Course 3 assignment Denys Sikorskyi

Exercise 8.1

Using next notation: P – a patch, $\tilde{P}$ – observed patch, $\bar{P}$ – mean of the observed patch, C_P – covariance matrix, u – noiseless image, $\tilde{u}$ – observed image, n or N – noise. We have $\tilde{P} = P + N$, and we need to prove using independence P and N that $C_{\tilde{P}} = C_P + \sigma^2 I$. If P and N are independent, then $var(\tilde{P}) = var(P) + var(N)$. $var(P) = C_P$ and because N is white noise, we have diagonal covariance matrix with every pixel of N is an observation of a gaussian with variance σ^2 and mean 0. $\Rightarrow var(N) = \sigma^2 I$. Then $C_{\tilde{P}} = C_P + \sigma^2 I$. For $E\tilde{P}$ we have $E(P+N) = E(P) + EN = EP + 0 = EP = \bar{P} \Rightarrow E\tilde{P} = \bar{P}$.

Exercise 8.3

Let's write the equations:

$$\hat{P}_1 = \bar{P} + [C_{\tilde{P}} - \sigma^2 I] C_{\tilde{P}}^{-1} (\tilde{P} - \bar{P})$$

$$\hat{P}_2 = \bar{P}^1 + C_P^1 [C_P^1 + \sigma^2 I]^{-1} (\tilde{P} - \bar{P}^1)$$

We can rewrite the first expression using eigenvalues G_i of $C_{\tilde{P}}$

$$\hat{P}_1 = \sum_{i=1}^{n} \langle \hat{P}_1, G_i \rangle G_i = \sum_{i=1}^{n} \langle \bar{P}, G_i \rangle G_i + \langle \tilde{P} - \bar{P}, ([C_{\tilde{P}} - \sigma^2 I] C_{\tilde{P}}^{-1})^T G_i \rangle G_i =$$

$$= \sum_{i=1}^{n} \langle \bar{P}, G_i \rangle G_i + \langle \tilde{P} - \bar{P}, \underbrace{(I - \sigma^2 C_{\tilde{P}}^{-1})^T}_{\text{can neglect transposition because of symmetry}} G_i \rangle G_i = \sum \langle \bar{P}, G_i \rangle G_i +$$

$$+ \langle \tilde{P} - \bar{P}, (I - \sigma^2 C_{\tilde{P}}^{-1}) G_i \rangle G_i = \sum \langle \bar{P} + \left(1 - \frac{\sigma^2}{\lambda_i}\right)(\tilde{P} - \bar{P}), G_i \rangle G_i =$$

where λ_i – eigenvalues

$= \sum_i \langle \tilde{P} - \frac{\sigma^2}{\lambda_i}(\tilde{P} - \bar{P}), G_i \rangle G_i = \sum_i (1 - \frac{\sigma^2}{\lambda_i}) \langle \tilde{P}, G_i \rangle +$
$+ \sum \frac{\sigma^2}{\lambda_i} \langle \bar{P}, G_i \rangle G_i$

Let's work with second expression

$\hat{P}_2^1 = \sum \langle \hat{P}_2, G_i \rangle G_i$ (it's different G_i base from previous formula) $= \sum \langle \bar{P}^1, G_i \rangle + \sum [C_{\tilde{P}}^1 [C_{\tilde{P}} + \sigma^2 I]^{-1}]$ $\cdot (\tilde{P} - \bar{P}^1), G_i \rangle G_i = \sum \langle \bar{P}^1, G_i \rangle G_i + \langle \tilde{P} - \bar{P}^1, [C_p^1 [C_p^1 + \sigma^2 I]^{-1}] G_i \rangle$ $\cdot G_i = \sum_i \langle \bar{P}^1, G_i \rangle G_i + \langle \tilde{P} - \bar{P}^1, \underbrace{C_p^1 (C_p^1 + \sigma^2 I)^{-1}}_{\text{symmetric}} G_i \rangle G_i =$

$\ominus \mid C_p^1 (C_p^1 + \sigma^2 I)^{-1} G_i = \frac{\lambda_i}{\lambda_i + \sigma^2} G_i \ominus \sum \langle \bar{P}^1, G_i \rangle G_i + (\frac{\lambda_i}{\lambda_i + \sigma^2}) \cdot$
$\cdot \langle \tilde{P} - \bar{P}^1, G_i \rangle G_i = \sum \langle \tilde{P}^1, G_i \rangle G_i \cdot (1 - \frac{\sigma^2}{\lambda_i + \sigma^2}) +$
$+ \sum_i \frac{\sigma^2}{\lambda_i + \sigma^2} \langle \bar{P}^1, G_i \rangle G_i$

So, we have the same structure: sum of coef multiplied by ~~patch~~ ~~past~~ part with observed patch plus the difference term. So, the two steps of the Bayesian method ~~are~~ an application of the Wiener empirical ~~and~~ oracular method and the coefficients in the second equations depends on the result of the first, but the difference depends on the mean of the patch.

Exercise 8.4

$MSE = \int P(\tilde{P}) \int (P(P|\tilde{P}) \|P - \hat{P}\| d\tilde{P}) dP = \int\int P(\tilde{P}) P(P|\tilde{P}) .$
$. \|P - \hat{P}\|^2 d\tilde{P} dP = \int\int P(\tilde{P}) P(P|\tilde{P}) \|P - \hat{P}\|^2 d\tilde{P} dP$. According to the Fubini-Tonelli's theorem the order of the integration can be changed (integrals in P or $\tilde{P}$ are L_2), so we have $\int P(\tilde{P}) \int (P(P|\tilde{P}) \|P - \tilde{P}\|^2 dP) d\tilde{P}$

Exercise 8.5

The purpose is to verify that MMSE minimizes the MSE.

$MMSE(\tilde{P}) = E[\|\hat{P} - \tilde{P}\|^2 | \tilde{P}] = \int P(P|\tilde{P})(P - \hat{P})^2 dP dP$

$\frac{\partial MMSE(\hat{P})}{\partial \hat{P}} = \frac{\partial}{\partial \hat{P}} \left(\int P(P|\tilde{P})(P - \hat{P})^2 dP \right) = -2 \int P(P|\tilde{P})(P - \hat{P}) dP$

$= 0. \int P \cdot P(P|\tilde{P}) dP = \int \hat{P} P(P|\tilde{P}) dP = \hat{P} \int P(P|\tilde{P}) dP$

$\int P \cdot P(\tilde{P}|P) P(P) (P(\tilde{P}))^{-1} dP = \hat{P} \int P(\tilde{P}|P) P(P) (P(\tilde{P}))^{-1} dP =$

$= \hat{P} (P(\tilde{P}))^{-1} \int P(\tilde{P}|P) P(P) dP = \hat{P} P(\tilde{P})^{-1} \underbrace{\int P(\tilde{P}|P) dP}_{= P(\tilde{P})}$

$= \hat{P}$. So, we can conclude that $\hat{P}$ is the optimal estimator of MMSE, where $\hat{P} = \int \frac{P(\tilde{P}|P)}{P(\tilde{P})} P(P) P dP$.