An Eggciting Dilemma

Problem

A farmer was taking her eggs to market in her cart, and she hit a pothole, which knocked over all the containers of eggs. Though she herself was unhurt, every egg was broken. So, she went to her insurance agent, who asked how many eggs she had. She said that she did not know, but remembered some things from the various ways she tried packing the eggs.

She knows when she put the eggs in groups of three, there was one egg left over. When she put them in groups of four, there was also one egg left over. When she put them in groups of five, there were two eggs left over and when she put them in groups of six, there was one egg left over. But when she put them in groups of seven, she ended up with complete groups of seven with no eggs left over.

Your task is to answer the questions: What can the farmer and insurance agent figure out from this information about how many eggs the farmer had? Is there more than one possibility? Carefully explain.

Translation into math

Let the number of eggs the farmer has be x. Since when she put the eggs into groups of three there is 1 left over, when x is divided by 3 there is a remainder of 1, or in mathematical notation

$x \bmod 3 \equiv 1$

Similarly, we can translate the other observations of the number of eggs. When she put them in groups of four, there was also one egg left over:

$x \bmod 4 \equiv 1$

When she put them in groups of five, there were two eggs left over and when she put them in groups of six, there was one egg left over:

$x \bmod 5 \equiv 2$

$x \bmod 6 \equiv 1$

When she put them in groups of seven, she ended up with complete groups of seven with no eggs left over:

$x \bmod 7 \equiv 0$

Then the system of equations that determines the number of eggs is:

$$\begin{cases} x \bmod 3 \equiv 1 \\ x \bmod 4 \equiv 1 \\ x \bmod 5 \equiv 2 \\ x \bmod 6 \equiv 1 \\ x \bmod 7 \equiv 0 \end{cases}$$

Answer

The farmer can have 217 + 420 n eggs, where $n \in \mathbb{Z}$ (n is an integer)

Solution

Solving simultaneous mod equations

When solving a mod equation a mod b ≡ c, we know that (a+bn) mod b ≡a mod b, $n \in \mathbb{Z}$ because bn is a multiple of b so a+bn will have the same remainder when divided by b as a. Then, if one solution a_1 is found, then since

$(a_1 + b\,n) \bmod b \equiv a_1 \bmod b \equiv c$

so the solution set of the equation a mod b ≡ c is $a_1 + b\,n,\ n \in \mathbb{Z}$.

If there are two equations now:

$$\begin{cases} a \bmod b \equiv c \\ a \bmod e \equiv f \end{cases}$$

and we find a number d which is a multiple of both b and e, then the above system also implies that

$$\begin{cases} (a + d) \bmod b \equiv c \\ (a + d) \bmod e \equiv f \end{cases},\ n \in \mathbb{Z}$$

This means that if we find a solution a_1 to the system then a_1+dn, $n \in \mathbb{Z}$ will also be a solution. However, this will not capture all solutions because d might could be any multiple of both b and e, so this may skip some solutions that would also work. In order to not skip any solutions, d must be the least common multiple of b and e in order for a_1+dn, $n \in \mathbb{Z}$ to capture all solutions to the system above.

The same logic applies for systems of many mod equations. Thus, if a solution x_1 to the system

$$\begin{cases} x \bmod 3 \equiv 1 \\ x \bmod 4 \equiv 1 \\ x \bmod 5 \equiv 2 \\ x \bmod 6 \equiv 1 \\ x \bmod 7 \equiv 0 \end{cases}$$ is found, then the full solutions will be $x, +n \cdot \mathrm{lcm}(3, 4, 5, 6, 7),\ n \in \mathbb{Z}$.

Solving the system

One method would be to guess numbers for x, until one works. Though this seems tedious, it would be simple to code because a computer simply has to iterate x and check if the mod equations hold true. In addition, the computer would only need to check the numbers between 1 and lcm(3,4,5,6,7) inclusive, because if a number outside that range works then one can subtract or add lcm(3,4,5,6,7) until it is in the range.

It is easy to see that 7=7+0=6+1=5+2 is a solution to the equations $\begin{cases} x \bmod 5 \equiv 2 \\ x \bmod 6 \equiv 1 \\ x \bmod 7 \equiv 0 \end{cases}$, so by the logic above the full solution to this smaller system would be $7 + n \cdot \mathrm{lcm}(5, 6, 7),\ n \in \mathbb{Z}$. Since 5, 6, and 7 are relatively prime, lcm(5,6,7)=$5 \times 6 \times 7$=210, so the general solution to this smaller system is $7 + 210n$. Some examples are 7, 7+210=217, 7+420=427, etc. However, we are looking at the full system which is

$$\begin{cases} x \bmod 3 \equiv 1 \\ x \bmod 4 \equiv 1 \\ x \bmod 5 \equiv 2 \\ x \bmod 6 \equiv 1 \\ x \bmod 7 \equiv 0 \end{cases}$$

so the earlier answer must also satisfy the equations x mod 3≡1 and x mod 4≡1. if we look at the example 7, it does not work, because 7 mod 3 ≡ 1 but 7 mod 4 ≡ 3. Now if we look at the example 217, it does work with the full system, because 217 mod 3 ≡ 1 and 217 mod 4 ≡ (216+1) mod 4 ≡ 1 mod 4 ≡ 1.

Now that we have a solution, 217, the general solution will be $217 + n \cdot \mathrm{lcm}(3, 4, 5, 6, 7),\ n \in \mathbb{Z}$ as discussed above. Since 6 is a multiple of 3 and 210=lcm(5,6,7)

lcm(3,4,5,6,7)=lcm(4,5,6,7)=lcm(4,210)

210 is a multiple of $2=2^1$ but not of $4=2^2$, so the lcm(4,210) needs one more 2 in its prime factorization than 210, so

lcm(3,4,5,6,7)=lcm(4,210)=210×2=420.

Plugging this into the original solution gives a general solution of $217 + 420\,n,\ n \in \mathbb{Z}$, so that is the number of eggs the farmer could have.

If we say that the farmer had a positive number of eggs, then the answer becomes more restricted, so she can have $217 + 420\,n,\ n \in \mathbb{N}_0$ (n is a whole number) ■

Alternate Solution

We could use the same solution as above to solve two mod equations, then add the third into the system, then the 4th, then the 5th, then the 6th. At first I tried this, but then I realized it would be easier if I started with the last three because 7 was an easy to find solution to the last three equations.

Conclusion

Extensions

The problem could be extended by adding some other condition, such as that the farmer has less than 400 eggs to make a unique solution possible. Alternatively, the farmer could have observed more about her eggs and thus make it a little longer.

Self-assessment

This work was helpful to practice with simultaneous mod equations and I think I did good work. When I first did the problem I worked on it with Bharath and he helped me check some cases, though he used the alternate solution (at first) which was longer.