On the influence of morphological operators on testing for a region of interest

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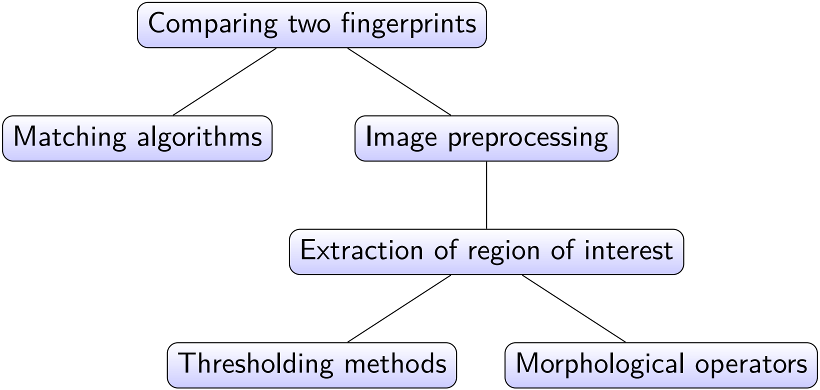
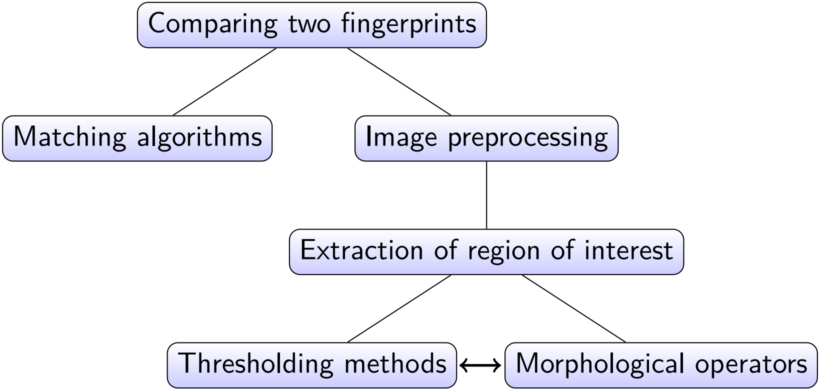
Motivation


Comparing two fingerprints
Matching algorithms
Image preprocessing
Extraction of region of interest
Thresholding methods
Morphological operators


Comparing two fingerprints
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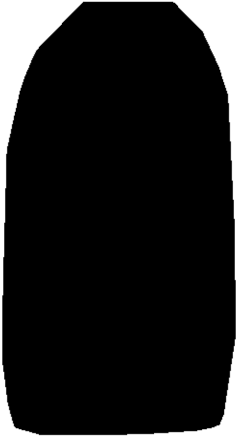
(a) Fingerprint



(b) Binarized region of interest

Figure: Example of region of interest extraction.

Thresholding methods

Provide a binarization of the image to categorize pixels into ROI and background.

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Morphological operators

Improve the categorization.

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Interpretation

Interpretation as a statistical test.

- Falsely classifying a background pixel as foreground $\Leftrightarrow$ Type I error
- Falsely classifying a foreground pixel as background $\Leftrightarrow$ Type II error
- Morphological operators: Lower the amount of the errors in this testing procedure.

Goal: Quantify the change of the probabilities of type I and II errors through application of morphological operators

Goal: Quantify the change of the probabilities of type I and II errors through application of morphological operators *for a specifix pixel*.

- Assume a simplified model: Rectangular region of interest with a checkerboard pattern surrounded by a constant gray background.
  - We can bound the probability of a type I error after thresholding.
  - The model still exhibits convexity of the ROI and an oscillatory pattern within the ROI, as in a fingerprint image.
- Only analyze morphological opening and closing.

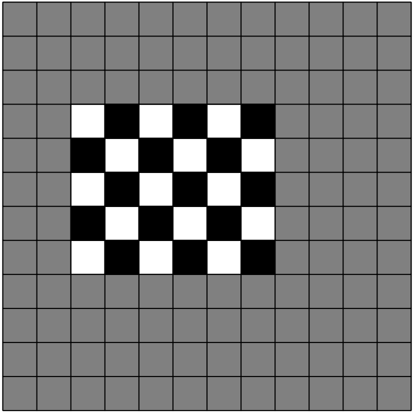


Figure: Example of an image, that contains a rectangular region of interest with a checkerboard pattern. ($m = 12$, $n = 12$)

Roadmap

1. Motivation
2. Thresholding
   - The statistical model
   - Find a statistical test to test for the ROI
   - Bound for the probability of a type I error
   - Analysis of the probability of a type II error
3. Morphological operators
   - Examples of the effect of morphological opening & closing
   - Main results: Changes of the error probabilities after morphological opening & closing
   - Numerical results
4. Outlook

Thresholding

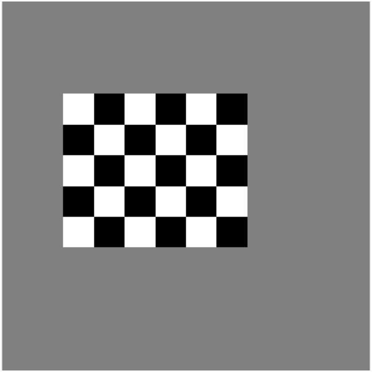
The statistical model

Let $m, n \in \mathbb{N}$ and $\Omega = \{1, \dots, m\} \times \{1, \dots, n\}$. Assume we are given data

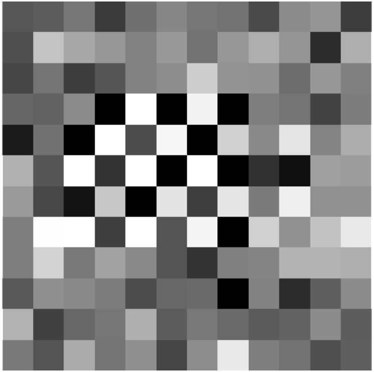
$$F(i,j) = c + V(i,j) + \varepsilon_{i,j}$$

- $(i,j) \in \Omega$
- $c \in \mathbb{R}$ is constant
- $V(i,j) \in \{0, \pm c\}$
- $\varepsilon_{i,j} \sim \mathcal{N}(0, \sigma^2)$ i.i.d. normal distributed random variables

Assumption 1: The image V contains a rectangular region of interest.
Assumption 2: The ROI has a checkerboard pattern.



(a) Example of a possible image V. ($m = 12$, $n = 12$)



(b) Example of a possible F. Same V as on the left side with noise added. ($\sigma = 50$)

Simplifications

1. Assume variance σ of the noise terms to be known beforehand.
2. Ignore the limitations of grayscale images and let the images in our model take all values in the real numbers.

Definition

Let $m, n \in \mathbb{N}$ and let $V \in \mathbb{R}^{m \times n}$. Let $(i,j) \in \Omega$. The *forward and backward discrete derivative of V evaluated at* (i,j) are defined as

$$\Delta^+ V(i,j) = \begin{pmatrix} V(i+1,j) - V(i,j) \\ V(i,j+1) - V(i,j) \end{pmatrix} \in \mathbb{R}^2$$

and

$$\Delta^- V(i,j) = \begin{pmatrix} V(i-1,j) - V(i,j) \\ V(i,j-1) - V(i,j) \end{pmatrix} \in \mathbb{R}^2,$$

respectively.

Testing for the ROI

Observation: Pixel (i,j) being a background pixel is equivalent to $\min\{\|\Delta^+ V(i,j)\|, \|\Delta^- V(i,j)\|\} = 0$.

The statistical test

The null hypothesis, alternative hypothesis and test statistic for our testing procedure are

$$
\begin{aligned}
H_0(i,j) &: \min\{\|\Delta^+ V(i,j)\|, \|\Delta^- V(i,j)\|\} = 0 \\
H_1(i,j) &: \min\{\|\Delta^+ V(i,j)\|, \|\Delta^- V(i,j)\|\} \neq 0 \\
T(i,j) &:= \min\{\|\Delta^+ F(i,j)\|, \|\Delta^- F(i,j)\|\}
\end{aligned}
$$

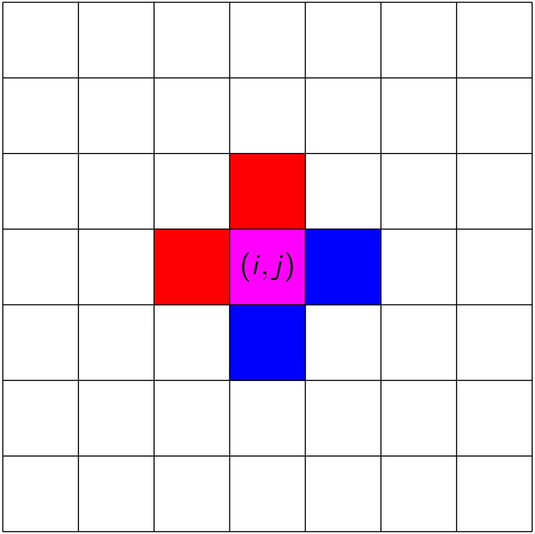


Figure: The random variable $\|\Delta^- F(i,j)\|$ depends on (i,j) and the red pixels. The random variable $\|\Delta^+ F(i,j)\|$ depends on (i,j) and the blue pixels.

Notation

$\mathcal{V}_c^{m,n}$	Set of images $V \in \{0, \pm c\}^{m\times n}$ that contain a rectangular region of interest with a checkerboard pattern.
$\mathbb{P}_V(\dots)$	Probability of an event given a *fixed* image V.
$\mathcal{H}_0(i,j)$	Subset of $\mathcal{V}_c^{m,n}$, such that the null hypothesis at (i,j) is true.
$\mathcal{H}_1(i,j)$	Subset of $\mathcal{V}_c^{m,n}$, such that the alternative hypothesis at (i,j) is true.
Δ^+, Δ^-	Forward and backward discrete derivative operator.
$\|.\|$	ℓ^2-norm

Bound for the probability of a type I error

Goal: Given a statistical significance α, find a threshold t_α, such that

$$\mathbb{P}_V(T(i,j) \geq t_\alpha) \leq \alpha$$

for every $V \in \mathcal{H}_0(i,j)$.

Bound for the probability of a type I error - Step 0

Define

$$\mathcal{H}_0^+(i,j) := \left\{ V \in \mathcal{V}_c^{m,n} \mid \|\Delta^+ V(i,j)\| = 0 \right\},$$
$$\mathcal{H}_0^-(i,j) := \left\{ V \in \mathcal{V}_c^{m,n} \mid \|\Delta^- V(i,j)\| = 0 \right\}.$$

Then $\mathcal{H}_0(i,j) = \mathcal{H}_0^+(i,j) \cup \mathcal{H}_0^-(i,j)$.

Bound for the probability of a type I error - Step 1

Lemma

Let $(i,j) \in \Omega$ *and* $t \in \mathbb{R}^+$. *Let* $V \in \mathcal{V}_c^{m,n}$. *Then*

$$\mathbb{P}_V(T(i,j) \geq t) \leq \min\left\{\mathbb{P}_V(\|\Delta^+ F(i,j)\| \geq t), \mathbb{P}_V(\|\Delta^- F(i,j)\| \geq t)\right\}.$$

Bound for the probability of a type I error - Step 2

Theorem

Let $(i,j) \in \Omega$ *and* $t \in \mathbb{R}^+$. *Let* $V_1 \in \mathcal{H}_0^+(i,j)$ *and* $V_2 \in \mathcal{H}_0^-(i,j)$. *Then*

$$\mathbb{P}_{V_1}(\|\Delta^+ F(i,j)\| \leq t) = p_\sigma(t) = \mathbb{P}_{V_2}(\|\Delta^- F(i,j)\| \leq t)$$

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$$\mathbb{P}_{V_1}(\|\Delta^+ F(i,j)\| \leq t) = p_\sigma(t) = \mathbb{P}_{V_2}(\|\Delta^- F(i,j)\| \leq t)$$

where

$$\begin{aligned} p_\sigma(t) := \frac{1}{\sqrt{3}} \left(\frac{3}{2} - \frac{3}{2} \exp\left(-\frac{t^2}{3\sigma^2} \right) I_0 \left(\frac{t^2}{6\sigma^2} \right) \right) - \sqrt{3} \\ - \frac{2-\sqrt{3}}{2} Q_1 \left(\frac{2-\sqrt{3}}{6} \sqrt{\frac{t}{\sigma}}, \frac{2+\sqrt{3}}{6} \sqrt{\frac{t}{\sigma}} \right) \\ + \frac{2+\sqrt{3}}{2} Q_1 \left(\frac{2+\sqrt{3}}{6} \sqrt{\frac{t}{\sigma}}, \frac{2-\sqrt{3}}{6} \sqrt{\frac{t}{\sigma}} \right) \end{aligned}$$

with I_0 *being the modified Bessel function of the first kind and* Q_M *denoting the Marcum Q-function.*

Bound for the probability of a type I error - Result

Corollary

Let $(i,j) \in \Omega$ and $t \in \mathbb{R}^+$. Then

$$\mathbb{P}_V(T(i,j) \geq t) \leq 1 - p_\sigma(t)$$

for every $V \in \mathcal{H}_0(i,j)$.

Note, that $p_\sigma(t_\alpha \cdot \sigma) = p_1(t_\alpha)$. Hence, it is sufficient to calculate a threshold t_α for $\sigma = 1$. This is done numerically by an easy trial and error algorithm.

Numerical results

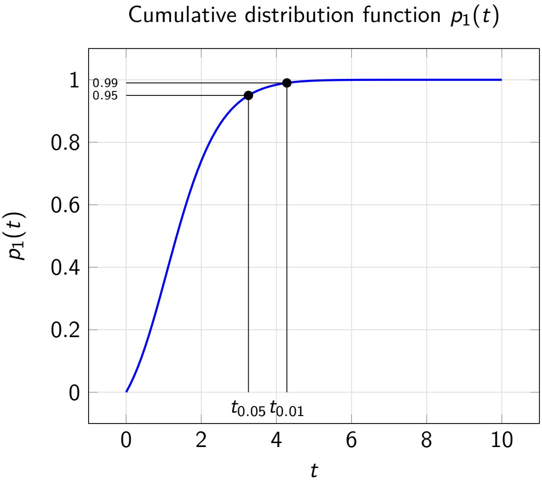


Figure: Plot of the cumulative distribution function $p_1(t)$. The numerically computed thresholds $t_{0.05} = 3.2555$ and $t_{0.01} = 4.2792$ are marked.

Analysis of the probability of a type II error - An approach

Theorem

Let $(i,j) \in \Omega$ and $t \in \mathbb{R}^+$. Let $V, V_1, V_2 \in \mathcal{H}_1(i,j)$. Assume V_1 is such, that $\|\Delta^+ V_1(i,j)\| = \sqrt{2}c$ and V_2 such, that $\|\Delta^+ V_2(i,j)\| = \sqrt{8}c$. Then the following inequalities hold:

$$\mathbb{P}_V\left(T(i,j) \leq t\right) \leq 2 \cdot \mathbb{P}_{V_1}\left(\|\Delta^+ F(i,j)\| \leq t\right)$$
$$\mathbb{P}_V\left(T(i,j) \leq t\right) \geq \mathbb{P}_{V_2}\left(\|\Delta^+ F(i,j)\| \leq t\right)$$

Numerical results

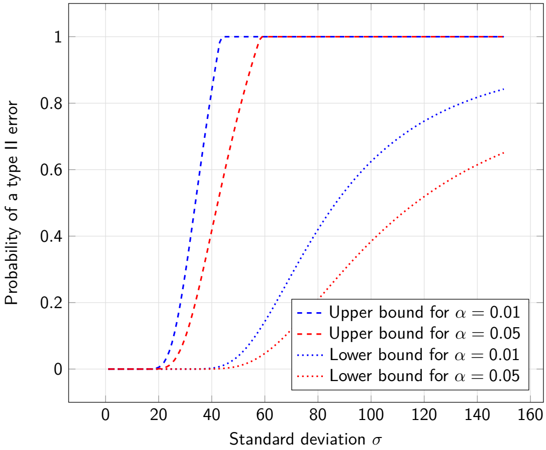


Figure: Simulated bounds for the probability of a type II error. We use the thresholds $t_{0.05} = 3.2555$ for $\alpha = 0.05$ and $t_{0.01} = 4.2792$ for $\alpha = 0.01$. (Sample size 1000000)

Morphological operators

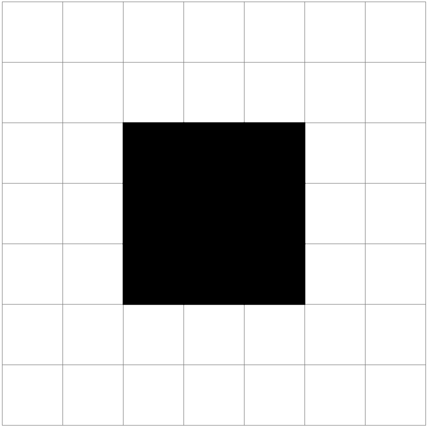
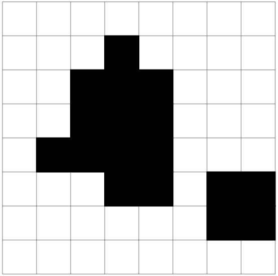
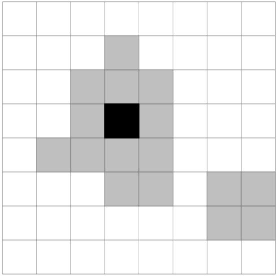


Figure: A 3×3 pixel structuring element.

Example of binary morphological opening - 1/2

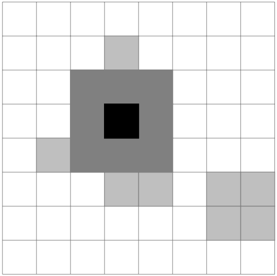


(a) Binary image, where the pixels with value one are black.

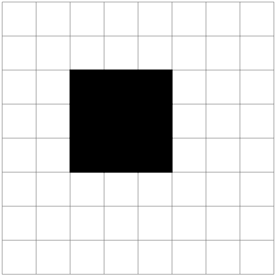


(b) Result of binary erosion of the image with a 3×3 pixel structuring element. The transparent pixels are now set to zero.

Example of binary morphological opening - 2/2

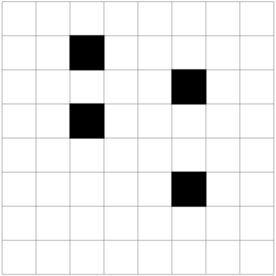


(a) Image after binary opening, i.e. erosion and dilation. The gray pixels were set to one again.



(b) Result of opening. Outliers have been eliminated and edges have been smoothed.

Example of binary morphological closing - 1/2



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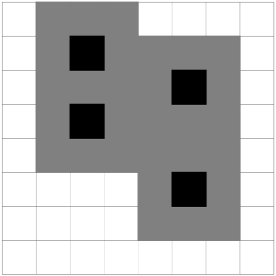
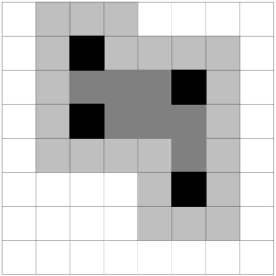
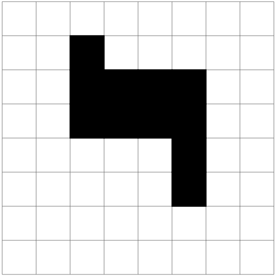


(b) Image after binary dilation with a 3×3 pixel structuring element. The gray pixels are now set to one.

Example of binary morphological closing - 2/2



(a) Image after binary closing, i.e. dilation and erosion. The transparent pixels are set to zero again.



(b) Result of closing. The gaps between pixels have been filled.

Notes on binary morphological opening & closing:

- The outcome is highly dependent on the *chosen* structuring element $\Psi \subseteq \mathbb{Z}^2$.
- Morphological opening smoothes edges and eliminates outliers.
- Morphological closing smoothes edges and fills gaps.
- $\mathfrak{I} \circ \Psi \subseteq \mathfrak{I} \subseteq \mathfrak{I} \bullet \Psi$

Lemma

Let $m, n \in \mathbb{N}$ and $\Psi \subseteq \mathbb{Z}^2$ be a structuring element. Let $(i,j) \in \Omega$. Then the following equalities hold:

$$\{\mathfrak{I} \in \{0,1\}^{m\times n} \mid (\mathfrak{I} \circ \Psi)(i,j) = 1\}$$
$$= \bigcup_{(k,l)\in\Psi} \bigcap_{(\tilde{k},\tilde{l})\in\Psi} \left\{\mathfrak{I} \in \{0,1\}^{m\times n} \mid \mathfrak{I}(i-k+\tilde{k}, j-l+\tilde{l}) = 1\right\}$$

$$\{\mathfrak{I} \in \{0,1\}^{m\times n} \mid (\mathfrak{I} \bullet \Psi)(i,j) = 1\}$$
$$= \bigcap_{(k,l)\in\Psi} \bigcup_{(\tilde{k},\tilde{l})\in\Psi} \left\{\mathfrak{I} \in \{0,1\}^{m\times n} \mid \mathfrak{I}(i+k-\tilde{k}, j+l-\tilde{l}) = 1\right\}$$

Question: What is the effect of opening and closing on the probabilities of a type I and II error?

Theorem

Let $\alpha \in (0,1)$ *and* t_α *a threshold, such that*

$$\min\left\{\mathbb{P}_V\left(\|\Delta^+ F(i,j)\| \geq t_\alpha\right), \mathbb{P}_V\left(\|\Delta^- F(i,j)\| \geq t_\alpha\right)\right\} \leq \alpha$$

for all $V \in \mathcal{H}_0(i,j)$.

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for all $V \in \mathcal{H}_0(i,j)$. *Let* $\mathfrak{I}_\alpha$ *be the binary image defined by*

$$\mathfrak{I}_\alpha(i,j) = \mathbb{1}_{\{T(i,j) \geq t_\alpha\}}$$

for all $(i,j) \in \Omega$.

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for all $(i,j) \in \Omega$.
Let $\varphi \in \mathbb{N}$ *be odd. Let* $\Phi_\varphi = \left\{-\frac{\varphi-1}{2}, -\frac{\varphi-3}{2}, \ldots, \frac{\varphi-3}{2}, \frac{\varphi-1}{2}\right\}$ *and* $\Psi_\varphi = \Phi_\varphi \times \Phi_\varphi$ *be a structuring element. Let* $(i,j) \in \Omega$ *and* $V \in \mathcal{H}_0(i,j)$.

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$$\mathbb{P}_V\left((\mathfrak{I}_\alpha \circ \Psi_\varphi)(i,j) = 1\right) \leq \varphi \alpha^{\frac{\varphi+1}{2}} \tag{1}$$

$$\mathbb{P}_V\left(((\mathfrak{I}_\alpha \circ \Psi_\varphi) \bullet \Psi_\varphi)(i,j) = 1\right) \leq \varphi^3 \alpha^{\frac{\varphi+1}{2}} \tag{2}$$

Main ideas of the proof:

- Note, that if $H_0(i, j)$ is true, then the null hypothesis is true for a whole row or column in the image.
- If we only take every second pixel in the row/column, they will be independent, which yields the exponent $\frac{\varphi+1}{2}$.

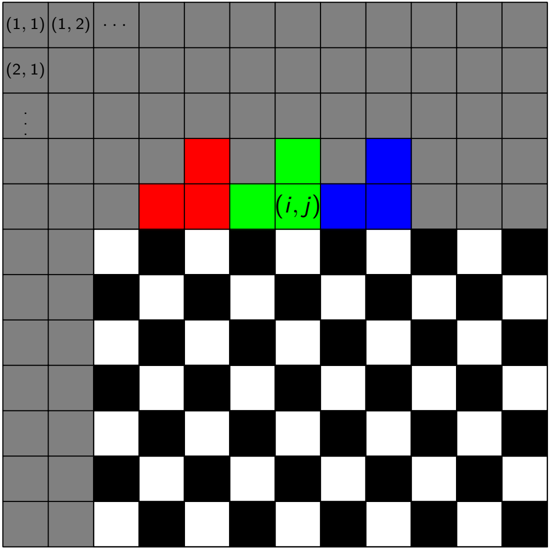

Figure: The random variable $\|\Delta^- F(i,j)\|$ only depends on the green pixels, $\|\Delta^- F(i,j-2)\|$ depends on the red pixels and $\|\Delta^- F(i,j+2)\|$ on the blue pixels. Since these are all distinct from another, the random variables are independent.

Theorem

Let t be a threshold, such that

$$\mathbb{P}_V\left(T(i,j) \leq t\right) \leq \beta$$

for all $V \in \mathcal{H}_1(i,j)$.

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Theorem

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$$\mathbb{P}_V\left((\mathfrak{I} \circ \Psi_\varphi)(i,j) = 0\right) \leq \varphi^2 \beta \tag{3}$$

$$\mathbb{P}_V\left(((\mathfrak{I} \circ \Psi_\varphi) \bullet \Psi_\varphi)(i,j) = 0\right) \leq \varphi^2 \beta \tag{4}$$

Main ideas of the proof:

- Note, that if $H_0(i,j)$ is true, then there exists a square with side length φ, such that the null hypothesis is true for every pixel inside that square.
- The square has φ^2 pixels and for each pixel the probability of a type II error is less than β.

Note: We expect a lower probability than $\varphi^2\beta$ after opening and closing. This does not seem provable with the methods used, since any independence is gone after opening. The expected improvement through closing can be observed in the numerical results.

Methodology

- Create 5 test images each for 128×128, 256×256 and 512×512 pixel.
- Determine the correct region of interest.
- Perform the following 50 times to even out outliers:
  - Create standard normal distributed noise.
  - Loop over the standard deviation $\sigma \in \{1, \ldots, 150\}$ and add the noise multiplied by the standard deviation to the image.
    - Binarize the image using the testing procedure from the first section with $\alpha = 0.05$ and $t_{0.05} = 3.2555\sigma$.
    - Perform binary morphological opening.
    - Perform binary morphological closing.
  - In each step, count the number of type I and II errors for each σ separately.
- Add up all type I and II errors for all images and divide by the total number of background/foreground pixel, respectively.

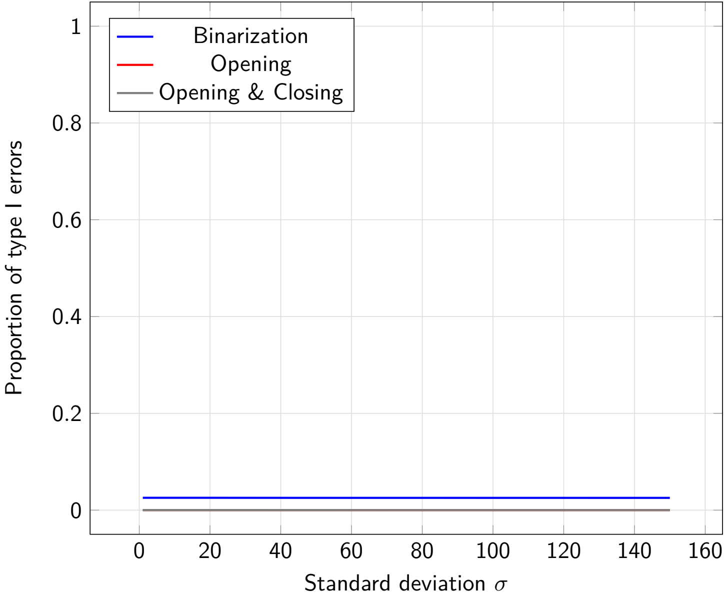

Figure: Proportion of type I errors after binarization, opening and closing.

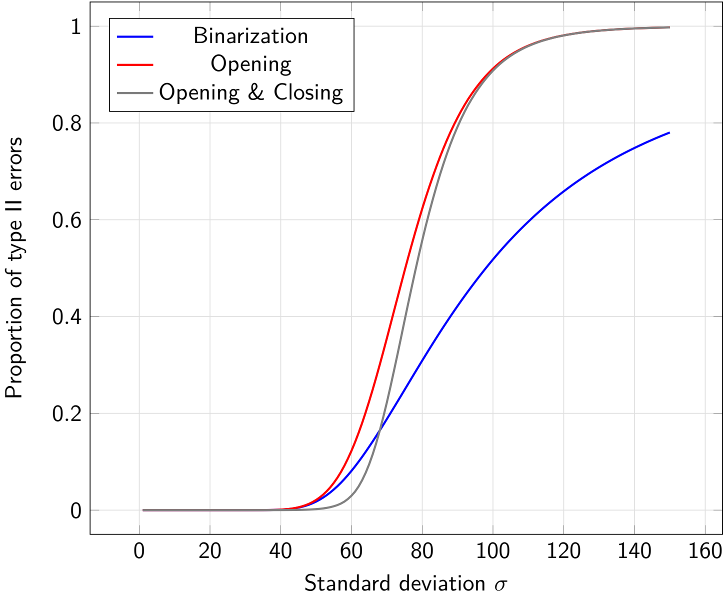

Figure: Proportion of type II errors after binarization, opening and closing.

Outlook

How did we obtain the exponent $\frac{\varphi+1}{2}$?

For a background pixel to be falsely identified as foreground after opening, at least $\frac{\varphi+1}{2}$ many pixels background pixels have to be falsely categorized by the binarization through the statistical test. This is the *minimum number of independent points in a structuring element* that can lead to a false categorization of a pixel. With this connection in mind, the techniques to prove the main results of this paper might present themselves useful in other cases as well.

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What can be gained by using multiple testing techniques?

The main results of this work only considered the categorization of a single pixel. Since we perform tests for all pixels at once, the analysis of techniques from multiple testing in connection with morphological operations would be an interesting field of study to expand the results of this paper.

Thank you for the attention!