**Indian Institute of Space Science and Technology**

Department of Aerospace Engineering


# Assignment 2: Glider Performance Analysis

*Atmospheric Flight Mechanics*


**Submitted By:**

Dishant Jain

SC23B061


December 14, 2025

# Introduction

The objective of this assignment is to simulate the equations of motion for a glider under various flight conditions. We will explore the aircraft's trajectory and flight characteristics under three key steady-state conditions:

1. **Minimum Glide Angle (Maximum Range)** : The flight condition that maximizes the horizontal distance covered for a given loss in altitude.
2. **Minimum Sink Rate (Maximum Endurance)** : The flight condition that maximizes the time spent in the air for a given loss in altitude.
3. **Maximum Lift Coefficient** ($C_{L_{max}}$) : The flight condition corresponding to the stall speed, which defines the lower limit of the flight envelope.

The analysis is based on a point-mass model restricted to motion in the vertical plane. Since the aircraft is unpowered, thrust is set to zero ($T = 0$), leaving gravity, lift, and drag as the only forces. For the analytical derivation, steady-state flight is assumed ($\dot{V} \approx 0, \dot{\gamma} \approx 0$), reducing the differential equations of motion to algebraic force-balance equations. Small flight path angles ($\sin\gamma \approx \gamma, \cos\gamma \approx 1$) are assumed to simplify these expressions, allowing for closed-form solutions of optimal lift coefficients and velocities. Specifically, maximum range occurs at $(L/D)_{max}$, where parasitic drag equals induced drag ($C_{D0} = KC_L^2$) which corresponds to Case 1 i.e. Min Glide Angle, while maximum endurance occurs when power required is minimized ($3C_{D0} = KC_L^2$) which refers to Case 2 i.e Min Sink rate or max time to stay airborne hence max endurance.

Numerical simulations are performed using MATLAB's `ode45` solver to integrate the full non-linear equations of motion. Unlike the analytical approach, the simulation captures transient behaviors such as phugoid oscillations and accounts for varying atmospheric density with altitude. By comparing the steady-state simulation results with theoretical predictions, this report validates the model and analyzes the dynamic stability of the glider under different operating conditions.

# Equations of Motion

The equations governing the glider's motion in the longitudinal plane are given by:

$$\frac{dV}{dt} = \frac{T - D - mg\sin\gamma}{m} \tag{1}$$

$$\frac{d\gamma}{dt} = \frac{L - mg\cos\gamma}{mV} \tag{2}$$

$$\frac{dH}{dt} = V\sin\gamma \tag{3}$$

$$\frac{dX_e}{dt} = V\cos\gamma \tag{4}$$

The motion is analysed using a quasi–steady, small–angle approach with constant lift coefficient $C_L$. Solving the Equations of Motion keeping all approximations in account we reduce the equations to the following manner.

$$D = -W\gamma \tag{5}$$
$$L = W \tag{6}$$
$$\frac{dH}{dt} = V\gamma \tag{7}$$
$$\frac{dX_e}{dt} = V \tag{8}$$

Hence the flight–path angle can be written as

$$\gamma \approx -\frac{D}{W} = -\frac{D}{L} = -\frac{1}{E},$$

where $E = L/D$ is the aerodynamic efficiency.

## Case 1: Minimum Glide Angle

The Minimum Glide Angle condition represents the shallowest possible path the aircraft can fly. Physically, this corresponds to the maximum horizontal distance (range) covered for a given loss of altitude.

$$\frac{dH}{dX_e} = \tan\gamma \approx \gamma$$

For a loss of altitude $\Delta H$, the horizontal range is

$$R = \frac{\Delta H}{|\tan\gamma|} \approx \frac{\Delta H}{|\gamma|} = \Delta H\, E$$

And the endurance can be calculated by

$$\frac{dH}{dt} = V\gamma$$

$$dt = -E\sqrt{\frac{1/2 C_L \rho_0 e^{-H/2\beta}}{W/S}}dH$$

$$T = 2\beta E\sqrt{\frac{\rho_0 C_L}{2(W/S)}}\left(1 - e^{-\frac{H_{\text{init}}}{2\beta}}\right)$$

where $H_{init}$ refers to the initial altitude.

Thus the range is directly proportional to the efficiency $E = L/D$, and maximum range is obtained when $E$ (or equivalently $L/D$) is maximized. Using the parabolic drag polar

$$C_D = C_{D0} + KC_L^2,$$

the lift–to–drag ratio becomes

$$\frac{L}{D} = \frac{C_L}{C_D}, \qquad \frac{D}{L} = \frac{C_D}{C_L} = \frac{C_{D0}}{C_L} + KC_L.$$

To maximize $L/D$ we minimize $D/L$ with respect to $C_L$:

$$\frac{\mathrm{d}}{\mathrm{d}C_L}\left(\frac{C_D}{C_L}\right) = \frac{\mathrm{d}}{\mathrm{d}C_L}\left(\frac{C_{D0}}{C_L} + KC_L\right) = -\frac{C_{D0}}{C_L^2} + K = 0.$$

Solving for $C_L$ yields

$$K = \frac{C_{D0}}{C_L^2} \quad \Rightarrow \quad C_L^2 = \frac{C_{D0}}{K} \quad \Rightarrow \quad C_L^\star = \sqrt{\frac{C_{D0}}{K}}.$$

At this operating point,

$$C_D^\star = C_{D0} + KC_L^{\star 2} = C_{D0} + C_{D0} = 2C_{D0},$$

so the maximum efficiency and corresponding minimum glide angle are

$$E_{max} = \frac{C_L^\star}{C_D^\star} = \frac{\sqrt{C_{D0}/K}}{2C_{D0}}, \qquad \gamma_{\min} \approx -\frac{1}{(E)_{\max}} \qquad R = \Delta H, E_{max} \qquad T = 2\beta E \sqrt{\frac{\rho_0 C_L}{2(W/S)}} \left(1 - e^{-\frac{H_{\text{init}}}{2\beta}}\right)$$

This condition corresponds to the $E_{max}$ condition which we know of. And as we know $R = \Delta H\,(L/D)$, this proves that the glider achieves maximum range when $L/D$, or equivalently the efficiency $E$, is maximized, which occurs at $C_L = \sqrt{C_{D0}/K}$.

Thus for Min Glide Angle we have to operate the aircraft at $E_m$ condition which dictates $C_L = C_L^\star$ and corresponding $v^\star$ and $\gamma_{min}$ as well.

| Altitude | $C_l^\star$ | $V^\star$ (m/s) | $E_m$ | $\gamma_{min}$ | $X_e$ (km) | Time (s) |
|---|---|---|---|---|---|---|
| 3000 m | 0.8924 | 38.6079 | 12.394 | -4.6228 | 0 | 0 |
| 0 m | 0.8924 | 32.8548 | 12.394 | -4.6228 | 37.182 | 1045.119 |

Table 1: Flight Parameters

## Case 2: Minimum Sink Rate

The Minimum Sink Rate condition corresponds to the flight state where the aircraft loses altitude as slowly as possible. Physically, this maximizes endurance, meaning the glider remains airborne for the longest possible duration for a given starting altitude.

$$\frac{dH}{dt} = V\gamma = \frac{-VD}{W} = -\sqrt{\frac{2W}{\rho S C_L}}\frac{C_D}{C_L} = -\sqrt{\frac{2W}{\rho S}}\frac{C_D}{C_L^{3/2}}$$

To minimize $\frac{dH}{dt}$, we must minimize the term $\frac{C_D}{C_L^{3/2}}$. Using the parabolic drag polar $C_D = C_{D0} + KC_L^2$, we differentiate the ratio with respect to $C_L$ and set it to zero:

$$\frac{d}{dC_L}\left(\frac{C_{D0} + KC_L^2}{C_L^{3/2}}\right) = 0$$

Solving this yields the condition for minimum sink rate:

$$3C_{D0} = KC_L^2 \implies C_L = \sqrt{\frac{3C_{D0}}{K}}$$

This indicates that maximum endurance occurs at a higher lift coefficient than maximum range, specifically where the induced drag is three times the zero-lift drag.

At this operating point,

$$C_D = C_{D0} + KC_L^2 = C_{D0} + 3C_{D0} = 4C_{D0},$$

so the efficiency and corresponding glide angle for Max Endurance/Min Sink Rate are

$$E = \frac{C_L}{C_D} = \frac{\sqrt{3C_{D0}/K}}{4C_{D0}}, \qquad \gamma \approx -\frac{1}{E} \qquad R = \Delta HE \qquad T = 2\beta E\sqrt{\frac{\rho_0 C_L}{2(W/S)}}\left(1 - e^{-\frac{H_{\text{init}}}{2\beta}}\right)$$

This condition corresponds to the $P_{req}$ condition which we know of. And as we know $\frac{dH}{dt} = \frac{-VD}{W}$, this proves that the glider achieves Min Sink Rate and Max Endurance when $C_L^{3/2}/C_D$ is maximized, which occurs at $C_L = \sqrt{3C_{D0}/K}$.

Thus for Min. Sink Rate condition we have to operate the aircraft at the following parametrs

| Altitude | $C_l$ | $V$ (m/s) | $E$ | $\gamma$ | $X_e$ (km) | Time (s) |
|---|---|---|---|---|---|---|
| 3000 m | 1.5457 | 29.3355 | 10.734 | -5.3377 | 0 | 0 |
| 0 m | 1.5457 | 24.9641 | 10.734 | -5.3377 | 32.202 | 1191.239 |

Table 2: Flight Parameters

**Note:** One thing to note here is that the $C_L$ required for this acse is not practically possible as $C_{L2} > C_{Lmax}$. So we can say that the aircraft will not be able to achieve the corresponding values and will rather achieve Max Endurance conditions at Max $C_L$ condition instead.

### Case 3: Maximum $C_L$

For this case we have the value of $C_L$ and using it we will solve for all other parameters like $C_D$, V, E, $\gamma$, Range and Endurance.

$$C_D = C_{D0} + KC_L^2 \qquad V = \sqrt{\frac{2W}{\rho SC_L}} \qquad E = \frac{C_L}{C_D} \qquad \gamma = -\frac{1}{E}$$

$$R = \Delta HE \qquad T = 2\beta E\sqrt{\frac{\rho_0 C_L}{2(W/S)}}\left(1 - e^{-\frac{H_{\text{init}}}{2\beta}}\right)$$

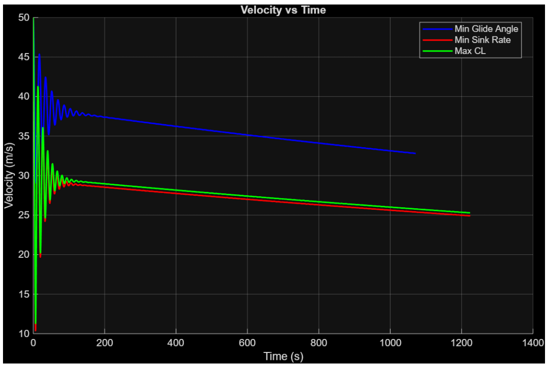
Aircraft Parametrs when flying at Max $C_L$:

| Altitude | $C_l$ | $V$ (m/s) | $E$ | $\gamma$ | $X_e$ (km) | Time (s) |
|---|---|---|---|---|---|---|
| 3000 m | 1.5 | 29.7790 | 10.8932 | -5.2598 | 0 | 0 |
| 0 m | 1.5 | 25.3415 | 10.8932 | -5.2598 | 32.679 | 1190.901 |

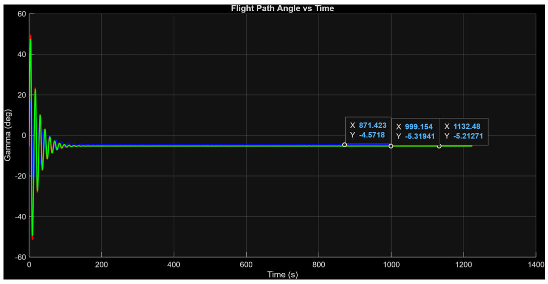
Table 3: Flight Parameters

## Simulation Setup and Results

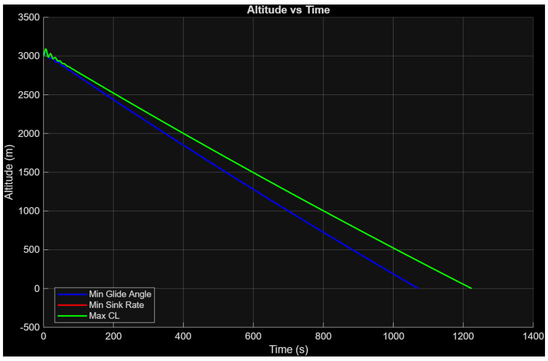
The simulation was performed using MATLAB's `ode45` solver. The initial conditions were given as per the problem statement and ut as initial conditions in the solver to solve the Differential Equations of motions. A solver function file was created with the original Equations of motions which was then called into the main file for solving the situation wise problems.



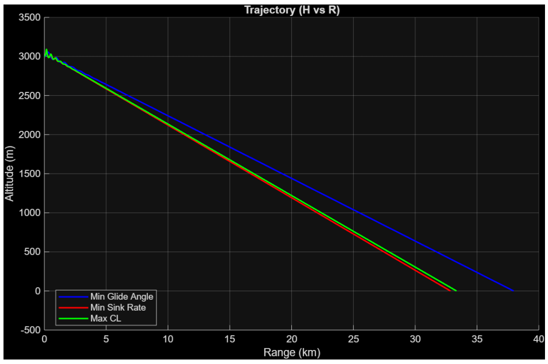

(a) Velocity vs Time



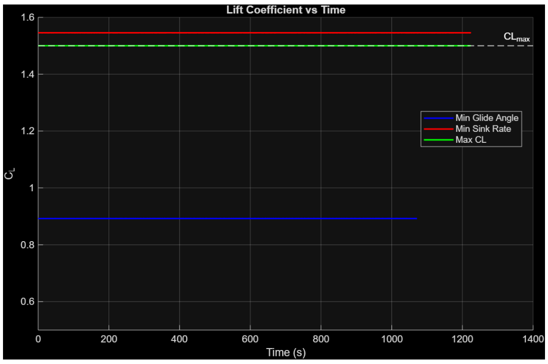

(b) $\gamma$ vs Time



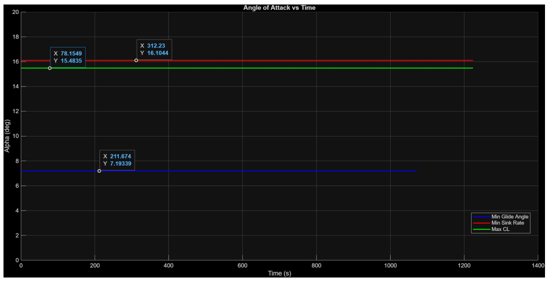

(c) Altitude vs Time




(d) Trajectory




(e) $C_L$ vs Time



(f) Angle of Attack vs Time

Figure 1: Simulation Results

Listing 1: Glide Performance Analysis Matlab Code Output

```
Case: Min Glide Angle | Range: 37.96 km | Time: 1071.47 s |
   CL: 0.892 | Alpha:  7.19 deg
Case: Min Sink Rate   | Range: 32.85 km | Time: 1224.37 s |
   CL: 1.546 | Alpha: 16.10 deg
Case: Max CL          | Range: 33.34 km | Time: 1223.83 s |
   CL: 1.500 | Alpha: 15.48 deg
```

# Analysis and Conclusion

- **Phugoid Oscillations:**The simulation results exhibit distinct oscillatory behavior in both velocity ($V$) and flight path angle ($\gamma$) before settling into steady-state trim conditions.

These are identified as phugoid oscillations, a classic long-period mode of motion common in lightly damped aircraft such as gliders.

- When the glider airspeed exceeds the trim velocity, the increased lift causes the aircraft to pitch up and climb, converting kinetic energy into potential energy (altitude). As it climbs, airspeed drops below the trim value, causing lift to decrease. The glider then noses down, accelerating under gravity and converting potential energy back into kinetic energy, restarting the cycle.
- As seen in the graphs, the velocity and flight path angle are out of phase by approximately 90 degrees. The maximum rate of climb (steepest $\gamma$) occurs roughly when the velocity is highest, driving the glider upward.
- **Case 1 (Min Glide Angle):** This condition provides the maximum range (37.96 km) by flying at a higher velocity and lower angle of attack ($\alpha \approx 7.2°$). This is optimal for distance tasks but results in a shorter flight duration (17.9 min).
- **Case 2 (Min Sink Rate):** This condition maximizes endurance (20.4 min) by flying at a slower speed and higher angle of attack ($\alpha \approx 16.1°$). While the glider stays airborne longer, it covers significantly less ground distance (32.85 km).
- **Case 3 (Max $C_L$):** Operating near the stall point ($\alpha \approx 15.5°$) yields results very similar to the minimum sink rate, but with reduced safety margins against stall. As the $C_L$ required for Case 2 is physically impossible as it exceeds $C_{LMax}$ so the Maximum Endurance for this particular aircraft will be exhibited by this case.
- The analytical and numerical solutions are approximately the same and the difference arises due to the use of approximation while solving analytically but still Analytical solutions forms the basis of this numerical solution
- The computational analysis successfully determined the optimal flight envelopes for the glider. The presence of phugoid oscillations validates the dynamic model, confirming that the aircraft behaves as a physically realistic point mass exchanging energy around a stable equilibrium.
- The variation in flight parameters across the three cases highlights the distinct aerodynamic trade-offs inherent in glider performance. In the Minimum Glide Angle case, the aircraft operates at a lower angle of attack ($\alpha \approx 7.2°$) and a lower lift coefficient ($C_L \approx 0.89$), necessitating a higher trim velocity to balance weight; this configuration minimizes drag relative to lift, thereby maximizing range at the expense of endurance. Conversely, the Minimum Sink Rate and Maximum $C_L$ cases require flying at much higher angles of attack ($\alpha > 15°$) to generate maximum lift at lower speeds. This shift significantly reduces the trim velocity and descent rate, extending flight duration (endurance) by over 2 minutes compared to the max range case, but the increased induced drag results in a steeper glide slope and reduced ground distance. Thus, the transition from maximum range to maximum endurance involves shifting the trim point from a high-speed, low-drag regime to a low-speed, high-lift regime.

# A Approximations for Analytical Solution

- $Thrust = 0$
- $\frac{dV}{dt} \approx 0$
- $\frac{d\gamma}{dt} \approx 0$
- $\gamma$ is small, Hence $\sin\gamma = \gamma$ and $\cos\gamma = 1$