CHAPTER

3

Second Order Linear Equations

3.1

1. Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r^2\,e^{rt}$. Direct substitution into the differential equation yields $(r^2 + 2r - 3)e^{rt} = 0$. Canceling the exponential, the characteristic equation is $r^2 + 2r - 3 = 0$. The roots of the equation are $r = -3\,, 1$. Hence the general solution is $y = c_1 e^t + c_2 e^{-3t}$.

2. Let $y = e^{rt}$. Substitution of the assumed solution results in the characteristic equation $r^2 + 3r + 2 = 0$. The roots of the equation are $r = -2\,, -1$. Hence the general solution is $y = c_1 e^{-t} + c_2 e^{-2t}$.

4. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $2r^2 - 3r + 1 = 0$. The roots of the equation are $r = 1/2\,, 1$. Hence the general solution is $y = c_1 e^{t/2} + c_2 e^t$.

6. The characteristic equation is $4r^2 - 9 = 0$, with roots $r = \pm 3/2$. Therefore the general solution is $y = c_1 e^{-3t/2} + c_2 e^{3t/2}$.

8. The characteristic equation is $r^2 - 2r - 2 = 0$, with roots $r = 1 \pm \sqrt{3}$. Hence the general solution is $y = c_1 e^{(1-\sqrt{3})t} + c_2 e^{(1+\sqrt{3})t}$.

9. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + r - 2 = 0$. The roots of the equation are $r = -2\,, 1$. Hence the general solution is $y = c_1 e^{-2t} + c_2 e^t$. Its derivative is $y' = -2c_1 e^{-2t} + c_2 e^t$. Based on the

first condition, $y(0) = 1$, we require that $c_1 + c_2 = 1$. In order to satisfy $y'(0) = 1$, we find that $-2c_1 + c_2 = 1$. Solving for the constants, $c_1 = 0$ and $c_2 = 1$. Hence the specific solution is $y(t) = e^t$.

11. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $6r^2 - 5r + 1 = 0$. The roots of the equation are $r = 1/3, 1/2$. Hence the general solution is $y = c_1 e^{t/3} + c_2 e^{t/2}$. Its derivative is $y' = c_1 e^{t/3}/3 + c_2 e^{t/2}/2$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 4$. In order to satisfy the condition $y'(0) = 1$, we find that $c_1/3 + c_2/2 = 0$. Solving for the constants, $c_1 = 12$ and $c_2 = -8$. Hence the specific solution is $y(t) = 12\, e^{t/3} - 8\, e^{t/2}$.

12. The characteristic equation is $r^2 + 3r = 0$, with roots $r = -3, 0$. Therefore the general solution is $y = c_1 + c_2 e^{-3t}$, with derivative $y' = -3\, c_2 e^{-3t}$. In order to satisfy the initial conditions, we find that $c_1 + c_2 = -2$, and $-3\, c_2 = 3$. Hence the specific solution is $y(t) = -1 - e^{-3t}$.

13. The characteristic equation is $r^2 + 5r + 3 = 0$, with roots

$$r_{1,2} = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}\,.$$

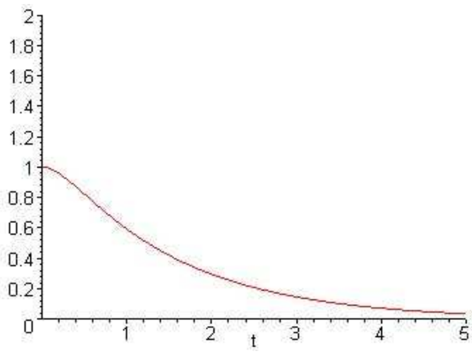
The general solution is $y = c_1 e^{(-5-\sqrt{13})t/2} + c_2 e^{(-5+\sqrt{13})t/2}$, with derivative

$$y' = \frac{-5-\sqrt{13}}{2}\, c_1 e^{(-5-\sqrt{13})t/2} + \frac{-5+\sqrt{13}}{2}\, c_2 e^{(-5+\sqrt{13})t/2}\,.$$

In order to satisfy the initial conditions, we require that

$$c_1 + c_2 = 1 \quad \text{and} \quad \frac{-5-\sqrt{13}}{2}\, c_1 + \frac{-5+\sqrt{13}}{2}\, c_2 = 0.$$

Solving for the coefficients, $c_1 = (1 - 5/\sqrt{13})/2$ and $c_2 = (1 + 5/\sqrt{13})/2$.



14. The characteristic equation is $2r^2 + r - 4 = 0$, with roots

$$r_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{33}}{4}\,.$$

The general solution is $y = c_1e^{(-1-\sqrt{33})t/4} + c_2e^{(-1+\sqrt{33})t/4}$, with derivative

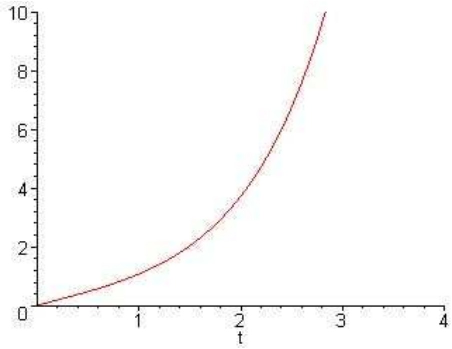
$$y' = \frac{-1-\sqrt{33}}{4}\,c_1e^{(-1-\sqrt{33})t/4} + \frac{-1+\sqrt{33}}{4}\,c_2e^{(-1+\sqrt{33})t/4}\,.$$

In order to satisfy the initial conditions, we require that

$$c_1 + c_2 = 0 \quad \text{and} \quad \frac{-1-\sqrt{33}}{4}\,c_1 + \frac{-1+\sqrt{33}}{4}\,c_2 = 1.$$

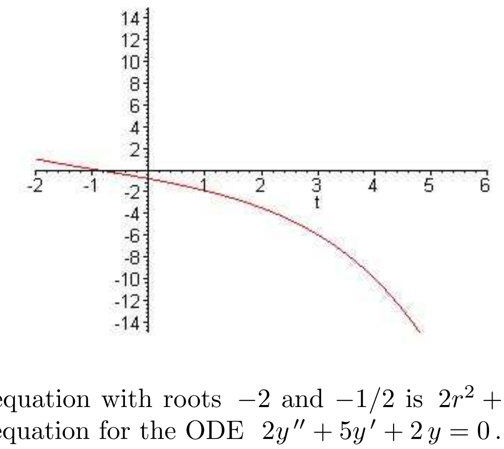
Solving for the coefficients, $c_1 = -2/\sqrt{33}$ and $c_2 = 2/\sqrt{33}$. The specific solution is

$$y(t) = -2\left[e^{(-1-\sqrt{33})t/4} - e^{(-1+\sqrt{33})t/4}\right]/\sqrt{33}\,.$$



16. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Therefore the general solution is $y = c_1e^{-t/2} + c_2e^{t/2}$. Since the initial conditions are specified at $t = -2$, is more convenient to write $y = d_1e^{-(t+2)/2} + d_2e^{(t+2)/2}$. The derivative is given by $y' = -\left[d_1e^{-(t+2)/2}\right]/2 + \left[d_2e^{(t+2)/2}\right]/2$. In order to satisfy the initial conditions, we find that $d_1 + d_2 = 1$, and $-d_1/2 + d_2/2 = -1$. Solving for the coefficients, $d_1 = 3/2$, and $d_2 = -1/2$. The specific solution is

$$y(t) = \frac{3}{2}e^{-(t+2)/2} - \frac{1}{2}e^{(t+2)/2} = \frac{3}{2e}e^{-t/2} - \frac{e}{2}e^{t/2}\,.$$



18. An algebraic equation with roots -2 and $-1/2$ is $2r^2 + 5r + 2 = 0$. This is the characteristic equation for the ODE $2y'' + 5y' + 2\,y = 0$.

20. The characteristic equation is $2r^2 - 3r + 1 = 0$, with roots $r = 1/2,\ 1$. Therefore the general solution is $y = c_1 e^{t/2} + c_2 e^t$, with derivative $y' = c_1 e^{t/2}/2 + c_2 e^t$. To satisfy the initial conditions, we require that $c_1 + c_2 = 2$ and $c_1/2 + c_2 = 1/2$. Solving for the coefficients, $c_1 = 3$ and $c_2 = -1$. This means that the specific solution is $y(t) = 3e^{t/2} - e^t$. To find the stationary point, set $y' = 3e^{t/2}/2 - e^t = 0$. There is a unique solution, with $t_1 = \ln(9/4)$. This implies that the maximum value is then $y(t_1) = 9/4$. To find the x-intercept, solve the equation $3e^{t/2} - e^t = 0$. The solution is readily found to be $t_2 = \ln 9 \approx 2.1972$.

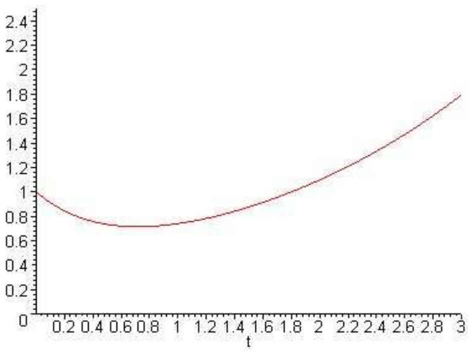
22. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Hence the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$ and $y' = -c_1 e^{-t/2}/2 + c_2 e^{t/2}/2$. Invoking the initial conditions, we require that $c_1 + c_2 = 2$ and $-c_1 + c_2 = 2\beta$. The specific solution is $y(t) = (1-\beta)e^{-t/2} + (1+\beta)e^{t/2}$. Based on the form of the solution, it is evident that as $t \to \infty$, $y(t) \to 0$ as long as $\beta = -1$.

23. The characteristic equation is $r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$. Examining the coefficients, the roots are $r = \alpha\ ,\ \alpha - 1$. Hence the general solution of the differential equation is $y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$. Assuming $\alpha \in \mathbb{R}$, all solutions will tend to zero as long as $\alpha < 0$. On the other hand, all solutions will become unbounded as long as $\alpha - 1 > 0$, that is, $\alpha > 1$.

25.(a) The characteristic equation is $2r^2 + 3r - 2 = 0$, with roots $r = 1/2$ and $r = -2$. The initial conditions give us the solution

$$y(t) = (2\beta + 1)e^{-2t}/5 + (4 - 2\beta)e^{t/2}/5.$$

(b) $y(t) = 2\,e^{t/2}/5 + 3\,e^{-2t}/5$.



The minimum occurs at $(t_0\,, y_0) \approx (0.7167\,, 0.7155)$. (The value of t_0 is $2\ln(6)/5$.)

(c) This happens when $\beta = 2$. (When the coefficient of the positive exponential power becomes negative.)

26.(a) The characteristic roots are $r = -3\,, -2$. The solution of the initial value problem is $y(t) = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$.

(b) The maximum point has coordinates $t_0 = \ln\left[\frac{3(4+\beta)}{2(6+\beta)}\right]$, $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2}$.

(c) $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2} \geq 4$, as long as $\beta \geq 6 + 6\sqrt{3}$.

(d) $\lim_{\beta\to\infty} t_0 = \ln(3/2)$, $\lim_{\beta\to\infty} y_0 = \infty$.

27.(a) Assuming that y is a constant, the ODE reduces to $c\,y = d$. Hence the only equilibrium solution is $y = d/c$.

(b) Setting $y = Y + d/c$, substitution into the differential equation results in

$$a\,Y'' + b\,Y' + c(Y + d/c) = d.$$

The equation satisfied by Y is

$$a\,Y'' + b\,Y' + cY = 0.$$

3.2

1.
$$W(e^{2t}, e^{-3t/2}) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{t/2}.$$

3.
$$W(e^{-2t}, t\,e^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}.$$

5.
$$W(e^t \sin t\,, e^t \cos t) = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t(\sin t + \cos t) & e^t(\cos t - \sin t) \end{vmatrix} = -e^{2t}.$$

6.
$$W(\cos^2\theta\,, 1 + \cos 2\theta) = \begin{vmatrix} \cos^2\theta & 1 + \cos 2\theta \\ -2\sin\theta\,\cos\theta & -2\sin 2\theta \end{vmatrix} = 0.$$

7. Write the equation as $y'' + (3/t)y' = 1$. $p(t) = 3/t$ is continuous for all $t > 0$. Since $t_0 > 0$, the IVP has a unique solution for all $t > 0$.

9. Write the equation as $y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$. The coefficients are not continuous at $t = 0$ and $t = 4$. Since $t_0 \in (0\,, 4)$, the largest interval is $0 < t < 4$.

10. The coefficient $3\ln|t|$ is discontinuous at $t = 0$. Since $t_0 > 0$, the largest interval of existence is $0 < t < \infty$.

11. Write the equation as $y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0$. The coefficients are discontinuous at $x = 0$ and $x = 3$. Since $x_0 \in (0\,, 3)$, the largest interval is $0 < x < 3$.

13. $y_1'' = 2$. We see that $t^2(2) - 2(t^2) = 0$. $y_2'' = 2\,t^{-3}$, with $t^2(y_2'') - 2(y_2) = 0$. Let $y_3 = c_1 t^2 + c_2 t^{-1}$, then $y_3'' = 2c_1 + 2c_2 t^{-3}$. It is evident that y_3 is also a solution.

16. No. Substituting $y = \sin(t^2)$ into the differential equation,

$$-4t^2 \sin(t^2) + 2\cos(t^2) + 2t\,\cos(t^2)p(t) + \sin(t^2)q(t) = 0\,.$$

At $t = 0$, this equation becomes $2 = 0$ (if we suppose that $p(t)$ and $q(t)$ are continuous), which is impossible.

17. $W(e^{2t}, g(t)) = e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$. Dividing both sides by e^{2t}, we find that g must satisfy the ODE $g' - 2g = 3e^{2t}$. Hence $g(t) = 3t\,e^{2t} + c\,e^{2t}$.

19. $W(f\,,g) = fg' - f'g$. Also, $W(u\,,v) = W(2f - g\,,f + 2g)$. Upon evaluation, $W(u\,,v) = 5fg' - 5f'g = 5W(f\,,g)$.

20. $W(f\,,g) = fg' - f'g = t\,\cos\,t - \sin\,t$, and $W(u\,,v) = -4fg' + 4f'g$. Hence $W(u\,,v) = -4t\,\cos\,t + 4\sin\,t$.

21. We compute

$$W(a_1y_1 + a_2y_2, b_1y_1 + b_2y_2) = \begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} =$$

$$= (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (b_1y_1 + b_2y_2)(a_1y_1' + a_2y_2') =$$

$$= a_1b_2(y_1y_2' - y_1'y_2) - a_2b_1(y_1y_2' - y_1'y_2) = (a_1b_2 - a_2b_1)W(y_1, y_2)$$

This now readily shows that y_3 and y_4 is a fundamental set of solutions if and only if $a_1b_2 - a_2b_1 \neq 0$.

23. The general solution is $y = c_1e^{-3t} + c_2e^{-t}$. $W(e^{-3t}, e^{-t}) = 2e^{-4t}$, and hence the exponentials form a fundamental set of solutions. On the other hand, the fundamental solutions must also satisfy the conditions $y_1(1) = 1$, $y_1'(1) = 0$; $y_2(1) = 0$, $y_2'(1) = 1$. For y_1, the initial conditions require $c_1 + c_2 = e$, $-3c_1 - c_2 = 0$. The coefficients are $c_1 = -e^3/2$, $c_2 = 3e/2$. For the solution y_2, the initial conditions require $c_1 + c_2 = 0$, $-3c_1 - c_2 = e$. The coefficients are $c_1 = -e^3/2$, $c_2 = e/2$. Hence the fundamental solutions are

$$y_1 = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)} \quad \text{and} \quad y_2 = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}.$$

24. Yes. $y_1'' = -4\,\cos\,2t$; $y_2'' = -4\,\sin\,2t$. $W(\cos\,2t\,,\sin\,2t) = 2$.

25. Clearly, $y_1 = e^t$ is a solution. $y_2' = (1 + t)e^t$, $y_2'' = (2 + t)e^t$. Substitution into the ODE results in $(2 + t)e^t - 2(1 + t)e^t + t\,e^t = 0$. Furthermore, $W(e^t, te^t) = e^{2t}$. Hence the solutions form a fundamental set of solutions.

27. Clearly, $y_1 = x$ is a solution. $y_2' = \cos\,x$, $y_2'' = -\sin\,x$. Substitution into the ODE results in $(1 - x\,\cot\,x)(-\sin\,x) - x(\cos\,x) + \sin\,x = 0$. We can compute that

$W(y_1, y_2) = x\cos x - \sin x$, which is nonzero for $0 < x < \pi$. Hence $\{x\,, \sin x\}$ is a fundamental set of solutions.

30. Writing the equation in standard form, we find that $P(t) = \sin t/\cos t$. Hence the Wronskian is $W(t) = c\,e^{-\int \frac{\sin t}{\cos t}dt} = c\,e^{\ln|\cos t|} = c\cos t$, in which c is some constant.

31. After writing the equation in standard form, we have $P(x) = 1/x$. The Wronskian is $W(x) = c\,e^{-\int \frac{1}{x}dx} = c\,e^{-\ln|x|} = c/x$, in which c is some constant.

32. Writing the equation in standard form, we find that $P(x) = -2x/(1-x^2)$. The Wronskian is $W(x) = c\,e^{-\int \frac{-2x}{1-x^2}dx} = c\,e^{-\ln\left|1-x^2\right|} = c/(1-x^2)$, in which c is some constant.

33. Rewrite the equation as $p(t)y'' + p'(t)y' + q(t)y = 0$. After writing the equation in standard form, we have $P(t) = p'(t)/p(t)$. Hence the Wronskian is

$$W(t) = c\,e^{-\int \frac{p'(t)}{p(t)}dt} = c\,e^{-\ln p(t)} = c/p(t)\,.$$

34. Multiply the equation by t and recognize that we can use the previous problem with $p(t) = t^2$. We identify $c = 2$ from $W(1) = 2$ and then $W(5) = 2/25$.

35. The Wronskian associated with the solutions of the differential equation is given by $W(t) = c\,e^{-\int \frac{-2}{t^2}dt} = c\,e^{-2/t}$. Since $W(2) = 3$, it follows that for the hypothesized set of solutions, $c = 3\,e$. Hence $W(4) = 3\sqrt{e}$.

36. For the given differential equation, the Wronskian satisfies the first order differential equation $W' + p(t)W = 0$. Given that W is constant, it is necessary that $p(t) \equiv 0$.

37. Direct calculation shows that

$$W(f\,g\,, f\,h) = (fg)(fh)' - (fg)'(fh) =$$
$$= (fg)(f'h + fh') - (f'g + fg')(fh) = f^2\,W(g\,, h)\,.$$

39. Since y_1 and y_2 are solutions, they are differentiable. The hypothesis can thus be restated as $y_1'(t_0) = y_2'(t_0) = 0$ at some point t_0 in the interval of definition. This implies that $W(y_1\,, y_2)(t_0) = 0$. But $W(y_1\,, y_2)(t_0) = c\,e^{-\int p(t)dt}$, which cannot be equal to zero, unless $c = 0$. Hence $W(y_1\,, y_2) \equiv 0$, which is ruled out for a fundamental set of solutions.

42. $P = 1$, $Q = x$, $R = 1$. We have $P'' - Q' + R = 0$. The equation is exact. Note that $(y')' + (xy)' = 0$. Hence $y' + xy = c_1$. This equation is linear, with integrating factor $\mu = e^{x^2/2}$. Therefore the general solution is

$$y(x) = c_1 e^{-x^2/2}\int_{x_0}^{x} e^{u^2/2}du + c_2 e^{-x^2/2}.$$

43. $P = 1$, $Q = 3x^2$, $R = x$. Note that $P'' - Q' + R = -5x$, and therefore the differential equation is not exact.

45. $P = x^2$, $Q = x$, $R = -1$. We have $P'' - Q' + R = 0$. The equation is exact. Write the equation as $(x^2 y')' - (xy)' = 0$. After integration, we conclude that $x^2 y' - xy = c$. Divide both sides of the ODE by x^2. The resulting equation is linear, with integrating factor $\mu = 1/x$. Hence $(y/x)' = c\,x^{-3}$. The solution is $y(t) = c_1 x^{-1} + c_2 x$.

47. $P = x^2$, $Q = x$, $R = x^2 - \nu^2$. Hence the coefficients are $2P' - Q = 3x$ and $P'' - Q' + R = x^2 + 1 - \nu^2$. The adjoint of the original differential equation is given by $x^2 \mu'' + 3x\,\mu' + (x^2 + 1 - \nu^2)\mu = 0$.

49. $P = 1$, $Q = 0$, $R = -x$. Hence the coefficients are given by $2P' - Q = 0$ and $P'' - Q' + R = -x$. Therefore the adjoint of the original equation is $\mu'' - x\,\mu = 0$.

3.3

2. $e^{2-3i} = e^2 e^{-3i} = e^2(\cos 3 - i \sin 3)$.

3. $e^{i\pi} = \cos \pi + i \sin \pi = -1$.

4. $e^{2-\frac{\pi}{2}i} = e^2(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}) = -e^2\, i$.

6. $\pi^{-1+2i} = e^{(-1+2i)\ln \pi} = e^{-\ln \pi} e^{2 \ln \pi\, i} = (\cos(2 \ln \pi) + i \sin(2 \ln \pi))/\pi$.

8. The characteristic equation is $r^2 - 2r + 6 = 0$, with roots $r = 1 \pm i\sqrt{5}$. Hence the general solution is $y = c_1 e^t \cos \sqrt{5}\,t + c_2\, e^t \sin \sqrt{5}\,t$.

9. The characteristic equation is $r^2 + 2r - 8 = 0$, with roots $r = -4, 2$. The roots are real and different, hence the general solution is $y = c_1 e^{-4t} + c_2\, e^{2t}$.

10. The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Hence the general solution is $y = c_1 e^{-t} \cos t + c_2\, e^{-t} \sin t$.

12. The characteristic equation is $4r^2 + 9 = 0$, with roots $r = \pm\frac{3}{2}\,i$. Hence the general solution is $y = c_1 \cos(3t/2) + c_2 \sin(3t/2)$.

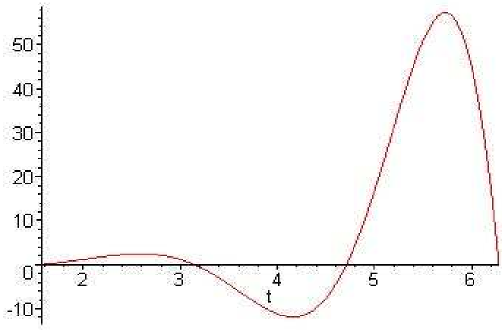
13. The characteristic equation is $r^2 + 2r + 1.25 = 0$, with roots $r = -1 \pm \frac{1}{2}i$. Hence the general solution is $y = c_1 e^{-t} \cos(t/2) + c_2\, e^{-t} \sin(t/2)$.

15. The characteristic equation is $r^2 + r + 1.25 = 0$, with roots $r = -\frac{1}{2} \pm i$. Hence the general solution is $y = c_1 e^{-t/2} \cos t + c_2\, e^{-t/2} \sin t$.

16. The characteristic equation is $r^2 + 4r + 6.25 = 0$, with roots $r = -2 \pm \frac{3}{2}\, i$. Hence the general solution is $y = c_1 e^{-2t} \cos(3t/2) + c_2\, e^{-2t} \sin(3t/2)$.

17. The characteristic equation is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence the general solution is $y = c_1 \cos 2t + c_2 \sin 2t$. Now $y' = -2c_1 \sin 2t + 2c_2 \cos 2t$. Based on the first condition, $y(0) = 0$, we require that $c_1 = 0$. In order to satisfy the condition $y'(0) = 1$, we find that $2c_2 = 1$. The constants are $c_1 = 0$ and $c_2 = 1/2$. Hence the specific solution is $y(t) = \frac{1}{2} \sin 2t$.

19. The characteristic equation is $r^2 - 2r + 5 = 0$, with roots $r = 1 \pm 2i$. Hence the general solution is $y = c_1 e^t \cos 2t + c_2\, e^t \sin 2t$. Based on the initial condition $y(\pi/2) = 0$, we require that $c_1 = 0$. It follows that $y = c_2\, e^t \sin 2t$, and so the first derivative is $y' = c_2\, e^t \sin 2t + 2c_2\, e^t \cos 2t$. In order to satisfy the condition $y'(\pi/2) = 2$, we find that $-2e^{\pi/2} c_2 = 2$. Hence we have $c_2 = -e^{-\pi/2}$. Therefore the specific solution is $y(t) = -e^{t-\pi/2} \sin 2t$.



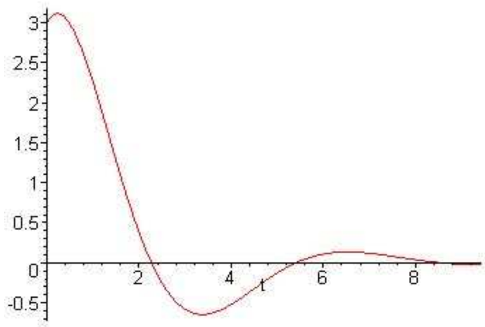
20. The characteristic equation is $r^2 + 1 = 0$, with roots $r = \pm i$. Hence the general solution is $y = c_1 \cos t + c_2 \sin t$. Its derivative is $y' = -c_1 \sin t + c_2 \cos t$. Based on the first condition, $y(\pi/3) = 2$, we require that $c_1 + \sqrt{3}\, c_2 = 4$. In order to satisfy the condition $y'(\pi/3) = -4$, we find that $-\sqrt{3}\, c_1 + c_2 = -8$. Solving these for the constants, $c_1 = 1 + 2\sqrt{3}$ and $c_2 = \sqrt{3} - 2$. Hence the specific solution is a steady oscillation, given by $y(t) = (1 + 2\sqrt{3}) \cos t + (\sqrt{3} - 2) \sin t$.

21. From Problem 15, the general solution is $y = c_1 e^{-t/2} \cos t + c_2\, e^{-t/2} \sin t$. Invoking the first initial condition, $y(0) = 3$, which implies that $c_1 = 3$. Substituting, it follows that $y = 3e^{-t/2} \cos t + c_2\, e^{-t/2} \sin t$, and so the first derivative is

$$y' = -\frac{3}{2} e^{-t/2} \cos t - 3e^{-t/2} \sin t + c_2\, e^{-t/2} \cos t - \frac{c_2}{2} e^{-t/2} \sin t\,.$$

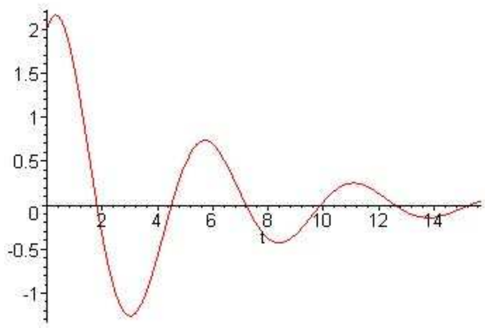
Invoking the initial condition, $y'(0) = 1$, we find that $-\frac{3}{2} + c_2 = 1$, and so $c_2 = \frac{5}{2}$.

Hence the specific solution is $y(t) = 3e^{-t/2}\cos t + \frac{5}{2}\, e^{-t/2}\sin t$.



24.(a) The characteristic equation is $5r^2 + 2r + 7 = 0$, with roots $r = -\frac{1}{5} \pm i\frac{\sqrt{34}}{5}$. The solution is $u = c_1 e^{-t/5}\cos\frac{\sqrt{34}}{5}t + c_2 e^{-t/5}\sin\frac{\sqrt{34}}{5}t$. Invoking the given initial conditions, we obtain the equations for the coefficients: $c_1 = 2$, $-2 + \sqrt{34}\, c_2 = 5$. That is, $c_1 = 2$, $c_2 = 7/\sqrt{34}$. Hence the specific solution is

$$u(t) = 2e^{-t/5}\cos\frac{\sqrt{34}}{5}t + \frac{7}{\sqrt{34}}e^{-t/5}\sin\frac{\sqrt{34}}{5}t\,.$$



(b) Based on the graph of $u(t)$, T is in the interval $14 < t < 16$. A numerical solution on that interval yields $T \approx 14.5115$.

26.(a) The characteristic equation is $r^2 + 2ar + (a^2+1) = 0$, with roots $r = -a \pm i$. Hence the general solution is $y(t) = c_1 e^{-at}\cos t + c_2 e^{-at}\sin t$. Based on the initial conditions, we find that $c_1 = 1$ and $c_2 = a$. Therefore the specific solution is given by

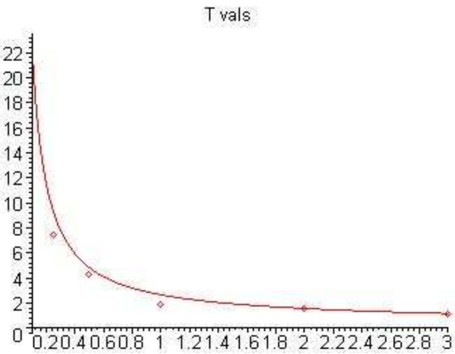
$$y(t) = e^{-at}\cos t + a\, e^{-at}\sin t = \sqrt{1+a^2}\; e^{-at}\cos(t-\phi),$$

in which $\phi = \arctan(a)$.

(b) For estimation, note that $|y(t)| \le \sqrt{1+a^2}\; e^{-at}$. Now consider the inequality $\sqrt{1+a^2}\; e^{-at} \le 1/10$. The inequality holds for $t \ge \frac{1}{a}\ln(10\sqrt{1+a^2})$. Therefore $T \le \frac{1}{a}\ln(10\sqrt{1+a^2})$. Setting $a = 1$, the numerical value is $T \approx 1.8763$.

(c) Similarly, $T_{1/4} \approx 7.4284$, $T_{1/2} \approx 4.3003$, $T_2 \approx 1.5116$, $T_3 \approx 1.1496$.

(d)



Note that the estimates T_a approach the graph of $\frac{1}{a}\ln(10\sqrt{1+a^2})$ as a gets large.

27. Direct calculation gives the result. On the other hand, it was shown in Problem 3.2.37 that $W(f\,g\,, f\,h) = f^2 W(g\,, h)$. Hence

$$W(e^{\lambda t}\cos\,\mu t\,, e^{\lambda t}\sin\,\mu t) = e^{2\lambda t}\,W(\cos\,\mu t\,, \sin\,\mu t) =$$

$$= e^{2\lambda t}\,[\cos\,\mu t(\sin\,\mu t)' - (\cos\,\mu t)'\sin\,\mu t] = \mu\, e^{2\lambda t}.$$

28.(a) Clearly, y_1 and y_2 are solutions. Also, $W(\cos\,t, \sin\,t) = \cos^2 t + \sin^2 t = 1$.

(b) $y' = i\,e^{it}$, $y'' = i^2\,e^{it} = -e^{it}$. Evidently, y is a solution and so $y = c_1 y_1 + c_2 y_2$.

(c) Setting $t = 0$, $1 = c_1\cos\,0 + c_2\sin\,0$, and $c_1 = 0$. Differentiating, $i\,e^{it} = c_2\cos\,t$. Setting $t = 0$, $i = c_2\cos\,0$ and hence $c_2 = i$. Therefore $e^{it} = \cos\,t + i\sin\,t$.

29. Euler's formula is $e^{it} = \cos\,t + i\sin\,t$. It follows that $e^{-it} = \cos\,t - i\sin\,t$. Adding these equation, $e^{it} + e^{-it} = 2\cos\,t$. Subtracting the two equations results in $e^{it} - e^{-it} = 2i\sin\,t$.

30. Let $r_1 = \lambda_1 + i\mu_1$, and $r_2 = \lambda_2 + i\mu_2$. Then

$$e^{(r_1+r_2)t} = e^{(\lambda_1+\lambda_2)t+i(\mu_1+\mu_2)t} = e^{(\lambda_1+\lambda_2)t}\,[\cos(\mu_1+\mu_2)t + i\sin(\mu_1+\mu_2)t] =$$

$$= e^{(\lambda_1+\lambda_2)t}\,[(\cos\,\mu_1 t + i\sin\,\mu_1 t)(\cos\,\mu_2 t + i\sin\,\mu_2 t)] =$$

$$= e^{\lambda_1 t}(\cos\,\mu_1 t + i\sin\,\mu_1 t)\cdot e^{\lambda_2 t}(\cos\,\mu_1 t + i\sin\,\mu_1 t)$$

Hence $e^{(r_1+r_2)t} = e^{r_1 t}\,e^{r_2 t}$.

32. If $\phi(t) = u(t) + i\,v(t)$ is a solution, then

$$(u+iv)'' + p(t)(u+iv)' + q(t)(u+iv) = 0\,,$$

and $(u'' + iv'') + p(t)(u' + iv') + q(t)(u + iv) = 0$. After expanding the equation and separating the real and imaginary parts,

$$u'' + p(t)u' + q(t)u = 0$$
$$v'' + p(t)v' + q(t)v = 0.$$

Hence both $u(t)$ and $v(t)$ are solutions.

34. Let $x = \ln t$. We differentiate, using the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\frac{1}{t}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{1}{t} + \frac{dy}{dx}\left(-\frac{1}{t^2}\right) = \frac{d^2y}{dx^2}\frac{1}{t^2} + \frac{dy}{dx}\left(-\frac{1}{t^2}\right).$$

Using these, we can see that

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0$$

transforms into

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + \alpha\frac{dy}{dx} + \beta y = \frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$

35. The equation transforms into $y'' + y = 0$. The characteristic roots are $r = \pm i$. The solution is

$$y = c_1\cos(x) + c_2\sin(x) = c_1\cos(\ln t) + c_2\sin(\ln t).$$

36. The equation transforms into $y'' + 3y' + 2y = 0$. The characteristic roots are $r = -1,\ -2$. The solution is

$$y = c_1e^{-x} + c_2e^{-2x} = c_1e^{-\ln t} + c_2e^{-2\ln t} = \frac{c_1}{t} + \frac{c_2}{t^2}.$$

37. The equation transforms into $y'' + 2y' + 1.25y = 0$. The characteristic roots are $r = -1 \pm i/2$. The solution is

$$y = c_1e^{-x}\cos(x/2) + c_2e^{-x}\sin(x/2) = c_1\frac{\cos(\frac{1}{2}\ln t)}{t} + c_2\frac{\sin(\frac{1}{2}\ln t)}{t}.$$

38. The equation transforms into $y'' - 5y' - 6y = 0$. The characteristic roots are $r = -1,\ 6$. The solution is

$$y = c_1e^{-x} + c_2e^{6x} = c_1e^{-\ln t} + c_2e^{6\ln t} = c_1\frac{1}{t} + c_2t^6.$$

39. The equation transforms into $y'' - 5y' + 6y = 0$. The characteristic roots are $r = 2,\ 3$. The solution is

$$y = c_1e^{2x} + c_2e^{3x} = c_1e^{2\ln t} + c_2e^{3\ln t} = c_1t^2 + c_2t^3.$$

40. The equation transforms into $y'' - 2y' + 5y = 0$. The characteristic roots are $r = 1 \pm 2i$. The solution is

$$y = c_1 e^x \cos(2x) + c_2 e^x \sin(2x) = c_1 t \cos(2 \ln t) + c_2 t \sin(2 \ln t).$$

41. The equation transforms into $y'' + 2y' - 3y = 0$. The characteristic roots are $r = 1, -3$. The solution is

$$y = c_1 e^x + c_2 e^{-3x} = c_1 e^{\ln t} + c_2 e^{-3 \ln t} = c_1 t + \frac{c_2}{t^3}.$$

42. The equation transforms into $y'' + 6y' + 10y = 0$. The characteristic roots are $r = -3 \pm i$. The solution is

$$y = c_1 e^{-3x} \cos(x) + c_2 e^{-3x} \sin(x) = c_1 \frac{1}{t^3} \cos(\ln t) + c_2 \frac{1}{t^3} \sin(\ln t).$$

43.(a) By the chain rule, $y'(x) = \frac{dy}{dx} x'$. In general, $\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt}$. Setting $z = \frac{dy}{dt}$, we have

$$\frac{d^2y}{dt^2} = \frac{dz}{dx}\frac{dx}{dt} = \frac{d}{dx}\left[\frac{dy}{dx}\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2y}{dx^2}\frac{dx}{dt}\right]\frac{dx}{dt} + \frac{dy}{dx}\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt}.$$

However, $\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2x}{dt^2}\right]\frac{dt}{dx} \cdot \frac{dx}{dt} = \frac{d^2x}{dt^2}$. Hence $\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}$.

(b) Substituting the results in part (a) into the general ODE, $y'' + p(t)y' + q(t)y = 0$, we find that

$$\frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} + p(t)\frac{dy}{dx}\frac{dx}{dt} + q(t)y = 0\,.$$

Collecting the terms,

$$\left[\frac{dx}{dt}\right]^2 \frac{d^2y}{dx^2} + \left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\frac{dy}{dx} + q(t)y = 0\,.$$

(c) Assuming $\left[\frac{dx}{dt}\right]^2 = k\,q(t)$, and $q(t) > 0$, we find that $\frac{dx}{dt} = \sqrt{k\,q(t)}$, which can be integrated. That is, $x = \xi(t) = \int \sqrt{k\,q(t)}\,dt$.

(d) Let $k = 1$. It follows that $\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} = \frac{d\xi}{dt} + p(t)\xi(t) = \frac{q'}{2\sqrt{q}} + p\sqrt{q}$. Hence

$$\left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right] \Big/ \left[\frac{dx}{dt}\right]^2 = \frac{q'(t) + 2p(t)q(t)}{2\,[q(t)]^{3/2}}\,.$$

As long as $dx/dt \neq 0$, the differential equation can be expressed as

$$\frac{d^2y}{dx^2} + \left[\frac{q'(t) + 2p(t)q(t)}{2\,[q(t)]^{3/2}}\right]\frac{dy}{dx} + y = 0\,.$$

For the case $q(t) < 0$, write $q(t) = -\left[-q(t)\right]$, and set $\left[\frac{dx}{dt}\right]^2 = -q(t)$.

45. $p(t) = 3t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2\,[q(t)]^{3/2}} = (1 + 3t^2)/t^2\,.$$

The ratio is not constant, and therefore the equation cannot be transformed.

46. $p(t) = t - 1/t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2\,[q(t)]^{3/2}} = 1\,.$$

The ratio is constant, and therefore the equation can be transformed. From Problem 43, the transformed equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $r^2 + r + 1 = 0$, with roots $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution is

$$y(x) = c_1 e^{-x/2} \cos \sqrt{3}\,x/2 + c_2\, e^{-x/2} \sin \sqrt{3}\,x/2\,.$$

Since $x = t^2/2$, the solution in the original variable t is

$$y(t) = e^{-t^2/4}\left[c_1 \cos\left(\sqrt{3}\,t^2/4\right) + c_2 \sin\left(\sqrt{3}\,t^2/4\right)\right]\,.$$

3.4

2. The characteristic equation is $9r^2 + 6r + 1 = 0$, with the double root $r = -1/3$. The general solution is $y(t) = c_1 e^{-t/3} + c_2 t\, e^{-t/3}$.

3. The characteristic equation is $4r^2 - 4r - 3 = 0$, with roots $r = -1/2\,,\ 3/2$. The general solution is $y(t) = c_1 e^{-t/2} + c_2 e^{3t/2}$.

4. The characteristic equation is $4r^2 + 12r + 9 = 0$, with double root $r = -3/2$. The general solution is $y(t) = (c_1 + c_2\, t)e^{-3t/2}$.

5. The characteristic equation is $r^2 - 2r + 10 = 0$, with complex roots $r = 1 \pm 3i$. The general solution is $y(t) = c_1 e^t \cos 3t + c_2 e^t \sin 3t$.

6. The characteristic equation is $r^2 - 6r + 9 = 0$, with the double root $r = 3$. The general solution is $y(t) = c_1 e^{3t} + c_2 t\, e^{3t}$.

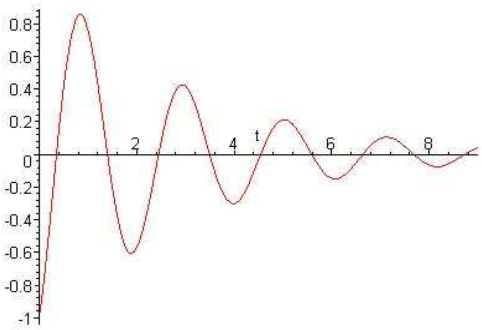
7. The characteristic equation is $4r^2 + 17r + 4 = 0$, with roots $r = -1/4\,,\ -4$. The general solution is $y(t) = c_1 e^{-t/4} + c_2 e^{-4t}$.

8. The characteristic equation is $16r^2 + 24r + 9 = 0$, with double root $r = -3/4$. The general solution is $y(t) = c_1 e^{-3t/4} + c_2 t\, e^{-3t/4}$.

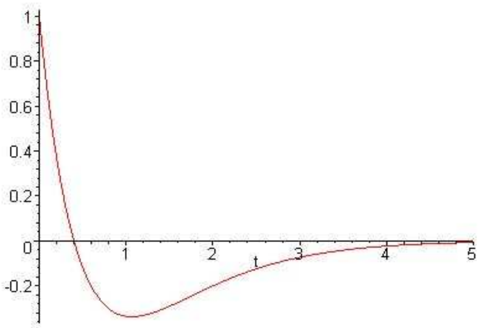
10. The characteristic equation is $2r^2 + 2r + 1 = 0$. We obtain the complex roots $r = -\frac{1}{2} \pm \frac{1}{2}\,i$. The general solution is $y(t) = c_1 e^{-t/2}\cos(t/2) + c_2 e^{-t/2}\sin(t/2)$.

11. The characteristic equation is $9r^2 - 12r + 4 = 0$, with the double root $r = 2/3$. The general solution is $y(t) = c_1 e^{2t/3} + c_2 t\, e^{2t/3}$. Invoking the first initial condition, it follows that $c_1 = 2$. Now $y'(t) = (4/3 + c_2)e^{2t/3} + 2c_2 t\, e^{2t/3}/3$. Invoking the second initial condition, $4/3 + c_2 = -1$, or $c_2 = -7/3$. Hence we obtain the solution $y(t) = 2e^{2t/3} - \frac{7}{3}te^{2t/3}$. Since the second term dominates for large t, $y(t) \to -\infty$.

13. The characteristic equation is $9r^2 + 6r + 82 = 0$. We obtain the complex roots $r = -\frac{1}{3} \pm 3\,i$. The general solution is $y(t) = c_1 e^{-t/3}\cos 3t + c_2 e^{-t/3}\sin 3t$. Based on the first initial condition, $c_1 = -1$. Invoking the second initial condition, we conclude that $1/3 + 3c_2 = 2$, or $c_2 = \frac{5}{9}$. Hence $y(t) = -e^{-t/3}\cos 3t + \frac{5}{9}e^{-t/3}\sin 3t$.



15.(a) The characteristic equation is $4r^2 + 12r + 9 = 0$, with double root $r = -\frac{3}{2}$. The general solution is $y(t) = c_1 e^{-3t/2} + c_2 t\, e^{-3t/2}$. Invoking the first initial condition, it follows that $c_1 = 1$. Now $y'(t) = (-3/2 + c_2)e^{-3t/2} - \frac{3}{2}c_2 t\, e^{-3t/2}$. The second initial condition requires that $-3/2 + c_2 = -4$, or $c_2 = -5/2$. Hence the specific solution is $y(t) = e^{-3t/2} - \frac{5}{2}t\, e^{-3t/2}$.



(b) The solution crosses the x-axis at $t = 2/5$.

(c) The solution has a minimum at the point $(16/15\,, -5e^{-8/5}/3)$.

(d) Given that $y'(0) = b$, we have $-3/2 + c_2 = b$, or $c_2 = b + 3/2$. Hence the solution is $y(t) = e^{-3t/2} + (b + \frac{3}{2})t\, e^{-3t/2}$. Since the second term dominates, the long-term solution depends on the sign of the coefficient $b + 3/2$. The critical value is $b = -3/2$.

16. The characteristic roots are $r_1 = r_2 = 1/2$. Hence the general solution is given by $y(t) = c_1 e^{t/2} + c_2 t\, e^{t/2}$. Invoking the initial conditions, we require that $c_1 = 2$, and that $1 + c_2 = b$. The specific solution is

$$y(t) = 2e^{t/2} + (b - 1)t\, e^{t/2}.$$

Since the second term dominates, the long-term solution depends on the sign of the coefficient $b - 1$. The critical value is $b = 1$.

18.(a) The characteristic roots are $r_1 = r_2 = -2/3$. Therefore the general solution is given by $y(t) = c_1 e^{-2t/3} + c_2 t\, e^{-2t/3}$. Invoking the initial conditions, we require that $c_1 = a$, and that $-2a/3 + c_2 = -1$. After solving for the coefficients, the specific solution is $y(t) = ae^{-2t/3} + (\frac{2a}{3} - 1)t\, e^{-2t/3}$.

(b) Since the second term dominates, the long-term solution depends on the sign of the coefficient $\frac{2a}{3} - 1$. The critical value is $a = 3/2$.

20.(a) The characteristic equation is $r^2 + 2a\, r + a^2 = (r + a)^2 = 0$.

(b) With $p(t) = 2a$, Abel's Formula becomes

$$W(y_1\, , y_2) = c\, e^{-\int 2a\, dt} = c\, e^{-2at}\, .$$

(c) $y_1(t) = e^{-at}$ is a solution. From part (b),

$$e^{-at}\, y_2'(t) + a\, e^{-at} y_2(t) = c\, e^{-2at}\, ,$$

which can be written as

$$\frac{d}{dt}\left[e^{at}\, y_2(t)\right] = c\, ,$$

resulting in

$$e^{at}\, y_2(t) = c\, t\, .$$

23. Set $y_2(t) = t^2\, v(t)$. Substitution into the ODE results in

$$t^2(t^2 v'' + 4tv' + 2v) - 4t(t^2 v' + 2tv) + 6t^2 v = 0\, .$$

After collecting terms, we end up with $t^4 v'' = 0$. Hence $v(t) = c_1 + c_2 t$, and thus $y_2(t) = c_1 t^2 + c_2 t^3$. Setting $c_1 = 0$ and $c_2 = 1$, we obtain $y_2(t) = t^3$.

24. Set $y_2(t) = t\, v(t)$. Substitution into the ODE results in

$$t^2(tv'' + 2v') + 2t(tv' + v) - 2tv = 0\, .$$

After collecting terms, we end up with $t^3v'' + 4t^2v' = 0$. This equation is linear in the variable $w = v'$. It follows that $v'(t) = c\,t^{-4}$, and $v(t) = c_1t^{-3} + c_2$. Thus $y_2(t) = c_1t^{-2} + c_2t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = t^{-2}$.

26. Set $y_2(t) = tv(t)$. Substitution into the ODE results in $v'' - v' = 0$. This ODE is linear in the variable $w = v'$. It follows that $v'(t) = c_1e^t$, and $v(t) = c_1e^t + c_2$. Thus $y_2(t) = c_1te^t + c_2t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = te^t$.

28. Set $y_2(x) = e^xv(x)$. Substitution into the ODE results in

$$v'' + \frac{x-2}{x-1}v' = 0\,.$$

This ODE is linear in the variable $w = v'$. An integrating factor is

$$\mu = e^{\int \frac{x-2}{x-1}dx} = \frac{e^x}{x-1}\,.$$

Rewrite the equation as $\left[\frac{e^xv'}{x-1}\right]' = 0$, from which it follows that $v'(x) = c(x-1)e^{-x}$. Hence $v(x) = c_1xe^{-x} + c_2$ and $y_2(x) = c_1x + c_2e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

29. Set $y_2(x) = y_1(x)\,v(x)$, in which $y_1(x) = x^{1/4}e^{2\sqrt{x}}$. It can be verified that y_1 is a solution of the ODE, that is, $x^2y_1'' - (x - 0.1875)y_1 = 0$. Substitution of the given form of y_2 results in the differential equation

$$2x^{9/4}v'' + (4x^{7/4} + x^{5/4})v' = 0\,.$$

This ODE is linear in the variable $w = v'$. An integrating factor is

$$\mu = e^{\int \left[2x^{-1/2} + \frac{1}{2x}\right]dx} = \sqrt{x}\,e^{4\sqrt{x}}.$$

Rewrite the equation as $\left[\sqrt{x}\,e^{4\sqrt{x}}\,v'\right]' = 0$, from which it follows that

$$v'(x) = c\,e^{-4\sqrt{x}}/\sqrt{x}\,.$$

Integrating, $v(x) = c_1e^{-4\sqrt{x}} + c_2$ and as a result,

$$y_2(x) = c_1x^{1/4}e^{-2\sqrt{x}} + c_2x^{1/4}e^{2\sqrt{x}}.$$

Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{1/4}e^{-2\sqrt{x}}$.

32. Direct substitution verifies that $y_1(t) = e^{-\delta x^2/2}$ is a solution of the ODE. Now set $y_2(x) = y_1(x)\,v(x)$. Substitution of y_2 into the ODE results in

$$v'' - \delta xv' = 0\,.$$

This ODE is linear in the variable $w = v'$. An integrating factor is $\mu = e^{-\delta x^2/2}$. Rewrite the equation as $\left[\,e^{-\delta x^2/2}v'\right]' = 0$, from which it follows that

$$v'(x) = c_1\,e^{\delta x^2/2}\,.$$

Integrating, we obtain

$$v(x) = c_1 \int_{x_0}^{x} e^{\delta u^2/2} du + v(x_0).$$

Hence

$$y_2(x) = c_1 e^{-\delta x^2/2} \int_{x_0}^{x} e^{\delta u^2/2} du + c_2 e^{-\delta x^2/2}.$$

Setting $c_2 = 0$, we obtain a second independent solution.

34. After writing the ODE in standard form, we have $p(t) = 3/t$. Based on Abel's identity, $W(y_1, y_2) = c_1 e^{-\int \frac{3}{t} dt} = c_1 t^{-3}$. As shown in Problem 33, two solutions of a second order linear equation satisfy

$$(y_2/y_1)' = W(y_1, y_2)/y_1^2 .$$

In the given problem, $y_1(t) = t^{-1}$. Hence $(t\, y_2)' = c_1 t^{-1}$. Integrating both sides of the equation, $y_2(t) = c_1 t^{-1} \ln t + c_2 t^{-1}$.

36. After writing the ODE in standard form, we have $p(x) = -x/(x-1)$. Based on Abel's identity, $W(y_1, y_2) = c\, e^{\int \frac{x}{x-1} dx} = c\ e^x (x-1)$. Two solutions of a second order linear equation satisfy

$$(y_2/y_1)' = W(y_1, y_2)/y_1^2 .$$

In the given problem, $y_1(x) = e^x$. Hence $(e^{-x}\, y_2)' = c\, e^{-x}(x-1)$. Integrating both sides of the equation, $y_2(x) = c_1 x + c_2 e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

37. Write the ODE in standard form to find $p(x) = 1/x$. Based on Abel's identity, $W(y_1, y_2) = c\, e^{-\int \frac{1}{x} dx} = c\, x^{-1}$. Two solutions of a second order linear ODE satisfy $(y_2/y_1)' = W(y_1, y_2)/y_1^2$. In the given problem, $y_1(x) = x^{-1/2} \sin x$. Hence

$$\left(\frac{\sqrt{x}}{\sin x}\, y_2\right)' = c\, \frac{1}{\sin^2 x} .$$

Integrating both sides of the equation, $y_2(x) = c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{-1/2} \cos x$.

39.(a) The characteristic equation is $ar^2 + c = 0$. If $a, c > 0$, then the roots are $r_{1,2} = \pm i\sqrt{c/a}$. The general solution is

$$y(t) = c_1 \cos \sqrt{\frac{c}{a}}\, t + c_2 \sin \sqrt{\frac{c}{a}}\, t ,$$

which is bounded.

(b) The characteristic equation is $ar^2 + br = 0$. The roots are $r_{1,2} = 0, -b/a$, and hence the general solution is $y(t) = c_1 + c_2 e^{-bt/a}$. Clearly, $y(t) \to c_1$. With the given initial conditions, $c_1 = y_0 + (a/b) y_0'$.

40. Note that $\cos t \sin t = \frac{1}{2}\sin 2t$. Then $1 - k\cos t \sin t = 1 - \frac{k}{2}\sin 2t$. Now if $0 < k < 2$, then $\frac{k}{2}\sin 2t < |\sin 2t|$ and $-\frac{k}{2}\sin 2t > -|\sin 2t|$. Hence

$$1 - k\cos t \sin t = 1 - \frac{k}{2}\sin 2t > 1 - |\sin 2t| \geq 0 .$$

41. The equation transforms into $y'' - 4y' + 4y = 0$. We obtain a double root $r = 2$. The solution is

$$y = c_1e^{2x} + c_2xe^{2x} = c_1e^{2\ln t} + c_2 \ln t e^{2\ln t} = c_1t^2 + c_2t^2 \ln t.$$

43. The equation transforms into $y'' - 7y'/2 + 5y/2 = 0$. The characteristic roots are $r = 1,\ 5/2$, so the solution is

$$y = c_1e^{x} + c_2e^{5x/2} = c_1e^{\ln t} + c_2e^{5\ln t/2} = c_1t + c_2t^{5/2}.$$

44. The equation transforms into $y'' + 2y' + y = 0$. We get a double root $r = -1$. The solution is

$$y = c_1e^{-x} + c_2xe^{-x} = c_1e^{-\ln t} + c_2 \ln t e^{-\ln t} = c_1t^{-1} + c_2t^{-1} \ln t.$$

45. The equation transforms into $y'' - 3y' + 9y/4 = 0$. We obtain the double root $r = 3/2$. The solution is

$$y = c_1e^{3x/2} + c_2xe^{3x/2} = c_1e^{3\ln t/2} + c_2 \ln t e^{3\ln t/2} = c_1t^{3/2} + c_2t^{3/2} \ln t.$$

46. The equation transforms into $y'' + 4y' + 13y = 0$. The characteristic roots are $r = -2 \pm 3i$. The solution is

$$y = c_1e^{-2x}\cos(3x) + c_2e^{-2x}\sin(3x) = c_1t^{-2}\cos(3\ln t) + c_2t^{-2}\sin(3\ln t).$$

3.5

2. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t$. Since the function $g(t) = 3\sin 2t$ is not proportional to the solutions of the homogeneous equation, set $Y = A\cos 2t + B\sin 2t$. Substitution into the given ODE, and comparing the coefficients, results in the system of equations $B - 4A = 3$ and $A + 4B = 0$. Hence $Y = -\frac{12}{17}\cos 2t + \frac{3}{17}\sin 2t$. The general solution is $y(t) = y_c(t) + Y$.

3. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, with roots $r = -1,\ 3$. Hence $y_c(t) = c_1e^{-t} + c_2e^{3t}$. Note that the assignment $Y = Ate^{-t}$ is not sufficient to match the coefficients. Try $Y = Ate^{-t} + Bt^2e^{-t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4A + 2B = 0$ and $-8B = -3$. This implies that $Y = \frac{3}{16}te^{-t} + \frac{3}{8}t^2e^{-t}$. The general solution is $y(t) = y_c(t) + Y$.

5. The characteristic equation for the homogeneous problem is $r^2+9=0$, with complex roots $r=\pm 3i$. Hence $y_c(t)=c_1\cos 3t+c_2\sin 3t$. To simplify the analysis, set $g_1(t)=6$ and $g_2(t)=t^2e^{3t}$. By inspection, we have $Y_1=2/3$. Based on the form of g_2, set $Y_2=Ae^{3t}+Bte^{3t}+Ct^2e^{3t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $18A+6B+2C=0$, $18B+12C=0$, and $18C=1$. Hence

$$Y_2=\frac{1}{162}e^{3t}-\frac{1}{27}te^{3t}+\frac{1}{18}t^2e^{3t}.$$

The general solution is $y(t)=y_c(t)+Y_1+Y_2$.

7. The characteristic equation for the homogeneous problem is $2r^2+3r+1=0$, with roots $r=-1,\ -1/2$. Hence $y_c(t)=c_1e^{-t}+c_2e^{-t/2}$. To simplify the analysis, set $g_1(t)=t^2$ and $g_2(t)=3\sin t$. Based on the form of g_1, set $Y_1=A+Bt+Ct^2$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $A+3B+4C=0$, $B+6C=0$, and $C=1$. Hence we obtain $Y_1=14-6t+t^2$. On the other hand, set $Y_2=D\cos t+E\sin t$. After substitution into the ODE, we find that $D=-9/10$ and $E=-3/10$. The general solution is $y(t)=y_c(t)+Y_1+Y_2$.

9. The characteristic equation for the homogeneous problem is $r^2+\omega_0^2=0$, with complex roots $r=\pm\,\omega_0\, i$. Hence $y_c(t)=c_1\cos\omega_0 t+c_2\sin\omega_0 t$. Since $\omega\neq\omega_0$, set $Y=A\cos\omega t+B\sin\omega t$. Substitution into the ODE and comparing the coefficients results in the system of equations $(\omega_0^2-\omega^2)A=1$ and $(\omega_0^2-\omega^2)B=0$. Hence

$$Y=\frac{1}{\omega_0^2-\omega^2}\cos\omega t\,.$$

The general solution is $y(t)=y_c(t)+Y$.

10. From Problem 9, $y_c(t)$ is known. Since $\cos\omega_0 t$ is a solution of the homogeneous problem, set $Y=At\cos\omega_0 t+Bt\sin\omega_0 t$. Substitution into the given ODE and comparing the coefficients results in $A=0$ and $B=\frac{1}{2\omega_0}$. Hence the general solution is $y(t)=c_1\cos\omega_0 t+c_2\sin\omega_0 t+\frac{t}{2\omega_0}\sin\omega_0 t$.

12. The characteristic equation for the homogeneous problem is $r^2-r-2=0$, with roots $r=-1,\,2$. Hence $y_c(t)=c_1e^{-t}+c_2\,e^{2t}$. Based on the form of the right hand side, that is, $\cosh(2t)=(e^{2t}+e^{-2t})/2$, set $Y=At\,e^{2t}+Be^{-2t}$. Substitution into the given ODE and comparing the coefficients results in $A=1/6$ and $B=1/8$. Hence the general solution is $y(t)=c_1e^{-t}+c_2\,e^{2t}+t\,e^{2t}/6+e^{-2t}/8$.

14. The characteristic equation for the homogeneous problem is $r^2+4=0$, with roots $r=\pm\,2i$. Hence $y_c(t)=c_1\cos 2t+c_2\sin 2t$. Set $Y_1=A+Bt+Ct^2$. Comparing the coefficients of the respective terms, we find that $A=-1/8$, $B=0$, $C=1/4$. Now set $Y_2=D\,e^t$, and obtain $D=3/5$. Hence the general solution is

$$y(t)=c_1\cos 2t+c_2\sin 2t-1/8+t^2/4+3\,e^t/5\,.$$

Invoking the initial conditions, we require that $19/40+c_1=0$ and $3/5+2c_2=2$. Hence $c_1=-19/40$ and $c_2=7/10$.

15. The characteristic equation for the homogeneous problem is $r^2 - 2r + 1 = 0$, with a double root $r = 1$. Hence $y_c(t) = c_1 e^t + c_2 t\, e^t$. Consider $g_1(t) = t\, e^t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1 = At^2 e^t + Bt^3 e^t$ (the first term is not sufficient for a match). Upon substitution, we obtain $Y_1 = t^3 e^t/6$. By inspection, $Y_2 = 4$. Hence the general solution is $y(t) = c_1 e^t + c_2 t\, e^t + t^3 e^t/6 + 4$. Invoking the initial conditions, we require that $c_1 + 4 = 1$ and $c_1 + c_2 = 1$. Hence $c_1 = -3$ and $c_2 = 4$.

17. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$. Since the function $\sin 2t$ is a solution of the homogeneous problem, set $Y = At \cos 2t + Bt \sin 2t$. Upon substitution, we obtain $Y = -\frac{3}{4}t \cos 2t$. Hence the general solution is $y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4}t \cos 2t$. Invoking the initial conditions, we require that $c_1 = 2$ and $2c_2 - \frac{3}{4} = -1$. Hence $c_1 = 2$ and $c_2 = -1/8$.

18. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$. Based on the form of $g(t)$, set $Y = At\, e^{-t} \cos 2t + Bt\, e^{-t} \sin 2t$. After comparing coefficients, we obtain $Y = t\, e^{-t} \sin 2t$. Hence the general solution is

$$y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t + t\, e^{-t} \sin 2t\,.$$

Invoking the initial conditions, we require that $c_1 = 1$ and $-c_1 + 2c_2 = 0$. Hence $c_1 = 1$ and $c_2 = 1/2$.

20. The characteristic equation for the homogeneous problem is $r^2 + 1 = 0$, with complex roots $r = \pm\, i$. Hence $y_c(t) = c_1 \cos t + c_2 \sin t$. Let $g_1(t) = t \sin t$ and $g_2(t) = t$. By inspection, it is easy to see that $Y_2(t) = t$. Based on the form of $g_1(t)$, set $Y_1(t) = At \cos t + Bt \sin t + Ct^2 \cos t + Dt^2 \sin t$. Substitution into the equation and comparing the coefficients results in $A = 0$, $B = 1/4$, $C = -1/4$, and $D = 0$. Hence $Y(t) = t + \frac{1}{4}t \sin t - \frac{1}{4}t^2 \cos t$.

21. The characteristic equation for the homogeneous problem is $r^2 - 5r + 6 = 0$, with roots $r = 2, 3$. Hence $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$. Consider $g_1(t) = e^{2t}(3t + 4) \sin t$, and $g_2(t) = e^t \cos 2t$. Based on the form of these functions on the right hand side of the ODE, set $Y_2(t) = e^t(A_1 \cos 2t + A_2 \sin 2t)$ and $Y_1(t) = (B_1 + B_2 t)e^{2t} \sin t + (C_1 + C_2 t)e^{2t} \cos t$. Substitution into the equation and comparing the coefficients results in

$$Y(t) = -\frac{1}{20}(e^t \cos 2t + 3e^t \sin 2t) + \frac{3}{2}te^{2t}(\cos t - \sin t) + e^{2t}(\frac{1}{2}\cos t - 5 \sin t).$$

23. We obtain the double characteristic root $r = 2$. Hence $y_c(t) = c_1 e^{2t} + c_2 te^{2t}$. Consider the functions $g_1(t) = 2t^2$, $g_2(t) = 4te^{2t}$, and $g_3(t) = t \sin 2t$. The corresponding forms of the respective parts of the particular solution are $Y_1(t) = A_0 + A_1 t + A_2 t^2$, $Y_2(t) = e^{2t}(B_2 t^2 + B_3 t^3)$, and $Y_3(t) = t(C_1 \cos 2t + C_2 \sin 2t) + (D_1 \cos 2t + D_2 \sin 2t)$. Substitution into the equation and comparing the coeffi-

cients results in

$$Y(t) = \frac{1}{4}(3 + 4t + 2t^2) + \frac{2}{3}t^3e^{2t} + \frac{1}{8}t \cos 2t + \frac{1}{16}(\cos 2t - \sin 2t).$$

24. The homogeneous solution is $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$. Since $\cos 2t$ and $\sin 2t$ are both solutions of the homogeneous equation, set

$$Y(t) = t(A_0 + A_1t + A_2t^2) \cos 2t + t(B_0 + B_1t + B_2t^2) \sin 2t.$$

Substitution into the equation and comparing the coefficients results in

$$Y(t) = (\frac{13}{32}t - \frac{1}{12}t^3) \cos 2t + \frac{1}{16}(28t + 13t^2) \sin 2t.$$

25. The homogeneous solution is $y_c(t) = c_1e^{-t} + c_2te^{-2t}$. None of the functions on the right hand side are solutions of the homogenous equation. In order to include all possible combinations of the derivatives, consider

$$Y(t) = e^t(A_0 + A_1t + A_2t^2) \cos 2t + e^t(B_0 + B_1t + B_2t^2) \sin 2t+$$

$$+e^{-t}(C_1 \cos t + C_2 \sin t) + De^t.$$

Substitution into the differential equation and comparing the coefficients results in

$$\begin{aligned} Y(t) &= e^t(A_0 + A_1t + A_2t^2) \cos 2t + e^t(B_0 + B_1t + B_2t^2) \sin 2t \\ &+ e^{-t}(-\frac{2}{3} \cos t + \frac{2}{3} \sin t) + 2e^t/3, \end{aligned}$$

in which $A_0 = -4105/35152$, $A_1 = 73/676$, $A_2 = -5/52$, $B_0 = -1233/35152$, $B_1 = 10/169$, $B_2 = 1/52$.

26. The homogeneous solution is $y_c(t) = c_1e^{-t} \cos 2t + c_2e^{-t} \sin 2t$. None of the terms on the right hand side are solutions of the homogenous equation. In order to include the appropriate combinations of derivatives, consider

$$Y(t) = e^{-t}(A_1t + A_2t^2) \cos 2t + e^{-t}(B_1t + B_2t^2) \sin 2t+$$

$$+e^{-2t}(C_0 + C_1t) \cos 2t + e^{-2t}(D_0 + D_1t) \sin 2t.$$

Substitution into the differential equation and comparing the coefficients results in

$$Y(t) = \frac{3}{16}te^{-t} \cos 2t + \frac{3}{8}t^2e^{-t} \sin 2t - \frac{1}{25}e^{-2t}(7 + 10t) \cos 2t+$$

$$+\frac{1}{25}e^{-2t}(1 + 5t) \sin 2t.$$

28. The homogeneous solution is $y_c(t) = c_1 \cos \lambda t + c_2 \sin \lambda t$. Since the differential operator does not contain a first derivative (and $\lambda \neq m\pi$), we can set

$$Y(t) = \sum_{m=1}^{N} C_m \sin m\pi t.$$

Substitution into the ODE yields

$$-\sum_{m=1}^{N} m^2\pi^2 C_m \sin m\pi t + \lambda^2 \sum_{m=1}^{N} C_m \sin m\pi t = \sum_{m=1}^{N} a_m \sin m\pi t\,.$$

Equating coefficients of the individual terms, we obtain

$$C_m = \frac{a_m}{\lambda^2 - m^2\pi^2}\,, \quad m = 1, 2 \ldots N.$$

30. The homogeneous solution is $y_c(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$. The input function is independent of the homogeneous solutions, on any interval. Since the right hand side is piecewise constant, it follows by inspection that

$$Y(t) = \begin{cases} 1/5\,, & 0 \le t \le \pi/2 \\ 0, & t > \pi/2 \end{cases}\,.$$

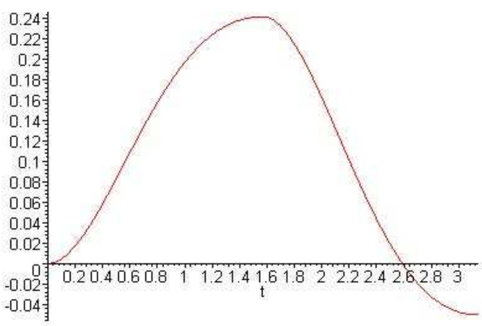
For $0 \le t \le \pi/2$, the general solution is $y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + 1/5$. Invoking the initial conditions $y(0) = y'(0) = 0$, we require that $c_1 = -1/5$, and that $c_2 = -1/10$. Hence

$$y(t) = \frac{1}{5} - \frac{1}{10}(2e^{-t}\cos 2t + e^{-t}\sin 2t)$$

on the interval $0 \le t \le \pi/2$. We now have the values $y(\pi/2) = (1 + e^{-\pi/2})/5$, and $y'(\pi/2) = 0$. For $t > \pi/2$, the general solution is $y(t) = d_1 e^{-t}\cos 2t + d_2 e^{-t}\sin 2t$. It follows that $y(\pi/2) = -e^{-\pi/2}d_1$ and $y'(\pi/2) = e^{-\pi/2}d_1 - 2e^{-\pi/2}d_2$. Since the solution is continuously differentiable, we require that

$$\begin{aligned} -\,e^{-\pi/2}d_1 &= (1 + e^{-\pi/2})/5 \\ e^{-\pi/2}d_1 - 2e^{-\pi/2}d_2 &= 0\,. \end{aligned}$$

Solving for the coefficients, $d_1 = 2d_2 = -(e^{\pi/2} + 1)/5$.



32. Since $a, b, c > 0$, the roots of the characteristic equation has negative real parts. That is, $r = \alpha \pm \beta i$, where $\alpha < 0$. Hence the homogeneous solution is

$$y_c(t) = c_1 e^{\alpha t}\cos \beta t + c_2 e^{\alpha t}\sin \beta t\,.$$

If $g(t) = d$, then the general solution is

$$y(t) = d/c + c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t .$$

Since $\alpha < 0$, $y(t) \to d/c$ as $t \to \infty$. If $c = 0$, then that characteristic roots are $r = 0$ and $r = -b/a$. The ODE becomes $ay'' + by' = d$. Integrating both sides, we find that $ay' + by = d\,t + c_1$. The general solution can be expressed as

$$y(t) = d\,t/b + c_1 + c_2 e^{-bt/a}.$$

In this case, the solution grows without bound. If $b = 0$, also, then the differential equation can be written as $y'' = d/a$, which has general solution $y(t) = d\,t^2/2a + c_1 + c_2$. Hence the assertion is true only if the coefficients are positive.

33.(a) Since D is a linear operator,

$$D^2y + bDy + cy = D^2y - (r_1 + r_2)Dy + r_1 r_2 y =$$
$$= D^2y - r_2 Dy - r_1 Dy + r_1 r_2 y = D(Dy - r_2 y) - r_1(Dy - r_2 y) =$$
$$= (D - r_1)(D - r_2)y.$$

(b) Let $u = (D - r_2)y$. Then the ODE (i) can be written as $(D - r_1)u = g(t)$, that is, $u' - r_1 u = g(t)$. The latter is a linear first order equation in u. Its general solution is

$$u(t) = e^{r_1 t} \int_{t_0}^{t} e^{-r_1 \tau} g(\tau) d\tau + c_1 e^{r_1 t} .$$

From above, we have $y' - r_2 y = u(t)$. This equation is also a first order ODE. Hence the general solution of the original second order equation is

$$y(t) = e^{r_2 t} \int_{t_0}^{t} e^{-r_2 \tau} u(\tau) d\tau + c_2 e^{r_2 t} .$$

Note that the solution $y(t)$ contains two arbitrary constants.

35. Note that $(2D^2 + 3D + 1)y = (2D + 1)(D + 1)y$. Let $u = (D + 1)y$, and solve the ODE $2u' + u = t^2 + 3\sin t$. This equation is a linear first order ODE, with solution

$$u(t) = e^{-t/2} \int_{t_0}^{t} e^{\tau/2} \left[\tau^2/2 + \frac{3}{2} \sin \tau \right] d\tau + c\, e^{-t/2} =$$
$$= t^2 - 4t + 8 - \frac{6}{5} \cos t + \frac{3}{5} \sin t + c\, e^{-t/2}.$$

Now consider the ODE $y' + y = u(t)$. The general solution of this first order ODE is

$$y(t) = e^{-t} \int_{t_0}^{t} e^{\tau} u(\tau) d\tau + c_2 e^{-t} ,$$

in which $u(t)$ is given above. Substituting for $u(t)$ and performing the integration,

$$y(t) = t^2 - 6t + 14 - \frac{9}{10} \cos t - \frac{3}{10} \sin t + c_1 e^{-t/2} + c_2 e^{-t}.$$

36. We have $(D^2+2D+1)y = (D+1)(D+1)y$. Let $u = (D+1)y$, and consider the ODE $u' + u = 2e^{-t}$. The general solution is $u(t) = 2t\,e^{-t} + c\,e^{-t}$. We therefore have the first order equation $u' + u = 2t\,e^{-t} + c_1 e^{-t}$. The general solution of the latter differential equation is

$$y(t) = e^{-t}\int_{t_0}^{t} [2\tau + c_1\,]\,d\tau + c_2 e^{-t} = e^{-t}(t^2 + c_1 t + c_2).$$

37. We have $(D^2+2D)y = D(D+2)y$. Let $u = (D+2)y$, and consider the equation $u' = 3 + 4\sin 2t$. Direct integration results in $u(t) = 3t - 2\cos 2t + c$. The problem is reduced to solving the ODE $y' + 2y = 3t - 2\cos 2t + c$. The general solution of this first order differential equation is

$$y(t) = e^{-2t}\int_{t_0}^{t} e^{2\tau}\,[3\tau - 2\cos 2\tau + c\,]\,d\tau + c_2 e^{-2t} =$$

$$= \frac{3}{2}t - \frac{1}{2}(\cos 2t + \sin 2t) + c_1 + c_2 e^{-2t}.$$

3.6

1. The solution of the homogeneous equation is $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$. The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{5t}$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{e^{3t}(2e^t)}{W(t)}dt = 2e^{-t}$$

$$u_2(t) = \int \frac{e^{2t}(2e^t)}{W(t)}dt = -e^{-2t}$$

Hence the particular solution is $Y(t) = 2e^t - e^t = e^t$.

3. The solution of the homogeneous equation is $y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$. The functions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-2t}$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{-t}(3e^{-t})}{W(t)}dt = -3t^2/2$$

$$u_2(t) = \int \frac{e^{-t}(3e^{-t})}{W(t)}dt = 3t$$

Hence the particular solution is $Y(t) = -3t^2 e^{-t}/2 + 3t^2 e^{-t} = 3t^2 e^{-t}/2$.

4. The functions $y_1(t) = e^{t/2}$ and $y_2(t) = te^{t/2}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{t}$. First write the equation in

standard form, so that $g(t) = 4e^{t/2}$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{t/2}(4e^{t/2})}{W(t)}dt = -2t^2$$

$$u_2(t) = \int \frac{e^{t/2}(4e^{t/2})}{W(t)}dt = 4t$$

Hence the particular solution is $Y(t) = -2t^2e^{t/2} + 4t^2e^{t/2} = 2t^2e^{t/2}$.

6. The solution of the homogeneous equation is $y_c(t) = c_1 \cos 3t + c_2 \sin 3t$. The two functions $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$ form a fundamental set of solutions, with $W(y_1, y_2) = 3$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{\sin 3t(9 \sec^2 3t)}{W(t)}dt = -\csc 3t$$

$$u_2(t) = \int \frac{\cos 3t(9 \sec^2 3t)}{W(t)}dt = \ln|\sec 3t + \tan 3t|$$

Hence the particular solution is $Y(t) = -1 + (\sin 3t) \ln|\sec 3t + \tan 3t|$. The general solution is given by

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + (\sin 3t) \ln|\sec 3t + \tan 3t| - 1 .$$

7. The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-4t}$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{-2t}(t^{-2}e^{-2t})}{W(t)}dt = -\ln t$$

$$u_2(t) = \int \frac{e^{-2t}(t^{-2}e^{-2t})}{W(t)}dt = -1/t$$

Hence the particular solution is $Y(t) = -e^{-2t}\ln t - e^{-2t}$. Since the second term is a solution of the homogeneous equation, the general solution is given by

$$y(t) = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}\ln t.$$

8. The solution of the homogeneous equation is $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$. The two functions $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{\sin 2t(3 \csc 2t)}{W(t)}dt = -3t/2$$

$$u_2(t) = \int \frac{\cos 2t(3 \csc 2t)}{W(t)}dt = \frac{3}{4}\ln|\sin 2t|$$

Hence the particular solution is $Y(t) = -\frac{3}{2}t\cos 2t + \frac{3}{4}(\sin 2t)\ln|\sin 2t|$. The general solution is given by

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{3}{2}t\cos 2t + \frac{3}{4}(\sin 2t)\ln|\sin 2t|\,.$$

9. The functions $y_1(t) = \cos(t/2)$ and $y_2(t) = \sin(t/2)$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = 1/2$. First write the ODE in standard form, so that $g(t) = \sec(t/2)/2$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{\cos(t/2)\,[\sec(t/2)]}{2W(t)}dt = 2\,\ln|\cos(t/2)|$$

$$u_2(t) = \int \frac{\sin(t/2)\,[\sec(t/2)]}{2W(t)}dt = t$$

The particular solution is $Y(t) = 2\cos(t/2)\ln|\cos(t/2)| + t\,\sin(t/2)$. The general solution is given by

$$y(t) = c_1 \cos(t/2) + c_2 \sin(t/2) + 2\,\cos(t/2)\,\ln|\cos(t/2)| + t\,\sin(t/2).$$

10. The solution of the homogeneous equation is $y_c(t) = c_1 e^t + c_2 t e^t$. The functions $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental set of solutions, with $W(y_1, y_2) = e^{2t}$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{te^t(e^t)}{W(t)(1+t^2)}dt = -\frac{1}{2}\ln(1+t^2)$$

$$u_2(t) = \int \frac{e^t(e^t)}{W(t)(1+t^2)}dt = \arctan t$$

The particular solution is $Y(t) = -\frac{1}{2}e^t\,\ln(1+t^2) + te^t\,\arctan(t)$. Hence the general solution is given by $y(t) = c_1 e^t + c_2 te^t - \frac{1}{2}e^t\,\ln(1+t^2) + te^t\,\arctan(t)$.

12. The functions $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\frac{1}{2}\int^t g(s)\,\sin 2s\,ds$$

$$u_2(t) = \frac{1}{2}\int^t g(s)\,\cos 2s\,ds$$

Hence the particular solution is

$$Y(t) = -\frac{1}{2}\cos 2t\int^t g(s)\,\sin 2s\,ds + \frac{1}{2}\sin 2t\int^t g(s)\,\cos 2s\,ds\,.$$

Note that $\sin 2t\,\cos 2s - \cos 2t\,\sin 2s = \sin(2t - 2s)$. It follows that

$$Y(t) = \frac{1}{2}\int^t g(s)\sin(2t-2s)ds\,.$$

The general solution of the differential equation is given by

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{2}\int^t g(s)\sin(2t-2s)ds\,.$$

13. Note first that $p(t) = 0\,, q(t) = -2/t^2$ and $g(t) = (3t^2-1)/t^2$. The functions $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is $W(y_1,y_2) = -3$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{t^{-1}(3t^2-1)}{t^2\,W(t)}dt = t^{-2}/6 + \ln\,t$$

$$u_2(t) = \int \frac{t^2(3t^2-1)}{t^2\,W(t)}dt = -t^3/3 + t/3$$

Therefore $Y(t) = 1/6\, + t^2 \ln\,t - t^2/3 + 1/3$. Hence the general solution is

$$y(t) = c_1 t^2 + c_2 t^{-1} + t^2 \ln\,t + 1/2\,.$$

15. Observe that $g(t) = t\,e^{2t}$. The functions $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions. The Wronskian of these two functions is $W(y_1,y_2) = t\,e^t$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{e^t(t\,e^{2t})}{W(t)}dt = -e^{2t}/2$$

$$u_2(t) = \int \frac{(1+t)(t\,e^{2t})}{W(t)}dt = t\,e^t$$

Therefore $Y(t) = -(1+t)e^{2t}/2 + t\,e^{2t} = -e^{2t}/2 + t\,e^{2t}/2\,.$

16. Observe that $g(t) = 2(1-t)\,e^{-t}$. Direct substitution of $y_1(t) = e^t$ and $y_2(t) = t$ verifies that they are solutions of the homogeneous equation. The Wronskian of the two solutions is $W(y_1,y_2) = (1-t)\,e^t$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{2t(1-t)e^{-t}}{W(t)}dt = te^{-2t} + e^{-2t}/2$$

$$u_2(t) = \int \frac{2(1-t)}{W(t)}dt = -2\,e^{-t}$$

Therefore $Y(t) = te^{-t} + e^{-t}/2 - 2t\,e^{-t} = -te^{-t} + e^{-t}/2\,.$

17. Note that $g(x) = \ln\,x$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 \ln\,x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = x^3$. Using the method of variation of parameters, the particular solution is

$$Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

in which

$$u_1(x) = -\int \frac{x^2 \ln x(\ln x)}{W(x)} dx = -(\ln x)^3/3$$

$$u_2(x) = \int \frac{x^2(\ln x)}{W(x)} dx = (\ln x)^2/2$$

Therefore $Y(x) = -x^2(\ln x)^3/3 + x^2(\ln x)^3/2 = x^2(\ln x)^3/6$.

19. First write the equation in standard form. Note that the forcing function becomes $g(x)/(1-x)$. The functions $y_1(x) = e^x$ and $y_2(x) = x$ are a fundamental set of solutions, as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = (1-x)e^x$. Using the method of variation of parameters, the particular solution is

$$Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x),$$

in which

$$u_1(x) = -\int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau$$

$$u_2(x) = \int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau$$

Therefore

$$Y(x) = -e^x \int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau + x \int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau =$$

$$= \int^x \frac{(xe^\tau - e^x\tau)g(\tau)}{(1-\tau)^2 e^\tau} d\tau\,.$$

20. First write the equation in standard form. The forcing function becomes $g(x)/x^2$. The functions $y_1(x) = x^{-1/2}\sin x$ and $y_2(x) = x^{-1/2}\cos x$ are a fundamental set of solutions. The Wronskian of the solutions is $W(y_1,y_2) = -1/x$. Using the method of variation of parameters, the particular solution is

$$Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x),$$

in which

$$u_1(x) = \int^x \frac{\cos\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau$$

$$u_2(x) = -\int^x \frac{\sin\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau$$

Therefore

$$Y(x) = \frac{\sin x}{\sqrt{x}} \int^x \frac{\cos\tau\,(g(\tau))}{\tau\sqrt{\tau}} dt - \frac{\cos x}{\sqrt{x}} \int^x \frac{\sin\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau =$$

$$= \frac{1}{\sqrt{x}} \int^x \frac{\sin(x-\tau)\, g(\tau)}{\tau\sqrt{\tau}} d\tau\,.$$

21. Let $y_1(t)$ and $y_2(t)$ be a fundamental set of solutions, and $W(t) = W(y_1, y_2)$ be the corresponding Wronskian. Any solution, $u(t)$, of the homogeneous equation is

a linear combination $u(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$. Invoking the initial conditions, we require that

$$y_0 = \alpha_1 \, y_1(t_0) + \alpha_2 \, y_2(t_0)$$
$$y_0' = \alpha_1 \, y_1'(t_0) + \alpha_2 \, y_2'(t_0)$$

Note that this system of equations has a unique solution, since $W(t_0) \neq 0$. Now consider the nonhomogeneous problem, $L[v] = g(t)$, with homogeneous initial conditions. Using the method of variation of parameters, the particular solution is given by

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(s)} ds \,.$$

The general solution of the IVP (iii) is

$$v(t) = \beta_1 y_1(t) + \beta_2 y_2(t) + Y(t) = \beta_1 y_1(t) + \beta_2 y_2(t) + y_1(t) u_1(t) + y_2(t) u_2(t)$$

in which u_1 and u_2 are defined above. Invoking the initial conditions, we require that

$$0 = \beta_1 y_1(t_0) + \beta_2 y_2(t_0) + Y(t_0)$$
$$0 = \beta_1 y_1'(t_0) + \beta_2 y_2'(t_0) + Y'(t_0)$$

Based on the definition of u_1 and u_2, $Y(t_0) = 0$. Furthermore, since $y_1 u_1' + y_2 u_2' = 0$, it follows that $Y'(t_0) = 0$. Hence the only solution of the above system of equations is the trivial solution. Therefore $v(t) = Y(t)$. Now consider the function $y = u + v$. Then $L[y] = L[u+v] = L[u] + L[v] = g(t)$. That is, $y(t)$ is a solution of the nonhomogeneous problem. Further, $y(t_0) = u(t_0) + v(t_0) = y_0$, and similarly, $y'(t_0) = y_0'$. By the uniqueness theorems, $y(t)$ is the unique solution of the initial value problem.

23. A fundamental set of solutions is $y_1(t) = \cos t$ and $y_2(t) = \sin t$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = 1$. By the result in Problem 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{\cos(s)\, \sin(t) - \cos(t)\, \sin(s)}{W(s)} g(s) ds \\ &= \int_{t_0}^{t} [\cos(s)\, \sin(t) - \cos(t)\, \sin(s)]\, g(s) ds \,. \end{aligned}$$

Finally, we have $\cos(s)\, \sin(t) - \cos(t)\, \sin(s) = \sin(t-s)$.

24. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = e^{bt}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = (b-a) e^{(a+b)t}$. By the result in Problem 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{e^{as} e^{bt} - e^{at} e^{bs}}{W(s)} g(s) ds \\ &= \frac{1}{b-a} \int_{t_0}^{t} \frac{e^{as} e^{bt} - e^{at} e^{bs}}{e^{(a+b)s}} g(s) ds \,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a} \int_{t_0}^{t} \left[e^{b(t-s)} - e^{a(t-s)} \right] g(s) ds \,.$$

26. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$. By the result in Problem 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{te^{as+at} - s\,e^{at+as}}{W(s)} g(s)ds \\ &= \int_{t_0}^{t} \frac{(t-s)e^{as+at}}{e^{2as}} g(s)ds\,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \int_{t_0}^{t} (t-s)e^{a(t-s)} g(s)ds\,.$$

27. The form of the kernel depends on the characteristic roots. If the roots are real and distinct,

$$K(t-s) = \frac{e^{b(t-s)} - e^{a(t-s)}}{b-a}\,.$$

If the roots are real and identical,

$$K(t-s) = (t-s)e^{a(t-s)}\,.$$

If the roots are complex conjugates,

$$K(t-s) = \frac{e^{\lambda(t-s)}\ \sin\mu(t-s)}{\mu}\,.$$

28. Let $y(t) = v(t)y_1(t)$, in which $y_1(t)$ is a solution of the homogeneous equation. Substitution into the given ODE results in

$$v''y_1 + 2v'y_1' + vy_1'' + p(t)\left[v'y_1 + vy_1'\right] + q(t)vy_1 = g(t)\,.$$

By assumption, $y_1'' + p(t)y_1 + q(t)y_1 = 0$, hence $v(t)$ must be a solution of the ODE

$$v''y_1 + \left[2y_1' + p(t)y_1\right]v' = g(t)\,.$$

Setting $w = v'$, we also have $w'y_1 + \left[2y_1' + p(t)y_1\right]w = g(t)$.

30. First write the equation as $y'' + 7t^{-1}y + 5t^{-2}\,y = t^{-1}$. As shown in Problem 28, the function $y(t) = t^{-1}v(t)$ is a solution of the given ODE as long as v is a solution of

$$t^{-1}\,v'' + \left[-2t^{-2} + 7t^{-2}\right]v' = t^{-1}\,,$$

that is, $v'' + 5t^{-1}\,v' = 1$. This ODE is linear and first order in v'. The integrating factor is $\mu = t^5$. The solution is $v' = t/6 + c\,t^{-5}$. Direct integration now results in $v(t) = t^2/12 + c_1 t^{-4} + c_2$. Hence $y(t) = t/12 + c_1 t^{-5} + c_2 t^{-1}$.

31. Write the equation as $y'' - t^{-1}(1+t)y + t^{-1}\,y = t\,e^{2t}$. As shown in Problem 28, the function $y(t) = (1+t)v(t)$ is a solution of the given ODE as long as v is a solution of

$$(1+t)\,v'' + \left[2 - t^{-1}(1+t)^2\right]v' = t\,e^{2t},$$

that is, $v'' - \frac{1+t^2}{t(t+1)}\,v' = \frac{t}{t+1}e^{2t}$. This equation is first order linear in v', with integrating factor $\mu = t^{-1}(1+t)^2e^{-t}$. The solution is $v' = (t^2e^{2t} + c_1te^t)/(1+t)^2$. Integrating, we obtain $v(t) = e^{2t}/2 - e^{2t}/(t+1) + c_1e^t/(t+1) + c_2$. Hence the solution of the original ODE is $y(t) = (t-1)e^{2t}/2 + c_1e^t + c_2(t+1)$.

32. Write the equation as $y'' + t(1-t)^{-1}y - (1-t)^{-1}\,y = 2(1-t)\,e^{-t}$. The function $y(t) = e^tv(t)$ is a solution to the given ODE as long as v is a solution of

$$e^tv'' + \left[2e^t + t(1-t)^{-1}e^t\right]v' = 2(1-t)\,e^{-t},$$

that is, $v'' + [(2-t)/(1-t)]\,v' = 2(1-t)\,e^{-2t}$. This equation is first order linear in v', with integrating factor $\mu = e^t/(t-1)$. The solution is

$$v' = (t-1)(2e^{-2t} + c_1e^{-t}).$$

Integrating, we obtain $v(t) = (1/2 - t)e^{-2t} - c_1te^{-t} + c_2$. Hence the solution of the original ODE is $y(t) = (1/2-t)e^{-t} - c_1t + c_2e^t$.

3.7

1. $R\cos\,\delta = 3$ and $R\sin\,\delta = 4$, so $R = \sqrt{25} = 5$ and $\delta = \arctan(4/3)$. Hence

$$u = 5\,\cos(2t - 0.9273).$$

3. $R\cos\,\delta = 4$ and $R\sin\,\delta = -2$, so $R = \sqrt{20} = 2\sqrt{5}$ and $\delta = -\arctan(1/2)$. So

$$u = 2\sqrt{5}\,\cos(3t + 0.4636).$$

4. $R\cos\,\delta = -2$ and $R\sin\,\delta = -3$, so $R = \sqrt{13}$ and $\delta = \pi + \arctan(3/2)$. Hence

$$u = \sqrt{13}\,\cos(\pi t - 4.1244).$$

5. The spring constant is $k = 2/(1/2) = 4$ lb/ft. Mass $m = 2/32 = 1/16$ lb-s^2/ft. Since there is no damping, the equation of motion is

$$\frac{1}{16}u'' + 4u = 0\,,$$

that is, $u'' + 64u = 0$. The initial conditions are $u(0) = 1/4$ ft, $u'(0) = 0$ ft/s. The general solution is $u(t) = A\,\cos\,8t + B\,\sin\,8t$. Invoking the initial conditions, we have $u(t) = \frac{1}{4}\,\cos\,8t$. $R = 3$ inches, $\delta = 0$ rad, $\omega_0 = 8$ rad/s, and $T = \pi/4$ s.

7. The spring constant is $k = 3/(1/4) = 12$ lb/ft. Mass $m = 3/32$ lb-s^2/ft. Since there is no damping, the equation of motion is

$$\frac{3}{32}u'' + 12u = 0\,,$$

that is, $u'' + 128u = 0$. The initial conditions are $u(0) = -1/12$ ft, $u'(0) = 2$ ft/s. The general solution is $u(t) = A\cos 8\sqrt{2}\,t + B\sin 8\sqrt{2}\,t$. Invoking the initial conditions, we have

$$u(t) = -\frac{1}{12}\cos 8\sqrt{2}\,t + \frac{1}{4\sqrt{2}}\sin 8\sqrt{2}\,t\,.$$

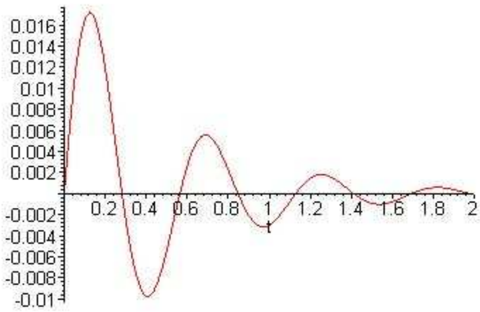
$R = \sqrt{11/288}$ ft, $\delta = \pi - \arctan(3/\sqrt{2})$ rad, $\omega_0 = 8\sqrt{2}$ rad/s, and $T = \pi/(4\sqrt{2})$ s.

10. The spring constant is $k = 16/(1/4) = 64$ lb/ft. Mass $m = 1/2$ lb-s^2/ft. The damping coefficient is $\gamma = 2$ lb-s/ft. Hence the equation of motion is

$$\frac{1}{2}u'' + 2u' + 64u = 0\,,$$

that is, $u'' + 4u' + 128u = 0$. The initial conditions are $u(0) = 0$ ft, $u'(0) = 1/4$ ft/s. The general solution is $u(t) = A\cos 2\sqrt{31}\,t + B\sin 2\sqrt{31}\,t$. Invoking the initial conditions, we have

$$u(t) = \frac{1}{8\sqrt{31}}e^{-2t}\sin 2\sqrt{31}\,t\,.$$



Solving $u(t) = 0$, on the interval $[0.2\,, 0.4]$, we obtain $t = \pi/2\sqrt{31} = 0.2821$ s. Based on the graph, and the solution of $u(t) = 0.01$, we have $|u(t)| \leq 0.01$ for $t \geq \tau = 0.2145$.

11. The spring constant is $k = 3/(.1) = 30$ N/m. The damping coefficient is given as $\gamma = 3/5$ N-s/m. Hence the equation of motion is

$$2u'' + \frac{3}{5}u' + 30u = 0\,,$$

that is, $u'' + 0.3u' + 15u = 0$. The initial conditions are $u(0) = 0.05$ m and $u'(0) = 0.01$ m/s. The general solution is $u(t) = A\cos\mu t + B\sin\mu t$, in which $\mu = 3.87008$ rad/s. Invoking the initial conditions, we have

$$u(t) = e^{-0.15t}(0.05\cos\mu t + 0.00452\sin\mu t)\,.$$

Also, $\mu/\omega_0 = 3.87008/\sqrt{15} \approx 0.99925$.

13. The frequency of the undamped motion is $\omega_0 = 1$. The quasi frequency of the damped motion is $\mu = \frac{1}{2}\sqrt{4-\gamma^2}$. Setting $\mu = \frac{2}{3}\omega_0$, we obtain $\gamma = \frac{2}{3}\sqrt{5}$.

14. The spring constant is $k = mg/L$. The equation of motion for an undamped system is $mu'' + \frac{mg}{L}u = 0$. Hence the natural frequency of the system is $\omega_0 = \sqrt{\frac{g}{L}}$. The period is $T = 2\pi/\omega_0$.

15. The general solution of the system is $u(t) = A\cos\gamma(t-t_0) + B\sin\gamma(t-t_0)$. Invoking the initial conditions, we have

$$u(t) = u_0\cos\gamma(t-t_0) + (u_0'/\gamma)\sin\gamma(t-t_0).$$

Clearly, the functions $v = u_0\cos\gamma(t-t_0)$ and $w = (u_0'/\gamma)\sin\gamma(t-t_0)$ satisfy the given criteria.

16. Note that $r\sin(\omega_0 t - \theta) = r\sin\omega_0 t\cos\theta - r\cos\omega_0 t\sin\theta$. Comparing the given expressions, we have $A = -r\sin\theta$ and $B = r\cos\theta$. That is, $r = R = \sqrt{A^2+B^2}$, and $\tan\theta = -A/B = -1/\tan\delta$. The latter relation is also $\tan\theta + \cot\delta = 1$.

18. The system is critically damped, when $R = 2\sqrt{L/C}$. Here $R = 1000$ ohms.

21.(a) Let $u = Re^{-\gamma t/2m}\cos(\mu t - \delta)$. Then attains a maximum when $\mu t_k - \delta = 2k\pi$. Hence $T_d = t_{k+1} - t_k = 2\pi/\mu$.

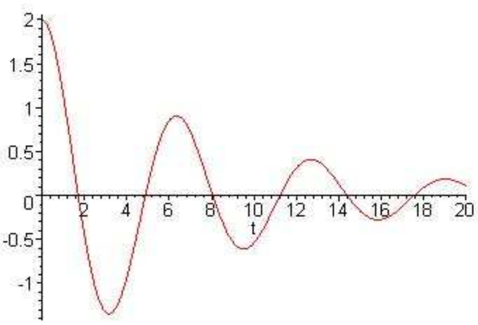
(b) $u(t_k)/u(t_{k+1}) = e^{-\gamma t_k/2m}/e^{-\gamma t_{k+1}/2m} = e^{(\gamma t_{k+1} - \gamma t_k)/2m}$. Hence

$$u(t_k)/u(t_{k+1}) = e^{\gamma(2\pi/\mu)/2m} = e^{\gamma T_d/2m}.$$

(c) $\Delta = \ln[u(t_k)/u(t_{k+1})] = \gamma(2\pi/\mu)/2m = \pi\gamma/\mu m$.

22. The spring constant is $k = 16/(1/4) = 64$ lb/ft. Mass $m = 1/2$ lb-s^2/ft. The damping coefficient is $\gamma = 2$ lb-s/ft. The quasi frequency is $\mu = 2\sqrt{31}$ rad/s. Hence $\Delta = \frac{2\pi}{\sqrt{31}} \approx 1.1285$.

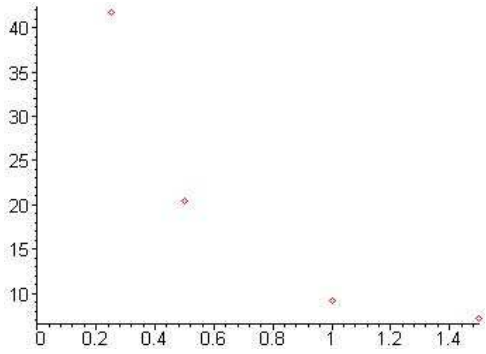
25.(a) The solution of the IVP is $u(t) = e^{-t/8}(2\cos\frac{3}{8}\sqrt{7}\,t + 0.252\sin\frac{3}{8}\sqrt{7}\,t)$.



Using the plot, and numerical analysis, $\tau \approx 41.715$.

(b) For $\gamma = 0.5$, $\tau \approx 20.402$; for $\gamma = 1.0$, $\tau \approx 9.168$; for $\gamma = 1.5$, $\tau \approx 7.184$.

(c)



(d) For $\gamma = 1.6$, $\tau \approx 7.218$; for $\gamma = 1.7$, $\tau \approx 6.767$; for $\gamma = 1.8$, $\tau \approx 5.473$; for $\gamma = 1.9$, $\tau \approx 6.460$. τ steadily decreases to about $\tau_{min} \approx 4.873$, corresponding to the critical value $\gamma_0 \approx 1.73$.

(e) We have $u(t) = \frac{4e^{-\gamma t/2}}{\sqrt{4-\gamma^2}} \cos(\mu t - \delta)$, where $\mu = \frac{1}{2}\sqrt{4-\gamma^2}$, $\delta = \tan^{-1} \frac{\gamma}{\sqrt{4-\gamma^2}}$. Hence $|u(t)| \leq \frac{4e^{-\gamma t/2}}{\sqrt{4-\gamma^2}}$.

26.(a) The characteristic equation is $mr^2 + \gamma r + k = 0$. Since $\gamma^2 < 4km$, the roots are $r_{1,2} = -\frac{\gamma}{2m} \pm i\frac{\sqrt{4mk-\gamma^2}}{2m}$. The general solution is

$$u(t) = e^{-\gamma t/2m}\left[A \cos \frac{\sqrt{4mk-\gamma^2}}{2m} t + B \sin \frac{\sqrt{4mk-\gamma^2}}{2m} t\right].$$

Invoking the initial conditions, $A = u_0$ and

$$B = \frac{(2mv_0 - \gamma u_0)}{\sqrt{4mk-\gamma^2}}.$$

(b) We can write $u(t) = R\,e^{-\gamma t/2m}\cos(\mu t - \delta)$, in which

$$R = \sqrt{u_0^2 + \frac{(2mv_0 - \gamma u_0)^2}{4mk-\gamma^2}},$$

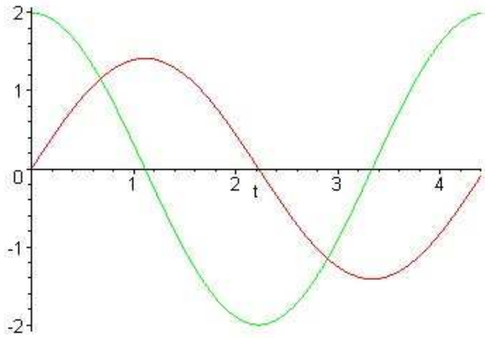
and

$$\delta = \arctan\left[\frac{(2mv_0 - \gamma u_0)}{u_0\sqrt{4mk-\gamma^2}}\right].$$

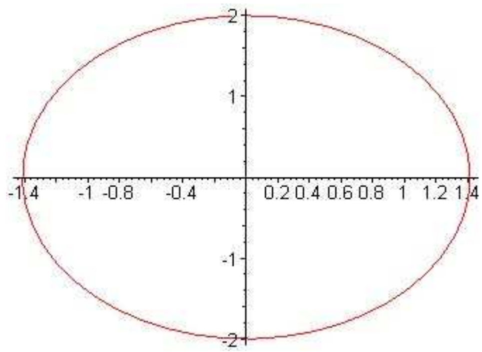
(c) $R = \sqrt{u_0^2 + \frac{(2mv_0-\gamma u_0)^2}{4mk-\gamma^2}} = 2\sqrt{\frac{m(ku_0^2+\gamma u_0 v_0+mv_0^2)}{4mk-\gamma^2}} = \sqrt{\frac{a+b\gamma}{4mk-\gamma^2}}$. It is evident that R increases (monotonically) without bound as $\gamma \to (2\sqrt{mk})^-$.

28.(a) The general solution is $u(t) = A\cos\sqrt{2}\,t + B\sin\sqrt{2}\,t$. Invoking the initial conditions, we have $u(t) = \sqrt{2}\,\sin\sqrt{2}\,t$.

(b)

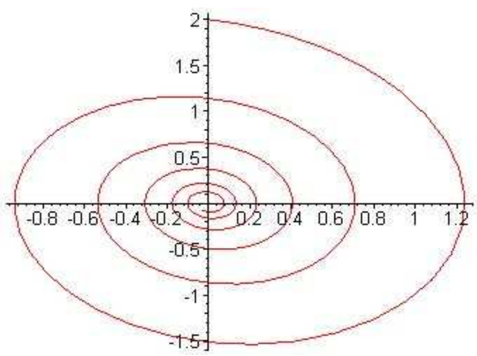


(c)



The condition $u'(0) = 2$ implies that $u(t)$ initially increases. Hence the phase point travels clockwise.

29. $u(t) = \frac{16}{\sqrt{127}} e^{-t/8} \sin\frac{\sqrt{127}}{8}t$.



31. Based on Newton's second law, with the positive direction to the right,

$$\sum F = mu''$$

where

$$\sum F = -ku - \gamma u'.$$

Hence the equation of motion is $mu'' + \gamma u' + ku = 0$. The only difference in this problem is that the equilibrium position is located at the unstretched configuration of the spring.

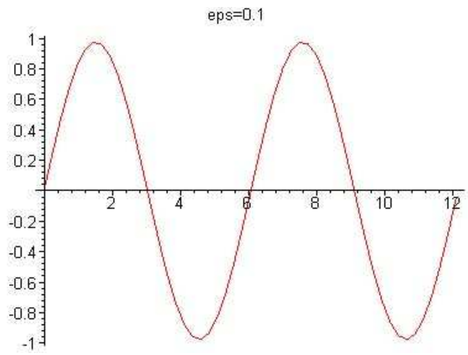
32.(a) The restoring force exerted by the spring is $F_s = -(ku + \epsilon u^3)$. The opposing viscous force is $F_d = -\gamma u'$. Based on Newton's second law, with the positive direction to the right,

$$F_s + F_d = mu''.$$

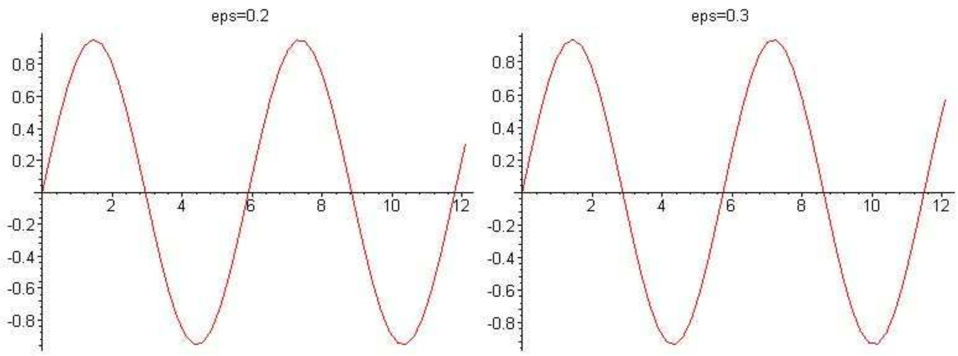
Hence the equation of motion is $mu'' + \gamma u' + ku + \epsilon u^3 = 0$.

(b) With the specified parameter values, the equation of motion is $u'' + u = 0$. The general solution of this ODE is $u(t) = A \cos t + B \sin t$. Invoking the initial conditions, the specific solution is $u(t) = \sin t$. Clearly, the amplitude is $R = 1$, and the period of the motion is $T = 2\pi$.

(c) Given $\epsilon = 0.1$, the equation of motion is $u'' + u + 0.1\,u^3 = 0$. A solution of the IVP can be generated numerically:

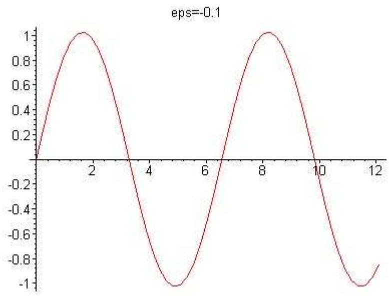


(d)

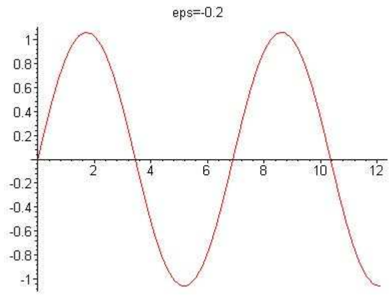


(e) The amplitude and period both seem to decrease.

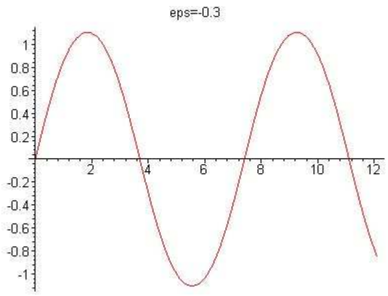
(f)



(a) $\varepsilon = -0.1$



(b) $\varepsilon = -0.2$



(c) $\varepsilon = -0.3$

3.8

2. We have

$$\sin(\alpha \pm \beta) = \sin\,\alpha\,\cos\,\beta\,\pm\,\cos\,\alpha\,\sin\,\beta\,.$$

Subtracting the two identities, we obtain

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\,\cos\,\alpha\,\sin\,\beta\,.$$

Setting $\alpha + \beta = 7t$ and $\alpha - \beta = 6t$, we get that $\alpha = 6.5t$ and $\beta = 0.5t$. This implies that $\sin\,7t - \sin\,6t = 2\,\sin\,(t/2)\cos\,(13t/2)$.

3. Consider the trigonometric identity

$$\cos(\alpha \pm \beta) = \cos\,\alpha\,\cos\,\beta \mp \sin\,\alpha\,\sin\,\beta\,.$$

Adding the two identities, we obtain $\cos(\alpha - \beta) + \cos(\alpha + \beta) = 2\,\cos\,\alpha\,\cos\,\beta$. Comparing the expressions, set $\alpha + \beta = 2\pi t$ and $\alpha - \beta = \pi t$. This means $\alpha = 3\pi t/2$ and $\beta = \pi t/2$. Upon substitution, we have $\cos(\pi t) + \cos(2\pi t) = 2\,\cos(3\pi t/2)\,\cos(\pi t/2)$.

4. Adding the two identities $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$, it follows that $\sin(\alpha-\beta)+\sin(\alpha+\beta) = 2\sin\alpha\cos\beta$. Setting $\alpha+\beta = 4t$ and $\alpha-\beta = 3t$, we have $\alpha = 7t/2$ and $\beta = t/2$. Hence $\sin 3t + \sin 4t = 2\sin(7t/2)\cos(t/2)$.

6. Using MKS units, the spring constant is $k = 5(9.8)/0.1 = 490$ N/m, and the damping coefficient is $\gamma = 2/0.04 = 50$ N-s/m. The equation of motion is

$$5u'' + 50u' + 490u = 10\sin(t/2)\,.$$

The initial conditions are $u(0) = 0$ m and $u'(0) = 0.03$ m/s.

8.(a) The homogeneous solution is $u_c(t) = Ae^{-5t}\cos\sqrt{73}\,t + Be^{-5t}\sin\sqrt{73}\,t$. Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{1}{153281}\left[-160\cos(t/2) + 3128\sin(t/2)\right].$$

Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that

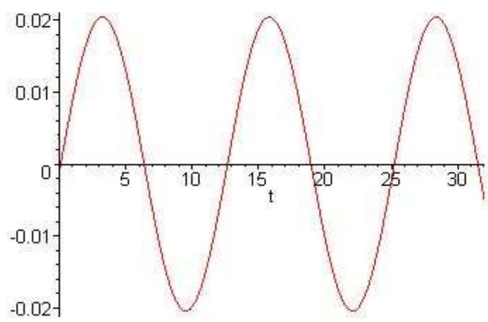
$$A = 160/153281 \text{ and } B = 383443\sqrt{73}\,/1118951300\,.$$

Hence the response is

$$u(t) = \frac{1}{153281}\left[160\,e^{-5t}\cos\sqrt{73}\,t + \frac{383443\sqrt{73}}{7300}e^{-5t}\sin\sqrt{73}\,t\right] + U(t).$$

(b) $u_c(t)$ is the transient part and $U(t)$ is the steady state part of the response.

(c)



(d) The amplitude of the forced response is given by $R = 2/\Delta$, in which

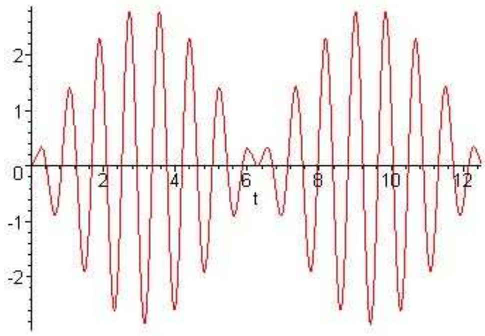
$$\Delta = \sqrt{25(98-\omega^2)^2 + 2500\,\omega^2}\,.$$

The maximum amplitude is attained when Δ is a minimum. Hence the amplitude is maximum at $\omega = 4\sqrt{3}$ rad/s.

9. The spring constant is $k = 12$ lb/ft and hence the equation of motion is

$$\frac{6}{32}u'' + 12u = 4\cos 7t\,,$$

that is, $u'' + 64u = \frac{64}{3}\cos 7t$. The initial conditions are $u(0) = 0$ ft, $u'(0) = 0$ ft/s. The general solution is $u(t) = A\cos 8t + B\sin 8t + \frac{64}{45}\cos 7t$. Invoking the initial conditions, we have $u(t) = -\frac{64}{45}\cos 8t + \frac{64}{45}\cos 7t = \frac{128}{45}\sin(t/2)\sin(15t/2)$.



12. The equation of motion is

$$2u'' + u' + 3u = 3\cos 3t - 2\sin 3t\,.$$

Since the system is damped, the steady state response is equal to the particular solution. Using the method of undetermined coefficients, we obtain

$$u_{ss}(t) = \frac{1}{6}(\sin 3t - \cos 3t).$$

Further, we find that $R = \sqrt{2}/6$ and $\delta = \arctan(-1) = 3\pi/4$. Hence we can write $u_{ss}(t) = \frac{\sqrt{2}}{6}\cos(3t - 3\pi/4)$.

13.(c) The amplitude of the steady-state response is given by

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2}}\,.$$

Since F_0 is constant, the amplitude is maximum when the denominator of R is minimum. Let $z = \omega^2$, and consider the function $f(z) = m^2(\omega_0^2 - z)^2 + \gamma^2 z$. Note that $f(z)$ is a quadratic, with minimum at $z = \omega_0^2 - \gamma^2/2m^2$. Hence the amplitude R attains a maximum at $\omega_{max}^2 = \omega_0^2 - \gamma^2/2m^2$. Furthermore, since $\omega_0^2 = k/m$, and therefore

$$\omega_{max}^2 = \omega_0^2\left[1 - \frac{\gamma^2}{2km}\right].$$

Substituting $\omega^2 = \omega_{max}^2$ into the expression for the amplitude,

$$R = \frac{F_0}{\sqrt{\gamma^4/4m^2 + \gamma^2\,(\omega_0^2 - \gamma^2/2m^2)}} =$$

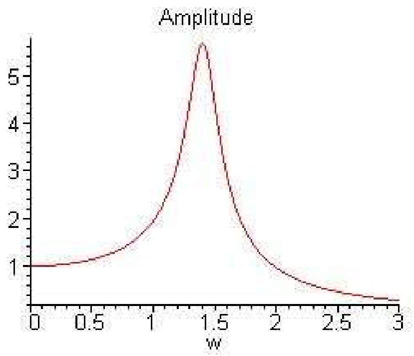
$$= \frac{F_0}{\sqrt{\omega_0^2\gamma^2 - \gamma^4/4m^2}} = \frac{F_0}{\gamma\omega_0\sqrt{1 - \gamma^2/4mk}}\,.$$

17.(a) Here $m = 1$, $\gamma = 0.25$, $\omega_0^2 = 2$, $F_0 = 2$. Hence $u_{ss}(t) = \frac{2}{\Delta}\cos(\omega t - \delta)$, where $\Delta = \sqrt{(2 - \omega^2)^2 + \omega^2/16} = \frac{1}{4}\sqrt{64 - 63\omega^2 + 16\,\omega^4}$, and $\tan\delta = \frac{\omega}{4(2-\omega^2)}$.

(b) The amplitude is

$$R = \frac{8}{\sqrt{64 - 63\omega^2 + 16\,\omega^4}}\,.$$

(c)



(d) See Problem 13. The amplitude is maximum when the denominator of R is minimum. That is, when $\omega = \omega_{max} = 3\sqrt{14}/8 \approx 1.4031$. Hence $R(\omega = \omega_{max}) = 64/\sqrt{127}$.

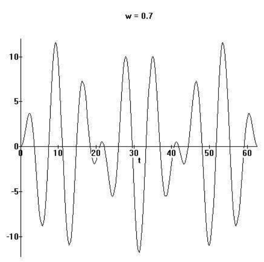
18.(a) The homogeneous solution is $u_c(t) = A\cos\, t + B\sin\, t$. Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{3}{1-\omega^2}\,\cos\,\omega t\,.$$

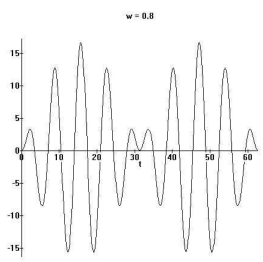
Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = 3/(\omega^2 - 1)$ and $B = 0$. Hence the response is

$$u(t) = \frac{3}{1-\omega^2}\,[\,\cos\,\omega t - \cos\, t\,]\,.$$

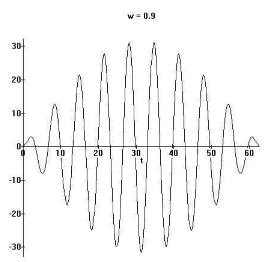
(b)



(a) $\omega = 0.7$



(b) $\omega = 0.8$



(c) $\omega = 0.9$

Note that

$$u(t) = \frac{6}{1-\omega^2}\,\sin\left[\frac{(1-\omega)t}{2}\right]\sin\left[\frac{(\omega+1)t}{2}\right].$$

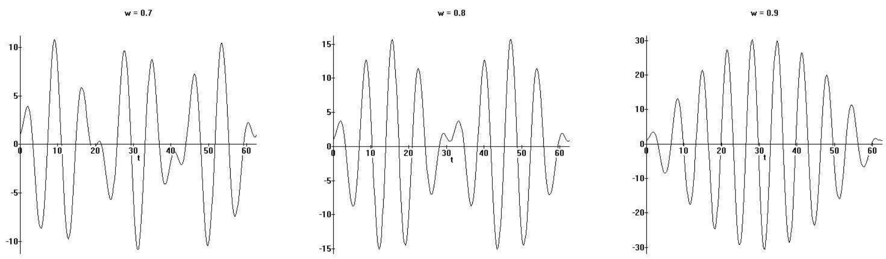
19.(a) The homogeneous solution is $u_c(t) = A\cos t + B\sin t$. Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{3}{1-\omega^2}\,\cos\omega t\,.$$

Hence the general solution is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = (\omega^2+2)/(\omega^2-1)$ and $B = 1$. Hence the response is

$$u(t) = \frac{1}{1-\omega^2}\left[\,3\,\cos\omega t - (\omega^2+2)\cos t\,\right] + \sin t\,.$$

(b)

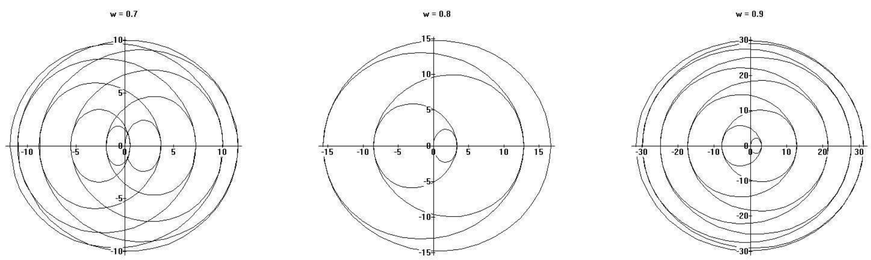


(a) $\omega = 0.7$ (b) $\omega = 0.8$ (c) $\omega = 0.9$

Note that

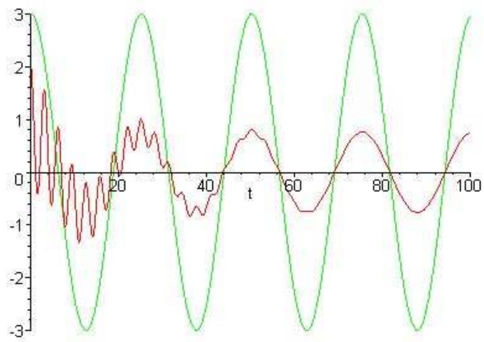
$$u(t) = \frac{6}{1-\omega^2}\,\sin\left[\frac{(1-\omega)t}{2}\right]\sin\left[\frac{(\omega+1)t}{2}\right] + \cos t + \sin t\,.$$

20.

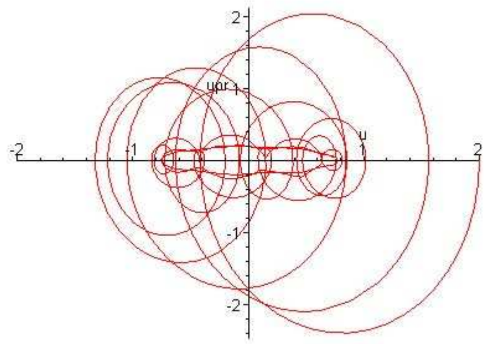


(a) $\omega = 0.7$ (b) $\omega = 0.8$ (c) $\omega = 0.9$

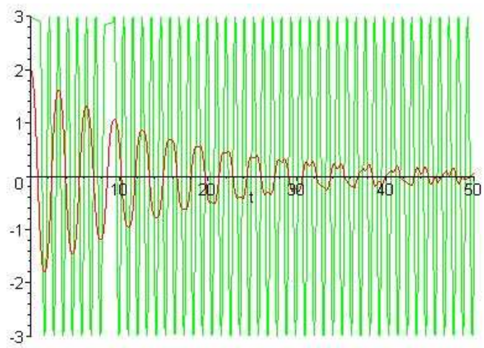
21.(a)



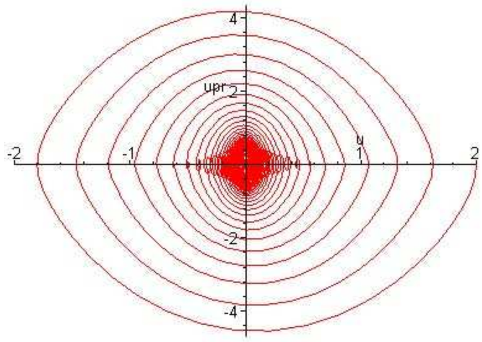
(b) Phase plot - u' vs u :



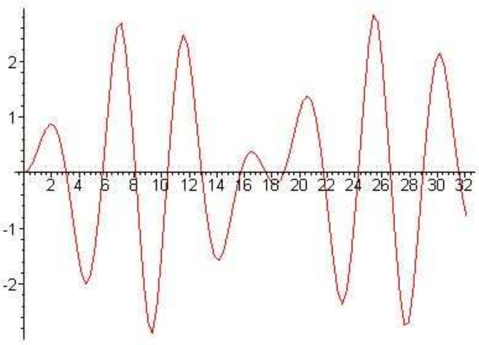
23.(a)



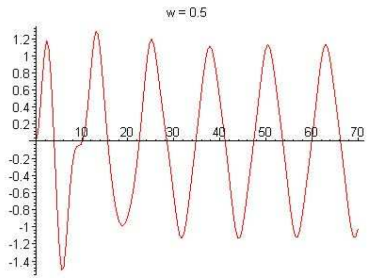
(b) Phase plot - u' vs u :



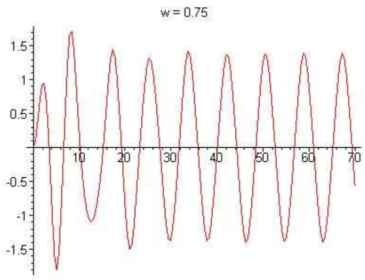
24.



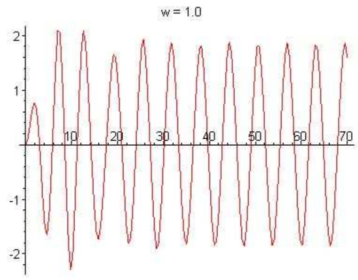
25.(a)



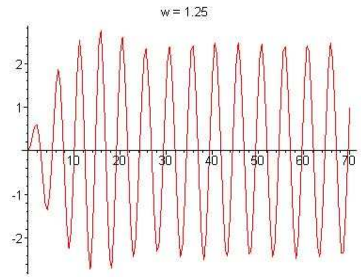
(a) $\omega = 0.5$



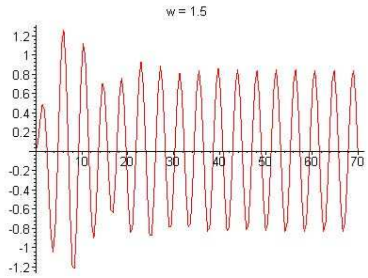
(b) $\omega = 0.75$



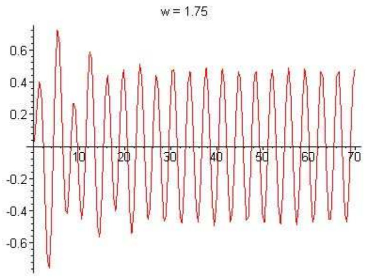
(c) $\omega = 1$



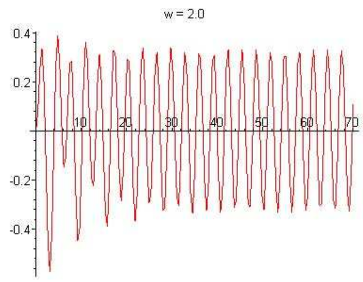
(d) $\omega = 1.25$



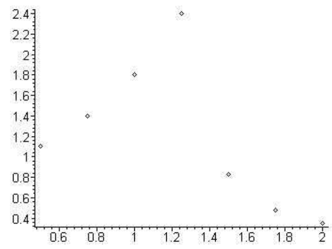
(e) $\omega = 1.5$



(f) $\omega = 1.75$



(g) $\omega = 2$



(h) A vs ω

(c) The amplitude for a similar system with a linear spring is given by

$$R = \frac{5}{\sqrt{25 - 49\omega^2 + 25\omega^4}} \, .$$