - \int 2x\frac{e^{-x}}{-1}dx]$$
$$= -cx^2 + ce^{x}[2x\frac{e^{-x}}{-1} - \int 2\frac{e^{-x}}{-1}dx]$$
$$= c(-x^2 - 2x - 2) = -2c(1 + x + x^2/2!).$$

Choosing $c = -1/2!$ gives the desired result. For the general case $c = -1/N!$

33. $(y_2/y_1)' = (y_1y_2' - y_1'y_2)/y_1^2 = W(y_1,y_2)/y_1^2$. Abel's identity is $W(y_1,y_2) = c\exp[-\int_{t_0}^{t}p(r)dr]$. Hence $(y_2/y_1)' = cy_1^{-2}\exp[-\int_{t_0}^{t}p(r)dr]$. Integrating and setting $c = 1$ (since a solution y_2 can be multiplied by any constant) and taking the constant of integration to be zero we obtain

$$y_2(t) = y_1(t)\int_{t_0}^{t}\frac{\exp[-\int_{s_0}^{s}p(r)dr]}{[y_1(s)]^2}ds.$$

35. From Problem 33 and Abel's formula we have
$$\left(\frac{y_2}{y_1}\right)' = \frac{\exp[\int(1/t)dt]}{\sin^2(t^2)} = \frac{e^{\ln t}}{\sin^2(t^2)} = t\csc^2(t^2).$$ Thus $y_2/y_1 = -(1/2)\cot(t^2)$ and hence we can choose $y_2 = \cos(t^2)$ since $y_1 = \sin^2(t^2)$.

38. The general solution of the D.E. is $y = c_1e^{r_1t} + c_2e^{r_2t}$ where $r_1,r_2 = (-b \pm \sqrt{b^2-4ac})/2a$ provided $b^2 - 4ac \neq 0$. In this case there are two possibilities. If $b^2 - 4ac > 0$ then $(b^2 - 4ac)^{1/2} < b$ and r_1 and r_2 are real and <u>negative</u>. Consequently $e^{r_1t} \to 0$ and $e^{r_2t} \to 0$; and hence

$y \to 0$, as $t \to \infty$. If $b^2 - 4ac < 0$ then r_1 and r_2 are complex conjugates with <u>negative</u> real part. Again $e^{r_1t} \to 0$ and $e^{r_2t} \to 0$; and hence $y \to 0$, as $t \to \infty$. Finally, if $b^2 - 4ac = 0$, then $y = c_1e^{r_1t} + c_2te^{r_1t}$ where $r_1 = -b/2a < 0$. Hence, again $y \to 0$ as $t \to \infty$. This conclusion does not hold if either $b = 0$ (since $y(t) = c_1\cos\omega t + c_2\sin\omega t$) or $c = 0$ (since $y_1(t) = c_1$).

42. Substituting $z = \ln t$ into the D.E. gives
$\dfrac{d^2y}{dz^2} + \dfrac{dy}{dz} + 0.25y = 0$, which has the solution
$y(z) = c_1e^{-z/2} + c_2ze^{-z/2}$ so that $y(t) = c_1t^{-1/2} + c_2t^{-1/2}\ln t$.

<u>Section 3.6, Page 178</u>

1. First we find the solution of the homogeneous D.E., which has the characteristic equation $r^2-2r-3 = (r-3)(r+1) = 0$. Hence $y_c = c_1e^{3t} + c_2e^{-t}$ and we can assume $Y = Ae^{2t}$ for the particular solution. Thus $Y' = 2Ae^{2t}$ and $Y'' = 4Ae^{2t}$ and substituting into the D.E. yields
$4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t}) = 3e^{2t}$. Thus $-3A = 3$ and $A = -1$, yielding $y = c_1e^{3t} + c_2e^{-t} - e^{2t}$.

4. Initially we assume $Y = A + B\sin 2t + C\cos 2t$. However, since a constant is a solution of the related homogeneous D.E. we must modify Y by multiplying the constant A by t and thus the correct form is $Y = At + B\sin 2t + C\cos 2t$.

6. Since $y_c = c_1e^{-t} + c_2te^{-t}$ we must assume $Y = At^2e^{-t}$, so that $Y' = 2Ate^{-t} - At^2e^{-t}$ and $y'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$. Substituting in the D.E. gives $(At^2-4At+2A)e^{-t} + 2(-At^2+2At)e^{-t} + At^2e^{-t} = 2e^{-t}$. Notice that all terms on the left involving t^2 and t add to zero and we are left with $2A = 2$, or $A = 1$. Hence $y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}$.

8. The assumed form is $Y = (At + B)\sin 2t + (Ct + D)\cos 2t$, which is appropriate for both terms appearing on the right side of the D.E. Since none of the terms appearing

in Y are solutions of the homogeneous equation, we do not need to modify Y.

11. First solve the homogeneous D.E. Substituting $y = e^{rt}$ gives $r^2 + r + 4 = 0$. Hence $y_c = e^{-t/2}[c_1\cos(\sqrt{15}\,t/2) + c_2\sin(\sqrt{15}\,t/2)]$. We replace sinht by $(e^t - e^{-t})/2$ and then assume $Y(t) = Ae^t + Be^{-t}$. Since neither e^t nor e^{-t} are solutions of the homogeneous equation, there is no need to modify our assumption for Y. Substituting in the D.E., we obtain $6Ae^t + 4Be^{-t} = e^t - e^{-t}$. Hence, A = 1/6 and B = -1/4. The general solution is
$y = e^{-t/2}[c_1\cos(\sqrt{15}\,t/2) + c_2\sin(\sqrt{15}\,t/2)] + e^t/6 - e^{-t}/4$.
[For this problem we could also have found a particular solution as a linear combination of sinht and cosht: Y(t) = Acosht + Bsinht. Substituting this in the D.E. gives (5A + B)cosht + (A + 5B)sinht = 2sinht. The solution is A = -1/12 and B = 5/12. A simple calculation shows that $-(1/12)\cosh t + (5/12)\sinh t = e^t/6 - e^{-t}/4$.]

13. $y_c = c_1e^{-2t} + c_2e^t$ so for the particular solution we assume Y = At + B. Since neither At or B are solutions of the homogeneous equation it is not necessary to modify the original assumption. Substituting Y in the D.E. we obtain 0 + A -2(At+B) = 2t or -2A = 2 and A-2B = 0. Solving for A and B we obtain $y = c_1e^{-2t} + c_2e^t - t - 1/2$ as the general solution. $y(0) = 0 \rightarrow c_1 + c_2 - 1/2 = 0$ and $y'(0) = 1 \rightarrow -2c_1 + c_2 - 1 = 1$, which yield $c_1 = -1/2$ and $c_2 = 1$. Thus $y = e^t - (1/2)e^{-2t} - t - 1/2$.

16. Since the nonhomogeneous term is the product of a linear polynomial and an exponential, assume Y of the same form: $Y = (At+B)e^{2t}$. Thus $Y' = Ae^{2t} + 2(At+B)e^{2t}$ and $Y'' = 4Ae^{2t}+4(At+B)e^{2t}$. Substituting into the D.E. we find -3At = 3t and 2A - 3B = 0, yielding A = -1 and B = -2/3. Since the characteristic equation is $r^2 - 2r - 3 = 0$, the general solution is
$y = c_1e^{3t} + c_2e^{-t} - \frac{2}{3}e^{2t} - te^{2t}$.

19a. The solution of the homogeneous D.E. is $y_c = c_1e^{-3t} + c_2$. After inspection of the nonhomogeneous term, for $2t^4$ we must assume a fourth order polynominial, for t^2e^{-3t} we must assume a quadratic polynomial times the exponential, and for sin3t we must assume Csin3t + Dcos3t. Thus
$Y(t) = (A_0t^4+A_1t^3+A_2t^2+A_3t+A_4) + (B_0t^2+B_1t+B_2)e^{-3t}+C\sin 3t+D\cos 3t.$
However, since e^{-3t} and a constant are solutions of the homogeneous D.E., we must multiply the coefficient of e^{-3t} and the polynomial by t. The correct form is

$$Y(t) = t(A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4) + t(B_0t^2 + B_1t + B_2)e^{-3t} + C\sin 3t + D\cos 3t.$$

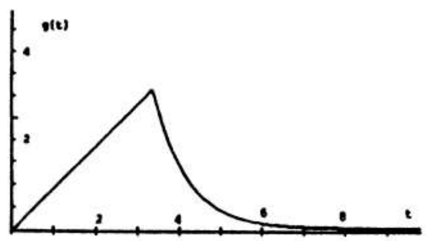
22a. The solution of the homgeneous D.E. is $y_c = e^{-t}[c_1\cos t + c_2\sin t]$. After inspection of the nonhomogeneous term, we assume $Y(t) = Ae^{-t} + (B_0t^2 + B_1t + B_2)e^{-t}\cos t + (C_0t^2 + C_1t + C_2)e^{-t}\sin t$. Since $e^{-t}\cos t$ and $e^{-t}\sin t$ are solutions of the homogeneous D.E., it is necessary to multiply both the last two terms by t. Hence the correct form is

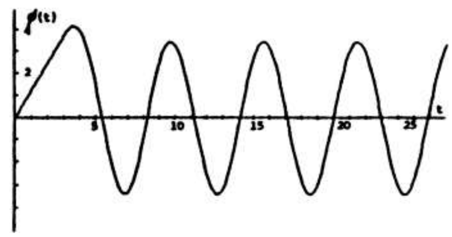
$$Y(t) = Ae^{-t} + t(B_0t^2 + B_1t + B_2)e^{-t}\cos t + t(C_0t^2 + C_1t + C_2)e^{-t}\sin t.$$

28. First solve the I.V.P. $y'' + y = t$, $y(0) = 0$, $y'(0) = 1$ for $0 \le t \le \pi$. The solution of the homogeneous D.E. is $y_c(t) = c_1\cos t + c_2\sin t$. The correct form for Y(t) is $y(t) = A_0t + A_1$. Substituting in the D.E. we find $A_0 = 1$ and $A_1 = 0$. Hence, $y = c_1\cos t + c_2\sin t + t$. Applying the I.C., we obtain $y = t$. For $t > \pi$ we have $y'' + y = \pi e^{\pi-t}$ so the form for Y(t) is $Y(t) = Ee^{\pi-t}$. Substituting Y(t) in the D.E., we obtain $Ee^{\pi-t} + Ee^{\pi-t} = \pi e^{\pi-t}$ so $E = \pi/2$. Hence the general solution for $t > \pi$ is $Y = D_1\cos t + D_2\sin t + (\pi/2)e^{\pi-t}$. If y and y′ are to be continuous at $t = \pi$, then the solutions and their derivatives for $t \le \pi$ and $t > \pi$ must have the same value at $t = \pi$. These conditions require $\pi = -D_1 + \pi/2$ and $1 = -D_2 - \pi/2$. Hence $D_1 = -\pi/2$, $D_2 = -(1 + \pi/2)$, and

$$y = \phi(t) = \begin{cases} t, & 0 \le t \le \pi \\ -(\pi/2)\cos t - (1 + \pi/2)\sin t + (\pi/2)e^{\pi-t}, & t > \pi. \end{cases}$$

The graphs of the nonhomogeneous term and ϕ follow.





30. According to Theorem 3.6.1, the difference of any two solutions of the linear second order nonhomogeneous D.E. is a solution of the corresponding homogeneous D.E. Hence $Y_1 - Y_2$ is a solution of $ay'' + by' + cy = 0$. In Problem 38 of Section 3.5 we showed that if $a > 0$, $b > 0$, and $c > 0$ then every solution of this D.E. goes to zero as $t \to \infty$. If $b = 0$, then y_c involves only sines and cosines, so $Y_1 - Y_2$ does not approach zero as $t \to \infty$.

33. From Problem 32 we write the D.E. as $(D-4)(D+1)y = 3e^{2t}$. Thus let $(D+1)y = u$ and then $(D-4)u = 3e^{2t}$. This last equation is the same as $du/dt - 4u = 3e^{2t}$, which may be solved by multiplying both sides by e^{-4t} and integrating (see section 2.1). This yields $u = (-3/2)e^{2t} + Ce^{4t}$. Substituting this form of u into $(D+1)y = u$ we obtain $dy/dt + y = (-3/2)e^{2t} + Ce^{4t}$. Again, multiplying by e^{t} and integrating gives $y = (-1/2)e^{2t} + C_1e^{4t} + C_2e^{-t}$, where $C_1 = C/5$.

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2. Two linearly independent solutions of the homogeneous D.E. are $y_1(t) = e^{2t}$ and $y_2(t) = e^{-t}$. Assume $Y = u_1(t)e^{2t} + u_2(t)e^{-t}$, then $Y'(t) = [2u_1(t)e^{2t} - u_2(t)e^{-t}] + [u_1'(t)e^{2t}$

$+ u_2'(t)e^{-t}]$. We set $u_1'(t)e^{2t} + u_2'(t)e^{-t} = 0$. Then $Y'' = 4u_1e^{2t} + u_2e^{-t} + 2u_1'e^{2t} - u_2'e^{-t}$ and substituting in the D.E. gives $2u_1'(t)e^{2t} - u_2'(t)e^{-t} = 2e^{-t}$. Thus we have two algebraic equations for $u_1'(t)$ and $u_2'(t)$ with the solution $u_1'(t) = 2e^{-3t}/3$ and $u_2'(t) = -2/3$. Hence $u_1(t) = -2e^{-3t}/9$ and $u_2(t) = -2t/3$. Substituting in the formula for $Y(t)$ we obtain $Y(t) = (-2e^{-3t}/9)e^{2t} + (-2t/3)e^{-t} = (-2e^{-t}/9) - (2te^{-t}/3)$. Since e^{-t} is a solution of the homogeneous D.E., we can choose $Y(t) = -2te^{-t}/3$.

5. Since cost and sint are solutions of the homogeneous D.E., we assume $Y = u_1(t)\cos t + u_2(t)\sin t$. Thus $Y' = -u_1(t)\sin t + u_2(t)\cos t$, after setting $u_1'(t)\cos t + u_2'(t)\sin t = 0$. Finding Y'' and substituting into the D.E. then yields $-u_1'(t)\sin t + u_2'(t)\cos t = \tan t$. The two equations for $u_1'(t)$ and $u_2'(t)$ have the solution: $u_1'(t) = -\sin^2 t/\cos t = -\sec t + \cos t$ and $u_2'(t) = \sin t$. Thus $u_1(t) = \sin t - \ln(\tan t + \sec t)$ and $u_2(t) = -\cos t$, which when substituted into the assumed form for Y, simplified, and added to the homogeneous solution yields
$y = c_1\cos t + c_2\sin t - (\cos t)\ln(\tan t + \sec t)$.

11. Two linearly independent solutions of the homogeneous D.E. are $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$. Applying Theorem 3.7.1 with $W(y_1,y_2)(t) = -e^{5t}$, we obtain

$$Y(t) = -e^{3t}\int\frac{e^{2s}g(s)}{-e^{5s}}\,ds + e^{2t}\int\frac{e^{3s}g(s)}{-e^{5s}}\,ds$$
$$= \int[e^{3(t-s)} - e^{2(t-s)}]g(s)ds.$$

The complete solution is then obtained by adding $c_1e^{3t} + c_2e^{2t}$ to $Y(t)$.

14. That t and te^t are solutions of the homogeneous D.E. can be verified by direction substitution. Thus we assume $Y = tu_1(t) + te^tu_2(t)$. Following the pattern of earlier problems we find $tu_1'(t) + te^tu_2'(t) = 0$ and

$u_1'(t) + (t+1)e^t u_2' = 2t$. [Note that $g(t) = 2t$, since the D.E. must be put into the form of Eq.(16)]. The solution of these equations gives $u_1'(t) = -2$ and $u_2'(t) = 2e^{-t}$. Hence, $u_1(t) = -2t$ and $u_2(t) = -2e^{-t}$, and $Y(t) = t(-2t) + te^t(-2e^{-t}) = -2t^2 - 2t$. However, since t is a solution of the homogeneous D.E. we can choose as our particular solution $Y(t) = -2t^2$.

18. For this problem, and for many others, it is probably easier to rederive Eqs.(26) without using the explicit form for $y_1(x)$ and $y_2(x)$ and then to substitute for $y_1(x)$ and $y_2(x)$ in Eqs.(26). In this case if we take $y_1 = x^{-1/2}\sin x$ and $y_2 = x^{-1/2}\cos x$, then $W(y_1,y_2) = -1/x$. If the D.E. is put in the form of Eq.(16), then $g(x) = 3x^{-1/2}\sin x$ and thus $u_1'(x) = 3\sin x\cos x$ and $u_2'(x) = -3\sin^2 x = 3(-1 + \cos 2x)/2$. Hence $u_1(x) = (3\sin^2 x)/2$ and $u_2(x) = -3x/2 + 3(\sin 2x)/4$, and

$$\begin{aligned} Y(x) &= \frac{3\sin^2 x}{2}\frac{\sin x}{\sqrt{x}} + \left(-\frac{3x}{2} + \frac{3\sin 2x}{4}\right)\frac{\cos x}{\sqrt{x}} \\ &= \frac{3\sin^2 x}{2}\frac{\sin x}{\sqrt{x}} + \left(-\frac{3x}{2} + \frac{3\sin x\cos x}{2}\right)\frac{\cos x}{\sqrt{x}} \\ &= \frac{3\sin x}{2\sqrt{x}} - \frac{3\sqrt{x}\cos x}{2}. \end{aligned}$$

The first term is a multiple of $y_1(x)$ and thus can be neglected for $Y(x)$.

22. Putting limits on the integrals of Eq.(28) and changing the integration variable to s yields

$$\begin{aligned} Y(t) &= -y_1(t)\int_{t_0}^{t}\frac{y_2(s)g(s)ds}{W(y_1,y_2)(s)} + y_2(t)\int_{t_0}^{t}\frac{y_1(s)g(s)ds}{W(y_1,y_2)(s)} \\ &= \int_{t_0}^{t}\frac{-y_1(t)y_2(s)g(s)ds}{W(y_1,y_2)(s)} + \int_{t_0}^{t}\frac{y_2(t)y_1(s)g(s)ds}{W(y_1,y_2)(s)} \\ &= \int_{t_0}^{t}\frac{[y_1(s)y_2(t) - y_1(t)y_2(s)]g(s)ds}{y_1(s)y_2'(s) - y_1'(s)y_2(s)}. \end{aligned}$$

To show that $Y(t)$ satisfies $L[y] = g(t)$ we must take the derivative of Y using Leibnitz's rule, which says that if $Y(t) = \int_{t_0}^{t}G(t,s)ds$, then $Y'(t) = G(t,t) + \int_{t_0}^{t}\frac{\partial G}{\partial t}(t,s)ds$. Letting $G(t,s)$ be the above integrand, then $G(t,t) = 0$

and $\dfrac{\partial G}{\partial t} = \dfrac{y_1(s)y_2'(t) - y_1'(t)y_2(s)}{W(y_1,y_2)(s)}\, g(s)$. Likewise

$$Y'' = \frac{\partial G(t,t)}{\partial t} + \int_{t_0}^{t} \frac{\partial^2 G}{\partial t^2}(t,s)ds$$

$$= g(t) + \int_{t_0}^{t} \frac{y_1(s)y_2''(t) - y_1''(t)y_2(s)}{W(y_1,y_2)(s)}ds.$$

Since y_1 amd y_2 are solutions of $L[y] = 0$, we have $L[Y] = g(t)$ since all the terms involving the integral will add to zero. Clearly $Y(t_0) = 0$ and $Y'(t_0) = 0$.

25. Note that $y_1 = e^{\lambda t}\cos\mu t$ and $y_2 = e^{\lambda t}\sin\mu t$ and thus $W(y_1,y_2) = \mu e^{2\lambda t}$. From Problem 22 we then have:

$$Y(t) = \int_{t_0}^{t} \frac{e^{\lambda s}\cos\mu s e^{\lambda t}\sin\mu t - e^{\lambda t}\cos\mu t e^{\lambda s}\sin\mu s}{\mu e^{2\lambda s}}\, g(s)ds$$

$$= \mu^{-1} \int_{t_0}^{t} e^{\lambda(t-s)}[\cos\mu s \sin\mu t - \cos\mu t \sin\mu s]g(s)ds$$

$$= \mu^{-1} \int_{t_0}^{t} e^{\lambda(t-s)}[\sin\mu(t-s)]g(s)ds.$$

29. First, we put the D.E. in standard form by dividing by t^2: $y'' - 2y'/t + 2y/t^2 = 4$. Assuming that $y = tv(t)$ and substituting in the D.E. we obtain $tv'' = 4$. Hence $v'(t) = 4\ln t + c_2$ and $v(t) = 4 \int \ln t\, dt + c_2 t = 4(t\ln t - t) + c_2 t$. The general solution is $c_1 y_1(t) + tv(t) = c_1 t + 4(t^2\ln t - t^2) + c_2 t^2$. Since $-4t^2$ is a multiple of $y_2 = c_2 t^2$ we can write $y = c_1 t + c_2 t^2 + 4t^2\ln t$.

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2. From Eq.(15) we have $R\cos\delta = -1$, and $R\sin\delta = \sqrt{13}$. Thus $R = \sqrt{1+3} = 2$ and $\delta = \tan^{-1}(-\sqrt{3}) + \pi = 2\pi/3 \cong 2.09440$. Note that we have to "add" π to the inverse tangent value since δ must be a second quadrant angle. Thus $u = 2\cos(t-2\pi/3)$.

6. The motion is an undamped free vibration. The units are in the CGS system. The spring constant $k = (100\text{ gm})(980\text{cm/sec}^2)/5\text{cm}$. Hence the D.E. for the motion is $100u'' + [(100 \cdot 980)/5]u = 0$ where u is measured in cm and time in sec. We obtain $u'' + 196u = 0$

so
$u = A\cos 14t + B\sin 14t$. The I.C. are $u(0) = 0 \to A = 0$ and $u'(0) = 10$ cm/sec $\to B = 10/14 = 5/7$. Hence $u(t) = (5/7)\sin 14t$, which first reaches equilibrium when $14t = \pi$, or $t = \pi/14$.

8. We use Eq.(33) without R and E(t) (there is no resistor or impressed voltage) and with L = 1 henry and $1/C = 4\text{x}10^6$ since $C = .25\text{x}10^{-6}$ farads. Thus the I.V.P. is $Q'' + 4\text{x}10^6\, Q = 0$, $Q(0) = 10^{-6}$ coulombs and $Q'(0) = 0$.

9. The spring constant is $k = (20)(980)/5 = 3920$ dyne/cm. The I.V.P. for the motion is $20u'' + 400u' + 3920u = 0$ or $u'' + 20u' + 196u = 0$ and $u(0) = 2$, $u'(0) = 0$. Here u is measured in cm and t in sec. The general solution of the D.E. is $u = Ae^{-10t}\cos 4\sqrt{6}\,t + Be^{-10t}\sin 4\sqrt{6}\,t$. The I.C. $u(0) = 2 \to A = 2$ and $u'(0) = 0 \to -10A + 4\sqrt{6}\,B = 0$. The solution is $u = e^{-10t}[2\cos 4\sqrt{6}\,t + 5(\sin 4\sqrt{6}\,t)/\sqrt{6}\,]$cm. The quasi frequency is $\mu = 4\sqrt{6}$, the quasi period is $T_d = 2\pi\mu = \pi/2\sqrt{6}$ and $T_d/T = 7/2\sqrt{6}$ since $T = 2\pi/14 = \pi/7$. To find an upper bound for τ, write u in the form of Eq.(30): $u(t) = \sqrt{4+25/6}\,e^{-10t}\cos(4\sqrt{6}\,t-\delta)$. Now, since $|\cos(4\sqrt{6}\,t-\delta)| \le 1$, we have $|u(t)| < .05 \Rightarrow \sqrt{4+25/6}\,e^{-10t} < .05$, which yields $\tau = .4046$. A more precise answer can be obtained with a computer algebra system, which in this case yields $\tau = .4045$. The original estimate was unusually close for this problem since $\cos(4\sqrt{6}\,t-\delta) = -0.9996$ for $t = .4046$.

12. Substituting the given values for L, C and R in Eq.(33), we obtain the D.E. $.2Q'' + 3\text{x}10^2\, Q' + 10^5\, Q = 0$. The I.C. are $Q(0) = 10^{-6}$ and $Q'(0) = I(0) = 0$. Assuming $Q = e^{rt}$, we obtain the roots of the characteristic equation as $r_1 = -500$ and $r_2 = -1000$. Thus $Q = c_1e^{-500t} + c_2e^{-1000t}$ and hence $Q(0) = 10^{-6} \to c_1 + c_2 = 10^{-6}$ and $Q'(0) = 0 \to -500c_1 - 1000c_2 = 0$. Solving for c_1 and c_2 yields the solution.

17. The mass is 8/32 lb-sec^2/ft, and the spring constant is 8/(1/8) = 64 lb/ft. Hence $(1/4)u'' + \gamma u' + 64u = 0$ or $u'' + 4\gamma u' + 256u = 0$, where u is measured in ft, t in sec and the units of γ are lb-sec/ft. We look for solutions of the D.E. of the form $u = e^{rt}$ and find $r^2 + 4\gamma r + 256 = 0$, so $r_1,r_2 = [-4\gamma \pm \sqrt{16\gamma^2 - 1024}\,]/2$. The system will be overdamped, critically damped or

underdamped as $(16\gamma^2 - 1024)$ is > 0, $=0$, or < 0, respectively. Thus the system is critically damped when $\gamma = 8$ lb-sec/ft.

19. The general solution of the D.E. is $u = Ae^{r_1 t} + Be^{r_2 t}$ where $r_1, r_2 = [-\gamma \pm (\gamma^2 - 4km)^{1/2}]/2m$ provided $\gamma^2 - 4km \neq 0$, and where A and B are determined by the I.C. When the motion is overdamped, $\gamma^2 - 4km > 0$ and $r_1 > r_2$. Setting $u = 0$, we obtain $Ae^{r_1 t} = -Be^{r_2 t}$ or $e^{(r_1 - r_2)t} = -B/A$. Since the exponential function is a monotone function, there is at most one value of t (when $B/A < 0$) for which this equation can be satisfied. Hence u can vanish at most once. If the system is critically damped, the general solution is $u(t) = (A + Bt)e^{-\gamma t/2m}$. The exponential function is never zero; hence u can vanish only if $A + Bt = 0$. If $B = 0$ then u never vanishes; if $B \neq 0$ then u vanishes once at $t = -A/B$ provided $A/B < 0$.

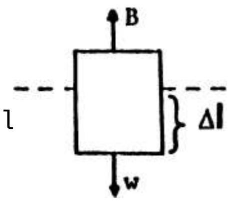
20. The general solution of Eq.(21) for the case of critical damping is $u = (A + Bt)e^{-\gamma t/2m}$. The I.C. $u(0) = u_0 \rightarrow A = u_0$ and $u'(0) = v_0 \rightarrow A(-\gamma/2m) + B = v_0$. Hence $u = [u_0 + (v_o + \gamma u_0/2m)t]e^{-\gamma t/2m}$. If $v_0 = 0$, then $u = u_0(1 + \gamma t/2m)e^{-\gamma t/2m}$, which is never zero since γ and m are postive. By L'Hopital's Rule $u \rightarrow 0$ as $t \rightarrow \infty$. Finally for $u_0 > 0$, we want the condition which will insure that $v = 0$ at least once. Since the exponential function is never zero we require $u_0 + (v_0 + \gamma u_0/2m)t = 0$ at a positive value of t. This requires that $v_0 + \gamma u_0/2m \neq 0$ and that $t = -u_0(v_0 + u_0\gamma/2m)^{-1} > 0$. We know that $u_0 > 0$ so we must have $v_0 + \gamma u_0/2m < 0$ or $v_0 < -\gamma u_0/2m$.

23. From Problem 21: $\Delta = \dfrac{2\pi\gamma}{\mu(2m)} = T_d\gamma/2m$. Substituting the known values we find $\gamma = \dfrac{(1/2)\ (3)}{.3} = 5$ lb sec/ft.

24. From Eq.(13) $\omega_0^2 = \dfrac{2k}{3}$ so $P = 2\pi/\sqrt{2k/3} = \pi \rightarrow k = 6$. Thus $u(t) = c_1\cos 2t + c_2\sin 2t$ and $u(0) = 2 \rightarrow c_1 = 2$ and $u'(0) = v \rightarrow c_2 = v/2$. Hence $u(t) = 2\cos 2t + \dfrac{v}{2}\sin 2t =$

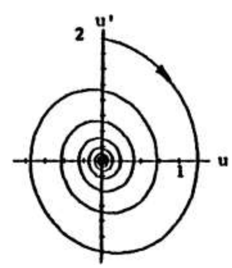
$\sqrt{4+\frac{v^2}{4}}\cos(2t-\gamma)$. Thus $\sqrt{4+\frac{v^2}{4}} = 3$ and $v = \pm 2\sqrt{5}$.

27. First, consider the static case. Let Δl denote the length of the block below the surface of the water. The weight of the block, which is a downward force, is $w = \rho l^3 l^3 g$. This is balanced by an equal and opposite buoyancy force B, which is equal to the weight of the displaced water. Thus $B = (\rho_0 l^2 \Delta l)g = \rho \Delta l^3 g$ so $\rho_0 \Delta l = \rho l$. Now let x be the displacement of the block from its equilibrium position. We take downward as the positive direction. In a displaced position the forces acting on the block are its weight, which acts downward and is unchanged, and the buoyancy force which is now $\rho_0 l^2(\Delta l + x)g$ and acts upward. The resultant force must be equal to the mass of the block times the acceleration, namely $\rho l^3 x''$. Hence $\rho l^3 g - \rho_0 l^2(\Delta l + x)g = \rho l^3 x''$. The D.E. for the motion of the block is $\rho l^3 x'' + \rho_0 l^2 gx = 0$. This gives a simple harmonic motion with frequency $(\rho_0 g/\rho l)^{1/2}$ and natural period $2\pi(\rho l/\rho_0 g)^{1/2}$.



29a. The characteristic equation is $4r^2 + r + 8 = 0$, so $r = (-1\pm\sqrt{127})/8$ and hence
$u(t) = e^{-t/8}(c_1\cos\frac{\sqrt{127}}{8}t + c_2\sin\frac{\sqrt{127}}{8}t$. $u(0) = 0 \rightarrow c_1 = 0$
and $u'(0) = 2 \rightarrow \frac{\sqrt{127}}{8}c_2 = 2$. Thus
$$u(t) = \frac{16}{\sqrt{127}}e^{-t/8}\sin\frac{\sqrt{127}}{8}t.$$

29c. The phase plot is the spiral shown and the direction of motion is clockwise since the graph starts at (0,2) and u increases initially.



30c. Using u(t) as found in part(b), show that $ku^2/2 + m(u')^2/2 = (ka^2 + mb^2)/2$ for all t.

Section 3.9, Page 205

1. We use the trigonometric identities
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ to obtain
$\cos(A + B) - \cos(A - B) = -2\sin A \sin B$. If we choose
$A + B = 9t$ and $A - B = 7t$, then $A = 8t$ and $B = t$.
Substituting in the formula just derived, we obtain
$\cos 9t - \cos 7t = -2\sin 8t \sin t$.

5. The mass $m = 4/32 = 1/8$ lb-sec^2/ft and the spring constant $k = 4/(1/8) = 32$ lb/ft. Since there is no damping, the I.V.P. is $(1/8)u'' + 32u = 2\cos 3t$, $u(0) = 1/6$, $u'(0) = 0$ where u is measured in ft and t in sec.

7a. From the solution to Problem 5, we have $m = 1/8$, $F_0 = 2$, $\omega_0^2 = 256$, and $\omega^2 = 9$, so Eq.(3) becomes
$u = c_1\cos 16t + c_2\sin 16t + \dfrac{16}{247}\cos 3t$. The I.C.
$u(0) = 1/6 \rightarrow c_1 + 16/247 = 1/6$ and $u'(0) = 0 \rightarrow 16c_2 = 0$, so the solution is $u = (151/1482)\cos 16t + (16/247)\cos 3t$ ft.

7c. Resonance occurs when the frequency ω of the forcing function $4\sin\omega t$ is the same as the natural frequency ω_0 of the system. Since $\omega_0 = 16$, the system will resonate when $\omega = 16$ rad/sec.

10. The I.V.P. is $.25u'' + 16u = 8\sin 8t$, $u(0) = 3$ and $u'(0) = 0$. Thus, the particular solution has the form $t(A\cos 8t + B\sin 8t)$ and resonance occurs.

11a. For this problem the mass $m = 8/32$ lb-sec^2/ft and the spring constant $k = 8/(1/2) = 16$ lb/ft, so the D.E. is $0.25u'' + 0.25u' + 16u = 4\cos 2t$ where u is measured in ft and t in sec. To determine the steady state response we need only compute a particular solution of the nonhomogeneous D.E. since the solutions of the homogeneous D.E. decay to zero as $t \rightarrow \infty$. We assume $u(t) = A\cos 2t + B\sin 2t$, and substitute in the D.E.:
$- A\cos 2t - B\sin 2t + (1/2)(-A\sin 2t + B\cos 2t) + 16(A\cos 2t + B\sin 2t) = 4\cos 2t$. Hence $15A + (1/2)B = 4$ and $-(1/2)A + 15B = 0$, from which we obtain $A = 240/901$ and $B = 8/901$. The steady state response is
$u(t) = (240\cos 2t + 8\sin 2t)/901$.

11b. In order to determine the value of m that maximizes the steady state response, we note that the present problem has exactly the form of the problem considered in the text. Referring to Eqs.(8) and (9), the response is a maximum when Δ is a minimum since F_0 is constant. Δ, as given in Eq.(10), will be a minimum when $f(m) = m^2(\omega_0{}^2 - \omega^2)^2 + \gamma^2\omega^2$, where $\omega_0{}^2 = k/m$, is a minimum. We calculate df/dm and set this quantity equal to zero to obtain $m = k/\omega^2$. We verify that this value of m gives a minimum of f(m) by the second derivative test. For this problem k = 16 lb/ft and $\omega = 2$ rad/sec so the value of m that maximizes the response of the system is m = 4 slugs.

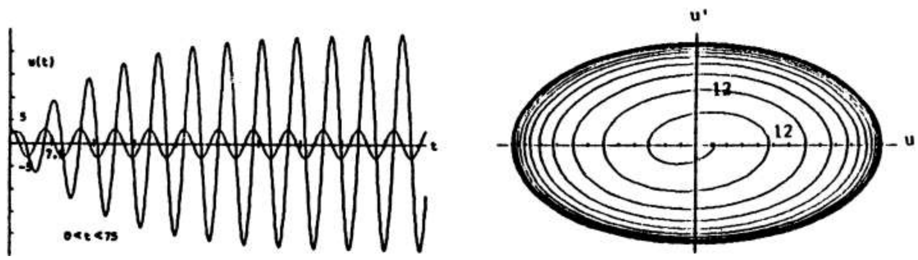
15. We must solve the three I.V.P.: (1)$u_1'' + u_1 = F_0 t$, $0 < t < \pi$, $u_1(0) = u_1'(0) = 0$; (2) $u_2'' + u_2 = F_0(2\pi - t)$, $\pi < t < 2\pi$, $u_2(\pi) = u_1(\pi)$, $u_2'(\pi) = u_1'(\pi)$; and (3) $u_3'' + u_3 = 0$, $2\pi < t$, $u_3(2\pi) = u_2(2\pi)$, $u_3'(2\pi) = u_2'(2\pi)$. The conditions at π and 2π insure the continuity of u and u′ at those points. The general solutions of the D.E. are $u_1 = b_1\cos t + b_2\sin t + F_0 t$, $u_2 = c_1\cos t + c_2\sin t + F_0(2\pi - t)$, and $u_3 = d_1\cos t + d_2\sin t$. The I.C. and matching conditions, in order, give $b_1 = 0$, $b_2 + F_0 = 0$, $-b_1 + \pi F_0 = -c_1 + \pi F_0$, $-b_2 + F_0 = -c_2 - F_0$, $c_1 = d_1$, and $c_2 - F_0 = d_2$. Solving these equations we obtain

$$u = F_0 \begin{cases} t - \sin t &, 0 \le t \le \pi \\ (2\pi - t) - 3\sin t &, \pi < t \le 2\pi \\ -4\sin t &, 2\pi < t. \end{cases}$$

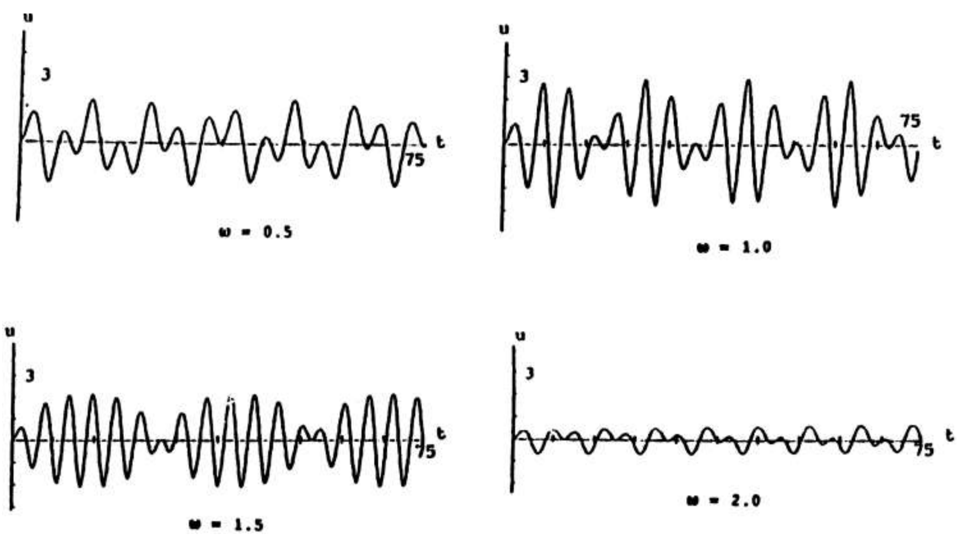
16. The I.V.P. is $Q'' + 5\text{x}10^3 Q' + 4\text{x}10^6 Q = 12$, $Q(0) = 0$, and $Q'(0) = 0$. The particular solution is of the form Q = A, so that upon substitution into the D.E. we obtain $4\text{x}10^6 A = 12$ or $A = 3\text{x}10^{-6}$. The general solution of the D.E. is $Q = c_1e^{r_1t} + c_2e^{r_2t} + 3\text{x}10^{-6}$, where r_1 and r_2 satisfy $r^2 + 5\text{x}10^3 r + 4\text{x}10^6 = 0$ and thus are $r_1 = -1000$ and $r_2 = -4000$. The I.C. yield $c_1 = -4\text{x}10^{-6}$ and $c_2 = 10^{-6}$ and thus $Q = 10^{-6}(e^{-4000t} - 4e^{-1000t} + 3)$ coulombs. Substituting t = .001 sec we obtain
$Q(.001) = 10^{-6}(e^{-4} - 4e^{-1} + 3) = 1.5468 \text{ x } 10^{-6}$ coulombs. Since exponentials are to a negative power $Q(t) \to 3\text{x}10^{-6}$ coulombs as $t \to \infty$, which is the steady state charge.

22. The amplitude of the steady state response is seven or eight times the amplitude (3) of the forcing term. This large an increase is due to the fact that the forcing function has the same frequency as the natural frequency, ω_0, of the system.

    There also appears to be a phase lag of approximately 1/4 of a period. That is, the maximum of the response occurs 1/4 of a period after the maximum of the forcing function. Both these results are substantially different than those of either Problems 21 or 23.



24.



From viewing the above graphs, it appears that the system exhibits a beat near $\omega = 1.5$, while the pattern for $\omega = 1.0$ is more irregular. However, the system exhibits the resonance characteristic of the linear system for ω near 1, as the amplitude of the response is the largest here.

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2. Writing the equation in standard form, we obtain $y''' + [(\sin t)/t]y'' + (3/t)y = \cos t/t$. The functions $p_1(t) = \sin t/t$, $p_3(t) = 3/t$ and $g(t) = \cos t/t$ have discontinuities at $t = 0$. Hence Theorem 4.1.1 guarantees that a solution exists for $t < 0$ and for $t > 0$.

8. We have $W(f_1, f_2, f_3) = \begin{vmatrix} 2t-3 & 2t^2+1 & 3t^2+t \\ 2 & 4t & 6t+1 \\ 0 & 4 & 6 \end{vmatrix} = 0$ for all t.
   Thus by the extension of Theorem 3.3.1 the given functions are linearly dependent. Thus $c_1(2t-3) + c_2(2t^2+1) + c_3(3t^2+t) = (2c_2+3c_3)t^2 + (2c_1+c_3)t + (-3c_1+c_2) = 0$ when $(2c_2 + 3c_3) = 0$, $2c_1 + c_3 = 0$ and $-3c_1 + c_2 = 0$. Thus $c_1 = 1$ $c_2 = 3$ and $c_3 = -2$.

13. That e^t, e^{-t}, and e^{-2t} are solutions can be verified by direct substitution. Computing the Wronskian we obtain,

$$W(e^t,e^{-t},e^{-2t}) = \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} = e^{-2t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{-2t}$$

17. To show that the given Wronskian is zero, it is helpful, in evaluating the Wronskian, to note that $(\sin^2 t)' = 2\sin t\cos t = \sin 2t$. This result can be obtained directly since $\sin^2 t = (1 - \cos 2t)/2 = \frac{1}{10}(5) + (-1/2)\cos 2t$ and hence $\sin^2 t$ is a linear combination of 5 and $\cos 2t$. Thus the functions are linearly dependent and their Wronskian is zero.

19c. If we let $L[y] = y^{iv} - 5y'' + 4y$ and if we use the result of Problem 19b, we have $L[e^{rt}] = (r^4 - 5r^2 + 4)e^{rt}$. Thus e^{rt} will be a solution of the D.E. provided $(r^2-4)(r^2-1) = 0$. Solving for r, we obtain the four solutions e^t, e^{-t}, e^{2t} and e^{-2t}. Since

Capítulo 4

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Thus by the extension of Theorem 3.3.1 the given functions are linearly dependent. Thus $c_1(2t-3) + c_2(2t^2+1) + c_3(3t^2+t) =$
$(2c_2+3c_3)t^2 + (2c_1+c_3)t + (-3c_1+c_2) = 0$ when $(2c_2 + 3c_3) = 0$, $2c_1 + c_3 = 0$ and $-3c_1 + c_2 = 0$. Thus $c_1 = 1$ $c_2 = 3$ and $c_3 = -2$.

13. That e^t, e^{-t}, and e^{-2t} are solutions can be verified by direct substitution. Computing the Wronskian we obtain,

$$W(e^t,e^{-t},e^{-2t})= \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} = e^{-2t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{-2t}$$

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19c. If we let $L[y] = y^{iv} - 5y'' + 4y$ and if we use the result of Problem 19b, we have $L[e^{rt}] = (r^4 - 5r^2 + 4)e^{rt}$. Thus e^{rt} will be a solution of the D.E. provided $(r^2-4)(r^2-1) = 0$. Solving for r, we obtain the four solutions e^t, e^{-t}, e^{2t} and e^{-2t}. Since

$W(e^t, e^{-t}, e^{2t}, e^{-2t}) \neq 0$, the four functions form a fundamental set of solutions.

21. Comparing this D.E. to that of Problem 20 we see that $p_1(t) = 2$ and thus from the results of Problem 20 we have $W = ce^{-\int 2dt} = ce^{-2t}$.

27. As in Problem 26, let $y = v(t)e^t$. Differentiating three times and substituting into the D.E. yields $(2-t)e^tv''' + (3-t)e^tv'' = 0$. Dividing by $(2-t)e^t$ and letting $w = v''$ we obtain the first order separable equation $w' = -\frac{t-3}{t-2}w = (-1 + \frac{1}{t-2})w$. Separating t and w, integrating, and then solving for w yields $w = v'' = c_1(t-2)e^{-t}$. Integrating this twice then gives $v = c_1te^{-t} + c_2t + c_3$ so that $y = ve^t = c_1t + c_2te^t + c_3e^t$, which is the complete solution, since it contains the given $y_1(t)$ and three constants.

Section 4.2, Page 219

2. If $-1 + i\sqrt{3} = Re^{i\theta}$, then $R = [(-1)^2 + (\sqrt{3})^2]^{1/2} = 2$. The angle θ is given by $R\cos\theta = 2\cos\theta = -1$ and $R\sin\theta = 2\sin\theta = \sqrt{3}$. Hence $\cos\theta = -1/2$ and $\sin\theta = \sqrt{3}/2$ which has the solution $\theta = 2\pi/3$. The angle θ is only determined up to an additive integer multiple of $\pm 2\pi$.

8. Writing (1-i) in the form $Re^{i\theta}$, we obtain $(1-i) = \sqrt{2}e^{i(-\pi/4+2m\pi)}$ where m is any integer. Hence, $(1-i)^{1/2} = [2^{1/2}e^{i(-\pi/4+2m\pi)}]^{1/2} = 2^{1/4}e^{i(-\pi/8+m\pi)}$. We obtain the two square roots by setting m = 0,1. They are $2^{1/4}e^{-i\pi/8}$ and $2^{1/4}e^{i7\pi/8}$. Note that any other integer value of m gives one of these two values. Also note that 1-i could be written as $1-i = \sqrt{2}e^{i(7\pi/4 + 2m\pi)}$.

12. We look for solutions of the form $y = e^{rt}$. Substituting in the D.E., we obtain the characteristic equation $r^3 - 3r^2 + 3r - 1 = 0$ which has roots r = 1,1,1. Since the roots are repeated, the general solution is $y = c_1e^t + c_2te^t + c_3t^2e^t$.

15. We look for solutions of the form $y = e^{rt}$. Substituting in the D.E. we obtain the characteristic equation $r^6 + 1 = 0$. The six roots of -1 are obtained by setting $m = 0,1,2,3,4,5$ in $(-1)^{1/6} = e^{i(\pi+2m\pi)/6}$. They are $e^{i\pi/6} = (\sqrt{3} + i)/2$, $e^{i\pi/2} = i$, $e^{i5\pi/6} = (-\sqrt{3} + i)/2$, $e^{i7\pi/6} = (-\sqrt{3} - i)/2$, $e^{i3\pi/2} = -i$, and $e^{i11\pi/6} = (\sqrt{3} - i)/2$. Note that there are three pairs of conjugate roots. The general solution is

$$y = e^{\sqrt{3}\,t/2}[c_1\cos(t/2) + c_2\sin(t/2)] + e^{-\sqrt{3}\,t/2}[c_3\cos(t/2) + c_4\sin(t/2)] + c_5\cos t + c_6\sin t.$$

23. The characteristic equation is $r^3 - 5r^2 + 3r + 1 = 0$. Using the procedure suggested following Eq. (12) we try $r = 1$ as a root and find that indeed it is. Factoring out $(r-1)$ we are then left with $r^2 - 4r - 1 = 0$, which has the roots $2 \pm \sqrt{5}$.

27. The characteristic equation in this case is $12r^4 + 31r^3 + 75r^2 + 37r + 5 = 0$. Using an equation solver we find $r = -\frac{1}{4}, -\frac{1}{3}, -1 \pm 2i$. Thus $y = c_1e^{-t/4} + c_2e^{-t/3} + e^{-t}(c_3\cos 2t + c_4\sin 2t)$. As in Problem 23, it is possible to find the first two of these roots without using an equation solver.

29. The characteristic equation is $r^3 + r = 0$ and hence $r = 0, +i, -i$ are the roots and the general solution is $y(t) = c_1 + c_2\cos t + c_3\sin t$. $y(0) = 0$ implies $c_1 + c_2 = 0$, $y'(0) = 1$ implies $c_3 = 1$ and $y''(0) = 2$ implies $-c_2 = 2$. Use this last equation in the first to find $c_1 = 2$ and thus $y(t) = 2 - 2\cos t + \sin t$, which continues to oscillate as $t \to \infty$.

30. The general solution is given by Eq. (21).

31. The general solution would normally be written $y(t) = c_1 + c_2t + c_3e^{2t} + c_4te^{2t}$. However, in order to evaluate the c's when the initial conditions are given at $t = 1$, it is advantageous to rewrite $y(t)$ as $y(t) = c_1 + c_2t + c_5e^{2(t-1)} + c_6(t-1)e^{2(t-1)}$.

34. The characteristic equation is $4r^3 + r + 5 = 0$, which has roots -1, $\frac{1}{2} \pm i$. Thus

$y(t) = c_1e^{-t} + e^{t/2}(c_2\cos t + c_3\sin t)$,

$y'(t) = -c_1e^{-t} + e^{t/2}[(c_2/2 + c_3)\cos t + (-c_2 + c_3/2)\sin t]$

and

$y''(t) = c_1e^{-t} + e^{t/2}[(-3c_2/4 + c_3)\cos t + (-c_2 - 3c_3/4)\sin t]$.

The I.C. then yield $c_1 + c_2 = 2$, $-c_1 + c_2/2 + c_3 = 1$ and $c_1 - 3c_2/4 + c_3 = -1$. Solving these last three equations give $c_1 = 2/13$, $c_2 = 24/13$ and $c_3 = 3/13$.

37. The approach developed in this section for solving the D.E. would normally yield $y(t) = c_1\cos t + c_2\sin t + c_5e^t + c_6e^{-t}$ as the solution. Now use the definition of coshx and sinht to yield the desired result. It is convenient to use cosht and sinht rather than e^t and e^{-t} because the I.C. are given at $t = 0$. Since cosht and sinht and all of their derivatives are either 0 or 1 at $t = 0$, the algebra in satisfying the I.C. is simplified.

38a. Since $p_1(t) = 0$, $W = ce^{-\int 0dt} = c$.

39a. As in Section 3.8, the force that the spring designated by k_1 exerts on mass m_1 is $-3u_1$. By an analysis similar to that shown in Section 3.8, the middle spring exerts a force of $-2(u_1-u_2)$ on mass m_1 and a force of $-2(u_2-u_1)$ on mass m_2. In all cases the positive direction is taken in the direction shown in Figure 4.2.4.

39c. From Eq.(i) we have $u_1''(0) = 2u_2(0) - 5u_1(0) = -1$ and $u_1'''(0) = 2u_2'(0) - 5u_1'(0) = 0$. From Prob.39b we have $u_1 = c_1\cos t + c_2\sin t + c_3\cos\sqrt{6}\,t + c_4\sin\sqrt{6}\,t$. Thus $c_1+c_3 = 1$, $c_2+\sqrt{6}\,c_4 = 0$, $-c_1-6c_3 = -1$ and $-c_2-6\sqrt{6}\,c_4 = 0$, which yield $c_1 = 1$ and $c_2 = c_3 = c_4 = 0$, so that $u_1 = \cos t$. The first of Eqs.(i) then gives u_2.

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1. First solve the homogeneous D.E. The characteristic

equation is $r^3 - r^2 - r + 1 = 0$, and the roots are $r = -1, 1, 1$; hence $y_c(t) = c_1e^{-t} + c_2e^t + c_3te^t$. Using the superposition principle, we can write a particular solution as the sum of particular solutions corresponding to the D.E. $y''' - y'' - y' + y = 2e^{-t}$ and $y''' - y'' - y' + y = 3$. Our initial choice for a particular solution, Y_1, of the first equation is Ae^{-t}; but e^{-t} is a solution of the homogeneous equation so we multiply by t. Thus, $Y_1(t) = Ate^{-t}$. For the second equation we choose $Y_2(t) = B$, and there is no need to modify this choice. The constants are determined by substituting into the individual equations. We obtain $A = 1/2$, $B = 3$. Thus, the general solution is
$y = c_1e^{-t} + c_2e^t + c_3te^t + 3 + (te^{-t})/2$.

5. The characteristic equation is $r^4 - 4r^2 = r^2(r^2-4) = 0$, so $y_c(t) = c_1 + c_2t + c_3e^{-2t} + c_4e^{2t}$. For the particular solution correspnding to t^2 we assume $Y_1 = t^2(At^2 + Bt + C)$ and for the particular solution corresponding to e^t we assume $Y_2 = De^t$. Substituting Y_1, in the D.E. yields $-48A = 1$, $B = 0$ and $24A-8C = 0$ and substituting Y_2 yields $-3D = 1$. Solving for A, B, C and D gives the desired solution.

9. The characteristic equation for the related homogeneous D.E. is $r^3 + 4r = 0$ with roots $r = 0, +2i, -2i$. Hence $y_c(t) = c_1 + c_2\cos 2t + c_3\sin 2t$. The initial choice for $Y(t)$ is $At + B$, but since B is a solution of the homogeneous equation we must multiply by t and assume $Y(t) = t(At+B)$. A and B are found by substituting in the D.E., which gives $A = 1/8$, $B = 0$, and thus the general solution is $y(t) = c_1 + c_2\cos 2t + c_3\sin 2t + (1/8)t^2$. Applying the I.C. we have $y(0) = 0 \to c_1 + c_2 = 0$, $y'(0) = 0 \to 2c_3 = 0$, and $y''(0) = 1 \to -4c_2 + 1/4 = 1$, which have the solution $c_1 = 3/16$, $c_2 = -3/16$, $c_3 = 0$. For small t the graph will approximate $3(1-\cos 2t)/16$ and for large t it will be approximated by $t^2/8$.

13. The characteristic equation for the homogeneous D.E. is $r^3 - 2r^2 + r = 0$ with roots $r = 0,1,1$. Hence the complementary solution is $y_c(t) = c_1 + c_2e^t + c_3te^t$. We

consider the differential equations $y''' - 2y'' + y' = t^3$ and $y''' - 2y'' + y' = 2e^t$ separately. Our initial choice for a particular solution, Y_1, of the first equation is $A_0t^3 + A_1t^2 + A_2t + A_3$; but since a constant is a solution of the homogeneous equation we must multiply by t. Thus $Y_1(t) = t(A_0t^3 + A_1t^2 + A_2t + A_3)$. For the second equation we first choose $Y_2(t) = Be^t$, but since both e^t and te^t are solutions of the homogeneous equation, we multiply by t^2 to obtain $Y_2(t) = Bt^2e^t$. Then $Y(t) = Y_1(t) + Y_2(t)$ by the superposition principle and $y(t) = y_c(t) + Y(t)$.

17. The complementary solution is $y_c(t) = c_1 + c_2e^{-t} + c_3e^t + c_4te^t$. The superposition principle allows us to consider separately the D.E. $y^{iv} - y''' - y'' + y' = t^2 + 4$ and $y^{iv} - y''' - y'' + y' = t\sin t$. For the first equation our initial choice is $Y_1(t) = A_0t^2 + A_1t + A_2$; but this must be multiplied by t since a constant is a solution of the homogeneous D.E. Hence $Y_1(t) = t(A_0t^2 + A_1t + A_2)$. For the second equation our initial choice that $Y_2 = (B_0t + B_1)\cos t + + (C_0t + C_1)\sin t$ does not need to be modified. Hence

$$Y(t) = t(A_0t^2 + A_1t + A_2) + (B_0t + B_1)\cos t + (C_0t + C_1)\sin t.$$

20. $(D-a)(D-b)f = (D-a)(Df-bf) = D^2f - (a+b)Df + abf$ and $(D-b)(D-a)f = (D-b)(Df-af) = D^2f - (b+a)Df + baf$. Since $a+b = b+a$ and $ab = ba$, we find the given equation holds for any function f.

22a. The D.E. of Problem 13 can be written as $D(D-1)^2y = t^3 + 2e^t$. Since D^4 annihilates t^3 and $(D-1)$ annihilates $2e^t$, we have $D^5(D-1)^3y = 0$, which corresponds to Eq.(ii) of Problem 21. The solution of this equation is $y(x) = A_1t^4 + A_2t^3 + A_3t^2 + A_4t + A_5 + (B_1t^2 + B_2t + B_3)e^{-t}$. Since $A_5 + (B_2t + B_3)e^{-t}$ are solutions of the homogeneous equation related to the original D.E., they may be deleted and thus $Y(t) = A_1t^4 + A_2t^3 + A_3t^2 + A_4t + B_1t^2e^{-t}$.

22b. $(D+1)^2(D^2+1)$ annihilates the right side of the D.E. of Problem 14.

22e. $D^3(D^2+1)^2$ annihilates the right side of the D.E. of Problem 17.

Section 4.4, Page 229

1. The complementary solution is $y_c = c_1 + c_2\cos t + c_3\sin t$ and thus we assume a particular solution of the form $Y = u_1(t) + u_2(t)\cos t + u_3(t)\sin t$. Differentiating and assuming Eq.(5), we obtain $Y' = -u_2\sin t + u_3\cos t$ and
$$u_1' + u_2'\cos t + u_3'\sin t = 0 \qquad \text{(a)}.$$
Continuing this process we obtain $Y'' = -u_2\cos t - u_3\sin t$, $Y''' = u_2\sin t - u_3\cos t - u_2'\cos t - u_3'\sin t$ and
$$-u_2'\sin t + u_3'\cos t = 0 \qquad \text{(b)}.$$
Substituting Y and its derivatives, as given above, into the D.E. we obtain the third equation:
$$-u_2'\cos t - u_3'\sin t = \tan t \qquad \text{(c)}.$$
Equations (a), (b) and (c) constitute Eqs.(10) of the text for this problem and may be solved to give $u_1' = \tan t$, $u_2' = -\sin t$, and $u_3' = -\sin^2 t/\cos t$. Thus $u_1 = -\ln\cos t$, $u_2 = \cos t$ and $u_3 = \sin t - \ln(\sec t + \tan t)$ and $Y = -\ln\cos t + 1 - (\sin t)\ln(\sec t + \tan t)$. Note that the constant 1 can be absorbed in c_1.

4. Replace $\tan t$ in Eq. (c) of Prob. 1 by $\sec t$ and use Eqs. (a) and (b) as in Prob. 1 to obtain $u_1' = \sec t$, $u_2' = -1$ and $u_3' = -\sin t/\cos t$.

5. Replace $\sec t$ in Problem 7 with $e^{-t}\sin t$.

7. Since e^t, $\cos t$ and $\sin t$ are solutions of the related homogenous equation we have $Y(t) = u_1e^t + u_2\cos t + u_3\sin t$. Eqs. (10) then are
$$u_1'e^t + u_2'\cos t + u_3'\sin t = 0$$
$$u_1'e^t - u_2'\sin t + u_3'\cos t = 0$$
$$u_1'e^t - u_2'\cos t - u_3'\sin t = \sec t.$$
Using Abel's identity, $W(t) = c\exp(-\int p_1(t)dt) = ce^t$.

Using the above equations, $W(0) = 2$, so $c = 2$ and $W(t) = 2e^t$. From Eq.(11), we have $u'_1(t) = \dfrac{\text{sect } W_1(t)}{2e^t}$,

where $W_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1$ and thus

$u'_1(t) = \frac{1}{2}e^{-t}/\cos t$. Likewise

$u'_2 = \dfrac{\text{sect } W_2(t)}{2e^t} = -\frac{1}{2}\text{sect}(\cos t - \sin t)$ and

$u'_3 = \dfrac{\text{sect } W_3(t)}{2e^t} = -\frac{1}{2}\text{sect}(\sin t + \cos t)$. Thus

$u_1 = \frac{1}{2}\int_{t_0}^{t} \frac{e^{-s}ds}{\cos s}$, $u_2 = -\frac{1}{2}t - \frac{1}{2}\ln(\cos t)$ and $u_3 = -\frac{1}{2}t + \frac{1}{2}\ln(\cos t)$ which, when substituted into the assumed form for Y, yields the desired solution.

11. Since the D.E. is the same as in Problem 7, we may use the complete solution from that, with $t_0 = 0$. Thus $y(0) = c_1 + c_2 = 2$, $y'(0) = c_1 + c_3 - \frac{1}{2} + \frac{1}{2} = -1$ and $y''(0) = c_1 - c_2 + \frac{1}{2} - 1 + \frac{1}{2} = 1$. Again, a computer algebra system may be used to yield the respective derivatives.

14. Since a fundamental set of solutions of the homogeneous D.E. is $y_1 = e^t$, $y_2 = \cos t$, $y_3 = \sin t$, a particular solution is of the form $Y(t) = e^t u_1(t) + (\cos t)u_2(t) + (\sin t)u_3(t)$. Differentiating and making the same assumptions that lead to Eqs.(10), we obtain

$$u'_1 e^t + u'_2\cos t + u'_3\sin t = 0$$
$$u'_1 e^t - u'_2\sin t + u'_3\cos t = 0$$
$$u'_1 e^t - u'_2\cos t - u'_3\sin t = g(t)$$

Solving these equations using either determinants or by elimination, we obtain $u'_1 = (1/2)e^{-t}g(t)$, $u'_2 = (1/2)(\sin t - \cos t)g(t)$, $u'_3 = -(1/2)(\sin t + \cos t)g(t)$. Integrating these and substituting into Y yields

$$Y(t) = \frac{1}{2}\{e^t\int_{t_0}^{t} e^{-s}g(s)ds + \cos t \int_{t_0}^{t}(\sin s - \cos s)g(s)ds$$
$$-\sin t\int_{t_0}^{t}(\sin s + \cos s)g(s)ds\}.$$

This can be written in the form

$$Y(t) = (1/2)\int_{t_0}^{t}(e^{t-s} + \cos t\sin s - \cos t\cos s$$
$$-\sin t\sin s - \sin t\cos s)g(s)ds.$$

If we use the trigonometric identities $\sin(A-B) = \sin A\cos B - \cos A\sin B$ and $\cos(A-B) = \cos A\cos B + \sin A\sin B$, we obtain the desired result. Note: Eqs.(11) and (12) of this section give the same result, but it is not recommended to memorize these equations.

16. The particular solution has the form $Y = e^t u_1(t) + te^t u_2(t) + t^2e^t u_3(t)$. Differentiating, making the same assumptions as in the earlier problems, and solving the three linear equations for u_1', u_2', and u_3' yields $u_1' = (1/2)t^2e^{-t}g(t)$, $u_2' = -te^{-t}g(t)$ and $u_3' = (1/2)e^{-t}g(t)$. Integrating and substituting into Y yields the desired solution. For instance

$$te^t u_2 = -te^t\int_{t_0}^{t} se^{-s}g(s)ds = -\frac{1}{2}\int_{t_0}^{t} 2tse^{(t-s)}g(s)ds, \text{ and}$$

likewise for u_1 and u_3. If $g(t) = t^{-2}e^t$ then $g(s) = e^s/s^2$ and the integration is accomplished using the power rule. Note that terms involving t_0 become part of the complimentary solution.

Capítulo 5

CHAPTER 5

Section 5.1, Page 237

2. Use the ratio test:

$$\lim_{n \to \infty} \frac{|(n+1)x^{n+1}/2^{n+1}|}{|nx^n/2^n|} = \lim_{n \to \infty} \frac{n+1}{n} \frac{1}{2} |x| = \frac{|x|}{2}.$$

Therefore the series converges absolutely for $|x| < 2$. For $x = 2$ and $x = -2$ the n^{th} term does not approach zero as $n \to \infty$ so the series diverge. Hence the radius of convergence is $\rho = 2$.

5. Use the ratio test:

$$\lim_{n \to \infty} \frac{|(2x+1)^{n+1}/(n+1)^2|}{|(2x+1)^n/n^2|} = \lim_{n \to \infty} \frac{n^2}{(n+1)^2}|2x+1| = |2x+1|.$$

Therefore the series converges absolutely for $|2x+1| < 1$, or $|x+1/2| < 1/2$. At $x = 0$ and $x = -1$ the series also converge absolutely. However, for $|x+1/2| > 1/2$ the series diverges by the ratio test. The radius of convergence is $\rho = 1/2$.

9. For this problem $f(x) = \sin x$. Hence $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x, \ldots$. Then $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1, \ldots$. The even terms in the series will vanish and the odd terms will alternate in sign. We obtain $\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$. From the ratio test it follows that $\rho = \infty$.

12. For this problem $f(x) = x^2$. Hence $f'(x) = 2x$, $f''(x) = 2$, and $f^{(n)}(x) = 0$ for $n > 2$. Then $f(-1) = 1$, $f'(-1) = -2$, $f''(-1) = 2$ and $x^2 = 1 - 2(x+1) + 2(x+1)^2/2! = 1 - 2(x+1) + (x+1)^2$. Since the series terminates after a finite number of terms, it converges for all x. Thus $\rho = \infty$.

13. For this problem $f(x) = \ln x$. Hence $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 1\cdot 2/x^3, \ldots$, and $f^{(n)}(x) = (-1)^{n+1}(n-1)!/x^n$. Then $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 1\cdot 2, \ldots$, $f^{(n)}(1) = (-1)^{n+1}(n-1)!$ The Taylor series is $\ln x = (x-1) - (x-1)^2/2 + (x-1)^3/3 - \ldots =$ $\sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^n/n$. It follows from the ratio test that

the series converges absolutely for $|x-1| < 1$. However, the series diverges at $x = 0$ so $\rho = 1$.

18. Writing the individual terms of y, we have
$y = a_0 + a_1x + a_2x^2 + \ldots + a_ax^n + \ldots$, so
$y' = a_1 + 2a_2x + 3a_3x^2 + \ldots + (n+1)a_{n+1}x^n + \ldots$, and
$y'' = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \ldots + (n+2)(n+1)a_{n+2}x^n + \ldots$.
If $y'' = y$, we then equate coefficients of like powers of x to obtain $2a_2 = a_0$, $3\cdot 2a_3 = a_1$, $4\cdot 3a_4 = a_2$, ... $(n+2)(n+1)a_{n+2} = a_n$, which yields the desired result for $n = 0,1,2,3\ldots$.

19. Set $m = n-1$ on the right hand side of the equation. Then $n = m+1$ and when $n = 1$, $m = 0$. Thus the right hand side becomes $\sum_{m=0}^{\infty} a_m(x-1)^{m+1}$, which is the same as the left hand side when m is replaced by n.

23. Multiplying each term of the first series by x yields
$$x\sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=1}^{\infty} na_nx^n = \sum_{n=0}^{\infty} na_nx^n,$$ where the last equality can be verified by writing out the first few terms. Changing the index from k to n (n=k) in the second series then yields
$$\sum_{n=0}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty}(n+1)a_nx^n.$$

25. $$\sum_{m=2}^{\infty} m(m-1)a_mx^{m-2} + x\sum_{k=1}^{\infty} ka_kx^{k-1} =$$
$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{k=1}^{\infty} ka_kx^k =$$
$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + na_n]x^n.$$ In the first case we have let $n = m - 2$ in the first summation and multiplied each term of the second summation by x. In the second case we have let $n = k$ and noted that for $n = 0$, $na_n = 0$.

28. If we shift the index of summation in the first sum by

letting m = n-1, we have

$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1}x^m$. Substituting this into the given equation and letting m = n again, we obtain:

$$\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0, \text{ or}$$

$$\sum_{n=0}^{\infty} [(n+1)a_{n+1} + 2a_n]x^n = 0.$$

Hence $a_{n+1} = -2a_n/(n+1)$ for n = 0,1,2,3,... . Thus $a_1 = -2a_0$, $a_2 = -2a_1/2 = 2^2a_0/2$, $a_3 = -2a_2/3 = -2^3a_0/2\cdot3 = -2^3a_0/3!$... and $a_n = (-1)^n2^na_0/n!$. Notice that for n = 0 this formula reduces to a_0 so we can write

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n 2^n a_0 x^n/n! = a_0\sum_{n=0}^{\infty} (-2x)^n/n! = a_0 e^{-2x}.$$

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2. $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ and since we must multiply y' by x in the D.E. we do not shift the index; and

$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$. Substituting in the D.E., we obtain

$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$. In order to have the starting point the same in all three summations, we let n = 0 in the first and third terms to obtain the following

$$(2\cdot1\ a_2 - a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)a_n]x^n = 0.$$

Thus $a_{n+2} = a_n/(n+2)$ for n = 1,2,3,... . Note that the recurrence relation is also correct for n = 0. We show how to calculate the odd a's:

$a_3 = a_1/3$, $a_5 = a_3/5 = a_1/5\cdot3$, $a_7 = a_5/7 = a_1/7\cdot5\cdot3, \ldots$.
Now notice that $a_3 = 2a_1/(2\cdot3) = 2a_1/3!$, that
$a_5 = 2\cdot4a_1/(2\cdot3\cdot4\cdot5) = 2^2\cdot2a_1/5!$, and that
$a_7 = 2\cdot4\cdot6a_1/(2\cdot3\cdot4\cdot5\cdot6\cdot7) = 2^3\cdot3!\ a_1/7!$. Likewise
$a_9 = a_7/9 = 2^3\cdot3!\ a_1/(7!)9 = 2^3\cdot3!\ 8a_1/9! = 2^4\cdot4!\ a_1/9!$.
Continuing we have $a_{2m+1} = 2^m\cdot m!\ a_1/(2m+1)!$. In the same way we find that the even a's are given by $a_{2m} = a_0/2^m\ m!$.
Thus

$$y = a_0\sum_{m=0}^{\infty}\frac{x^{2m}}{2^m m!} + a_1\sum_{m=0}^{\infty}\frac{2^m m!\ x^{2m+1}}{(2m+1)!}.$$

3. $y = \sum_{n=0}^{\infty}a_n(x-1)^n$; $y' = \sum_{n=1}^{\infty}na_n(x-1)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$,
and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n.$$

Substituting in the D.E. and setting $x = 1 + (x-1)$ we obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n - \sum_{n=1}^{\infty}na_n(x-1)^n$$
$$- \sum_{n=0}^{\infty}a_n(x-1)^n = 0,$$

where the third term comes from:

$$-(x-1)y' = \sum_{n=1}^{\infty}(n+1)a_{n+1}(x-1)^{n+1} = -\sum_{n=1}^{\infty}na_n(x-1)^n.$$

Letting $n = 0$ in the first, second, and the fourth sums, we obtain

$$(2\cdot1\cdot a_2 - 1\cdot a_1 - a_0)(x-1)^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2}$$
$$- (n+1)a_{n+1} - (n+1)a_n](x-1)^n = 0.$$

Thus $(n+2)a_{n+2} - a_{n+1} - a_n = 0$ for $n = 0,1,2,\ldots$. This recurrance relation can be used to solve for a_2 in terms of a_0 and a_1, then for a_3 in terms of a_0 and a_1, etc. In many cases it is easier to first take $a_0 = 0$ and generate one solution and then take $a_1 = 0$ and generate the second linearly independent solution. Thus, choosing $a_0 = 0$ we find that $a_2 = a_1/2$, $a_3 = (a_2+a_1)/3 = a_1/2$, $a_4 = (a_3+a_2)/4 = a_1/4$, $a_5 = (a_4+a_3)/5 = 3a_1/20, \ldots$. This yields the solution $y_2(x) = a_1[(x-1) + (x-1)^2/2 + (x-1)^3/2 + (x-1)^4/4 + 3(x-1)^5/20 + \ldots]$. The second independent solution may be obtained by choosing $a_1 = 0$. Then $a_2 = a_0/2$, $a_3 = (a_2+a_1)/3 = a_0/6$, $a_4 = (a_3+a_2)/4 = a_0/6$, $a_5 = (a_4+a_3)/5 = a_0/15, \ldots$. This yields the solution $y_1(x) = a_0[1+(x-1)^2/2+(x-1)^3/6+(x-1)^4/6+(x-1)^5/15+\ldots]$.

5. $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$; and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

Substituting in the D.E. and shifting the index in both summations for y'' gives

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n+1)n\, a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n =$$

$$(2\cdot1\cdot a_2 + a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n]x^n = 0.$$

Thus $a_2 = -a_0/2$ and $a_{n+2} = na_{n+1}/(n+2) - a_n/(n+2)(n+1)$, $n = 1,2,\ldots$. Choosing $a_0 = 0$ yields $a_2 = 0$, $a_3 = -a_1/6$, $a_4 = 2a_3/4 = -a_1/12, \ldots$ which gives one solution as $y_2(x) = a_1(x - x^3/6 - x^4/12 + \ldots)$. A second linearly independent solution is obtained by choosing $a_1 = 0$. Then $a_2 = -a_0/2$, $a_3 = a_2/3 = -a_0/6$, $a_4 = 2a_3/4 - a_2/12 = -a_0/24, \ldots$ which gives $y_1(x) = a_0(1 - x^2/2 - x^3/6 - x^4/24 + \ldots)$.

8. If $y = \sum_{n=1}^{\infty} a_n(x-1)^n$ then

$$xy = [1+(x-1)]y = \sum_{n=1}^{\infty} a_n(x-1)^n + \sum_{n=1}^{\infty} a_n(x-1)^{n+1},$$

$$y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1}, \text{ and}$$

$xy'' = [1+(x-1)]y''$

$$= \sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-1}.$$

14. You will need to rewrite x+1 as 3 + (x-2) in order to multiply x+1 times y' as a power series about $x_0 = 2$.

16a. From Problem 6 we have

$$y(x) = c_1(1 - x^2 + \frac{1}{6}x^4+\ldots) + c_2(x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \ldots).$$

Now $y(0) = c_1 = -1$ and $y'(0) = c_2 = 3$ and thus

$$y(x) = -1+x^2-\frac{1}{6}x^4+3x-\frac{3}{4}x^3 = -1+3x+x^2-\frac{3}{4}x^3-\frac{1}{6}x^4 + \ldots .$$

16c. By plotting $f = -1 + 3x + x^2 - 3x^3/4$ and $g = f - x^4/6$ between -1 and 1 it appears that f is a reasonable approximation for $|x| < 0.7$.

19. The D.E. transforms into $u''(t) + t^2u'(t) + (t^2+2t)u(t) = 0$. Assuming that $u(t) = \sum_{n=0}^{\infty} a_nt^n$, we have $u'(t) = \sum_{n=1}^{\infty} na_nt^{n-1}$ and $u''(t) = \sum_{n=2}^{\infty} n(n-1)a_nt^{n-2}$. Substituting in the D.E. and shifting indices yields

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty} (n-1)a_{n-1}t^n + \sum_{n=2}^{\infty} a_{n-2}t^n + \sum_{n=1}^{\infty} 2a_{n-1}t^n = 0,$$

$$2\cdot1\cdot a_2t^0 + (3\cdot2\cdot a_3 + 2\cdot a_0)t^1 + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} + a_{n-2}]t^n = 0.$$

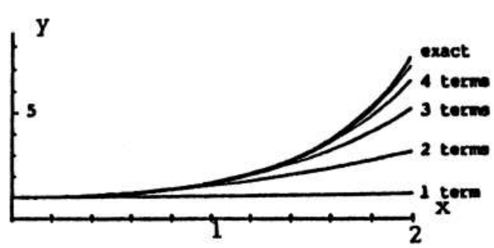
It follows that $a_2 = 0$, $a_3 = -a_0/3$ and $a_{n+2} = -a_{n-1}/(n+2) - a_{n-2}/[(n+2)(n+1)]$, $n = 2,3,4\ldots$. We obtain one solution by choosing $a_1 = 0$. Then $a_4 = -a_0/12$, $a_5 = -a_2/5 -a_1/20 = 0$, $a_6 = -a_3/6 - a_2/30 = a_0/18,\ldots$. Thus one solution is $u_1(t) = a_0(1 - t^3/3 - t^4/12 + t^6/18 + \ldots)$ so

$$y_1(x) = u_1(x-1) = a_0[1 - (x-1)^3/3 - (x-1)^4/12 + (x-1)^6/18 + \ldots].$$

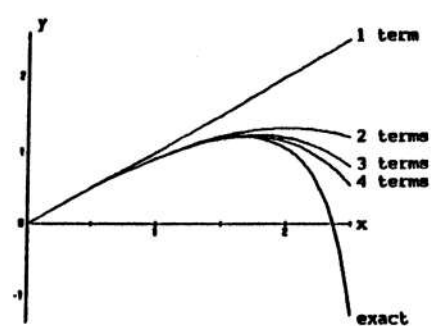
We obtain a second solution by choosing $a_0 = 0$. Then
$a_4 = -a_1/4$, $a_5 = -a_2/5 - a_1/20 = -a_1/20$,
$a_6 = -a_3/6 - a_2/30 = 0$, $a_7 = -a_4/7 - a_3/42 = a_1/28, \ldots$.
Thus a second linearly independent solution is
$u_2(t) = a_1[t - t^4/4 - t^5/20 + t^7/28 + \ldots]$ or
$y_2(x) = u_2(x-1)$
$\quad = a_1[(x-1) - (x-1)^4/4 - (x-1)^5/20 + (x-1)^7/28 + \ldots]$.
The Taylor series for $x^2 - 1$ about $x = 1$ may be obtained by writing $x = (x-1) + 1$ so $x^2 = (x-1)^2 + 2(x-1) + 1$ and $x^2 - 1 = (x-1)^2 + 2(x-1)$. The D.E. now appears as $y'' + (x-1)^2y' + [(x-1)^2 + 2(x-1)]y = 0$ which is identical to the transformed equation with $t = x - 1$.

22b. $y = a_0 + a_1x + a_2x^2 + \ldots$, $y^2 = a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)x^2 + \ldots$, $y' = a_1 + 2a_2x + 3a_3x^2 + \ldots$, and $(y')^2 = a_1^2 + 4a_1a_2x + (6a_1a_3 + 4a_2^{\,2})x^2 + \ldots$. Substituting these into $(y')^2 = 1 - y^2$ and collecting coefficients of like powers of x yields $(a_1^2 + a_0^2 - 1) + (4a_1a_2 + 2a_0a_1)x + (6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_1^2)x^2 + \ldots = 0$. As in the earlier problems, each coefficient must be zero. The I.C. $y(0) = 0$ requires that $a_0 = 0$, and thus $a_1^2 + a_0^2 - 1 = 0$ gives $a_1^2 = 1$. However, the D.E.indicates that y' is always positive, so $y'(0) = a_1 > 0$ implies $a_1 = 1$. Then $4a_1a_2 + 2a_0a_1 = 0$ implies that $a_2 = 0$; and $6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_1^2 = 6a_1a_3 + a_1^2 = 0$ implies that $a_3 = -1/6$. Thus $y = x - x^3/3! + \ldots$.

23.



26.



26. We have $y(x) = a_0y_1 + a_1y_2$, where y_1 and y_2 are found in

Problem 10. Now $y(0) = a_0 = 0$ and $y'(0) = a_1 = 1$. Thus
$y(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240}$.

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1. The D.E. can be solved for y'' to yield $y'' = -xy' - y$. If $y = \phi(x)$ is a solution, then $\phi''(x) = -x\phi'(x) - \phi(x)$ and thus setting $x = 0$ we obtain $\phi''(0) = -0 - 1 = -1$. Differentiating the equation for y'' yields $y''' = -xy'' - 2y'$ and hence setting $y = \phi(x)$ again yields $\phi'''(0) = -0 - 0 = 0$. In a similar fashion $y^{iv} = -xy''' - 3y''$ and thus $\phi^{iv}(0) = -0 - 3(-1) = 3$. The process can be continued to calculate higher derivatives of $\phi(x)$.

6. The zeros of $P(x) = x^2 - 2x - 3$ are $x = -1$ and $x = 3$. For $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$ the distance to the nearest zero of $P(x)$ is 1,3, and 1, respectively. Thus a lower bound for the radius of convergence for series solutions in powers of $(x-4)$, $(x+4)$, and x is $\rho = 1$, $\rho = 3$, and $\rho = 1$, respectively.

9a. Since $P(x) = 1$ has no zeros, the radius of convergence for $x_0 = 0$ is $\rho = \infty$.

9f. Since $P(x) = x^2 + 2$ has zeros at $x = \pm\sqrt{2}\,i$, the lower bound for the radius of convergence of the series solution about $x_0 = 0$ is $\rho = \sqrt{2}$.

9h. Since $x_0 = 1$ and $P(x) = x$ has a zero at $x = 0$, $\rho = 1$.

10a. If we assume that $y = \sum_{n=2}^{\infty} a_n x^n$, then $y' = \sum_{n=2}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Substituting in the D.E., shifting indices of summation, and collecting coefficients of like powers of x yields the equation
$(2\cdot1\cdot a_2 + \alpha^2 a_0)x^0 + [3\cdot2\cdot a_3 + (\alpha^2-1)a_1]x^1$

$$+\sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (\alpha^2-n^2)a_n]x^n = 0.$$

Hence the recurrence relation is
$a_{n+2} = (n^2-\alpha^2)a_n/(n+2)(n+1)$, $n = 0, 1,2,\ldots$. For the first solution we choose $a_1 = 0$. We find that
$a_2 = -\alpha^2 a_0/2\cdot1$, $a_3 = 0$, $a_4 = (2^2-\alpha^2)a_2/4\cdot3 = -(2^2-\alpha^2)\alpha^2 a_0/4!$
$\ldots$, $a_{2m} = -[(2m-2)^2 - \alpha^2]\ldots(2^2-\alpha^2)\alpha^2 a_0/(2m)!$,
and $a_{2m+1} = 0$, so $y_1(x) = 1 - \frac{\alpha^2}{2!}x^2 - \frac{(2^2-\alpha^2)\alpha^2}{4!}x^4 - \ldots$

$$- \frac{[(2m-2)^2-\alpha^2]\ldots(2^2-\alpha^2)\alpha^2}{(2m)!}x^{2m} - \ldots,$$

where we have set $a_0 = 1$. For the second solution we take $a_0 = 0$ and $a_1 = 1$ in the recurrence relation to obtain the desired solution.

10b. If α is an even integer $2k$ then $(2m-2)^2 - \alpha^2 = (2m-2)^2 - 4k^2 = 0$. Thus when $m = k+1$ all terms in the series for $y_1(x)$ are zero after the x^{2k} term. A similar argument shows that if $\alpha = 2k+1$ then all terms in $y_2(x)$ are zero after the x^{2k+1}.

11. The Taylor series about $x = 0$ for sinx is
$\sin x = x - x^3/3! + x^5/5! - \ldots$. Assuming that
$y = \sum_{n=2}^{\infty} a_n x^n$ we find $y'' + (\sin x)y = 2a_2 + 6a_3 x + 12a_4 x^2$
$+ 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \ldots$
$+ (x-x^3/3!+x^5/5!-\ldots)(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots)$
$= 2a_2 + (6a_3+a_0)x + (12a_4+a_1)x^2 + (20a_5+a_2-a_0/6)x^3 +$
$(30a_6+ a_3-a_1/6)x^4 + (42a_7+a_4+a_0/5!)x^5 + \ldots = 0$. Hence
$a_2 = 0$, $a_3 = -a_0/6$, $a_4 = -a_1/12$, $a_5 = a_0/120$,
$a_6 = (a_1+a_0)/180$, $a_7 = -a_0/7! + a_1/504$, $\ldots$. We set
$a_0 = 1$ and $a_1 = 0$ and obtain
$y_1(x) = (1 - x^3/6 + x^5/120 + x^6/180 + \ldots)$. Next we set
$a_0 = 0$ and $a_1 = 1$ and obtain
$y_2(x) = (x - x^4/12 + x^6/180 + x^7/504 +\ldots)$. Since

$p(x) = 1$ and $q(x) = \sin x$ both have $\rho = \infty$, the solution in this case converges for all x, that is, $\rho = \infty$

18. We know that $e^x = 1 + x + x^2/2! + X^3/3! + \ldots$, and therefore $e^{x^2} = 1 + x^2 + x^4/2! + x^6/3! + \ldots$. Hence, if $y = \Sigma a_n x^n$, we have
$a_1 + 2a_2x + 3a_3x^2 + \ldots = (1+x^2 + x^4/2+\ldots)(a_0+a_1x+a_2x^2+\ldots)$
$= a_0 + a_1x + (a_0+a_2)x^2 + \ldots$:
Thus, $a_1 = a_0$ $2a_2 = a_1$ and $3a_3 = a_0 + a_2$, which yield the desired solution.

20. Substituting $y = \sum_{n=2}^{\infty} a_n x^n$ into the D.E. we obtain
$\sum_{n=2}^{\infty} na_n x^{n-1} - \sum_{n=2}^{\infty} a_n x^n = x^2$. Shifting indices in the summation yields $\sum_{n=2}^{\infty}[(n+1)a_{n+1} - a_n]x^n = x^2$. Equating coefficients of both sides then gives: $a_1 - a_0 = 0$, $2a_2 - a_1 = 0$, $3a_3 - a_2 = 1$ and $(n+1)a_{n+1} = a_n$ for $n = 3,4,\ldots$. Thus $a_1 = a_0$, $a_2 = a_1/2 = a_0/2$, $a_3 = 1/3 + a_2/3 = 1/3 + a_0/2\cdot3$, $a_4 = a_3/4 = 1/3\cdot4 + a_0/2\cdot3\cdot4$, ..., $a_n = a_{n-1}/n = 2/n! + a_0/n!$ and hence

$$y(x) = a_0\left(1 + x + \frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\ldots\right) + 2\left(\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^n}{n!}+\ldots\right).$$

Using the power series for e^x, the first and second sums can be rewritten as $a_0e^x + 2(e^x - 1 - x - x^2/2)$.

22. Substituting $y = \sum_{n=2}^{\infty} a_n x^n$ into the Legendre equation, shifting indices, and collecting coefficients of like powers of x yields
$[2\cdot1\cdot a_2 + \alpha(\alpha+1)a_0]x^0 + \{3\cdot2\cdot a_3 - [2\cdot1 - \alpha(\alpha+1)]a_1\}x^1 +$
$\sum_{n=2}^{\infty}\{(n+2)(n+1)a_{n+2} - [n(n+1) - \alpha(\alpha+1)]a_n\}x^n = 0$. Thus
$a_2 = -\alpha(\alpha+1)a_0/2!$, $a_3 = [2\cdot1 - \alpha(\alpha+1)]a_1/3! =$
$-(\alpha-1)(\alpha+2)a_1/3!$ and the recurrence relation is

$(n+2)(n+1)a_{n+2} = -[\alpha(\alpha+1) - n(n+1)]a_n = -(\alpha-n)(\alpha+n+1)a_n$, $n = 2,3,\ldots$. Setting $a_1 = 0$, $a_0 = 1$ yields a solution with $a_3 = a_5 = a_7 = \ldots = 0$ and

$a_4 = \alpha(\alpha-2)(\alpha+1)(\alpha+3)/4!,\ldots$, $a_{2m} = (-1)^m\alpha(\alpha-2)(\alpha-4)\ldots(\alpha-2m+2)(\alpha+1)(\alpha+3)\ldots(\alpha+2m-1)/(2m)!,\ldots$. The second linearly independent solution is obtained by setting $a_0 = 0$ and $a_1 = 1$. The coefficients are $a_2 = a_4 = a_6 = \ldots = 0$ and $a_3 = -(\alpha-1)(\alpha+2)/3!$, $a_5 = -(\alpha-3)(\alpha+4)a_3/5\cdot4 = (\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)/5!,\ldots$.

26. Using the chain rule we have:

$$\frac{dF(\phi)}{d\phi} = \frac{dF[\phi(x)]}{dx}\frac{dx}{d\phi} = -f'(x)\sin\phi(x) = -f'(x)\sqrt{1-x^2},$$

$$\frac{d^2F(\phi)}{d\phi^2} = \frac{d}{dx}[-f'(x)\sqrt{1-x^2}]\frac{dx}{d\phi} = (1-x^2)f''(x) - xf'(x),$$

which when substituted into the D.E. yields the desired result.

28. Since $[(1-x^2)y']' = (1-x^2)y'' - 2xy'$, the Legendre Equation, from Problem 22, can be written as shown. Thus, carrying out the steps indicated yields the two equations:

$$P_m[(1-x^2)P_n']' = -n(n+1)P_nP_m$$

$$P_n[(1-x^2)P_m']' = -m(m+1)P_nP_m.$$

As long as $n \neq m$ the second equation can be subtracted from the first and the result integrated from -1 to 1 to obtain

$$\int_{-1}^{1}\{P_m[(1-x^2)P_n']'-P_n[(1-x^2)P_m']'\}dx = [m(m+1)-n(n+1)]\int_{-1}^{1}P_nP_m dx$$

The left side may be integrated by parts to yield

$$[P_m(1-x^2)P_n' - P_n(1-x^2)P_m']_{-1}^{1} + \int_{-1}^{1}[P_m'(1-x^2)P_n' - P_n'(1-x^2)P_m']dx,$$

which is zero. Thus $\int_{-1}^{1}P_n(x)P_m(x)dx = 0$ for $n \neq m$.

Section 5.4, Page 259

1. Since the coefficients of y, y' and y'' have no common factors and since $P(x)$ vanishes only at $x = 0$ we conclude that $x = 0$ is a singular point. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we obtain $p(x) = (1-x)/x$ and $q(x) = 1$. Thus for the singular point we have

$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} 1-x = 1$, $\lim_{x\to 0} x^2q(x) = 0$ and thus $x = 0$ is a regular singular point.

5. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we find $p(x) = x/(1-x)(1+x)^2$ and $q(x) = 1/(1-x^2)(1+x)$. Therefore $x = \pm 1$ are singular points. Since $\lim_{x\to 1} (x-1)p(x)$ and $\lim_{x\to 1} (x-1)^2q(x)$ both exist, we conclude $x = 1$ is a regular singular point. Finally, since $\lim_{x\to -1} (x+1)p(x)$ does not exist, we find that $x = -1$ is an irregular singular point.

12. Writing the D.E. in the form $y + p(x)y' + q(x)y = 0$, we see that $p(x) = e^x/x$ and $q(x) = (3\cos x)/x$. Thus $x = 0$ is a singular point. Since $xp(x) = e^x$ is analytic at $x = 0$ and $x^2q(x) = 3x\cos x$ is analytic at $x = 0$ the point $x = 0$ is a regular singular point.

17. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we see that $p(x) = \dfrac{x}{\sin x}$ and $q(x) = \dfrac{4}{\sin x}$. Since $\lim_{x\to 0} q(x)$ does not exist, the point $x_0 = 0$ is a singular point and since neither $\lim_{x\to\pm n\pi} p(x)$ nor $\lim_{x\to\pm n\pi} q(x)$ exist either the points $x_0 = \pm n\pi$ are also singular points. To determine whether the singular points are regular or irregular we must use Eq.(8) and the result #7 of multiplication and division of power series from Section 5.1. For $x_0 = 0$, we have

$$xp(x) = \frac{x^2}{\sin x} = \frac{x^2}{x-\frac{x^3}{6}+\ldots} = x[1 + \frac{x^2}{6} + \ldots]$$

$$= x + \frac{x^3}{6} + \ldots,$$

which converges about $x_0 = 0$ and thus $xp(x)$ is analytic at $x_0 = 0$. $x^2q(x)$, by similar steps, is also analytic at $x_0 = 0$ and thus $x_0 = 0$ is a regular singular point. For $x_0 = n\pi$, we have

$$(x-n\pi)p(x) = \frac{(x-n\pi)x}{\sin x} = \frac{(x-n\pi)[(x-n\pi) + n\pi]}{\pm(x-n\pi)\mp\frac{(x-n\pi)^3}{6} \pm \ldots}$$

$$= [(x-n\pi)+n\pi][\pm 1 \pm \frac{(x-n\pi)^2}{6} \pm \ldots],$$ which

converges about $x_0 = n\pi$ and thus $(x-n\pi)p(x)$ is analytic at $x = n\pi$. Similarly $(x+n\pi)p(x)$ and $(x\pm n\pi)^2 q(x)$ are analytic and thus $x_0 = \pm n\pi$ are regular singular points.

19. Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the D.E. yields

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + 3\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$ The last sum becomes $\sum_{n=2}^{\infty} a_{n-2}x^{n-1}$ by replacing n+1 by n-1, the first term of the middle sum is $3a_1$, and thus we have

$$3a_1 + \sum_{n=2}^{\infty}\{[2n(n-1)+3n]a_n + a_{n-2}\}x^{n-1} = 0.$$ Hence $a_1 = 0$ and

$a_n = \dfrac{-a_{n-2}}{n(2n+1)}$, which is the desired recurrance relation. Thus all even coefficients are found in terms of a_0 and all odd coefficients are zero, thereby yielding only one solution of the desired form.

21. If $\xi = 1/x$ then

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = -\frac{1}{x^2}\frac{dy}{d\xi} = -\xi^2\frac{dy}{d\xi},$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\xi}(-\xi^2\frac{dy}{d\xi})\frac{d\xi}{dx} = (-2\xi\frac{dy}{d\xi} - \xi^2\frac{d^2y}{d\xi^2})(-\frac{1}{x^2})$$

$$= \xi^4\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi}.$$

Substituting in the D.E. we have

$$P(1/\xi)[\xi^4\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi}] + Q(1/\xi)[-\xi^2\frac{dy}{d\xi}] + R(1/\xi)y = 0,$$

$$\xi^4 P(1/\xi)\frac{d^2y}{d\xi^2} + [2\xi^3 P(1/\xi) - \xi^2 Q(1/\xi)]\frac{dy}{d\xi} + R(1/\xi)y = 0.$$

The result then follows from the theory of singular points at $\xi = 0$.

23. Since $P(x) = x^2$, $Q(x) = x$ and $R(x) = -4$ we have

$f(\xi) = [2P(1/\xi)/\xi - Q(1/\xi)/\xi^2]/P(1/\xi) = 2/\xi - 1/\xi = 1/\xi$ and $g(\xi) = R(1/\xi)/\xi^4P(1/\xi) = -4/\xi^2$. Thus the point at infinity is a singular point. Since both $\xi f(\xi)$ and $\xi^2g(\xi)$ are analytic at $\xi = 0$, the point at infinity is a regular singular point.

25. Since $P(x) = x^2$, $Q(x) = x$, and $R(x) = x^2 - \upsilon^2$, $f(\xi) = [2P(1/\xi)/\xi - Q(1/\xi)/\xi^2]/P(1/\xi) = 2/\xi - 1/\xi = 1/\xi$ and $g(\xi) = R(1/\xi)/\xi^4P(1/\xi) = (1/\xi^2 - \upsilon^2)/\xi^2 = 1/\xi^4 - \upsilon^2/\xi^2$. Thus the point at infinity is a singular point. Although $\xi f(\xi) = 1$ is analytic at $\xi = 0$, $\xi^2g(\xi) = 1/\xi^2 - \upsilon^2$ is not, so the point at infinity is an irregular singular point.

Section 5.5, Page 265

2. Comparing the D.E. to Eq.(27), we seek solutions of the form $y = (x+1)^r$ for $x + 1 > 0$. Substitution of y into the D.E. yields $[r(r-1) + 3r + 3/4](x+1)^r = 0$. Thus $r^2 + 2r + 3/4 = 0$, which yields $r = -3/2, -1/2$. The general solution of the D.E. is then
$y = c_1|x+1|^{-1/2} + c_2|x+1|^{-3/2}$, $x \neq -1$.

4. If $y = x^r$ then $r(r-1) + 3r + 5 = 0$. So $r^2 + 2r + 5 = 0$ and $r = (-2 \pm \sqrt{4-20})/2 = -1 \pm 2i$. Thus the general solution of the D.E. is
$y = c_1x^{-1}\cos(2\ln|x|) + c_2x^{-1}\sin(2\ln|x|)$, $x \neq 0$.

9. Again let $y = x^r$ to obtain $r(r-1) - 5r + 9 = 0$, or $(r-3)^2 = 0$. Thus the roots are $x = 3,3$ and $y = c_1x^3 + c_2x^3\ln|x|$, $x \neq 0$, is the solution of the D.E.

13. If $y = x^r$, then $F(r) = 2r(r-1) + r - 3 = 2r^2 - r - 3 = (2r-3)(r+1) = 0$, so $y = c_1x^{3/2} + c_2x^{-1}$ and $y' = \frac{3}{2}c_1x^{1/2} - c_2x^{-2}$. Setting $x = 1$ in y and y′ we obtain $c_1 + c_2 = 1$ and $\frac{3}{2}c_1 - c_2 = 4$, which yield $c_1 = 2$ and $c_2 = -1$. Hence $y = 2x^{3/2} - x^{-1}$. As $x \to 0^+$ we have $y \to -\infty$ due to the second term.

16. We have $F(r) = r(r-1) + 3r + 5 = r^2 + 2r + 5 = 0$. Thus

$r_1, r_2 = -1 \pm 2i$ and $y = x^{-1}[c_1\cos(2\ln x) + c_2\sin(2\ln x)]$.
Then $y(1) = c_1 = 1$ and $y' = -x^{-2}[\cos(2\ln x) + c_2\sin(2\ln x)]$ $+ x^{-1}[-\sin(2\ln x)2/x + c_2\cos(2\ln x)2/x]$ so that $y'(1) = -1-2c_2 = -1$, or $c_2 = 0$.

17. Substituting $y = x^r$, we find that $r(r-1) + \alpha r + 5/2 = 0$ or $r^2 + (\alpha-1)r + 5/2 = 0$. Thus $r_1, r_2 = [-(\alpha-1) \pm \sqrt{(\alpha-1)^2-10}]/2$. In order for solutions to approach zero as $x\to 0$ it is necessary that the real parts of r_1 and r_2 be positive. Suppose that $\alpha > 1$, then $\sqrt{(\alpha-1)^2-10}$ is either imaginary or real and less than $\alpha - 1$; hence the real parts of r_1 and r_2 will be negative. Suppose that $\alpha = 1$, then $r_1, r_2 = \pm i\sqrt{10}$ and the solutions are oscillatory. Suppose that $\alpha < 1$, then $\sqrt{(\alpha-1)^2-10}$ is either imaginary or real and less than $|\alpha-1| = 1 - \alpha$; hence the real parts of r_1 and r_2 will be positive. Thus if $\alpha < 1$ the solutions of the D.E. will approach zero as $x\to 0$.

21. In all cases the roots of $F(r) = 0$ are given by Eq.(5) and the forms of the solution are given in Theorem 5.5.1.

21a. The real part of the root must be positive so, from Eq.(5), $\alpha < 0$. Also $\beta > 0$, since the $\sqrt{(\alpha-1)^2-4\beta}$ term must be less than $|\alpha-1|$.

22. Assume that $y = v(x)x^{r_1}$. Then $y' = v(x)r_1x^{r_1-1} + v'(x)x^{r_1}$ and $y'' = v(x)r_1(r_1-1)x^{r_1-2} + 2v'(x)r_1x^{r_1-1} + v''(x)x^{r_1}$. Substituting in the D.E. and collecting terms yields $x^{r_1+2}v'' + (\alpha + 2r_1)x^{r_1+1}v' + [r_1(r_1-1) + \alpha r_1 + \beta]x^{r_1}v = 0$. Now we make use of the fact that r_1 is a double root of $f(r) = r(r-1) + \alpha r + \beta$. This means that $f(r_1) = 0$ and $f'(r_1) = 2r_1 - 1 + \alpha = 0$. Hence the D.E. for v reduces to $x^{r_1+2}v'' + x^{r_1+1}v'$. Since $x > 0$ we may divide by x^{r_1+1} to obtain $xv'' + v' = 0$. Thus $v(x) = \ln x$ and a second solution is $y = x^{r_1}\ln x$.

25. The change of variable $x = e^z$ transforms the D.E. into $u'' - 4u' + 4u = z$, which has the solution

$u(z) = c_1e^{2z} + c_2ze^{2z} + (1/4)z + 1/4$. Hence
$y(x) = c_1x^2 + c_2x^2\ln x + (1/4)\ln x + 1/4$.

31. If $x > 0$, then $|x| = x$ and $|x|^{r_1} = x^{r_1}$ so we can choose $c_1 = k_1$. If $x < 0$, then $|x| = -x$ and $|x|^{r_1} = (-x)^{r_1} = (-1)^{r_1}x^{r_1}$ and we can choose $c_1 = (-1)^{r_1}k_1$, or $k_1 = (-1)^{-r_1}c_1 = (-1)^{r_1}c_1$. In both cases we have $c_2 = k_2$.

Section 5.6, Page 271

2. If the D.E. is put in the standard form $y'' + p(x)y + q(x)y = 0$, then $p(x) = x^{-1}$ and $q(x) = 1 - 1/9x^2$. Thus $x = 0$ is a singular point. Since $xp(x) \to 1$ and $x^2q(x) \to -1/9$ as $x \to 0$ it follows that $x = 0$ is a regular singular point. In determining a series solution of the D.E. it is more convenient to leave the equation in the form given rather than divide by the x^2, the coefficient of y''. If we substitute $y = \sum_{n=0}^{\infty} a_nx^{n+r}$, we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_nx^{n+r} + \sum_{n=0}^{\infty} (n+r)a_nx^{n+r} + (x^2 - \frac{1}{9})\sum_{n=0}^{\infty} a_nx^{n+r} = 0.$$

Note that $x^2 \sum_{n=0}^{\infty} a_nx^{n+r} = \sum_{n=0}^{\infty} a_nx^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2}x^{n+r}$. Thus we have $[r(r-1) + r - \frac{1}{9}]a_0x^r + [(r+1)r + (r+1) - \frac{1}{9}]a_1x^{r+1} +$
$\sum_{n=2}^{\infty} \{[(n+r)(n+r-1) + (n+r) - \frac{1}{9}]a_n + a_{n-2}\} x^{n+r} = 0$. From the first term, the indicial equation is $r^2 - 1/9 = 0$ with roots $r_1 = 1/3$ and $r_2 = - 1/3$. For either value of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation is $[(n+r)^2 - 1/9]a_n = -a_{n-2}$. For $r = 1/3$ we have

$$a_n = \frac{-a_{n-2}}{(n + \frac{1}{3})^2-(\frac{1}{3})^2} = - \frac{a_{n-2}}{(n + \frac{2}{3})n}, \; n = 2,3,4,\ldots .$$

Since $a_1 = 0$ it follows from the recurrence relation that $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients it is convenient to let $n = 2m$, $m = 1,2,3,\ldots$. Then $a_{2m} = -a_{2m-2}/2^2m(m + \frac{1}{3})$. The first few coefficients are given by

$$a_2 = \frac{(-1)a_0}{2^2(1 + \frac{1}{3})1},\ a_4 = \frac{(-1)a_2}{2^2(2 + \frac{1}{3})2} = \frac{a_0}{2^4(1 + \frac{1}{3})(2 + \frac{1}{3})2!}$$

$$a_6 = \frac{(-1)a_4}{2^2(3 + \frac{1}{3})3} = \frac{(-1)a_0}{2^6(1 + \frac{1}{3})(2 + \frac{1}{3})(3 + \frac{1}{3})3!},$$ and the

coefficent of x^{2m} for $m = 1, 2,\ldots$ is

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}m!(1 + \frac{1}{3})(2 + \frac{1}{3})\ \ldots\ (m + \frac{1}{3})}.$$ Thus one

solution (on setting $a_0 = 1$) is

$$y_1(x) = x^{1/3}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\ (1 + \frac{1}{3})(2 + \frac{1}{3})\ldots(m + \frac{1}{3})}\left(\frac{x}{2}\right)^{2m}\right].$$

Since $r_2 = -1/3 \neq r_1$ and $r_1 - r_2 = 2/3$ is not an integer, we can calculate a second series solution corresponding to $r = -1/3$. The recurrence relation is $n(n-2/3)a_n = -a_{n-2}$, which yields the desired solution following the steps just outlined. Note that $a_1 = 0$, as in the first solution, and thus all the odd coefficients are zero.

4. Putting the D.E. in standard form $y''+p(x)y'+q(x)y = 0$, we see that $p(x) = 1/x$ and $q(x) = -1/x$. Thus $x = 0$ is a singular point, and since $xp(x) \to 1$ and $x^2q(x) \to 0$, as $x \to 0$, $x = 0$ is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in $xy'' + y' - y = 0$ and shifting indices we obtain

$$\sum_{n=-1}^{\infty} a_{n+1}(r+n+1)(r+n)x^{n+r} + \sum_{n=-1}^{\infty} a_{n+1}(r+n+1)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0,$$

$$[r(r-1) + r]a_0x^{-1+r} + \sum_{n=0}^{\infty} [(r+n+1)^2a_{n+1} - a_n]x^{n+r} = 0.$$ The indicial equation is $r^2 = 0$ so $r = 0$ is a double root. Thus we will obtain only one series of the form $y = x^r \sum_{n=0}^{\infty} a_nx^n$. The recurrence relation is $(n+1)^2a_{n+1} = a_n$, $n = 0,1,2,\ldots$. The coefficients are $a_1 = a_0$, $a_2 = a_1/2^2 = a_0/2^2$, $a_3 = a_2/3^2 = a_0/3^2 \cdot 2^2$, $a_4 = a_3/4^2 = a_0/4^2 \cdot 3^2 \cdot 2^2, \ldots$ and $a_n = a_0/(n!)^2$. Thus one solution (on setting $a_0 = 1$) is $y = \sum_{n=0}^{\infty} x^n/(n!)^2$.

11. If we make the change of variable $t = x-1$ and let $y = u(t)$, then the Legendre equation transforms to $(t^2 + 2t)u''(t) + 2(t+1)u'(t) - \alpha(\alpha+1)u(t) = 0$. Since $x = 1$ is a regular singular point of the original equation, we know that $t = 0$ is a regular singular point of the transformed equation. Substituting $u = \sum_{n=0}^{\infty} a_nt^{n+r}$ in the transformed equation and shifting indices, we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_nt^{n+r} + 2\sum_{n=-1}^{\infty} (n+r+1)(n+r)a_{n+1}t^{n+r}$$
$$+ 2 \sum_{n=0}^{\infty} (n+r)a_nt^{n+r} + 2 \sum_{n=-1}^{\infty}(n+r+1)a_{n+1}t^{n+r}$$
$$- \alpha(\alpha+1) \sum_{n=0}^{\infty} a_nt^{n+r} = 0, \text{ or}$$

$$[2r(r-1) + 2r]a_0 t^{r-1} + \sum_{n=0}^{\infty} \{2(n+r+1)^2a_{n+1}$$
$$+ [(n+r)(n+r+1) - \alpha(\alpha+1)]a_n\}t^{n+r} = 0.$$

The indicial equation is $2r^2 = 0$ so $r = 0$ is a double root. Thus there will be only one series solution of the form $y = \sum_{n=0}^{\infty} a_n t^{n+r}$. The recurrence relation is

$2(n+1)^2 a_{n+1} = [\alpha(\alpha+1) - n(n+1)]a_n$, $n = 0,1,2,\ldots$. We have
$a_1 = [\alpha(\alpha+1)]a_0/2\cdot1^2$, $a_2 = [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot2]a_0/2^2\cdot2^2\cdot1^2$,
$a_3 = [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot2][\alpha(\alpha+1) - 2\cdot3]a_0/2^3\cdot3^2\cdot2^2\cdot1^2,\ldots,$
and $a_n = [\alpha(\alpha+1)][\alpha(\alpha+1)-1\cdot2]..[\alpha(\alpha+1)-(n-1)n]a_0/2^n(n!)^2$.
Reverting to the variable x it follows that one solution of the Legendre equation in powers of x-1 is

$$y_1(x) = \sum_{n=0}^{\infty} [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot2] \ldots$$

$[\alpha(\alpha+1) - (n-1)n](x-1)^n/2^n(n!)^2$ where we have set $a_0 = 1$, which is equivalent to the answer in the text if a (-1) is taken out of each square bracket.

14. The standard form is $y'' + p(x)y' + q(x)y = 0$, with $p(x) = 1/x$ and $q(x) = 1$. Thus $x = 0$ is a singular point; and since $xp(x) \to 1$ and $x^2q(x) \to 0$ as $x \to 0$, $x = 0$ is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into $x^2y'' + xy' + x^2y = 0$ and shifting indices appropriately, we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0,$$

or

$$[r(r-1)+r]a_0x^r + [(1+r)r+1+r]a_1x^{r+1}$$
$$+\sum_{n=2}^{\infty} [(n+r)^2a_n + a_{n-2}]x^{n+r} = 0.$$ The indicial equation is $r^2 = 0$ so $r = 0$ is a double root. It is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation in $n^2a_n = -a_{n-2}$, $n = 2,3,\ldots$. Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients we let $n = 2m$, $m = 1,2,\ldots$. Then $a_{2m} = -a_{2m-2}/2^2m^2$ so $a_2 = -a_0/2^2\cdot1^2$, $a_4 = a_0/2^2\cdot2^2\cdot1^2\cdot2^2,\ldots$, and $a_{2m} = (-1)^m a_0/2^{2m}(m!)^2$. Thus one solution of the Bessel

equation of order zero is $J_0(x) = 1 + \sum_{m=1}^{\infty} (-1)^m x^{2m}/2^{2m}(m!)^2$ where we have set $a_0 = 1$. Using the ratio test it can be shown that the series converges for all x. Also note that $J_0(x) \to 1$ as $x \to 0$.

15. In order to determine the form of the integral for x near zero we must study the integrand for x small. Using the above series for J_0, we have

$$\frac{1}{x[J_0(x)]^2} = \frac{1}{x[1 - x^2/2 + \ldots]^2} = \frac{1}{x[1 - x^2 + \ldots]} =$$

$\frac{1}{x}[1 + x^2 + \ldots]$ for x small. Thus

$$y_2(x) = J_0(x)\int\frac{dx}{x[J_0(x)]^2} = J_0(x)\int[\frac{1}{x} + x + \ldots]dx$$

$= J_0(x)[\ln x + \frac{x^2}{x} + \ldots]$, and it is clear that $y_2(x)$ will contain a logarithmic term.

16a. Putting the D.E. in the standard form $y'' + p(x)y' + q(x)y = 0$ we see that $p(x) = 1/x$ and $q(x) = (x^2-1)/x^2$. Thus $x = 0$ is a singular point and since $xp(x) \to 1$ and $x^2q(x) \to -1$ as $x \to 0$, $x = 0$ is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into $x^2y'' + xy' + (x^2-1)y = 0$, shifting indices appropriately, and collecting coefficients of common powers of x we obtain $[r(r-1) + r - 1]a_0x^r + [(1+r)r + 1 + r -1]a_1x^{r+1}$

$$+ \sum_{n=2}^{\infty} \{[(n+r)^2 - 1]a_n + a_{n-2}\}x^{n+r} = 0.$$

The indicial equation is $r^2-1 = 0$ so the roots are $r_1 = 1$ and $r_2 = -1$. For either value of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation is $[(n+r)^2 - 1]a_n = -a_{n-2}$, $n = 2,3,4\ldots$. For $r = 1$ we have $a_n = -a_{n-2}/[n(n+2)]$, $n = 2,3,4,\ldots$. Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = \ldots = 0$. Let $n = 2m$. Then $a_{2m} = -a_{2m-2}/2^2m(m+1)$, m =

$1,2,\ldots,$ so $a_2 = -a_0/2^2 \cdot 1\cdot 2$, $a_4 = -a_2/2^2 \cdot 1\cdot 2\cdot 3 =$
$a_0/2^2 \cdot 2^2 \cdot 1\cdot 2\cdot 2\cdot 3,\ldots,$ and $a_{2m} = (-1)^m a_0/2^{2m} m!(m+1)!$. Thus one solution (set $a_0 = 1/2$) of the Bessel equation of order one is $J_1(x) = (x/2)\sum_{n=0}^{\infty} (-1)^n x^{2n}/(n+1)!n!2^{2n}$. The ratio test shows that the series converges for all x. Also note that $J_1(x) \to 0$ as $x \to 0$.

16b. For $r = -1$ the recurrence relation is $[(n-1)^2 - 1]a_n = -a_{n-2}$, $n = 2,3,\ldots$. Substituting $n = 2$ into the relation yields $[(2-1)^2 - 1]a_2 = 0\ a_2 = -a_0$. Hence it is impossible to determine a_2 and consequently impossible to find a series solution of the form $x^{-1}\sum_{n=0}^{\infty} b_n x^n$.

Secton 5.7, Page 278

1. The D.E. has the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x$, $Q(x) = 2x$, and $R(x) = 6e^x$. From this we find $p(x) = Q(x)/P(x) = 2$ and $q(x) = R(x)/P(x) = 6e^x/x$ and thus $x = 0$ is a singular point. Since $xp(x) = 2x$ and $x^2q(x) = 6xe^x$ are analytic at $x = 0$ we conclude that $x = 0$ is a regular singular point. Next, we have $xp(x) \to 0 = p_0$ and $x^2q(x) \to 0 = q_0$ as $x \to 0$ and thus the indicial equation is $r(r-1) + 0\cdot r + 0 = r^2 - r = 0$, which has the roots $r_1 = 1$ and $r_2 = 0$.

3. The equation has the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x(x-1)$, $Q(x) = 6x^2$ and $R(x) = 3$. Since $P(x)$, $Q(x)$, and $R(x)$ are polynomials with no common factors and $P(0) = 0$ and $P(1) = 0$, we conclude that $x = 0$ and $x = 1$ are singular points. The first point, $x = 0$, can be shown to be a regular singular point using steps similar to those to shown in Problem 1. For $x = 1$, we must put the D.E. in a form similar to Eq.(1) for this case. To do this, divide the D.E. by x and multiply by $(x-1)$ to obtain $(x-1)^2y'' + 6x(x-1)y + \frac{3}{x}(x-1)y = 0$. Comparing this to Eq.(1) we find that $(x-1)p(x) = 6x$ and $(x-1)^2q(x) = 3(x-1)/x$ which are both analytic at

$x = 1$ and hence $x = 1$ is a regular singular point. These last two expressions approach $p_0 = 6$ and $q_0 = 0$ respectively as $x \to 1$, and thus the indicial equation is $r(r-1) + 6r + 0 = r(r+5) = 0$.

9. For this D.E., $p(x) = \dfrac{-(1+x)}{x^2(1-x)}$ and $q(x) = \dfrac{2}{x(1-x)}$ and thus $x = 0, -1$ are singular points. Since $xp(x)$ is not analytic at $x = 0$, $x = 0$ is not a regular singular point. Looking at $(x-1)p(x) = \dfrac{1+x}{x^2}$ and $(x-1)^2q(x) = \dfrac{2(1-x)}{x}$ we see that $x = 1$ is a regular singular point and that $p_0 = 2$ and $q_0 = 0$.

17a. We have $p(x) = \dfrac{\sin x}{x^2}$ and $q(x) = -\dfrac{\cos x}{x^2}$, so that $x = 0$ is a singular point. Note that $xp(x) = (\sin x)/x \to 1 = p_0$ as $x \to 0$ and $x^2q(x) = -\cos x \to -1 = q_0$ as $x \to 0$. In order to assert that $x = 0$ is a regular singular point we must demonstrate that $xp(x)$ and $x^2q(x)$, with $xp(x) = 1$ at $x = 0$ and $x^2q(x) = -1$ at $x = 0$, have convergent power series (are analytic) about $x = 0$. We know that $\cos x$ is analytic so we need only consider $(\sin x)/x$. Now

$$\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)! \text{ for } -\infty < x < \infty \text{ so}$$

$$(\sin x)/x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n+1)! \text{ and hence is analytic.}$$

Thus we may conclude that $x = 0$ is a regular singular point.

17b. From part a) it follows that the indicial equation is $r(r-1) + r - 1 = r^2 - 1 = 0$ and the roots are $r_1 = 1$, $r_2 = -1$.

17c. To find the first few terms of the solution corresponding to $r_1 = 1$, assume that

$y = x(a_0 + a_1x + a_2x^2 + \dots) = a_0x + a_1x^2 + a_2x^3 + \dots$.

Substituting this series for y in the D.E. and expanding $\sin x$ and $\cos x$ about $x = 0$ yields

$x^2(2a_1 + 6a_2x + 12a_3x^2 + 20a_4x^3 + \dots) +$

$(x - x^3/3! + x^5/5! - \dots)(a_0 + 2a_1x + 3a_2x^2 + 4a_3x^3 + 5a_4x^4 +$

$\ldots) - (1 - x^2/2! + x^4/4! - \ldots)(a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + \ldots) = 0$. Collecting terms, $(2a_1 + 2a_1 - a_1)x^2 + (6a_2 + 3a_2 - a_0/6 - a_2 + a_0/2)x^3 + (12a_3 + 4a_3 - 2a_1/6 - a_3 + a_1/2)x^4 + (20a_4 + 5a_4 - 3a_2/6 + a_0/120 - a_4 + a_2/2 - a_0/24)x^5 + \ldots = 0$. Simplifying, $3a_1x^2 + (8a_2 + a_0/3)x^3 + (15a_3 + a_1/6)x^4 + (24a_4 - a_0/30)x^5 + \ldots = 0$. Thus, $a_1 = 0$, $a_2 = -a_0/4!$, $a_3 = 0$, $a_4 = a_0/6!, \ldots$. Hence
$y_1(x) = x - x^3/4! + x^5/6! + \ldots$ where we have set $a_0 = 1$.
From Eq. (24) the second solution has the form

$$y_2(x) = ay_1(x)\ln x + x^{-1}\left(1+\sum_{n=1}^{\infty} c_nx^n\right)$$

$$= ay_1(x)\ln x + \frac{1}{x} + c_1 + c_2x + c_3x^2 + c_4x^3 + \ldots, \text{ so}$$

$y_2' = ay_1'\ln x + ay_1x^{-1} - x^{-2} + c_2 + 2c_3x + 3c_4x^2 + \ldots$, and
$y_2'' = ay_1'' \ln x + 2ay_1'x^{-1} - ay_1x^{-2} + 2x^{-3} + 2c_3 + 3c_4x + \ldots$.
When these are substituted in the given D.E. the terms including $\ln x$ will appear as
$a[x^2y_1'' + (\sin x)y_1' - (\cos x)y_1]$, which is zero since y_1 is a solution. For the remainder of the terms, use
$y_1 = x - x^3/24 + x^5/720$ and the $\cos x$ and $\sin x$ series as shown earlier to obtain
$-c_1 + (2/3+2a)x + (3c_3+c_1/2)x^2 + (4/45+c_2/3+8c_4)x^3 + \ldots = 0$.
These yield $c_1 = 0$, $a = -1/3$, $c_3 = 0$, and
$c_4 = -c_2/24 - 1/90$. We may take $c_2 = 0$, since this term will simply generate $y_1(x)$ over again. Thus
$y_2(x) = -\frac{1}{3}y_1(x)\ln x + x^{-1} - \frac{1}{90}x^3$. If a computer algebra system is used, then additional terms in each series may be obtained without much additional effort. The next terms, in each case, are shown here:

$$y_1(x) = x - \frac{x^3}{24} + \frac{x^5}{720} - \frac{43x^7}{1451520} + \ldots \text{ and}$$

$$y_2(x) = -\frac{1}{3}y_1(x)\ln x + \frac{1}{x}\left[1 - \frac{x^4}{90} + \frac{41x^6}{120960} - \ldots\right].$$

18. We first write the D.E. in the standard form as given for Theorem 5.7.1 except that we are expanding in powers of $(x-1)$ rather than powers of x:

$(x-1)^2y'' + (x-1)[(x-1)/2\ln x]y' + [(x-1)^2/\ln x]y = 0$. since $\ln 1 = 0$, $x = 1$ is a singular point. To show it is a regular singular point of this D.E. we must show that $(x-1)/\ln x$ is analytic at $x = 1$; it will then follow that $(x-1)^2/\ln x = (x-1)[(x-1)/\ln x]$ is also analytic at $x = 1$. If we expand $\ln x$ in a Taylor series about $x = 1$ we find that $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$.

Thus

$$(x-1)/\ln x = [1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - \ldots]^{-1} = 1 + \frac{1}{2}(x-1)+\ldots$$

has a power series expansion about $x = 1$, and hence is analytic. We can use the above result to obtain the indicial equation at $x = 1$. We have $(x-1)^2y'' + (x-1)[\frac{1}{2} + \frac{1}{4}(x-1) + \ldots]y' + [(x-1) + \frac{1}{2}(x-1)^2 + \ldots]y = 0$. Thus $p_0 = 1/2$, $q_0 = 0$ and the indicial equation is $r(r-1) + r/2 = 0$. Hence $r = 1/2$ and $r = 0$. In order to find the first three non-zero terms in a series solution corresponding to $r = 1/2$, it is better to keep the differential equation in its original form and to substitute the above power series for $\ln x$:

$$[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots]y'' + \frac{1}{2}y' + y = 0.$$

Next we substitute $y = a_0(x-1)^{1/2} + a_1(x-1)^{3/2} + a_2(x-1)^{5/2} + \ldots$ and collect coefficients of like powers of $(x-1)$ which are then set equal to zero. This requires some algebra before we find that $6a_1/4 + 9a_0/8 = 0$ and $5a_2 + 5a_1/8 - a_0/12 = 0$. These equations yield $a_1 = -3a_0/4$ and $a_2 = 53a_0/480$. With $a_0 = 1$ we obtain the solution

$y_1(x) = (x-1)^{1/2} - \frac{3}{4}(x-1)^{3/2} + \frac{53}{480}(x-1)^{5/2} + \ldots$. Since the radius of convergence of the series for $\ln x$ is 1, we would expect $\rho = 1$.

20a. If we write the D.E. in the standard form as given in Theorem 5.7.1 we obtain $x^2y'' + x[\alpha/x]y' + [\beta/x]y = 0$ where $xp(x) = \alpha/x$ and $x^2q(x) = \beta/x$. Neither of these terms are analytic at $x = 0$ so $x = 0$ is an irregular singular point.

20b. Substituting $y = x^r\sum_{n=0}^{\infty} a_nx^n$ in $x^3y'' + \alpha xy' + \beta y = 0$ gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r+1} + \alpha\sum_{n=0}^{\infty}(n+r)a_nx^{n+r} + \beta\sum_{n=0}^{\infty} a_nx^{n+r} = 0.$$

Shifting the index in the first series and collecting coefficients of common powers of x we obtain $(\alpha r + \beta)a_0x^r$

$$+ \sum_{n=1}^{\infty} (n+r-1)(n+r-2)a_{n-1} + [\alpha(n+r) + \beta]a_nx^{n+r} = 0.$$ Thus

the indicial equation is $\alpha r + \beta = 0$ with the single root $r = -\beta/\alpha$.

20c. From part b, the recurrence relation is

$$a_n = \frac{(n+r-1)(n+r-2)a_{n-1}}{\alpha(n+r) + \beta}, \quad n = 1,2,...$$

$$= \frac{(n - \frac{\beta}{\alpha} -1)(n - \frac{\beta}{\alpha} -2)a_{n-1}}{\alpha n}, \text{ for } r = -\beta/\alpha.$$

For $\frac{\beta}{\alpha} = -1$, then, $a_n = \frac{n(n-1)a_{n-1}}{\alpha n}$, which is zero for $n = 1$ and thus $y(x) = x$ is the solution. Similarly for $\frac{\beta}{\alpha} = 0$, $a_n = \frac{(n-1)(n-2)}{\alpha n}$ and again for $n = 1$ $a_1 = 0$ and $y(x) = 1$ is the solution. Continuing in this fashion, we see that the series solution will terminate for β/α any positive integer as well as 0 and -1. For other values of β/α, we have $\frac{a_n}{a_{n-1}} = \frac{(n-\frac{\beta}{2}-1)(n-\frac{\beta}{\alpha}-2)}{\alpha n}$, which approaches ∞ as $n \to \infty$ and thus the ratio test yields a zero radius of convergence.

21b. Substituting $y = \sum_{n=0}^{\infty} a_nx^{n+r}$ in the D.E. in standard form gives

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_nx^{n+r} + \alpha\sum_{n=0}^{\infty} (n+r)a_nx^{n+r+1-s}$$

$$+ \beta \sum_{n=0}^{\infty} a_n x^{n+r+2-t} = 0.$$

If s = 2 and t = 2 the first term in each of the three series is $r(r-1)a_0x^r$, αra_0x^{r-1}, and βa_0x^r, respectively. Thus we must have $\alpha ra_0 = 0$ which requires r = 0. Hence there is at most one solution of the assumed form.

21d. In order for the indicial equation to be quadratic in r it is necessary that the first term in the first series contribute to the indicial equation. This means that the first term in the second and the third series cannot appear before the first term of the first series. The first terms are $r(r-1)a_0x^r$, αra_0x^{r+1-s}, and βa_0x^{r+2-t}, respectively. Thus if $s \leq 1$ and $t \leq 2$ the quadratic term will appear in the indicial equation.

<u>Section 5.8, Page 289</u>

1. It is clear that x = 0 is a singular point. The D.E. is in the standard form given in Theorem 5.7.1 with $xp(x) = 2$ and $x^2q(x) = x$. Both are analytic at x = 0, so x = 0 is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E., shifting indices appropriately, and collecting coefficients of like powers of x yields

$$[r(r-1) + 2r]a_0x^r + \sum_{n=1}^{\infty} [(r+n)(r+n+1)a_n + a_{n-1}]x^{r+n} = 0.$$

The indicial equation is $F(r) = r(r+1) = 0$ with roots $r_1 = 0$, $r_2 = -1$. Treating a_n as a function of r, we see that $a_n(r) = -a_{n-1}(r)/F(r+n)$, n = 1,2,... if $F(r+n) \neq 0$. Thus $a_1(r) = -a_0/F(r+1)$, $a_2(r) = a_0/F(r+1)F(r+2)$,..., and $a_n(r) = (-1)^n a_0/F(r+1)F(r+2)...F(r+n)$, provided $F(r+n) \neq 0$ for n = 1,2,... . For the case $r_1 = 0$, we have $a_n(0) = (-1)^n a_0/F(1)F(2) \ldots F(n) = (-1)^n a_0/n!(n+1)!$ so one solution is $y_1(x) = \sum_{n=0}^{\infty} (-1)^n x^n/n!(n+1)!$ where we have set $a_0 = 1$.

If we try to use the above recurrence relation for the case $r_2 = -1$ we find that $a_n(-1) = -a_{n-1}/n(n-1)$, which is undefined for $n = 1$. Thus we must follow the procedure described at the end of Section 5.7 to calculate a second solution of the form given in Eq.(24). Specifically, we use Eqs.(19) and (20) of that section to calculate a and $c_n(r_2)$, where $r_2 = -1$. Since $r_1 - r_2 = 1 = N$, we have $a_N(r) = a_1(r) = -1/F(r+1)$, with $a_0 = 1$. Hence

$$a = \lim_{r\to -1} [(r+1)(-1)/F(r+1)] = \lim_{r\to -1}[-(r+1)/(r+1)(r+2)] = -1.$$

Next

$$c_n(-1) = \frac{d}{dr}[(r+1)a_n(r)]\Big|_{r=-1} = (-1)^n\frac{d}{dr}\left[\frac{(r+1)}{F(r+1)\ \dots\ F(r+n)}\right]\Big|_{r=-1},$$

where we again have set $a_0 = 1$. Observe that $(r+1)/F(r+1)\dots F(r+n)=1/[(r+2)^2(r+3)^2\dots(r+n)^2(r+n+1)]=1/G_n(r)$. Hence $c_n(-1) = (-1)^{n+1}G_n'(-1)/G_n^2(-1)$. Notice that $G_n(-1) = 1^2\cdot 2^2\cdot 3^2\dots(n-1)^2 n = (n-1)!n!$ and $G_n'(-1)/G_n(-1) = 2[1/1 + 1/2 + 1/3 +\dots+ 1/(n-1)] + 1/n = H_n + H_{n-1}$. Thus $c_n(-1) = (-1)^{n+1}(H_n + H_{n-1})/(n-1)!n!$. From Eq.(24) of Section 5.7 we obtain the second solution

$$y_2(x) = - y_1(x)\ln x + x^{-1}[1 - \sum_{n=1}^{\infty}(-1)^n(H_n + H_{n-1})x^n/n!(n-1)!].$$

2. It is clear that $x = 0$ is a singular point. The D.E. is in the standard form given in Theorem 5.7.1 with $xp(x) = 3$ and $x^2q(x) = 1+x$. Both are analytic at $x = 0$, so $x = 0$ is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E., shifting indices appropriately, and collecting coefficients of like powers of x yields

$$[r(r-1) + 3r + 1]a_0x^r + \sum_{n=1}^{\infty} \{[(r+n)(r+n+2) + 1]a_n + a_{n-1}\} x^{n+r} = 0.$$

The indicial equation is $F(r) = r^2 + 2r + 1 = (r+1)^2 = 0$ with the double root $r_1 = r_2 = -1$. Treating a_n as a function of r, we see that $a_n(r) = -a_{n-1}(r)/F(r+n)$,

$n = 1,2,\ldots$. Thus $a_1(r) = -a_0/F(r+1)$, $a_2(r) = a_0/F(r+1)F(r+2),\ldots$, and $a_n(r) = (-1)^n a_0/F(r+1)F(r+2)\ldots F(r+n)$. Setting $r = -1$ we find that $a_n(-1)=(-1)^n a_0/(n!)^2$, $n = 1,2,\ldots$. Hence one solution is $y_1(x) = x^{-1}\sum_{n=0}^{\infty}(-1)^n x^n/(n!)^2$ where we have set $a_0 = 1$. To find a second solution we follow the procedure described in Section 5.7 for the case when the roots of the indicial equation are equal. Specifically, the second solution will have the form given in Eq.(17) of that section. We must calculate $a_n'(-1)$. If we let $G_n(r) = F(r+1)\ldots F(r+n) = (r+2)^2(r+3)^2\ldots(r+n+1)^2$ and take $a_0 = 1$, then $a_n'(-1) = (-1)^n[1/G_n(r)]'$ evaluated $r = -1$. Hence $a_n'(-1) = (-1)^{n+1}G_n'(-1)/G_n^2(-1)$. But $G_n(-1) = (n!)^2$ and $G_n'(-1)/G_n(-1) = 2[1/1 + 1/2 + 1/3 + \ldots + 1/n] = 2H_n$. Thus a second solution is

$$y_2(x) = y_1(x)\ln x - 2x^{-1}\sum_{n=1}^{\infty}(-1)^n H_n x^n/(n!)^2.$$

3. The roots of the indicial equation are r_1 and $r_2 = 0$ and thus the analysis is similar to that for Problem 2.

4. The roots of the indicial equation are $r_1 = -1$ and $r_2 = -2$ and thus the analysis is similar to that for Problem 1.

5. Since $x = 0$ is a regular singular point, substitute $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E., shift indices appropriately, and collect coefficients of like powers of x to obtain $[r^2 - 9/4]a_0x^r + [(r+1)^2 - 9/4]a_1x^{r+1}$

$$+ \sum_{n=2}^{\infty} \{[(r+n)^2 - 9/4]a_n + a_{n-2}\}\, x^{n+r} = 0.$$

The indicial equation is $F(r) = r^2 - 9/4 = 0$ with roots $r_1 = 3/2$, $r_2 = -3/2$. Treating a_n as a function of r we see that $a_n(r)= -a_{n-2}(r)/F(r+n)$, $n = 2,3,..$ if $F(r+n) \neq$

0. For the case $r_1 = 3/2$, $F(r_1+1)$, which is the coefficient of x^{r_1+1} is $\neq 0$ so we must set $a_1 = 0$. It follows that $a_3 = a_5 = \ldots = 0$. For the even coefficients, set $n = 2m$ so
$a_{2m}(3/2) = -a_{2m-2}(3/2)/F(3/2 + 2m) = -a_{2m-2}/2^2m(m+3/2)$,
$m = 1,2\ldots$. Thus $a_2(3/2) = -a_0/2^2\cdot 1(1 + 3/2)$,
$a_4(3/2) = a_0/2^4\cdot 2!(1 + 3/2)(2 + 3/2),\ldots$, and
$a_{2m}(3/2) = (-1)^m/2^{2m}m!\cdot(1 + 3/2)\ldots(m + 3/2)$. Hence one solution is

$$y_1(x) = x^{3/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1 + 3/2)(2 + 3/2)\ldots(m + 3/2)}\left(\frac{x}{2}\right)^{2m}\right],$$

where we have set $a_0 = 1$. For this problem, the roots r_1 and r_2 of the indicial equation differ by an integer: $r_1 - r_2 = 3/2 - (-3/2) = 3$. Hence we can anticipate that there may be difficulty in calculating a second solution corresponding to $r = r_2$. This difficulty will occur in calculating $a_3(r) = -a_1(r)/F(r+3)$ since when $r = r_2 = -3/2$ we have $F(r_2+3) = F(r_1) = 0$. However, in this problem we are fortunate because $a_1 = 0$ and it will not be necessary to use the theory described at the end of Section 5.7. Notice for $r = r_2 = -3/2$ that the coefficient of x^{r_2+1} is $[(r_2+1)^2 - 9/4]a_1$, which does not vanish unless $a_1 = 0$. Thus the recurrence relation for the odd coefficients yields $a_5 = -a_3/F(7/2)$, $a_7 = -a_5/F(11/2) = a_3/F(11/2)F(7/2)$ and so forth. Substituting these terms into the assumed form we see that a multiple of $y_1(x)$ has been obtained and thus we may take $a_3 = 0$ without loss of generality. Hence $a_3 = a_5 = a_7 = \ldots = 0$. The even coefficients are given by $a_{2m}(-3/2) = -a_{2m-2}(-3/2)/F(2m - 3/2)$, $m = 1,2\ldots$.
Thus $a_2(-3/2) = -a_0/2^2\cdot 1\cdot(1 - 3/2)$,
$a_4(-3/2) = a_0/2^4\cdot 2!(1 - 3/2)(2 - 3/2),\ldots$, and
$a_{2m}(-3/2) = (-1)^m a_0/2^{2m}m!(1 - 3/2)(2 - 3/2)\ldots(m - 3/2)$.
Thus a second solution is

$$y_2(x) = x^{-3/2}\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^m}{m!(1 - 3/2)(2 - 3/2)\ldots(m - 3/2)}\left(\frac{x}{2}\right)^{2m}\right].$$

7. Apply the ratio test:

$$\lim_{m \to \infty} \frac{|(-1)^{m+1}\ x^{2m+2}/2^{2m+2}[(m+1)!]^2|}{|(-1)^m\ x^{2m}/2^{2m}(m!)^2|} = |x^2| \lim_{m \to \infty} \frac{1}{2^2(m+1)^2} = 0$$

for every x. Thus the series for $J_0(x)$ converges absolutely for all x.

12. If $\xi = \alpha x^\beta$, then $dy/dx = \frac{1}{2}x^{-1/2}f + x^{1/2}f'\alpha\beta x^{\beta-1}$ where f' denotes $df/d\xi$. Find d^2y/dx^2 in a similar fashion and use algebra to show that f satisfies the D.E. $\xi^2 f'' + \xi f' + [\xi^2 - \upsilon^2]f = 0$.

13. To compare $y'' - xy = 0$ with the D.E. of Problem 12, we must multiply by x^2 to get $x^2y'' - x^3y = 0$. Thus $2\beta = 3$, $\alpha^2\beta^2 = -1$ and $1/4 - \upsilon^2\beta^2 = 0$. Hence $\beta = 3/2$, $\alpha = 2i/3$ and $\upsilon^2 = 1/9$ which yields the desired result.

14. First we verify that $J_0(\lambda_j x)$ satisfies the D.E. We know that $J_0(t)$ is a solution of the Bessel equation of order zero:

$$t^2J_0''(t) + tJ_0'(t) + t^2J_0(t) = 0 \text{ or}$$

$$J_0''(t) + t^{-1}J_0'(t) + J_0(t) = 0.$$

Let $t = \lambda_j x$. Then

$$\frac{d}{dx}\ J_0(\lambda_j x) = \frac{d}{dt}\ J_0(t)\frac{dt}{dx} = \lambda_j J_0'(t)$$

$$\frac{d^2}{dx^2}\ J_0(\lambda_j x) = \lambda_j \frac{d}{dt}[J_0'(t)]\frac{dt}{dx} = \lambda_j^2 J_0''(t).$$

Substituting $y = J_0(\lambda_j x)$ in the given D.E. and making use of these results, we have

$$\lambda_j^2 J_0''(t) + (\lambda_j/t)\ \lambda_j J_0'(t) + \lambda_j^2 J_0(t) =$$

$$\lambda_j^2[J_0''(t) + t^{-1}J_0'(t) + J_0(t)] = 0.$$

Thus $y = J_0(\lambda_j x)$ is a solution of the given D.E. For the second part of the problem we follow the hint. First, rewrite the D.E. by multiplying by x to yield $xy'' + y' + \lambda_j^2 xy = 0$, which can be written as $(xy')' = -\lambda_j^2 xy$. Now let $y_i(x) = J_0(\lambda_i x)$ and $y_j(x) = J_0(\lambda_j x)$ and we have, respectively: $(xy_i')' = -\lambda_i^2 xy_i$

$$(xy_j')' = -\lambda_j^2 xy_j.$$

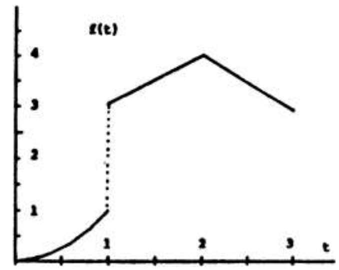
Now multiply the first equation by y_j, the second by y_i,

Capítulo 6

CHAPTER 6

Section 6.1, Page 298

1. The graph of $f(t)$ is shown. Since the function is continuous on each interval, but has a jump discontinuity at $t = 1$, $f(t)$ is piecewise continuous.



2. Note that $\lim\limits_{t\to 1^+} (t-1)^{-1} = \infty$.

5b. Since t^2 is continuous for $0 \le t \le A$ for any positive A and since $t^2 \le e^{at}$ for any $a > 0$ and for t sufficiently large, it follows from Theorem 6.1.2 that $\pounds\{t^2\}$ exists for $s > 0$. $\pounds\{t^2\} = \int_0^\infty e^{-st}t^2dt = \lim\limits_{M\to\infty}\int_0^M e^{-st}t^2dt$

$$= \lim_{M\to\infty}\left[\frac{-t^2}{s}e^{-st}\Big|_0^M + \frac{2}{s}\int_0^M e^{-st}tdt\right]$$
$$= \frac{2}{s}\lim_{M\to\infty}\left[-\frac{1}{s}te^{-st}\Big|_0^M + \frac{1}{s}\int_0^M e^{-st}dt\right]$$
$$= \frac{2}{s^2}\lim_{M\to\infty} -\frac{1}{s}e^{-st}\Big|_0^M = \frac{2}{s^3}.$$

6. That $f(t) = \cos at$ satisfies the hypotheses of Theorem 6.1.2 can be verified by recalling that $|\cos at| \le 1$ for all t. To determine $\pounds\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt$ we must integrate by parts twice to get $\int_0^\infty e^{-st}\cos at\,dt =$ $\lim\limits_{M\to\infty}[(-s^{-1}e^{-st}\cos at + as^{-2}e^{-st}\sin at)\Big|_0^M$

$- (a^2/s^2)\int_0^M e^{-st}\cos at\,dt]$. Evaluating the first two terms, letting $M \to \infty$, and adding the third term to both sides, we obtain $[1 + a^2/s^2]\int_0^\infty e^{-st}\cos at\,dt = 1/s$, $s > 0$. Division by $[1 + a^2/s^2]$ and simplification yields the desired solution.

9. From the definition for $\cosh bt$ we have $\pounds\{e^{at}\cosh bt\} = \pounds\{\frac{1}{2}[e^{(a+b)t} + e^{(a-b)t}]\}$. Using the linearity

property of £, Eq.(5), the right side becomes $\frac{1}{2}£\{e^{(a+b)t}\} + \frac{1}{2}£\{e^{(a-b)t}\}$ which can be evaluated using the result of Example 5 and thus

$$£\{e^{at}\cosh bt\} = \frac{1/2}{s-(a+b)} + \frac{1/2}{s-(a-b)}$$
$$= \frac{s-a}{(s-a)^2-b^2}, \text{ for } s-a > |b|.$$

13. We write $\sin at = (e^{iat} - e^{-iat})/2i$, then the linearity of the Laplace transform operator allows us to write $£\{e^{at}\sin bt\} = (1/2i)£\{e^{(a+ib)t}\}-(1/2i)£\{e^{(a-ib)t}\}$. Each of these two terms can be evaluated by using the result of Example 5, where we now have to require s to be greater than the real part of the complex numbers $a \pm ib$ in order for the integrals to converge. Complex algebra then gives the desired result. An alternate method of evaluation would be to use integration on the integral appearing in the definition of $£\{e^{at}\sin bt\}$, but that method requires integration by parts twice.

16. As in Problem 13, $£\{t\sin at\} = (1/2i)£\{te^{iat}\} - (1/2i)£\{te^{-iat}\}$. Using the result of Problem 15 we obtain $£\{t\sin at\} = (1/2i)[(s-b)^{-2} - (s+b)^{-2}]$ where $b = ia$ and $s > 0$. Hence $£\{t\sin at\} = 2as/(s^2+a^2)^2$, $s > 0$.

19. Use the approach shown in Problem 16 with the result of Problem 18, for $n = 2$. A computer algebra system may also be used.

21. The integral $\int_0^A (t^2 + 1)^{-1}dt$ can be evaluated in terms of the arctan function and then Eq. (3) can be used. To illustrate Theorem 6.1.1, however, consider that $\frac{1}{t^2+1} < \frac{1}{t^2}$ for $t \geq 1$ and, from Example 3, $\int_1^\infty t^{-2}dt$ converges and hence $\int_1^\infty (t^2 + 1)^{-1}dt$ also converges. $\int_0^1 (t^2 + 1)^{-1}dt$ is finite and hence does not affect the convergence of $\int_0^\infty (t^2 + 1)^{-1}dt$ at infinity.

25. If we let $u = f$ and $dv = e^{-st}dt$ then $F(s) = \int_0^\infty e^{-st}f(t)dt$
$= \lim_{M \to \infty} -\frac{1}{s}e^{-st}f(t)\Big|_0^M + \frac{1}{s}\int_0^\infty e^{-st}f'(t)dt$. Now use an argument similar to that given to establish Theorem 6.1.2.

27a. Make a transformation of variables with $x = st$ and $dx = sdt$. Then use the definition of $\Gamma(P+1)$ from Problem 26.

27b. From part a, $\pounds\{t^n\} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n dx = \frac{n}{s^{n+1}}\int_0^\infty e^{-x}x^{n-1}dx$
$= \frac{n!}{s^{n+1}}\int_0^\infty e^{-x}dx$, using integration by parts successively. Evaluation of the last integral yields the desired answer.

27c. From part a, $\pounds\{t^{-1/2}\} = \frac{1}{\sqrt{s}}\int_0^\infty e^{-x}x^{-1/2}dx$. Let $x = y^2$, then $2dy = x^{-1/2}dx$ and thus $\pounds\{t^{-1/2}\} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-y^2}dy$.

27d. Use the definition of $\pounds\{t^{1/2}\}$ and integrate by parts once to get $\pounds\{t^{1/2}\} = (1/2s)\pounds\{t^{-1/2}\}$. The result follows from part c.

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Problems 1 through 10 are solved by using partial fractions and algebra to manipulate the given function into a form matching one of the functions appearing in the middle column of Table 6.2.1.

2. We have $\frac{4}{(s-1)^3} = 2\frac{2!}{(s-1)^{2+1}}$ and thus the inverse Laplace transform is $2t^2e^t$, using line 11.

4. We have $\frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{9/5}{s-3} + \frac{6/5}{s+2}$ using partial fractions. Thus $(9/5)e^{3t} + (6/5)e^{-2t}$ is the inverse transform, from line 2.

7. We have $\frac{2s+1}{s^2-2s+2} = \frac{2s+1}{(s-1)^2+1} = \frac{2(s-1)}{(s-1)^2+1} + \frac{3}{(s-1)^2+1}$, where

we first used the concept of completing the square (in the denominator) and then added and subtracted appropriately to put the numerator in the desired form. Lines 9 and 10 may now be used to find the desired result.

In each of the Problems 11 through 23 it is assumed that the I.V.P. has a solution $y = \phi(t)$ which, with its first two derivatives, satisfies the conditions of the Corollary to Theorem 6.2.1.

11. Take the Laplace transform of the D.E., using Eq.(1) and Eq.(2), to get
$s^2Y(s) - sy(0) - y'(0) - [sY(s) - y(0)] - 6Y(s) = 0$.
Using the I.C. and solving for Y(s) we obtain
$Y(s) = \frac{s-2}{s^2-s-6}$. Following the pattern of Eq.(12) we have
$\frac{s-2}{s^2-s-6} = \frac{a}{s+2} + \frac{b}{s-3} = \frac{a(s-3)+b(s+2)}{(s+2)(s-3)}$. Equating like powers in the numerators we find $a+b = 1$ and $-3a + 2b = -2$. Thus $a = 4/5$ and $b = 1/5$ and
$Y(s) = \frac{4+5}{s+2} + \frac{1/5}{s-3}$, which yields the desired solution using Table 6.2.1.

14. Taking the Laplace transform we have $s^2Y(s) - sy(0) -y'(0) - 4[sY(s)-y(0)] + 4Y(s) = 0$. Using the I.C. and solving for Y(s) we find $Y(s) = \frac{s-3}{s^2-4s+4}$. Since the denominator is a perfect square, the partial fraction form is $\frac{s-3}{s^2-4s+4} = \frac{a}{(s-2)^2} + \frac{b}{s-2}$. Solving for a and b, as shown in examples of this section or in Problem 11, we find $a = -1$ and $b = 1$.
Thus $Y(s) = \frac{1}{s-2} - \frac{1}{(s-2)^2}$, from which we find
$y(t) = e^{2t} - te^{2t}$ (lines 2 and 11 in Table 6.2.1).

15. Note that $Y(s) = \frac{2s-4}{s^2-2s-2} = \frac{2s-4}{(s-1)^2-3} = \frac{2(s-1)}{(s-1)^2-3} - \frac{2}{(s-1)^2-3}$.
Three formulas in Table 6.2.1 are now needed: F(s-c) in line 14 in conjunction with the ones for coshat and sinhat, lines 7 and 8.

17. The Laplace transform of the D.E. is $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4[s^3Y(s)-s^2y(0) -sy'(0) - y''(0)] + 6[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] +Y(s) = 0$. Using the I.C. and solving for Y(s) we find

$$Y(s) = \frac{s^2 - 4s + 7}{s^4-4s^3+6s^2-4s+1}.$$ The correct partial fraction form for this is $$\frac{a}{(s-1)^4} + \frac{b}{(s-1)^3} + \frac{c}{(s-1)^2} + \frac{d}{s-1}.$$ Setting this equal to Y(s) above and equating the numerators we have $s^2-4s+7 = a + b(s-1) + c(s-1)^2 + d(s-1)^3$. Solving for a,b,c, and d and use of Table 6.2.1 yields the desired solution.

20. The Laplace transform of the D.E. is $s^2Y(s) - sy(0) - y'(0) + \omega^2Y(s) = s/(s^2+4)$. Applying the I.C. and solving for Y(s) we get $Y(s) = s/[(s^2+4)(s^2+\omega^2)] + s/(s^2+\omega^2)$. Decomposing the first term by partial fractions we have

$$Y(s) = \frac{s}{(\omega^2-4)(s^2+4)} - \frac{s}{(\omega^2-4)(s^2+\omega^2)} + \frac{s}{s^2+\omega^2}$$

$$= (\omega^2-4)^{-1}\left[\frac{(\omega^2-5)s}{s^2+\omega^2} + \frac{s}{s^2+4}\right].$$

Then, using Table 6.1.2, we have $y = (\omega^2-4)^{-1}[(\omega^2-5)\cos\omega t + \cos 2t]$.

22. Solving for Y(s) we find $Y(s) = 1/[(s-1)^2 + 1] + 1/(s+1)[(s-1)^2 + 1]$. Using partial fractions on the second term we obtain

$$Y(s) = 1/[(s-1)^2 + 1] + \{1/(s+1) - (s-3)/[(s-1)^2 + 1]\}/5$$
$$= (1/5)\{(s+1)^{-1}-(s-1)[(s-1)^2 + 1]^{-1} + 7[(s-1)^2 + 1]^{-1}\}.$$

Hence, $y = (1/5)(e^{-t} - e^t\cos t + 7e^t\sin t)$.

24. Under the standard assumptions, the Lapace transform of the left side of the D.E. is $s^2Y(s) - sy(0) - y'(0) + 4Y(s)$. To transform the right side we must revert to the definition of the Laplace trasnform to determine $\int_0^\infty e^{-st}f(t)dt$. Since f(t) is piecewise continuous we are able to calculate $\mathcal{L}\{f(t)\}$ by

$$\int_0^\infty e^{-st}f(t)dt = \int_0^\pi e^{-st}\,dt + \lim_{M\to\infty}\int_\pi^M (e^{-st})(0)dt$$
$$= \int_0^\pi e^{-st}dt = (1 - e^{-\pi s})/s.$$

Hence, the Laplace transform Y(s) of the solution is given by $Y(s) = s/(s^2+4) + (1 - e^{-\pi s})/s(s^2+4)$.

27b. The Taylor series for f about t = 0 is

$f(t) = \sum_{n=0}^{\infty}(-1)^n t^{2n}/(2n+1)!$, which is obtained from part(a) by dividing each term of the sine series by t. Also, f is continuous for t > 0 since $\lim_{t\to 0+}(\sin t)/t = 1$. Assuming that we can compute the Laplace transform of f term by term, we obtain $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sum_{n=0}^{\infty}(-1)^n t^{2n}/(2n+1)!\}$

$$= \sum_{n=0}^{\infty}[(-1)^n/(2n+1)!L\{t^{2n}\}$$
$$= \sum_{n=0}^{\infty}[(-1)^n(2n)!/(2n+1)!]s^{-(2n+1)}$$
$$= \sum_{n=0}^{\infty}[(-1)^n/(2n+1)]s^{-(2n+1)},$$ which converges for s > 1.

The Taylor series for arctan x is given by

$\sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)$, for $|x| < 1$. Comparing $\mathcal{L}\{f(t)\}$ with the Taylor series for arctanx, we conclude that $\mathcal{L}\{f(t)\} = \arctan(1/s)$, s > 1.

30. Setting n = 2 in Problem 28b, we have

$\mathcal{L}\{t^2\sin bt\} = \dfrac{d^2}{ds^2}[b/(s^2+b^2)] = \dfrac{d}{ds}[-2bs/(s^2+b^2)^2] =$

$-2b/(s^2+b^2)^2 + 8bs^2/(s^2+b^2)^3 = 2b(3s^2-b^2)/(s^2+b^2)^3$.

32. Using the result of Problem 28a. we have

$\mathcal{L}\{te^{at}\} = -\dfrac{d}{ds}(s-a)^{-1} = (s-a)^{-2}$

$\mathcal{L}\{t^2e^{at}\} = -\dfrac{d}{ds}(s-a)^{-2} = 2(s-a)^{-3}$.

$\mathcal{L}\{t^3e^{at}\} = -\dfrac{d}{ds}2(s-a)^{-3} = 3!(s-a)^{-4}$. Continuing in this fashion, or using induction, we obtain the desired result.

36a. Taking the Laplace transform of the D.E. we obtain
$\pounds\{y''\} - \pounds\{ty\} = \pounds\{y''\} + \pounds\{-ty\}$
$= s^2Y(s) - sy(0) - y'(0) + Y'(s) = 0.$
Hence, Y satisfies $Y' + s^2Y = s$.

38a. From Eq(i) we have $A_k = \lim\limits_{s\to r_k} (s-r_k)\dfrac{P(r_k)}{Q(r_k)}$, since Q has distinct zeros. Thus $A_k = P(r_k)\lim\limits_{s\to r_k}\dfrac{s-r_k}{Q(r_k)} = \dfrac{P(r_k)}{Q'(r_k)}$, by L'Hopital's Rule.

38b. Since $\pounds^{-1}\left\{\dfrac{1}{s-r_k}\right\} = e^{r_k t}$, the result follows.

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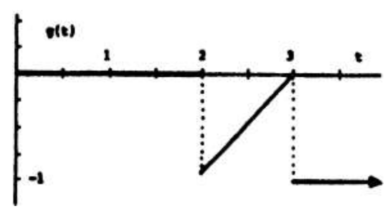
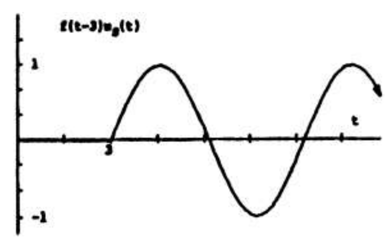
2. From the definition of $u_c(t)$ we have:
$g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
$$= \begin{cases} 0 - 0 = 0, & 0\le t<2 \\ (t-3) - 0 = t-3, & 2\le t<3. \\ (t-3) - (t-2) = -1, & 3\le t \end{cases}$$



4. As indicated in the discussion following Example 1, the unit step function can be used to translate a given function f, with domain $t\ge0$, a distance c to the right by the multiplication $u_c(t)f(t-c)$. Hence the required graph of $y = u_3(t)f(t-3)$ for $f(t) = \sin t$ is shown.



8. In order to use Theorem 6.3.1 we must write f(t) in terms of $u_c(t)$. Since $t^2- 2t + 2 = (t-1)^2 + 1$ (by completing the square), we can thus write $f(t) = u_1(t)g(t-1)$, where $g(t) = t^2+1$. Now applying Theorem 6.3.1 we have
$\pounds\{f(t)\} = \pounds\{u_1(t)g(t-1)\} = e^{-s}\,\pounds\{g(t)\} = e^{-s}(2/s^3 + 1/s).$

14. Use partial fractions to write $F(s) = e^{-2s}[(s-1)^{-1} - (s+2)^{-1}]/3$. For ease in calculations let us define $G(s) = (s-1)^{-1}$ and $H(s) = (s+2)^{-1}$. Then $F(s) = [e^{-2s} G(s) - e^{-2s} H(s)]/3$. Using the fact that $\pounds\{e^{at}\} = (s-a)^{-1}$ and applying Theorem 6.3.1, we have $F(s) = [e^{-2s} \pounds\{e^{t}\} - e^{-2s} \pounds\{e^{-2t}\}]/3$. Thus $F(s) = [\pounds\{u_2(t)e^{(t-2)}\} - \pounds\{u_2(t)e^{-2(t-2)}\}]/3$. Using the linearity of the Laplace transform, we have $\pounds\{f(t)\} = \pounds\{u_2(t)[e^{t-2} - e^{-2(t-2)}]/3\}$. Hence, $f(t) = [u_2(t)(e^{t-2} - e^{-2(t-2)})]/3$. An alternate method is to complete the square in the denominator: $F(s) = \dfrac{e^{-2s}}{(s+1/2)^2 - 9/4}$. This gives $f(t) = (2/3)u_2(t)e^{-(t-2)/2} \sinh\frac{3}{2}(t-2)$, which can be shown to be the same as that found above.

21. By completing the square in the denominator of F we can write $F(s) = (2s+1)/[(2s+1)^2 + 4]$. This has the form $G(2s+1)$ where $G(u) = u/(u^2+4)$. We must find $\pounds^{-1}\{G(2s+1)\}$. Applying the results of Problem 19(c), we have $\pounds^{-1}\{F(s)\} = \frac{1}{2}e^{-t/2}\cos(\frac{2t}{2})$.

22. If the approach of Problem 21 is used we find $f(t) = (1/3)e^{2t/3}\sinh(t/3)$, which is equivalent to the given answer using the definition of sinht.

27. Assuming that term-by-term integration of the infinite series is permissible and recalling that $\pounds\{u_c(t)\} = e^{-cs}/s$ for $s > 0$, we have $\pounds\{f(t)\} = (1/s) + \sum_{k=1}^{\infty}(-1)^k \pounds\{u_k(t)\}$
$= (1/s) + \sum_{k=1}^{\infty}(-1)^k e^{-ks}/s = [\sum_{k=0}^{\infty}(-e^{-s})^k]/s$. We recognize the last infinite series as the geometric series, $\sum_{k=0}^{\infty}ar^k$, with $a = 1$ and $r = -e^{-s}$. This series converges to $[1/(1+e^{-s})]$ if $|r| < 1$. Hence, $\pounds\{f(t)\} = (1/s)[1/(1+e^{-s})]$, $s > 0$.

28. Using the definition of the Laplace transform we have $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$. Since f is periodic with period T, we have $f(t+T) = f(t)$. This suggests that we rewrite the improper integral as $\int_0^\infty e^{-st}f(t)dt =$
$\sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st}f(t)dt$. The periodicity of f also suggests that we make the change of variable $t = r + nT$. Hence,
$F(s) = \sum_{n=0}^{\infty} \int_0^T e^{-s(r+nT)}f(r+nT)dr = \sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-rs}f(r)dr$,
where we have used the fact that
$f(r+nT) = f(r+(n-1)T) = \ldots = f(r+T) = f(r)$ from the definition that f is periodic. We recognize this last series as the geometric series, $\sum_{n=0}^{\infty} au^n$, with
$a = \int_0^T e^{-rs}f(r)dr$ and $u = e^{-sT}$. The geometric series converges to $a/(1-u)$ for $|u| < 1$ and consequently we obtain
$F(s) = (1 - e^{-sT})^{-1} \int_0^T e^{-rs}f(r)dr$, $s > 0$.

30. The function f is periodic with period 2. The result of Problem 28 gives us $\mathcal{L}\{f(t)\} = \int_0^2 e^{-st}f(t)dt/(1-e^{-2s})$.
Calculating the integral we have

$$\begin{aligned}\int_0^2 e^{-st}f(t)dt &= \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt \\ &= (1-e^{-s})/s + (e^{-2s}-e^{-s})/s \\ &= (e^{-2s}-2e^{-s}+1)/s \\ &= (1-e^{-s})^2/s.\end{aligned}$$

Since the denominator of $\mathcal{L}\{f(t)\}$, $1 - e^{-2s}$, may be written as $(1-e^{-s})(1+e^{-s})$ we obtain the desired answer.

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1. f(t) can be written in the form $f(t) = 1 - u_{\pi/2}(t)$ and thus the Laplace transforms of the D.E. is
$(s^2+1)Y(s) - sy(0) - y'(0) = (1/s) - e^{-\pi s/2}/s$. Introducing the I.C. and solving for Y(s), we obtain
$Y(s) = (s^2+1)^{-1} + [s(s^2+1)]^{-1} - e^{-\pi s/2}/s(s^2+1)$. Using partial fractions on the second and third terms we find

$Y(s)=(1/s) + (s^2+1)^{-1} - s/(s^2+1) - e^{-\pi s/2}/s + e^{-\pi s/2}s/(s^2+1)$. The inverse transform of the first three terms can be obtained directly from Table 6.2.1. Using Theorem 6.3.1 to find the inverse transform of the last two terms we have $\mathcal{L}^{-1}\{e^{-\pi s/2}/s\} = u_{\pi/2}(t)g(t - \pi/2)$ where $g(t) = \mathcal{L}^{-1}\{1/s\} = 1$ and $\mathcal{L}^{-1}\{e^{-\pi s/2}s/(s^2+1)\} = u_{\pi/2}(t)h(t - \pi/2)$ where $h(t) = \mathcal{L}^{-1}\{s/(s^2+1)\} = \cos t$. Hence,

$y = 1 + \sin t - \cos t + u_{\pi/2}(t)[\cos(t - \pi/2) - 1]$

$= 1 + \sin t - \cos t - u_{\pi/2}(t)[1 - \sin t]$. The graph of the forcing function is a unit pulse for $0 \le t < \pi/2$ and 0 thereafter. The graph of the solution will be composed of two segments. The first, for $0 \le t < \pi/2$, is a sinusoid oscillating about 1, which represents the system response to a unit forcing function and the given initial conditions. For $t \ge \pi/2$, the forcing function, $f(t)$, is zero and the "initial" conditions are

$y(\pi/2) = \lim_{t\to\pi^-/2} 1 + \sin t - \cos t = 2$ and

$y'(\pi/2) = \lim_{t\to\pi^-/2} \cos t + \sin t = 1$. In this case the system response is $y(t) = 2\sin t - \cos t$, which is a sinusoid oscillating about zero.

3. According to Theorem 6.3.1, $\mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\} = e^{-2\pi s}\,\mathcal{L}\{\sin t\} = e^{-2\pi s}/(s^2+1)$. Transforming the D.E., we have $(s^2+4)Y(s) - sy(0) - y'(0) = 1/(s^2+1) - e^{-2\pi s}/(s^2+1)$. Introducing the I.C. and solving for $Y(s)$, we obtain $Y(s) = (1-e^{-2\pi s})/(s^2+1)(s^2+4)$. We apply partial fractions to write

$Y(s) = [(s^2+1)^{-1} - (s^2+4)^{-1} - e^{-2\pi s}(s^2+1)^{-1} + e^{-2\pi s}(s^2+4)^{-1}]/3$.

We compute the inverse transform of the first two terms directly from Table 6.2.1 after noting that $(s^2+4)^{-1} = (1/2)[2/(s^2+4)]$. We apply Theorem 6.3.1 to the last two terms to obtain the solution,

$$y =(1/3)\{\sin t-(1/2)\sin 2t-u_{2\pi}(t)[\sin(t-2\pi)-(1/2)\sin 2(t-2\pi)]\}.$$

This may be simplified, using trigonometric identities, to $y = [(2\sin t - \sin 2t)(1-u_{2\pi}(t))]/6$. Note that the forcing function is $\sin t - \sin(t-2\pi) = 0$ for $t \ge 2\pi$. The solution is $y(t) = 2\sin t - \sin 2t$ for $0 \le t < 2\pi$. Thus $y(2\pi^-) = 0$ and $y'(2\pi^-) = 2\cos 2\pi - 2\cos 4\pi = 0$. Hence the "initial" value problem for $t \ge 2\pi$ is $y'' + 4y = 0$, $y(2\pi) = 0$, $y'(2\pi) = 0$, which has the trivial solution $y \equiv 0$.

8. Taking the Laplace transform, applying the I.C. and using Theorem 6.3.1 we have $(s^2+s+5/4)Y(s) = (1-e^{-\pi s/2})/s^2$. Thus

$$Y(s) = \frac{1-e^{-s/2}}{s^2(s^2+s+5/4)}$$

$$= (1-e^{-\pi s/2})\left\{\frac{4/5}{s^2} - \frac{16/25}{s} + \frac{(16/25)s-4/25}{(s+1/2)^2+1}\right\}$$

$= (1-e^{-\pi s/2})H(s)$, where we have used partial fractions and completed the square in the denominator of the last term. Since the numerator of the last term of H can be written as $\frac{16}{25}[(s+1/2) - 3/4]$, we see that

$£^{-1}\{H(s)\} = (4/25)(5t - 4 + 4e^{-t/2}\cos t - 3e^{-t/2}\sin t)$, which yields the desired solution. The graph of the forcing function is a ramp ($f(t) = t$) for $0 \le t < \pi/2$ and a constant ($f(t) = \pi/2$) for $t \ge \pi/2$. The solution will be a damped sinusoid oscillating about the "ramp" $(20t-16)/25$ for $0 \le t < \pi/2$ and oscillating about $2\pi/5$ for $t \ge \pi/2$.

10. Note that $g(t) = \sin t - u_\pi(t)\sin t = \sin t + u_\pi(t)\sin(t-\pi)$. Proceeding as in Problem 8 we find

$Y(s) = (1+e^{-\pi s})\dfrac{1}{(s^2+1)(s^2+s+5/4)}$. The correct partial fraction expansion of the quotient is $\dfrac{as+b}{s^2+1} + \dfrac{cs+d}{s^2+s+5/4}$, where

$a+c = 0$, $a+b+d = 0$, $(5/4)a+b+c = 0$ and $(5/4)b+d = 1$ by equating coefficients. Solving for the constants yields the desired solution.

16b. Taking the Laplace transform of the D.E. we obtain $U(s^2+s/4+1) = k(e^{-3s/2}-e^{-5s/2})/s$, since the I.C. are zero. Solving for U and using partial fractions yields

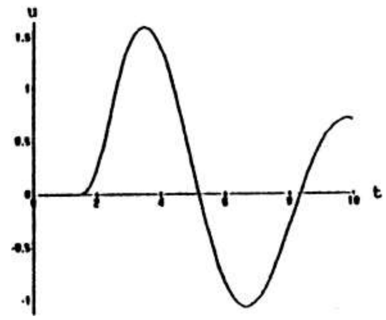
$U(s) = k(e^{-3s/2}-e^{-5s/2})\left(\dfrac{1}{s}-\dfrac{s+1/4}{s^2+s/4+1}\right)$. Thus, if

$H(s) = \left(\dfrac{1}{s}-\dfrac{s+1/4}{s^2+s/4+1}\right)$, then

$h(t) = 1 - e^{-t/8}\left(\cos\dfrac{3\sqrt{7}}{8}t+\dfrac{\sqrt{7}}{21}\sin\dfrac{3\sqrt{7}}{8}t\right)$ and

$u(t) = ku_{3/2}(t)h(t-3/2) - ku_{5/2}(t)h(t-5/2)$.

16c. In all cases the plot will be zero for $0 \le t < 3/2$. For $3/2 \le t < 5/2$ the plot will be the system response (damped sinusoid) to a step input of magnitude k. For $t \geq 5/2$, the plot will be the system response to the I.C. $u(5^-/2)$, $u'(5^-/2)$ with no forcing function. The graph shown is for k = 2. Varying k will just affect the amplitude. Note that the amplitude never reaches 2, which would be the steady state response for the step input $2u_{3/2}(t)$. Note also that the solution and its derivative are continuous at t = 5/2.



19a. The graph on $0 \leq t < 6\pi$ will depend on how large n is. For instance, if n = 2 then

$$f(t) = \begin{cases} 1 & 0 \leq t < \pi,\ 2\pi \leq t < 6\pi \\ -1 & -\pi \leq t < 2\pi \end{cases}.$$ For

$$n \geq 6,\ f(t) = \begin{cases} 1 & 0 \leq t < \pi,\ 2\pi \leq t < 3\pi,\ 4\pi \leq t < 5\pi \\ -1 & \pi \leq t < 2\pi,\ 3\pi \leq t < 4\pi,\ 5\pi \quad t < 6\pi \end{cases}.$$

19b. Taking the Laplace transform of the D.E. and using the I.C. we have $Y(s) = \dfrac{1}{s(s^2+1)}[1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-\pi ks}]$, since $\mathcal{L}\{u_{\pi k}(t)\} = \dfrac{e^{-\pi ks}}{s}$. Since $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$, we then obtain

$$y(t) = 1 - \cos t + 2\sum_{k=1}^{\infty}(-1)^{-k}u_{\pi k}(t)[1 - \cos(t-\pi k)],$$ using line 13 in Table 6.2.1.

19d. Since $\cos(t-\pi k) = (-1)^k \cos t$, the solution in part b can be written as

$$y(t) = 1 - \cos t + 2\sum_{k=1}^{\infty}(-1)^k u_{\pi k}(t) - 2\sum_{k=1}^{\infty}(-1)^{2k} \cos t$$

$$= 1 - \cos t - 2n\cos t + 2\sum_{k=1}^{\infty}(-1)^k u_{\pi k}(t)$$ which diverges for $n \to \infty$.

20. In this case

$$Y(s) = \frac{1}{s(s^2+.1s+1)}\left[1 + 2\sum_{n=1}^{\infty}(-1)^k e^{-\pi ks}\right].$$ Using partial fractions we have

$$H(s) = \frac{1}{s(s^2+.1s+1)} = \frac{1}{s} - \frac{s+.1}{s^2+.1s+1}$$
$$= \frac{1}{s} - \frac{s+.05}{(s+.05)^2+b^2} - \frac{.05}{(s+.05)^2+b^2},$$ where

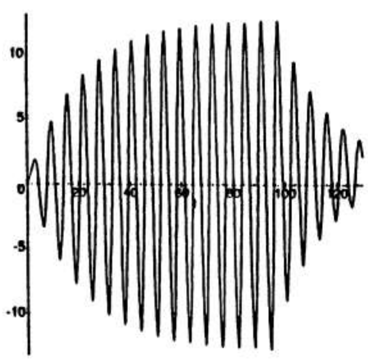
$b^2 = [1-(.05)^2] = .9975$. Now let

$h(t) = \pounds^{-1}\{H(s)\} = 1 - e^{-.05t}\cos bt - \frac{.05}{b}e^{-.05t}\sin bt$. Hence,

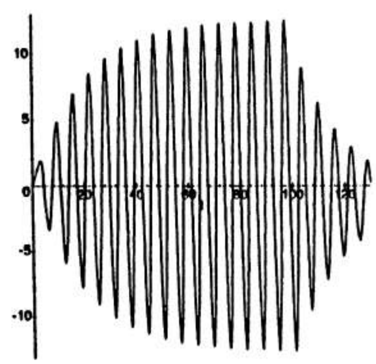
$y(t) = h(t) + 2\sum_{n=1}^{\infty}(-1)^k u_{\pi k}(t)\, h(t-\pi k)$, and thus, for $t > n$ the solution will be approximated by

$\pm 1 - Ae^{-.05(t-n\pi)}\cos[b(t-n\pi) + \delta]$, and therefore converges as $t\to\infty$.

20a. $y(t)$ for $n = 30$

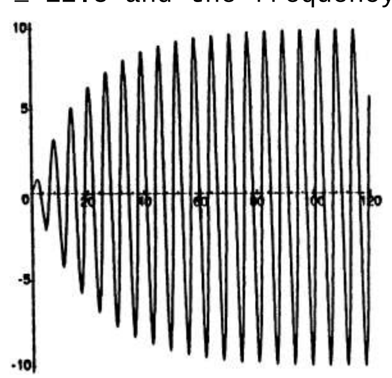


$y(t)$ for $n = 31$



20b. From the graph of part a, $A \cong 12.5$ and the frequency is 2π.

20c. From the graph (or analytically) $A = 10$ and the frequency is 2π.



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1. Proceeding as in Example 1, we take the Laplace transform of the D.E. and apply the I.C.:
$(s^2 + 2s + 2)Y(s) = s + 2 + e^{-\pi s}$. Thus,
$Y(s) = (s+2)/[(s+1)^2 + 1] + e^{-\pi s}/[(s+1)^2 + 1]$. We write the first term as $(s+1)/[(s+1)^2 + 1] + 1/[(s+1)^2 + 1]$. Applying Theorem 6.3.1 and using Table 6.2.1, we obtain the solution, $y = e^{-t}\cos t + e^{-t}\sin t - u_\pi(t)e^{-(t-\pi)}\sin t$.
Note that $\sin(t-\pi) = -\sin t$.

3. Taking the Laplace transform and using the I.C. we have
$(s^2+ 3s+2)Y(s) = \dfrac{1}{2} + e^{-5s} + \dfrac{e^{-10s}}{s}$. Thus
$Y(s) = \dfrac{1/2}{s^2+3s+2} + \dfrac{e^{-5s}}{s^2+3s+2} + e^{-10s}\left(\dfrac{1/2}{s}+\dfrac{1/2}{s+2}-\dfrac{1}{s+1}\right)$ and hence
$y(t) = \dfrac{1}{2}h(t) + u_5(t)h(t-5) + u_{10}(t)[\dfrac{1}{2}+\dfrac{1}{2}e^{-2(t-10)}-e^{-(t-10)}]$
where $h(t) = e^{-t} - e^{-2t}$.

5. The Laplace transform of the D.E. is
$(s^2+2s+3)Y(s) = \dfrac{1}{s^2+1} + e^{-3\pi s}$, so
$Y(s) = \dfrac{1}{(s^2+1)(s^2+2s+3)} + e^{-3\pi s}[\dfrac{1}{s^2+2s+3}]$. Using partial fractions or a computer algebra system we obtain
$y(t) = \dfrac{1}{4}\sin t - \dfrac{1}{4}\cos t + \dfrac{1}{4}e^{-t}\cos\sqrt{2}\,t + \dfrac{1}{\sqrt{2}}u_{3\pi}(t)h(t-3\pi)$,
where $h(t) = e^{-t}\sin\sqrt{2}\,t$.

7. Taking the Laplace transform of the D.E. yields
$(s^2+1)Y(s) - y'(0) = \int_0^\infty e^{-st}\delta(t-2\pi)\cos t\,dt$. Since $\delta(t-2\pi) = 0$ for $t \neq 2\pi$ the integral on the right is equal to $\int_{-\infty}^{\infty} e^{-st}\delta(t-2\pi)\cos t\,dt$ which equals $e^{-2\pi s}\cos 2\pi$ from Eq.(16). Substituting for $y'(0)$ and solving for $Y(s)$ gives $Y(s) = \dfrac{1}{s^2+1} + \dfrac{e^{-2\pi s}}{s^2+1}$ and hence
$$y(t) = \sin t + u_{2\pi}(t)\sin(t-2\pi) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 2\sin t & 2\pi \le t \end{cases}$$

10. See the solution for Problem 7.

13a. From Eq. (22) y(t) will complete one cycle when $\sqrt{15}\,(t-5)/4 = 2\pi$ or $T = t - 5 = 8\pi/\sqrt{15}$, which is consistent with the plot in Fig. 6.5.3. Since an impulse causes a discontinuity in the first derivative, we need to find the value of y' at $t = 5$ and $t = 5 + T$. From Eq. (22) we have, for $t \geq 5$,

$$y' = e^{-(t-5)/4}\left[\frac{-1}{2\sqrt{15}}\sin\frac{\sqrt{15}}{4}(t-5) + \frac{1}{2}\cos\frac{\sqrt{15}}{4}(t-5)\right].$$ Thus

$y'(5) = \frac{1}{2}$ and $y'(5+T) = \frac{1}{2}e^{-T/4}$. Since the original impulse, $\delta(t-5)$, caused a discontinuity in y' of 1/2, we must choose the impulse at $t = 5 + T$ to be $-e^{-T/4}$, which is equal and opposite to y' at $5 + T$.

13b. Now consider $2y'' + y' + 2y = \delta(t-5) + k\delta(t-5-T)$ with $y(0) = 0$, $y'(0) = 0$. Using the results of Example 1 we have

$$y(t) = \frac{2}{\sqrt{15}}u_5(t)e^{-(t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5)$$

$$+ \frac{2k}{\sqrt{15}}u_{5+T}(t)e^{-(t-5-T)/4}\sin\frac{\sqrt{15}}{4}(t-5-T)$$

$$= \frac{2}{\sqrt{15}}e^{-(t-5)/4}\left[u_5(t)\sin\frac{\sqrt{15}}{4}(t-5)+ku_{5+T}(t)e^{T/4}\sin\frac{\sqrt{15}}{4}(t-5-T)\right]$$

$$= \frac{2}{\sqrt{15}}e^{-(t-5)/4}\left[u_5(t)+ke^{T/4}u_{5+T}(t)\right]\sin\frac{\sqrt{15}}{4}(t-5).$$ If

$y(t) \equiv 0$ for $t \geq 5 + T$, then $1 + ke^{T/4} = 0$, or $k = -e^{-T/4}$, as found in part (a).

17b. We have $(s^2+1)Y(s) = \sum_{k=1}^{20} e^{-k\pi s}$ so that $Y(s) = \sum_{k=1}^{20} \frac{e^{-ks}}{s^2+1}$

and hence $y(t) = \sum_{k=1}^{20} u_{k\pi}(t)\sin(t-k\pi)$

$= u_\pi(t)\sin(t-\pi) + u_{2\pi}(t)\sin(t-2\pi) + \ldots + u_{10\pi}\sin(t-10\pi)$.

For $0 \leq t < \pi$, $y(t) \equiv 0$. For $\pi \leq t < 2\pi$, $y(t) = \sin(t-\pi) = -\sin t$. For $2\pi \leq t < 3\pi$, $y(t) = \sin(t-\pi) + \sin(t-2\pi) = -\sin t + \sin t \equiv 0$. Due to the periodicity of $\sin t$, the solution will exhibit this behavior in alternate intervals for $0 \leq t < 20\pi$. After $t = 20\pi$ the solution remains at zero.

21b. Taking the transform and using the I.C. we have

$$(s^2+1)Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi} \text{ so that } Y(s) = \sum_{k=1}^{15} \frac{e^{-(2k-1)\pi}}{s^2+1}.$$

Thus $y(t) = \displaystyle\sum_{k=1}^{15} u_{(2k-1)\pi}(t)\sin[t-(2k-1)\pi]$

$= \sin(t-\pi) + \sin(t-3\pi) \ldots + \sin(t-29\pi)$

$= -\sin t - \sin t \ldots -\sin t$

$= -15\sin t.$

25b. Substituting for f(t) we have $y = \int_0^t e^{-(t-\tau)}\delta(\tau-\pi)\sin(t-\tau)d\tau$. We know that the integration variable is always less than t (the upper limit) and thus for $t < \pi$ we have $\tau < \pi$ and thus $\delta(\tau-\pi) = 0$. Hence $y = 0$ for $t < \pi$. For $t > \pi$ utilize Eq.(16).

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1c. Using the format of Eqs.(2) and (3) we have

$$\begin{aligned} f*(g*h) &= \int_0^t f(t-\tau)(g*h)(\tau)d\tau \\ &= \int_0^t f(t-\tau)\left[\int_0^\tau g(\tau-\eta)h(\eta)d\eta\right]d\tau \\ &= \int_0^t \left[\int_\eta^t f(t-\tau)g(\tau-\eta)d\tau\right]h(\eta)(d\eta). \end{aligned}$$

The last double integral is obtained from the previous line by interchanging the order of the η and τ integrations. Making the change of variable $\omega = \tau - \eta$ on the inside integral yields

$$\begin{aligned} f*(g*h) &= \int_0^t \left[\int_0^{t-\eta} f(t-\eta-\omega)g(\omega)d\omega\right]h(\eta)d\eta \\ &= \int_0^t (f*g)(t-\eta)h(\eta)d\eta = (f*g)*h. \end{aligned}$$

4. It is possible to determine f(t) explicitly by using integration by parts and then find its transform F(s). However, it is much more convenient to apply Theorem 6.6.1. Let us define $g(t) = t^2$ and $h(t) = \cos 2t$. Then, $f(t) = \int_0^t g(t-\tau)h(\tau)d\tau$. Using Table 6.2.1, we have $G(s) = \mathcal{L}\{g(t)\} = 2/s^3$ and $H(s) = \mathcal{L}\{h(t)\} = s/(s^2+4)$. Hence, by Theorem 6.6.1, $\mathcal{L}\{f(t)\} = F(s) = G(s)H(s) = 2/s^2(s^2+4)$.

8. As was done in Example 1 think of F(s) as the product of s^{-4} and $(s^2+1)^{-1}$ which, according to Table 6.2.1, are the transforms of $t^3/6$ and sint, respectively. Hence, by Theorem 6.6.1, the inverse transform of F(s) is
$f(t) = (1/6)\int_0^t (t-\tau)^3 \sin\tau d\tau$.

13. We take the Laplace transform of the D.E. and apply the I.C.: $(s^2 + 2s + 2)Y(s) = \alpha/(s^2 + \alpha^2)$. Solving for Y(s), we have $Y(s) = [\alpha/(s^2+\alpha^2)][(s+1)^2 + 1]^{-1}$, where the second factor has been written in a convenient way by completing the square. Thus Y(s) is seen to be the product of the transforms of $\sin\alpha t$ and $e^{-t}\sin t$ respectively. Hence, according to Theorem 6.6.1, $y = \int_0^t e^{-(t-\tau)}\sin(t-\tau)\sin\alpha\tau d\tau$.

15. Proceeding as in Problem 13 we obtain
$$Y(s) = \frac{s}{s^2+s+5/4} + \frac{1-e^{-s}}{s(s^2+s+5/4)}$$
$$= \frac{(s+1/2) - 1/2}{(s+1/2)^2+1} + \frac{1-e^{-s}}{s} \cdot \frac{1}{(s+1/2)^2+1},$$
where the first term is obtained by completing the square in the denominator and the second term is written as the product of two terms whose inverse transforms are known, so that Theorem 6.6.1 can be used. Note that $\mathcal{L}^{-1}\{(1-e^{-s})/s\} = 1 - u_\pi(t)$. Also note that a different form of the same solution would be obtained by writing the second term as $(1-e^{-\pi s})(\frac{a}{s} + \frac{bs + c}{(s+1/2)^2+1})$ and solving for a, b and c. In this case $\mathcal{L}^{-1}\{1-e^{-s}\} = \delta(t) - \delta(t-\pi)$ from Section 6.5.

17. Taking the Laplace transform, using the I.C. and solving, we have $Y(s) = (s+3)/(s+1)(s+2) + s/(s^2+\alpha^2)(s+1)(s+2)$. As in Problem 15, there are several correct ways the second term can be treated in order to use the convolution integral. In order to obtain the desired answer, write the second term as
$\frac{s}{s^2+\alpha^2}(\frac{a}{s+1} + \frac{b}{s+2})$ and solve for a and b.

20. To find $\Phi(s)$ you must recognize the integral that

appears in the equation as a convolution integral. Taking the transform of both sides then yields

$\Phi(s) + K(s)\Phi(s) = F(s)$, or $\Phi(s) = \dfrac{F(s)}{1+K(s)}$.

Capítulo 7

CHAPTER 7

Section 7.1, Page 344

2. As in Example 1, let $x_1 = u$ and $x_2 = u'$, then $x_1' = x_2$ and $x_2' = u'' = 3\sin t - .5x_2 - 2x_1$.

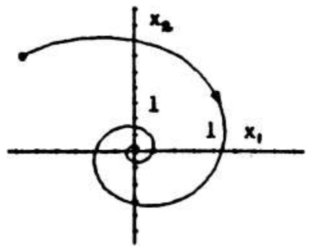
4. In this case let $x_1 = u$, $x_2 = u'$, $x_3 = u''$, and $x_4 = u'''$.

5. Let $x_1 = u$ and $x_2 = u'$; then $x_1' = x_2$ is the first of the desired pair of equations. The second equation is obtained by substituting $u'' = x_2'$, $u' = x_2$, and $u = x_1$ in the given D.E. The I.C. become $x_1(0) = u_0$, $x_2(0) = u_0'$.

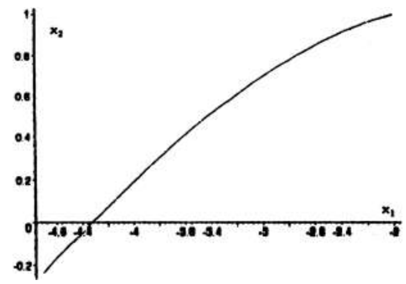
8. Follow the steps outlined in Problem 7. Solve the first D.E. for x_2 to obtain $x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1'$. Substitute this into the second D.E. to obtain $x_1'' - x_1' - 2x_1 = 0$, which has the solution $x_1 = c_1e^{2t} + c_2e^{-t}$. Differentiating this and substituting into the above equation for x_2 yields $x_2 = \frac{1}{2}c_1e^{2t} + 2c_2e^{-t}$. The I.C. then give $c_1 + c_2 = 3$ and $\frac{1}{2}c_1 + 2c_2 = \frac{1}{2}$, which yield $c_1 = \frac{11}{3}$, $c_2 = -\frac{2}{3}$. Thus $x_1 = \frac{11}{3}e^{2t} - \frac{2}{3}e^{-t}$ and $x_2 = \frac{11}{6}e^{2t} - \frac{4}{3}e^{-t}$. Note that for large t, the second term in each solution vanishes and we have $x_1 \cong \frac{11}{3}e^{2t}$ and $x_2 \cong \frac{11}{6}e^{2t}$, so that $x_1 \cong 2x_2$. This says that the graph will be asymptotic to the line $x_1 = 2x_2$ for large t.

9. Solving the first D.E. for x_2 gives $x_2 = \frac{4}{3}x_1' - \frac{5}{3}x_1$, which substituted into the second D.E. yields $x_1'' - 2.5x_1' + x_1 = 0$. Thus $x_1 = c_1e^{t/2} + c_2e^{2t}$ and $x_2 = -c_1e^{t/2} + c_2e^{2t}$. Using the I.C. yields $c_1 = -3/2$ and $c_2 = -1/3$. For large t, $x_1 \cong (-1/2)e^{2t}$ and $x_2 \cong (-1/2)e^{2t}$ and thus the graph is asymptotic to $x_1 = x_2$ in the third quadrant. The graph is shown on the right.

12.



9.



12. Solving the first D.E. for x_2 gives $x_2 = \frac{1}{2}x_1' + \frac{1}{4}x_1$ and substitution into the second D.E. gives $x_1'' + x_1' + \frac{17}{4}x_1 = 0$. Thus $x_1 = e^{-t/2}(c_1\cos 2t + c_2\sin 2t)$ and $x_2 = e^{-t/2}(c_2\cos 2t - c_1\sin 2t)$. The I.C. yields $c_1 = -2$ and $c_2 = 2$.

14. If $a_{12} \neq 0$, then solve the first equation for x_2, obtaining $x_2 = [x_1' - a_{11}x_1 - g_1(t)]/a_{12}$. Upon substituting this expression into the second equation, we have a second order linear O.D.E. for x_1. One I.C. is $x_1(0) = x_1^0$. The second I.C. is $x_2(0) = [x_1'(0) - a_{11}x_1(0) - g_1(0)]/a_{12} = x_2^0$. Solving for $x_1'(0)$ gives $x_1'(0) = a_{12}x_2^0 + a_{11}x_1^0 + g_1(0)$. These results hold when $a_{11}, \ldots, a_{22}$ are functions of t as long as the derivatives exist and $a_{12}(t)$ and $a_{21}(t)$ are not both zero on the interval. The initial conditions will involve $a_{11}(0)$ and $a_{12}(0)$.

19. Let us number the nodes 1,2, and 3 clockwise beginning with the top right node in Figure 7.1.4. Also let I_1, I_2, I_3, and I_4 denote the currents through the resistor $R = 1$, the inductor $L = 1$, the capacitor $C = \frac{1}{2}$, and the resistor $R = 2$, respectively. Let V_1, V_2, V_3, and V_4 be the corresponding voltage drops. Kirchhoff's first law applied to nodes 1 and 2, respectively, gives (i) $I_1 - I_2 = 0$ and (ii) $I_2 - I_3 - I_4 = 0$. Kirchhoff's second law applied to each loop gives (iii) $V_1 + V_2 + V_3 = 0$ and (iv) $V_3 - V_4 = 0$. The current-

voltage relation through each circuit element yields four more equations: (v) $V_1 = I_1$, (vi) $I_2' = v_2$, (vii) $(1/2)V_3' = I_3$ and (viii) $V_4 = 2I_4$. We thus have a system of eight equations in eight unknowns, and we wish to eliminate all of the variables except I_2 and V_3 from this system of equations. For example, we can use Eqs.(i) and (iv) to eliminate I_1 and V_4 in Eqs.(v) and (viii). Then use the new Eqs.(v) and (viii) to eliminate V_1 and I_4 in Eqs.(ii) and (iii). Finally, use the new Eqs. (ii) and (iii) in Eqs.(vi) and (vii) to obtain $I_2' = - I_2 - V_3$, $V_3' = 2I_2 - V_3$. These equations are identical (when subscripts on the remaining variables are dropped) to the equations given in the text.

21a. Note that the amount of water in each tank remains constant. Thus $Q_1(t)/30$ and $Q_2(t)/20$ represent oz./gal of salt in each tank. As in Example 1 of Section 2.3, we assume the mixture in each tank is well stirred. Then, for the first tank we have $\frac{dQ_1}{dt} = 1.5 - 3\frac{Q_1(t)}{30} + 1.5\frac{Q_2(t)}{20}$, where the first term on the right represents the amount of salt per minute entering the mixture from an external source, the second term represents the loss of salt per minute going to Tank 2 and the third term represents the gain of salt per minute entering from Tank 2. Similarly, we have $\frac{dQ_2}{dt} = 3 + 3\frac{Q_1(t)}{30} - 4\frac{Q_2(t)}{20}$ for Tank 2.

21b. Solve the second equation for $Q_1(t)$ to obtain $Q_1(t) = 10Q_2' + 2Q_2 - 30$. Substitution into the first equation then yields $10Q_2'' + 3Q_2' + \frac{1}{8}Q_2 = \frac{9}{2}$. The steady state solution for this is $Q_2^E = 8(9/2) = 36$. Substituting this value into the equation for Q_1 yields $Q_1^E = 72 - 30 = 42$.

21c. Substitute $Q_1 = x_1 + 42$ and $Q_2 = x_2 + 36$ into the equations found in part a.

Section 7.2, Page 355

1a. $2\mathbf{A} = \begin{pmatrix} 2 & -4 & 0 \\ 6 & 4 & -2 \\ -4 & 2 & 6 \end{pmatrix}$ so that

$$2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2+4 & -4-2 & 0+3 \\ 6-1 & 4+5 & -2+0 \\ -4+6 & 2+1 & 6+2 \end{pmatrix} = \begin{pmatrix} 6 & -6 & 3 \\ 5 & 9 & -2 \\ 2 & 3 & 8 \end{pmatrix}$$

1c. Using Eq.(9) and following Example 1 we have

$$\mathbf{AB} = \begin{pmatrix} 4+2+0 & -2-10+0 & 3+0+0 \\ 12-2-6 & -6+10-1 & 9+0-2 \\ -8-1+18 & 4+5+3 & -6+0+6 \end{pmatrix},$$

which yields the correct answer.

6. $\mathbf{AB} = \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix}$ and $\mathbf{BC} = \begin{pmatrix} 5 & 3 & 3 \\ -1 & 7 & 3 \\ 2 & 3 & -2 \end{pmatrix}$ so that

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 7 & -11 & -3 \\ 11 & 20 & 17 \\ -4 & 3 & -12 \end{pmatrix}.$$

In problems 10 through 19 the method of row reduction, as illustrated in Example 2, can be used to find the inverse matrix or else to show that none exists. We start with the original matrix augmented by the indentity matrix, describe a suitable sequence of elementary row operations, and show the result of applying these operations.

10. Start with the given matrix augmented by the identity matrix. $\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right)$

Add 2 times the first row to the second row.

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 11 & 2 & 1 \end{array}\right)$$

Multiply the second row by (1/11).

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2/11 & 1/11 \end{array}\right)$$

Add (-4) times the second row to the first row.

$$\left(\begin{array}{cc|cc} 1 & 0 & 3/11 & -4/11 \\ 0 & 1 & 2/11 & 1/11 \end{array}\right)$$

Since we have performed the same operation on the given matrix and the identity matrix, the 2 x 2 matric appearing on the right side of this augmented matrix is the desired inverse matrix. The answer can be checked by multiplying it by the given matrix; the result should be the indentity matrix.

12. The augmented matrix in this case is:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right)$$

Add (-2) times the first row to the second row and (-3) times the first row to the third row.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right)$$

Multiply the second and third rows by (-1) and interchange them.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

Add (-3) times the third row to the first and second rows. $\begin{pmatrix} 1 & 2 & 0 & \vdots & -5 & 3 & 0 \\ 0 & 1 & 0 & \vdots & -3 & 3 & -1 \\ 0 & 0 & 1 & \vdots & 2 & -1 & 0 \end{pmatrix}$

Add (-2) times the second row to the first row.

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 1 & -3 & 2 \\ 0 & 1 & 0 & \vdots & -3 & 3 & -1 \\ 0 & 0 & 1 & \vdots & 2 & -1 & 0 \end{pmatrix}$$

The desired answer appears on the right side of this augmented matrix.

14. Again, start with the given matrix augmented by the identity matrix. $\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ -2 & 1 & 8 & \vdots & 0 & 1 & 0 \\ 1 & -2 & -7 & \vdots & 0 & 0 & 1 \end{pmatrix}$

Add (2) times the first row to the second row and add(-1) times the first row to the third row.

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 5 & 10 & \vdots & 2 & 1 & 0 \\ 0 & -4 & -8 & \vdots & -1 & 0 & 1 \end{pmatrix}$$

Add (4/5) times the second row to the third row.

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 5 & 10 & \vdots & 2 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 3/5 & 4/5 & 0 \end{pmatrix}$$

Since the third row of the left matrix is all zeros, no further reduction can be performed, and the given matrix is singular.

22. $\mathbf{x}' = \begin{pmatrix} 4 \\ 2 \end{pmatrix} 2e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$; and

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 12-4 \\ 8-4 \end{pmatrix} e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}.$$

25. $\mathbf{\Psi}' = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}.$

Section 7.3, Page 366

1. Form the augmented matrix, as in Example 1, and use row reduction.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{array}\right)$$

Add (-3) times the first row to the second and add the first row to the third.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

Add (-1) times the second row to the third.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

The third row is equivalent to $-3x_3 = 1$ or $x_3 = -1/3$. Likewise the second row is equivalent to $x_2 + 4x_3 = 1$, so $x_2 = 7/3$. Finally, from the first row, $x_1 - x_3 = 0$, so $x_1 = -1/3$. The answer can be checked by substituting into the original equations.

2. The augmented matrix is $\begin{pmatrix} 1 & 2 & -1 & \vdots & 1 \\ 2 & 1 & 1 & \vdots & 1 \\ 1 & -1 & 2 & \vdots & 1 \end{pmatrix}$. Row reduction then

yields $\begin{pmatrix} 1 & 2 & -1 & \vdots & 1 \\ 0 & -3 & 3 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 1 \end{pmatrix}$.

The last row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 1$, and there is no choice of x_1, x_2, and x_3 that satisfies this equation. Hence the given system of equations has no solution.

3. Form the augmented matrix and use row reduction.

$$\begin{pmatrix} 1 & 2 & -1 & \vdots & 2 \\ 2 & 1 & 1 & \vdots & 1 \\ 1 & -1 & 2 & \vdots & -1 \end{pmatrix}$$

Add (-2) times the first row to the second and add (-1) times the first row to the third.

$$\begin{pmatrix} 1 & 2 & -1 & \vdots & 2 \\ 0 & -3 & 3 & \vdots & -3 \\ 0 & -3 & 3 & \vdots & -3 \end{pmatrix}$$

Add (-1) times the second row to the third row and then multiply the second row by (-1/3).

$$\begin{pmatrix} 1 & 2 & -1 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

Since the last row has only zero entries, it may be dropped. The second row corresponds to the equation $x_2 - x_3 = 1$. We can assign an arbitrary value to either x_2 or x_3 and use this equation to solve for the other. For example, let $x_3 = c$, where c is arbitrary. Then $x_2 = 1 + c$. The first row corresponds to the equation $x_1 + 2x_2 - x_3 = 2$, so $x_1 = 2 - 2x_2 + x_3 = 2 - 2(1+c)+c = -c$.

6. To determine whether the given set of vectors is linearly independent we must solve the system $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$ for c_1, c_2, and c_3. Writing this in scalar form, we have

$$\begin{aligned} c_1 \quad\;\; &+ c_3 = 0 \\ c_1 + c_2 \quad\;\; &= 0, \\ c_2 &+ c_3 = 0 \end{aligned}$$

so the augmented matrix is

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & 0 \\ 1 & 1 & 0 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \end{pmatrix}$$

Row reduction yields

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 2 & \vdots & 0 \end{pmatrix}.$$

From the third row we have $c_3 = 0$. Then from the second row, $c_2 - c_3 = 0$, so $c_2 = 0$. Finally from the first row $c_1 + c_3 = 0$, so $c_1 = 0$. Since $c_1 = c_2 = c_3 = 0$, we conclude that the given vectors are linearly independent.

8. As in Problem 6 we wish to solve the system $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$ for c_1, c_2, c_3, and c_4. Form the augmented matrix and use row reduction.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 2 & 0 & -1 & 0 & 0 \\ 2 & 3 & 1 & -1 & 0 \\ 3 & 1 & 0 & 3 & 0 \end{array}\right)$$

Add (-2) times the first row to the second, add (-2) times the first row to the third, and add (-3) times the first row to the fourth.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 2 & 3 & 6 & 0 \\ 0 & 5 & 5 & 5 & 0 \\ 0 & 4 & 6 & 12 & 0 \end{array}\right)$$

Multiply the second row by (1/2) and then add (-5) times the second row to the third and add (-4) times the second row to the fourth.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 1 & 3/2 & 3 & 0 \\ 0 & 0 & -5/2 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The third row is equivalent to the equation $c_3 + 4c_4 = 0$. One way to satisfy this equation is by choosing $c_4 = -1$; then $c_3 = 4$. From the second row we then have $c_2 = -(3/2)c_3 - 3c_4 = -6 + 3 = -3$. Then, from the first row, $c_1 = c_2 + 2c_3 + 3c_4 = -3 + 8 - 3 = 2$. Hence the given vectors are linearly dependent, and satisfy $2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} + 4\mathbf{x}^{(3)} - \mathbf{x}^{(4)} = \mathbf{0}$.

14. Let $t = t_0$ be a fixed value of t in the interval $0 \le t \le 1$. To determine whether $\mathbf{x}^{(1)}(t_0)$ and $\mathbf{x}^{(2)}(t_0)$ are

linearly dependent we must solve $c_1\mathbf{x}^{(1)}(t_0)+c_2\mathbf{x}^{(2)}(t_0)=\mathbf{0}$. We have the augmented matrix

$$\begin{pmatrix} e^{t_0} & 1 & . & 0 \\ t_0e^{t_0} & t_0 & . & 0 \end{pmatrix}.$$

Multiply the first row by $(-t_0)$ and add to the second row to obtain $\begin{pmatrix} e^{t_0} & 1 & . & 0 \\ 0 & 0 & . & 0 \end{pmatrix}$.

Thus, for example, we can choose $c_1 = 1$ and $c_2 = -e^{t_0}$, and hence the given vectors are linearly dependent at t_0. Since t_0 is arbitrary the vectors are linearly dependent at each point in the interval. However, there is no linear relation between $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ that is valid throughout the interval $0 \le t \le 1$. For example, if $t_1 \ne t_0$, and if c_1 and c_2 are chosen as above, then

$$c_1\mathbf{x}^{(1)}(t_1) + c_2\mathbf{x}^{(2)}(t_1) = \begin{pmatrix} e^{t_1} \\ t_1e^{t_1} \end{pmatrix} + -e^{t_0}\begin{pmatrix} 1 \\ t_1 \end{pmatrix} = \begin{pmatrix} e^{t_1} - e^{t_0} \\ t_1e^{t_1} - t_1e^{t_0} \end{pmatrix} \ne \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence the given vectors must be linearly independent on $0 \le t \le 1$. In fact, the same argument applies to any interval.

15. To find the eigenvalues and eigenvectors of the given matrix we must solve $\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(5-\lambda)(1-\lambda) - (-1)(3) = 0$, or $\lambda^2 - 6\lambda + 8 = 0$. Hence $\lambda_1 = 2$ and $\lambda_2 = 4$ are the eigenvalues. The eigenvector corresponding to λ_1 must satisfy $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $3x_1 - x_2 = 0$. If we let $x_1 = 1$, then $x_2 = 3$ and the eigenvector is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, or any constant multiple of this vector. Similarly, the eigenvector corresponding to λ_2 must satisfy

$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $x_1 - x_2 = 0$. Hence $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, or a multiple thereof.

18. Since $\bar{a}_{12} = a_{21}$, the given matrix is Hermitian and we know in advance that its eigenvalues are real. To find the eigenvalues and eigenvectors we must solve $\begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(1-\lambda)^2 - i(-i) = \lambda^2 - 2\lambda$, so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$; observe that they are indeed real even though the given matrix has imaginary entries. The eigenvector corresponding to λ_1 must satisfy $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $x_1 + ix_2 = 0$. Note that the second equation $-ix_1 + x_2 = 0$ is a multiple of the first. If $x_1 = 1$, then $x_2 = i$, and the eigenvector is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. In a similar way we find that the eigenvector associated with λ_2 is $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

21. The eigenvalues and eigenvectors satisfy $\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(1-\lambda)[(1-\lambda)^2 + 4] = 0$, which has roots $\lambda = 1, 1 \pm 2i$. For $\lambda = 1$, we then have $2x_1 - 2x_3 = 0$ and $3x_1 + 2x_2 = 0$. Choosing $x_1 = 2$ then yields $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ as the eigenvector corresponding to $\lambda = 1$. For $\lambda = 1 + 2i$ we have $-2ix_1 = 0$, $2x_1 - 2ix_2 - 2x_3 = 0$ and $3x_1 + 2x_2 - 2ix_3 = 0$, yielding $x_1 = 0$ and $x_3 = -ix_2$. Thus $\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 1 + 2i$. A similar calculation shows that

$\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 1 - 2i$.

24. Since the given matrix is real and symmetric, we know that the eigenvalues are real. Further, even if there are repeated eigenvalues, there will be a full set of three linearly independent eigenvectors. To find the eigenvalues and eigenvectors we must solve

$$\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ The determinant of

coefficients is $(3-\lambda)[-\lambda(3-\lambda)-4] - 2[2(3-\lambda) -8] + 4[4+4\lambda] = -\lambda^3 + 6\lambda^2 + 15\lambda + 8$. Setting this equal to zero and solving we find $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 8$. The eigenvectors corresponding to λ_1 and λ_2 must satisfy

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$ hence there is only the single

relation $2x_1 + x_2 + 2x_3 = 0$ to be satisfied. Consequently, two of the variables can be selected arbitrarily and the third is then determined by this equation. For example, if $x_1 = 1$ and $x_3 = 1$, then $x_2 = -4$, and we obtain the eigenvector $x^{(1)} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$. Similarly, if $x_1 = 1$ and $x_2 = 0$, then $x_3 = -1$, and we have the eigenvector $x^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, which is linearly independent of $x^{(1)}$. There are many other choices that could have been made; however, by Eq.(38) there can be no more than two linearly independent eigenvectors corresponding to the eigenvalue -1. To find the eigenvector corresponding to λ_3 we must solve

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ Interchange the first and second rows and use row reduction to obtain the equivalent system $x_1 - 4x_2 + x_3 = 0$, $2x_2 - x_3 = 0$. Since there are two equations to satisfy only one variable can be assigned an arbitrary value. If we let $x_2 = 1$, then $x_3 = 2$ and $x_1 = 2$, so we find that $\mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

27. We are given that $\mathbf{Ax} = \mathbf{b}$ has solutions and thus we have $(\mathbf{Ax},\mathbf{y}) = (\mathbf{b},\mathbf{y})$. From Problem 26, though, $(\mathbf{Ax},\mathbf{y}) = (\mathbf{x}, \mathbf{A^*y}) = 0$. Thus $(\mathbf{b},\mathbf{y}) = \mathbf{0}$. For Example 2, $\mathbf{A}^* = \bar{\mathbf{A}}^T = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & -2 & 3 \end{pmatrix}$ and, using row reduction, the augmented matrix for $\mathbf{A^*y} = \mathbf{0}$ becomes $\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Thus $y = c\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and hence $(\mathbf{b},\mathbf{y}) = b_1 + 3b_2 + b_3 = 0$.

<u>Section 7.4, Page 371</u>

1. Use Mathematical Induction. It has already been proven that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions, then so is $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$. Assume that if $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ are solutions, then $\mathbf{x} = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$ is a solution. Then use Theorem 7.4.1 to conclude that $\mathbf{x} + c_{k+1}\mathbf{x}^{(k+1)}$ is also a solution and thus $c_1\mathbf{x}^{(1)} + \cdots + c_{k+1}\mathbf{x}^{(k+1)}$ is a solution if $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k+1)}$ are solutions.

2a. From Eq.(10) we have $W = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = x_1^{(1)}x_2^{(2)} - x_2^{(1)}x_1^{(2)}$. Taking the derivative of these two products yields four terms which may be written as

$$\frac{dW}{dt} = [\frac{dx_1^{(1)}}{dt}x_2^{(2)} - x_2^{(1)}\frac{dx_1^{(2)}}{dt}] + [x_1^{(1)}\frac{dx_2^{(2)}}{dt} - \frac{dx_2^{(1)}}{dt}x_1^{(2)}].$$

The terms in the square brackets can now be recognized as the respective determinants appearing in the desired solution. A similar result was mentioned in Problem 20 of Section 4.1.

2b. If $\mathbf{x}^{(1)}$ is substituted into Eq.(3) we have

$$\frac{dx_1^{(1)}}{dt} = p_{11}\,x_1^{(1)} + p_{12}\,x_2^{(1)}$$

$$\frac{dx_2^{(1)}}{dt} = p_{21}\,x_1^{(1)} + p_{22}\,x_2^{(1)}.$$

Substituting the first equation above and its counterpart for $\mathbf{x}^{(2)}$ into the first determinant appearing in dW/dt and evaluating the result yields $p_{11}\begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = p_{11}W$.

Similarly, the second determinant in dW/dt is evaluated as $p_{22}W$, yielding the desired result.

2c. From prt b we have $\frac{dW}{W} = [p_{11}(t) + p_{22}(t)]dt$ which gives $W(t) = c\exp\int[p_{11}(t) + p_{22}(t)]dt$.

6a. $W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2.$

6b. Pick $t = t_0$, then $c_1\mathbf{x}^{(1)}(t_0) + c_2\mathbf{x}^{(2)}(t_0) = \mathbf{0}$ implies $c_1\begin{pmatrix} t_0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} t_0^2 \\ 2t_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has a non-zero solution for c_1 and c_2 if and only if $\begin{vmatrix} t_0 & t_0^2 \\ 1 & 2t_0 \end{vmatrix} = 2t_0^2 - t_0^2 = t_0^2 = \mathbf{0}$.

Thus $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent at each point except $t = 0$. Thus they are linearly independent on every interval.

6c. From part a we see that the Wronskian vanishes at $t = 0$, but not at any other point. By Theorem 7.4.3, if $p(t)$, from Eq.(3), is continuous, then the Wronskian is either identically zero or else never vanishes. Hence, we conclude that the D.E. satisfied by $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ must have at least one discontinuous coefficient at $t = 0$.

6d. To obtain the system satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ we consider

$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$, or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} t \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$.

Taking the derivative we obtain $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2t \\ 2 \end{pmatrix}$.

Solving this last system for c_1 and c_2 we find $c_1 = x_1' - tx_2'$ and $c_2 = x_2'/2$. Thus

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1' - tx_2') \begin{pmatrix} t \\ 1 \end{pmatrix} + \dfrac{x_2'}{2} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$, which yields

$x_1 = tx_1' - \dfrac{t^2}{2}\, x_2'$ and $x_2 = x_1'$. Writing this system in

matrix form we have $\mathbf{x} = \begin{pmatrix} t & -t^2/2 \\ 1 & 0 \end{pmatrix}\mathbf{x}'$. Finding the

inverse of the matrix multiplying $\mathbf{x}'$ yields the desired solution.

<u>Section 7.5, Page 381</u>

1. Assuming that there are solutions of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$, we substitute into the D.E. to find

$r\boldsymbol{\xi}e^{rt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{\xi}e^{rt}$. Since $\boldsymbol{\xi} = \mathbf{I}\boldsymbol{\xi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi}$, we can

write this equation as $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{\xi} - r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$ and

thus we must solve $\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for r, ξ_1, ξ_2.

The determinant of the coefficients is $(3-r)(-2-r) + 4 = r^2 - r - 2$, so the eigenvalues are $r = -1, 2$. The eigenvector corresponding to $r = -1$

satisfies $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which yields $2\xi_1 - \xi_2 = 0$.

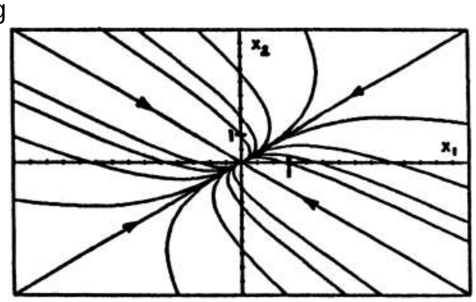
Thus $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{-t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$, where we have set $\xi_1 = 1$.

(Any other non zero choice would also work). In a

similar fashion, for $r = 2$, we have $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

or $\xi_1 - 2\xi_2 = 0$. Hence $\mathbf{x}^{(2)}(t) = \boldsymbol{\xi}^{(2)}e^{2t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ by setting $\xi_2 = 1$. The general solution is then $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$. To sketch the trajectories we follow the steps illustrated in Examples 1 and 2. Setting $c_2 = 0$ we have $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$ or $x_1 = c_1e^{-t}$ and $x_2 = 2c_1e^{-t}$ and thus one asymptote is given by $x_2 = 2x_1$. In a similar fashion $c_1 = 0$ gives $x_2 = (1/2)x_1$ as a second asymptote. Since the roots differ in sign, the trajectories for this problem are similar in nature to those in Example 1. For $c_2 \neq 0$, all solutions will be asymptotic to $x_2 = (1/2)x_1$ as $t \to \infty$. For $c_2 = 0$, the solution approaches the origin along the line $x_2 = 2x_1$.

5. Proceeding as in Problem 1 we assume a solution of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$, where r, ξ_1, ξ_2 must now satisfy $\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Evaluating the determinant of the coefficients set equal to zero yields $r = -1, -3$ as the eigenvalues. For $r = -1$ we find $\xi_1 = \xi_2$ and thus $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $r = -3$ we find $\xi_2 = -\xi_1$ and hence $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The general solution is then $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$. Since there are two negative eigenvalues, we would expect the trajectories to be similar to those of Example 2. Setting $c_2 = 0$ and eliminating t (as in Problem 1) we find that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ approaches the origin along the line $x_2 = x_1$. Similarly $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$ approaches the origin along the line



$x_2 = -x_1$. As long as $c_1 \neq 0$ (since e^{-t} is the dominant term as $t \to 0$), all trajectories approach the origin asymptotic to $x_2 = x_1$. For $c_1 = 0$, the trajectory approaches the origin along $x_2 = -x_1$, as shown in the graph.

6. The characteristic equation is $(5/4 - r)^2 - 9/16 = 0$, so $r = 2, 1/2$. Since the roots are of the same size, the behavior of the solutions is similar to Problem 5, except the trajectories are reversed since the roots are positive.

7. Again assuming $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ we find that r, ξ_1, ξ_2 must satisfy $\begin{pmatrix} 4-r & -3 \\ 8 & -6-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of the coefficients set equal to zero yields $r = 0, -2$. For $r = 0$ we find $4\xi_1 = 3\xi_2$. Choosing $\xi_2 = 4$ we find $\xi_1 = 3$ and thus $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Similarly for $r = -2$ we have $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and thus $\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$. To sketch the trajectories, note that the general solution is equivalent to the simultaneous equations $x_1 = 3c_1 + c_2 e^{-2t}$ and $x_2 = 4c_1 + 2c_2 e^{-2t}$. Solving the first equation for $c_2 e^{-2t}$ and substituting into the second yields $x_2 = 2x_1 - 2c_1$ and thus the trajectories are parallel straight lines.

9. The eigvalues are given by $\begin{vmatrix} 1-r & i \\ -i & 1-r \end{vmatrix} = (1-r)^2 + i^2 = r(r-2) = 0$. For $r = 0$ we have $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$ or $-i\xi_1 + \xi_2 = 0$ and thus $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is one eigenvector. Similarly $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is the eigenvector for $r = 2$.

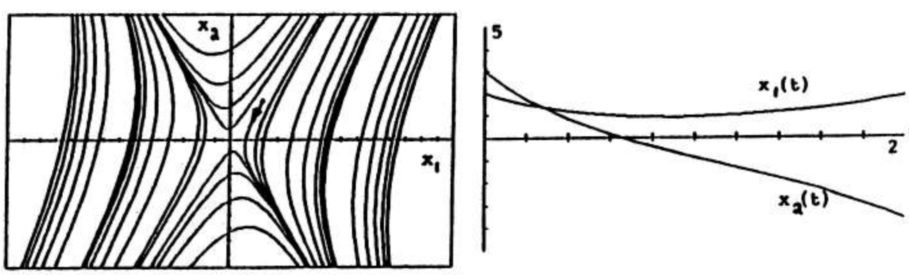
14. The eigenvalues and eigenvectors of the coefficient

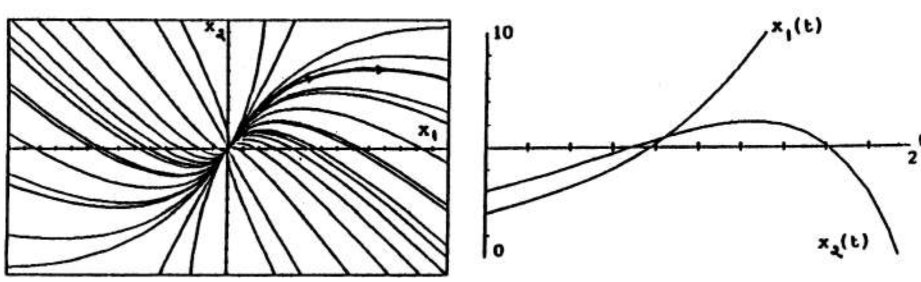
matrix satisfy $\begin{pmatrix} 1-r & -1 & 4 \\ 3 & 2-r & -1 \\ 2 & 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients set equal to zero reduces to $r^3 - 2r^2 - 5r + 6 = 0$, so the eigenvalues are $r_1 = 1$, $r_2 = -2$, and $r_3 = 3$. The eigenvector corresponding to r_1 must satisfy $\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.
Using row reduction we obtain the equivalent system $\xi_1 + \xi_3 = 0$, $\xi_2 - 4\xi_3 = 0$. Letting $\xi_1 = 1$, it follows that
$\xi_3 = -1$ and $\xi_2 = -4$, so $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$. In a similar way the eigenvectors corresponding to r_2 and r_3 are found to be $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\boldsymbol{\xi}^{(3)} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, respectively. Thus the general solution of the given D.E. is
$\mathbf{x} = c_1\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} e^{t} + c_2\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-2t} + c_3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$. Notice that the "trajectories" of this solution would lie in the $x_1\, x_2\, x_3$ three dimensional space.

16. The eigenvalues and eigenvectors of the coefficient matrix are found to be $r_1 = -1$, $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $r_2 = 3$, $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. Thus the general solution of the given D.E. is $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$. The I.C. yields the system of equations $c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The augmented

matrix of this system is $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 5 & 3 \end{array}\right)$ and by row reduction we obtain $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1/2 \end{array}\right)$. Thus $c_2 = 1/2$ and $c_1 = 1/2$. Substituting these values in the general solution gives the solution of the I.V.P. As $t \to \infty$, the solution becomes asymptotic to $\mathbf{x} = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$, or $x_2 = 5x_1$.

20. Substituting $\mathbf{x} = \boldsymbol{\xi} t^r$ into the D.E. we obtain $r\boldsymbol{\xi} t^r = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{\xi} t^r$. For $t \neq 0$ this equation can be written as $\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The eigenvalues and eigenvectors are $r_1 = 1$, $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $r_2 = -1$, $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Substituting these in the assumed form we obtain the general solution $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}$.

25.


27.


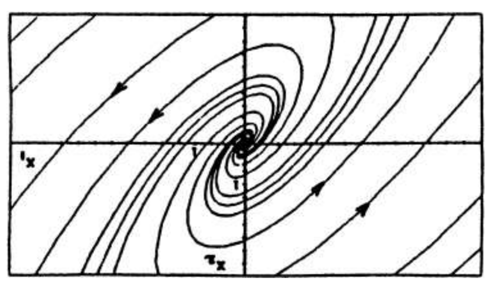
31c. The eigevalues are given by

$$\begin{vmatrix} -1-r & -1 \\ -\alpha & -1-r \end{vmatrix} = r^2 + 2r + 1 - \alpha = 0. \quad \text{Thus } r_{1,2} = -1 \pm \sqrt{\alpha}.$$

Note that in Part (a) the eigenvalues are both negative while in Part (b) they differ in sign. Thus, in this part, if we choose $\alpha = 1$, then one eigenvalue is zero, which is the transition of the one root from negative to positive. This is the desired bifurcation point.

Section 7.6, Page 390

1. We assume a solution of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$ thus r and $\boldsymbol{\xi}$ are solutions of $\begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(r^2-2r-3) + 8 = r^2 - 2r + 5$, so the eigenvalues are $r = 1 \pm 2i$. The eigenvector corresponding to $1 + 2i$ satisfies $\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $(2-2i)\xi_1 - 2\xi_2 = 0$. If $\xi_1 = 1$, then $\xi_2 = 1-i$ and $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ and thus one complex-valued solution of the D.E. is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t}$.



To find real-valued solutions (see Eqs.8 and 9) we take the real and imaginary parts, respectively of $\mathbf{x}^{(1)}(t)$. Thus $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^t(\cos 2t + i\sin 2t)$

$$= e^t\begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t + \sin 2t + i(\sin 2t - \cos 2t) \end{pmatrix}$$

$$= e^t\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + ie^t\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

Hence the general solution of the D.E. is

$\mathbf{x} = c_1e^t\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2e^t\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$. The solutions spiral to ∞ as $t \to \infty$ due to the e^t terms.

7. The eigenvalues and eigenvectors of the coefficient matrix satisfy $\begin{pmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients reduces to $(1-r)(r^2 - 2r + 5)$ so the eigenvalues are $r_1 = 1$, $r_2 = 1 + 2i$, and $r_3 = 1 - 2i$. The eigenvector corresponding to r_1 satisfies
$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; hence $\xi_1 - \xi_3 = 0$ and $3\xi_1 + 2\xi_2 = 0$. If we let $\xi_2 = -3$ then $\xi_1 = 2$ and $\xi_3 = 2$, so one solution of the D.E. is $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}e^t$. The eigenvector corresponding to r_2 satisfies $\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence $\xi_1 = 0$ and $i\xi_2 + \xi_3 = 0$. If we let $\xi_2 = 1$, then $\xi_3 = -i$. Thus a complex-valued solution is
$\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^t(\cos 2t + i \sin 2t)$. Taking the real and imaginary parts, see prob. 1, we obtain $\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} e^t$ and $\begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix} e^t$, respectively. Thus the general solution is

$\mathbf{x} = c_1\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}e^t + c_2e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}$, which spirals to ∞ about the x_1 axis in the $x_1x_2x_3$ space as $t \to \infty$.

9. The eigenvalues and eigenvectors of the coefficient matrix satisfy $\begin{pmatrix} 1-r & -5 \\ 1 & -3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $r^2 + 2r + 2$ so that the eigenvalues are $r = -1 \pm i$. The eigenvector corresponding to $r = -1 + i$

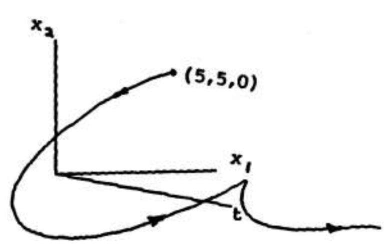
is given by $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \mathbf{0}$ so that $\xi_1 = (2+i)\xi_2$ and thus one complex-valued solution is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t}$. Finding the real and complex parts of $\mathbf{x}^{(1)}$ leads to the general solution $\mathbf{x} = c_1e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$. Setting $t = 0$ we find $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is equivalent to the system $\begin{matrix} 2c_1 + c_2 = 1 \\ c_1 + 0 = 1 \end{matrix}$. Thus $c_1 = 1$ and $c_2 = -1$ and

$$\mathbf{x}(t) = e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$
$$= e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix},$$ which spirals to zero as

$t \to \infty$ due to the e^{-t} term.

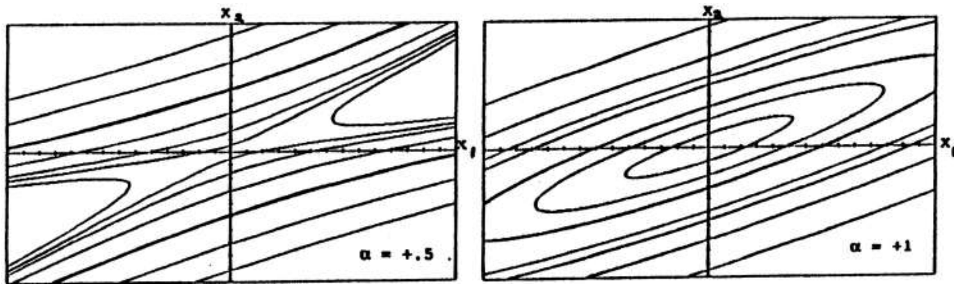
11a. The eigenvalues are given by
$$\begin{vmatrix} 3/4-r & -2 \\ 1 & -5/4-r \end{vmatrix} = r^2 + r/2 + 17/16 = 0.$$

11d. The trajectory starts at (5,5) in the x_1x_2 plane and spirals around and converges to the t axis as $t \to \infty$.



15a. The eigenvalues satisfy $\begin{vmatrix} 2-r & -5 \\ \alpha & -2-r \end{vmatrix} = r^2 - 4 + 5\alpha = 0$, so $r_1, r_2 = \pm\sqrt{4-5\alpha}$.

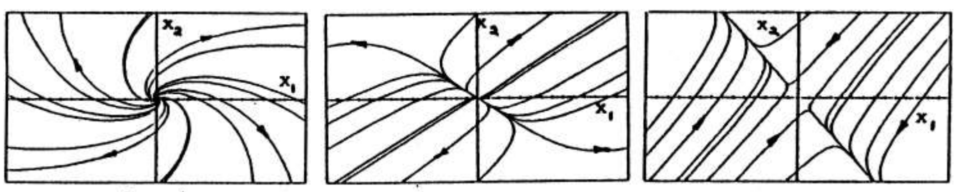
15b. The critical value of α yields $r_1 = r_2 = 0$, or $\alpha = 4/5$.

15c. 

16a. $\begin{vmatrix} 5/4-r & 3/4 \\ \alpha & 5/4-r \end{vmatrix} = r^2 - 5r/2 + (25/16 - 3\alpha/4) = 0$, so
$r_{1,2} = 5/4 \pm \sqrt{3\alpha}/2$.

16b. There are two critical values of α. For $\alpha < 0$ the eigenvalues are complex, while for $\alpha > 0$ they are real. There will be a second critical value of α when $r_2 = 0$, or $\alpha = 25/12$. In this case the second real eigenvalue goes from positive to negative.

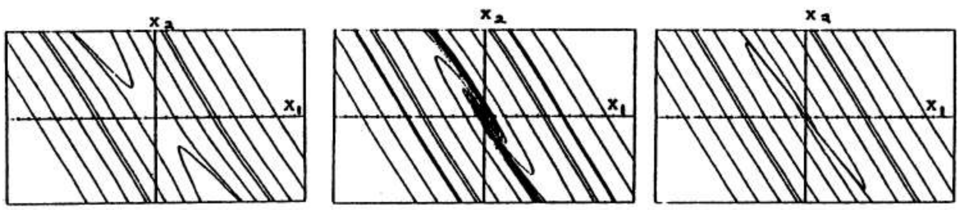
16c.



α = -1 α = 1 α = 3.0

18a. We have $\begin{vmatrix} 3-r & \alpha \\ -6 & -4-r \end{vmatrix} = r^2 + r - 12 + 6\alpha = 0$, so
$r_1, r_2 = -1/2 \pm \sqrt{49-24\alpha}/2$.

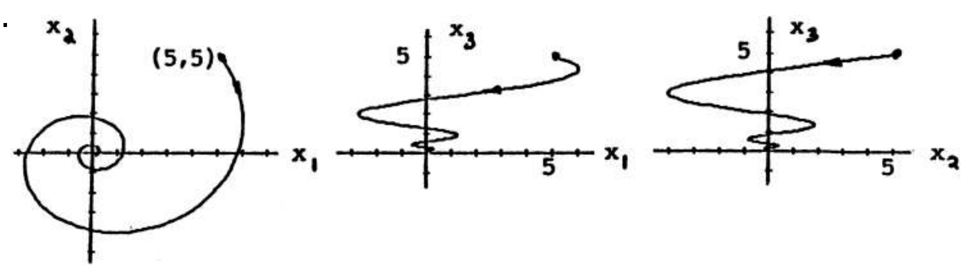
18b. The critical values occur when $49 - 24\alpha = 1$ (in which case
$r_2 = 0$) and when $49 - 24\alpha = 0$, in which case $r_1 = r_2 = -1/2$. Thus $\alpha = 2$ and $\alpha = 49/24 \cong 2.04$.

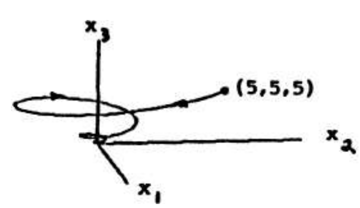
18c. 

α = 1.8 α = 2.1 α = 2.03

21. If we seek solutions of the form $\mathbf{x} = \boldsymbol{\xi}t^r$, then r must be an eigenvalue and $\boldsymbol{\xi}$ a corresponding eigenvector of the coefficient matrix. Thus r and $\boldsymbol{\xi}$ satisfy
$\begin{pmatrix} -1-r & -1 \\ 2 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(-1-r)^2 + 2 = r^2 + 2r + 3$, so the eigenvalues are $r = -1 \pm \sqrt{2}\,i$. The eigenvector corresponding to $-1 + \sqrt{2}\,i$ satisfies $\begin{pmatrix} -\sqrt{2}\,i & -1 \\ 2 & -\sqrt{2}\,i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\sqrt{2}\,i\xi_1 + \xi_2 = 0$. If we let $\xi_1 = 1$, then $\xi_2 = -\sqrt{2}\,i$, and $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix}$. Thus a complex-valued solution of the given D.E. is $\begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix} t^{-1+\sqrt{2}\,i}$. From Eq. (15) of Section 5.5 we have (since $t^{\sqrt{2}\,i} = e^{\ln t^{\sqrt{2}\,i}} = e^{\sqrt{2}\,i\,\ln t}$) $t^{-1+\sqrt{2}\,i} = t^{-1}[\cos(\sqrt{2}\ln t) + i\sin(\sqrt{2}\ln t)]$ for $t > 0$. Separating the complex valued solution into real and imaginary parts, we obtain the two real-valued solutions
$\mathbf{u} = t^{-1}\begin{pmatrix} \cos(\sqrt{2}\ln t) \\ \sqrt{2}\,\sin(\sqrt{2}\ln t) \end{pmatrix}$ and $\mathbf{v} = t^{-1}\begin{pmatrix} \sin(\sqrt{2}\ln t) \\ -\sqrt{2}\cos(\sqrt{2}\ln t) \end{pmatrix}$.

23a. The eigenvalues are given by $(r+1/4)[(r+1/4)^2 + 1] = 0$.

23b.


23c. Graph starts in the first octant and spirals around the x_3 axis, converging to zero.


29a. We have $y_1' = x_1' = y_2$, $y_3' = x_2' = y_4$, $y_2' = -2y_1 + y_3$, and $y_4' = y_1 - 2y_3$. Thus

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} \mathbf{y}.$$

29b. The eigenvalues are given by $r^4 + 4r^2 + 3 = 0$, which yields $r^2 = -1, -\sqrt{3}$, so $r = \pm i, \pm\sqrt{3}\,i$.

29c. For $r = \pm\, i$ the eigenvectors are given by
$$\begin{pmatrix} -i & 1 & 0 & 0 \\ -2 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ 1 & 0 & -2 & -i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = 0.$$ Choosing $\xi_1 = 1$ yields $\xi_2 = i$ and choosing $\xi_3 = 1$ yields $\xi_4 = i$, so $(1, i, 1, i)^T(\cos t + i\sin t)$ is a solution. Finding the real and imaginary parts yields
$\mathbf{w}_1 = (\cos t, -\sin t, \cos t, -\sin t)^T$ and
$\mathbf{w}_2 = (\sin t, \cos t, \sin t, \cos t)^T$ as two real solutions.
In a similar fashion, for $r = \pm\sqrt{3}\,i$, we obtain
$\boldsymbol{\xi} = (1, \sqrt{3}\,i, -1, -\sqrt{3}\,i)$ and
$\mathbf{w}_3 = (\cos\sqrt{3}\,t, -\sqrt{3}\sin\sqrt{3}\,t, -\cos\sqrt{3}\,t, \sqrt{3}\sin\sqrt{3}\,t)^T$ and
$\mathbf{w}_4 = (\sin\sqrt{3}\,t, \sqrt{3}\cos\sqrt{3}\,t, -\sin\sqrt{3}\,t, -\sqrt{3}\cos\sqrt{3}\,t)^T$.
Thus $\mathbf{y}=c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + c_4\mathbf{w}_4$, so $\mathbf{y}^T(0) = (2, 1, 2, 1)$ yields $c_1 + c_3 = 2$, $c_2 + \sqrt{3}\,c_4 = 1$, $c_1 - c_3 = 2$, and $c_2 - \sqrt{3}\,c_4 = 1$, which yields $c_1 = 2$, $c_2 = 1$, and $c_3 = c_4 = 0$. Hence $\mathbf{y} = \begin{pmatrix} 2\cos t + \sin t \\ -2\sin t + \cos t \\ 2\cos t + \sin t \\ -2\sin t + \cos t \end{pmatrix}$.

29e. The natural frequencies are $\omega_1 = 1$ and $\omega_2 = \sqrt{3}$, which are the absolute value of the eigenvalues. For any other choice of I.C., both frequencies will be present, and thus another mode of oscillation with a different frequency (depending on the I.C.) will be present.

Section 7.7, Page 400

Each of the Problems 1 through 10, except 2 and 8, has been solved in one of the previous sections. Thus a fundamental matrix for the given systems can be readily written down. The fundamental matrix $\mathbf{\Phi}(t)$ satisfying $\mathbf{\Phi}(0) = \mathbf{I}$ can then be found, as shown in the following problems.

2. The characteristic equation is given by $\begin{vmatrix} -3/4-r & 1/2 \\ 1/8 & -3/4-r \end{vmatrix} =$ $r^2 + 3r/2 + 1/2 = 0$, so $r = 1, 1/2$. For $r = 1$ we have $\begin{pmatrix} 1/4 & 1/2 \\ 1/8 & 1/4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and thus $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Likewise $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and thus $\mathbf{x}^{(1)}(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-t}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-t/2}$. To find the first column of $\mathbf{\Phi}$ we choose c_1 and c_2 so that $c_1\mathbf{x}^{(1)}(0) + c_2\mathbf{x}^{(2)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which yields $-2c_1 + 2c_2 = 1$ and $c_1 + c_2 = 0$. Thus $c_1 = -1/4$ and $c_2 = 1/4$ and the first column of $\mathbf{\Phi}$ is $\begin{pmatrix} 1/2e^{-t/2} + 1/2e^{t} \\ 1/4e^{-t/2} - 1/4e^{-t/2} \end{pmatrix}$. The second colunm of $\mathbf{\Phi}$ is determined by $d_1\mathbf{x}^{(1)}(0) + d_2\mathbf{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which yields $d_1 = d_2 =$ $1/2$ and thus the second column of $\mathbf{\Phi}$ is $\begin{pmatrix} e^{-t/2} - e^{-t} \\ 1/2e^{-t/2} + 1/2e^{-t} \end{pmatrix}$.

4. From Problem 4 of Section 7.5 we have the two linearly independent solutions $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$. Hence a fundamental matrix $\mathbf{\Psi}$ is given by $\mathbf{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$. To find the fundamental matrix $\mathbf{\Phi}(t)$ satisfying the I.C. $\mathbf{\Phi}(0) = \mathbf{I}$ we can proceed in either of two ways. One way is to find $\mathbf{\Psi}(0)$, invert it to obtain $\mathbf{\Psi}^{-1}(0)$, and then to form the product $\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)$, which is $\mathbf{\Phi}(t)$. Alternatively, we can find the first column of $\mathbf{\Phi}$ by determining the linear combination $c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$ that satisfies the I.C. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This requires that $c_1 + c_2 = 1$, $-4c_1 + c_2 = 0$, so we obtain $c_1 = 1/5$ and $c_2 = 4/5$. Thus the first column of $\mathbf{\Phi}(t)$ is

$\begin{pmatrix} (1/5)e^{-3t} + (4/5)e^{2t} \\ -(4/5)e^{-3t} + (4/5)e^{2t} \end{pmatrix}$. Similarly, the second column of $\mathbf{\Phi}$ is that linear combination of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ that satisfies the I.C. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus we must have $c_1 + c_2 = 0$, $-4c_1 + c_2 = 1$; therefore $c_1 = -1/5$ and $c_2 = 1/5$. Hence the second column of $\mathbf{\Phi}(t)$ is $\begin{pmatrix} -(1/5)e^{-3t} + (1/5)e^{2t} \\ (4/5)e^{-3t} + (1/5)e^{2t} \end{pmatrix}$.

6. Two linearly independent real-valued solutions of the given D.E. were found in Problem 2 of Section 7.6. Using the result of that problem, we have $\mathbf{\Psi}(t) = \begin{pmatrix} -2e^{-t}\sin 2t & 2e^{-t}\cos 2t \\ e^{-t}\cos 2t & e^{-t}\sin 2t \end{pmatrix}$. To find $\mathbf{\Phi}(t)$ we determine the linear combinations of the columns of $\mathbf{\Psi}(t)$ that satisfy the I.C. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. In the first case c_1 and c_2 satisfy $0c_1 + 2c_2 = 1$ and $c_1 + 0c_2 = 0$. Thus $c_1 = 0$ and $c_2 = 1/2$. In the second case we have $0c_1 + 2c_2 = 0$ and $c_1 + 0c_2 = 1$, so $c_1 = 1$ and $c_2 = 0$. Using these values of c_1 and c_2 to form the first and second columns of $\mathbf{\Phi}(t)$ respectively, we obtain $\mathbf{\Phi}(t) = \begin{pmatrix} e^{t}\cos 2t & -2e^{-t}\sin 2t \\ (1/2)e^{-t}\sin 2t & e^{-t}\cos 2t \end{pmatrix}$.

10. From Problem 14 Section 7.5 we have $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}e^{t}$, $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}e^{-2t}$ and $\mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}e^{3t}$. For the first column of $\mathbf{\Phi}$ we want to choose c_1, c_2, c_3 such that $c_1\mathbf{x}^{(1)}(0) + c_2\mathbf{x}^{(2)}(0) + c_3\mathbf{x}^{(3)}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Thus $c_1 + c_2 + c_3 = 1$, $-4c_1 - c_2 + 2c_3 = 0$ and $-c_1 - c_2 + c_3 = 0$, which yield $c_1 = 1/6$, $c_2 = 1/3$ and $c_3 = 1/2$. The first column of $\mathbf{\Phi}$ is then $(1/6e^{t} + 1/3e^{-2t} + 1/2e^{3t}, -2/3e^{t} - 1/3e^{-2t} + e^{3t}, -1/6e^{t} - 1/3e^{-2t} + 1/2e^{3t})^T$. Likewise, for the second

column we have $d_1\mathbf{x}^{(1)}(0) + d_2\mathbf{x}^{(2)}(0) + d_3\mathbf{x}^{(3)}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,

which yields $d_1 = -1/3$, $d_2 = 1/3$ and $d_3 = 0$ and thus $(-1/3e^t + 1/3e^{-2t},\ 4/3e^t - 1/3e^{-2t},\ 1/3e^t - 1/3e^{-2t})^T$ is the second column of $\mathbf{\Phi}(t)$. Finally, for the third column

we have $e_1\mathbf{x}^{(1)}(0) + e_2\mathbf{x}^{(2)}(0) + e_3\mathbf{x}^{(3)}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, which

gives $e_1 = 1/2$, $e_2 = -1$ and $e_3 = 1/2$ and hence $(1/2e^t - e^{-2t} + 1/2e^{3t},\ -2e^t + e^{-2t} + e^{3t},$ $-1/2e^t + e^{-2t} + 1/2e^{3t})^T$ is the third column of $\mathbf{\Phi}(t)$.

11. From Eq. (14) the solution is given by $\mathbf{\Phi}(t)\mathbf{x}^0$. Thus

$$\mathbf{x} = \begin{pmatrix} 3/2e^t-1/2e^{-t} & -1/2e^t+1/2e^{-t} \\ 3/2e^t-3/2e^{-t} & -1/2e^t+3/2e^{-t} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7/2e^t-3/2e^t \\ 7/2e^t-9/2e^{-t} \end{pmatrix} = \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t - \frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}.$$

Section 7.8, Page 407

1. The eigenvalues and eigenvectors of the given coefficient matrix satisfy $\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(3-r)(-1-r) + 4 = r^2 - 2r + 1 = (r-1)^2$ so $r_1 = 1$ and $r_2 = 1$. The eigenvectors corresponding to this double eigenvalue satisfy $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $\xi_1 - 2\xi_2 = 0$. Thus the only eigenvectors are multiples of $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. One solution of the given D.E. is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^t$, but there is no second solution of this form. To find a second solution we assume, as in Eq. (13), that $\mathbf{x} = \boldsymbol{\xi}te^t + \boldsymbol{\eta}e^t$ and substitute this expression into the D.E. As in Example 2 we find that $\boldsymbol{\xi}$ is an eigenvector, so we choose $\boldsymbol{\xi} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then $\boldsymbol{\eta}$ must

satisfy $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which verifies Eq.(16).

Solving these equations yields $\eta_1 - 2\eta_2 = 1$. If $\eta_2 = k$, where k is an arbitrary constant, then $\eta_1 = 1 + 2k$. Hence the second solution that we obtain is

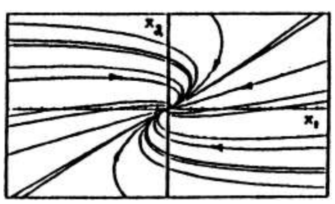
$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}te^t + \begin{pmatrix} 1+2k \\ k \end{pmatrix}e^t = \begin{pmatrix} 2 \\ 1 \end{pmatrix}te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t + k\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^t.$$

The last term is a multiple of the first solution $\mathbf{x}^{(1)}(t)$ and may be neglected, that is, we may set $k = 0$. Thus

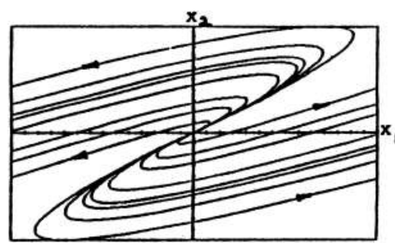
$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$ and the general solution is

$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$. All solutions diverge to infinity as $t \to \infty$. The graph is shown on the right.

3. The origin is attracting



1.



5. Substituting $\mathbf{x} = \boldsymbol{\xi}e^{rt}$ into the given system, we find that the eigenvalues and eigenvectors satisfy

$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients

is $-r^3 + 3r^2 - 4$ and thus $r_1 = -1$, $r_2 = 2$ and $r_3 = 2$. The eigenvector corresponding to r_1 satisfies

$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ which yields $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ and

$\mathbf{x}^{(1)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t}$. The eigenvectors corresponding to the

double eigenvalue must satsify $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which yields the single eigenvector $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and hence

$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$. The second solution corresponding to the double eigenvalue will have the form specified by Eq.(13), which yields $\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \boldsymbol{\eta}e^{2t}$.

Substituting this into the given system, or using Eq.(16), we find that $\boldsymbol{\eta}$ satisfies $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Using row reduction we find that $\eta_1 = 1$ and $\eta_2 + \eta_3 = 1$, where either η_2 or η_3 is arbitrary. If we choose $\eta_2 = 0$, then $\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and thus $\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$. The general solution is then $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)}$.

9. We have $\begin{vmatrix} 2-r & 3/2 \\ -3/2 & -1-r \end{vmatrix} = (r-1/2)^2 = 0$. For $r = 1/2$, the eigenvector is given by $\begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$, so $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}$ is one solution. For the second solution we have $\mathbf{x} = \boldsymbol{\xi}te^{t/2} + \boldsymbol{\eta}e^{t/2}$, where $(\mathbf{A} - \frac{1}{2}\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$, $\mathbf{A}$ being the coefficient matrix for this problem. This last equation reduces to $3\eta_1/2 + 3\eta_2/2 = 1$ and $-3\eta_1/2 - 3\eta_2/2 = -1$. Choosing $\eta_2 = 0$ yields $\eta_1 = 2/3$ and hence

$\mathbf{x} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} + c_2\begin{pmatrix} 2/3 \\ 0 \end{pmatrix} e^{t/2} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{t/2}$. $\mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ gives $c_1 + 2c_2/3 = 3$ and $-c_1 = -2$, and hence $c_1 = 2$, $c_2 = 3/2$. Substituting these into the above $\mathbf{x}$ yields the solution.

11. The eigenvalues are $r = 1,1,2$. For $r = 2$, we have $\begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which yields $\boldsymbol{\xi} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, so one solution is $\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{2t}$. For $r = 1$, we have

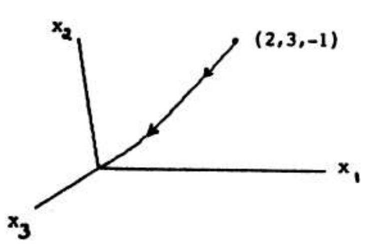
$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which yields the second solution

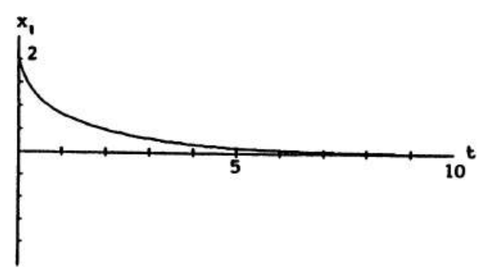
$\mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}e^{t}$. The third solution is of the form

$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}te^{t} + \eta e^{t}$, where $\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\boldsymbol{\eta} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$ and thus $\eta_1 = -1/4$ and $6\eta_2 + \eta_3 = -21/4$. Choosing $\eta_2 = 0$ gives $\eta_3 = -21/4$ and hence

$\mathbf{x}(t) = c_1\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}e^{t} + c_2\left[\begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix}e^{t} + \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}te^{t}\right] + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{2t}$. The I.C. then yield $c_1 = 2$, $c_2 = 4$ and $c_3 = 3$ and hence

$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ -33 \end{pmatrix}e^{t} + 4\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}te^{t} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{2t}$, which become unbounded as $t \to \infty$.

12. 



14. Assuming $\mathbf{x} = \boldsymbol{\xi}t^{r}$ and substituting into the given system, we find r and $\boldsymbol{\xi}$ must satisfy $\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has the double eigenvalue $r = -3$ and single eigenvector

$\begin{pmatrix}1\\1\end{pmatrix}$. Hence one solution of the given D.E. is

$\mathbf{x}^{(1)}(t) = \begin{pmatrix}1\\1\end{pmatrix} t^{-3}$. By analogy with the scalar case considered in Section 5.5 and Example 2 of this section, we seek a second solution of the form $\mathbf{x} = \boldsymbol{\eta}t^{-3}\ln t + \boldsymbol{\zeta}t^{-3}$. Substituting this expression into the D.E. we find that $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ satisfy the equations $(\mathbf{A} + 3\mathbf{I})\boldsymbol{\eta} = \mathbf{0}$ and

$(\mathbf{A} + 3\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$, where $\mathbf{A} = \begin{pmatrix}1 & -4\\4 & -7\end{pmatrix}$ and $\mathbf{I}$ is the identity

matrix. Thus $\boldsymbol{\eta} = \begin{pmatrix}1\\1\end{pmatrix}$, from above, and $\boldsymbol{\zeta}$ is found to be

$\begin{pmatrix}0\\-1/4\end{pmatrix}$. Thus a second solution is

$\mathbf{x}^{(2)}(t) = \begin{pmatrix}1\\1\end{pmatrix} t^{-3}\ln t + \begin{pmatrix}1\\-1/4\end{pmatrix} t^{-3}$.

15. All solutions of the given system approach zero as $t \to \infty$ if and only if the eigenvalues of the coefficient matrix either are real and negative or else are complex with negative real part. Write down the determinantal equation satisfied by the eigenvalues and determine when the eigenvalues are as stated.

17a. The eigenvalues and eigenvectors of the coefficient

matrix satisfy $\begin{pmatrix}1-r & 1 & 1\\2 & 1-r & -1\\-3 & 2 & 4-r\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\\\xi_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. The determinant

of coefficients is $8 - 12r + 6r^2 - r^3 = (2-r)^3$, so the eigenvalues are $r_1 = r_2 = r_3 = 2$. The eigenvectors corresponding to this triple eigenvalue satisfy

$\begin{pmatrix}-1 & 1 & 1\\2 & -1 & -1\\-3 & 2 & 2\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\\\xi_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. Using row reduction we can reduce

this to the equivalent system $\xi_1 - \xi_2 - \xi_3 = 0$, and $\xi_2 + \xi_3 = 0$. If we let $\xi_2 = 1$, then $\xi_3 = -1$ and $\xi_1 = 0$,

so the only eigenvectors are multiples of $\boldsymbol{\xi} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$.

17b. From part a, one solution of the given D.E. is

$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$, but there are no other linearly independent solutions of this form.

17c. We now seek a second solution of the form $\mathbf{x} = \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t}$. Thus $\mathbf{Ax} = \mathbf{A}\boldsymbol{\xi} te^{2t} + \mathbf{A}\boldsymbol{\eta} e^{2t}$ and $\mathbf{x}' = 2\boldsymbol{\xi} te^{2t} + \boldsymbol{\xi} e^{2t} + 2\boldsymbol{\eta} e^{2t}$. Equating like terms, we then have $(\mathbf{A}-2\mathbf{I})\boldsymbol{\xi} = 0$ and $(\mathbf{A}-2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$. Thus $\boldsymbol{\xi}$ is as in part a and the second equation yields

$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. By row reduction this is

equivalent to the system $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. If we

choose $\eta_3 = 0$, then $\eta_2 = 1$ and $\eta_1 = 1$, so $\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Hence

a second solution of the D.E. is

$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$

17d. Assuming $\mathbf{x} = \boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta} te^{2t} + \boldsymbol{\zeta} e^{2t}$, we have $\mathbf{Ax} = \mathbf{A}\boldsymbol{\xi}(t^2/2)e^{2t} + \mathbf{A}\boldsymbol{\eta} te^{2t} + \mathbf{A}\boldsymbol{\xi} e^{2t}$ and $\mathbf{x}' = \boldsymbol{\xi} te^{2t} + 2\boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta} e^{2t} + 2\boldsymbol{\eta} te^{2t} + 2\boldsymbol{\xi} e^{2t}$ and thus $(\mathbf{A}-2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, $(\mathbf{A}-2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ and $(\mathbf{A}-2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$. Again, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are as found previously and the last equation is equivalent to

$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. By row reduction we find the

equivalent system $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$. If we let

$\zeta_2 = 0$, then $\zeta_3 = 3$ and $\zeta_1 = 2$, so $\boldsymbol{\zeta} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ and

$$\mathbf{x}^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (t^2/2)e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} e^{2t}.$$

17e. $\boldsymbol{\Psi}$ is the matrix with $\mathbf{x}^{(1)}$ as the first column, $\mathbf{x}^{(2)}$ as the second column and $\mathbf{x}^{(3)}$ as the third column.

17f. $\mathbf{T} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and using row operations on $\mathbf{T}$ and $\mathbf{I}$, or a computer algebra system, $\mathbf{T}^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}$ and thus

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{J}.$$

19a. $\mathbf{J}^2 = \mathbf{JJ} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$

$$\mathbf{J}^3 = \mathbf{JJ}^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$$

19b. Based upon the results of part a, assume

$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$, then

$$\mathbf{J}^{n+1} = \mathbf{JJ^n} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$= \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix}$, which is the same as $\mathbf{J}^n$ with n replaced by n+1. Thus, by mathematical induction, $\mathbf{J}^n$ has the desired form.

19c. From Eq.(23), Section 7.7, we have

$$\exp(\mathbf{J}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{J}^n t^n}{n!}$$

$$= \mathbf{I} + \sum_{n=1}^{\infty} \begin{pmatrix} \dfrac{\lambda^n t^n}{n!} & \dfrac{n\lambda^{n-1}t^n}{n!} \\ 0 & \dfrac{\lambda^n t^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \displaystyle\sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!} & \displaystyle\sum_{n=1}^{\infty} \frac{\lambda^{n-1}t^n}{(n-1)!} \\ 0 & 1 + \displaystyle\sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}, \text{ since}$$

$$\sum_{n=1}^{\infty} \frac{\lambda^{n-1}t^n}{(n-1)!} = t(1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!}) = te^{\lambda t}.$$

19d. From Eq. (28), Section 7.7, we have

$$\mathbf{x} = \exp(\mathbf{J}t)\mathbf{x}^0 = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} x_1^0 e^{\lambda t} + x_2^0 te^{\lambda t} \\ x_2^0 e^{\lambda t} \end{pmatrix}$$

$$= \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} e^{\lambda t} + \begin{pmatrix} x_2^0 \\ 0 \end{pmatrix} te^{\lambda t}.$$

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1. From Section 7.5 Problem 3 we have

   $\mathbf{x}^{(c)} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$. Note that

   $\mathbf{g}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$ and that $r = 1$ is an eigenvalue of the coefficient matrix. Thus if the method of undetermined coefficients is used, the assumed form is given by Eq.(18).

2. Using methods of previous sections, we find that the eigenvalues are $r_1 = 2$ and $r_2 = -2$, with corresponding eigenvectors $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$. Thus

$\mathbf{x}^{(c)} = c_1\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}e^{-2t}$. Writing the nonhomogeneous term as $\begin{pmatrix}1\\0\end{pmatrix}e^{t} + \begin{pmatrix}0\\\sqrt{3}\end{pmatrix}e^{-t}$ we see that we can assume $\mathbf{x}^{(p)} = \mathbf{a}e^{t} + \mathbf{b}e^{-t}$. Substituting this in the D.E., we obtain

$\mathbf{a}e^{t} - \mathbf{b}e^{-t} = \mathbf{Aa}e^{t} + \mathbf{Ab}e^{-t} + \begin{pmatrix}1\\0\end{pmatrix}e^{t} + \begin{pmatrix}0\\\sqrt{3}\end{pmatrix}e^{-t}$, where $\mathbf{A}$ is the given coefficient matrix. All the terms involving e^{t} must add to zero and thus we have $\mathbf{Aa} - \mathbf{a} + \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

This is equivalent to the system

$\sqrt{3}\,a_2 = -1$ and $\sqrt{3}\,a_1 - 2a_2 = 0$, or $a_1 = -2/3$ and $a_2 = -1/\sqrt{3}$. Likewise the terms involving e^{-t} must add to zero, which yields $\mathbf{Ab} + \mathbf{b} + \begin{pmatrix}0\\\sqrt{3}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. The solution of this system is $b_1 = -1$ and $b_2 = 2/\sqrt{3}$. Substituting these values for $\mathbf{a}$ and $\mathbf{b}$ into $\mathbf{x}^{(p)}$ and adding $\mathbf{x}^{(p)}$ to $\mathbf{x}^{(c)}$ yields the desired solution.

3. The method of undetermined coefficients is not straight forward since the assumed form of $\mathbf{x}^{(p)} = \mathbf{a}\cos t + \mathbf{b}\sin t$ leads to singular equations for $\mathbf{a}$ and $\mathbf{b}$. From Problem 3 of Section 7.6 we find that a fundamental matrix is

$\mathbf{\Psi}(t) = \begin{pmatrix}5\cos t & 5\sin t\\2\cos t + \sin t & -\cos t + 2\sin t\end{pmatrix}$. The inverse matrix is

$\mathbf{\Psi}^{-1}(t) = \begin{pmatrix}\dfrac{\cos t - 2\sin t}{5} & \sin t\\\dfrac{2\cos t + \sin t}{5} & -\cos t\end{pmatrix}$, which may be found as in Section 7.2 or by using a computer algebra system. Thus we may use the method of variation of parameters where $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$ and $\mathbf{u}(t)$ is given by $\mathbf{u}'(t) = \mathbf{\Psi}^{-1}(t)g(t)$ from Eq.(27). For this problem

$\mathbf{g}(t) = \begin{pmatrix}-\cos t\\\sin t\end{pmatrix}$ and thus

$$\mathbf{u}'(t) = \begin{pmatrix} \frac{\cos t - 2\sin t}{5} & \sin t \\ \frac{2\cos t + \sin t}{5} & -\cos t \end{pmatrix} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$
$$= \frac{1}{5} \begin{pmatrix} 2 - 3\cos 2t + \sin 2t \\ -1 - \cos 2t - 3\sin 2t \end{pmatrix},$$

after multiplying and using appropriate trigonometric identities. Integration and multiplication by $\mathbf{\Psi}$ yields the desired solution.

4. In this problem we use the method illustrated in Example 1. From Problem 4 of Section 7.5 we have the transformation matrix $\mathbf{T} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$. Inverting $\mathbf{T}$ we find that $\mathbf{T}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$. If we let $\mathbf{x} = \mathbf{Ty}$ and substitute into the D.E., we obtain

$$\mathbf{y}' = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\mathbf{y} + \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$
$$= \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{y} + \frac{1}{5}\begin{pmatrix} e^{-2t} + 2e^{t} \\ 4e^{-2t} - 2e^{t} \end{pmatrix}.$$

This corresponds to the two scalar equations

$$y_1' + 3y_1 = (1/5)e^{-2t} + (2/5)e^{t},$$
$$y_2' - 2y_2 = (4/5)e^{-2t} - (2/5)e^{t},$$

which may be solved by the methods of Section 2.1. For the first equation the integrating factor is e^{3t} and we obtain $(e^{3t}y_1)' = (1/5)e^{t} + (2/5)e^{4t}$, so $e^{3t}y_1 = (1/5)e^{t} + (1/10)e^{4t} + c_1$. For the second equation the integrating factor is e^{-2t}, so $(e^{-2t}y_2)' = (4/5)e^{-4t} - (2/5)e^{-t}$. Hence $e^{-2t}y_2 = -(1/5)e^{-4t} + (2/5)e^{-t} + c_2$. Thus

$$\mathbf{y} = \begin{pmatrix} 1/5 \\ -1/5 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/10 \\ 2/5 \end{pmatrix} e^{t} + \begin{pmatrix} c_1e^{-3t} \\ c_2e^{2t} \end{pmatrix}.$$

Finally, multiplying by $\mathbf{T}$, we obtain

$$\mathbf{x} = \mathbf{Ty} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{t} + c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The last two terms are the general solution of the corresponding homogeneous system, while the first two terms constitute a particular solution of the nonhomogeneous system.

12. Since the coefficient matrix is the same as that of Problem 3, use the same procedure as done in that problem, including the $\mathbf{\Psi}^{-1}$ found there. In the interval $\pi/2 < t < \pi$ $\sin t > 0$ and $\cos t < 0$; hence $|\sin t| = \sin t$, but $|\cos t| = -\cos t$.

14. To verify that the given vector is the general solution of the corresponding system, it is sufficient to substitute it into the D.E. Note also that the two terms in $\mathbf{x}^{(c)}$ are linearly independent. If we seek a solution of the form $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$ then we find that the equation corresponding to Eq.(26) is $t\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$, where $\mathbf{\Psi}(t) = \begin{pmatrix} t & 1/t \\ t & 3/t \end{pmatrix}$ and $\mathbf{g}(t) = \begin{pmatrix} 1-t^2 \\ 2t \end{pmatrix}$. Thus $\mathbf{u}' = (1/t)\mathbf{\Psi}^{-1}(t)\mathbf{g}(t)$. Using a computer algebra system or row operations on $\mathbf{\Psi}$ and $\mathbf{I}$, we find that $\mathbf{\Psi}^{-1} = \begin{pmatrix} 3/2t & -1/2t \\ -t/2 & t/2 \end{pmatrix}$ and hence $u_1' = \dfrac{3}{2t^2} - \dfrac{3}{2} - \dfrac{1}{t}$ and $u_2' = \dfrac{-1}{2} + \dfrac{t^2}{2} + t$, which yields $u_1 = \dfrac{-3}{2t} - \dfrac{3t}{2} - \ln t + c_1$ and $u_2 = -\dfrac{1}{2}t + \dfrac{t^3}{6} + \dfrac{t^2}{2} + c_2$. Multiplication of $\mathbf{u}$ by $\mathbf{\Psi}(t)$ yields the desired solution.

Capítulo 8

CHAPTER 8

Section 8.1, Page 427

In the following problems that ask for a large number of numerical calculations the first few steps are shown. It is then necessary to use these samples as a model to format a computer program or calculator to find the remaining values.

1a. The Euler formulas is $y_{n+1} = y_n + h(3 + t_n - y_n)$ for $n = 0,1,2,3...$ and with $t_0 = 0$ and $y_0 = 1$. Thus
$y_1 = 1 + .05(3 + 0 - 1) = 1.1$
$y_2 = 1.1 + .05(3 + .05 - 1.1) = 1.1975 \cong y(.1)$
$y_3 = 1.1975 + .05(3 + .1 - 1.1975) = 1.29263$
$y_4 = 1.29263 + .05(3 + .15 - 1.29263) = 1.38549 \cong y(.2)$.

1c. The backward Euler formula is $y_{n+1} = y_n + h(3 + t_{n+1} - y_{n+1})$. Solving this for y_{n+1} we find $y_{n+1}= [y_n + h(3 + t_{n+1})]/(1+n)$.
Thus $y_1 = \dfrac{1 + .05(3.05)}{1.05} = 1.097619$ and
$y_2 = \dfrac{1.097619 + .05(3.1)}{1.05} = 1.192971$.

5a. $y_1 = y_0 + h\dfrac{y_0^2 + 2t_0y_0}{3 + t_0^2} = .5 + .05\dfrac{(.5)^2 + 0}{3 + 0} = .504167$
$y_2 = .504167 + .05\dfrac{(.504167)^2 + 2(.05)(.504167)}{3 + (.05)^2} = .509239$

5c. $y_1 = .5 + .05\dfrac{y_1^2 + 2(.05)y_1}{3 + (.05)^2}$, which is a quadratic equation in y_1. Using the quadratic formula, or an equation solver, we obtain $y_1 = .5050895$. Thus
$y_2 = .5050895 + .05\dfrac{y_2^2 + 2(.1)y_2}{3 + (.1)^2}$ which is again quadratic in y_2, yielding $y_2 = .5111273$.

7a. For part a eighty steps must be taken, that is, $n = 0,1,...79$ and for part b 160 steps must taken with $n = 0,1,...159$. Thus use of a programmable calculator or a computer is required.

7c. We have $y_{n+1} = y_n + h(.5-t_{n+1} + 2y_{n+1})$, which is linear in y_{n+1} and thus we have $y_{n+1} = \dfrac{y_n + .5h - ht_{n+1}}{1 - 2h}$. Again, 80 steps are needed here and 160 steps in part d. In This case a spreadsheet is very useful. The first few, the middle three and last two lines are shown for $h = .025$:

n	y_n	t_n	y_{n+1}
0	1	0	1.06513
1	1.06513	.025	1.13303
2	1.13303	.050	1.20381
⋮			
38	7.49768	.950	7.87980
39	7.87980	.975	8.28137
40	8.28137	1.000	8.70341
⋮			
78	55.62105	1.950	58.50966
79	58.50966	1.975	61.54964
80	61.54964	2.000	

At least eight decimal places were used in all calculations.

9c. The backward Euler formula gives $y_{n+1} = y_n + h\sqrt{t_{n+1} + y_{n+1}}$. Subtracting y_n from both sides, squaring both sides, and solving for y_{n+1} yields $y_{n+1} = y_n + \dfrac{h^2}{2} + h\sqrt{y_n + t_{n+1} + h^2/4}$. Alternately, an equation solver can be used to solve $y_{n+1} = y_n + h\sqrt{t_{n+1} + y_{n+1}}$ for y_{n+1}. The first few values, for $h = 0.25$, are $y_1 = 3.043795$, $y_2 = 2.088082$, $y_3 = 3.132858$ and $y_4 = 3.178122 \cong y(.1)$.

15. If $y' = 1 - t + 4y$ then $y'' = -1 + 4y' = -1 + 4(1-t+4y) = 3 - 4t + 16y$. In Eq.(12) we let y_n, y_n' and y_n'' denote the approximate values of $\phi(t_n)$, $\phi'(t_n)$, and $\phi''(t_n)$, respectively. Keeping the first three terms in the Taylor series we have

$$y_{n+1} = y_n + y_n'h + y_n''\, h^2/2$$
$$= y_n + (1 - t_n + 4y_n)h + (3 - 4t_n + 16y_n)h^2/2.$$

For $n = 0$, $t_0 = 0$ and $y_0 = 1$ we have

$$y_1 = 1 + (1 - 0 + 4)(.1) + (3 - 0 + 16)\frac{(.1)^2}{2} = 1.595.$$

16. If $y = \phi(t)$ is the exact solution of the I.V.P., then $\phi'(t) = 2\phi(t) - 1$ and $\phi''(t) = 2\phi'(t) = 4\phi(t) - 2$. From Eq.(21), $e_{n+1} = [2\phi(\bar{t}_n) - 1]h^2$, $t_n < \bar{t}_n < t_n + h$. Thus a

bound for e_{n+1} is $|e_{n+1}| \le [1 + 2\max_{0\le t\le 1}|\phi(t)|]h^2$. Since the exact solution is $y = \phi(t) = [1 + \exp(2t)]/2$,
$e_{n+1} = h^2\exp(2\bar{t}_n)$. Therefore
$|e_1| \le (0.1)^2\exp(0.2) = 0.012$ and
$|e_4| \le (0.1)^2\exp(0.8) = 0.022$, since the maximum value of $\exp(2\bar{t}_n)$ occurs at $t = .1$ and $t = .4$ respectively. From Problem 2 of Section 2.7, the actual error in the first step is .0107.

19. The local truncation error is $e_{n+1} = \phi''(\bar{t}_n)h^2/2$. For this problem $\phi'(t) = 5t - 3\phi^{1/2}(t)$ and thus
$\phi''(t) = 5 - (3/2)\phi^{-1/2}\phi' = 19/2 - (15/2)t\phi^{-1/2}$.
Substituting this last expression into e_{n+1} yields the desired answer.

22d. Since $y'' = -5\pi\sin5\pi t$, Eq.(21) gives $e_{n+1} = -(5\pi/2)\sin(5\pi\bar{t}_n)h^2$.
Thus $|e_{n+1}| < \dfrac{5\pi}{2}h^2 < .05$, or $h < \dfrac{1}{\sqrt{50\pi}} \cong .08$.

23a. From Eq.(14) we have $E_n = \phi(t_n) - y_n$. Using this in Eq.(20) we obtain
$E_{n+1} = E_n + h\{f[t_n,\phi(t_n)] - f(t_n,y_n)\} + \phi''(\bar{t}_n)h^2/2$. Using the given inequality involving L we have
$|f[t_n,\phi(t_n)] - f(t_n,y_n)| \le L\,|\phi(t_n) - y_n| = L|E_n|$ and thus
$|E_{n+1}| \le |E_n| + hL|E_n| + \max_{t_0\le t\le t_n}|\phi''(t)|h^2/2 = \alpha|E_n| + \beta h^2$.

23b. Since $\alpha = 1 + hL$, $\alpha - 1 = hL$. Hence $\beta h^2(\alpha^n-1)/(\alpha-1) =$
$\beta h^2[(1+hL)^n - 1]/hL = \beta h[(1+hL)^n - 1]/L$.

23c. $(1+hL)^n \le \exp(nhL)$ follows from the observation that
$\exp(nhL) = [\exp(nL)]^n = (1 + hL + h^2L^2/2! + \dots)^n$.
Noting that $nh = t_n - t_0$, the rest follows from Eq.(ii).

24. The Taylor series for $\phi(t)$ about $t = t_{n+1}$ is
$\phi(t) = \phi(t_{n+1}) + \phi'(t_{n+1})(t-t_{n+1}) + \phi''(t_{n+1})\dfrac{(t-t_{n+1})^2}{2} + \dots$.
Letting $\phi'(t) = f(t,\phi(t))$, $t = t_n$ and $h = t_{n+1} - t_n$ we have $\phi(t_n) = \phi(t_{n+1}) - f(t_{n+1},\phi(t_{n+1}))h + \phi''(\bar{t}_n)h^2/2$, where $t_n < \bar{t}_n < t_{n+1}$. Thus
$\phi(t_{n+1}) = \phi(t_n) + f(t_{n+1},\phi(t_{n+1}))h - \phi''(\bar{t}_n)h^2/2$. Comparing this to Eq. 13 we then have $e_{n+1} = -\phi''(\bar{t}_n)h^2/2$.

25b. From Problem 1 we have $y_{n+1} = y_n + h(3 + t_n - y_n)$, so
$y_1 = 1 + .05(3 + 0 - 1) = 1.1$
$y_2 = 1.1 + .05(3 + .05 - 1.1) = 1.20 \cong y(.1)$
$y_3 = 1.20 + .05(3 + .1 - 1.20) = 1.30$
$y_4 = 1.30 + .05(3 + .15 - 1.30) = 1.39 \cong y(.2)$.

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1a. The improved Euler formula is
$y_{n+1} = y_n + [y'_n + f(t_n + h, y_n + hy'_n)]h/2$ where
$y' = f(t,y) = 3 + t - y$. Hence $y'_n = 3 + t_n - y_n$ and
$f(t_n + h, y_n + hy'_n) = 3 + t_{n+1} - (y_n + hy'_n)$. Thus we obtain

$$y_{n+1} = y_n + (3 + t_n - y_n)h + \frac{h^2}{2}(1 - y'_n)$$

$$= y_n + (3 + t_n - y_n)h + \frac{h^2}{2}(-2 - t_n + y_n).$$ Thus

$$y_1 = 1 + (3-1)(.05) + \frac{(.05)^2}{2}(-2+1) = 1.098750$$ and

$$y_2 = y_1 + (3 +.05 - y_1)(.05) + \frac{(.05)^2}{2}(-2 -.05 + y_1) = 1.19512$$

are the first two steps. In this case, the equation specifying y_{n+1} is somewhat more complicated when $y'_n = 3 + t_n - y_n$ was substituted. When designing the steps to calculate y_{n+1} on a computer, y'_n can be calculated first and thus the simpler formula for y_{n+1} can be used. The exact solution is $y(t) = 2 + t - e^{-t}$, so $y(.1) = 1.19516$, $y(.2) = 1.38127$, $y(.3) = 1.55918$ and $y(.04) = 1.72968$, so the approximations using $h = .0125$ are quite accurate to five decimal places.

4. In this case $y'_n = 2t_n + e^{-t_n y_n}$ and thus the improved Euler formula is

$$y_{n+1} = y_n + \frac{[(2t_n + e^{-t_n y_n}) + 2t_{n+1} + e^{-t_{n+1}(y_n + hy'_n)}]h}{2}.$$ For

$n = 0, 1, 2$ we get $y_1 = 1.05122$, $y_2 = 1.10483$ and $y_3 = 1.16072$ for $h = .05$.

10. See Problem 4.

11. The improved Euler formula is

$$y_{n+1} = y_n + \frac{f(t_n,y_n) + f(t_{n+1}, y_n + hf(t_n,y_n))}{2}h.$$ As suggested

in the text, it's best to perform the following steps when implementing this formula: let $k_1 = (4 - t_n y_n)/(1 + y_n^2)$, $k_2 = y_n + hk_1$ and $k_3 = (4 - t_{n+1}k_2)/(1 + k_2^2)$. Then $y_{n+1} = y_n + (k_1 + k_3)h/2$.

14a. Since $\phi(t_n + h) = \phi(t_{n+1})$ we have, using the first part of Eq.(5) and the given equation,
$$e_{n+1} = \phi(t_{n+1}) - y_{n+1} = [\phi(t_n)-y_n] + [\phi'(t_n) - \frac{y'_n + f(t_n+h,\ y_n+hy'_n)}{2}]h + \phi''(t_n)h^2/2! + \phi'''(\bar{t}_n)h^3/3!.$$
Since $y_n = \phi(t_n)$ and $y'_n = \phi'(t_n) = f(t_n,y_n)$ this reduces to
$$e_{n+1} = \phi''(t_n)h^2/2! - \{f[t_n+h,\ y_n + hf(t_n,y_n)] - f(t_n,y_n)\}h/2! + \phi'''(\bar{t}_n)h^3/3!,$$
which can be written in the form of Eq.(i).

14b. First observe that $y' = f(t,y)$ and $y'' = f_t(t,y) + f_y(t,y)y'$. Hence $\phi''(t_n) = f_t(t_n,y_n) + f_y(t_n,y_n)f(t_n,y_n)$. Using the given Taylor series, with $a = t_n$, $h = h$, $b = y_n$ and $k = hf(t_n,y_n)$ we have

$$f[t_n+h,y_n+hf(t_n,y_n)] = f(t_n,y_n)+f_t(t_n,y_n)h+f_y(t_n,y_n)hf(t_n,y_n) + [f_{tt}(\xi,\eta)h^2+2f_{ty}(\xi,\eta)h^2f(t_n,y_n)+f_{yy}(\xi,\eta)h^2f^2(t_n,y_n)]/2!$$

where $t_n < \xi < t_n + h$ and $|\eta-y_n| < h|f(t_n,y_n)|$. Substituting this in Eq.(i) and using the earlier expression for $\phi''(t_n)$ we find that the first term on the right side of Eq.(i) reduces to
$$-[f_{tt}(\xi,\eta) + 2f_{ty}(\xi,\eta)f(t_n,y_n) + f_{yy}(\xi,\eta)f^2(t_n,y_n)]h^3/4,$$
which is proportional to h^3 plus, possibly, higher order terms. The reason that there may be higher order terms is because ξ and η will, in general, depend upon h.

14c. If $f(t,y)$ is linear in t and y, then $f_{tt} = f_{ty} = f_{yy} = 0$ and the terms appearing in the last formula of part (b) are all zero.

15. Since $\phi(t) = [4t - 3 + 19\exp(4t)]/16$ we have $\phi'''(t) = 76\exp(4t)$ and thus from Problem 14c, since f is linear in t and y, we find
$e_{n+1} = 38[\exp(4\bar{t}_n)]h^3/3$. Thus
$|e_{n+1}| \le (38h^3/3)\exp(8) = 37{,}758.8h^3$ on $0 \le t \le 2$. For $n = 1$, we have $|e_1| = |\phi(t_1) - y_1| \le (38/3)\exp(0.2)(.05)^3 = .001934$, which is approximately 1/15 of the error indicated in Eq.(27) of the previous section.

19. The Euler method gives
$y_1 = y_0 + h(5t_0 - 3\sqrt{y_0}) = 2 + .1(-3\sqrt{2}) = 1.57574$ and the improved Euler method gives

$$y_1 = y_0 + \frac{f(t_0,y_0) + f(t_1,y_1)}{2} h$$
$$= 2 + [-3\sqrt{2} + (.5 - 3\sqrt{1.57574})].05 = 1.62458.$$

Thus, the estimated error in using the Euler method is $1.62458 - 1.57574 = .04884$. Since we want our error tolerance to be no greater than .0025 we need to adjust the step size downward by a factor of $\sqrt{.0025/.04884} \cong .226$. Thus a step size of $h = (.1)(.23) = .023$ would be needed for the required local truncation error bound of .0025.

24. The modified Euler formula is
$y_{n+1} = y_n + hf[t_n + h/2, y_n + (h/2)f(t_n,y_n)]$ where
$f(t,y) = 5t - 3\sqrt{y}$. Thus
$y_1 = 2 + .05[5(t_0 + .025) - 3\text{sqrt}(2 + .025(5t_0 - 3\sqrt{2}))]$
$= 1.79982$ for $t_0 = 0$. The values obtained here are between the values for $h = .05$ and for $h = .025$ using the Euler method in Problem 2.

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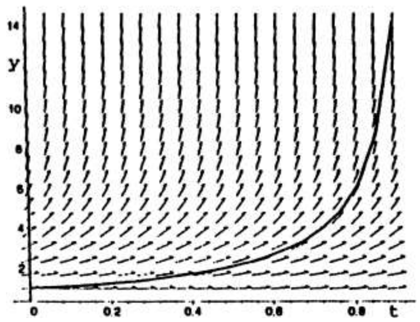
4. The Runge-Kutta formula is
$y_{n+1} = y_n + h(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})/6$ where k_{n1}, k_{n2} etc. are given by Eqs.(3). Thus for
$f(t,y) = 2t + e^{-ty}$, $(t_0,y_0) = (0,1)$ and $h = .1$ we have
$k_{01} = 0 + e^0 = 1$
$k_{02} = 2(0 + .05) + e^{-(0+.05)(1+.05k_{01})} = 1.048854$
$k_{03} = 2(.05) + e^{-(.05)(1+.05k_{02})} = 1.048738$
$k_{04} = 2(.1) + e^{-(.1)(1+.1k_{03})} = 1.095398$ and hence
$y(.1) \cong y_1 = 1 + .1(k_{01} + 2k_{02} + 2k_{03} + k_{04})/6 = 1.104843.$

11. We have $f(t_n,y_n) = (4 - t_ny_n)/(1 + y_n^2)$. Thus for $t_0 = 0$, $y_0 = -2$ and $h = .1$ we have
$k_{01} = f(0,-2) = .8$
$k_{02} = f(.05, - 2 + .05(.8)) = f(.05, - 1.96) = .846414,$
$k_{03} = f(.05, - 2 + .05k_{02}) = f(.05, - 1.957679) = .847983,$
$k_{04} = f(.1, - 2 + .1k_{03}) = f(.1, - 1.915202) = .897927,$ and
$y_1 = -2 + .1(k_{01} + 2k_{02} + 2k_{03} + k_{04})/6 = -1.915221.$ For comparison, see Problem 11 in Sections 8.1 and 8.2.

14a.



14b. We have $f(t_n,y_n) = t_n^2 + y_n^2$, $t_0 = 0$, $y_0 = 1$ and $h = .1$ so
$k_{01} = 0^2 + 1^2 = 1$
$k_{02} = (.05)^2 + (1 + .05)^2 = 1.105$
$k_{03} = (.05)^2 + [1 + .05(1.105)]^2 = 1.11605$
$k_{04} = (.1)^2 + [1 + .1(1.11605)]^2 = 1.245666$ and thus
$y_1 = 1 + .1(k_{01} + 2k_{02} + 2k_{03} + k_{04})/6 = 1.11146$. Using these steps in a computer program, we obtain the following values for y:

t	h =.1	h = .05	h = .025	h = 0.125
.8	5.842	5.8481	5.8483	5.8486
.9	14.0218	14.2712	14.3021	14.3046
.95		46.578	49.757	50.3935

14c. No accurate solution can be obtained for y(1), as the values at t = .975 for h = .025 and h = .0125 are 1218 and 23,279 respectively. These are caused by the slope field becoming vertical as $t \to 1$.

Section 8.4, Page 444

4a. The predictor formula is
$y_{n+1} = y_n + h(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})/24$
and the corrector formula is
$y_{n+1} = y_n + h(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})/24$, where
$f_n = 2t_n + \exp(-t_ny_n)$. Using the Runge-Kutta method, from Section 8.3, Problem 4a, we have for $t_0 = 0$ and $y_0 = 1$,
$y_1 = 1.1048431$, $y_2 = 1.2188411$ and $y_3 = 1.3414680$. Thus the predictor formula gives $y_4 = 1.4725974$, so $f_4 = 1.3548603$ and the corrector formula then gives $y_4 = 1.4726173$, which is the desired value. These results, and the next step, are summarized in the following table:

n	y_n	f_n	y_{n+1}	f_{n+1}	y_{n+1} Corrected
0	1	1			
1	1.1048431	1.0954004			
2	1.2188411	1.1836692			
3	1.3414680	1.2686862	1.4725974	1.3548603	1.4726173
4	1.4726173	1.3548559	1.6126246	1.4465016	1.6126215
5	1.6126215				

where f_n is given above, y_{n+1} is given by the predictor formula, and the corrected y_{n+1} is given by the corrector formula. Note that the value for f_4 on the line for $n = 4$ uses the corrected value for y_4, and differs slightly from the f_4 on the line for $n = 3$, which uses the predicted value for y_4.

4b. The fourth order Adams-Moulton method is given by Eq. (10): $y_{n+1} = y_n + (h/24)(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})$. Substituting $h = .1$ we obtain $y_{n+1} = y_n + (.0375)(19f_n - 5f_{n-1} + f_{n-2}) + .0375f_{n+1}$. For $n = 2$ we then have

$y_3 = y_2 + (.0375)(19f_2 - 5f_1 + f_0) + .0375f_3$
$\quad = 1.293894103 + .0375(.6 + e^{-.3y_3})$, using values for y_2, f_0, f_1, f_2 from part a. An equation solver then yields $y_3 = 1.341469821$. Likewise

$y_4 = y_3 + (.0375)(19f_3 - 5f_2 + f_1) + .0375f_4$
$\quad = 1.421811841 + .0375(.8 + e^{-.4y_4})$, where f_3 is calculated using the y_3 found above. This last equation yields $y_4 = 1.472618922$. Finally

$y_5 = y_4 + (.0375)(19f_4 - 5f_3 + f_2) + .0375f_5$
$\quad = 1.558379316 + .0375(1.0 + e^{-.5y_5})$, which gives $y_5 = 1.612623138$.

4c. We use Eq. (16):
$y_{n+1} = (1/25)(48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf_{n+1})$.
Thus $y_4 = .04(48y_3 - 36y_2 + 16y_1 - 3y_0) + .048f_4$
$\quad = 1.40758686 + .048(.8 + e^{-.4y_4})$, using values for y_0, y_1. y_2, y_3 from part a. An equation solver then yields $y_4 = 1.472619913$. Likewise
$y_5 = .04(48y_4 - 36y_3 + 16y_2 - 3y_1) + .048f_5$
$\quad = 1.54319349 + .048(1 + e^{-.5y_5})$, which gives $y_5 = 1.612625556$.

7a. Using the predictor and corrector formulas (Eqs.6 and 10) with $f_n = .5 - t_n + 2y_n$ and using the Runge-Kutta method to calculate y_1,y_2 and y_3, we obtain the following table for $h = .05$, $t_0 = 0$, $y_0 = 1$:

n	y_n	f_n	y_{n+1}	f_{n+1}	y_{n+1} corrected
0	1	2.5			
1	1.130171	2.710342			
2	1.271403	2.9420805			
3	1.424858	3.199717	1.591820	3.483640	1.591825
4	1.591825	3.483649	1.773716	3.797433	1.773721
5	1.773721	3.797443	1.972114	4.144227	1.972119
6	1.972119	4.144238	2.188747	4.527495	2.188753
7	2.188753	4.527507	2.425535	4.951070	2.425542
8	2.425542	4.951084	2.684597	5.419194	2.684604
9	2.684604	5.419209	2.968276	5.936551	2.968284
10	2.968284				

7b. From Eq.(10) we have

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})$$

$$= y_n + \frac{h}{24}[9(.5 - t_{n+1} + 2y_{n+1}) + 19f_n - 5f_{n-1} + f_{n-2}].$$

Solving for y_{n+1} we obtain

$$y_{n+1} = [y_n + \frac{h}{24}(19f_n - 5f_{n-1} + f_{n-2} + 4.5 - 9t_{n+1})]/(1-.75h).$$

For $h = .05$, $t_0 = 0$, $y_0 = 1$ and using y_1 and y_2 as calculated using the Runge-Kutta formula, we obtain the following table:

n	y_n	f_n	y_{n+1}
0	1	2.5	
1	1.130171	2.710342	
2	1.271403	2.942805	1.424859
3	1.424859	3.199718	1.591825
4	1.591825	3.483650	1.773722
5	1.773722	3.797444	1.972120
6	1.972120	4.144241	2.188755
7	2.188755	4.527510	2.425544
8	2.425544	4.951088	2.684607
9	2.684607	5.419214	2.968287
10	2.968287		

7c. From Eq (16) we have

$$y_{n+1} = (48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf_{n+1})/25$$

$$= [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12h(.5-t_{n+1})]/25+(24/25)hy_{n+1}.$$

Solving for y_{n+1} we have

$$y_{n+1} = [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12h(.5-t_{n+1})]/(25-24h).$$

Again, using Runge-Kutta to find y_1 and y_2, we then obtain the following table:

n	y_n	y_{n+1}
0	1	
1	1.130170833	
2	1.271402571	
3	1.424858497	1.591825573
4	1.591825573	1.773724801
5	1.773724801	1.972125968
6	1.972125968	2.188764173
7	2.188764173	2.425557376
8	2.425557376	2.684625416
9	2.684625416	2.968311063
10	2.968311063	

The exact solution is $y(t) = e^t + t/2$ so $y(.5) = 2.9682818$ and $y(2) = 55.59815$, so we see that the predictor-corrector method in part a is accurate through three decimal places.

16. Let $P_2(t) = At^2 + Bt + C$. As in Eqs. (12) and (13) let $P_2(t_{n-1}) = y_{n-1}$, $P_2(t_n) = y_n$, $P_2(t_{n+1}) = y_{n+1}$ and $P_2'(t_{n+1}) = f(t_{n+1},y_{n+1}) = f_{n+1}$. Recall that $t_{n-1} = t_n - h$ and $t_{n+1} = t_n + h$ and thus we have the four equations:

$$\begin{aligned}
A(t_n-h)^2 + B(t_n-h) + C &= y_{n-1} && \text{(i)}\\
At_n^2 + Bt_n + C &= y_n && \text{(ii)}\\
A(t_n+h)^2 + B(t_n+h) + C &= y_{n+1} && \text{(iii)}\\
2A(t_n+h) + B &= f_{n+1} && \text{(iv)}
\end{aligned}$$

Subtracting Eq. (i) from Eq. (ii) to get Eq. (v) (not shown) and subtracting Eq. (ii) from Eq. (iii) to get Eq. (vi) (not shown), then subtracting Eq. (v) from Eq. (vi) yields $y_{n+1} - 2y_n + y_{n-1} = 2Ah^2$, which can be solved for A. Thus $B = f_{n+1} - 2A(t_n+h)$ [from Eq. (iv)] and $C = y_n - t_nf_{n+1} + At_n^2 + 2At_nh$ [from Eq. (ii)]. Using these values for A, B and C in Eq. (iv) yields $y_{n+1} = (1/3)(4y_n-y_{n-1}+2hf_{n+1})$, which is Eq. (15).

Section 8.5, Page 454

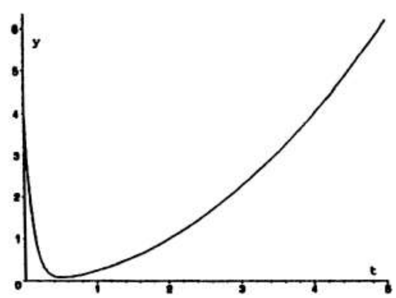
2a. If $0 \le t \le 1$ then we know $0 \le t^2 \le 1$ and hence $e^y \le t^2 + e^y \le 1 + e^y$. Since each of these terms represents a slope, we may conclude that the solution of Eq.(i) is bounded above by the solution of Eq.(iii) and is bounded below by the solution of Eq.(iv).

2b. $\phi_1(t)$ and $\phi_2(t)$ can each be found by separation of

variables. For $\phi_1(t)$ we have $\frac{1}{1+e^y}dy = dt$, or
$\frac{e^{-y}}{e^{-y}+1}dy = dt$. Integrating both sides yields
$-\ln(e^{-y}+1) = t + c$. Solving for y we find
$y = \ln[1/(c_1e^{-t}-1)]$. Setting $t = 0$ and $y = 0$, we obtain
$c_1 = 2$ and thus $\phi_1(t) = \ln[e^t/(2-e^t)]$. As $t \to \ln 2$, we see that $\phi_1(t) \to \infty$. A similar analysis shows that
$\phi_2(t) = \ln[1/(c_2-t)]$, where $c_2 = 1$ when the I.C. are used. Thus $\phi_2(t) \to \infty$ as $t \to 1$ and thus we conclude that $\phi(t) \to \infty$ for some t such that $\ln 2 \le t \le 1$.

2c. From Part b: $\phi_1(.9) = \ln[1/(c_1e^{-.9}-1)] = 3.4298$ yields
$c_1 = 2.5393$ and thus $\phi_1(t) \to \infty$ when $t \cong .9319$.
Similarly for $\phi_2(t)$ we have $c_2 = .9324$ and thus
$\phi_2(t) \to \infty$ when $t \cong .932$.

4a. The D.E. is $y' + 10y = 2.5t^2 + .5t$. So $y_h = ce^{-10t}$ is the solution of the related homogeneous equation and the particular solution, using undetermined coefficients, is
$y_p = At^2 + Bt + C$.
Substituting this into the D.E. yields
$A = 1/4$, $B = C = 0$.
To satisfy the I.C., $c = 4$, so
$y(t) = 4e^{-10t} + (t^2/4)$,
which is shown in the graph.



4b. From the discussion following Eq (15), we see that h must be less than $\frac{2}{|r|}$ for the Euler method to be stable.
Thus, for $r = 10$, $h < .2$. For $h = .2$ we obtain the following values:

t =	4	4.2	4.4	4.6	4.8	5.0
y =	8	.4	8.84	1.28	9.76	2.24

and for $h = .18$ we obtain:

t =	4.14	4.32	4.50	4.68	4.86	5.04
y =	4.26	4.68	5.04	5.48	5.89	6.35.

Clearly the second set of values is stable, although far from accurate.

4c. For a step size of .25 we find

$t = 4$	4.25	4.75	5.00
$y = 4.018$	4.533	5.656	6.205,

for a step size of .28 we find

$t = 4.2$	4.48	4.76	5.00
$y = 10.14$	10.89	11.68	12.51,

and for a step size of .3 we find

$t = 4.2$	4.5	4.8	5.1
$y = 353$	484	664	912.

Thus instability appears to occur about $h = .28$ and certainly by $h = .3$. Note that the exact solution for $t = 5$ is $y = 6.2500$, so for $h = .25$ we do obtain a good approximation.

4d. For $h = .5$ the error at $t = 5$ is .013, while for $h = .385$, the error at $t = 5.005$ is .01.

5a. The general solution of the D.E. is $y(t) = t + ce^{\lambda t}$, where $y(0) = 0 \rightarrow c = 0$ and thus $y(t) = t$, which is independent of λ.

5c. Your result in Part b will depend upon the particular computer system and software that you use. If there is sufficient accuracy, you will obtain the solution $y = t$ for t on $0 \leq t \leq 1$ for each value of λ that is given, since there is no discretization error. If there is not sufficient accuracy, then round-off error will affect your calculations. For the larger values of λ, the numerical solution will quickly diverge from the exact solution, $y = t$, to the general solution $y = t + ce^{\lambda t}$, where the value of c depends upon the round-off error. If the latter case does not occur, you may simulate it by computing the numerical solution to the I.V.P. $y' - \lambda y = 1 - \lambda t$, $y(.1) = .10000001$. Here we have assumed that the numerical solution is exact up to the point $t = .09$ [i.e. $y(.09) = .09$] and that at $t = .1$ round-off error has occurred as indicated by the slight error in the I.C. It has also been found that a larger step size ($h = .05$ or $h = .1$) may also lead to round-off error.

Section 8.6, Page 457

2a. The Euler formula is $\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}_n$, where

$$\mathbf{f}_n = \begin{pmatrix} 2x_n+t_ny_n \\ x_ny_n \end{pmatrix},\ x_0 = 1 \text{ and } y_0 = 1. \text{Thus } f_0 = \begin{pmatrix} 2-0 \\ (1)(1) \end{pmatrix} =$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix},\ \mathbf{x}_1 = \begin{pmatrix} 1+.1(2) \\ 1+.1(1) \end{pmatrix} = \begin{pmatrix} 1.2 \\ 1.1 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 2.4+.1(1.1) \\ (1.2)(1.1) \end{pmatrix} = \begin{pmatrix} 2.51 \\ 1.32 \end{pmatrix}$$

and $\mathbf{x}_2 = \begin{pmatrix} 1.2+.1(2.51) \\ 1.1+.1(1.32) \end{pmatrix} = \begin{pmatrix} 1.451 \\ 1.232 \end{pmatrix} \cong \begin{pmatrix} \phi(.2) \\ \psi(.2) \end{pmatrix}$

2b. Eqs. (7) give:

$$\mathbf{k}_{01} = \begin{pmatrix} f(0,1,1) \\ g(0,1,1) \end{pmatrix} = \begin{pmatrix} 2+0 \\ (1)(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\mathbf{k}_{02} = \begin{pmatrix} 2.4+.1(1.1) \\ (1.2)(1.1) \end{pmatrix} = \begin{pmatrix} 2.51 \\ 1.32 \end{pmatrix}$$

$$\mathbf{k}_{03} = \begin{pmatrix} 2.502+.1(1.132) \\ (1.251)(1.132) \end{pmatrix} = \begin{pmatrix} 2.6152 \\ 1.41613 \end{pmatrix}$$

$$\mathbf{k}_{04} = \begin{pmatrix} 3.04608+.2(1.28323) \\ (1.52304)(1.28323) \end{pmatrix} = \begin{pmatrix} 3.30273 \\ 1.95441 \end{pmatrix}$$

Using Eq. (6) in scalar form, we then have

$x_1 = 1+(.2/6)[2+2(2.51)+2(2.6152)+3.30273] = 1.51844$

$y_1 = 1+(.2/6)[1+2(1.32)+2(1.41613)+1.95440] = 1.28089.$

7. Write a computer program to do this problem as there are twenty steps or more for $h \le .05$.

8. If we let $y = x'$, then $y' = x''$ and thus we obtain the system $x' = y$ and $y' = t-3x-t^2y$, with $x(0) = 1$ and $y(0) = x'(0) = 2$. Thus $f(t,x,y) = y$, $g(t,x,y) = t - 3x - t^2y$, $t_0 = 0$, $x_0 = 1$ and $y_0 = 2$. If a program has been written for an earlier problem, then its best to use that. Otherwise, the first two steps are as follows:

$$\mathbf{k}_{01} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\mathbf{k}_{02} = \begin{pmatrix} 2+(-.15) \\ .05-3(1.1)-(.05)^2(1.85) \end{pmatrix} = \begin{pmatrix} 1.85 \\ -3.25463 \end{pmatrix}$$

$$\mathbf{k}_{03} = \begin{pmatrix} 2+(-.16273) \\ .05-3(1.0925)-(.05)^2(1.83727) \end{pmatrix} = \begin{pmatrix} 1.83727 \\ -3.23209 \end{pmatrix}$$

$$\mathbf{k}_{04} = \begin{pmatrix} 2+(-.32321) \\ .1-3(1.18373)-(.1)^2(1.67679) \end{pmatrix} = \begin{pmatrix} 1.67679 \\ -3.46796 \end{pmatrix}$$

and thus

$$x_1 = 1+(.1/6)[2 + 2(1.85)+2(1.83727)+(1.67679)]=1.18419,$$
$$y_1 = 2+(.1/6)[-3-2(3.25463)-2(3.23209)-3.46796]=1.67598,$$

which are approximations to $x(.1)$ and $y(.1) = x'(.1)$. In a similar fashion we find

$$\mathbf{k}_{11} = \begin{pmatrix} 1.67598 \\ -3.46933 \end{pmatrix} \qquad \mathbf{k}_{12} = \begin{pmatrix} 1.50251 \\ -3.68777 \end{pmatrix}$$

$$\mathbf{k}_{13} = \begin{pmatrix} 1.49159 \\ -3.66151 \end{pmatrix} \qquad \mathbf{k}_{14} = \begin{pmatrix} 1.30983 \\ -3.85244 \end{pmatrix}$$

and thus

$$x_2=x_1+(.1/6)[1.67598+2(1.50251)+2(1.49159)+1.30983]=1.33376$$
$$y_2=y_1-(.1/6)[3.46933+2(3.68777)+2(3.66151)+3.85244]=1.30897.$$

Three more steps must be taken in order to approximate $x(.5)$ and $y(.5) = x'(.5)$. The intermediate steps yield $x(.3) \cong 1.44489$, $y(.3) \cong .9093062$ and $x(.4) \cong 1.51499$, $y(.4) \cong .4908795$.

Capítulo 9

CHAPTER 9

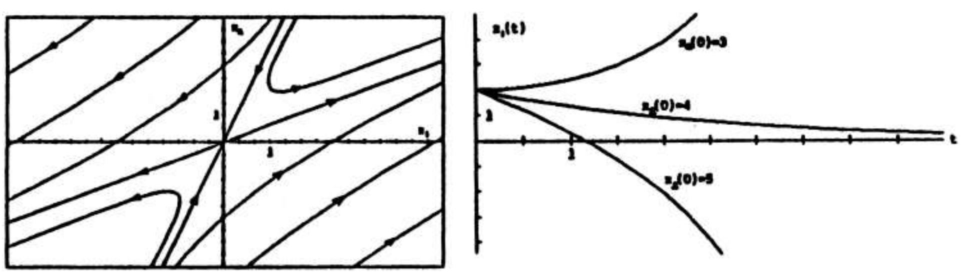
Section 9.1, Page 468

For Problems 1 through 16, once the eigenvalues have been found, Table 9.1.1 will, for the most part, quickly yield the type of critical point and the stability. In all cases it can be easily verified that **A** is nonsingular.

1a. The eigenvalues are found from the equation det($\mathbf{A}$-r$\mathbf{I}$)=0. Substituting the values for **A** we have $\begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix}$ = $r^2 - r - 2 = 0$ and thus the eigenvalues are $r_1 = -1$ and $r_2 = 2$. For $r_1 = -1$, we have $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and thus $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and for r_2 we have $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and thus $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

1b. Since the eigenvalues differ in sign, the critical point is a saddle point and is unstable.

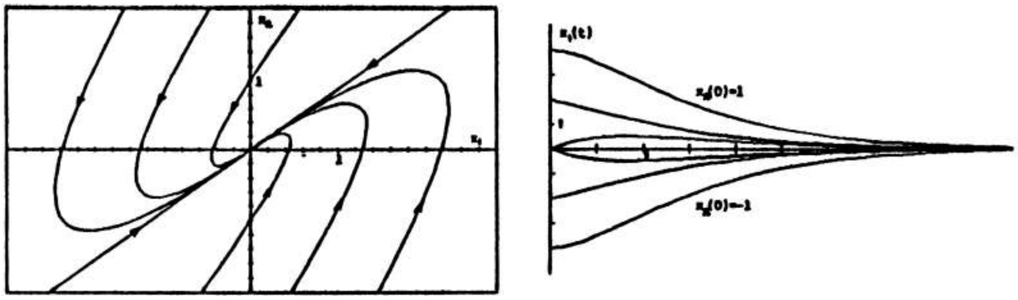
1d.



4a. Again the eigenvalues are given by $\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix}$ = $r^2 + 6r + 9 = 0$ and thus $r_1 = r_2 = -3$. The eigenvectors are solutions of $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and hence there is just one eigenvector $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4b. Since the eigenvalues are negative, (0,0) is an improper node which is asymptotically stable. If we had found that there were two independent eigenvectors then (0,0) would have been a proper node, as indicated in Case 3a.

4d.



7a. In this case $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 2r + 5$ and thus the eigenvalues are $r_{1,2} = 1 \pm 2i$. For $r_1 = 1 + 2i$ we have

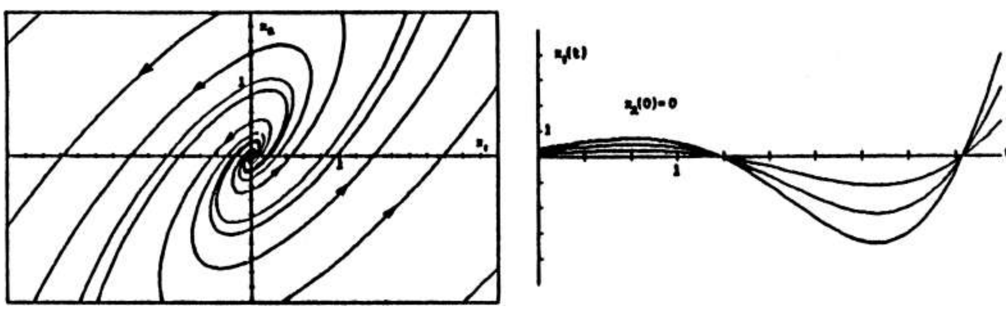
$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2-2i & -2 \\ 8-8i & -8 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and thus}$$

$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$. Similarly for $r_2 = 1-2i$ we have

$$\begin{pmatrix} 2+2i & -2 \\ 4 & -2+2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and hence } \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}.$$

7b. Since the eigenvalues are complex with positive real part, we conclude that the critical point is a spiral point and is unstable.

7d.



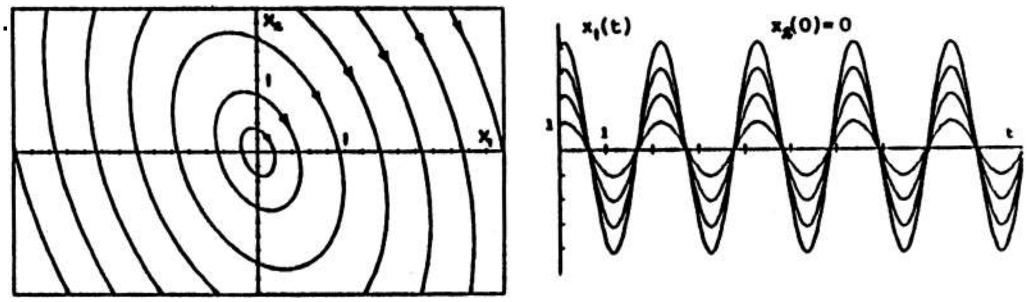
10a. Again, $\det(\mathbf{A}-r\mathbf{I}) = r^2 + 9$ and thus we have $r_{1,2} = \pm 3i$.

For $r_1 = 3i$ we have $\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and thus

$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}$. Likewise for $r_2 = -3i$,

$$\begin{pmatrix} 1+3i & 2 \\ -5 & -1+3i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ so that } \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ -1-3i \end{pmatrix}.$$

10b. Since the eigenvalues are pure imaginary the critical point is a center, which is stable.

10d.


13. If we let $\mathbf{x} = \mathbf{x^0} + \mathbf{u}$ then $\mathbf{x}' = \mathbf{u}'$ and thus the system becomes $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x^0} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{u} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ which will be in the form of Eq.(2) if $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x^0} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Using row operations, this last set of equations is equivalent to $\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x^0} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and thus $x_1^0 = 1$ and $x_2^0 = 1$. Since $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{u}$ has (0,0) as the critical point, we conclude that (1,1) is the critical point of the original system. As in the earlier problems, the eigenvalues are given by $\begin{vmatrix} 1-r & 1 \\ 1 & -1-r \end{vmatrix} = r^2 - 2 = 0$ and thus $r_{1,2} = \pm\sqrt{2}$. Hence the critical point (1,1) is an unstable saddle point.

17. The equivalent system is $dx/dt = y, dy/dt = -(k/m)x-(c/m)y$ which is written in the form of Eq.(2) as $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. The point (0,0) is clearly a critical point, and since $\mathbf{A}$ is nonsingular, it is the only one. The characteristic equation is $r^2+(c/m)r+k/m=0$ so $r_1, r_2 = [-c \pm (c^2 - 4km)^{1/2}]/2m$. In the underdamped case $c^2 - 4km < 0$, the characteristic roots are complex with negative real parts and thus the critical point (0,0) is an asymptotically stable spiral point. In the overdamped case $c^2 - 4km > 0$, the characteristic roots are real, unequal, and negative and hence the critical point (0,0) is asymptotically stable node. In the critically damped case $c^2 - 4km = 0$, the characteristic roots are equal and negative. As indicated in the solution to Problem 4, to determine whether this is an improper or proper node we must determine whether there

are one or two linearly independent eigenvectors. The eigenvectors satisfy the equation
$\begin{pmatrix} c/2m & 1 \\ -k/m & -c/2m \end{pmatrix}\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, which has just one solution if $c^2 - 4km = 0$. Thus the critical point (0,0) is an asymptotically stable improper node.

18a. If $\mathbf{A}$ has one zero eigenvalue then for $r = 0$ we have $\det(\mathbf{A}-r\mathbf{I}) = \det\mathbf{A} = 0$. Hence $\mathbf{A}$ is singular which means $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions and consequently there are infinitely many critical points.

18b. From Chapter 7, the solution is $\mathbf{x}(t) = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}e^{r_2t}$, which can be written in scalar form as $x_1 = c_1\xi_1^{(1)} + c_2\xi_1^{(2)}e^{r_2t}$ and $x_2 = c_1\xi_2^{(1)} + c_2\xi_2^{(2)}e^{r_2t}$. Assuming $\xi_1^{(2)} \neq 0$, the first equation can be solved for $c_2e^{r_2t}$, which is then substituted into the second equation to yield $x_2 = c_1\xi_2^{(1)} + [\xi_2^{(2)}/\xi_1^{(2)}][x_1-c_1\xi_1^{(1)}]$. These are straight lines parallel to the vector $\boldsymbol{\xi}^{(2)}$. Note that the family of lines is independent of c_2. If $\xi_1^{(2)} = 0$, then the lines are vertical. If $r_2 > 0$, the direction of motion will be in the same direction as indicated for $\boldsymbol{\xi}^{(2)}$. If $r_2 < 0$, then it will be in the opposite direction.

19a. $\text{Det}(A-rI) = r^2 - (a_{11}+a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$. If $a_{11} + a_{22} = 0$, then $r^2 = -(a_{11}a_{22} - a_{21}a_{12}) < 0$ if $a_{11}a_{22} - a_{21}a_{12} > 0$.

19b. Eq.(i) can be written in scalar form as $dx/dt = a_{11}x + a_{12}y$ and $dy/dt = a_{21}x + a_{22}y$, which then yields Eq.(iii). Ignoring the middle quotient in Eq.(iii), we can rewrite that equation as $(a_{21}x + a_{22}y)dx - (a_{11}x + a_{12}y)dy = 0$, which is exact since $a_{22} = -a_{11}$ from Eq.(ii)..

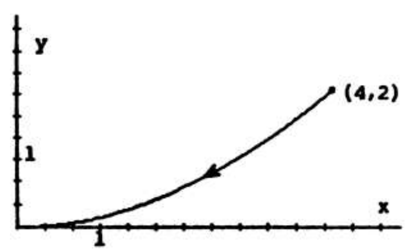
19c. Integrating $\phi_x = a_{21}x + a_{22}y$ we obtain $\phi = a_{21}x^2/2 + a_{22}xy + g(y)$ and thus $a_{22}x + g' = -a_{11}x - a_{12}y$ or $g' = -a_{12}y$ using Eq.(ii). Hence $a_{21}x^2/2 + a_{22}xy - a_{12}y^2/2 = k/2$ is the solution to Eq.(iii). The quadratic equation $Ax^2 + Bxy + Cy^2 = D$ is an ellipse provided $B^2 - 4AC < 0$. Hence for our problem if

$a_{22}^2 + a_{21}a_{12} < 0$ then Eq.(iv) is an ellipse. Using $a_{11} + a_{22} = 0$ we have $a_{22}^2 = -a_{11}a_{22}$ and hence $-a_{11}a_{22} + a_{21}a_{12} < 0$ or $a_{11}a_{22} - a_{21}a_{12} > 0$, which is true by Eqs.(ii). Thus Eq.(iv) is an ellipse under the conditions of Eqs.(ii).

20. The given system can be written as $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Thus the eigenvalues are given by $r^2 - (a_{11}+a_{22})r + a_{11}a_{22} - a_{12}a_{21} = 0$ and using the given definitions we rewrite this as $r^2 - pr + q = 0$ and thus $r_{1,2} = (p \pm \sqrt{p^2-4q})/2 = (p \pm \sqrt{\Delta})/2$. The results are now obtained using Table 9.1.1.

<u>Section 9.2, Page 477</u>

1. Solutions of the D.E. for x are y are $x = Ae^{-t}$ and $y = Be^{-2t}$ respectively. $x(0) = 4$ and $y(0) = 2$ yield $A = 4$ and $B = 2$, so $x = 4e^{-t}$ and $y = 2e^{-2t}$. Solving the first equation for e^{-t} and then substituting into the second yields $y = 2[x/4]^2 = x^2/8$, which is a parabola. From the original D.E., or from the parametric solutions, we find that $0 < x \le 4$ and $0 < y \le 2$ for $t \ge 0$ and thus only the portion of the parabola shown is the trajectory, with the direction of motion indicated.

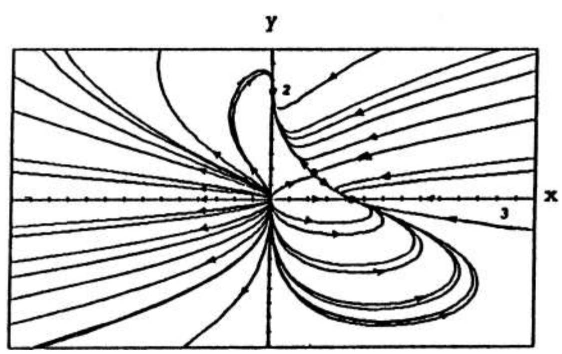


3. Utilizing the approach indicated in Eq.(14), we have $dy/dx = -x/y$, which separates into $xdx + ydy = 0$. Integration then yields the circle $x^2 + y^2 = c^2$, where $c^2 = 16$ for both sets of I.C. The direction of motion can be found from the original D.E. and is counterclockwise for both I.C. To obtain the parametric equations, we write the system in the form $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, which has the characteristic equation $\begin{vmatrix} -r & -1 \\ 1 & -r \end{vmatrix} = r^2 + 1 = 0$, or $r = \pm i$. Following

the procedures of Section 7.6, we find that one solution of the above system is $\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i\sin t \\ \sin t - i\cos t \end{pmatrix}$ and thus two real solutions are $\mathbf{u}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ and $\mathbf{v}(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$. The general solution of the system is then $\begin{pmatrix} x \\ y \end{pmatrix} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$ and hence the first I.C. yields $c_1 = 4$, $c_2 = 0$, or $x = 4\cos t$, $y = 4\sin t$. The second I.C. yields $c_1 = 0$, $c_2 = -4$, or $x = -4\sin t$, $y = 4\cos t$. Note that both these parametric representations satsify the form of the trajectories found in the first part of this problem.

7a. The critical points are given by the solutions of $x(1-x-y) = 0$ and $y(1/2 - y/4 - 3x/4) = 0$. The solutions corresponding to either $x = 0$ or $y = 0$ are seen to be $x = 0, y = 0$; $x = 0, y = 2$; $x = 1, y = 0$. In addition, there is a solution corresponding to the intersection of the lines $1 - x - y = 0$ and $1/2 - y/4 - 3x/4 = 0$ which is the point $x = 1/2$, $y = 1/2$. Thus the critical points are (0,0), (0,2), (1,0), and (1/2,1/2).

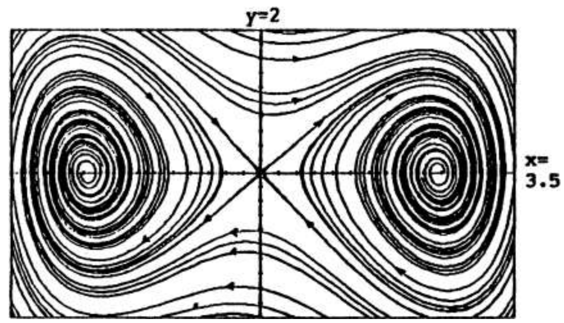
7b.



7c. For (0,0) since all trajectories leave this point, this is an unstable node. For (0,2) and (1,0) since the trajectories tend to these points, respectively, they are asymptotically stable nodes. For (1/2,1/2), one trajectory tends to (1/2,1/2) while all others tend to infinity, so this is an unstable saddle point.

12a. The critical points are given by $y = 0$ and $x(1 - x^2/6 - y/5) = 0$, so $(0,0)$, $(\sqrt{6},0)$ and $(-\sqrt{6},0)$ are the only critical points.

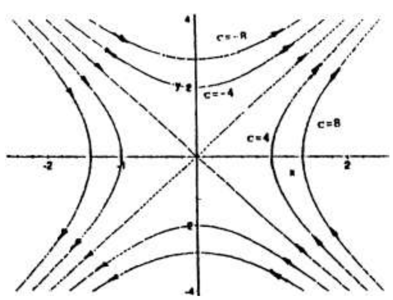
12b.



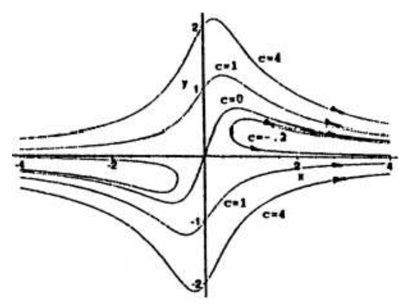
12c. Clearly $(\sqrt{6},0)$ and $(-\sqrt{6},0)$ are spiral points, and are asymptotically stable since the trajectories tend to each point, respectively. (0,0) is a saddle point, which is unstable, since the trajectories behave like the ones for (1/2,1/2) in Problem 7.

15a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{8x}{2y}$, so $4xdx - ydy = 0$ and thus $4x^2 - y^2 = c$, which are hyperbolas for $c \neq 0$ and straight lines $y = \pm 2x$ for $c = 0$.

15b.



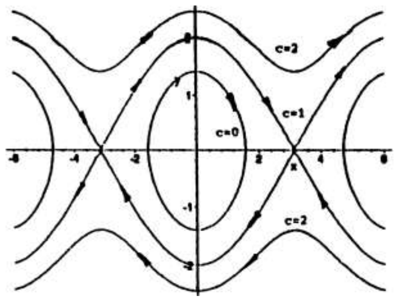
19b.



19a. $\dfrac{dy}{dx} = \dfrac{y-2xy}{-x+y+x^2}$, so $(y-2xy)dx + (x-y-x^2)dy = 0$, which is an exact D.E. Therefore $\phi(x,y) = xy - x^2y + g(y)$ and hence $\dfrac{\partial\phi}{\partial y} = x - x^2 + g'(y) = x - y - x^2$, so $g'(y) = -y$ and $g(y) = -y^2/2$. Thus $2x^2y - 2xy + y^2 = c$ (after multiplying by -2) is the desired solution.

21a. $\dfrac{dy}{dx} = \dfrac{-\sin x}{y}$, so $ydy + \sin xdx = 0$ and thus $y^2/2 - \cos x = c$.

21b.



23. We know that $\phi'(t) = F[\phi(t), \psi(t)]$ and $\psi'(t) = G[\phi(t), \psi(t)]$ for $\alpha < t < \beta$. By direct substitution we have
$\Phi'(t) = \phi'(t-s) = F[\phi(t-s), \psi(t-s)] = F[\Phi(t), \Psi(t)]$ and
$\Psi'(t) = \psi'(t-s) = G[\phi(t-s), \psi(t-s)] = G[\Phi(t), \Psi(t)]$ for
$\alpha < t-s < \beta$ or $\alpha+s < t < \beta+s$.

24. Suppose that $t_1 > t_0$. Let $s = t_1 - t_0$. Since the system is autonomous, the result of Problem 23, with s replaced by -s shows that $x = \phi_1(t+s)$ and $y = \psi_1(t+s)$ generates the same trajectory (C_1) as $x = \phi_1(t)$ and $y = \psi_1(t)$. But at $t = t_0$ we have $x = \phi_1(t_0+s) = \phi_1(t_1) = x_0$ and $y = \psi_1(t_0+s) = \psi_1(t_1) = y_0$. Thus the solution $x = \phi_1(t+s)$, $y = \psi_1(t+s)$ satisfies exactly the same initial conditions as the solution $x = \phi_0(t)$, $y = \psi_0(t)$ which generates the trajectory C_0. Hence C_0 and C_1 are the same.

25. From the existence and uniqueness theorem we know that if the two solutions $x = \phi(t)$, $y = \psi(t)$ and $x = x_0$, $y = y_0$ satisfy $\phi(a) = x_0$, $\psi(a) = y_0$ and $x = x_0$, $y = y_0$ at $t = a$, then these solutions are identical. Hence $\phi(t) = x_0$ and $\psi(t) = y_0$ for all t contradicting the fact that the trajectory generated by $[\phi(t), \psi(t)]$ started at a noncritical point.

26. By direct substitution
$\Phi'(t) = \phi'(t+T) = F[\phi(t+T), \psi(t+T)] = F[\Phi(t), \Psi(t)]$ and
$\Psi'(t) = \psi'(t+T) = G[\phi(t+T), \psi(t+T)], G[\Phi(t), \Psi(t)]$.
Furthermore $\Phi(t_0) = x_0$ and $\Psi(t_0) = y_0$. Thus by the existence and uniqueness theorem $\Phi(t) = \phi(t)$ and $\Psi(t) = \psi(t)$ for all t.

Section 9.3, Page 487

In Problems 1 through 4, write the system in the form of

Eq.(4). Then if $\mathbf{g(0)} = \mathbf{0}$ we may conclude that (0,0) is a critical point. In addition, if $\mathbf{g}$ satisfies Eq.(5) or Eq.(6), then the system is almost linear. In this case the linear system, Eq.(1), will determine, in most cases, the type and stability of the critical point (0,0) of the almost linear system. These results are summarized in Table 9.3.1.

3. In this case the system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x)\sin y \\ 1 - \cos y \end{pmatrix}.$$ However, the coefficient matrix is singular and $g_1(x,y) = (1+x)\sin y$ does not satisfy Eq.(6). However, if we consider the Taylor series for $\sin y$, we see that $(1+x)\sin y - y = \sin y - y + x\sin y = -y^3/3! + y^5/5! + \cdots + x(y - y^3/3! + \cdots)$, which does satisfy Eq.(6), using $x = r\cos\theta$, $y = r\sin\theta$. Thus the first equation now becomes

$$\frac{dx}{dt} = y + [(1+x)\sin y - y]$$ and hence

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x)\sin y - y \\ 1-\cos y \end{pmatrix},$$ where the coefficient matrix is now nonsingular and

$$\mathbf{g}(x,y) = \begin{pmatrix} (1+x)\sin y - y \\ 1-\cos y \end{pmatrix}$$ satisfies Eq.(6).

4. In this case the system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$$ and thus $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and

$\mathbf{g} = \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$. Since $\mathbf{g(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we conclude that (0,0) is a critical point. Following the procedure of Example 1, we let $x = r\cos\theta$ and $y = r\sin\theta$ and thus

$$g_1(x,y)/r = \frac{r^2\sin^2\theta}{r} \rightarrow 0 \text{ as } r \rightarrow 0$$ and thus the system is almost linear. Since $\det(\mathbf{A}-r\mathbf{I}) = (r-1)^2$, we find that the eigenvalues are $r_1 = r_2 = 1$. Since the roots are equal, we must determine whether there are one or two eigenvectors to classify the type of critical point. The eigenvectors are determined by $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and hence

there is only one eigenvector $\boldsymbol{\xi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus the critical point for the linear system is an unstable improper node. From Table 9.3.1 we then conclude that the given system, which is almost linear, has a critical point near (0,0) which is either a node or spiral point (depending on how the roots bifurcate) which is unstable.

6a. The critical points are the solutions of $x(1-x-y) = 0$ and $y(3-x-2y) = 0$. Solutions are $x = 0$, $y = 0$; $x = 0$, $3 - 2y = 0$ which gives $y = 3/2$; $y = 0$ and $1 - x = 0$ which give $x = 1$; and $1 - x - y = 0$, $3 - x - 2y = 0$ which give $x = -1$, $y = 2$. Thus the critical points are (0,0), (0,3/2), (1,0) and (-1,2).

6b, 6c. For the critical point (0,0) the D.E. is already in the form of an almost linear system; and the corresponding linear system is $du/dt = u$, $dv/dt = 3v$ which has the eigenvalues $r_1 = 1$ and $r_2 = 3$. Thus the critical point (0,0) is an unstable node. Each of the other three critical points is dealt with in the same manner; we consider only the critical point (-1,2). In order to translate this critical point to the origin we set $x(t) = -1 + u(t)$, $y(t) = 2 + v(t)$ and substitute in the D.E. to obtain

$du/dt = -1 + u - (-1+u)^2 - (-1+u)(2+v) = u + v - u^2 - uv$

and

$dv/dt = 3(2+v) - (-1+u)(2+v) - 2(2+v)^2 = -2u - 4v - uv - 2v^2$.

Writing this in the form of Eq.(4) we find that $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix}$ and $\mathbf{g} = -\begin{pmatrix} u^2 + uv \\ uv + 2v^2 \end{pmatrix}$ which is an almost linear system. The eigenvalues of the corresponding linear system are $r = (-3 \pm \sqrt{9 + 8})/2$ and hence the critical point (-1,2), of the original system, is an unstable saddle point.

10a. The critical points are solutions of $x + x^2 + y^2 = 0$ and $y(1-x) = 0$, which yield (0,0) and (-1,0).

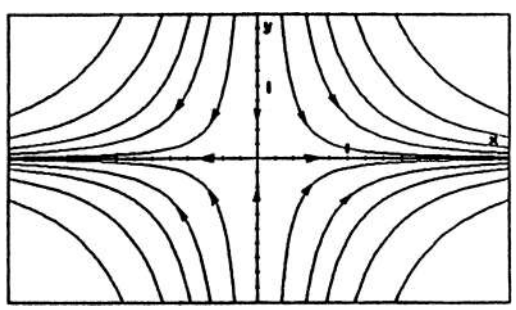
10b. For (0,0) the D.E. is already in the form of an almost linear system and thus $du/dt = u$ and $dv/dt = v$. For (-1,0) we let $u = x+1$, $v = y$ so that substituting $x = u-1$ and $y = v$ into the D.E. we obtain $\dfrac{du}{dt} = -u + u^2 + v^2$ and

$\frac{dv}{dt} = 2v - uv$. Thus the corresponding linear system is $u' = -u$ and $v' = -2v$.

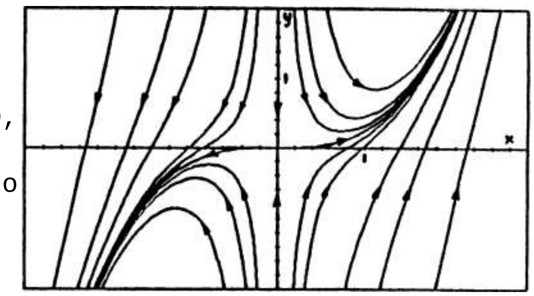
10c. For (0,0) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which has $r_1 = r_2 = 1$, so that (0,0), for the nonlinear system, will be either a node or spiral point, depending on how the roots bifurcate. In any case, since r_1 and r_2 are positive, the system will be unstable. For (-1,0) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ and thus $r_1 = -1$ and $r_2 = 2$, and hence the nonlinear system, from Table 9.3.1, has an unstable saddle point at (-1,0).

18a. The system is $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ x^3 \end{pmatrix}$ and thus is almost linear using the procedures outlined in the earlier problems. The corresponding linear system has the eigenvalues $r_1 = 1$, $r_2 = -2$ and thus (0,0) is an unstable saddle point for both the linear and almost linear systems.

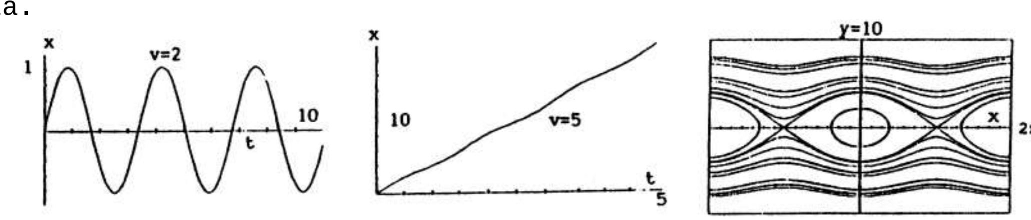
18b. The trajectories of the linear system are the solutions of $dx/dt = x$ and $dy/dt = -2y$ and thus $x(t) = c_1e^t$ and $y(t) = c_2e^{-2t}$. To sketch these, solve the first equation for e^t and substitute into the second to obtain $y = c_1^2c_2/x^2$, $c_1 \neq 0$. Several trajectories are shown in the figure. Since $x(t) = c_1e^t$, we must pick $c_1 = 0$ for $x \to 0$ and $t \to \infty$. Thus $x = 0$, $y = c_2e^{-2t}$ (the vertical axis) is the only trajectory for which $x \to 0$, $y \to 0$ as $t \to \infty$.



18c. For $x \neq 0$ we have $dy/dx = (dy/dt)/(dx/dt) = (-2y+x^3)/x$. This is a linear equation, and the general solution is $y = x^3/5 + k/x^2$, where k is an arbitrary constant. In addition the system of equations has the solution $x = 0$, $y = Be^{-2t}$. Any solution with its initial point on the y-axis (x=0) is given by the latter solution. The trajectories corresponding to these solutions approach the origin as $t \to \infty$. The trajectory that passes through the origin and divides the family of curves is given by $k = 0$, namely $y = x^3/5$. This trajectory corresponds to the trajectory $y = 0$ for the linear problem. Several trajectories are sketched in the figure.

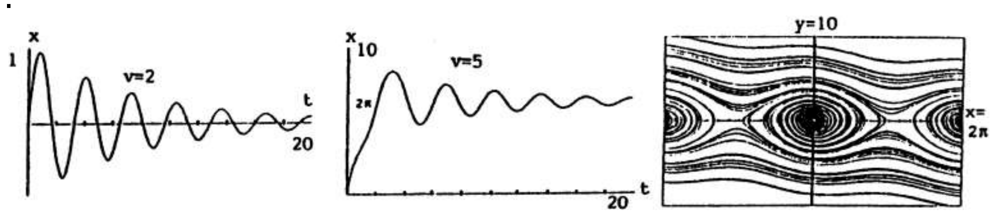


22a.



22b. From the graphs in part a, we see that v_c is between $v = 2$ and $v = 5$. Using several values for v, we estimate $v_c \cong 4.00$.

23a.



For $v = 2$, the motion is damped oscillatory about $x = 0$. For $v = 5$, the pendulum swings all the way around once and then is a damped oscillation about $x = 2\pi$ (after one full rotation). For problem 22, this later case is not damped, so x continues to increase, as shown earlier.

27a. Setting $c = 0$ in Eq.(10) of Section 9.2 we obtain $mL^2 d^2\theta/dt^2 + mgL\sin\theta = 0$. Considering $d\theta/dt$ as a function of θ and using the chain rule we have

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d}{d\theta}\left(\frac{d\theta}{dt}\right)\frac{d\theta}{dt} = \frac{1}{2}\frac{d}{d\theta}\left(\frac{d\theta}{dt}\right)^2.$$ Thus

$(1/2)mL^2 d[(d\theta/dt)^2]/d\theta = -mgL\sin\theta$. Now integrate both sides from α to θ where $d\theta/dt = 0$ at $\theta = \alpha$: $(1/2)mL^2(d\theta/dt)^2 = mgL(\cos\theta - \cos\alpha)$. Thus $(d\theta/dt)^2 = (2g/L)(\cos\theta - \cos\alpha)$. Since we are releasing the pendulum with zero velocity from a positive angle α, the angle θ will initially be decreasing so $d\theta/dt < 0$. If we restrict attention to the range of θ from $\theta = \alpha$ to $\theta = 0$, we can assert $d\theta/dt = -\sqrt{2g/L}\sqrt{\cos\theta - \cos\alpha}$. Solving for dt gives $dt = -\sqrt{L/2g}\,d\theta/\sqrt{\cos\theta - \cos\alpha}$.

27b. Since there is no damping, the pendulum will swing from its initial angle α through 0 to $-\alpha$, then back through 0 again to the angle α in one period. It follows that $\theta(T/4) = 0$. Integrating the last equation and noting that as t goes from 0 to T/4, θ goes from α to 0 yields $T/4 = -\sqrt{L/2g}\int_\alpha^0 (1/\sqrt{\cos\theta - \cos\alpha})d\theta$.

28a. If $\dfrac{dx}{dt} = y$, then $\dfrac{d^2x}{dt^2} = \dfrac{dy}{dt} = -g(x) - c(x)y$.

28b. Under the given assumptions we have $g(x) = g(0) + g'(0)x + g''(\xi_1)x^2/2$ and $c(x) = c(0) + c'(\xi_2)x$, where $0 < \xi_1, \xi_2 < x$ and $g(0) = 0$. Hence

$$\frac{dy}{dt} = (-g(0) - g'(0)x) - c(0)y - [g''(\xi_1)x^2/2 - c'(\xi_2)xy]$$

and thus the system can be written as

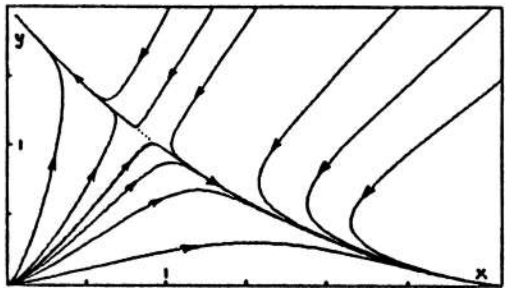
$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g'(0) & -c(0) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ -g''(\xi_1)x^2/2 - c'(\xi_2)xy \end{pmatrix},$$

from which the results follow.

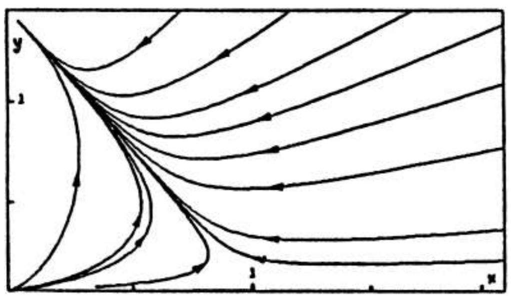
<u>Section 9.4, Page 501</u>

3b. $x(1.5 - .5x - y) = 0$ and $y(2 - y - 1.125x) = 0$ yield (0,0), (0,2) and (3,0) very easily. The fourth critical point is the intersection of $.5x + y = 1.5$ and $1.125x + y = 2$, which is (.8,1.1).

3c. From Eq (5) we get $\dfrac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1.5-x_0-y_0 & -x_0 \\ -1.125y_0 & 2-2y_0-1.125x_0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$. For (0,0) we get $u' = 1.5u$ and $v' = 2v$, so $r = 3/2$ and $r = 2$, and thus (0,0) is an unstable node. For (0,2) we have $u' = -.5u$ and $v' = -2.25u-2v$, so $r = -.5, -2$ and thus (0,2) is an asymptotically stable node. For (3,0) we get $u' = -1.5u-3v$ and $v' = -1.375v$, so $r = -1.5, -1.375$ and hence (3,0) is an symptotically stable node. For (.8,1.1) we have $u' = -.4u - .8v$ and $v' = -1.2375u - 1.1v$ which give $r = -1.80475, .30475$ and thus (.8,1.1) is an unstable saddle point.

3e. 

5b. The critical points are found by setting $dx/dt = 0$ and $dy/dt = 0$ and thus we need to solve $x(1 - x - y) = 0$ and $y(1.5 - y - x) = 0$. The first yields $x = 0$ or $y = 1 - x$ and the second yields $y = 0$ or $y = 1.5 - x$. Thus $(0,0)$, $(0,3/2)$ and $(1,0)$ are the only critical points since the two straight lines do not intersect in the first quadrant (or anywhere in this case). This is an example of one of the cases shown in Figure 9.4.5 a or b.

5e. 

6b. The critical points are found by setting $dx/dt = 0$ and $dy/dt = 0$ and thus we need to solve $x(1-x + y/2) = 0$ and $y(5/2 - 3y/2 + x/4) = 0$. The first yields $x = 0$ or $y = 2x - 2$ and the second yields $y = 0$ or $y = x/6 + 5/3$. Thus we find the critical points $(0,0)$, $(1,0)$, $(0,5/3)$ and $(2,2)$. The last point is the intersection of the two straight lines, which will be used again in part d.

6c. For $(0,0)$ the linearized system is $x' = x$ and $y' = 5y/2$, which has the eigenvalues $r_1 = 1$ and $r_2 = 5/2$. Thus the origin is an unstable node. For $(2,2)$ we let $x = u + 2$ and $y = v + 2$ in the given system to find (since $x' = u'$ and $y' = v'$) that

$du/dt = (u+2)[1 - (u+2) + (v+2)/2] = (u+2)(-u+v/2)$ and

$dv/dt = (v+2)[5/2 - 3(v+2)/2 + (u+2)/4] = (v+2)(u/4 - 3v/2)$.

Hence the linearized equations are $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -2 & 1 \\ 1/2 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

which has the eigenvalues $r_{1,2} = (-5 \pm \sqrt{3})/2$. Since these are both negative we conclude that $(2,2)$ is an asymptotically stable node. In a similar fashion for

(1,0) we let $x = u + 1$ and $y = v$ to obtain the linearized system $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -1 & 1/2 \\ 0 & 11/4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$. This has $r_1 = -1$ and $r_2 = 11/4$ as eigenvalues and thus (1,0) is an unstable saddle point. Likewise, for (0,5/3) we let $x = u$, $y = v + 5/3$ to find $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 11/6 & 0 \\ 5/12 & -5/2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ as the corresponding linear system. Thus $r_1 = 11/6$ and $r_2 = -5/2$ and thus (0,5/3) is an unstable saddle point.

6d. To sketch the required trajectories, we must find the eigenvectors for each of the linearized systems and then analyze the behavior of the linear solution near the critical point. Using this approach we find that the solution near (0,0) has the form $\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{5t/2}$ and thus the origin is approached only for large negative values of t. In this case e^t dominates $e^{5t/2}$ and hence in the neighborhood of the origin all trajectories are tangent to the x-axis except for one pair ($c_1 = 0$) that lies along the y-axis.

For (2,2) we find the eigenvector corresponding to $r = (-5 + \sqrt{3})/2 = -1.63$ is given by $(1-\sqrt{3})\xi_1/2 + \xi_2 = 0$ and thus $\begin{pmatrix} 1 \\ (\sqrt{3}-1)/2 \end{pmatrix} = \begin{pmatrix} 1 \\ .37 \end{pmatrix}$ is one eigenvector. For $r = (-5 -\sqrt{3})/2 = -3.37$ we have $(1 +\sqrt{3})\xi_1/2 + \xi_2 = 0$ and thus $\begin{pmatrix} 1 \\ -(\sqrt{3}+1)/2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1.37 \end{pmatrix}$ is the second eigenvector.

Hence the linearized solution is

$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ .37 \end{pmatrix} e^{-1.63t} + c_2 \begin{pmatrix} 1 \\ -1.37 \end{pmatrix} e^{-3.37t}$. For large positive values of t the first term is the dominant one and thus we conclude that all trajectories but two approach (2,2) tangent to the straight line with slope .37. If $c_1 = 0$, we see that there are exactly two ($c_2 > 0$ and $c_2 < 0$) trajectories that lie on the straight line with slope -1.37.

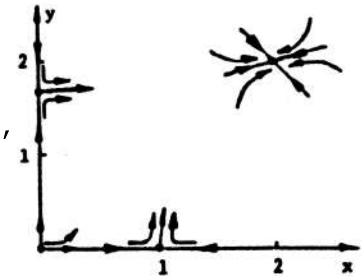
In similar fashion, we find the linearized solutions near (1,0) and (0,5/3) to be, respectively,

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 15/2 \end{pmatrix} e^{11t/4}$$

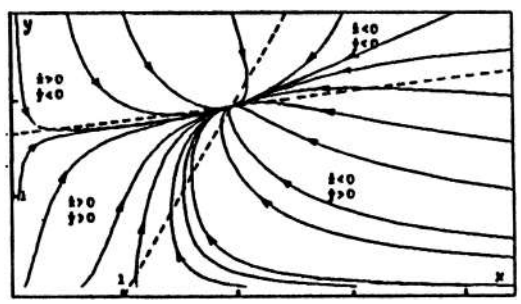
and

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t/2} + c_2 \begin{pmatrix} 1 \\ 5/52 \end{pmatrix} e^{11t/6},$$

which, along with the above analysis, yields the sketch shown.



6e.
6f. From the above sketch, it appears that $(x,y) \to (2,2)$ as $t \to \infty$ as long as (x,y) starts in the first quadrant. To ascertain this, we need to prove that x and y cannot become unbounded as $t \to \infty$. From the given system, we can observe that, since $x > 0$ and $y > 0$, that dx/dt and dy/dt have the same sign as the quantities $1 - x + y/2$ and $5/2 - 3y/2 + x/4$ respectively. If we set these quantities equal to zero we get the straight lines $y = 2x - 2$ and $y = x/6 + 5/3$, which divide the first quadrant into the four sectors shown. The signs of x' and y' are indicated, from which it can be concluded that x and y must remain bounded [and in fact approach (2,2)] as $t \to \infty$. The discussion leading up to Fig.9.4.4 is also useful here.



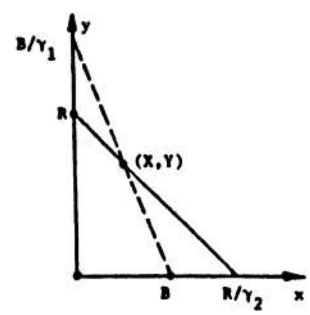
8a. Setting the right sides of the equations equal to zero gives the critical points (0,0), $(0, \varepsilon_2/\sigma_2)$, $(\varepsilon_1/\sigma_1, 0)$, and possibly $([\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1]/[\sigma_1\sigma_2 - \alpha_1\alpha_2], [\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2]/[\sigma_1\sigma_2 - \alpha_1\alpha_2])$. (The last point can be obtained from Eq.(36) also). The conditions $\varepsilon_2/\alpha_2 > \varepsilon_1/\sigma_1$ and $\varepsilon_2/\sigma_2 > \varepsilon_1/\alpha_1$ imply that $\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2 > 0$ and $\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1 < 0$. Thus either the x coordinate or the y coordinate of the last critical point is negative so a mixed state is not possible. The linearized system for (0,0) is $x' = \varepsilon_1 x$ and $y' = \varepsilon_2 y$ and thus (0,0) is an unstable equilibrium point. Similarly, it can be shown [by linearizing the given system or by using Eq.(35)] that $(0, \varepsilon_2/\sigma_2)$ is an asymptotically stable critical point and that $(\varepsilon_1\sigma_1, 0)$ is an unstable critical point. Thus the fish represented by y(redear) survive.

8b. The conditions $\epsilon_1/\sigma_1 > \epsilon_2/\alpha_2$ and $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$ imply that $\epsilon_2\sigma_1 - \epsilon_1\alpha_2 < 0$ and $\epsilon_1\sigma_2 - \epsilon_2\alpha_1 > 0$ so again one of the coordinates of the fourth point in 8a. is negative and hence a mixed state is not possible. An analysis similar to that in part(a) shows that (0,0) and $(0,\epsilon_2/\sigma_2)$ are unstable while $(\epsilon_1/\sigma_1,0)$ is stable. Hence the bluegill (represented by x) survive in this case.

9a. $x' = \epsilon_1 x(1 - \frac{\sigma_1}{\epsilon_1}x - \frac{\alpha_1}{\epsilon_1}y) = \epsilon_1 x(1 - \frac{1}{B}x - \frac{\gamma_1}{B}y)$

$y' = \epsilon_2 y(1 - \frac{\sigma_2}{\epsilon_2}y - \frac{\alpha_2}{\epsilon_2}x) = \epsilon_2 y(1 - \frac{1}{R}y - \frac{\gamma_2}{R}x)$. The coexistence equilibrium point is given by $\frac{1}{B}x + \frac{\gamma_1}{B}y = 1$ and $\frac{\gamma_2}{R}x + \frac{1}{R}y = 1$. Solving these (using determinants) yields $X = (B - \gamma_1 R)/(1 - \gamma_1\gamma_2)$ and $Y = (R - \gamma_2 B)/(1-\gamma_1\gamma_2)$.

9b. If B is reduced, it is clear from the answer to part(a) that X is reduced and Y is increased. To determine whether the bluegill will die out, we give an intuitive argument which can be confirmed by doing the analysis. Note that $B/\gamma_1 = \epsilon_1/\alpha_1 > \epsilon_2/\sigma_2 = R$ and $R/\gamma_2 = \epsilon_2/\alpha_2 > \epsilon_1/\sigma_1 = B$ so that the graph of the lines $1 - x/B - \gamma_1 y/B = 0$ and $1 - y/R - \gamma_2 x/R = 0$ must appear as indicated in the figure, where critical points are inidcated by heavy dots. As B is decreased, X decreases, Y increases (as indicated above) and the point of intersection moves closer to (0,R). If $B/\gamma_1 < R$ coexistence is not possible, and the only critical points are (0,0),(0,R)and (B,0).



It can be shown that (0,0) and (B,0) are unstable and (0,R) is asymptotically stable. Hence we conlcude, when coexistence is no longer possible, that $x \to 0$ and $y \to R$ and thus the bluegill population will die out.

12a. Setting each equation equal to zero, we obtain x = 0 or $(4 - x - y) = 0$ and y = 0 or $(2 + 2\alpha - y - \alpha x) = 0$. Thus we have (0,0), (4,0), $(0,2 + 2\alpha)$, and the intersection of $x + y = 4$ and $\alpha x + y = 2 + 2\alpha$. If $\alpha \neq 1$, this yields (2,2) as the fourth critical point.

12b. For $\alpha = .75$ the linear system is $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -2 & -2 \\ -1.5 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, which has the characteristic equation $r^2 + 4r + 1 = 0$ so that $r = -2 \pm \sqrt{3}$. Thus the critical point is an asymptotically stable node. For $\alpha = 1.25$, we have $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -2 & -2 \\ -2.5 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, so $r^2 + 4r - 1 = 0$ and $r = -2 \pm \sqrt{5}$. Thus (2,2) is an unstable saddle point.

12c. Letting $x = u+2$ and $y = v+2$ yields
$u' = (u+2)(4 - u-2 - v-2) = -2u - 2v - u^2 - uv$ and
$v' = (v+2)(2 + 2\alpha - v-2 - \alpha u - 2\alpha) = -2\alpha u - 2v - v^2 - \alpha uv$.
Thus the approximate linear system is $u' = -2u - 2v$ and $v' = -2\alpha u - 2v$.

12d. The eigenvalues are given by
$$\begin{vmatrix} -2-r & -2 \\ -2\alpha & -2-r \end{vmatrix} = r^2 + 4r + 4 - 4\alpha = 0, \text{ or } r = -2 \pm 2\sqrt{\alpha}.$$
Thus for $0 < \alpha < 1$ there are 2 negative real roots (asymptotially stable node) and for $\alpha > 1$ the real roots differ in sign, yielding an unstable saddle point. $\alpha = 1$ is the bifurcation point.

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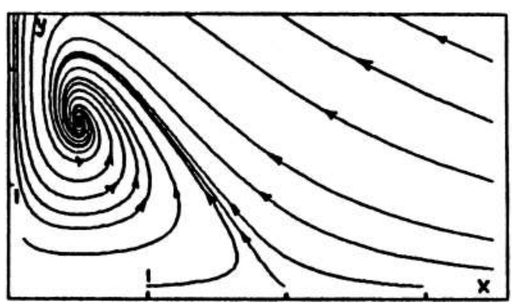
3b. We have $x = 0$ or $(1 - .5x - .5y) = 0$ and $y = 0$ or $(-.25 + .5x) = 0$ and thus we have three critical points: (0,0), (2,0) and (1/2,3/2).

3c. For (0,0) the linear system is $dx/dt = x$ and $dy/dt = -.25y$ and hence $A = \begin{pmatrix} 1 & 0 \\ 0 & -1/4 \end{pmatrix}$ which has eigenvalues $r_1 = 1$ and $r_2 = -1/4$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus (0,0) is an unstable saddle point.
For (2,0), we let $x = 2 + u$ and $y = v$ in the given equations and obtain $\dfrac{du}{dt} = -(u+v) - \dfrac{1}{2}u(u+v)$ and $\dfrac{dv}{dt} = \dfrac{3}{4}v + \dfrac{1}{2}uv$. The linear portion of this has matrix

$A = \begin{pmatrix} -1 & -1 \\ 0 & 3/4 \end{pmatrix}$, which has the eigenvalues $r_1 = -1$, $r_2 = 3/4$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$. Thus (2,0) is also an unstable saddle point.

For $\left(\frac{1}{2}, \frac{3}{2}\right)$ we let $x = 1/2 + u$ and $y = 3/2 + v$ in the given equations, which yields $\frac{du}{dt} = -\frac{1}{4}u - \frac{1}{4}v$, $\frac{dv}{dt} = \frac{3}{4}u$ as the linear portion. Thus $A = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & 0 \end{pmatrix}$, which has eigenvalues $r_{1,2} = (-1 \pm \sqrt{11}\,i)/8$. Thus $\left(\frac{1}{2}, \frac{3}{2}\right)$ is an asymptotically stable spiral point since the eigenvalues are complex with negative real part. Using $r_1 = (-1 + \sqrt{11}\,i)/8$ we find that one eigenvector is $\begin{pmatrix} -2 \\ 1 + \sqrt{11}\,i \end{pmatrix}$ and by Section 7.6 the second eigenvector is $\begin{pmatrix} -2 \\ 1 - \sqrt{11}\,i \end{pmatrix}$.

3e.



3f. For (x,y) above the line $x + y = 2$ we see that $x' < 0$ and thus x must remain bounded. For (x,y) to the right of $x = 1/2$, $y' > 0$ so it appears that y could grow large asymptotic to x = constant. However, this implies a contradiction (x = constant implies $x' = 0$, but as y gets larger, x' gets increasingly negative) and hence we conclude y must remain bounded and hence $(x,y) \to (1/2,3/2)$ as $t \to \infty$, again assuming they start in the first quadrant.

7a. The amplitude ratio is $(cK/\gamma)/(\sqrt{ac}\,K/\alpha) = \alpha\sqrt{c}/\gamma\sqrt{a}$.

7b. From Eq (2) $\alpha = .5$, $a = 1$, $\gamma = .25$ and $c = .75$, so the ratio is $.5\sqrt{.75}/.25\sqrt{1} = 2\sqrt{.75} = \sqrt{3} \cong 1.732$.

7c. A rough measurement of the amplitudes is $(6.1 - 1)/2 = 2.55$ and $(3.8 -. 9)/2 = 1.45$ and thus the ratio is approximately 1.76. In this case the linear approximation is a good predictor.

11. The presence of a trapping company actually would require a modification of the equations, either by altering the coefficients or by including nonhomogeneous terms on the right sides of the D.E. The effects of indiscreminate trapping could decrease the populations of both rabbits and fox significantly or decrease the fox population which could possibly lead to a large increase in the rabbit population. Over the long run it makes sense for a trapping company to operate in such a way that a consistent supply of pelts is available and to disturb the predator-prey system as little as possible. Thus, the company should trap fox only when their population is increasing, trap rabbits only when their population is increasing, trap rabbits and fox only during the time when both their populations are increasing, and trap neither during the time both their populations are decreasing. In this way the trapping company can have a moderating effect on the population fluctuations, keeping the trajectory close to the center.

13. The critical points of the system are the solutions of the algebraic equations $x(a - \sigma x - \alpha y) = 0$, and $y(-c + \gamma x) = 0$. the critical points are $x = 0$, $y = 0$; $x = a/\sigma$, $y = 0$; and $x = c/\gamma$, $y = a/\alpha - c\sigma/\alpha\gamma = \sigma A/\alpha$ where $A = a/\sigma - c/\gamma > 0$.
To study the critical point (0,0) we discard the nonlinear terms in the system of D.E. to obtain the corresponding linear system $dx/dt = ax$, $dy/dt = -cy$. The characteristic equation is $r^2 - (a+c)r - ac = 0$ so $r_1 = a$, $r_2 = -c$. Thus the critical point (0,0) is an unstable saddle point.
To study the critical point $(a/\sigma,0)$ we let $x = (a/\sigma) + u$, $y = 0 + v$ and substitute in the D.E. to obtain the almost linear system $du/dt = -au - (a\alpha/\sigma)v - \sigma u^2 - \alpha uv$, $dv/dt = \gamma Av + \gamma uv$. The corresponding linear system is $du/dt = -au - (a\alpha/\sigma)v$, $dv/dt = \gamma Av$. The characteristic equation is $r^2 + (a - \gamma A)r - a\gamma A = 0$ so $r_1 = -a$, $r_2 = \gamma A$.

Thus the critical point $(a/\sigma,0)$ is an unstable saddle point.
To study the critical point $(c/\gamma, \sigma A/\alpha)$ we let $x = (c/\gamma) + u$, $y = (\sigma A/\alpha) + v$ and substitute in the D.E. to obtain the almost linear system

$$\frac{du}{dt} = -(c\sigma/\gamma)u - (ac/\gamma)v - \sigma u^2 - \alpha uv$$

$$\frac{dv}{dt} = (\sigma A\gamma/\alpha)u + \gamma uv$$

The corresponding linear system is $du/dt = -(c\sigma/\gamma)u - (\alpha c/\gamma)v$, $dv/dt = (\sigma A\gamma/\alpha)u$. The characteristic equation is $r^2 + (c\sigma/\gamma)r + c\sigma A = 0$, so $r_1,r_2 = [-(c\sigma/\gamma) \pm \sqrt{(c\sigma/\gamma)^2 - 4c\sigma A}\,]/2$. Thus, depending on the sign of the discriminant we have that $(c/\gamma, \sigma A/\alpha)$ is either an asymptotically stable spiral point or an asymptotically stable node. Thus for nonzero initial data $(x,y) \to (c/\gamma, \sigma A/\alpha)$ as $t \to \infty$.

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1. Assuming that $V(x,y) = ax^2 + cy^2$ we find $V_x(x,y) = 2ax$, $V_y = 2cy$ and thus Eq.(7) yields $\dot{V}(x,y) = 2ax(-x^3 + xy^2) + 2cy(-2x^2y - y^3) = -[2ax^4 + 2(2c-a)x^2y^2 + 2cy^4]$. If we choose a and c to be any positive real numbers with $2c > a$, then $\dot{V}$ is a negative definite. Also, V is positive definite by Theorem 9.6.4. Thus by Theorem 9.6.1 the origin is an asymptotically stable critical point.

3. Assuming the same form for $V(x,y)$ as in Problem 1, we have
$\dot{V}(x,y) = 2ax(-x^3 + 2y^3) + 2cy(-2xy^2) = -2ax^4 + 4(a-c)xy^3$.
If we choose $a = c > 0$, then $\dot{V}(x,y) = -2ax^4 \le 0$ in any neighborhood containing the origin and thus $\dot{V}$ is negative semidefinite and V is positive definite. Theorem 9.6.1 then concludes that the origin is a stable critical point. Note that the origin may still be asymptotically stable, however, the $V(x,y)$ used here is not sufficient to prove that.

6a. The correct system is $dx/dt = y$ and $dy/dt = -g(x)$. Since $g(0) = 0$, we conclude that $(0,0)$ is a critical point.

6b. From the given conditions, the graph of g must be positive for $0 < x < k$ and negative for $-k < x < 0$. Thus

if $0 < x < k$ then $\int_0^x g(s)ds > 0$,

if $-k < x < 0$ then $\int_0^x g(s)ds = -\int_x^0 g(s)ds > 0$.

Since $V(0,0) = 0$ it follows that $V(x,y) = y^2/2 + \int_0^x g(s)ds$ is positive definite for $-k < x < k$, $-\infty < y < \infty$. Next, we have $\dot{V}(x,y) = V_x\frac{dx}{dt} + V_y\frac{dy}{dt} = g(x)y + y[-g(x)] = 0$.

Since $\dot{V}(x,y)$ is never positive, we may conclude that it is negative semidefinite and hence by Theorem 9.6.1 (0,0) is at least a stable critical point.

7b. V is positive definite by Theorem 9.6.4. Since $V_x(x,y) = 2x$, $V_y(x,y) = 2y$, we obtain

$\dot{V}(x,y) = 2xy - 2y^2 - 2y\sin x = 2y[-y + (x - \sin x)]$. If $x < 0$, then $\dot{V}(x,y) < 0$ for all $y > 0$. If $x > 0$, choose y so that $0 < y < x - \sin x$. Then $\dot{V}(x,y) > 0$. Hence V is not a Liapunov function.

7c. Since $V(0,0) = 0$, $1 - \cos x > 0$ for $0 < |x| < 2\pi$ and $y^2 > 0$ for $y \neq 0$, it follows that $V(x,y)$ is positive definite in a neighborhood of the origin. Next $V_x(x,y) = \sin x$, $V_y(x,y) = y$, so

$\dot{V}(x,y) = (\sin x)(y) + y(-y - \sin x) = -y^2$. Hence $\dot{V}$ is negative semidefinite and (0,0) is a stable critical point by Theorem 9.6.1.

7d. $V(x,y) = (x+y)^2/2 + x^2 + y^2/2 = 3x^2/2 + xy + y^2$ is positive definite by Theorem 9.6.4. Next $V_x(x,y) = 3x + y$, $V_y(x,y) = x + 2y$ so

$$\begin{aligned}
\dot{V}(x,y) &= (3x+y)y - (x+2y)(y+\sin x) \\
&= 2xy - y^2 - (x+2y)\sin x \\
&= 2xy - y^2 - (x+2y)(x - \alpha x^3/6) \\
&= -x^2 - y^2 + \alpha(x+2y)x^3/6 \\
&= -r^2 + \alpha r^4 (\cos\theta + 2\sin\theta)(\cos^3\theta)/6 < -r^2 + r^4/2 \\
&= -r^2(1-r^2/2).
\end{aligned}$$

Thus $\dot{V}$ is negative definite for $r < \sqrt{2}$. From Theorem 9.6.1 it follows that the origin is an asymptotically stable critical point.

8. Let $x = u$ and $y = du/dt$ to obtain the system $dx/dt = y$ and $dy/dt = -c(x)y - g(x)$. Now consider $V(x,y) = y^2/2 + \int_0^x g(s)ds$, which yields

$$\dot{V} = g(x)y + y[-c(x)y - g(x)] = -y^2c(x).$$

10b. Since $V_x(x,y) = 2Ax + By$, $V_y(x,y) = Bx + 2Cy$, we have $\dot{V}(x,y) = (2Ax + By)(a_{11}x + a_{12}y) + (Bx + 2Cy)(a_{21}x + a_{22}y) = (2Aa_{11} + Ba_{21})x^2 + [2(Aa_{12} + Ca_{21}) + B(a_{11}+a_{22})]xy + (2Ca_{22} + Ba_{12})y^2$. We choose A, B, and C so that $2Aa_{11} + Ba_{21} = -1$, $2(Aa_{12} + Ca_{21}) + B(a_{11}+a_{22}) = 0$, and $2Ca_{22} + Ba_{12} = -1$. The first and third equations give us A and C in terms of B, respectively. We substitute in the second equation to find B and then calculate A and C. The result is given in the text.

10c. Since $a_{11}a_{22} - a_{12}a_{21} > 0$ and $a_{11} + a_{22} < 0$, we see that $\Delta < 0$ and so $A > 0$. Using the expressions for A, B, and C found in part (b) we obtain

$$\begin{aligned}(4AC-B^2)\Delta^2 &= [a_{21}^2+a_{22}^2 + (a_{11}a_{22}-a_{12}a_{21})][a_{11}^2+a_{12}^2 + (a_{11}a_{22}-a_{12}a_{21})] \\ &\qquad - (a_{12}a_{22}+a_{11}a_{21})^2 \\ &= (a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2)(a_{11}a_{22}-a_{12}a_{21}) + (a_{11}^2+a_{12}^2)(a_{21}^2+a_{22}^2) \\ &\qquad + (a_{11}a_{22}-a_{12}a_{21})^2 - (a_{12}a_{22}+a_{11}a_{21})^2 \\ &= (a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2)(a_{11}a_{22}-a_{12}a_{21}) + 2(a_{11}a_{22}-a_{12}a_{21})^2.\end{aligned}$$

Since $a_{11}a_{22} - a_{12}a_{21} > 0$ it follows that $4AC - B^2 > 0$.

11a. For $V(x,y) = Ax^2 + Bxy + Cy^2$ we have

$$\begin{aligned}\dot{V} &= (2Ax + By)(a_{11}x+a_{12}y + F_1(x,y)) + (Bx+2Cy)(a_{21}x+a_{22}y + G_1(x,y)) \\ &= (2Ax+By)(a_{11}x+a_{12}y)+(Bx+2Cy)(a_{21}x+a_{22}y) \\ &\qquad + (2Ax+By)F_1(x,y) + (Bx+2Cy)G_1(x,y) \\ &= -x^2-y^2 + (2Ax+By)F_1(x,y) + (Bx+2Cy)G_1(x,y),\end{aligned}$$

if A,B and C are chosen as in Problem 10.

11b. Substituting $x = r\cos\theta$, $y = r\sin\theta$ we find that $\dot{V}[x(r,\theta), y(r,\theta)] = -r^2+r(2A\cos\theta+B\sin\theta)F_1[x(r,\theta), y(r,\theta)] + r(B\cos\theta + 2C\sin\theta)G_1[x(r,\theta), y(r,\theta)]$. Now we make use of the facts that (1) there exists an M such that $|2A| \le M$, $|B| \le M$, and $|2C| \le M$; and (2) given any $\varepsilon > 0$ there exists a circle $r = R$ such that $|F_1(x,y)| < \varepsilon r$ and $|G_1(x,y)| < \varepsilon r$ for $0 \le r < R$. We have $|2A\cos\theta + B\sin\theta| \le 2M$ and $|B\cos\theta + 2C\sin\theta| \le 2M$. Hence $\dot{V}[x(r,\theta), y(r,\theta)] \le -r^2 + 2Mr(\varepsilon r)+2Mr(\varepsilon r) = -r^2(1 - 4M\varepsilon)$.

If we choose $\varepsilon = M/8$ we obtain $\dot{V}[x(r,\theta), y(r,\theta)] \le -r^2/2$ for $0 \le r < R$. Hence $\dot{V}$ is negative definite in $0 \le r < R$ and from Problem 10c V is positive definite and thus V is a Liapunov function for the almost linear system.

Section 9.7, Page 530

1. Note that $r = 1$, $\theta = t + t_0$ satisfy the two equations for all t and is thus a periodic solution. If $r < 1$, then $dr/dt > 0$, and the direction of motion on a trajectory is outward. If $r > 1$, then the direction of motion is inward. It follows that the periodic solution $r = 1$, $\theta = t + t_0$ is a stable limit cycle.

2. $r = 1$, $\theta = -t + t_0$ is a periodic solution. If $r < 1$, then $dr/dt > 0$, and the direction of motion on a trajectory is outward. If $r > 1$, the $dr/dt > 0$, and the direction of motion is still outward. It follows that the solution $r = 1$, $\theta = -t + t_0$ is a semistable limit cycle.

4. $r = 1$, $\theta = -t + t_0$ and $r = 2$, $\theta = -t + t_0$ are periodic solutions. If $r < 1$, then $dr/dt < 0$, and the direction of motion on a trajectory is inward. If $1 < r < 2$, then $dr/dt > 0$, and the direction of motion is outward. Similarly, if $r > 2$, the direction of motion is inward. It follows that the periodic solution $r = 1$, $\theta = -t + t_0$ is unstable and the periodic solution $r = 2$, $\theta = -t + t_0$ is a stable limit cycle.

7. Differentiating x and y with respect to t we find that $dx/dt = (dr/dt)\cos\theta - (r\sin\theta)d\theta/dt$ and $dy/dt = (dr/dt)\sin\theta + (r\cos\theta)d\theta/dt$. Hence
$$\begin{aligned} y\,dx/dt - x\,dy/dt &= (r\sin\theta\cos\theta)dr/dt - (r^2\sin^2\theta)d\theta/dt - \\ &\quad (r\cos\theta\sin\theta)dr/dt - (r^2\cos\theta)d\theta/dt \\ &= -\,r^2 d\theta/dt. \end{aligned}$$

8a. Multiplying the first equation by x and the second by y and adding yields $x\,dx/dt + y\,dy/dt = (x^2+y^2)f(r)/r$, or $r\,dr/dt = rf(r)$, as in the derivation of Eq.(8), and thus $dr/dt = f(r)$. To obtain an equation for θ multiply the first equation by y, the second by x and substract to obtain $y\,dx/dt - x\,dy/dt = -x^2-y^2$, or $-r^2 d\theta/dt = -r^2$, using the results of Problem 7. Thus $d\theta/dt = 1$. It follows that periodic solutions are given by $r = c$, $\theta = t + t_0$ where $f(c) = 0$. Since $\theta = t + t_0$, the motion is counterclockwise.

8b. First note that $f(r) = r(r-2)^2(r-3)(r-1)$. Thus $r = 1$, $\theta = t + t_0$; $r = 2$, $\theta = t + t_0$; and $r = 3$, $\theta = t + t_0$ are periodic solutions. If $r < 1$, then $dr/dt > 0$, and the direction of motion on a trajectory is outward. If $1 < r < 2$, then $dr/dt < 0$ and the direction of motion is inward. Thus the periodic solution $r = 1$, $\theta = t + t_0$ is a stable limit cycle. If $2 < r < 3$, then $dr/dt < 0$, and the direction of motion is inward. Thus the periodic solution $r = 2$, $\theta = t + t_0$ is a semistable limit cycle. If $r > 3$, then $dr/dt > 0$, and the direction of motion is outward. Thus the periodic solution $r = 3$, $\theta = t + t_0$ is unstable.

9. Setting $x = r\cos\theta$, $y = r\sin\theta$ and using the techniques of Problem 8 the equations transform to $dr/dt = r^2 - 2$, $d\theta/dt = -1$. This system has a periodic solution $r = \sqrt{2}$, $\theta = -t + t_0$. If $r < \sqrt{2}$, then $dr/dt < 0$, and the direction of motion along a trajectory is inward. If $r > \sqrt{2}$, then $dr/dt > 0$, and the direction of motion is outward. Thus the periodic solution $r = \sqrt{2}$, $\theta = -t + t_0$ is unstable.

11. If $F(x,y) = x+y+x^3-y^2$, $G(x,y) = -x+2y+x^2y+y^3/3$, then $F_x(x,y) + G_y(x,y) = 1+3x^2+2+x^2+y^2 = 3+4x^2+y^2$. Since the conditions of Theorem 9.7.2 are satisfied for all x and y, and since $F_x + G_y > 0$ for all x and y, it follows that the system has no periodic nonconstant solution.

13. Since $x = \phi(t)$, $y = \psi(t)$ is a solution of Eqs.(15), we have $d\phi/dt = F[\phi(t),\psi(t)]$, $d\psi/dt = G[\phi(t),\psi(t)]$. Hence on the curve C,
$F(x,y)dy - G(x,y)dx = \phi'(t)\psi'(t)dt - \psi'(t)\phi'(t)dt = 0$. It follows that the line integral around C is zero. However, if $F_x + G_y$ has the same sign throughout D, then the double integral cannot be zero. This gives a contradiction. Thus either the solution of Eqs.(15) is not periodic or if it is, it cannot lie entirely in D.

16a. Setting $x' = 0$ and solving for y yields $y = x^3/3 - x + k$. Substituting this into $y' = 0$ then gives
$x + .8(x^3/3 - x + k) - .7 = 0$ Using an equation solver we obtain $x = 1.1994$, $y = -.62426$ for $k = 0$ and $x = .80485$, $y = -.13106$ for $k = .5$. To determine the type of critical points these are, we use Eq.(13) of Section 9.3 to find the

linear coefficient matrix to be $\mathbf{A} = \begin{pmatrix} 3(1-x_c^2) & 3 \\ -1/3 & -.8/3 \end{pmatrix}$, where x_c is the critical point. For $x_c = 1.1994$ we obtain complex conjugate eigenvalues with a negative real part, and therefore k = 0 yields an asymptotically stable spiral point. For $x_c = .80485$ the eigenvalues are also complex conjugates, but with positive real parts, so k = .5 yields an unstable spiral point.

16b. Letting k = .1, .2, .3, .4 in the cubic equation of part (a) and finding the corresponding eigenvalues from the matrix in part (a), we find that the real part of the eigenvalues change sign between k = .3 and k = .4. Continuing to iterate in this fashion we find that for k = .3464 that the real part of the eigenvalue is -.0002 while for k = .3465 the real part is .00005, which indicates $k_0 = .3465$ is the critical point for which the system changes from stable to unstable.

16d. You must plot the solution for values of k slightly less than k_0, found in part (c), to determine whether a limit cycle exists.

<u>Section 9.8, Page 538</u>

1a. From Eq (6), $\lambda = -8/3$ is clearly one eigenvalue and the other two may be found from $\lambda^2 + 11\lambda - 10(r-1) = 0$ using the quadratic formula.

1b. For $\lambda = \lambda_1$ we have

$$\begin{pmatrix} -10+8/3 & -10 & 0 \\ r & -1+8/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$ which requires $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 = 0$ and $\boldsymbol{\xi}_3$ arbitrary and thus $\boldsymbol{\xi}^{(1)} = (0,0,1)^T$.

For $\lambda = \lambda_3 = (-11 + \alpha)/2$, where $\alpha = \sqrt{81+40r}$, we have

$$\begin{pmatrix} -10+(11-\alpha)/2 & 10 & 0 \\ r & -1+(11-\alpha)/2 & 0 \\ 0 & 0 & -8/3+(11-\alpha)/2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The last line implies $\xi_3 = 0$ and multiplying the first line by

$(-9+\alpha)/2$ we obtain $\begin{pmatrix} (81-\alpha^2)/4 & 10(-9+\alpha)/2 \\ r & (9-\alpha)/2 \end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

Substituting $\alpha^2 = 81+40r$ we have

$\begin{pmatrix} -10r & -10(9-\alpha)/2 \\ r & (9-\alpha)/2 \end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. Thus $\xi^{(3)} = \begin{pmatrix} 9 - \sqrt{81+40r} \\ -2r \\ 0 \end{pmatrix}$,

which is proportional to the answer given in the text. Similar calculations give $\boldsymbol{\xi}^{(2)}$.

1c. Simply substitute $r = 28$ into the answers in parts (a) and (b).

2a. The calculations are somewhat simplified if you let $x = \beta + u$, $y = \beta + v$, and $z = (r-1)+w$, where $\beta = \sqrt{8(r-1)/3}$. An alternate approach is to extend Eq.(13) of Section 9.3, which is:

$$\begin{pmatrix}u\\v\\w\end{pmatrix}' = \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{pmatrix}_{(x_0,y_0,z_0)} \begin{pmatrix}u\\v\\w\end{pmatrix}.$$

In this example $F = -10x + 10y$, $G = rx - y - xz$ and $H = -8z/3 + xy$ and thus

$\begin{pmatrix}u\\v\\w\end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & -x_0 \\ y_0 & x_0 & -8/3 \end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}$, which is Eq.(8) for P_2 since $x_0 = y_0 = \sqrt{8(r-1)/3}$.

2b. Eq.(9) is found by evaluating $\begin{vmatrix} -10-\lambda & 10 & 0 \\ 1 & -1-\lambda & -\beta \\ \beta & \beta & -8/3-\lambda \end{vmatrix} = 0.$

2c. If $r = 28$, then Eq.(9) is $3\lambda^3 + 41\lambda^2 + 304\lambda + 4320 = 0$, which has the roots -13.8546 and $.093956 \pm 10.1945i$.

3b. If r_1,r_2,r_3 are the three roots of a cubic polynomial, then the polynomial can be factored as $(x-r_1)(x-r_2)(x-r_3)$. Expanding this and equating to the given polynomial we have $A = -(r_1+r_2+r_3)$, $B = r_1r_2 + r_1r_3 + r_2r_3$ and $C = -r_1r_2r_3$. We are interested in the case when the real part of the complex conjugate roots changes sign. Thus let $r_2 = \alpha+i\beta$ and $r_3 = \alpha-i\beta$, which yields

$A = -(r_1+2\alpha)$, $B = 2\alpha r_1 + \alpha^2 + \beta^2$ and $C = -r_1(\alpha^2+\beta^2)$.

Hence, if $AB = C$, we have

$-(r_1+2\alpha)(2\alpha r_1+\alpha^2+\beta^2) = -r_1(\alpha^2+\beta^2)$ or
$-2\alpha[r_1^2 + 2\alpha r_1 + (\alpha^2+\beta^2)] = 0$ or $-2\alpha[(r_1+\alpha)^2+\beta^2] = 0$.
Since the square bracket term is positive, we conclude that if $AB = C$, then $\alpha = 0$. That is, the conjugate complex roots are pure imaginary. Note that the converse is also true. That is, if the conjugate complex roots are pure imaginary then $AB = C$.

3c. Comparing Eq.(9) to that of part b, we have $A = 41/3$, $B = 8(r+10)/3$ and $C = 160(r-1)/3$. Thus $AB = C$ yields $r = 470/19$.

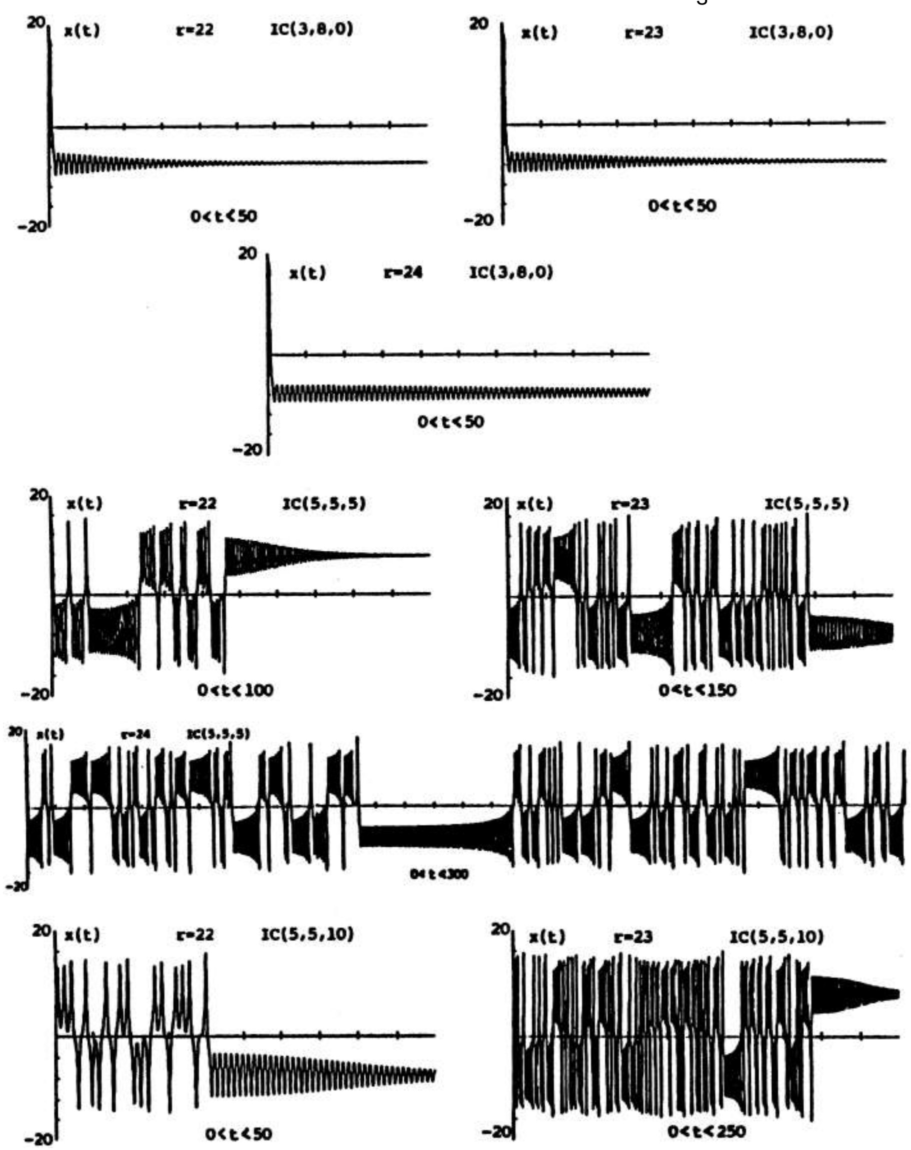
4. We have $\dot{V} = 2x[\sigma(-x+y)] + 2\sigma y[rx-y-xz] + 2\sigma z[-bz+xy]$
$= -2\sigma x^2 + 2\sigma xy + 2\sigma rxy - 2\sigma y^2 - 2\sigma bz^2$
$= 2\sigma\{-[x^2-(r+1)xy+y^2]-bz^2\}$. For $r < 1$, the term in the square brackets remains positive for all values of x and y, by Theorem 9.6.4, and thus $\dot{V}$ is negative definite. Thus, by the extension of Theorem 9.6.1 to three equations, we conclude that the origin is an asymptotically stable critical point.

5a. $V = rx^2 + \sigma y^2 + \sigma(z-2r)^2 = c > 0$ yields
$$\frac{dv}{dt} = 2rx[\sigma(-x+y)] + 2\sigma y(rx-y-xz) + 2\sigma(z-2r)(-bz+xy).$$ Thus
$$\dot{V} = -2\sigma[rx^2+y^2 + b(z^2 - 2rz)] = -2\sigma[rx^2 + y^2 + b(z-r)^2 - br^2].$$

5b. From the proof of Theorem 9.6.1, we find that we need to show that $\dot{V}$, as found in part a, is always negative as it crosses $V(x,y,z) = c$. (Actually, we need to use the extension of Theorem 9.6.1 to three equations, but the proof is very similar using the vector calculus approach.) From part a we see that
$\dot{V} < 0$ if $rx^2 + y^2 + b(z-r)^2 > br^2$, which holds if (x,y,z) lies outside the ellipsoid $\dfrac{x^2}{br} + \dfrac{y^2}{br^2} + \dfrac{(z-r)^2}{r^2} = 1$. (i)
Thus we need to choose c such that $V = c$ lies outside Eq.(i). Writing $V = c$ in the form of Eq.(i) we obtain the ellipsoid $\dfrac{x^2}{c/r} + \dfrac{y^2}{c/\sigma} + \dfrac{(z-2r)^2}{c/\sigma} = 1$. (ii) Now let $M = \max(\sqrt{br}, r\sqrt{b}, r)$, then the ellipsoid (i) is contained inside the sphere S1: $\dfrac{x^2}{M^2} + \dfrac{y^2}{M^2} + \dfrac{(z-r)^2}{M^2} = 1$.
Let S2 be a sphere centered at $(0,0,2r)$ with radius

M+r: $\dfrac{x^2}{(M+r)^2} + \dfrac{y^2}{(M+r)^2} + \dfrac{(z-2r)}{(M+r^2)} = 1$, then S1 is contained in S2. Thus, if we choose c, in Eq.(ii), such that $\dfrac{c}{r} > (M+r)^2$ and $\dfrac{c}{\sigma} > (M+r)^2$, then $\dot{V} < 0$ as the trajectory crosses V(x,y,z) = c. Note that this is a sufficient condition and there may be many other “better” choices using different techniques.

8b. Several cases are shown. Results may vary, particularly for r = 24, due to the closeness of r to $r_3 \cong 24.06$.



Capítulo

10

CHAPTER 10

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2. $y(x) = c_1\cos\sqrt{2}\,x + c_2\sin\sqrt{2}\,x$ is the general solution of the D.E. Thus $y'(x) = -\sqrt{2}\,c_1\sin\sqrt{2}\,x + \sqrt{2}\,c_2\cos\sqrt{2}\,x$ and hence $y'(0) = \sqrt{2}\,c_2 = 1$, which gives $c_2 = 1/\sqrt{2}$. Now, $y'(\pi) = -\sqrt{2}\,c_1\sin\sqrt{2}\,\pi + \cos\sqrt{2}\,\pi = 0$ then yields $c_1 = \dfrac{\cos\sqrt{2}\,\pi}{\sqrt{2}\sin\sqrt{2}\,\pi} = \cot\sqrt{2}\,\pi/\sqrt{2}$. Thus the desired solution is $y = (\cot\sqrt{2}\,\pi\cos\sqrt{2}\,x + \sin\sqrt{2}\,x)/\sqrt{2}$.

3. We have $y(x) = c_1\cos x + c_2\sin x$ as the general solution and hence $y(0) = c_1 = 0$ and $y(L) = c_2\sin L = 0$. If $\sin L \neq 0$, then $c_2 = 0$ and $y(x) = 0$ is the only solution. If $\sin L = 0$, then $y(x) = c_2\sin x$ is a solution for arbitrary c_2.

7. $y(x) = c_1\cos 2x + c_2\sin 2x$ is the solution of the related homogeneous equation and $y_p(x) = \frac{1}{3}\cos x$ is a particular solution, yielding $y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\cos x$ as the general solution of the D.E. Thus $y(0) = c_1 + \frac{1}{3} = 0$ and $y(\pi) = c_1 - \frac{1}{3} = 0$ and hence there is no solution since there is no value of c_1 that will satisfy both boundary conditions.

11. If $\lambda < 0$, the general solution of the D.E. is $y = c_1\sinh\sqrt{\mu}\,x + c_2\cosh\sqrt{\mu}\,x$ where $-\lambda = \mu$. The two B.C. require that $c_2 = 0$ and $c_1 = 0$ so $\lambda < 0$ is not an eigenvalue. If $\lambda = 0$, the general solution of the D.E. is $y = c_1 + c_2x$. The B.C. require that $c_1 = 0$, $c_2 = 0$ so again $\lambda = 0$ is not an eigenvalue. If $\lambda > 0$, the general solution of the D.E. is $y = c_1\sin\sqrt{\lambda}\,x + c_2\cos\sqrt{\lambda}\,x$. The B.C. require that $c_2 = 0$ and $\sqrt{\lambda}\;c_1\cos\sqrt{\lambda}\,\pi = 0$. The second condition is satisfied for $\lambda \neq 0$ and $c_1 \neq 0$ if $\sqrt{\lambda}\,\pi = (2_{n-1})\pi/2$, $n = 1,2,\ldots$. Thus the eigenvalues are $\lambda_n = (2n-1)^2/4$, $n = 1,2,3\ldots$ with the corresponding eigenfunctions $y_n(x) = \sin[(2n-1)x/2]$, $n = 1,2,3\ldots$.

15. For $\lambda < 0$ there are no eigenvalues, as shown in Problem 11. For $\lambda = 0$ we have $y(x) = c_1 + c_2x$, so $y'(0) = c_2 = 0$ and

$y'(\pi) = c_2 = 0$, and thus $\lambda = 0$ is an eigenvalue, with $y_0(x) = 1$ as the eigenfunction. For $\lambda > 0$ we again have $y(x) = c_1\sin\sqrt{\lambda}\,x + c_2\cos\sqrt{\lambda}\,x$, so $y'(0) = \sqrt{\lambda}\,c_1 = 0$ and $y'(L) = -c_2\sqrt{\lambda}\,\sin\sqrt{\lambda}\,L = 0$. We know $\lambda > 0$, in this case, so the eigenvalues are given by $\sin\sqrt{\lambda}\,L = 0$ or $\sqrt{\lambda}\,L = n\pi$. Thus $\lambda_n = (n\pi/L)^2$ and $y_n(x) = \cos(n\pi x/L)$ for $n = 1,2,3...$.

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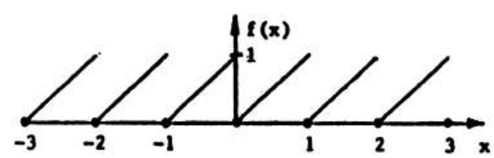
3. We look for values of T for which $\sinh 2(x+T) = \sinh 2x$ for all x. Expanding the left side of this equation gives $\sinh 2x\cosh 2T + \cosh 2x\sinh 2T = \sinh 2x$, which will be satisfied for all x if we can choose T so that $\cosh 2T = 1$ and $\sinh 2T = 0$. The only value of T satisfying these two constraints is $T = 0$. Since T is not positive we conclude that the function $\sinh 2x$ is not periodic.

5. We look for values of T for which $\tan\pi(x+T) = \tan\pi x$. Expanding the left side gives $\tan\pi(x+T) = (\tan\pi x + \tan\pi T)/(1-\tan\pi x\tan\pi T)$ which is equal to $\tan\pi x$ only for $\tan\pi T = 0$. The only positive solutions of this last equation are $T = 1,2,3...$ and hence $\tan\pi x$ is periodic with fundamental period $T = 1$.

7. To start, let $n = 0$, then $f(x) = \begin{cases} 0 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \end{cases}$; for $n = 1$, $f(x) = \begin{cases} 0 & 1 \le x < 2 \\ 1 & 2 \le x < 3 \end{cases}$; and for $n = 2$, $f(x) = \begin{cases} 0 & 3 \le x < 4 \\ 1 & 4 \le x < 5 \end{cases}$. By continuing in this fashion, and drawing a graph, it can be seen that $T = 2$.

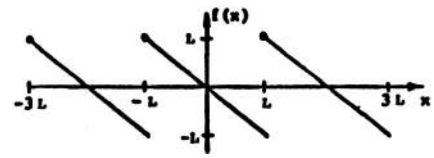
10. The graph of f(x) is:



We note that f(x) is a straight line with a slope of one in any interval. Thus f(x) has the form x+b in any interval for the correct value of b. Since $f(x+2) = f(x)$, we may set $x = -1/2$ to obtain $f(3/2) = f(-1/2)$. Noting that 3/2 is on the interval $1 < x < 2[f(3/2) = 3/2 + b]$ and that $-1/2$ is on the interval $-1 < x < 0[f(-1/2) = -1/2 + 1]$, we conclude that $3/2 + b = -1/2 + 1$, or $b = -1$ for the interval $1 < x < 2$. For the interval $8 < x < 9$ we have $f(x+8) = f(x+6) = ... = f(x)$ by successive applications of the periodicity condition. Thus for $x = 1/2$ we have $f(17/2) = f(1/2)$ or $17/2 + b = 1/2$ so $b = -8$ on the interval $8 < x < 9$.

In Problems 13 through 18 it is often necessary to use integration by parts to evaluate the coefficients, although all the details will not be shown here.

13a. The function represents a sawtooth wave. It is periodic with period 2L.



13b. The Fourier series is of the form

$$f(x) = a_0/2 + \sum_{m=1}^{\infty} (a_m\cos m\pi x/L + b_m\sin m\pi x/L\ ),$$ where the coefficients are computed from Eqs. (13) and (14). Substituting for f(x) in these equations yields $a_0 = (1/L)\int_{-L}^{L}(-x)dx = 0$ and $a_m = (1/L)\int_{-L}^{L}(-x)\cos(m\pi x/L)dx = 0$, $m = 1,2...$ (these can be shown by direct integration, or using the fact that $\int_{-a}^{a}g(x)dx = 0$ when g(x) is an odd function). Finally,

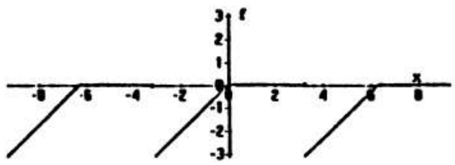
$$b_m = (1/L)\int_{-L}^{L}(-x)\sin(m\pi x/L)dx$$
$$= (x/m\pi)\cos(m\pi x/L)\Big|_{-L}^{L} - (1/m\pi)\int_{-L}^{L}\cos(m\pi x/L)dx$$
$$= (2L\cos m\pi)/m\pi - (L/m^2\pi^2)\sin(m\pi x/L)\Big|_{-L}^{L} = 2L(-1)^m/m\pi$$

Substituting these terms in the above Fourier series for f(x) yields the desired answer.

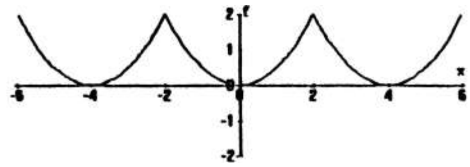
15a. See the next page.

15b. In this case f(x) is periodic of period 2π and thus $L = \pi$ in Eqs. (9), (13,) and (14). The constant a_0 is found to be $a_0 = (1/\pi)\int_{-\pi}^{0}xdx = -\pi/2$ since f(x) is zero on the interval $[0,\pi]$. Likewise $a_n = (1/\pi)\int_{-\pi}^{0}x\cos nx dx = [1 - (-1)^n]/n^2\pi$, using integration by parts and recalling that $\cos n\pi = (-1)^n$. Thus $a_n = 0$ for n even and $a_n = 2/n^2\pi$ for n odd, which may be written as $a_{2n-1} = 2/(2n-1)^2\pi$ since 2n-1 is always an odd number. In a similar fashion $b_n = (1/\pi)\int_{-\pi}^{0}x\sin nx dx = (-1)^{n+1}/n$ and thus the desired solution is obtained. Notice that in this case both cosine and sine terms appear in the Fourier series for the given f(x).

15a.



21a.

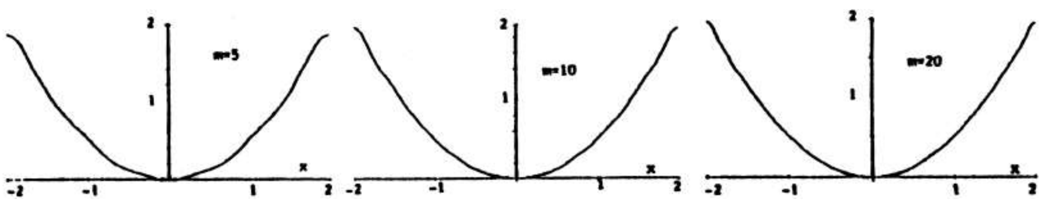


21b. $a_0 = \dfrac{1}{2}\displaystyle\int_{-2}^{2}\dfrac{x^2}{2}dx = \dfrac{1}{12}x^3\Big|_{-2}^{2} = \dfrac{4}{3}$, so $\dfrac{a_0}{2} = \dfrac{2}{3}$ and

$$a_n = \frac{1}{2}\int_{-2}^{2}\frac{x^2}{2}\cos\frac{n\pi x}{2}dx$$
$$= \frac{1}{4}\left[\frac{2x^2}{n\pi}\sin\frac{n\pi x}{2} + \frac{8x}{n^2\pi^2}\cos\frac{n\pi x}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}\right]\Bigg|_{-2}^{2}$$
$$= (8/n^2\pi^2)\cos(n\pi) = (-1)^n 8/n^2\pi^2$$

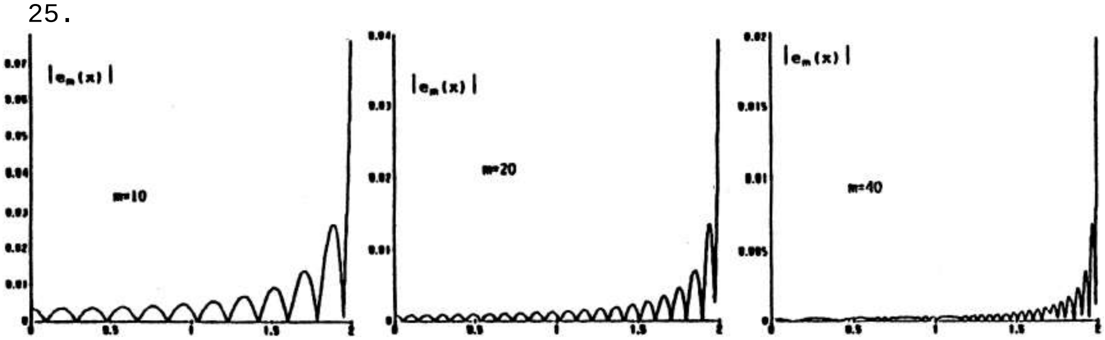
where the second line for a_n is found by integration by parts or a computer algebra system. Similarly, $b_n = \dfrac{1}{2}\displaystyle\int_{-2}^{2}\dfrac{x^2}{2}\sin\dfrac{n\pi x}{2}dx = 0$, since $x^2\sin\dfrac{n\pi x}{2}$ is an odd function. Thus $f(x) = \dfrac{2}{3} + \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos\dfrac{n\pi x}{2}$.

21c. As in Eq. (27), we have $s_m(x) = \dfrac{2}{3} + \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{m}\dfrac{(-1)^n}{n^2}\cos\dfrac{n\pi x}{2}$



21d. Observing the graphs we see that the Fourier series converges quite rapidly, except, at x = -2 and x = 2, where there is a sharp “point” in the periodic function.

25.

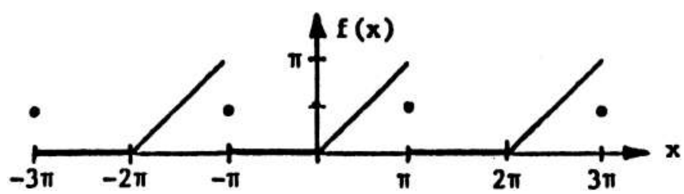


27a. First we have $\int_T^{a+T} g(x)dx = \int_0^a g(s)ds$ by letting $x = s + T$ in the left integral. Now, if $0 \le a \le T$, then from elementary calculus we know that
$\int_a^{a+T} g(x)dx = \int_a^T g(x)dx + \int_T^{a+T} g(x)dx = \int_a^T g(x)dx + \int_0^a g(x)dx$
using the equality derived above. This last sum is $\int_0^T g(x)dx$ and thus we have the desired result.

<u>Section 10.3, Page 562</u>

2a. Substituting for $f(x)$ in Eqs.(2) and (3) with $L = \pi$ yields $a_0 = (1/\pi)\int_0^\pi xdx = \pi/2$;
$a_m = (1/\pi)\int_0^\pi x\cos mx dx = (\cos m\pi - 1)/\pi m^2 = 0$ for m even and
$= -2/\pi m^2$ for m odd; and
$b_m = (1/\pi)\int_0^\pi x\sin mx dx = -(\pi\cos m\pi)/m\pi = (-1)^{m+1}/m$,
$m = 1,2...$. Substituting these values into Eq.(1) with $L = \pi$ yields the desired solution.

2b. The function to which the series converges is indicated in the figure and is periodic with period 2π. Note that



the Fourier series converges to $\pi/2$ at $x = -\pi, \pi$, etc., even though the function is defined to be zero there. This value ($\pi/2$) represents the mean value of the left and right hand limits at those points. In $(-\pi, 0)$, $f(x) = 0$ and $f'(x) = 0$ so both f and f′ are continuous and have finite limits as $x \to -\pi$ from the right and as $x \to 0$ from the left. In $(0, \pi)$, $f(x) = x$ and $f'(x) = 1$ and again both f and f′ are continuous and have limits as $x \to 0$ from the right and as $x \to \pi$ from the left. Since f and f′ are piecewise continuous on $[-\pi, \pi]$ the conditions of the Fourier theorem are satisfied.

4a. Substituting for $f(x)$ in Eqs.(2) and (3), with $L = 1$

yields $a_0 = \int_{-1}^1 (1-x^2)dx = 4/3$;

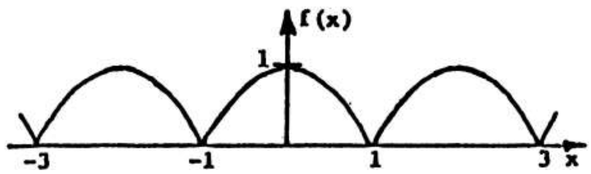
$a_n = \int_{-1}^{1}(1-x^2)\cos n\pi x dx = (2/n\pi)\int_{-1}^{1}x\sin n\pi x dx$

$= (-2/n^2\pi^2)[x\cos n\pi x\Big|_{-1}^{1} - \int_{-1}^{1}\cos n\pi x dx]$

$= 4(-1)^{n+1}/n^2\pi^2$; and

$b_n = \int_{-1}^{1}(1-x^2)\sin n\pi x dx = 0$. Substituting these values into Eq.(1) gives the desired series.

4b. The function to which the series converges is shown in the figure and is periodic of fundamental period 2. In $[-1,1]$ $f(x)$ and $f'(x) = -2x$ are both continuous and have finite limits as the endpoints of the interval are approached from within the interval.



7a. As in Problem 15, Section 10.2, we have

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{2\cos(2n-1)x}{\pi(2n-1)^2} + \frac{(-1)^{n+1}\sin nx}{n}\right].$$

7b. $$e_n(x) = f(x) + \frac{\pi}{4} - \sum_{k=1}^{n}\left[\frac{2\cos(2k-1)x}{\pi(2k-1)^2} + \frac{(-1)^{k+1}\sin kx}{k}\right].$$

Using a computer algebra system, we find that for $n = 5$, 10 and 20 the maximum error occurs at $x = -\pi$ in each case and is 1.6025, 1.5867 and 1.5787 respectively. Note that the author's n values are 10, 20 and 40, since he has included the zero cosine coefficient terms and the sine terms are all zero at $x = -\pi$.

7c. It's not possible in this case, due to Gibb's phenomenon, to satisfy $|e_n(x)| \le 0.01$ for all x.

12a. $a_0 = \int_{-1}^{1}(x-x^3)dx = 0$ and $a_n = \int_{-1}^{1}(x-x^3)\cos n\pi x dx = 0$ since $(x-x^3)$ and $(x-x^3)\cos n\pi x$ are odd functions.

$b_n = \int_{-1}^{1}(x-x^3)\sin n\pi x dx$

$$= \left[\frac{x^3}{n\pi}\cos n\pi x - \frac{3x^2}{n^2\pi^2}\sin n\pi x - \frac{(n^2\pi^2+6)}{n^3\pi^3}x\cos n\pi x + \frac{(n^2\pi^2+6)}{n^4\pi^4}\sin n\pi x\right]_{-1}^{1}$$

$$= \frac{-12}{n^3\pi^3}\cos n\pi, \text{ so } f(x) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x.$$

12b. $e_n(x) = f(x) + \frac{12}{\pi^3}\sum_{k=1}^{n}\frac{(-1)^k}{k^3}\sin k\pi x$. These errors will be much smaller than in the earlier problems due to the n^3 factor in the denominator. Convergence is much more rapid in this case.

14. The solution to the corresponding homogeneous equation is found by methods presented in Section 3.4 and is $y(t) = c_1\cos\omega t + c_2\sin\omega t$. For the nonhomogeneous terms we use the method of superposition and consider the sequence of equations $y_n'' + \omega^2 y_n = b_n\sin nt$ for $n = 1,2,3\ldots$. If $\omega > 0$ is not equal to an integer, then the particular solution to this last equation has the form $Y_n = a_n\cos nt + d_n\sin nt$, as previously discussed in Section 3.6. Substituting this form for Y_n into the equation and solving, we find $a_n = 0$ and $d_n = b_n/(\omega^2-n^2)$. Thus the formal general solution of the original nonhomogeneous D.E. is

$$y(t) = c_1\cos\omega t + c_2\sin\omega t + \sum_{n=1}^{\infty} b_n(\sin nt)/(\omega^2-n^2),$$ where

we have superimposed all the Y_n terms to obtain the infinite sum. To evalute c_1 and c_2 we set $t = 0$ to obtain $y(0) = c_1 = 0$ and

$$y'(0) = \omega c_2 + \sum_{n=1}^{\infty} nb_n/(\omega^2-n^2) = 0$$ where we have formally

differentiated the infinite series term by term and evaluated it at $t = 0$. (Differentiation of a Fourier Series has not been justified yet and thus we can only consider this method a formal solution at this point).

Thus $c_2 = -(1/\omega)\sum_{n=1}^{\infty} nb_n/(\omega^2-n^2)$, which when substituted into the above series yields the desired solution.

If $\omega = m$, a positive integer, then the particular solution of $y_m'' + m^2 y_m = b_m\sin mt$ has the form $Y_m = t(a_m\cos mt + d_m\sin mt)$ since $\sin mt$ is a solution of the related homogeneous D.E. Substituting Y_m into the D.E. yields $a_m = -b_m/2m$ and $d_m = 0$ and thus the general solution of the D.E. (when $\omega = m$) is now $y(t) = c_1\cos mt$

$$+ c_2\sin mt - b_m t(\cos mt)/2m + \sum_{n=1,n\neq m}^{\infty} b_n(\sin nt)/(m^2-n^2).$$

To evaluate c_1 and c_2 we set $y(0) = 0 = c_1$ and

$y'(0) = c_2 m - b_m/2m + \sum_{n=1, n\neq m}^{\infty} b_n n/(m^2-n^2) = 0$. Thus

$c_2 = b_m/2m^2 - \sum_{n=1, n\neq m}^{\infty} b_n n/m(m^2-n^2)$, which when substituted into the equation for y(t) yields the desired solution.

15. In order to use the results of Problem 14, we must first find the Fourier series for the given f(t). Using Eqs.(2) and (3) with $L = \pi$, we find that

$a_0 = (1/\pi)\int_0^{\pi} dx - (1/\pi)\int_{\pi}^{2\pi} dx = 0$;

$a_n = (1/\pi)\int_0^{\pi} \cos nx dx - (1/\pi)\int_{\pi}^{2\pi} \cos nx dx = 0$; and

$b_n = (1/\pi)\int_0^{\pi} \sin nx dx - (1/\pi)\int_{\pi}^{2\pi} \sin nx dx = 0$ for n even and $= 4/n\pi$ for n odd. Thus

$f(t) = (4/\pi) \sum_{n=1}^{\infty} \sin(2n-1)t/(2n-1)$. Comparing this to the forcing function of Problem 14 we see that b_n of Problem 14 has the specific values $b_{2n} = 0$ and $b_{2n-1} = (4/\pi)/(2n-1)$ in this example. Substituting these into the answer to Problem 14 yields the desired solution. Note that we have asumed ω is not a positive integer. Note also, that if the solution to Problem 14 is not available, the procedure for solving this problem would be exactly the same as shown in Problem 14.

16. From Problem 8, the Fourier series for f(t) is given by

$f(t) = 1/2 + (4/\pi^2)\sum_{n=1}^{\infty} \cos(2n-1)\pi t/(2n-1)^2$ and thus we may not use the form of the answer in Problem 14. The procedure outlined there, however, is applicable and will yield the desired solution.

18a. We will assume f(x) is continuous for this part. For the case where f(x) has jump discontinuities, a more detailed proof can be developed, as shown in part b. From Eq.(3) we have $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$. If we let $u = f(x)$ and $dv = \sin\frac{n\pi x}{L}dx$, then $du = f'(x)dx$ and $v = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}$.

Thus

$$b_n = \frac{1}{L}\left[\frac{-L}{n\pi}f(x)\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}dx\right]$$

$$= -\frac{1}{n\pi}[f(L)\cos n\pi - f(-L)\cos(-n\pi)] + \frac{1}{n\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}dx$$

$$= \frac{1}{n\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}dx,$$ since $f(L) = f(-L)$ and

$\cos(-n\pi) = \cos n\pi$. Hence $nb_n = \frac{1}{\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}dx$, which exists for all n since $f'(x)$ is piecewise continuous. Thus nb_n is bounded as $n \to \infty$. Likewise, for a_n, we obtain $na_n = -\frac{1}{\pi}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}dx$ and hence na_n is also bounded as $n \to \infty$.

18b. Note that f and f′ are continuous at all points where f″ is continuous. Let $x_1, \ldots, x_m$ be the points in $(-L,L)$ where f″ is not continuous. By splitting up the interval of integration at these points, and integrating Eq.(3) by parts twice, we obtain

$$n^2b_n = \frac{n}{\pi}\sum_{i=1}^{m}[f(x_i+)-f(x_i-)]\cos\frac{n\pi x_i}{L} - \frac{n}{\pi}[f(L-)-f(-L+)]\cos n\pi$$

$$-\frac{L}{\pi^2}\sum_{i=1}^{m}[f'(x_i+)-f'(x_1-)]\sin\frac{n\pi x_i}{L} - \frac{L}{\pi^2}\int_{-L}^{L} f''(x)\sin\frac{n\pi x}{L}dx,$$ where

we have used the fact that cosine is continuous. We want the first two terms on the right side to be zero, for otherwise they grow in magnitude with n. Hence f must be continuous throughout the closed interval $[-L,L]$. The last two terms are bounded, by the hypotheses on f′ and f″. Hence n^2b_n is bounded; similarly n^2a_n is bounded. Convergence of the Fourier series then follows by comparison with $\sum_{n=1}^{\infty} n^{-2}$.

<u>Section 10.4, Page 570</u>

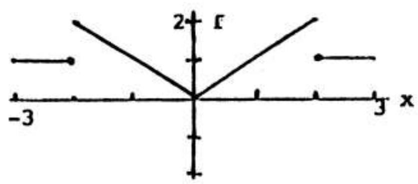
3. Let $f(x) = \tan 2x$, then

$$f(-x) = \tan(-2x) = \frac{\sin(-2x)}{\cos(-2x)} = \frac{-\sin(2x)}{\cos(2x)} = -\tan 2x = -f(x)$$

and thus $\tan 2x$ is an odd function.

6. Let $f(x) = e^{-x}$, then $f(-x) = e^{x}$ so that $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$ and thus e^{-x} is neither even nor odd.

7.



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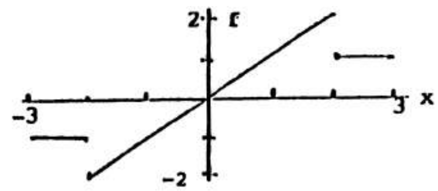
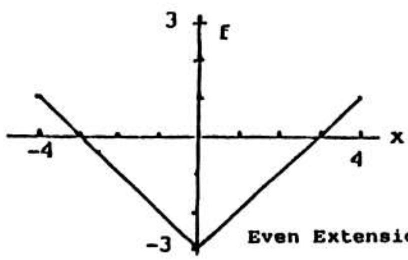


Odd Extension

10.



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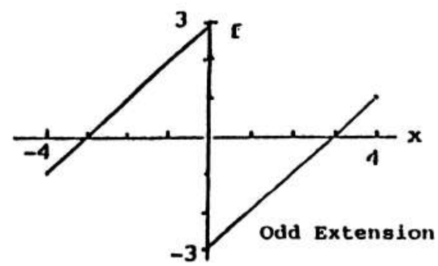


Odd Extension

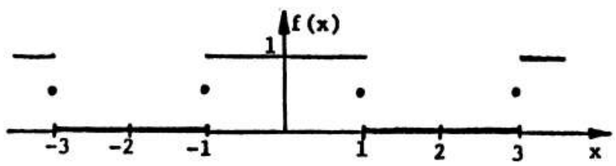
13. By the hint $f(-x) = g(-x) + h(-x) = g(x) - h(x)$, since g is an even function and f is an odd function. Thus $f(x) + f(-x) = 2g(x)$ and hence $g(x) = [f(x) + f(-x)]/2$ defines $g(x)$. Likewise $f(x)-f(-x) = g(x)-g(-x) + h(x)-h(-x) = 2h(x)$ and thus $h(x) = [f(x) - f(-x)]/2$.

All functions and their derivatives in Problems 14 through 30 are piecewise continuous on the given intervals and their extensions. Thus the Fourier Theorem applies in all cases.

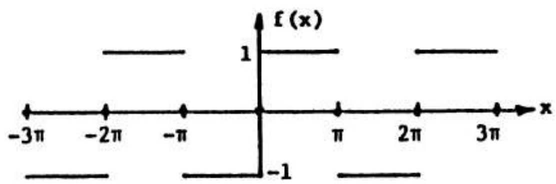
14. For the cosine series we use the even extension of the function given in Eq.(13) and hence
$f(x) = \begin{cases} 0 & -2 \le x < -1 \\ 1+x & -1 \le x < 0 \end{cases}$ on the interval $-2 \le x < 0$.
However, we don't really need this, as the coefficients in this case are given by Eqs.(7), which just use the original values for $f(x)$ on $0 < x \le 2$. Applying Eqs.(7) we have $L = 2$ and thus
$a_0 = (2/2)\int_0^1 (1-x)dx + (2/2)\int_1^2 0dx = 1/2$. Similarly,

$a_n = (2/2)\int_0^1 (1-x)\cos(n\pi x/2)dx = 4[1-\cos(n\pi/2)]/n^2\pi^2$ and
$b_n = 0$. Substituting these values in the Fourier series yields the desired results.
For the sine series, we use Eqs.(8) with $L = 2$. Thus
$a_n = 0$ and
$b_n = (2/2)\int_0^1 (1-x)\sin(n\pi x/2)dx = 4[n\pi/2 - \sin(n\pi/2)]/n^2\pi^2$.

15. The graph of the function to which the series converges is shown in the figure. Using Eqs.(7) with $L = 2$ we have $a_0 = \int_0^1 dx = 1$ and $a_n = \int_0^1 \cos(n\pi x/2)dx = 2\sin(n\pi/2)/n\pi$. Thus $a_n = 0$ for n even, $a_n = 2/n\pi$ for $n = 1,5,9,...$and $a_n = -2/n\pi$ for $n = 3,7,11,...$. Hence we may write $a_{2n} = 0$ and $a_{2n-1} = 2(-1)^{n+1}/(2n-1)\pi$, which when substituted into the series gives the desired answer.

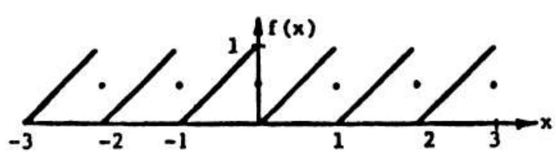


18. The graph of the function to which the series converges is indicated in the figure.



Since we want a sine series, we use Eqs.(8) to find, with $L = \pi$, that $b_n = (2/\pi)\int_0^\pi \sin nx dx = 2[1-(-1)^n]/n\pi$ and thus $b_n = 0$ for n even and $b_n = 4/n\pi$ for n odd.

20. The graph of the function to which the series converges is shown in the figure.



We note that f(x) is specified over its entire fundamental period ($T = 1$) and hence we cannot extend f to make it either an odd or an even function. Using Eqs.(2) and (3) from Section 10.3 we have ($L = 1/2$) $a_0 = 2\int_0^1 x dx = 1$, $a_n = 2\int_0^1 x\cos(2n\pi x)dx = 0$ and

$b_n = 2\int_0^1 x\sin(2n\pi x)dx = -1/n\pi$. [Note: We have used the results of Problem 27 of Section 10.2 in writing these integrals. That is, if f(x) is periodic of period T, then every integral of f over an interval of length T has the same value. Thus we integrate from 0 to 1 here, rather than $-1/2$ to $1/2$.] Substituting the above values into Eq.(1) of Section 10.3 yields the desired solution. It can also be observed from the above graph that $g(x) = f(x) - 1/2$ is an odd function. If Eqs.(8) are used with g(x), then it is found that

$g(x) = (-1/\pi)\sum_{n=1}^{\infty}\sin(2n\pi x)/n$ and thus we obtain the same series for f(x) as found above.

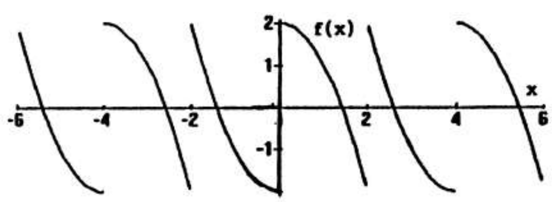
25a. $$b_n = \frac{2}{2}\int_0^2 (2-x^2)\sin\frac{n\pi x}{2}dx$$

$$= \left[\frac{-2}{n\pi}(2-x^2)\cos\frac{n\pi x}{2} - \frac{8x}{n^2\pi^2}\sin\frac{n\pi x}{2} - \frac{16}{n^3\pi^3}\cos\frac{n\pi x}{2}\right]\Big|_0^2$$

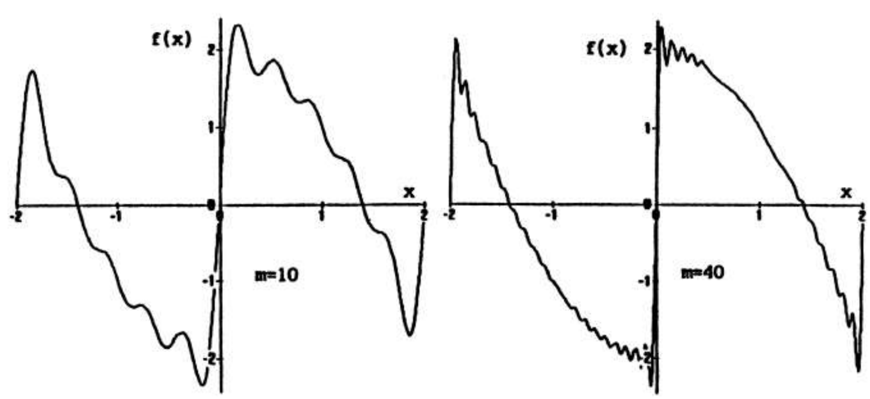
$$= \frac{4}{n\pi}(1+\cos n\pi) + \frac{16}{n^3\pi^3}(1-\cos n\pi)$$ and thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{4n^2\pi^2(1+\cos n\pi) + 16(1-\cos n\pi)}{n^3\pi^3}\right)\sin\frac{n\pi x}{2}$$

25b.



25c.



28b. For the cosine series(even extension) we have

$$a_0 = \frac{2}{2}\int_0^1 x dx = \frac{1}{2}$$

$$a_n = \frac{2}{2}\int_0^1 x\cos\frac{n\pi x}{2}dx = \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]\Big|_0^1$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2},\ \text{so}$$

$$g(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4\cos(n\pi/2) + 2n\pi\sin(n\pi/2) - 4}{n^2\pi^2}\cos\frac{n\pi x}{2}.$$

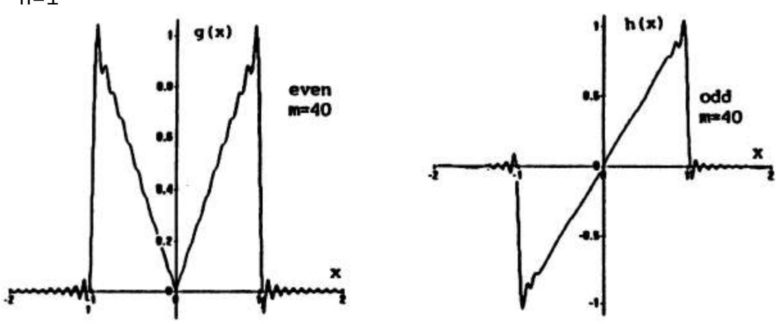
For the sine series (odd extension) we have

$$b_n = \frac{2}{2}\int_0^1 x\sin\frac{n\pi x}{2}dx = \left[\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Big|_0^1$$

$$= \frac{-2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2},\ \text{so}$$

$$h(x) = \sum_{n=1}^{\infty}\frac{4\sin(n\pi/2) - 2n\pi\cos(n\pi/2)}{n^2\pi^2}\sin\frac{n\pi x}{2}.$$

28c.



28d. The maximum error does not approach zero in either case, due to Gibb's phenomenon. Note that the coefficients in both series behave like $1/n$ as $n \to \infty$ since there is an n in the numerator.

31. We have $\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_0^L f(x)dx$. Now, if we let $x = -y$ in the first integral on the right, then $\int_{-L}^{0} f(x)dx = \int_L^0 f(-y)(-dy) = \int_0^L f(-y)dy = -\int_0^L f(y)dy$. Thus $\int_{-L}^{L} f(x)dx = -\int_0^L f(y)dy + \int_0^L f(x)dx = 0$.

32. To prove property 2 let f_1 and f_2 be odd functions and let $f(x) = f_1(x) \pm f_2(x)$. Then $f(-x) = f_1(-x) \pm f_2(-x) = -f_1(x) \pm [-f_2(x)] = -f_1(x) \mp f_2(x) = -f(x)$, so $f(x)$ is odd. Now let $g(x)= f_1(x)f_2(x)$, then

$$g(-x) = f_1(-x)f_2(-x) = [-f_1(x)][-f_2(x)] = f_1(x)f_2(x) = g(x)$$

and thus $g(x)$ is even. Finally, let $h(x) = f_1(x)/f_2(x)$ and hence $h(-x) = f_1(-x)/f_2(-x) = [-f_1(x)]/[-f_2(x)] = f_1(x)/f_2(x) = h(x)$, which says $h(x)$ is also even. Property 3 is proven in a similar manner.

34. Since $F(x) = \int_0^x f(t)dt$ we have
$F(-x) = \int_0^{-x} f(t)dt = -\int_0^x f(-s)ds$ by letting $t = -s$. If f is an even function, $f(-s) = f(s)$, we then have
$F(-x) = -\int_0^x f(s)ds = -F(x)$ from the original definition of F. Thus F(x) is an odd function. The argument is similar if f is odd.

35. Set $x = L/2$ in Eq.(6) of Section 10.3. Since we know f is continuous at L/2, we may conclude, by the Fourier theorem, that the series will converge to $f(L/2) = L$ at this point. Thus we have
$L = L/2 + (2L/\pi)\sum_{n=1}^{\infty}(-1)^{n+1}/(2n-1)$, since
$\sin[(2n-1)\pi/2] = (-1)^{n+1}$. Dividing by L and simplifying yields $\dfrac{\pi}{4} = \sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)} = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}$.

37a. Multiplying both sides of the equation by f(x) and integrating from 0 to L gives

$$\int_0^L [f(x)]^2dx = \int_0^L [f(x)\sum_{n=1}^{\infty} b_n\sin(n\pi x/L)]dx$$

$$= \sum_{n=1}^{\infty} b_n\int_0^L f(x)\sin(n\pi x/L)dx = (L/2)\sum_{n=1}^{\infty} b_n^2, \text{ by Eq.(8).}$$

This result is identical to that of Problem 17 of Section 10.3 if we set $a_n = 0$, $n = 0,1,2,\ldots$, since
$\dfrac{1}{L}\int_{-L}^L [f(x)]^2dx = \dfrac{2}{L}\int_0^L [f(x)]^2dx$. In a similar manner, it can be shown that

$$(2/L)\int_0^L [f(x)]^2dx = a_0^2/2 + \sum_{n=1}^{\infty} a_n^2.$$

37b. Since $f(x) = x$ and $b_n = 2L(-1)^{n+1}/n\pi$ (from Eq.(9)), we obtain

$$(2/L)\int_0^L [f(x)]^2dx = (2/L)\int_0^L x^2dx = 2L^2/3 = \sum_{n=1}^{\infty} b_n^2 =$$

$$\sum_{n=1}^{\infty} 4L^2/n^2\pi^2 = 4L^2/\pi^2\sum_{n=1}^{\infty}(1/n^2) \text{ or } \pi^2/6 = \sum_{n=1}^{\infty}(1/n^2).$$

38. We assume that the extensions of f and f' are piecewise continuous on [-2L,2L]. Since f is an odd periodic function of fundamental period 4L it follows from properties 2 and 3 that $f(x)\cos(n\pi x/2L)$ is odd and $f(x)\sin(n\pi x/2L)$ is even. Thus the Fourier coefficients of f are given by Eqs.(8) with L replaced by 2L; that is $a_n = 0$, $n = 0,1,2,\ldots$ and $b_n = (2/2L)\int_0^{2L} f(x)\sin(n\pi x/2L)dx$, $n = 1,2,\ldots$. The Fourier sine series for f is $f(x) = \sum_{n=1}^{\infty} b_n\sin(n\pi x/2L)$.

39. From Problem 38 we have $b_n = (1/L)\int_0^{2L} f(x)\sin(n\pi x/2L)dx$

$$= (1/L)\int_0^{L} f(x)\sin(n\pi x/2L)dx + (1/L)\int_L^{2L} f(2L-x)\sin(n\pi x/2L)dx$$

$$= (1/L)\int_0^{L} f(x)\sin(n\pi x/2L)dx - (1/L)\int_L^{0} f(s)\sin[n\pi(2L-s)/2L]ds$$

$$= (1/L)\int_0^{L} f(x)\sin(n\pi x/2L)dx - (1/L)\int_0^{L} f(s)\cos(n\pi)\sin(n\pi s/2L)ds$$

and thus $b_n = 0$ for n even and $b_n = (2/L)\int_0^{L} f(x)\sin(n\pi x/2L)dx$ for n odd. The Fourier series for f is given in Problem 38, where the b_n are given above.

Section 10.5, Page 579

3. We seek solutions of the form $u(x,t) = X(x)T(t)$. Substituting into the P.D.E. yields $X''T + X'T' + XT' = X''T + (X' + X)T' = 0$. Formally dividing by the quantity $(X' + X)T$ gives the equation $X''/(X' + X) = -T'/T$ in which the variables are separated. In order for this equation to be valid on the domain of u it is necessary that both sides be equal to the same constant λ. Hence $X''/(X' + X) = -T'/T = \lambda$ or equivalently, $X'' - \lambda(X' + X) = 0$ and $T' + \lambda T = 0$.

5. We seek solutions of the form $u(x,y) = X(x)Y(y)$. Substituting into the P.D.E. yields $X''Y + (x+y)XY'' = X''Y + xXY'' + yXY'' = 0$. Formally dividing by XY yields $X''/X + xY''/Y + yY''/Y = 0$. From this equation we see that the presence of the independent variable x multiplying the term u_{yy} in the original equation leads to the term xY''/Y when we attempt to separate the variables. It follows that the argument for a separation constant does not carry through and we cannot replace the P.D.E. by two O.D.E.

8. Following the procedures of Eqs.(5) through (8), we set $u(x,y) = X(x)T(t)$ in the P.D.E. to obtain $X''T = 4XT'$, or $X''X = 4T'/T$, which must be a constant. As stated in the text this separation constant must be $-\lambda^2$ (we choose $-\lambda^2$ so that when a square root is used later, the symbols are simpler) and thus $X'' + \lambda^2 X = 0$ and $T' + (\lambda^2/4)T = 0$. Now $u(0,t) = X(0)T(t) = 0$, for all $t > 0$, yields $X(0) = 0$, as discussed after Eq.(11) and similarly $u(2,t) = X(2)T(t) = 0$, for all $t > 0$, implies $X(2) = 0$. The D.E. for X has the solution $X(x) = C_1\cos\lambda x + C_2\sin\lambda x$ and $X(0) = 0$ yields $C_1 = 0$. Setting $x = 2$ in the remaining form of X yields $X(2) = C_2\sin 2\lambda = 0$, which has the solutions $2\lambda = n\pi$ or $\lambda = n\pi/2$, $n = 1,2,...$. Note that we exclude $n = 0$ since then $\lambda = 0$ would yield $X(x) = 0$, which is unacceptable. Hence $X(x) = \sin(n\pi x/2)$, $n = 1,2,...$. Finally, the solution of the D.E. for T yields $T(t) = \exp(-\lambda^2 t/4) = \exp(-n^2\pi^2 t/16)$. Thus we have found $u_n(x,t) = \exp(-n^2\pi^2 t/16)\sin(n\pi x/2)$. Setting $t = 0$ in this last expression indicates that $u_n(x,0)$ has, for the correct choices of n, the same form as the terms in $u(x,0)$, the initial condition. Using the principle of superposition we know that $u(x,t) = c_1u_1(x,t) + c_2u_2(x,t) + c_4u_4(x,t)$ satisfies the P.D.E. and the B.C. and hence we let $t = 0$ to obtain $u(x,0) = c_1u_1(x,0) + c_2u_2(x,0) + c_4u_4(x,0) = c_1\sin\pi x/2 + c_2\sin\pi x + c_4\sin 2\pi x$. If we choose $c_1 = 2$, $c_2 = -1$ and $c_4 = 4$ then $u(x,0)$ here will match the given initial condition, and hence substituting these values in $u(x,t)$ above then gives the desired solution.

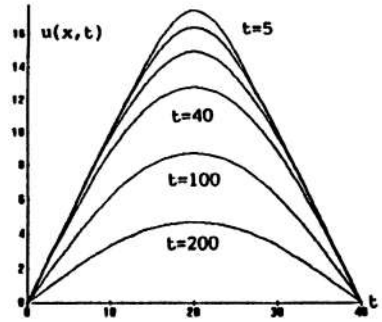
10. Since the B.C. for this heat conduction problem are $u(0,t) = u(40,t) = 0$, $t > 0$, the solution $u(x,t)$ is given by Eq.(19) with $\alpha^2 = 1\ \text{cm}^2/\text{sec}$, $L = 40$ cm, and the coefficients c_n determined by Eq.(21) with the I.C. $u(x,0) = f(x) = x$, $0 \le x \le 20$; $= 40 - x$, $20 \le x \le 40$.

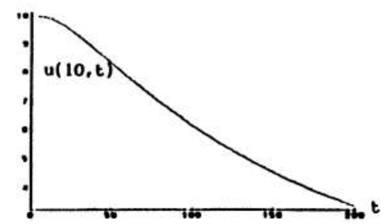
$$\text{Thus } c_n = \frac{1}{20}\left[\int_0^{20} x\sin\frac{n\pi x}{40}dx + \int_{20}^{40}(40-x)\sin\frac{n\pi x}{40}dx\right]$$

$$= \frac{160}{n^2\pi^2}\sin\frac{n\pi}{2}. \text{ It follows that}$$

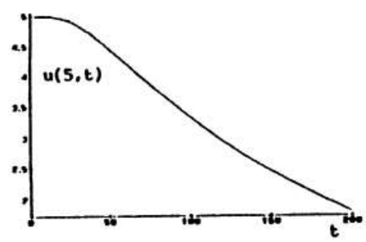
$$u(x,t) = \frac{160}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}e^{-n^2\pi^2 t/1600}\sin\frac{n\pi x}{40}.$$

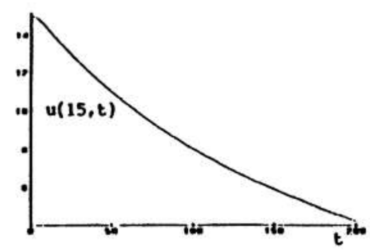
15a.



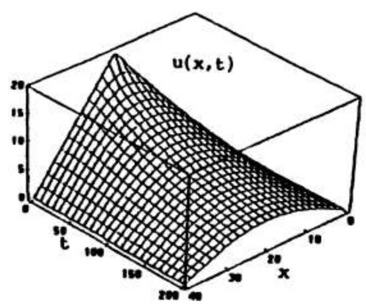


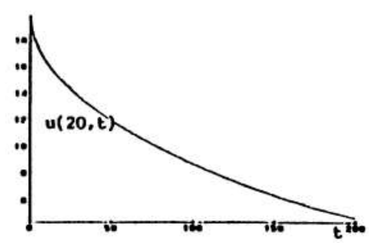
15b.





15c.





15d. As in Example 1, the maximum temperature will be at the midpoint, $x = 20$, and we use just the first term, since the others will be negligible for this temperature, since t is so large. Thus

$u(20,t) = 1 = \dfrac{160}{\pi^2}\sin(\pi/2)e^{-\pi^2 t/1600}\sin(20\pi/40)$. Solving for t, we obtain $e^{-\pi^2 t/1600} = \pi^2/160$, or

$t = \dfrac{1600}{\pi^2}\ln\dfrac{160}{\pi^2} = 451.60$ sec.

18a. Since the B.C. for this heat conduction problem are $u(0,t) = u(20,t) = 0$, $t > 0$, the solution $u(x,t)$ is given by Eq.(19) with $L = 20$ cm, and the coefficients c_n determined by Eq.(21) with the I.C. $u(x,0) = f(x) = 100^{\circ}C$. Thus $c_n = (1/10)\int_0^{20}(100)\sin(n\pi x/20)dx = -200[(-1)^n-1]/n\pi$ and hence

$c_{2n} = 0$ and $c_{2n-1} = 400/(2n-1)\pi$. Substituting these values into Eq.(19) yields

$$u(x,t) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\pi^2\alpha^2 t/400}}{2n-1}\sin\frac{(2n-1)\pi x}{20}$$

18b. For aluminum, we have $\alpha^2 = .86\ cm^2/sec$ (from Table 10.5.1) and thus the first two terms give

$$u(10,30) = \frac{400}{\pi}\{e^{-\pi^2(.86)30/400} - \frac{1}{3}e^{-9\pi^2(.86)30/400}\}$$

$= 67.228^oC$. If an additional term is used, the temperature is increased by

$$\frac{80}{\pi}e^{-25\pi^2(.86)30/400} = 3\times 10^{-6} \text{ degrees C.}$$

19b. Using only one term in the series for $u(x,t)$, we must solve the equation $5 = (400/\pi)\exp[-\pi^2(.86)t/400]$ for t. Taking the logarithm of both sides and solving for t yields $t \cong 400\ln(80/\pi)/\pi^2(.86) = 152.56$ sec.

20. Applying the chain rule to partial differentiation of u with respect to x we see that $u_x = u_\xi\xi_x = u_\xi(l/L)$ and $u_{xx} = u_{\xi\xi}(l/L)^2$. Substituting $u_{\xi\xi}/L^2$ for u_{xx} in the heat equation gives $\alpha^2 u_{\xi\xi}/L^2 = u_t$ or $u_{\xi\xi} = (L^2/\alpha^2)u_t$. In a similar manner, $u_t = u_\tau\tau_t = u_\tau(\alpha^2/L^2)$ and hence $\dfrac{L^2}{\alpha^2}u_t = u_\tau$ and thus $u_{\xi\xi} = u_\tau$.

22. Substituting $u(x,y,t) = X(x)Y(y)T(t)$ in the P.D.E. yields $\alpha^2(X''YT + XY''T) = XYT'$, which is equivalent to $\dfrac{X''}{X} + \dfrac{Y''}{Y} = \dfrac{T'}{\alpha^2 T}$. By keeping the independent variables x and y fixed and varying t we see that T'/α^2T must equal some constant σ_1 since the left side of the equation is fixed. Hence, $X''/X + Y''/Y = T'/\alpha^2T = \sigma_1$, or $X''/X = \sigma_1 - Y''/Y$ and $T' - \sigma_1\alpha^2T = 0$. By keeping x fixed and varying y in the equation involving X and Y we see that $\sigma_1 - Y''/Y$ must equal some constant σ_2 since the left side of the equation is fixed. Hence, $X''/X = \sigma_1 - Y''/Y = \sigma_2$ so $X'' - \sigma_2X = 0$ and $Y'' -(\sigma_1 - \sigma_2)Y = 0$. For $T' - \sigma_1\alpha^2T = 0$ to have solutions that remain bounded as $t \to \infty$ we must have $\sigma_1 < 0$. Thus, setting $\sigma_1 = -\lambda^2$, we have $T' + \alpha^2\lambda^2T = 0$. For $X'' - \sigma_2X = 0$ and homogeneous B.C., we conclude, as in Sect. 10.1, that $\sigma_2 < 0$ and, if we let

$\sigma_2 = -\mu^2$, then $X'' + \mu^2 X = 0$. With these choices for σ_1 and σ_2 we then have $Y'' + (\lambda^2-\mu^2)Y = 0$.

Section 10.6, Page 588

3. The steady-state temperature distribution $v(x)$ must satisfy Eq.(9) and also satsify the B.C. $v_x(0) = 0$, $v(L) = 0$. The general solution of $v'' = 0$ is $v(x) = Ax + B$. The B.C. $v_x(0) = 0$ implies $A = 0$ and then $v(L) = 0$ implies $B = 0$, so the steady state solution is $v(x) = 0$.

7. Again, $v(x)$ must satisfy $v'' = 0$, $v'(0) - v(0) = 0$ and $v(L) = T$. The general solution of $v'' = 0$ is $v(x) = ax + b$, so $v(0) = b$, $v'(0) = a$ and $v(L) = T$. Thus $a - b = 0$ and $aL + b = T$, which give $a = b = T/(1+L)$. Hence $v(x) = T(x+1)/(L+1)$.

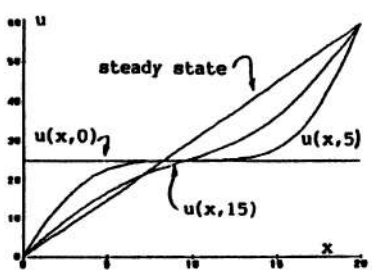
9a. Since the B.C. are not homogeneous, we must first find the steady state solution. Using Eqs.(9) and (10) we have $v'' = 0$ with $v(0) = 0$ and $v(20) = 60$, which has the solution $v(x) = 3x$. Thus the transient solution $w(x,t)$ satisfies the equations $\alpha^2 w_{xx} = w_t$, $w(0,t) = 0$, $w(20,t) = 0$ and $w(x,0) = 25 - 3x$, which are obtained from Eqs.(13) to (15). The solution of this problem is given by Eq.(4) with the c_n given by Eq.(6):

$$c_n = \frac{1}{10}\int_0^{20}(25-3x)\sin\frac{n\pi x}{20}dx = (70\cos n\pi + 50)/n\pi, \text{ and thus}$$

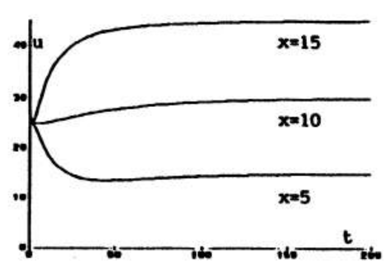
$$u(x,t) = 3x + \sum_{n=1}^{\infty}\frac{70\cos n\pi + 50}{n\pi}e^{-0.86n^2\pi^2 t/400}\sin\frac{n\pi x}{20} \text{ since}$$

$\alpha^2 = .86$ for aluminum.

9b.



9c.



9d. Using just the first term of the sum, we have

$$u(5,t) = 15 - \frac{20}{\pi}e^{-0.86\pi^2 t/400}\sin\frac{\pi}{4} = 15 \pm .15. \text{ Thus}$$

$\frac{20}{\pi}e^{-0.86\pi^2 t/400}\sin\frac{\pi}{4} = .15$, which yields t = 160.30 sec.

To obtain the answer in the text, the first two terms of the sum must be used, which requires an equation solver to solve for t. Note that this reduces t by only .01 seconds.

12a. Since the B.C. are $u_x(0,t) = u_x(L,t) = 0$, $t > 0$, the solution $u(x,t)$ is given by Eq.(35) with the coefficients c_n determined by Eq.(37). Substituting the I.C. $u(x,0) = f(x) = \sin(\pi x/L)$ into Eq.(37) yields

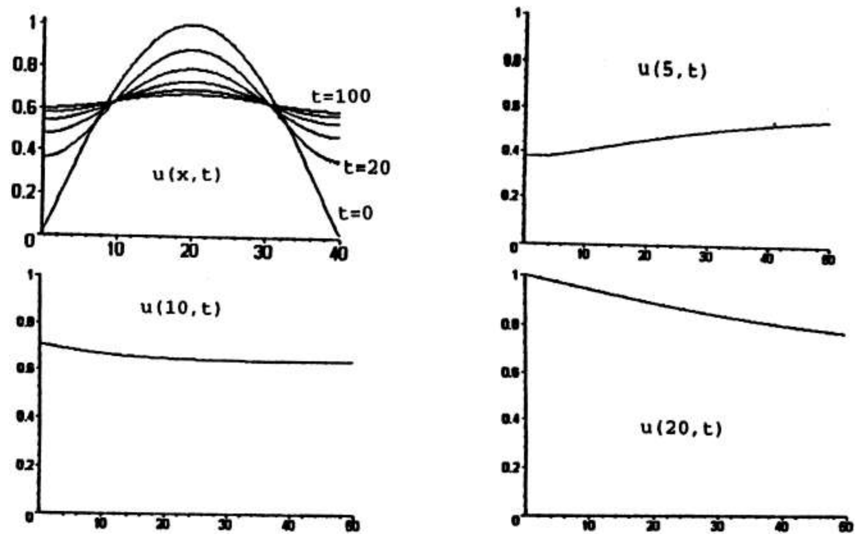
$$c_0 = (2/L)\int_0^L \sin(\pi x/L)dx = 4/\pi \text{ and}$$

$$\begin{aligned} c_n &= (2/L)\int_0^L \sin(\pi x/L)\cos(n\pi x/L)dx \\ &= (1/L)\int_0^L \{\sin[(n+1)\pi x/L] - \sin[(n-1)\pi x/L]\}dx \\ &= (1/\pi)\{[1 - \cos(n+1)\pi]/(n+1) - [1 - \cos(n-1)\pi]/(n-1)\} \\ &= 0, \ n \text{ odd}; = -4/(n^2-1)\pi, \ n \text{ even}. \text{ Thus} \end{aligned}$$

$$u(x,t) = 2/\pi-(4/\pi)\sum_{n=1}^{\infty}\exp[-4n^2\pi^2\alpha^2 t/L^2]\cos(2n\pi x/L)/(4n^2-1)$$

where we are now summing over even terms by setting $n = 2n$.

12b. As $t \to \infty$ we see that all terms in the series decay to zero except the constant term, $2/\pi$. Hence $\lim_{t\to\infty} u(x,t) = 2/\pi$.

12c. 

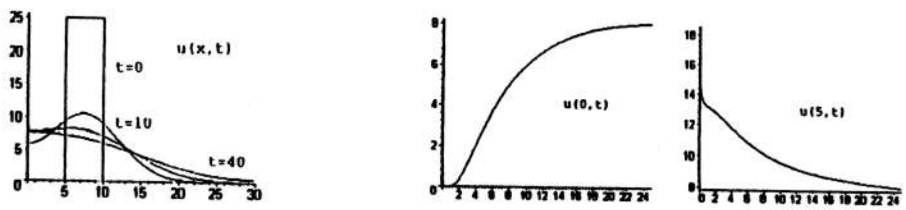
12d. The original heat in the rod is redistributed to give the final temperature distribution, since no heat is lost.

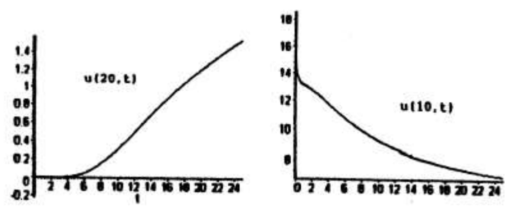
14a. Since the ends are insulated, the solution to this problem is given by Eq.(35), with $\alpha^2 = 1$ and $L = 30$, and Eq.(37). Thus $u(x,t) = \dfrac{c_0}{2} + \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2 t/900)\cos(n\pi x/30)$, where

$$c_0 = \frac{2}{30}\int_0^{30} f(x)dx = \frac{1}{15}\int_5^{10} 25dx = \frac{25}{3} \text{ and}$$

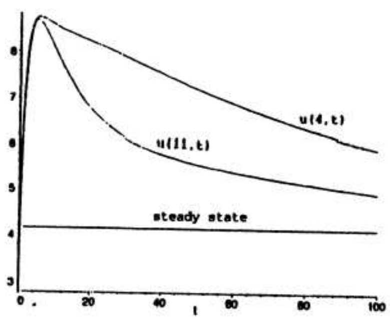
$$c_n = \frac{2}{30}\int_0^{3} f(x)\cos\frac{n\pi x}{30}dx = \frac{1}{15}\int_5^{10} 25\cos\frac{n\pi x}{30}dx = \frac{50}{n\pi}\left[\sin\frac{n\pi}{3} - \sin\frac{n\pi}{6}\right].$$

14b.





14c.



Although $x = 4$ and $x = 1$ are symmetrical to the initial temperature pulse, they are not symmetrical to the insulated end points.

15a. Substituting $u(x,t) = X(x)T(t)$ into Eq.(1) leads to the two O.D.E. $X'' - \sigma X = 0$ and $T' - \alpha^2\sigma T = 0$. An argument similar to the one in the text implies that we must have $X(0) = 0$ and $X'(L) = 0$. Also, by assuming σ is real and considering the three cases $\sigma < 0$, $\sigma = 0$, and $\sigma > 0$ we can show that only the case $\sigma < 0$ leads to nontrivial solutions of $X'' - \sigma X = 0$ with $X(0) = 0$ and $X'(L) = 0$. Setting $\sigma = -\lambda^2$, we obtain $X(x) = k_1\sin\lambda x + k_2\cos\lambda x$. Now, $X(0) = 0 \rightarrow k_2 = 0$ and thus $X(x) = k_1\sin\lambda x$.

Differentiating and setting $x = L$ yields $\lambda k_1\cos\lambda L = 0$. Since $\lambda = 0$ and $k_1 = 0$ lead to $u(x,t) = 0$, we must choose λ so that $\cos\lambda L = 0$, or $\lambda = (2n-1)\pi/2L$, $n = 1,2,3,\ldots$. These values for λ imply that $\sigma = -(2n-1)^2\pi^2/4L^2$ so the solutions $T(t)$ of $T' - \alpha^2\sigma T = 0$ are proportional to $\exp[-(2n-1)^2\pi^2\alpha^2t/4L^2]$. Combining the above results leads to the desired set of fundamental solutions.

15b. In order to satisfy the I.C. $u(x,0) = f(x)$ we assume that $u(x,t)$ has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n\exp[-(2n-1)^2\pi^2\alpha^2t/4L^2]\sin[(2n-1)\pi x/2L].$$

The coefficients c_n are determined by the requirement that

$$u(x,0) = \sum_{n=1}^{\infty} c_n\sin[(2n-1)\pi x/2L] = f(x).$$

Referring to Problem 39 of Section 10.4 reveals that such a representation for $f(x)$ is possible if we choose the coefficients $c_n = (2/L)\int_0^L f(x)\sin[(2n-1)\pi x/2L]dx$.

19. We must solve $v_1''(x) = 0$, $0 \le x \le a$ and $v_2''(x) = 0$, $a \le x \le L$ subject to the B.C. $v_1(0) = 0$, $v_2(L) = T$ and the continuity conditions at $x = a$. For the temperature to be continuous at $x = a$ we must have $v_1(a) = v_2(a)$ and for the rate of heat flow to be continuous we must have $-\kappa_1A_1v_1'(a) = -\kappa_2A_2v_2'(a)$, from Eq.(2) of Appendix A. The general solutions to the two O.D.E. are $v_1(x) = C_1x + D_1$ and $v_2(x) = C_2x + D_2$. By applying the boundary and continuity conditions we may solve for C_1, D_1, and C_2 and D_2 to obtain the desired solution.

Section 10.7, Page 600

1a. Since the initial velocity is zero, the solution is given by Eq.(20) with the coefficients c_n given by Eq.(22). Substituting $f(x)$ into Eq.(22) yields

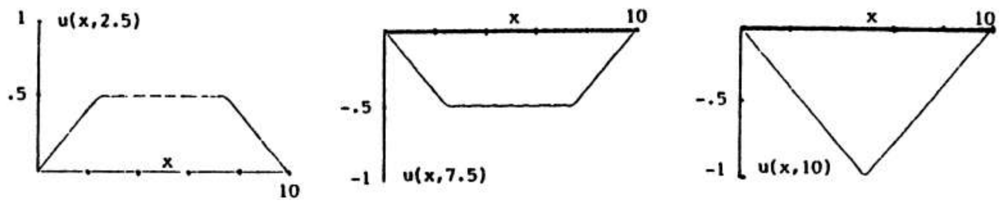
$$c_n = \frac{2}{L}\left[\int_0^{L/2}\frac{2x}{L}\sin\frac{n\pi x}{L}dx + \int_{L/2}^{L}\frac{2(L-x)}{L}\sin\frac{n\pi x}{L}dx\right]$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

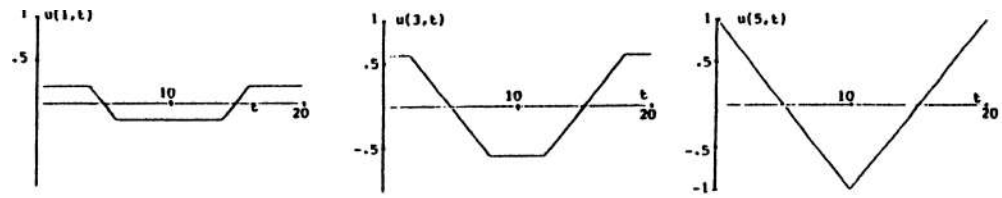
Thus Eq. (20) becomes

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}.$$

1b.



1c.



1e. The graphs in part b can best be understood using Eq.(28) (or the results of Problems 13 and 14). The original triangular shape is composed of two similar triangles of 1/2 the height, one of which moves to the right, $h(x-at)$, and the other to the left, $h(x+at)$. Recalling that the series are periodic then gives the results shown. The graphs in part c can then be visualized from those in part b.

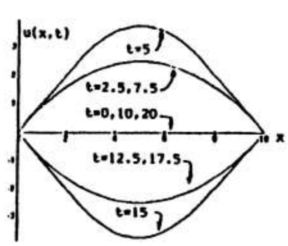
6a. The motion is governed by Eqs.(1), (3) and (31), and thus the solution is given by Eq.(34) where the k_n are given by Eq.(36):

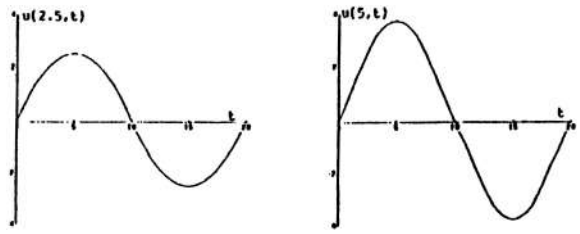
$$k_n = \frac{2}{n\pi a}\left[\int_0^{L/4}\frac{4x}{L}\sin\frac{n\pi x}{L}dx + \int_{L/4}^{3L/4}\sin\frac{n\pi x}{L}dx + \int_{3L}^{L}\frac{4(L-x)}{L}\sin\frac{n\pi x}{L}dx\right]$$

$$= \frac{8L}{n^3\pi^3 a}\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right).$$ Substituting this in Eq.(34)

yields $$u(x,t) = \frac{8L}{a\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}}{n^3}\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}.$$

6b.



6c. 

9. Assuming that $u(x,t) = X(x)T(t)$ and substituting for u in Eq.(1) leads to the pair of O.D.E. $X'' + \sigma X = 0$, $T'' + a^2\sigma T = 0$. Applying the B.C. $u(0,t) = 0$ and $u_x(L,t) = 0$ to $u(x,t)$ we see that we must have $X(0) = 0$ and $X'(L) = 0$. By considering the three cases $\sigma < 0$, $\sigma = 0$, and $\sigma > 0$ it can be shown that nontrivial solutions of the problem $X'' + \sigma X = 0$, $X(0) = 0$, $X'(L) = 0$ are possible if and only if $\sigma = (2n-1)^2\pi^2/4L^2$, $n = 1,2,\ldots$ and the corresponding solutions for $X(x)$ are proportional to $\sin[(2n-1)\pi x/2L]$. Using these values for σ we find that $T(t)$ is a linear combination of $\sin[(2n-1)\pi at/2L]$ and $\cos[(2n-1)\pi at/2L]$. Now, the I.C. $u_t(x,0)$ implies that $T'(0) = 0$ and thus functions of the form $u_n(x,t) = \sin[(2n-1)\pi x/2L]\cos[(2n-1)\pi at/2L]$, $n = 1,2,\ldots$ satsify the P.D.E. (1), the B.C. $u(0,t) = 0$, $u_x(L,t) = 0$, and the I.C. $u_t(x,0) = 0$. We now seek a superposition of these fundamental solutions u_n that also satisfies the I.C. $u(x,0) = f(x)$. Thus we assume that

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin[(2n-1)\pi x/2L]\cos[(2n-1)\pi at/2L].$$

The I.C. now implies that we must have

$$f(x) = \sum_{n=1}^{\infty} c_n \sin[(2n-1)\pi x/2L].$$

From Problem 39 of Section 10.4 we see that $f(x)$ can be represented by such a series and that

$$c_n = (2/L)\int_0^L f(x)\sin[(2n-1)\pi x/2L]dx, \quad n = 1,2,\ldots .$$

Substituting these values into the above series for $u(x,t)$ yields the desired solution.

10a. From Problem 9 we have

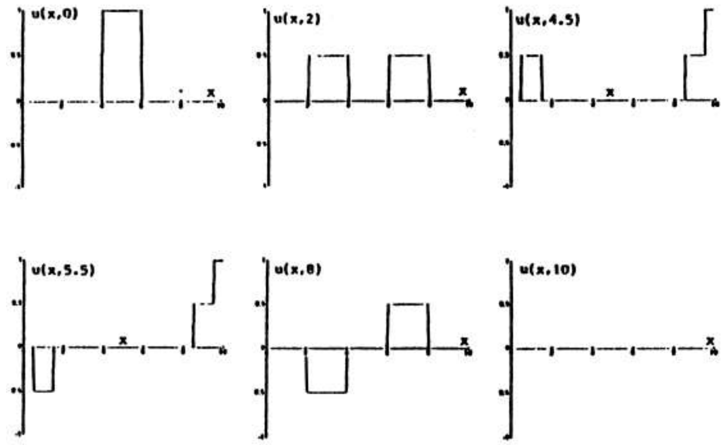
$$c_n = \frac{2}{L}\int_{(L-2)/2}^{(L+2)/2} \sin\frac{(2n-1)\pi x}{2L}dx$$

$$= \frac{-4}{(2n-1)\pi}\left[\cos\left(\frac{(2n-1)\pi(L+2)}{4L}\right) - \cos\left(\frac{(2n-1)\pi(L-2)}{4L}\right)\right]$$

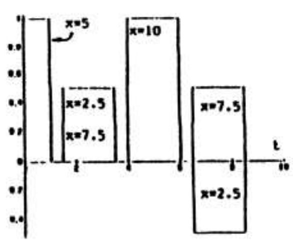
$$= \frac{8}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{4}\sin\frac{(2n-1)\pi}{2L}$$ using the

trigonometric relations for $\cos(A \pm B)$. Substituting this value of c_n into $u(x,t)$ in Problem 9 yields the desired solution.

10b.

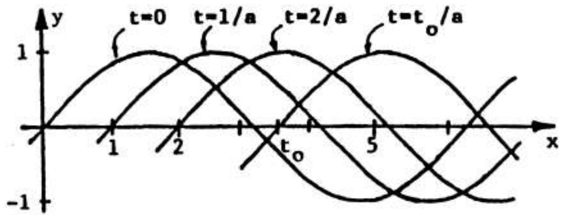


10c.



13. Using the chain rule we obtain $u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$ since $\xi_x = \eta_x = 1$. Differentiating a second time gives $u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$. In a similar way we obtain $u_t = u_\xi \xi_t + u_\eta \eta_t = -au_\xi + au_\eta$, since $\xi_t = -a$, $\eta_t = a$. Thus $u_{tt} = a^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$. Substituting for u_{xx} and u_{tt} in the wave equation, we obtain $u_{\xi\eta} = 0$. Integrating both sides of $u_{\xi\eta} = 0$ with respect to η yields $u_\xi(\xi,\eta) = \gamma(\xi)$ where γ is an arbitrary function of ξ. Integrating both sides of $u_\xi(\xi,\eta) = \gamma(\xi)$ with respect to ξ yields $u(\xi,\eta) = \int\gamma(\xi)d\xi + \psi(\eta) = \phi(\xi) + \psi(\eta)$ where $\psi(\eta)$ is an arbitrary function of η and $\int\gamma(\xi)d\xi$ is some function of ξ denoted by $\phi(\xi)$. Thus $u(x,t) = u(\xi(x,t),\eta(x,t)) = \phi(x - at) + \psi(x + at)$.

14. The graph of $y = \sin(x-at)$ for the various values of t is indicated in the figure on the next page. Note that the graph of $y = \sin x$ is displaced to the right by the distance "at" for each value of t.



Similarly, the graph of $y = \phi(x + at)$ would be displaced to the left by a distance "at" for each t. Thus $\phi(x + at)$ represents a wave moving to the left.

16. Write the equation as $a^2u_{xx} = u_{tt} + \alpha^2 u$ and assume $u(x,t) = X(x)T(t)$. This gives $a^2X''T = XT'' + \alpha^2 XT$, or $\frac{X''}{X} = \frac{1}{a^2}(\frac{T''}{T} + \alpha^2) = \sigma$. The separation constant σ is $-\lambda^2$ using the same arguments as in the text and earlier problems. Thus $X'' + \lambda^2 X = 0$, $X(0) = 0$, $X(L) = 0$ and $T'' + (\alpha^2 + z^2\lambda^2)T = 0$, $T'(0) = 0$. If we let $\beta_n^2 = \lambda_n^2 a^2 + \alpha^2$, we then have $u_n(x,t) = \cos\beta_n t\sin\frac{n\pi x}{L}$, where $\lambda_n = \frac{n\pi x}{L}$.
Using superposition we obtain $u(x,t) = \sum_{n=1}^{\infty} c_n\cos\beta_n t\sin\frac{n\pi x}{L}$ and thus $u(x,0) = \sum_{n=1}^{\infty} c_n\sin\frac{n\pi x}{L} = f(x)$. Hence c_n are given by Eq. (22).

17a. We have $u(x,t) = \phi(x-at) + \psi(x+at)$ and thus $u_t(x,t) = -a\phi'(x-at) + a\chi'(x+at)$. Hence $u(x,0) = \phi(x) + \psi(x) = f(x)$ and $u_t(x,0) = -a\phi'(x) + a\psi'(x) = 0$. Dividing the last equation by a yields the desired result.

17b. Using the hint and the first equation obtained in part (a) leads to $\phi(x) + \psi(x) = 2\phi(x) + c = f(x)$ so $\phi(x) = (1/2)f(x) - c/2$ and $\psi(x) = (1/2)f(x) + c/2$. Hence $u(x,t) = \phi(x - at) + \psi(x + at) = (1/2)[f(x - at) - c] + (1/2)[f(x + at) + c] = (1/2)[f(x - at) + f(x + at)]$.

17c. Substituting x + at for x in f(x) yields

$$f(x + at) = \begin{cases} 2 & -1 < x + at < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Subtracting "at" from both sides of the inequality then yields

$$f(x + at) = \begin{cases} 2 & -1 - at < x < 1 - at \\ 0 & \text{otherwise} \end{cases}.$$

18a. As in Problem 17a, we have $u(x,0) = \phi(x) + \psi(x) = 0$ and $u_t(x,0) = -a\phi'(x) + a\psi'(x) = g(x)$.

18b. From part (a) we have $\psi(x) = -\phi(x)$ which yields $-2a\phi(x) = g(x)$ from the second equation in part a. Integration then yields $\phi(x) - \phi(x_0) = \dfrac{-1}{2a}\int_{x_0}^{x} g(\xi)d\xi$ and hence
$\psi(x) = (1/2a)\int_{x_0}^{x} g(\xi)d\xi - \phi(x_0)$.

18c. $u(x,t) = \phi(x-at) + \psi(x+at)$

$$= -(1/2a)\int_{x_0}^{x-at} g(\xi)d\xi + \phi(x_0) + (1/2a)\int_{x_0}^{x+at} g(\xi)d\xi - \phi(x_0)$$

$$= (1/2a)[\int_{x_0}^{x+at} g(\xi)d\xi - \int_{x_0}^{x-at} g(\xi)d\xi]$$

$$= (1/2a)[\int_{x_0}^{x+at} g(\xi)d\xi + \int_{x-at}^{x_0} g(\xi)d\xi]$$

$$= (1/2a\int_{x-at}^{x+at} g(\xi)d\xi.$$

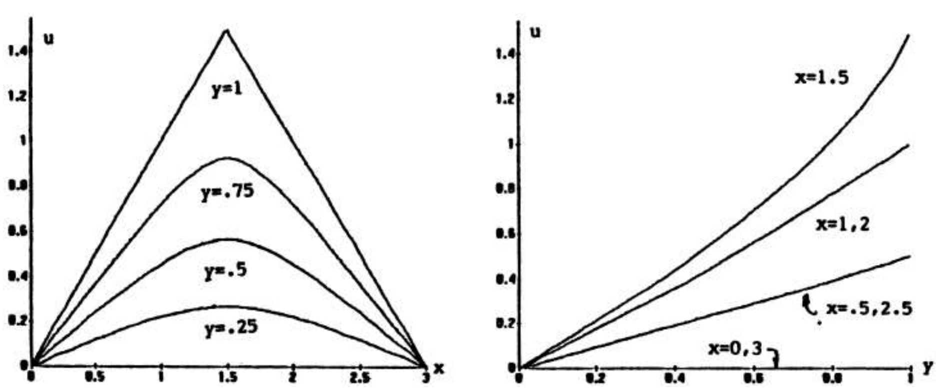
24. Substituting $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ into the P.D.E. yields $R''\Theta T + R'\Theta T/r + R\Theta''T/r^2 = R\Theta T''/a^2$ or equivalently $R''/R + R'/rR + \Theta''/\Theta r^2 = T''/a^2T$. In order for this equation to be valid for $0 < r < r_0$, $0 \le \theta \le 2\pi$, $t > 0$, it is necessary that both sides of the equation be equal to the same constant $-\sigma$. Otherwise, by keeping r and θ fixed and varying t, one side would remain unchanged while the other side varied. Thus we arrive at the two equations $T'' + \sigma a^2T = 0$ and $r^2R''/R + rR'/R + \sigma r^2 = -\Theta''/\Theta$. By an argument similar to the one above we conclude that both sides of the last equation must be equal to the same constant δ. This leads to the two equations $r^2R'' + rR' + (\sigma r^2 - \delta)R = 0$ and $\Theta'' + \delta\Theta = 0$. Since the circular membrane is continuous, we must have $\Theta(2\pi) = \Theta(0)$, which requires $\delta = \mu^2$, μ a non-negative integer. The condition $\Theta(2\pi) = \Theta(0)$ is also known as the periodicity condition. Since we also desire solutions which vary periodically in time, it is clear that the separation constant σ should be positive, $\sigma = \lambda^2$. Thus we arrive at the three equations $r^2R'' + rR' + (\lambda^2r^2 - \mu^2)R = 0$, $\Theta'' + \mu^2\Theta = 0$, and $T'' + \lambda^2a^2T = 0$.

Section 10.8, Page 611

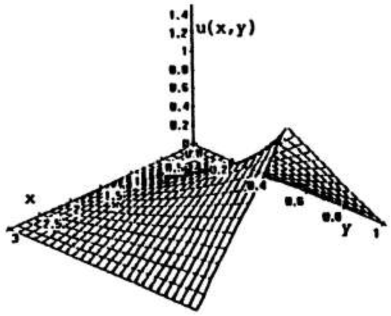
1a. Assuming that $u(x,y) = X(x)Y(y)$ leads to the two O.D.E. $X'' - \sigma X = 0$, $Y'' + \sigma Y = 0$. The B.C. $u(0,y) = 0$, $u(a,y) = 0$ imply that $X(0) = 0$ and $X(a) = 0$. Thus nontrivial solutions to $X'' - \sigma X = 0$ which satisfy these boundary conditions are possible only if $\sigma = -(n\pi/a)^2$, $n = 1,2...$; the corresponding solutions for $X(x)$ are proportional to $\sin(n\pi x/a)$. The B.C. $u(x,0) = 0$ implies that $Y(0) = 0$. Solving $Y'' - (n\pi/a)^2Y = 0$ subject to this condition we find that $Y(y)$ must be proportional to $\sinh(n\pi y/a)$. The fundamental solutions are then $u_n(x,y) = \sin(n\pi x/a)\sinh(n\pi y/a)$, $n = 1,2,...$, which satisfy Laplace's equation and the homogeneous B.C. We assume that $u(x,y) = \sum_{n=1}^{\infty} c_n\sin(n\pi x/a)\sinh(n\pi y/a)$, where the coefficients c_n are determined from the B.C. $u(x,b) = g(x) = \sum_{n=1}^{\infty} c_n\sin(n\pi x/a)\sinh(n\pi b/a)$. It follows that
$c_n\sinh(n\pi b/a) = (2/a)\int_0^a g(x)\sin(n\pi x/a)dx$, $n = 1,2,...$.

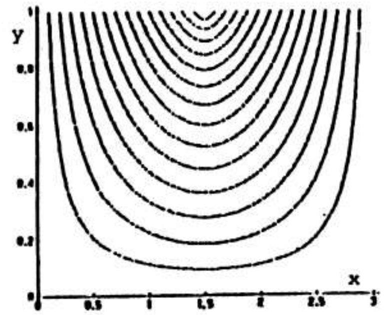
1b. Substituting for $g(x)$ in the equation for c_n we have $c_n\sinh(n\pi b/a) = (2/a)[\int_0^{a/2} x\sin(n\pi x/a)dx + \int_{a/2}^a (a-x)\sin(n\pi x/a)dx] = [4a \sin(n\pi/2)]/n^2\pi^2$, $n = 1,2,...,$ so $c_n = [4a \sin(n\pi/2)]/[n^2\pi^2\sinh(n\pi b/a)]$. Substituting these values for c_n in the above series yields the desired solution.

1c.



1d.





2. In solving the D.E. $Y'' - \lambda^2 Y = 0$, one normally writes $Y(y) = c_1\sinh\lambda y + c_2\cosh\lambda y$. However, since we have $Y(b) = 0$, it is advantageous to rewrite Y as $Y(y) = d_1\sinh\lambda(b-y) + d_2\cosh\lambda(b-y)$, where d_1, d_2 are also arbitrary constants and can be related to c_1, c_2 using the appropriate hyperbolic trigonometric identities. The important thing, however, is that the second form also satisfies the D.E. and thus $Y(y) = d_1\sinh\lambda(b-y)$ satisfies the D.E. and the homogeneous B.C. $Y(b) = 0$. The rest of the problem follows the pattern of Problem 1.

3a. Let $u(x,y) = v(x,y) + w(x,y)$, where u, v and w all satisfy Laplace's Eq., $v(x,y)$ satisfies the conditions in Eq. (4) and $w(x,y)$ satisfies the conditions of Problem 2.

4. Following the pattern of Problem 3, one could consider adding the solutions of four problems, each with only one non-homogeneous B.C. It is also possible to consider adding the solutions of only two problems, each with only two non-homogeneous B.C., as long as they involve the same variable. For instance, one such problem would be $u_{xx} + u_{yy} = 0$, $u(x,0) = 0$, $u(x,b) = 0$, $u(0,y) = k(y)$, $u(a,y) = f(y)$, which has the fundamental solutions $u_n(x,y) = [c_n\sinh(n\pi x/b) + d_n\cosh(n\pi x/b)]\sin(n\pi y/b)$.

Assuming $u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$ and using the B.C.

$u(0,y) = k(y)$ we obtain $d_n = (2/b)\int_0^b k(y)\sin(n\pi y/b)dy$.

Using the B.C. $u(a,y) = f(y)$ we obtain

$c_n\sinh(n\pi a/b) + d_n\cosh(n\pi a/b) = (2/b)\int_0^b f(y)\sin(n\pi y/b)dy$, which can be solved for c_n, since d_n is already known. The second problem, in this approach, would be $u_{xx} + u_{yy} = 0$, $u(x,0) = h(x)$, $u(x,b) = g(x)$, $u(0,y) = 0$ and $u(a,y) = 0$. This has the fundamental solutions

$u_n(x,y) = [a_n\sinh(n\pi y/a) + b_n\cosh(n\pi y/a)]\sin(n\pi x/a$, so that
$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$. Thus $u(x,0) = h(x)$ gives
$b_n = (2/a)\int_0^a h(x)\sin(n\pi x/a)dx$ and $u(x,b) = g(x)$ gives
$a_n\sinh(n\pi b/a) + b_n\cosh(n\pi b/a) = (2/a)\int_0^a g(x)\sin(n\pi x/a)dx$, which can be solved for a_n since b_n is known.

5. Using Eq.(20) and following the same arguments as presented in the text, we find that $R(r) = k_1r^n + k_2r^{-n}$ and $\Theta(\theta) = c_1\sin n\theta + c_2\cos n\theta$, for n a positive integer, and $u_0(r,\theta) = 1$ for $n = 0$. Since we require that $u(r,\theta)$ be bounded as $r \to \infty$, we conclude that $k_1 = 0$. The fundamental solutions are therefore
$u_n(r,\theta) = r^{-n}\cos n\theta$, $v_n(r,\theta) = r^{-n}\sin n\theta$, $n = 1,2,...$ together with $u_0(r,\theta) = 1$. Assuming that u can be expressed as a linear combination of the fundamental solutions we have
$u(r,\theta) = c_0/2 + \sum_{n=1}^{\infty} r^{-n}(c_n\cos n\theta + k_n\sin n\theta)$. The B.C. requires that
$u(a,\theta) = c_0/2 + \sum_{n=1}^{\infty} a^{-n}(c_n\cos n\theta + k_n\sin n\theta) = f(\theta)$ for $0 \le \theta < 2\pi$. This is precisely the Fourier series representation for $f(\theta)$ of period 2π and thus
$a^{-n}c_n = (1/\pi)\int_0^2 f(\theta)\cos n\theta d\theta$, $n = 0,1,2,...$ and

$a^{-n}k_n = (1/\pi)\int_0^2 f(\theta)\sin n\theta d\theta$, $n = 1,2...$.

7. Again we let $u(r,\theta) = R(r)\Theta(\theta)$ and thus we have $r^2R'' + rR' - \sigma R = 0$ and $\Theta'' + \sigma\Theta = 0$, with $R(0)$ bounded and the B.C. $\Theta(0) = \Theta(\alpha) = 0$. For $\sigma \le 0$ we find that $\Theta(\theta) \equiv 0$, so we let $\sigma = \lambda^2$ (λ^2 real) and thus $\Theta(\theta) = c_1\cos\lambda\theta + c_2\sin\lambda\theta$. The B.C. $\Theta(0) = 0 \to c_1 = 0$ and the B.C. $\Theta(\alpha) = 0 \to \lambda = n\pi/\alpha$, $n = 1,2,...$. Substituting these values into Eq.(31) we obtain $R(r) = k_1r^{n\pi/\alpha} + k_2r^{-n\pi/\alpha}$. However $k_2 = 0$ since $R(0)$ must be bounded, and thus the fundamental solutions are $u_n(r,\theta) = r^{n\pi/\alpha}\sin(n\pi\theta/\alpha)$. The desired solution may now be formed using previously discussed procedures.

8a. Separating variables, as before, we obtain $X'' + \lambda^2 X = 0$, $X(0) = 0$, $X(a) = 0$ and $Y'' - \lambda^2 Y = 0$, $Y(y)$ bounded as $y \to \infty$. Thus $X(x) = \sin(n\pi x/a)$, and $\lambda^2 = (n\pi/a)^2$. However, since neither sinhy nor coshy are bounded as $y \to \infty$, we must write the solution to $Y'' - (n\pi/a)^2 Y = 0$ as $Y(y) = c_1\exp[n\pi y/a] + c_2\exp[-n\pi y/a]$. Thus we must choose $c_1 = 0$ so that $u(x,y) = X(x)Y(y) \to 0$ as $y \to \infty$. The fundamental solutions are then $u_n(x,t) = e^{-n\pi y/a}\sin(n\pi x/a)$.

$u(x,y) = \sum_{n=1}^{\infty} c_n u_n(x,y)$ then gives

$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/a) = f(x)$ so that $c_n = \frac{2}{a}\int_0^1 f(x)\sin\frac{n\pi x}{a}dx$.

8b. $c_n = \frac{2}{a}\int_0^a x(a-x)\sin\frac{n\pi x}{a}dx = \frac{4a^2}{n^3\pi^3}(1-\cos n\pi)$

8c. Using just the first term and letting $a = 5$, we have $u(x,y) = \frac{200}{\pi^3}e^{-\pi y/5}\sin\frac{\pi x}{5}$, which, for a fixed y, has a maximum at $x = 5/2$ and thus we need to find y such that $u(5/2,y) = \frac{200}{\pi^3}e^{-\pi y/5} = .1$. Taking the logarithm of both sides and solving for y yields $y_0 = 6.6315$. With an equation solver, more terms can be used. However, to four decimal places, three terms yield the same result as above.

13a. Assuming that $u(x,y) = X(x)Y(y)$ and substituting into Eq.(1) leads to the two O.D.E. $X'' - \sigma X = 0$, $Y'' + \sigma Y = 0$. The B.C. $u(x,0) = 0$, $u_y(x,b) = 0$ imply that $Y(0) = 0$ and $Y'(b) = 0$. For nontrivial solutions to exist for $Y'' + \sigma Y = 0$ with these B.C. we find that σ must take the values $(2n-1)^2\pi^2/4b^2$, $n = 1,2,...$; the corresponding solutions for $Y(y)$ are proportional to $\sin[(2n-1)\pi y/2b]$. Solutions to $X'' - [(2n-1)^2\pi^2/4b^2]X = 0$ are of the form $X(x) = A\sinh[(2n-1)\pi x/2b] + B\cosh[(2n-1)\pi x/2b]$. However, the boundary condition $u(0,y) = 0$ implies that $X(0) = B = 0$. It follows that the fundamental solutions are $u_n(x,y) = c_n\sinh[(2n-1)\pi x/2b]\sin[(2n-1)\pi y/2b]$, $n = 1,2,...$. To satisfy the remaining B.C. at $x = a$ we assume that we can represent the solution $u(x,y)$ in the form $u(x,y) = \sum_{n=1}^{\infty} c_n\sinh[(2n-1)\pi x/2b]\sin[(2n-1)\pi y/2b]$.
The coefficients c_n are determined by the B.C.

$$u(a,y) = \sum_{n=1}^{\infty} c_n \sinh[(2n-1)\pi a/2b]\sin[(2n-1)\pi y/2b] = f(y).$$

By properly extending f as a periodic function of period 4b as in Problem 39, Section 10.4, we find that the coefficients c_n are given by

$$c_n \sinh[(2n-1)\pi a/2b] = (2/b)\int_0^b f(y)\sin[(2n-1)\pi y/2b]dy,$$

$n = 1,2,\ldots$.

Capítulo

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