Theorem 0.1. *Let $m, n \in \mathbb{N}$, $c \in \mathbb{R} \setminus \{0\}$ and $\Omega = \{1, \ldots, m\} \times \{1, \ldots, n\}$. Assume that F follows the statistical model given in (??) and let $T(i,j)$ be the test statistic as defined in (??) and $H_1(i,j)$ be the alternative hypothesis as defined in (??). Let t be a threshold, such that*

$$\mathbb{P}_V\left(T(i,j) \leq t\right) \leq \beta$$

for all $V \in \mathcal{H}_1(i,j)$. Let $\mathfrak{I}$ be the binary image defined by

$$\mathfrak{I}(i,j) = \mathbb{1}_{\{T(i,j) \geq t\}} \tag{1}$$

for all $(i,j) \in \Omega$.
Let $\varphi \in \mathbb{N}$ be odd. Let $\Phi_\varphi = \{-\frac{\varphi-1}{2}, -\frac{\varphi-3}{2}, \ldots, \frac{\varphi-3}{2}, \frac{\varphi-1}{2}\}$ and $\Psi_\varphi = \Phi_\varphi \times \Phi_\varphi$ be a structuring element. Let $(i,j) \in \Omega$ and $V \in \mathcal{H}_1(i,j)$.
Denote by $\Lambda = \{\kappa_1, \ldots, \kappa_2\} \times \{\lambda_1, \ldots, \lambda_2\}$ the rROI contained in V. Let $\min\{\kappa_2 - \kappa_1 + 1, \lambda_2 - \lambda_1 + 1\} \geq \varphi$. Then the following inequalities hold:

$$\mathbb{P}_V((\mathfrak{I} \circ \Psi_\varphi)(i,j) = 0) \leq \varphi^2 \beta \tag{2}$$

$$\mathbb{P}_V(((\mathfrak{I} \circ \Psi_\varphi) \bullet \Psi_\varphi)(i,j) = 0) \leq \varphi^2 \beta \tag{3}$$

Proof. We use $\Psi_\varphi = \Phi_\varphi \times \Phi_\varphi$ and get

$$\begin{aligned}
&\mathbb{P}_V(((\mathfrak{I} \circ \Psi_\varphi) \bullet \Psi_\varphi)(i,j) = 0) \\
&\quad = \mathbb{P}_V\left(\bigcup_{(k,l)\in\Psi_\varphi} \bigcap_{(\tilde{k},\tilde{l})\in\Psi_\varphi} \bigcap_{(r,s)\in\Psi_\varphi} \bigcup_{(\tilde{r},\tilde{s})\in\Psi_\varphi} \{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r}, j+l-\tilde{l}-s+\tilde{s}) = 0\}\right) \\
&\quad = \mathbb{P}_V\left(\bigcup_{k,l\in\Phi_\varphi} \bigcap_{\tilde{k},\tilde{l}\in\Phi_\varphi} \bigcap_{r,s\in\Phi_\varphi} \bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi} \{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r}, j+l-\tilde{l}-s+\tilde{s}) = 0\}\right)
\end{aligned}$$

Using sub-additivity we obtain

$$\begin{aligned}
&\mathbb{P}_V(((\mathfrak{I} \circ \Psi_\varphi) \bullet \Psi_\varphi)(i,j) = 0) \\
&\quad = \mathbb{P}_V\left(\bigcup_{k,l\in\Phi_\varphi} \bigcap_{\tilde{k},\tilde{l}\in\Phi_\varphi} \bigcap_{r,s\in\Phi_\varphi} \bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi} \{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r}, j+l-\tilde{l}-s+\tilde{s}) = 0\}\right)
\end{aligned}$$

$$\leq \sum_{k,l\in\Phi_\varphi} \mathbb{P}_V\left(\bigcap_{\tilde{k},\tilde{l}\in\Phi_\varphi}\bigcap_{r,s\in\Phi_\varphi}\bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi}\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r},j+l-\tilde{l}-s+\tilde{s})=0\}\right)$$

We can pull the two intersections together and get

$$\begin{aligned}
&\mathbb{P}_V(((\mathfrak{I}\circ\Psi_\varphi)\bullet\Psi_\varphi)(i,j)=0)\\
&\quad\leq \sum_{k,l\in\Phi_\varphi} \mathbb{P}_V\left(\bigcap_{\tilde{k},\tilde{l}\in\Phi_\varphi}\bigcap_{r,s\in\Phi_\varphi}\bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi}\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r},j+l-\tilde{l}-s+\tilde{s})=0\}\right)\\
&\quad= \sum_{k,l\in\Phi_\varphi} \mathbb{P}_V\left(\bigcap_{\tilde{k},\tilde{l},r,s\in\Phi_\varphi}\bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi}\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r},j+l-\tilde{l}-s+\tilde{s})=0\}\right)
\end{aligned}$$

We drop every term in the intersection besides $r-\tilde{k}, s-\tilde{l}\in\{-(\varphi-1),\varphi-1\}$. This yields

$$\begin{aligned}
&\mathbb{P}_V(((\mathfrak{I}\circ\Psi_\varphi)\bullet\Psi_\varphi)(i,j)=0)\\
&\quad\leq \sum_{k,l\in\Phi_\varphi} \mathbb{P}_V\left(\bigcap_{\tilde{k},\tilde{l},r,s\in\Phi_\varphi}\bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi}\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r},j+l-\tilde{l}-s+\tilde{s})=0\}\right)\\
&\quad\leq \sum_{k,l\in\Phi_\varphi} \mathbb{P}_V\left(\bigcap_{r-\tilde{k},s-\tilde{l}\in\{-(\varphi-1),\varphi-1\}}\bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi}\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r},j+l-\tilde{l}-s+\tilde{s})=0\}\right)
\end{aligned}$$

The sets $\bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi}\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r},j+l-\tilde{l}-s+\tilde{s})=0\}$ are mutually independent for $r-\tilde{k}, s-\tilde{l}\in\{-(\varphi-1),\varphi-1\}$ and fixed $k,l\in\Phi_\varphi$. Thus we obtain

$$\begin{aligned}
&\mathbb{P}_V(((\mathfrak{I}\circ\Psi_\varphi)\bullet\Psi_\varphi)(i,j)=0)\\
&\quad\leq \sum_{k,l\in\Phi_\varphi} \mathbb{P}_V\left(\bigcap_{r-\tilde{k},s-\tilde{l}\in\{-(\varphi-1),\varphi-1\}}\bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi}\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r},j+l-\tilde{l}-s+\tilde{s})=0\}\right)\\
&\quad= \sum_{k,l\in\Phi_\varphi}\prod_{r-\tilde{k},s-\tilde{l}\in\{-(\varphi-1),\varphi-1\}} \mathbb{P}_V\left(\bigcup_{\tilde{r},\tilde{s}\in\Phi_\varphi}\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r},j+l-\tilde{l}-s+\tilde{s})=0\}\right)
\end{aligned}$$

Again, by using sub-additivy, we get

$$
\begin{aligned}
&\mathbb{P}_V(((\mathfrak{I} \circ \Psi_\varphi) \bullet \Psi_\varphi)(i,j) = 0) \\
&\quad \leq \sum_{k,l \in \Phi_\varphi} \prod_{r-\tilde{k}, s-\tilde{l} \in \{-(\varphi-1), \varphi-1\}} \mathbb{P}_V \left(\bigcup_{\tilde{r}, \tilde{s} \in \Phi_\varphi} \{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r}, j+l-\tilde{l}-s+\tilde{s}) = 0\} \right) \\
&\quad = \sum_{k,l \in \Phi_\varphi} \prod_{r-\tilde{k}, s-\tilde{l} \in \{-(\varphi-1), \varphi-1\}} \sum_{\tilde{r}, \tilde{s} \in \Phi_\varphi} \mathbb{P}_V \Big(\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r}, j+l-\tilde{l}-s+\tilde{s}) = 0\} \Big)
\end{aligned}
$$

Using the assumption $\mathbb{P}_V\left(T(i,j) \leq t\right) \leq \beta$ we obtain the upper bound

$$
\begin{aligned}
&\mathbb{P}_V(((\mathfrak{I} \circ \Psi_\varphi) \bullet \Psi_\varphi)(i,j) = 0) \\
&\quad \leq \sum_{k,l \in \Phi_\varphi} \prod_{r-\tilde{k}, s-\tilde{l} \in \{-(\varphi-1), \varphi-1\}} \sum_{\tilde{r}, \tilde{s} \in \Phi_\varphi} \mathbb{P}_V \Big(\{\mathfrak{I}(i+k-\tilde{k}-r+\tilde{r}, j+l-\tilde{l}-s+\tilde{s}) = 0\} \Big) \\
&\quad \leq \sum_{k,l \in \Phi_\varphi} \prod_{r-\tilde{k}, s-\tilde{l} \in \{-(\varphi-1), \varphi-1\}} \sum_{\tilde{r}, \tilde{s} \in \Phi_\varphi} \beta \\
&\quad = \sum_{k,l \in \Phi_\varphi} \prod_{r-\tilde{k}, s-\tilde{l} \in \{-(\varphi-1), \varphi-1\}} \varphi^2 \beta \\
&\quad = \sum_{k,l \in \Phi_\varphi} (\varphi^2 \beta)^4 \\
&\quad = \varphi^2 (\varphi^2 \beta)^4 \\
&\quad = \varphi^{10} \beta^4
\end{aligned}
$$

This finishes the proof. $\square$

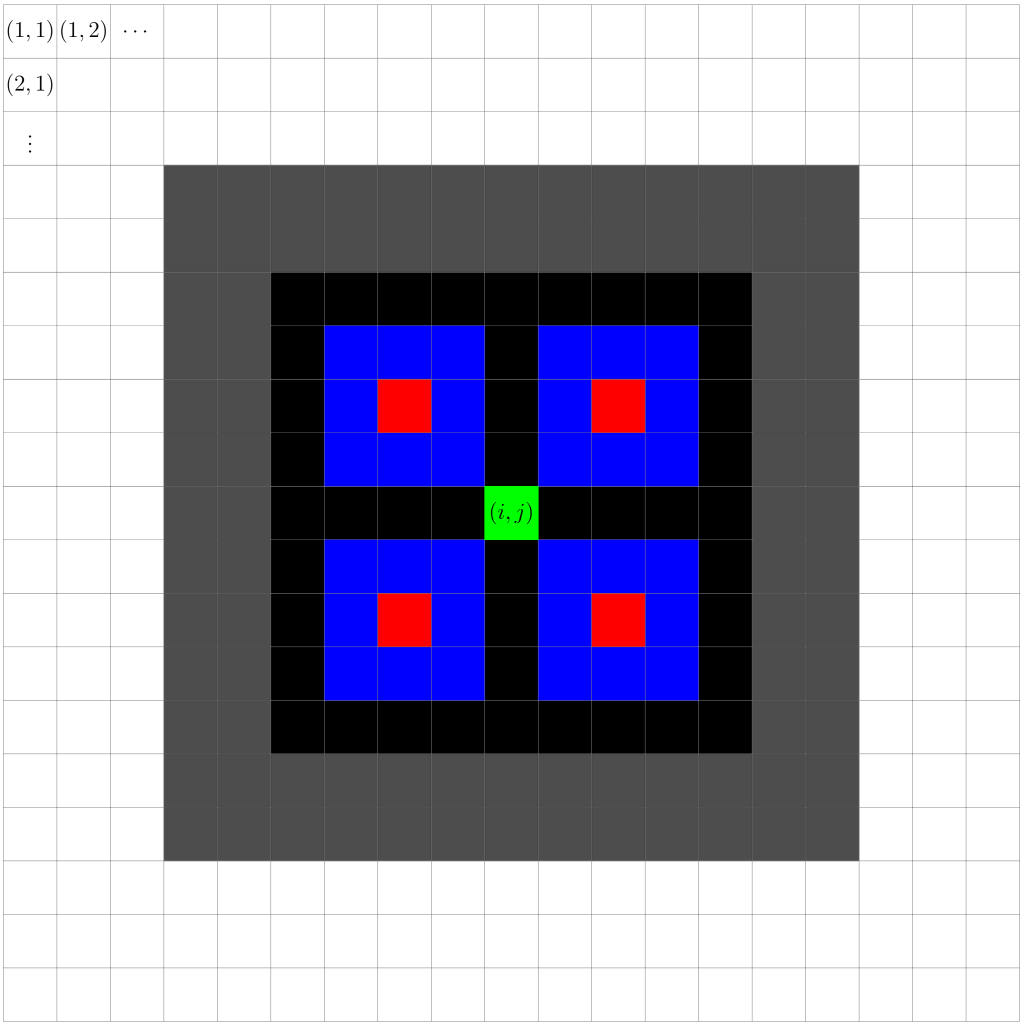

Figure 1: The black area is the pixels that contribute to $(\mathfrak{I} \circ \Psi_\varphi) \bullet \Psi_\varphi)(i, j)$. The blue squares are mutually independent. The red pixels are the pixels that we reduce the intersection in the proof to.