ROI Testing

Dominik Blank

Georg-August-Universität Göttingen

November 12, 2019

1. ROI Testing
   - The statistical model
   - Testing for the ROI
     - Preparations
     - Distribution of $\mathbb{P}(\tilde{d}^{\pm}(i,j) \leq t \mid d^{\pm}(i,j) = d)$
     - Statistical significance
     - Power of the test
     - Numerical results

2. Morphological operations
   - Opening and closing
   - Hypothesis testing and opening

ROI Testing

Let $M, N \in \mathbb{N}$ and $G = \{0, \dots, M-1\} \times \{0, \dots, N-1\}$. Assume we are given data

$$f(i,j) = c + v(i,j) + \varepsilon_{i,j}$$

- $(i,j) \in G$
- $c \in \mathbb{R}$ is constant
- $v : G \to \{0, \pm c\}$
- $\varepsilon_{m,n} \sim \mathcal{N}(0, \sigma^2)$ i.i.d. normal distributed random variables

Assumption 1: The image f contains a rectangular region of interest.

Assumption 2: The ROI has a checkerboard pattern.

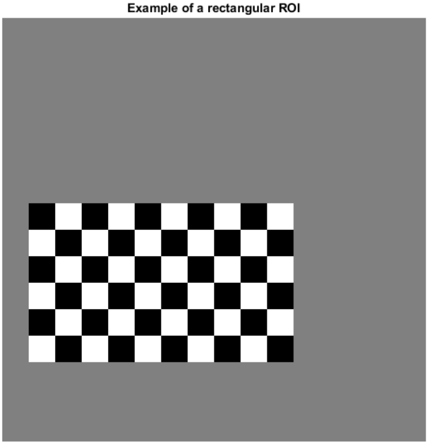


Figure: Example of a possible region of interest. ($M = 16$, $N = 16$)

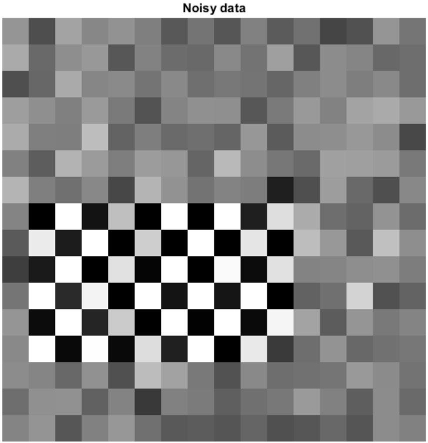


Figure: Same region of interest with noise added. ($\sigma = 30$)

Goal: Construct a statistical test for the region of interest.

Observation: If a pixel is background, its top and left neighbour pixels or bottom and right neighbour pixels are background as well.

For each pair $(i, j) \in G$ we define four non-observable values

$$d_1^{\pm}(i,j) = v(i \pm 1, j) - v(i,j)$$
$$d_2^{\pm}(i,j) = v(i, j \pm 1) - v(i,j)$$

and combine them to two new values

$$d^{\pm}(i,j) = \sqrt{(v(i \pm 1, j) - v(i,j))^2 + (v(i, j \pm 1) - v(i,j))^2}$$

The definition of $d^{\pm}(i,j)$ helps us define the null hypothesis:

$$H_0 : \min\{d^+(i,j), d^-(i,j)\} = 0$$

Since by definition $d^{\pm}(i,j) \geq 0$, our alternative hypothesis becomes

$$H_1 : \min\{d^+(i,j), d^-(i,j)\} > 0$$

We also define four observable values

$$\begin{aligned}\tilde{d}_1^{\pm}(i,j) &= f(i \pm 1, j) - f(i,j)\\ \tilde{d}_2^{\pm}(i,j) &= f(i, j \pm 1) - f(i,j)\end{aligned}$$

and combine them to

$$\tilde{d}^{\pm}(i,j) = \sqrt{\tilde{d}_1^{\pm}(i,j)^2 + \tilde{d}_2^{\pm}(i,j)^2}$$

We use $T = \min\{\tilde{d}^{+}(i,j), \tilde{d}^{-}(i,j)\}$ as our test statistic.

Since v only takes values in $\{0, -c, c\}$, $d^{\pm}$ also can only attain values in

$$\mathcal{D} = \{0, c, 2c, \sqrt{2}c, \sqrt{5}c, \sqrt{8}c\}$$

We want to determine the distribution of $\tilde{d}^{\pm}(i, j)$ conditioned on $d^{\pm}(i, j) = d$ for some $d \in \mathcal{D}$.

We get

$$\mathbb{P}(\tilde{d}^{\pm}(i,j) \leq t \mid d^{\pm}(i,j) = d) = \mathbb{P}\left(\sqrt{2}\sigma\sqrt{\left(\frac{X_1}{\sqrt{2}\sigma}\right)^2 + \left(\frac{X_2}{\sqrt{2}\sigma}\right)^2} \leq t\right)$$

with

$$X_1 = d_1^{\pm}(i,j) + \varepsilon_{m\pm 1,n} - \varepsilon_{m,n} \sim \mathcal{N}(d_1^{\pm}(i,j), 2\sigma^2)$$
$$X_2 = d_2^{\pm}(i,j) + \varepsilon_{m,n\pm 1} - \varepsilon_{m,n} \sim \mathcal{N}(d_2^{\pm}(i,j), 2\sigma^2)$$

Using the cumulative distribution function of the non-central chi distribution, we obtain

$$\begin{aligned}\mathbb{P}(\tilde{d}^{\pm}(i,j) \leq t \mid d^{\pm}(i,j) = d) &= \mathbb{P}\left(\sqrt{\left(\frac{X_1}{\sqrt{2}\sigma}\right)^2 + \left(\frac{X_2}{\sqrt{2}\sigma}\right)^2} \leq \frac{t}{\sqrt{2}\sigma}\right) \\ &= 1 - Q_1\left(\frac{d}{\sqrt{2}\sigma}, \frac{t}{\sqrt{2}\sigma}\right)\end{aligned}$$

For $d = 0$ this simplifies a lot and we get

$$\mathbb{P}(\tilde{d}^{\pm}(i,j) \leq t \mid d^{\pm}(i,j) = 0) = 1 - \exp\left(-\frac{t^2}{4\sigma^2}\right)$$

Using this, we get an upper bound for the probability of a type I error:

$$\mathbb{P}(T \geq t \mid H_0) \leq \exp\left(-\frac{t^2}{4\sigma^2}\right)$$

By taking $t = 2\sigma\sqrt{-\log(\alpha)}$ we thus can assure a statistical significance of α.

We are also interested in bounds for the probability of a type II error. Using results and notations from the previous sections, we get the lower bound

$$\beta = \mathbb{P}(T \leq t \mid H_1) \geq 1 - Q_1\left(\frac{2c}{\sigma}, \frac{t}{\sqrt{2}\sigma}\right)$$

On the other hand, we get the upper bound

$$\beta = \mathbb{P}(T \leq t \mid H_1) \leq 2 \cdot \left(1 - Q_1\left(\frac{c}{\sqrt{2}\sigma}, \frac{t}{\sqrt{2}\sigma}\right)\right)$$

Thus we can conclude, that

$$\beta \in \left[1 - Q_1\left(\frac{2c}{\sigma}, \frac{t}{\sqrt{2}\sigma}\right), \min\left\{2 \cdot \left(1 - Q_1\left(\frac{c}{\sqrt{2}\sigma}, \frac{t}{\sqrt{2}\sigma}\right)\right), 1\right\}\right]$$

In the case of a grayscale image, we assume $c = 127.5$. For $t = 2\sigma\sqrt{-\log(\alpha)}$ and $\alpha = 0.05$ we get the following bounds dependent on the standard deviation σ.

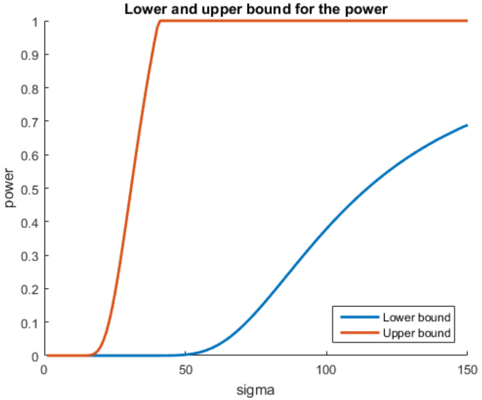


Figure: For $\alpha = 0.05$ this graph shows the lower and upper bounds for the power of the test for $\sigma \in \{1, 2, \ldots, 150\}$.

- No type II errors for $\sigma \in \{1, 2, \dots, 8\}$.
- Until $\sigma = 21$ the probability of a type II error stays below $\alpha = 0.05$.
- Starting at $\sigma = 41$ we can only use the trivial upper bound, i.e. 1.
- The lower bound stays at 0 until $\sigma = 23$.
- At $\sigma = 115$ the lower bound becomes bigger than 0.5.

Morphological operations

We start by defining erosion and dilation of binary images.

Definition

Let $A, B \subseteq \mathbb{R}^m$. The binary erosion of A by B is defined as

$$A \ominus_b B = \{x \in \mathbb{R}^m \mid x + b \in A \text{ for every } b \in B\}$$

Definition

Let $A, B \subseteq \mathbb{R}^m$. The binary dilation of A by B is defined as

$$A \oplus_b B = \{c \in \mathbb{R}^m \mid c = a + b \text{ for some } a \in A \text{ and } b \in B\}$$

Now we can define binary opening and closing.

Definition

The opening of an image A by a structuring element B is defined as

$$A \circ B = (A \ominus B) \oplus B$$

Definition

The closing of an image A by a structuring element B is defined as

$$A \bullet B = (A \oplus B) \ominus B$$

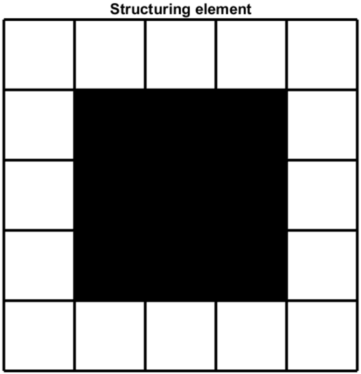
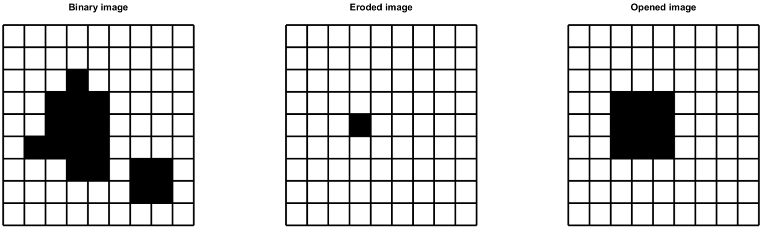
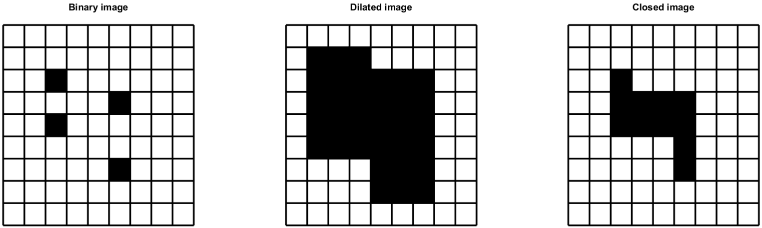


Figure: A 3×3 structuring element.



Example of a binary image (black boxes represent 1). The second image is the erosion of the image by a 3×3 structuring element. The third image is the dilation of the erosion, i.e. the opening of the image.



Example of a binary image (black boxes represent 1). The second image is the dilation of the image by a 3×3 structuring element. The third image is the erosion of the dilation, i.e. the closing of the image.

Question: What is the effect of opening and closing on statistical significance and power?

We will now take a look at the effect of opening on the significance level.

Theorem

Let f be an image that contains a rectangular ROI. Assume that we are given a binarized image f_{bin} with

$$\mathbb{P}(f_{bin}(i,j) = 1 \mid H_0(i,j)) \leq \alpha$$

where $H_0(i,j)$ denotes the null hypothesis for the pixel (i,j), which is, that it is a background pixel and thus should be set to zero.
Let $k \in \mathbb{N}$ be odd and B be a square structuring element with side length k. Then the following inequality holds:

$$\mathbb{P}((f_{bin} \circ B)(i,j) = 1 \mid H_0(i,j)) \leq k^2 \alpha^k$$