Georg-August-Universität Göttingen

Bachelor-Arbeit zum Thema

Gleichmäßige obere Schranken beim Kreisproblem

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Abstract

In dieser Arbeit wird eine asymptotische Formel für die ganzzahligen Darstellungen der natürlichen Zahlen $m \leq R$ durch eine positiv-definite quadratische Form hergeleitet. Die implizite Konstante des Fehlerterms wird dabei nicht von der jeweiligen quadratischen Form abhängen.

Satz 1. *Assume the following statistical model:*
Let $M, N \in \mathbb{N}$ *and* $G = \{0, \dots, M-1\} \times \{0, \dots, N-1\}$. *We are given data*

$$F(i,j) = c + V(i,j) + \varepsilon_{i,j} \tag{1}$$

where $(i,j) \in G$, $c \in \mathbb{R}$ *is constant,* $V(i,j) \in \{0, \pm c\}$ *and* $\varepsilon_{i,j} \sim \mathcal{N}(0, \sigma^2)$ *are i.i.d. normal distributed random variables for some* $\sigma > 0$ *and for all* $(i,j) \in G$.
We assume that V *contains a rectangular region of interest. That means, that there are coordinates* $(i_{tlc}, j_{tlc}), (i_{brc}, j_{brc}) \in G$ *with* $i_{tlc} \leq i_{brc}$ *and* $j_{tlc} \leq j_{brc}$, *such that* $V(i,j) \neq 0$ *if and only if* $i_{tlc} \leq i \leq i_{brc}$ *and* $j_{tlc} \leq j \leq j_{brc}$.
Furthermore assume, that the aforementioned region of interest has a checkerboard pattern, i.e. one of the following relations is true:

$$V(i,j) = c \Leftrightarrow i + j \text{ is odd} \tag{2a}$$

$$V(i,j) = c \Leftrightarrow i + j \text{ is even} \tag{2b}$$

for all $(i,j) \in R := \{i_{tlc}, \dots, i_{brc}\} \times \{j_{tlc}, \dots, j_{brc}\}$.
for all (i,j) *with* $i < i_{tlc}$ *or* $j < j_{tlc}$ *or* $i > i_{brc}$ *or* $j > j_{brc}$. *Let* f *be an image that contains a rectangular ROI. Assume that we are given a binarized image* f_{bin} *with*

$$\mathbb{P}(f_{bin}(i,j) = 1 \mid H_0(i,j)) \leq \alpha$$

where $H_0(i,j)$ *denotes the null hypothesis for the pixel* (i,j), *which is, that it is a background pixel and thus should be set to zero.*
Let $k \in \mathbb{N}$ *be odd and* B *be a square structuring element with side length* k. *For* $\tilde{m}, \tilde{n} \in \{-\frac{k-1}{2}, \dots, \frac{k-1}{2}\}$ *we denote by* $\mathcal{G}^k_{(\tilde{m},\tilde{n})}(i,j)$ *the set of all possible ground truths in the square with side length* k, *where the pixel* (i,j) *has offest* $(\tilde{m}, \tilde{n})$ *from the center of the square and assuming that the null hypothesis for the pixel* (i,j) *is true. Then the following inequality holds:*

$$\mathbb{P}((f_{bin} \circ B)(i,j) = 1 \mid H_0(i,j)) \leq k^2 \alpha^k$$

Proof. First we notice that for fixed $\tilde{m}, \tilde{n}$ the set $\mathcal{G}^k_{(\tilde{m},\tilde{n})}(i,j)$ contains ALL possible ground truths given that the null hypothesis for the pixel (i,j) is true, thus

$$\sum_{G \in \mathcal{G}^k_{(\tilde{m},\tilde{n})}(i,j)} \mathbb{P}(G \mid H_0(i,j)) = 1$$

Second we notice, that any element $G \in \mathcal{G}^k_{(\tilde{m},\tilde{n})}(i,j)$ already contains the null hypothesis for the pixel (i,j), thus giving

$$\mathbb{P}(G \mid H_0(i,j)) = \mathbb{P}(G)$$

Third we see that for any possible ground truth $G \in \mathcal{G}^k_{(\tilde{m},\tilde{n})}(i,j)$ for the whole k by k square to be set to one in f_{bin}, there are at least k falsely identified pixels in that square.
Let $K = \{-\frac{k-1}{2}, \dots, \frac{k-1}{2}\}$. Using above observations, we get

$$\begin{aligned}
&\mathbb{P}((f_{bin} \circ B)(i,j) = 1 \mid H_0(i,j)) \\
&\quad = \mathbb{P}\left(\bigcup_{\tilde{m},\tilde{n} \in K} \bigcap_{m,n \in K} (f_{bin}(i+m-\tilde{m}, j+n-\tilde{n}) = 1) \mid H_0(i,j) \right) \\
&\quad = \sum_{\tilde{m},\tilde{n} \in K} \mathbb{P}\left(\bigcap_{m,n \in K} (f_{bin}(i+m-\tilde{m}, j+n-\tilde{n}) = 1) \mid H_0(i,j) \right) \\
&\quad = \sum_{\tilde{m},\tilde{n} \in K} \sum_{G \in \mathcal{G}^k_{(\tilde{m},\tilde{n})}(i,j)} \mathbb{P}(G \mid H_0(i,j)) \cdot \mathbb{P}\left(\bigcap_{m,n \in K} (f_{bin}(i+m-\tilde{m}, j+n-\tilde{n}) = 1) \mid G, H_0(i,j) \right) \\
&\quad = \sum_{\tilde{m},\tilde{n} \in K} \sum_{G \in \mathcal{G}^k_{(\tilde{m},\tilde{n})}(i,j)} \mathbb{P}(G) \cdot \underbrace{\mathbb{P}\left(\bigcap_{m,n \in K} (f_{bin}(i+m-\tilde{m}, j+n-\tilde{n}) = 1) \mid G, H_0(i,j) \right)}_{\leq \alpha^k} \\
&\quad \leq \sum_{\tilde{m},\tilde{n} \in K} \sum_{G \in \mathcal{G}^k_{(\tilde{m},\tilde{n})}(i,j)} \mathbb{P}(G) \cdot \alpha^k \\
&\quad = \alpha^k \cdot \sum_{\tilde{m},\tilde{n} \in K} \underbrace{\sum_{G \in \mathcal{G}^k_{(\tilde{m},\tilde{n})}(i,j)} \mathbb{P}(G)}_{=1} = \alpha^k \cdot \sum_{\tilde{m},\tilde{n} \in K} 1 = \alpha^k \cdot |K|^2 = k^2 \alpha^k
\end{aligned}$$

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