Aula 21 Cálculo 3B | Aula 30 Cálculo 3A

Aula passada
- aplicação da transformada de Laplace para solucionar EDO's
- função degrau

Aula Hoje
- aplicação EDO's com termo não homogêneo descontínuo

6.4 Forçamento descontínuo

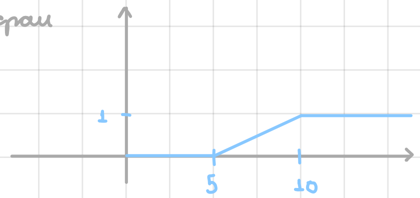
termo não homogêneo

Exemplo Resolva
$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

onde
$$g(t) = \begin{cases} 0, & 0 \le t < 5 \\ \frac{(t-5)}{5}, & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases}$$

Solução:

Passo 1: Escrever $g(t)$ em termo de funções degrau



$$g(t) = u_5(t).\frac{(t-5)}{5} - u_{10}(t)\frac{(t-5)}{5} + u_{10}(t)$$

Recorde:
$$\mathcal{L}\{u_c(t).f(t-c)\} = e^{-cs}.\mathcal{L}\{f(t)\}$$

fazendo aparecer a translação:

$$g(t) = \frac{1}{5}u_5(t).(t-5) - \frac{1}{5}u_{10}(t)(t-5-5+5) + u_{10}(t)$$
$$= \frac{1}{5}u_5(t).(t-5) - \frac{1}{5}u_{10}(t)(t-10) - \cancel{\frac{1}{5}u_{10}(t).5} + \cancel{u_{10}(t)}$$

Passo 2: Aplicando a transformada

$$\mathcal{L}\{y''+4y\} = \frac{1}{5}\mathcal{L}\{u_5(t).(t-5)\} - \frac{1}{5}\mathcal{L}\{u_{10}(t)(t-10)\}$$

$$s^2\mathcal{L}\{y\} - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 4\mathcal{L}\{y\} = \frac{1}{5}\left[\frac{1}{s^2}.e^{-5s} - \frac{1}{s^2}e^{-10s}\right]$$

$$\mathcal{L}\{y\} = \frac{1}{5}.\frac{1}{(s^2+4).s^2}.\left(e^{-5s} - e^{-10s}\right)$$

Passo 3: Calcular a transformada inversa

$$y(t) = \frac{1}{5}.\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(s^2+4)s^2}}_{(*)\text{ frações parciais}}.\left(e^{-5s} - e^{-10s}\right)\right\}$$

porque s^2+4 não tem raízes reais

$$(*)\quad \frac{1}{(s^2+4)s^2} = \frac{As+B}{s^2+4} + \frac{C}{s^2}$$
$$= \frac{As^3+Bs^2+Cs^2+4C}{(s^2+4)s^2}$$

$$\Rightarrow \left.\begin{array}{l} A = 0 \\ B + C = 0 \\ 4C = 1 \end{array}\right\} \Rightarrow \begin{array}{l} C = \frac{1}{4} \\ B = -\frac{1}{4} \end{array}$$

$$y(t) = \frac{1}{5}.\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)s^2}.\left(e^{-5s} - e^{-10s}\right)\right\}$$

$$= \frac{1}{20}.\mathcal{L}^{-1}\left\{\left(\frac{-1}{s^2+4} + \frac{1}{s^2}\right)\left(e^{-5s} - e^{-10s}\right)\right\}$$

$$= \frac{1}{20}.\mathcal{L}^{-1}\left\{-\frac{1}{s^2+4}.e^{-5s} + \frac{1}{s^2}.e^{-5s} + \frac{1}{s^2+4}.e^{-10s} - \frac{1}{s^2}.e^{-10s}\right\}$$

$$= -\frac{1}{20}\frac{1}{2}\mathcal{L}^{-1}\Big\{\underbrace{\frac{2}{s^2+4}.e^{-5s}}_{\mathcal{L}\{\text{sen}2t\}.e^{-5s}}\Big\} + \frac{1}{20}\mathcal{L}^{-1}\Big\{\underbrace{\frac{1}{s^2}.e^{-5s}}_{\mathcal{L}\{t\}.e^{-5s}}\Big\} + \frac{1}{20.2}\mathcal{L}^{-1}\Big\{\underbrace{\frac{2}{s^2+4}.e^{-10s}}_{\mathcal{L}\{\text{sen}2t\}.e^{-10s}}\Big\}$$

$$-\frac{1}{20}.\mathcal{L}^{-1}\Big\{\underbrace{\frac{1}{s^2}.e^{-10s}}_{\mathcal{L}\{t\}.e^{-10s}}\Big\}$$

$$\boxed{y(t) = -\frac{1}{40}u_5(t)\,\text{sen}(2(t-5)) + \frac{1}{20}u_5(t)(t-5) + \frac{1}{40}u_{10}(t)\,\text{sen}(2(t-10)) - \frac{1}{20}u_{10}(t)(t-10)}$$

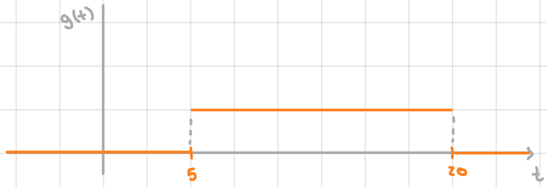
Exemplo Resolva $2y'' + y' + 2y = g(t)$ onde

$y(0) = y'(0) = 0$

$$g(t) = \begin{cases} 1 & 5 \le t < 20 \\ 0, & 0 \le t < 5 \text{ e } t \ge 20 \end{cases}$$

Solução:

Escrever $g(t)$ em termos de função degrau



$$g(t) = u_5(t) - u_{10}(t)$$

$$\mathcal{L}\{2y'' + y' + 2y\} = \mathcal{L}\{u_5(t) - u_{10}(t)\}$$

$$2s^2\mathcal{L}\{y\} - 2s\cancel{y(0)} - 2\cancel{y'(0)} + s\mathcal{L}\{y\} - \cancel{y(0)} + 2\mathcal{L}\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-10s}}{s}$$

$$(2s^2 + s + 2)\mathcal{L}\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-10s}}{s}$$

$$\mathcal{L}\{y\} = \underbrace{\frac{1}{(2s^2+s+2)s}}\,.\,(e^{-5s} - e^{-10s})$$

↳ frações parciais
sem raízes reais

$$\frac{1}{(2s^2+s+2)s} = \frac{As+B}{2s^2+s+2} + \frac{C}{s} = \frac{As^2 + Bs + 2Cs^2 + Cs + 2C}{(2s^2+s+2)s}$$

$$\Rightarrow \begin{cases} A + 2C = 0 \\ B + C = 0 \\ 2C = 1 \end{cases} \Rightarrow \begin{cases} C = \frac{1}{2} \\ B = -\frac{1}{2} \\ A = -1 \end{cases}$$

$$\mathcal{L}\{y\} = \left(-\frac{s+\frac{1}{2}}{2s^2+s+2} + \frac{1}{2}\cdot\frac{1}{s}\right)(e^{-5s} - e^{-10s})$$

$$y(t) = -\overbrace{\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{2s^2+s+2}\,.\,e^{-5s}\right\}}^{*} + \overbrace{\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{2s^2+s+2}\,.\,e^{-10s}\right\}}^{**}$$

$$+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\,.\,e^{-5s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\,.\,e^{-10s}\right\}$$

(*) $2s^2 + s + 2 = 2\left(s^2 + \frac{1}{2}s + 1\right) = 2\left[\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}\right]$

(*) $\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{2s^2+s+2}\,.\,e^{-5s}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\overset{\frac{1}{4}+\frac{1}{4}}{\frac{1}{2}}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\,.\,e^{-5s}\right\}$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\,.\,e^{-5s}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\underbrace{\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\underbrace{\frac{15}{16}}_{=a^2}}}\,.\,e^{-5s}\right\} + \frac{4}{\sqrt{15}}\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\underbrace{\frac{15}{16}}_{=a^2}}\,.\,e^{-5s}\right\}$$

$$F(s - 1/4)\,.\,e^{-5s}$$

onde $F(s) = \mathcal{L}\left\{\cos\left(\frac{\sqrt{15}}{4}\,.\,t\right)\right\}$

$$\boxed{\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t)}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\,.\,e^{-cs}\} = u_c(t)\,.\,f(t-c)}$$

$$\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\,.\,e^{-5s}\right\} = u_5(t)\,.\,e^{-\frac{1}{4}(t-5)}\,.\cos\left(\frac{\sqrt{15}}{4}(t-5)\right)$$

(**) $= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\,.\,e^{-10s}\right\}$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\,.\,e^{-10s}\right\} + \frac{4}{\sqrt{15}}\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\underbrace{\frac{15}{16}}_{=a^2}}\,.\,e^{-10s}\right\}$$

os outros três termos ficam similares usando as duas propriedades de translação juntas

Assim

$$y(t) = -\frac{1}{2}u_5(t)\,.\,e^{-\frac{1}{4}(t-5)}\,.\cos\left[\frac{\sqrt{15}}{4}(t-5)\right]$$

$$-\frac{1}{2\sqrt{15}}u_5(t)\,.\,e^{-\frac{1}{4}(t-5)}\,\text{sen}\left[\frac{\sqrt{15}}{4}(t-5)\right]$$

$$+\frac{1}{2}u_{10}(t)\,.\,e^{-\frac{1}{4}(t+10)}\,.\cos\left[\frac{\sqrt{15}}{4}(t-10)\right]$$

$$+\frac{1}{2\sqrt{15}}u_{10}(t)\,.\,e^{-\frac{1}{4}(t-10)}\,.\,\text{sen}\left[\frac{\sqrt{15}}{4}(t-10)\right]$$

$$+\frac{1}{2}u_5(t)(t-5) - \frac{1}{2}u_{10}(t)(t-10)$$