Aula 18 Cálculo 3B | Aula 27 Cálculo 3A

Aula passada Eq de ordem n

Aula Hoje transformada de Laplace

Cap 6 Transformada de Laplace

↳ Resolver EDO's

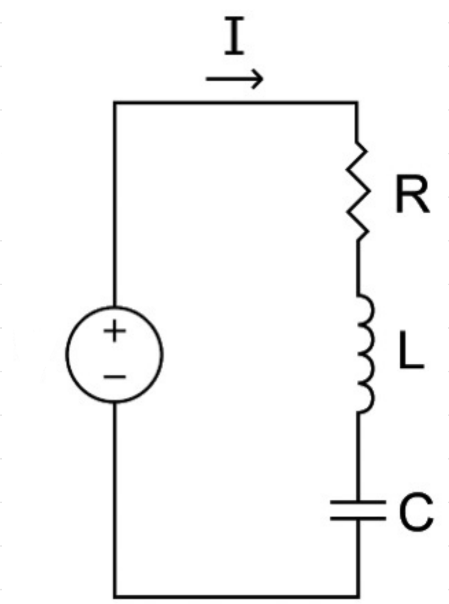
$$y'' + p(t)y' + q(t)y = G(t)$$

onde $G(t)$ é descontínua

↳ achar uma solução no sentido fraco

a função pode ser descontínua de 1ª espécie

Motivação Circuito RLC



$I(t)$ corrente elétrica no circuito

R (ohms) resistência → constante positiva

L (henry) indutância → constante positiva

$E(t)$ (volts) Tensão aplicada

$Q(t)$ (coulombs) carga total no capacitor

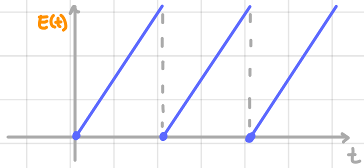
$$\boxed{I = \frac{dQ}{dt}}$$

Lei de Kirchhoff Tensão aplicada = a soma das quedas de tensão

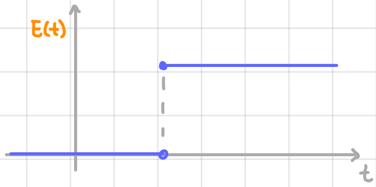
- queda de tensão no resistor $= RI$
- queda de tensão no capacitor $= \frac{1}{C}\cdot Q$
- queda de tensão no indutor $= L\cdot\frac{dI}{dt}$

obtemos $\boxed{LQ'' + RQ' + \frac{1}{C}Q = E(t)}$ linear de coeficientes constantes

O adicional aqui é que a tensão aplicada podem ser descontínua por exemplo:

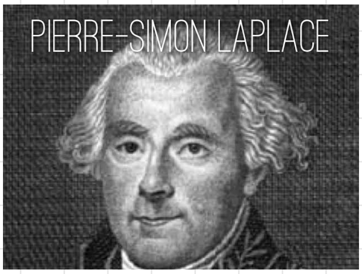


"dente de serra"



"função degrau"

6.1 Definição da transformada de Laplace



Definição Seja $f(t)$ uma função definimos a Transformada de Laplace a função

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\cdot f(t)\,dt}$$

$F(s)$ está definida para os valores de s tal que a integral converge

Vamos calcular algumas transformadas

Exemplo Calcule $\mathcal{L}\{f(t)\}$ das seguintes funções

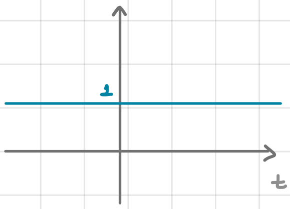
a) $f(t) = 1$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\cdot 1\,dt = \lim_{u\to\infty}\int_0^u e^{-st}\,dt$$

$$= \lim_{u\to\infty} \frac{e^{-st}}{-s}\Bigg|_0^u = \lim_{u\to\infty}\frac{e^{-su}}{-s} + \frac{1}{s}$$

$e^{-su} \to 0$ quando $s>0$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}\ ,\ s>0}$$



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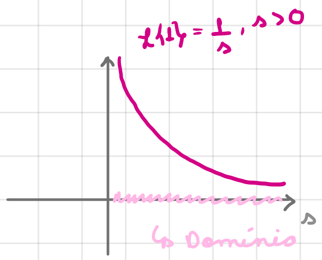


$\mathcal{L}\{1\}$

Transforma a função
$f(t) \equiv 1$ na função
$F(s) = \frac{1}{s}$, $s>0$

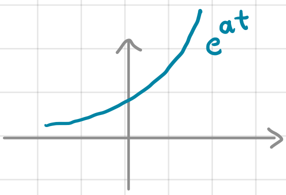
b) $f(t) = e^{at}$

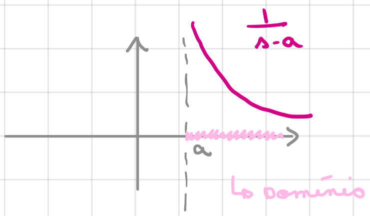
$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-s.t}e^{at}dt = \lim_{u\to\infty}\int_0^u e^{-(s-a)t}.dt$$

$$= \lim_{u\to\infty}\frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^u = \lim_{u\to\infty}\frac{e^{-(s-a)u}}{-(s-a)} + \frac{1}{s-a}$$

→ 0 quando $s>a$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}\ ,\ s>a}$$





c) $f(t) = \cos at$

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}.\cos at.dt$$

$$= \lim_{u\to\infty}\underbrace{\int_0^u e^{-st}\cos at\, dt}_{*}$$

$$* \int \underbrace{e^{-st}}_{dv}\underbrace{\cos at}_{u}\, dt$$

$u = \cos at$ $\quad dv = e^{-st}.dt$
$du = -a\,\text{sen}\,at$ $\quad v = \frac{1}{-s}e^{-st}$

$$= -\frac{1}{s}e^{-st}\cos at - \int \frac{a}{s}e^{-st}\underbrace{\text{sen}\,at}_{u}.dt$$

$u = \text{sen}\,at$
$du = a\cos t$
$dv = \frac{a}{s}e^{-st}.dt$
$v = -\frac{a}{s^2}e^{-st}.dt$

$$= -\frac{1}{s}e^{-st}\cos at + \frac{a}{s^2}e^{-st}\,\text{sen}\,at - \int \frac{a^2}{s^2}e^{-st}\cos at.dt$$

$$\left(1+\frac{a^2}{s^2}\right)\int e^{-st}\cos at.dt = \frac{a}{s^2}e^{-st}\,\text{sen}\,at - \frac{1}{s}e^{-st}\cos at$$

$$\int e^{-st}\cos at.dt = \frac{s^2}{s^2+a^2}\cdot\left[\frac{a}{s}e^{-st}\,\text{sen}\,at - \frac{1}{s}e^{-st}\cos at\right]$$

Voltando

$$\mathcal{L}\{\cos at\} = \lim_{u\to\infty}\left[\frac{s^2}{s^2+a}\left[\frac{a}{s}e^{-su}\,\text{sen}\,at - \frac{1}{s}e^{-st}\cos at\right] + \frac{s}{s^2+a^2}\right]$$

<0 $\quad <0$

quando $s>0$

$$\boxed{\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}\ ,\ s>0}$$

d) $f(x) = \text{sen}\,at$ Exercício

$$\boxed{\mathcal{L}\{\text{sen}\,at\} = \frac{a}{s^2+a^2}\ ,\ s>0}$$

e) $f(t) = t^n$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}.t\,dt = \lim_{u\to\infty}\int_0^u \underbrace{e^{-st}}_{dv}.\underbrace{t}_{u}dt$$ por partes

$$= \lim_{u\to\infty}\left(\frac{e^{-st}}{-s}\Big|_0^u - \int_0^u e^{-st}.dt\right)$$

$$= \lim_{u\to\infty}\frac{e^{-us}}{-s} + \frac{e^{-st}}{-s^2}\Big|_0^u = \frac{1}{s^2}$$

$\to 0,\ s>0$

$$\mathcal{L}\{t^2\} = \lim_{u\to\infty}\left(\int_0^u \underbrace{e^{-st}}_{dv}.\underbrace{t^2}_{u}dt\right)$$

$$= \lim_{u\to\infty}\frac{e^{-st}}{-s}.t\Big|_0^u + \int_0^u e^{-st}.2t.dt = \frac{2}{s^3}$$

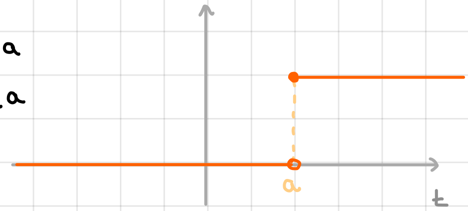
Em geral

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}\ ,\ s>0}$$

f) (Função Degrau)

$$u_a(t) = \begin{cases} 0 & t<a \\ 1 & t\geq a \end{cases}$$

↳ notação



Calculando a Transformada de Laplace

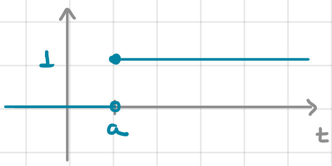
$$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st}\cdot u_a(t)\,dt$$

$$= \int_0^a e^{-st}\cdot \cancelto{0}{u_a(t)}\,dt + \int_a^\infty e^{-st}\cdot \cancelto{1}{u_a(t)}\,dt$$

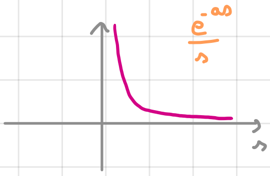
$u>0$

$$= \lim_{u\to\infty} \frac{\cancelto{0}{e^{-su}}}{-s} + \frac{e^{-sa}}{s} = \frac{e^{-sa}}{s}$$

$$\boxed{\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}, \quad s>0}$$



$u_a(t)$



$\mathcal{L}(u_a(t))$