ROI Testing

Dominik Blank

Georg-August-Universität Göttingen

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ROI Testing

Let $M, N \in \mathbb{N}$ and $\Omega = \{0, \ldots, M-1\} \times \{0, \ldots, N-1\}$. Assume we are given data

$$F(i,j) = c + V(i,j) + \varepsilon_{i,j}$$

- $(i,j) \in \Omega$
- $c \in \mathbb{R}$ is constant
- $V(i,j) \in \{0, \pm c\}$
- $\varepsilon_{i,j} \sim \mathcal{N}(0, \sigma^2)$ i.i.d. normal distributed random variables

Assumption 1: The image F contains a rectangular region of interest.

Assumption 2: The ROI has a checkerboard pattern.

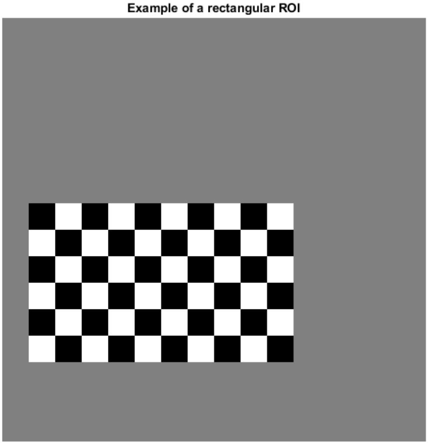


Figure: Example of a possible region of interest. ($M = 16$, $N = 16$)

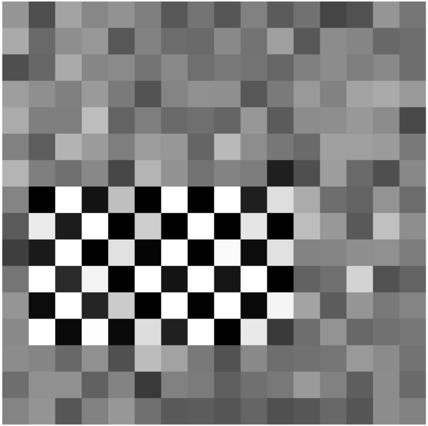


Figure: Same region of interest with noise added. ($\sigma = 30$)

Goal: Construct a statistical test for the region of interest.

For each pixel $(i,j) \in \Omega$ we define four non-observable and four observable values

$$\text{non-observable} \begin{cases} D_1^{\pm}(i,j) = V(i \pm 1, j) - V(i,j) \\ D_2^{\pm}(i,j) = V(i, j \pm 1) - V(i,j) \end{cases}$$

$$\text{observable} \begin{cases} \tilde{D}_1^{\pm}(i,j) = F(i \pm 1, j) - F(i,j) \\ \tilde{D}_2^{\pm}(i,j) = F(i, j \pm 1) - F(i,j) \end{cases}$$

and combine them to new values

$$D^{\pm}(i,j) = \sqrt{D_1^{\pm}(i,j)^2 + D_2^{\pm}(i,j)^2}$$

$$\tilde{D}^{\pm}(i,j) = \sqrt{\tilde{D}_1^{\pm}(i,j)^2 + \tilde{D}_2^{\pm}(i,j)^2}$$

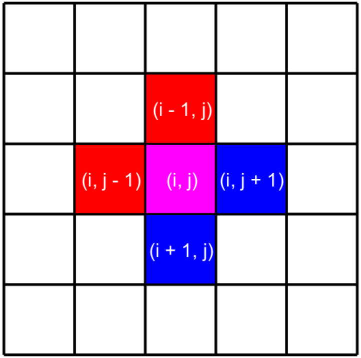


Figure: Pixel (i, j) with its neigbour pixels. The blue neighbours are used to calculate $\tilde{D}^{+}(i, j)$ and the red neighbours are used to calculate $\tilde{D}^{-}(i, j)$. The purple pixel is used for both $\tilde{D}^{+}(i, j)$ and $\tilde{D}^{-}(i, j)$.

We now test the null hypothesis

$$H_0 : \min\{D^+(i,j), D^-(i,j)\} = 0$$

against the alternative hypothesis

$$H_1 : \min\{D^+(i,j), D^-(i,j)\} > 0$$

using the test statistic

$$T = \min\{\tilde{D}^+(i,j), \tilde{D}^-(i,j)\}$$

Question: How do we determine the distribution of the test statistic T under the null hypothesis H_0?

We observe that

$$\mathbb{P}(T \geq t \mid H_0) \leq \mathbb{P}(\tilde{D}^{\pm}(i,j) \geq t \mid D^{\pm}(i,j) = 0)$$

Thus, if we can find a threshold t_α, s.t.

$$\mathbb{P}(\tilde{D}^{\pm}(i,j) \geq t_\alpha \mid D^{\pm}(i,j) = 0) \leq \alpha$$

for a given α, we have ensured a statistical significance of α.

We have

$$
\begin{aligned}
&\mathbb{P}(\tilde{D}^{\pm}(i,j) \leq t \mid D^{\pm}(i,j) = 0) \\
&= \mathbb{P}\left((\varepsilon_{i\pm1,j} - \varepsilon_{i,j})^2 + (\varepsilon_{i,j\pm1} - \varepsilon_{i,j})^2 \leq t^2 \right) \\
&= \mathbb{P}\left(\sqrt{\left(\frac{X_1}{\sigma}\right)^2 + \left(\frac{X_2}{\sigma}\right)^2} \leq \frac{t}{\sigma} \right)
\end{aligned}
$$

with $X_1 = \varepsilon_{i\pm1,j} - \varepsilon_{i,j}$, $X_2 = \varepsilon_{i,j\pm1} - \varepsilon_{i,j}$ and $Y = \varepsilon_{i,j}$. Then

$$
\begin{aligned}
X_1 &\sim \mathcal{N}(-Y, \sigma^2) \\
X_2 &\sim \mathcal{N}(-Y, \sigma^2) \\
Y &\sim \mathcal{N}(0, \sigma^2)
\end{aligned}
$$

Then we can see, that the square root inside has non-central chi distribution with two degrees of freedom and non-centrality parameter

$$
\lambda = \frac{\sqrt{2}|Y|}{\sigma}
$$

Thus, we have a compound probability distribution:

$$\mathbb{P}(\tilde{D}^{\pm}(i,j) \leq t \mid D^{\pm}(i,j) = 0)$$

$$= \int_0^{\frac{t}{\sigma}} \int_0^{\infty} \underbrace{x \exp\left(-\frac{x^2}{2} - \frac{y^2}{\sigma^2}\right) I_0\left(\frac{\sqrt{2}y}{\sigma}x\right)}_{\text{pdf of } \chi_2\left(\frac{\sqrt{2}y}{\sigma}\right) \text{ for fixed } y} \underbrace{\frac{2}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right)}_{\text{pdf of absolute value of } \mathcal{N}(0,\sigma^2)} dydx$$

$$= \frac{1}{\sqrt{3}}\left(\frac{3}{2} - \frac{3}{2}\exp\left(-\frac{t^2}{3\sigma^2}\right) I_0\left(\frac{t^2}{6\sigma^2}\right)\right) - \sqrt{3}$$

$$- \frac{2-\sqrt{3}}{2} Q_1\left(\frac{2-\sqrt{3}}{6}\sqrt{\frac{t}{\sigma}}, \frac{2+\sqrt{3}}{6}\sqrt{\frac{t}{\sigma}}\right)$$

$$+ \frac{2+\sqrt{3}}{2} Q_1\left(\frac{2+\sqrt{3}}{6}\sqrt{\frac{t}{\sigma}}, \frac{2-\sqrt{3}}{6}\sqrt{\frac{t}{\sigma}}\right)$$

With an easy program the threshold for a given α can be calculated. As can be seen from the CDF, it is enough to calculate the threshold t_α for $\sigma = 1$ and multiply the outcome by σ.

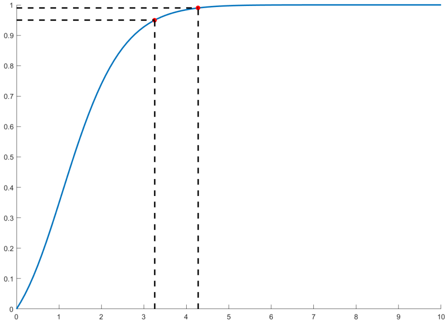


Figure: Cumulative distribution function of $\mathbb{P}(\tilde{D}^{\pm}(i,j) \leq t \mid D^{\pm}(i,j) = 0)$ with thresholds for $\alpha = 0.05$ and $\alpha = 0.01$. ($t_{0.05} = 3.2554$, $t_{0.01} = 4.2791$, $\sigma = 1$)

We are also interested in bounds for the power of this test.
Using results and notations from the previous sections, we get the upper bound

$$1-\beta = \mathbb{P}(T \geq t \mid H_1) \leq \mathbb{P}(\tilde{D}^+(i,j) \geq t \mid D^+(i,j) = \sqrt{8}c)$$

On the other hand, we get the lower bound

$$1-\beta = \mathbb{P}(T \geq t \mid H_1) \geq 2 \cdot \mathbb{P}(\tilde{D}^+(i,j) \geq t \mid D^+(i,j) = c) - 1$$

Thus we can conclude, that

$$1-\beta \geq \max\left\{2 \cdot \mathbb{P}(\tilde{D}^+(i,j) \geq t \mid D^+(i,j) = c) - 1, 0\right\}$$

By simulating the probabilities $\mathbb{P}(\tilde{D}^+(i,j) \geq t \mid D^+(i,j) = \sqrt{8}c)$ and $\mathbb{P}(\tilde{D}^+(i,j) \geq t \mid D^+(i,j) = c)$, we can get empirical bounds for the power of our test.

We have the following relation

$$\begin{aligned} D^+(i,j) = \sqrt{8}c &\Rightarrow D_1^+(i,j) = D_2^+(i,j) = 2c \\ D^+(i,j) = c \quad &\Rightarrow D_1^+(i,j) = D_2^+(i,j) = c \end{aligned}$$

Simulation of the probabilities can be done by following these steps:

- Simulate $\mathcal{N}(0, \sigma^2)$ random variables.
- Calculate $\tilde{D}^+(i,j)$ conditioned on either $D^+(i,j) = \sqrt{8}c$ or $D^+(i,j) = c$.
- Count proportion of simulations where $D^+(i,j) \geq t_\alpha \sigma$.

In the case of a grayscale image, we assume $c = 127.5$. For $t_{0.05} = 3.2554$ and 10.000.000 simulations of the noise terms, we get the following empirical bounds dependent on the standard deviation σ.

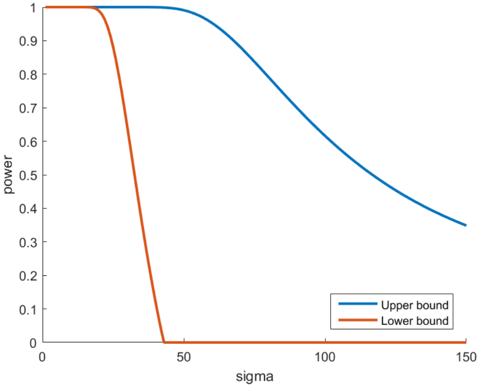


Figure: For $\alpha = 0.05$ this graph shows the lower and upper bounds for the power of the test for $\sigma \in \{1, 2, \dots, 150\}$.

Since $D^{\pm}(i,j)$ is the sum of dependent squared normal distributed random variables, it is not a simple Chi distributed random variable. If it were, we would get

$$1 - \beta \in \left[\max\left\{2 \cdot Q_1\left(\frac{c}{\sqrt{2}\sigma}, \frac{t}{\sqrt{2}\sigma}\right) - 1, 0\right\}, Q_1\left(\frac{2c}{\sigma}, \frac{t}{\sqrt{2}\sigma}\right)\right]$$

In the independent case we would take $t = 2\sigma\sqrt{-\log(\alpha)} \approx 3.3616\sigma$ and get the following bounds for the power dependent on σ. It is remarkable, how similar these bounds are to the empirical bounds.

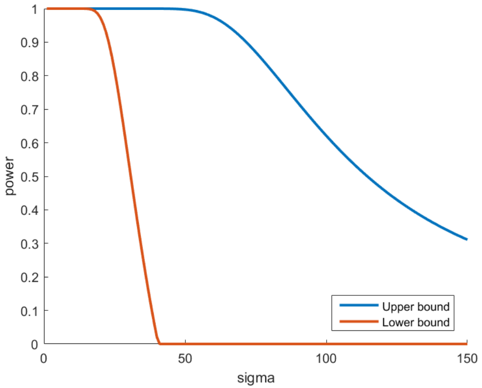


Figure: For $\alpha = 0.05$ this graph shows the lower and upper bounds for the power of the test for $\sigma \in \{1, 2, \ldots, 150\}$, if the terms of $\tilde{D}^{\pm}(i,j)$ were independent.

Morphological operations

Reminder: We always consider structuring elements with odd side length.

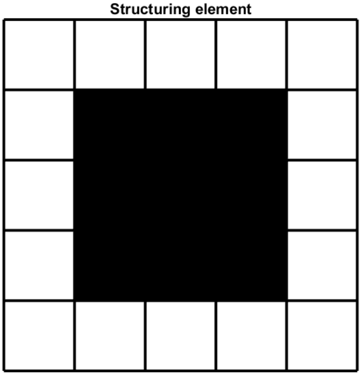
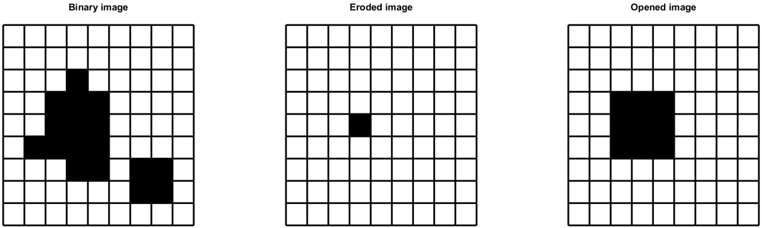
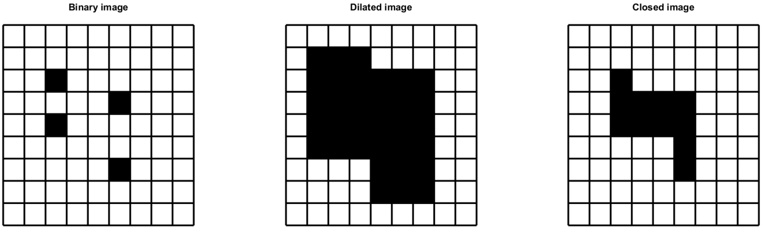


Figure: A 3×3 structuring element.



Example of a binary image (black boxes represent 1). The second image is the erosion of the image by a 3×3 structuring element. The third image is the dilation of the erosion, i.e. the opening of the image.



Example of a binary image (black boxes represent 1). The second image is the dilation of the image by a 3×3 structuring element. The third image is the erosion of the dilation, i.e. the closing of the image.

Question: What is the effect of opening and closing on statistical significance and power?

We will now take a look at the effect of opening on the significance level.

Theorem

Let F be an image that contains a rectangular ROI. Assume that we are given a binarized image F_{bin} with

$$\mathbb{P}(F_{bin}(i,j) = 1 \mid H_0(i,j)) \leq \alpha$$

where $H_0(i,j)$ denotes the null hypothesis for the pixel (i,j).
Let $k \in \mathbb{N}$ be odd and B be a square structuring element with side length k. Then the following inequality holds:

$$\mathbb{P}((F_{bin} \circ B)(i,j) = 1 \mid H_0(i,j)) \leq k\alpha^{\frac{k-1}{2}}$$

Outline of the proof:

- Note, that if H_0 is true for (i, j), then H_0 is true for a whole row or column in the image.
- With our notation from the test, this means, that $D^+ = 0$ or $D^+ = 0$ for the whole row/column.
- If we only take every second pixel in the row/column, they will be independent, which yields the exponent $\frac{k-1}{2}$.

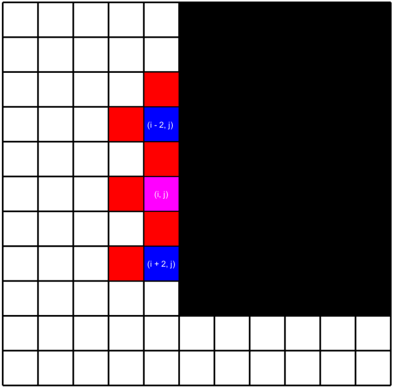


Figure: If we consider only every second pixel in the column, these will be independent.

We will also take a look at the effect of closing on the significance level.

Theorem

Let F be an image that contains a rectangular ROI. Assume that we are given a binarized image F_{bin} with

$$\mathbb{P}(F_{bin}(i,j) = 1 \mid H_0(i,j)) \leq \alpha$$

where $H_0(i,j)$ denotes the null hypothesis for the pixel (i,j).
Let $k \in \mathbb{N}$ be odd and B be a square structuring element with side length k. Then the following inequality holds:

$$\mathbb{P}((F_{bin} \bullet B)(i,j) = 1 \mid H_0(i,j)) \leq k^2\alpha$$

Outline of the proof:

- Note, that if H_0 is true for (i,j), then there exists a square with side length k, such that H_0 is true for every pixel inside that square.
- The square has k^2 pixels and for each pixel the null hypothesis holds, so each pixel has probability $\leq \alpha$ for a type I error.

Putting both results together, one gets

$$\mathbb{P}(((F_{bin} \circ B) \bullet B)(i,j) = 1 \mid H_0(i,j)) \leq k^3 \alpha^{\frac{k-1}{2}}$$

For $\alpha = 0.05$ one has the following statistical significance after opening and closing:

k	3	5	7	9	11
$k^3\alpha^{\frac{k-1}{2}}$	1.35	0.3125	0.042875	0.00455625	0.0004159375

Numerical results

Methodology

- Create 5 test images each for 128×128, 256×256 and 512×512 pixel.
- Determine the correct region of interest.
- Perform the following 50 times to even out outliers:
  - Create standard normal distributed noise.
  - Loop over the standard deviation $\sigma \in \{1, \ldots, 150\}$ and add the noise multiplied by the standard deviation to the image.
    - Binarize the image using the testing procedure from the first section with $\alpha = 0.05$ and $t_{0.05} = 3.2554\sigma$.
    - Perform binary morphological opening.
    - Perform binary morphological closing.
  - In each step, count the number of type I and II errors for each σ separately.
- Add up all type I and II errors for all images and divide by the number of background/foreground pixel, respectively.

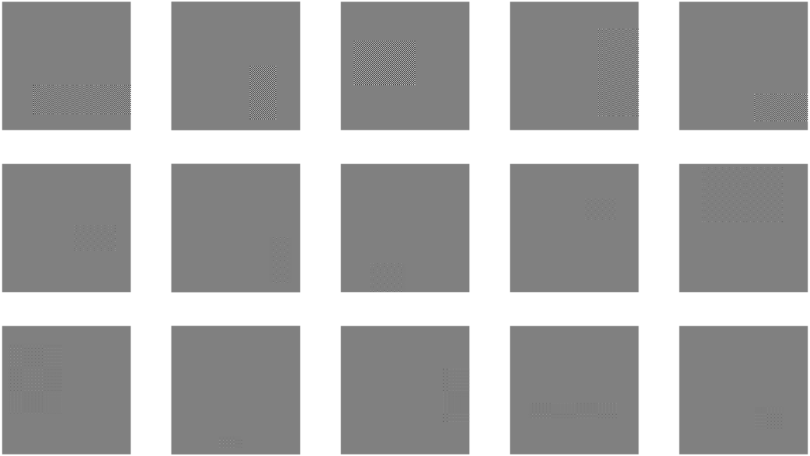


Figure: Test cases used to compute type I and II errors. Top row: 128x128 pixel. Middle row: 256x256 pixel. Bottom row: 512x512 pixel.

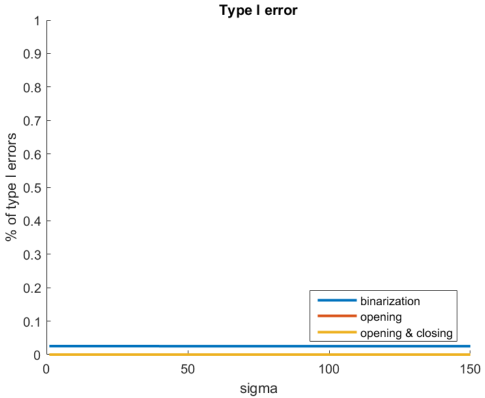


Figure: Numerical results of the percentage of type I errors dependent on σ.

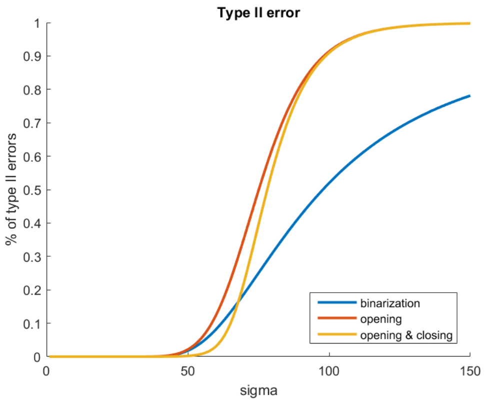


Figure: Numerical results of the percentage of type II errors dependent on σ.