Definition 1. Let $M, N \in \mathbb{N}$ and $V \in \mathbb{R}^{M\times N}$ be a matrix. Assume there are two pairs of indices $(i_{tlc}, j_{tlc}), (i_{brc}, j_{brc})$ with $1 \leq i_{tlc} \leq i_{brc} \leq M$ and $1 \leq j_{tlc} \leq j_{brc} \leq N$, such that

$$V(i,j) \neq 0 \text{ if and only if } (i,j) \in \{i_{tlc}, \dots, i_{brc}\} \times \{j_{tlc}, \dots, j_{brc}\} \tag{1}$$

We call $R = \{i_{tlc}, \dots, i_{brc}\} \times \{j_{tlc}, \dots, j_{brc}\}$ a *rectangular region of interest (rROI)* and say that V contains the rROI R.
Furthermore, we call (i_{tlc}, j_{tlc}) the *top left corner* and (i_{brc}, j_{brc}) the *bottom right corner* of the rROI.

Definition 2. Let $M, N \in \mathbb{N}$ and $c \in \mathbb{R} \setminus \{0\}$. Let $V \in \mathbb{R}^{M\times N}$ be a matrix, that only takes values in the set $\{0, \pm c\}$ and that contains a rectangular region of interest R. We say that R has a *checkerboard pattern*, if one of the following relations is true:

$$\text{For all } (i,j) \in R : V(i,j) = c \text{ if and only if } i+j \text{ is odd.} \tag{2a}$$

$$\text{For all } (i,j) \in R : V(i,j) = c \text{ if and only if } i+j \text{ is even.} \tag{2b}$$

Remark 1. If the first relation in the definition is true, that immediately implies, that

$$\text{for all } (i,j) \in R : V(i,j) = -c \text{ if and only if } i+j \text{ is even,}$$

since V takes only values in $\{0, \pm c\}$, but for $(i,j) \in R$ we have $V(i,j) \neq 0$. Similarly, if the second relation is true, it implies that

$$\text{for all } (i,j) \in R : V(i,j) = -c \text{ if and only if } i+j \text{ is odd.}$$

In both cases the values of V alternate between $+c$ and $-c$ along the rows and columns of R. This is similar to the classical checkerboard pattern.

Theorem 1. *Assume the following statistical model:*
Let $M, N \in \mathbb{N}$ *and* $G = \{1, \ldots, M\} \times \{1, \ldots, N\}$. *We are given data*

$$F(i,j) = c + V(i,j) + \varepsilon_{i,j} \tag{3}$$

where $(i,j) \in G$, $c \in \mathbb{R}$ *is constant,* $V(i,j) \in \{0, \pm c\}$ *and* $\varepsilon_{i,j} \sim \mathcal{N}(0, \sigma^2)$ *are i.i.d. normal distributed random variables for some* $\sigma > 0$ *and for all* $(i,j) \in G$.
Assume that V *contains a rectangular region of interest* R *and that* R *has a checkerboard pattern.*
Let $\varphi_\alpha \in \{0,1\}^{M \times N}$ *be the binary matrix, that represents the decision of the testing procedure to a given statistical significance* α.
Let $k \in \mathbb{N}$ *be odd and* B *be a square structuring element with side length* k.
Then the following inequality holds:

$$\mathbb{P}((\varphi_\alpha \circ B)(i,j) = 1 \mid H_0(i,j)) \leq k\alpha^{\frac{k+1}{2}}$$

Proof. We aim to find an upper bound for the probability

$$\mathbb{P}((\varphi_\alpha \circ B)(i,j) = 1 \mid H_0(i,j))$$

To do this, we first notice that $H_0(i,j)$ is equivalent to $V(i,j) = 0$, but since V contains a rectangular region of interest, this means that $i < i_{tlc}$ or $i > i_{brc}$ or $j < j_{tlc}$ or $j > j_{brc}$. We need to differentiate cases here. These four cases are not mutually exclusive, but have different implications for the row/column of the index (i,j) and a neighbouring row/column:

case	row/column of (i,j)	neighbouring row/column
$i < i_{tlc}$	$V(i,1) = \cdots = V(i,N) = 0$	$V(i-1,1) = \cdots = V(i-1,N) = 0$
$i > i_{brc}$	$V(i,1) = \cdots = V(i,N) = 0$	$V(i+1,1) = \cdots = V(i+1,N) = 0$
$j < j_{tlc}$	$V(1,j) = \cdots = V(M,j) = 0$	$V(1,j-1) = \cdots = V(M,j-1) = 0$
$j > j_{brc}$	$V(1,j) = \cdots = V(M,j) = 0$	$V(1,j+1) = \cdots = V(M,j+1) = 0$

Without loss of generality we assume the first case. This means that the null hypotheses $H(i,1), \ldots, H(i,N)$ and $H(i-1,1), \ldots, H(i-1,N)$ are true.

To be even more precise, it implies that $D^-(i,1) = \cdots = D^-(i,N) = 0$. We have assumed the side length k of the structuring element B to be odd. We define the two index sets $K = \{-\frac{k-1}{2}, -\frac{k-3}{2}, \ldots, \frac{k-3}{2}, \frac{k-1}{2}\}$ and $K_1 = \{-\frac{k-1}{2}, -\frac{k-5}{2}, \ldots, \frac{k-5}{2}, \frac{k-1}{2}\}$. This yields

$$
\begin{aligned}
&\mathbb{P}((\varphi_\alpha \circ B)(i,j) = 1 \mid H_0(i,j)) \\
&= \mathbb{P}\left(\bigcup_{\tilde{m},\tilde{n}\in K} \bigcap_{m,n\in K} \{\varphi_\alpha(i+m-\tilde{m}, j+n-\tilde{n}) = 1\} \mid D^-(i,1) = \cdots = D^-(i,N) = 0\right) \\
&= \sum_{\tilde{n}\in K} \mathbb{P}\left(\bigcup_{\tilde{m}\in K} \bigcap_{m,n\in K} \{\varphi_\alpha(i+m-\tilde{m}, j+n-\tilde{n}) = 1\} \mid D^-(i,1) = \cdots = D^-(i,N) = 0\right) \\
&\leq \sum_{\tilde{n}\in K} \mathbb{P}\left(\bigcup_{\tilde{m}\in K} \bigcap_{n\in K, m=\tilde{m}} \{\varphi_\alpha(i+m-\tilde{m}, j+n-\tilde{n}) = 1\} \mid D^-(i,1) = \cdots = D^-(i,N) = 0\right) \\
&= \sum_{\tilde{n}\in K} \mathbb{P}\left(\bigcup_{\tilde{m}\in K} \bigcap_{n\in K} \{\varphi_\alpha(i, j+n-\tilde{n}) = 1\} \mid D^-(i,1) = \cdots = D^-(i,N) = 0\right) \\
&= \sum_{\tilde{n}\in K} \mathbb{P}\left(\bigcap_{n\in K} \{\varphi_\alpha(i, j+n-\tilde{n}) = 1\} \mid D^-(i,1) = \cdots = D^-(i,N) = 0\right) \\
&\leq \sum_{\tilde{n}\in K} \mathbb{P}\left(\bigcap_{n\in K_1} \{\varphi_\alpha(i, j+n-\tilde{n}) = 1\} \mid D^-(i,1) = \cdots = D^-(i,N) = 0\right) \\
&\leq \sum_{\tilde{n}\in K} \prod_{n\in K_1} \mathbb{P}\left(\{\varphi_\alpha(i, j+n-\tilde{n}) = 1\} \mid D^-(i,1) = \cdots = D^-(i,N) = 0\right) \\
&= \sum_{\tilde{n}\in K} \prod_{n\in K_1} \mathbb{P}\left(T(i, j+n-\tilde{n}) \geq t_\alpha \mid D^-(i,1) = \cdots = D^-(i,N) = 0\right) \\
&\leq \sum_{\tilde{n}\in K} \prod_{n\in K_1} \mathbb{P}\left(\tilde{D}^-(i, j+n-\tilde{n}) \geq t_\alpha \mid D^-(i,1) = \cdots = D^-(i,N) = 0\right) \\
&= \sum_{\tilde{n}\in K} \prod_{n\in K_1} \mathbb{P}\left(\tilde{D}^-(i, j+n-\tilde{n}) \geq t_\alpha \mid D^-(i, j+n-\tilde{n}) = 0\right) \\
&\leq \sum_{\tilde{n}\in K} \prod_{n\in K_1} \alpha \\
&= |K| \alpha^{|K_1|} \\
&= k \alpha^{\frac{k+1}{2}}
\end{aligned}
$$

The other three cases can be proven in a similar way by swapping the roles of m or $\tilde{m}$ with n or $\tilde{n}$, respectively and/or by replacing D^- by D^+. $\square$