European Journal of Physics

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To cite this article: Taha Massalha and Rafael M Digilov 2015 *Eur. J. Phys.* **36** 065045

IOP Publishing European Journal of Physics

Eur. J. Phys. **36** (2015) 065045 (9pp) doi:10.1088/0143-0807/36/6/065045

Capillary viscometer with a pressure sensor: a subject for student projects

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Received 29 July 2015, revised 15 August 2015
Accepted for publication 25 August 2015
Published 8 October 2015

Abstract

We describe a vacuum-based pressure-detecting capillary viscometer for the viscosity determination of Newtonian fluids without known density. The viscometer operates on the principle that the variation in air pressure of the vacuum vessel $p(t)$ replaces the flow rate and pressure drop measurements which are usually required for the operation of a capillary tube viscometer. The mathematical expression for $p(t)$, found in the terms of the Lambert-W function, is used to fit the experimental data for viscosity determination. The results for viscosities of distilled water and 50 wt.% glycerol aqueous solution obtained under ambient temperature condition were compared to reference data and a good agreement was observed. The viscometer is suitable for undergraduate laboratories due to its low cost and simplicity in experimental setup. Moreover, the experimental with the vacuum vessel setup provides an in-depth understanding of fluid flow.

S Online supplementary data available from stacks.iop.org/EJP/36/065045/mmedia

Keywords: capillary flow, Lambert function, viscometer, viscosity

(Some figures may appear in colour only in the online journal)

1. Introduction

Viscosity is a measure of the resistance of a fluid (liquid or gas) to flow. This quantity is of considerable interest in physical chemistry, hydrodynamics, biomedical science and other scientific fields [1, 2] for understanding the nature of a fluid and its flow behavior. Many

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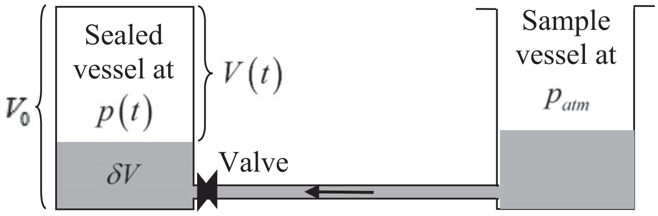

Figure 1. Schematic diagram of a communicating discharge device.

processes in the petroleum, chemical industries, metallurgy, food technologies and building materials and medicine are governed by viscosity. Viscosity measurements are used in quality control, equipment design and unit operations such as blending, cooling, pumping, homogenization, pasteurization, sterilization, evaporation etc. In daily life, it is observed that coating with shaving cream, the flow of water in town pipes, and the clinical analysis of blood samples are also related to the viscosity.

Determination of viscosity by means of a viscometer is a common practice in undergraduate laboratories [3–10]. The most popular viscometers are capillary, rotational and falling-ball ones. In capillary tube viscometers [11], the fluid viscosity η is determined from the Hagen–Poiseuille equation [12]

$$Q = \frac{\pi R^4}{8\eta L}\Delta p \tag{1}$$

where Q is the flow rate and Δp is the pressure difference through a capillary tube of radius R and length $L >> R$. Equation (1) is valid for a laminar steady state flow, that is for Reynolds' number, $\mathrm{Re} = \rho v R/\eta < 2300$, where ρ is the density, and $v = Q/\pi R^2$ is the average velocity of the fluid flow.

A number of experiments have been described for determining the viscosity of liquids by their flow rate through a capillary tube [4–6, 8, 11]. However, all these experiments are limited to low-viscosity fluids like water (of the order of 1 mPa s). For high viscosity liquids such as glycerol, oil, etc, the experiment is not easy to do using simple tools.

In the present paper, we describe a vacuum-based pressure-detecting capillary viscometer, in which high-viscosity liquid is sucked through a horizontal capillary tube into a vacuum vessel from the sample vessel. The flow rate, which is usually required for determining the viscosity, is replaced by measuring the pressure variation versus time $p(t)$ in a vacuum vessel using a pressure sensor. The advantage of this approach is that it is applicable to both high and low viscosity fluids.

2. Theoretical background

Consider the communicating discharge device in figure 1, consisting of two vessels coupled with a valve at the bottom by a horizontal capillary of radius R and length $L\,(\gg R)$. The open vessel with the test liquid is at atmospheric pressure, p_{atm}. The airtight vessel of volume V_0 comprises a residual air under a pressure of $p_0 < p_{\mathrm{atm}}$. When the valve is opened, the liquid begins to flow through the capillary into the airtight vessel under the pressure drop Δp between both ends of the capillary tube. Neglecting the hydrostatic head, one assumes that $\Delta p \approx p_{\mathrm{atm}} - p(t)$, where $p(t)$ is the air pressure in the airtight vessel at time t, which gradually increases from its initial value p_0 to the equilibrium value p_{atm}, at which the flow ceases. The volume of the fluid passes through a capillary tube at time t is $\delta V = V_0 - V(t)$, where $V(t)$ is the air volume, which decreases during the flow from V_0 to $\left(V_0 - V_{\text{liq}}\right)$, where V_{liq} is the accumulated volume of liquid in the airtight vessel when the flow ceases.

Assuming that the air behaves as an ideal gas and the temperature is constant, the ideal gas law can be applied and written that the product of pressure-volume at any instant is a constant

$$p_0 V_0 = \begin{cases} p_{\text{atm}}\left(V_0 - V_{\text{liq}}\right), & \text{when the flow ceases} \\ p(t)V(t), & \text{at any time } t. \end{cases} \tag{2}$$

From the first equation, we get

$$V_{\text{liq}} = V_0\left(1 - \frac{p_0}{p_{\text{atm}}}\right). \tag{3}$$

We use the second equation to write the expression for the volumetric flow rate $Q(t)$ as

$$Q(t) = \frac{\mathrm{d}}{\mathrm{d}t}\delta V(t) = -\frac{\mathrm{d}V(t)}{\mathrm{d}t} = -p_0 V_0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p(t)}\right) = \frac{p_0 V_0}{p^2(t)}\frac{\mathrm{d}p(t)}{\mathrm{d}t} \tag{4}$$

Under the assumption of quasi steady state flow and negligible friction loss due to a sudden contraction between the sample vessel and the tube, we can substitute equation (1) into equation (4) to obtain the differential equation for $p(t)$

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = \frac{\pi R^4}{8\eta L p_0 V_0} p^2(t)\left[p_{\text{atm}} - p(t)\right]. \tag{5}$$

By setting dimensionless variables

$$\tilde{p}(t) = p(t)/p_{\text{atm}} \text{ and } \tilde{t} = \beta t \text{ where } \beta = \frac{\pi R^4 p_{\text{atm}}^2}{8\eta L p_0 V_0} = \frac{\pi R^4 p_{\text{atm}}}{8\eta L V_{\text{liq}}}\left(\frac{p_{\text{atm}}}{p_0} - 1\right) \tag{6}$$

we obtain equation (5) in the dimensionless form

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{t}} = -\tilde{p}^2\left(\tilde{p} - 1\right) \tag{7}$$

which can be integrated with the initial condition $\tilde{p}(0) = \tilde{p}_0$ to give

$$\frac{1}{\tilde{p}} + \ln\left(\frac{1}{\tilde{p}} - 1\right) = -\tilde{t} + \frac{1}{\tilde{p}_0} + \ln\left(\frac{1}{\tilde{p}_0} - 1\right). \tag{8}$$

With $1/\tilde{p}_0 - 1 = \alpha$, the solution of equation (8) explicitly for $\tilde{p}(\tilde{t})$ can be written in the terms of the Lambert-W function [11]

$$\tilde{p}\left(\tilde{t}\right) = \frac{1}{1 + W\left(\alpha e^{\alpha - \tilde{t}}\right)} \tag{9}$$

where $W(x)$ is the Lambert-W function [13–15] (see the appendix). Since $0 < \tilde{p}_0 < \tilde{p}(t) \leqslant 1$ the principal branch of $W(x)$ (see the appendix) should be used.

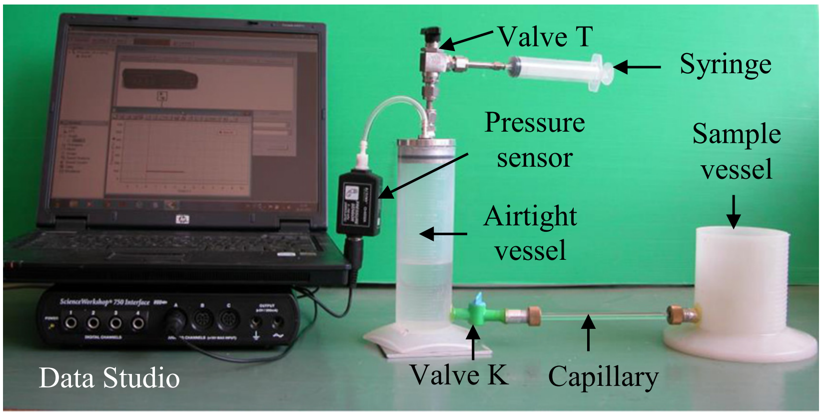

Figure 2. Experimental setup of the capillary viscometer with a pressure sensor.

In rescaling the variables, $\alpha = p_{atm}/p_0 - 1$ and equation (9) reads

$$p(t) = \frac{p_{atm}}{1 + W\left(\alpha e^{\alpha - \beta t}\right)} \tag{10}$$

which satisfies the initial condition $p(0) = p_0$ and provides the limit, $p(t \to \infty) = p_{atm}$, per the Lambert function definition. Equation (10) links the viscous behavior of the fluid with the temporal variation of air pressure in the airtight vessel through the parameter β and may be used for viscosity determination. Parameters α and β can be determined from the non-linear curve fitting of experimental data $p(t)$ to equation (10). Then, under certain testing conditions, knowing the fitting parameters, the geometry of the capillary and accumulated volume of liquid V_{liq}, the viscosity η can be calculated from equation (6) for β as

$$\eta = \frac{\pi R^4 p_{atm} \alpha}{8 L V_{liq} \beta}. \tag{11}$$

The relative error in the viscosity determination by equation (11) arises from the uncertainties in the measurement of the magnitudes involved in this equation. So, the total error in the determination of the variation parameters α and β is less than 0.5% . The uncertainty in the atmospheric pressure determination is $\left|\Delta p_{atm}/p_{atm}\right| < 0.5\%$. The radius of the precision capillary tube is given with an accuracy of $|\Delta R/R| \approx 1\%$. The error in the capillary length measured with a caliper is $|\Delta L/L| \approx 1\%$. The uncertainty in the determination of the fluid volume with the graduated cylinder is $\left|\Delta V_{liq}/V_{liq}\right| \approx 1\%$. Using the familiar error propagation formula

$$|\Delta\eta/\eta| \leqslant 4\,|\Delta R/R| + |\Delta L/L| + \left|\Delta p_{atm}/p_{atm}\right| + \left|\Delta V_{liq}/V_{liq}\right| + |\Delta\alpha/\alpha| + |\Delta\beta/\beta| \tag{12}$$

one finds that the relative error in viscosity determination does not exceed 7%.

3. Experimental setup and test results

The experimental setup, shown in figure 2, consists of two plastic cylinders that are joined by the valve K with a precision glass capillary tube ACE 8700-17 [16] having an inside diameter

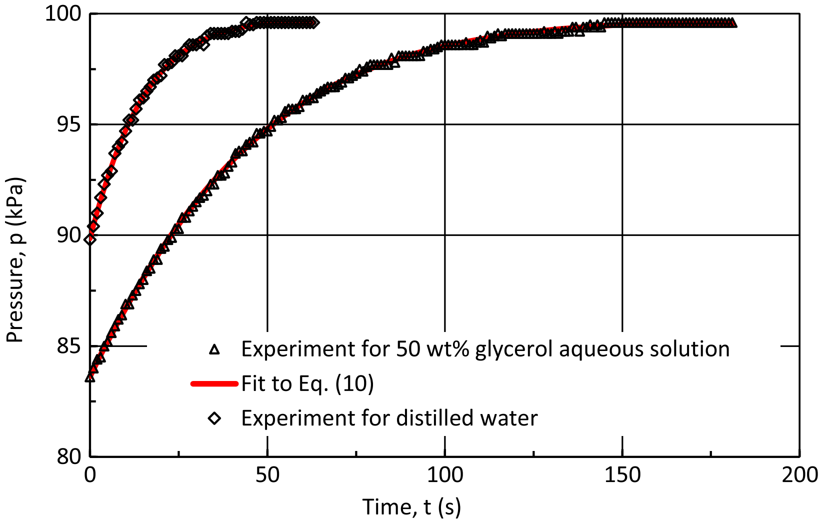

Figure 3. The temporal variation of the air pressure in the airtight vessel for the distilled water and the 50 wt.% glycerol aqueous solution; symbols represent the experimental data. Red solid lines are the best nonlinear curves fitting to equation (10).

of $d = 1 \pm 0.016$ mm. Open to the ambient cylinder of volume 300 mL contains the test liquid. The airtight transparent graduated cylinder of volume 280 mL is sealed with an aluminum flange with the inlet valve T, 50 mL syringe and pressure sensor CI-6532A connected through the DataStudio of Pasco to a computer. The principle of measurement is as follows. First, the valve K is closed and T is opened and about 200 mL of the test liquid is introduced into the open cylinder. The data acquisition system is activated and ambient atmospheric pressure p_{atm} is fixed with the pressure sensor. The syringe piston moves upward to remove the part of air from the airtight vessel and reduce the air pressure therein to $p_0 \approx 0.8 p_{\text{atm}}$. Then, the valve T is closed through a test, and K is opened allowing the liquid to flow through a capillary tube under the pressure difference, $\Delta p(t) = p_{\text{atm}} - p(t)$. The air space in the airtight vessel $V(t)$ gradually decreases, and $p(t)$ increases approaching to p_{atm}, at which the flow of the fluid stops. At this instant, the fluid volume V_{liq} which is accumulated in the hermetic vessel, is read by a ruled scale of the cylinder. The series of data $p(t)$ versus time detected at the sample rate 1 Hz are read out to a PC and fed further for processing.

The plot of the temporal variation of the air pressure in the airtight vessel during the flow of the distilled water through a capillary tube with an internal diameter of $d = 1 \pm 0.016$ mm and the length of 14.11 cm under ambient temperature $\approx$24 °C and atmospheric pressure $p_{\text{atm}} \approx 100$ kPa is shown in figure 3. The flow takes about 50 s, during which the air pressure $p(t)$ gradually increases from the initial value $p_0 \approx 90$ kPa to the equilibrium value p_{atm}, at which the flow ceases. The volume of the liquid accumulated in the hermetic vessel is read to be $V_{\text{liq}} = 25 \pm 0.25$ ml. With a Mathematica 9 file [17], the best fit of experimental data $p(t)$ to equation (10) with $\alpha = 0.1109 \pm 0.0006$ and $\beta = (0.0779 \pm 0.0011)\ \text{s}^{-1}$ indicates the validity of equation (8). The obtained experimental parameters and size of used capillaries are presented in table 1.

The viscosity of distilled water calculated by equation (11) is $\eta = 0.989 \pm 0.069$ mPa · s. Within the estimated uncertainty 7%, this result is compared with the value 0.956mPa · s obtained by the extrapolation of reference data [18] to

Table 1. Experimental parameters and results of the viscosity calculation.

Test liquids	V_{liq} (mL)	L (cm)	α	β (s^{-1})	η (m Pa · s)	η_{ref} (m Pa · s)
Distilled water	25	14.11	0.1109	0.0779	0.989	0.956
50 wt.% glycerol water mixture	38	9.31	0.1934	0.0288	4.964	5.068

temperature 24 °C, using Arrhenius' law

$$\eta = A\,\exp(B/T) \tag{13}$$

where T is the absolute temperature, and $A = (0.0011388 \pm 0.0001498)$ m Pa · s and $B = (1999.36 \pm 38.26)$ K are empirical constants determined from the fitting of the reference data for water viscosity at different temperatures, from 0 to 100 °C[18].

The plot of air pressure versus time $p(t)$ in the airtight vessel during the flow of 50 wt.% glycerol aqueous solution through the capillary tube of radius $d = 1 \pm 0.016$ mm and length $L = 9.31$ cm under the ambient temperature ≈26 °C and the atmospheric pressure $p_{\text{atm}} \approx 100$ k Pa is shown in figure 3. The flow takes about 3 min, the air pressure $p(t)$ increases from the initial value $p_0 = 83.6$ k Pa to the atmospheric pressure $p_{\text{atm}} \approx 99.8$k Pa. The best nonlinear curve fitting of experimental data $p(t)$ to equation (10) with the fitting parameters, $\alpha = 0.1934 \pm 0.0004$ and $\beta = (0.02878 \pm 0.00013)\,\text{s}^{-1}$ is excellent. The viscosity calculated by equation (11) using the experimental parameters in table 1, is $\eta = (4.964 \pm 0.347)$ m Pa · s, which correlates well with the value 5.068m Pa · s, obtained by extrapolation of reference data [18] to temperature 26 °C using equation (13) with $A = (0.0000942 \pm 0.0000343)$ m Pa · s and $B = (3258.3 \pm 102.4)$ K, obtained by fitting of reference data for the viscosity of 50 wt% glycerol aqueous solution at different temperatures to equation (13).

4. Conclusion

In conclusion, we have presented a relatively simple and inexpensive vacuum suction-type pressure-detecting capillary viscometer for viscosity measurement of Newtonian liquids. Experimentation with the viscometer provides a deeper understanding the fluid dynamics and gas laws. The viscometer may be easily used for quick unsophisticated tests of high and low viscous liquids without the need to know their density. It requires no further calibration once it is set and can be easily built in any undergraduate laboratory in the framework of the student project. The simplicity of this viscometer makes it ideal for students to explore more about the fluid viscosity through individual projects and extensions of this experiment.

Acknowledgment

We would like to thank Dr Nazih Kassis from the Academic Arab College for Education in Haifa, for his generous linguistic review and proofreading of the article.

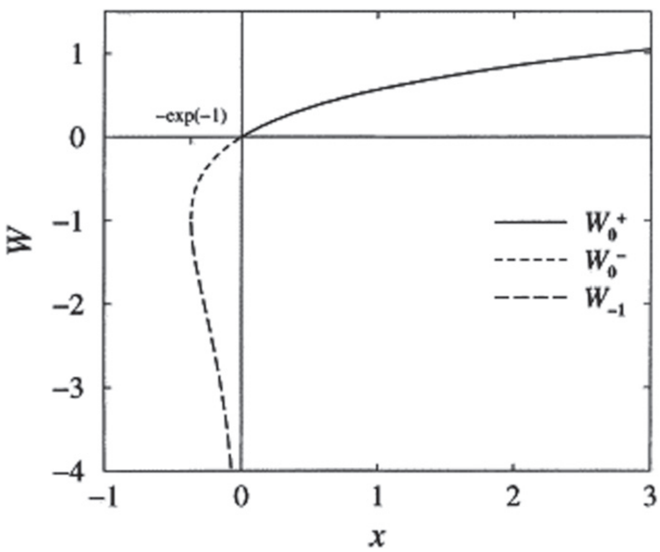

Figure A1. The real branches of the Lambert $W(x)$ function. The *principal branch* $W_0(x)$ is divided into $W_0^+(x)$ defined for $0 \leqslant x \leqslant \infty$ of our interest and $W_0^-(x)$ for the interval $-e^{-1} \leqslant x \leqslant 0$, where the *alternative branch* $W_{-1}(x)$ is also defined.

Appendix

A.1. The Lambert-W function

The Lambert-W function, $W(x)$ (see figure A1) is a multi-valued transcendental function defined as the solution of the equation:

$$W(x)\exp(W(x)) = x \tag{A1}$$

where x, in general, is a complex number. It belongs to the family of exponential and logarithmic functions and is similar to the trigonometric functions in the sense that it has no explicit closed form, but a very large number of physical problems are solved with relative ease using it in the solution [13, 14]. To see a graph of $W(x)$ for real x, one may issue the Mathematica [15] command Plot[ProductLog[x], {x, $-1/\mathrm{e}$, 4}]. Another popular physical computing packages such as Maple, Matlab and Macsyma include full support for the Lambert-W function, and utilize efficient algorithms to calculate its value at any point in its domain. For real values of the argument x, the principal branch $W_0(x)$ defined for $-e^{-1} \leqslant x < \infty$ is analytical at $x = 0$. For $x > 0$, there is just one real solution $W_0^+(x) \geqslant 0$. For $-e^{-1} \leqslant x \leqslant 0$, there are two real branches (see figure A1). The *principal branch* denoted by $W_0^-(x)$, satisfies $-1 \leqslant W(x) < 0$ and the *alternative branch* denoted by $W_{-1}(x)$, satisfies $W(x) < -1$. There are not real solutions for $x < -\exp(-1)$.

1. The first property of the Lambert-W function that follows directly from equation (A1) is:

$$W\left(ae^{a}\right) = a\ W(0) = 0 \tag{A2}$$

a special case of which $W(-e^{-1}) = -1$ (see also figure A1). The $W(1) = 0.56714$ is called the *omega constant* and is considered as a sort of '*golden ratio*' of exponents.

2. Differentiating equation (A1) for $W(x)$ and solving for $\mathrm{d}(W(x))/\mathrm{d}x$, we obtain the expression for the derivative of the Lambert $W(x)$ function:

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}x} &= \frac{W(x)}{x(1+W(x))}, \quad \text{if } x \neq 0 \\ &= \frac{\exp(-W(x))}{1+W(x)}.\end{aligned} \tag{A3}$$

3. With the natural substitution, $u = W(x)$ in equation (A1), i.e., $ue^{u} = x$, the Lambert $W(x)$ function can be integrated as follows:

$$\int W(x)\mathrm{d}x = x(W(x) + 1/W(x) - 1) = \Big(W(x)^2 + 1 - W(x)\Big)e^{W(x)}. \tag{A4}$$

4. For $x > 0$, the Lambert function $W(x) > 0$ and we can take the natural logarithms of both sides of equation (A1) and rearranging terms, we obtain the formula for logarithms of Lambert $W(x)$ function

$$\log(W(x)) = \ln x - W(x). \tag{A5}$$

For $x < 0$, $W(x) < 0$ and we can multiply both sides of equation (A1) with -1, take the logarithm and rewrite it to get a similar expansion for $W_{-1}(x)$ the branch

$$\log(-W(x)) = \ln(-x) - W(x). \tag{A6}$$

5. The Taylor series expansion of the Lambert function $W_0(x)$ is

$$W_0(x) = \sum_{n\geqslant 1}\frac{(-n)^{n-1}}{n!}x^{n} \tag{A7}$$

with the radius convergence $|x| < e^{-1}$

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