**Aula 22** Cálculo 3B 2021.2

Aula passada: transformada inversa

Aula Hoje: Aplicação em EDO, função degrau

## 6.2 Solução de Problemas de Valor inicial

**Propriedade 5** (teorema 6.2.1)

$$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)}$$

Solução: Suponha $f'$ contínua

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,ds \quad \text{por partes} \quad \left|\begin{array}{l} u = e^{-st} \\ du = -se^{-st} \\ dv = f'(t)dt \\ v = f(t)\end{array}\right.$$

$$= \lim_{z\to\infty} \int_0^z e^{-st} f'(t)\,dt$$

$$= \lim_{z\to\infty} e^{-st} f(t)\Big|_0^z - \int_0^z -se^{-st} f(t)\,dt$$

$$= \lim_{z\to\infty} \cancel{e^{-sz} f(z)} - f(0) + \underbrace{\lim_{z\to\infty}\int_0^z e^{-st} f(t)\,dt}_{\mathcal{L}\{f(t)\}}$$

$$\mathcal{L}\{f'(t)\} = \mathcal{L}\{f(t)\} - f(0) \quad \blacksquare$$

**Consequência:**

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

$$\mathcal{L}\{f''(t)\} = s\left[s\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$

$$\boxed{\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)}$$

↳ a transformada de Laplace "quebra" as derivadas

Em geral

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

**Exemplo** Resolva por Transformada de Laplace

$$\begin{cases} y'' + y = \operatorname{sen} 2t \\ y(0) = 2 \\ y'(0) = 1 \end{cases}$$

Solução

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\operatorname{sen} 2t\}$$

$$s^2\mathcal{L}\{y(t)\} - s\underbrace{y(0)}_{=2} - \underbrace{y'(0)}_{=1} + \mathcal{L}\{y(t)\} = \frac{2}{s^2+4}$$

$$(s^2+1)\mathcal{L}\{y(t)\} = \frac{2}{s^2+4} + 2s + 1$$

$$\mathcal{L}\{y(t)\} = \frac{2 + (s^2+4)(2s+1)}{(s^2+4).(s^2+1)}$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{2 + (s^2+4)(2s+1)}{(s^2+4).(s^2+1)}\right\}}_{\text{resolver esta transformada inversa}}$$

(*) frações parciais

$$\frac{2 + (s^2+4)(2s+1)}{(s^2+4).(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

porque não tem raízes reais

$$\frac{2s^3+s^2+8s+6}{(s^2+4).(s^2+1)} = \frac{(As+B)(s^2+1) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+1)}$$

$$= \frac{(A+C)s^3 + (B+D)s^2 + (A+4C)s + B+4D}{(s^2+4).(s^2+1)}$$

$$\begin{cases} A+C = 2 \\ B+D = 1 \\ A+4D = 8 \\ B+4D = 6 \end{cases} \qquad A = 0 \quad B = -2/3 \qquad C = 2 \quad D = 5/3$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2 + (s^2+4)(2s+1)}{(s^2+4).(s^2+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-2/3}{s^2+4} + \frac{2s}{s^2+1} + \frac{5/3}{s^2+1}\right\}$$

$$= -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$y(t) = -\frac{1}{3}\operatorname{sen} 2t + 2\cos t + \frac{5}{3}\operatorname{sen} t$$

## 6.3 Função degrau

Como antes — função degrau

$$\boxed{u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}}$$

↳ Notação

**Objetivo** Escrever funções descontínua com uso da função degrau

Por que fazer isso?

R: • modelos de eletrônica envolvem funções descontínua no termo não homogêneo $G(t)$

• $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$ $\quad s>0$

• $\mathcal{L}\{u_c(t).f(t-c)\} = e^{-cs}.\mathcal{L}\{f(t)\}$

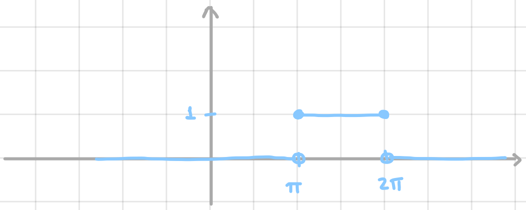
**Exemplo** Esboce a função $h(t) = u_\pi(t) - u_{2\pi}(t)$

Solução

$$u_\pi(t) = \begin{cases} 0, & t<\pi \\ 1, & t\geq \pi \end{cases}$$

$$u_{2\pi}(t) = \begin{cases} 0, & t<2\pi \\ 1, & t\geq 2\pi \end{cases}$$

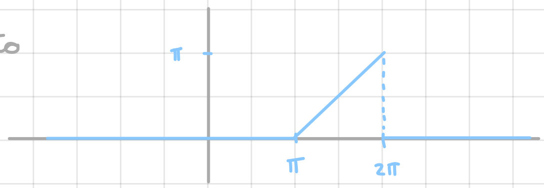
$$h(t) = \begin{cases} 0, & t<\pi \\ 1, & \pi\leq t\leq 2\pi \\ 0, & t>2\pi \end{cases}$$



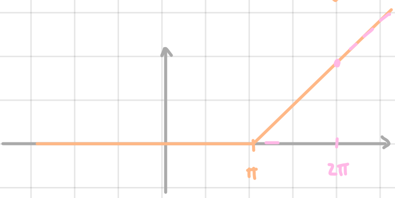

**Exemplo** Escreva em termos de função degrau e calcule $\mathcal{L}\{f(t)\}$

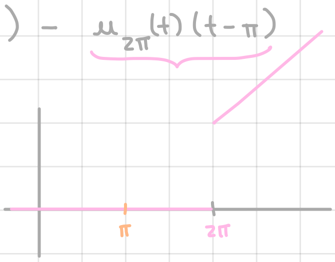
$$f(t) = \begin{cases} 0, & t<\pi \\ t-\pi, & \pi\leq t\leq 2\pi \\ 0, & t>2\pi \end{cases}$$

Solução




$$f(t) = \underbrace{u_\pi(t)(t-\pi)} - \underbrace{u_{2\pi}(t)(t-\pi)}$$




$$\boxed{\mathcal{L}\{u_c(t).f(t-c)\} = e^{-cs}\,\mathcal{L}\{f(t)\}}$$

$$f(t) = \underbrace{u_\pi(t)(t-\pi)}_{\text{OK}} - u_{2\pi}(t)(t-\pi \; -\pi+\pi)$$

$$- u_{2\pi}(t-2\pi) - \pi.u_{2\pi}(t)$$

$$\boxed{\mathcal{L}\{f\} = e^{-\pi s}.\frac{1}{s^2} - e^{-2\pi s}.\frac{1}{s^2} - \pi\frac{e^{-2\pi s}}{s}}$$

**Em geral:**

$$\boxed{u_a(t).f(t) = \begin{cases} 0, & t<a \\ f(t), & t\geq 0 \end{cases}}$$

**Exemplo** Escreva em termos de função degrau e calcule $\mathcal{L}\{f(t)\}$.

$$f(t) = \begin{cases} t^2 & 0\leq t<1 \\ 2t & t\geq 1 \end{cases}$$

Solução


$$f(t) = \underbrace{t^2 - u_1(t)t^2}_{\text{pedaço parábola}} + \underbrace{u_1(t)2t}_{\text{pedaço de reta}}$$

$$f(t) = t^2 - u_1(t)(t-1+1)^2 + u_1(t)2(t-1+1)$$

$u^2+2u+1$

$$= t^2 - u_1(t)\left[(t-1)^2 + 2(t-1) + 1\right]$$

$$+ u_1(t)\left[2(t-1) + 1\right]$$

$2u+1$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} - e^{-s}.\mathcal{L}\{u^2+2u+1\} + e^{-s}\mathcal{L}\{2u+1\}$$

$$= \frac{2}{s^3} - e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right] + e^{-s}\left[\frac{2}{s^2} + \frac{1}{s}\right]$$

$$= \frac{2}{s^3} - \frac{2e^{-s}}{s^3}$$

## 6.4 $\underbrace{\text{Forçamento}}_{}$ descontínuo

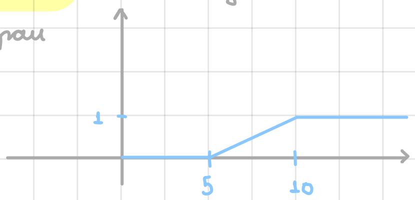
termo não homogêneo

**Exemplo** Resolva
$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

onde
$$g(t) = \begin{cases} 0, & 0 \le t < 5 \\ \dfrac{(t-5)}{5}, & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases}$$

**Solução:**

**Passo 1:** Escrever $g(t)$ em termos de funções degrau




$$g(t) = u_5(t).\frac{(t-5)}{5} - u_{10}(t)\frac{(t-5)}{5} + u_{10}(t)$$

Recorde:
$$\mathcal{L}\{u_c(t).f(t-c)\} = e^{-cs}.\ \mathcal{L}\{f(t)\}$$

fazendo aparecer a translação:

$$g(t) = \frac{1}{5}u_5(t).(t-5) - \frac{1}{5}u_{10}(t)(t-5-5+5) + u_{10}(t)$$

$$= \frac{1}{5}u_5(t).(t-5) - \frac{1}{5}u_{10}(t)(t-10) - \cancel{\frac{1}{5}u_{10}(t).5} + \cancel{u_{10}(t)}$$

**Passo 2:** Aplicando a transformada

$$\mathcal{L}\{y''+4y\} = \frac{1}{5}\mathcal{L}\{u_5(t).(t-5)\} - \frac{1}{5}\mathcal{L}\{u_{10}(t)(t-10)\}$$

$$s^2\mathcal{L}\{y\} - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,0} + 4\mathcal{L}\{y\} = \frac{1}{5}\left[\frac{1}{s^2}.e^{-5s} - \frac{1}{s^2}e^{-10s}\right]$$

$$\mathcal{L}\{y\} = \frac{1}{5}.\frac{1}{(s^2+4).s^2}.\left(e^{-5s} - e^{-10s}\right)$$

**Passo 3:** Calcular a transformada inversa

$$y(t) = \frac{1}{5}.\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(s^2+4)s^2}}_{(*)\ \text{frações parciais}}.\left(e^{-5s} - e^{10s}\right)\right\}$$

porque $s^2+4$ não tem raízes reais

$$(*)\quad \frac{1}{(s^2+4)s^2} = \frac{As+B}{s^2+4} + \frac{C}{s^2}$$

$$= \frac{As^3 + Bs^2 + Cs^2 + 4C}{(s^2+4)s^2}$$

$$\Rightarrow \left.\begin{matrix} A = 0 \\ B + C = 0 \\ 4C = 1 \end{matrix}\right\} \Rightarrow \begin{matrix} C = \frac{1}{4} \\ B = -\frac{1}{4} \end{matrix}$$

$$y(t) = \frac{1}{5}.\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)s^2}.\left(e^{-5s} - e^{10s}\right)\right\}$$

$$= \frac{1}{20}.\mathcal{L}^{-1}\left\{\left(-\frac{1}{s^2+4} + \frac{1}{s^2}\right)\left(e^{-5s} - e^{-10s}\right)\right\}$$

$$= \frac{1}{20}.\mathcal{L}^{-1}\left\{-\frac{1}{s^2+4}.e^{-5s} + \frac{1}{s^2}.e^{-5s} + \frac{1}{s^2+4}.e^{-10s} - \frac{1}{s^2}.e^{-10s}\right\}$$

$$= -\frac{1}{20}\frac{1}{2}\mathcal{L}^{-1}\left\{\underbrace{\frac{2}{s^2+4}.e^{-5s}}_{\mathcal{L}\{\text{sen}\,2t\}.e^{-5s}}\right\} + \frac{1}{20}\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2}e^{-5s}}_{\mathcal{L}\{t\}.e^{-5s}}\right\} + \frac{1}{20.2}\mathcal{L}^{-1}\left\{\underbrace{\frac{2}{s^2+4}.e^{-10s}}_{\mathcal{L}\{\text{sen}\,2t\}.e^{-10s}}\right\}$$

$$-\frac{1}{20}.\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2}.e^{-10s}}_{\mathcal{L}\{t\}.e^{-10s}}\right\}$$

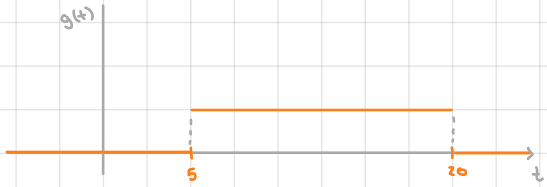
$$\boxed{y(t) = -\frac{1}{40}u_5(t)\,\text{sen}(2(t-5)) + \frac{1}{20}u_5(t)(t-5) + \frac{1}{40}u_{10}(t)\,\text{sen}(2(t-10)) - \frac{1}{20}u_{10}(t)(t-10)}$$

**Exemplo** Resolva $2y'' + y' + 2y = g(t)$ onde
$y(0) = y'(0) = 0$

$$g(t) = \begin{cases} 1 & 5 \le t < 20 \\ 0, & 0 \le t < 5 \text{ e } t \ge 20 \end{cases}$$

Solução:

Escrever $g(t)$ em termos de função degrau




$$g(t) = u_5(t) - u_{20}(t)$$

$$\mathcal{L}\{2y'' + y' + 2y\} = \mathcal{L}\{u_5(t) - u_{20}(t)\}$$

$$2s^2\mathcal{L}\{y\} - 2s\cancel{y(0)} - 2\cancel{y'(0)} + s\mathcal{L}\{y\} - \cancel{y(0)} + 2\mathcal{L}\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$(2s^2 + s + 2)\mathcal{L}\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$\mathcal{L}\{y\} = \underbrace{\frac{1}{(2s^2+s+2)s}} \cdot (e^{-5s} - e^{-20s})$$

↳ frações parciais sem raízes reais

$$\frac{1}{(2s^2+s+2)s} = \frac{As+B}{2s^2+s+2} + \frac{C}{s} = \frac{As^2 + Bs + 2Cs^2 + Cs + 2C}{(2s^2+s+2)s}$$

$$\Rightarrow \begin{cases} A + 2C = 0 \\ B + C = 0 \\ 2C = 1 \end{cases} \Rightarrow \begin{cases} C = \frac{1}{2} \\ B = -\frac{1}{2} \\ A = -1 \end{cases}$$

$$\mathcal{L}\{y\} = \left(-\frac{s+\frac{1}{2}}{2s^2+s+2} + \frac{1}{2}\cdot\frac{1}{s}\right)(e^{-5s} - e^{-20s})$$

$$y(t) = -\overbrace{\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{2s^2+s+2}\cdot e^{-5s}\right\}}^{*} + \overbrace{\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{2s^2+s+2}\cdot e^{-20s}\right\}}^{**}$$

$$+ \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot e^{-5s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot e^{-20s}\right\}$$

(*) $2s^2 + s + 2 = 2\left(s^2 + \frac{1}{2}s + 1\right) = 2\left[\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}\right]$

(*) $$\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{2s^2+s+2}\cdot e^{-5s}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\overset{\frac{1}{4}+\frac{1}{4}}{\frac{1}{2}}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}}\cdot e^{-5s}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}}\cdot e^{-5s}\right\}$$

$$= \frac{1}{2}\underbrace{\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \underbrace{\frac{15}{16}}_{=a^2}}\cdot e^{-5s}\right\}} + \frac{4}{\sqrt{15}}\frac{1}{8}\underbrace{\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2 + \underbrace{\frac{15}{16}}_{=a^2}}\cdot e^{-5s}\right\}}_{(※)}$$

linha 10 ← $\mathcal{L}\left\{e^{-\frac{1}{4}t}\cdot\cos\left(\frac{\sqrt{15}}{4}\right)t\right\}\cdot e^{-5s}$

$$\boxed{\mathcal{L}^{-1}\{F(s)\cdot e^{-cs}\} = u_c(t)\cdot f(t-c)}$$

$$\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}}\cdot e^{-5s}\right\} = u_5(t)\cdot e^{-\frac{1}{4}(t-5)}\cdot\cos\left(\frac{\sqrt{15}}{4}(t-5)\right)$$

$$(※) = u_5(t)\cdot e^{-1/4(t-5)}\,\text{sen}\left(\frac{\sqrt{15}}{4}(t-5)\right)$$

(**) $$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}}\cdot e^{-20s}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}}\cdot e^{-20s}\right\} + \frac{4}{\sqrt{15}}\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2 + \underbrace{\frac{15}{16}}_{=a^2}}\cdot e^{-20s}\right\}$$

os outros três termos ficam similares

Assim

$$y(t) = -\frac{1}{2}u_5(t)\cdot e^{-\frac{1}{4}(t-5)}\cdot\cos\left[\frac{\sqrt{15}}{4}(t-5)\right]$$

$$-\frac{1}{2\sqrt{15}}u_5(t)\cdot e^{-\frac{1}{4}(t-5)}\,\text{sen}\left[\frac{\sqrt{15}}{4}(t-5)\right]$$

$$+\frac{1}{2}u_{20}(t)\cdot e^{-\frac{1}{4}(t-10)}\cdot\cos\left[\frac{\sqrt{15}}{4}(t-20)\right]$$

$$+\frac{1}{2\sqrt{15}}u_{20}(t)\cdot e^{-\frac{1}{4}(t-20)}\cdot\text{sen}\left[\frac{\sqrt{15}}{4}(t-20)\right]$$

$$+\frac{1}{2}u_5(t)(t-5) - \frac{1}{2}u_{20}(t)(t-20)$$