Cálculo 2 2027.2

$\boxed{\text{Aula 18}}$

Aula passada curvas polares

Aula hoje Cálculo

TANGENTES PARA CURVAS POLARES

considere $r = f(\theta)$ curva em coordenadas polares

Como $x = r\cos\theta$
$y = r\,\text{sen}\theta$

$$\boxed{\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\,\text{sen}\theta \end{cases}}$$ parametrização natural

Como

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\,\text{sen}\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\,\text{sen}\theta}$$

Assim

$$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\,\text{sen}\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\,\text{sen}\theta}}$$

10.4. Área e comprimento de curvas polares

COMPRIMENTO

$$L = \int_\alpha^\beta \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}_{\text{para polares}}\, dt$$

$$\left(\frac{dr}{d\theta}\cos\theta - r\,\text{sen}\theta\right)^2 + \left(\frac{dr}{d\theta}\,\text{sen}\theta + r\cos\theta\right)^2$$

$$= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\,\text{sen}\theta\cos\theta + r^2\,\text{sen}^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2\text{sen}^2\theta + 2r\frac{dr}{d\theta}\,\text{sen}\theta\cos\theta + r^2\cos^2\theta$$

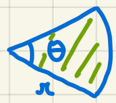
$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

$$\boxed{L = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta}$$ $r = f(\theta)$

ÁREA

Recorde

setor

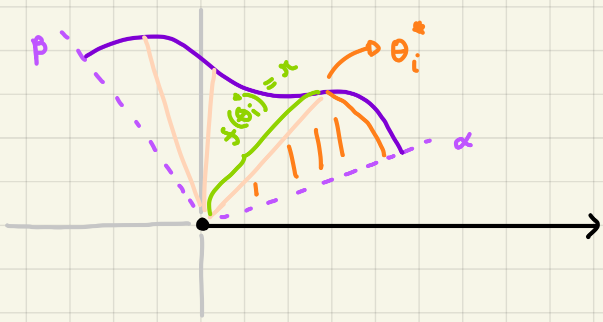


$$A = \frac{\theta}{2} r^2$$

$2\pi \to 2\pi r^2$

θ — x

Em geral seja $r = f(\theta)$, $\alpha \le \theta \le \beta$

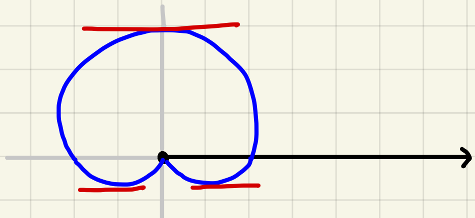


$$A \approx \sum_{i=1}^{n} \frac{f(\theta_i^*)^2}{2} \cdot \Delta\theta_i$$

Assim

$$\boxed{\text{Área} = \int_\alpha^\beta \frac{f(\theta)^2}{2}\, d\theta}$$

Exemplo Considere a cardióide $r = 1 + \text{sen}\theta$



(a) Encontre os pontos na cardioide onde a reta tangente é horizontal

(b) Calcule o comprimento da cardioide

Solução $\qquad r = 1 + \operatorname{sen}\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\operatorname{sen}\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\operatorname{sen}\theta}$$

$$= \frac{\cos\theta \cdot \operatorname{sen}\theta + (1+\operatorname{sen}\theta)\cos\theta}{\cos\theta \cdot \cos\theta - (1+\operatorname{sen}\theta)\operatorname{sen}\theta}$$

$$= \frac{2\cos\theta\operatorname{sen}\theta + \cos\theta}{\cos^2\theta - \operatorname{sen}^2\theta - \operatorname{sen}\theta} = 0$$

$\Leftrightarrow \quad 2\cos\theta\operatorname{sen}\theta + \cos\theta = 0$

e $\quad \cos^2\theta - \operatorname{sen}^2\theta - \operatorname{sen}\theta \neq 0$

$\cos\theta\,(2 + \operatorname{sen}\theta) = 0$

$\cos\theta = 0 \quad \rightarrow \quad \theta = \pi/2,\ 3\pi/2$

$\operatorname{sen}\theta = \frac{-1}{2} \quad \rightarrow \quad \theta = \frac{7\pi}{6},\ \frac{11\pi}{6}$

$$\frac{dx}{d\theta} = \cos^2\theta - \operatorname{sen}^2\theta - \operatorname{sen}\theta \Big|_{\theta=\pi/2}$$

$= -2 \neq 0$

$\frac{dx}{d\theta}\Big|_{\theta=3\pi/2} = 0 \qquad \frac{dx}{d\theta}\Big|_{\theta=7\pi/6} = \frac{3}{4} - \frac{1}{4} + \frac{1}{2} = 1$

$\frac{dx}{d\theta}\Big|_{\theta=\frac{11\pi}{6}} = \frac{3}{4} - \frac{1}{4} + \frac{1}{2} = 1$

Vale para $\pi/2,\ 7\pi/6,\ 11\pi/6$

$r = 1 + \operatorname{sen}\theta \Big|_{\theta=\pi/2} = 2$

$r = 1 + \operatorname{sen}\theta \Big|_{\theta=\frac{7\pi}{6}} = 1/2$

$r = 1 + \operatorname{sen}\theta \Big|_{\theta=\frac{11\pi}{6}} = 1/2$

$P_1 = (\pi/2, 2)$, $P_2 = (7\pi/6, 1/2)$ $P_3 = (\frac{11\pi}{6}, 1/2)$

(b) a cardioide inteira está em $0 \leq \theta \leq 2\pi$

$$L = \int_0^{2\pi} \sqrt{\frac{dr}{d\theta}^2 + r^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{\cos^2\theta + (1+\operatorname{sen}\theta)^2}\,.d\theta$$

$$= \int_0^{2\pi} \sqrt{\cos^2\theta + \operatorname{sen}^2\theta + 2\operatorname{sen}\theta + 1}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\operatorname{sen}\theta}\,.d\theta \cdot \frac{\sqrt{2-2\operatorname{sen}\theta}}{\sqrt{2-2\operatorname{sen}\theta}}$$

$$= \int_0^{2\pi} \frac{\sqrt{4 - 4\operatorname{sen}^2\theta}}{\sqrt{2-2\operatorname{sen}\theta}}\,.d\theta = \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\operatorname{sen}\theta}}\,.d\theta$$

$$\int_0^{2\pi} \frac{2\,|\cos\theta\,|}{\sqrt{2-2\operatorname{sen}\theta}}\, d\theta = \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\operatorname{sen}\theta}}\,.d\theta$$

$$- \int_{\pi/2}^{3\pi/2} \frac{2\cos\theta}{\sqrt{2-2\operatorname{sen}\theta}}\,.d\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\operatorname{sen}\theta}}\,.d\theta$$

$u = 2\operatorname{sen}\theta \qquad \theta = 0 \rightarrow u = 0$

$du = 2\cos\theta\, d\theta \qquad \theta = \pi/2 \rightarrow u = 2$

$\theta = 3\pi/2 \rightarrow u = -2$

$\theta = 2\pi \rightarrow u = 0$

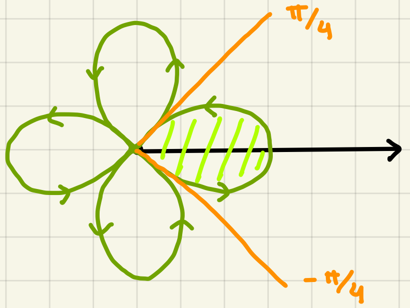
$$= \int_0^2 \frac{du}{\sqrt{2-u}} - \int_2^{-2} \frac{du}{\sqrt{2-u}} + \int_{-2}^0 \frac{du}{\sqrt{2-u}}.$$

$$-2\sqrt{2-u}\Big|_0^2 - 2\sqrt{2-u}\Big|_{-2}^2$$

$$-2\sqrt{2-u}\Big|_{-2}^0$$

$$0 + 2\sqrt{2} - 0 + 4 - 2\sqrt{2} + 4 = 8$$

Exemplo Calcule a área de uma pétala da rosácea $r = \cos 2\theta$.



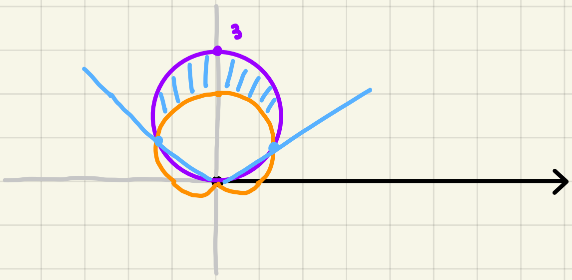
Solução uma pétala esta entre $-\frac{\pi}{4} \leq \theta \leq \pi/4$ então

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \int_{-\pi/4}^{\pi/4} \frac{\cos^2 2\theta}{2} d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}(1+\cos 4\theta)d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\operatorname{sen} 4\theta\right]_{-\pi/4}^{\pi/4}$$

$$\frac{1}{4}\cdot 2\frac{\pi}{4} + \frac{1}{4}\cancel{\operatorname{sen}\pi}^{0} - \frac{1}{4}\cancel{\operatorname{sen}-\pi}^{0} = \pi/8$$

Exemplo Calcule a área dentro o círculo $r = 3\operatorname{sen}\theta$ e fora da cardióide $r = 1 + \operatorname{sen}\theta$.



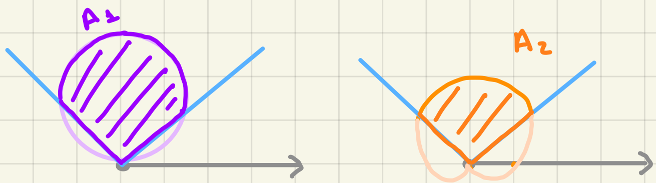
Para identificar o intervalo de integração precisamos saber onde as curvas se interceptam:

isto é, : $\underbrace{3\operatorname{sen}\theta}_{\text{círculo}} = \underbrace{1 + \operatorname{sen}\theta}_{\text{cardióide}}$

$$2\operatorname{sen}\theta = 1$$

$$\operatorname{sen}\theta = 1/2 \Rightarrow \theta = \pi/6, 5\pi/6$$

Note que como em coordenadas cartesianas a área que queremos é a área do círculo neste intervalo $\frac{\pi}{6} \leq \theta \leq 5\pi/6$ menos a da cardióide



$$A = A_1 - A_2 = \int_{\pi/6}^{5\pi/6} \frac{1}{2} 9\operatorname{sen}^2\theta\, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\operatorname{sen}\theta)^2 d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \frac{9}{2}\operatorname{sen}^2\theta - \frac{1}{2} - \operatorname{sen}\theta - \frac{1}{2}\operatorname{sen}^2\theta\, d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 4\operatorname{sen}^2\theta - \frac{1}{2} - \operatorname{sen}\theta\, .d\theta$$

$$\int_{\pi/6}^{5\pi/6} 2(1-\cos 2\theta)\, d\theta - \frac{1}{2}\theta\Big|_{\pi/6}^{5\pi/6} + \cos\theta\Big|_{\pi/6}^{5\pi/6}$$

$$2\theta\Big|_{\pi/6}^{5\pi/6} - \operatorname{sen} 2\theta\Big|_{\pi/6}^{5\pi/6} - \left(\frac{5\pi}{12} - \frac{\pi}{12}\right) - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$

$$\frac{10\pi}{6} - \frac{2\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - \frac{\pi}{3} = \frac{5\pi}{3} - \frac{2\pi}{3} = \pi$$