Exercise 1. Which of the following sets are convex?

1) A rectangle, i.e., a set of form $\{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i, i=1,\dots,n\}$.

Let's take defition of the half space: $\{x \mid a^T x \le b\}$

Our set $S = \{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i\}$ can be presented as intersection of finite number of halfspaces like $\{x \mid a^T x \le b$, where $b = \beta_i$ and $a = (0, 0, \dots, \overset{i\text{-th}}{1}, \dots, 0)\} = \{x \in \mathbb{R}^n \mid x_i \le \beta_i\}$ and $\{x \in \mathbb{R}^n \mid x_i \le -\alpha_i\}$, so S is convex.

2) The hyperbolic set $\{x \in \mathbb{R}^2_+ \mid x_1 x_2 \ge 1\} = S$

Let's prove by definition. Taking two points $(x_1, y_1) = x_1'$ and $(x_2, y_2) = x_2'$. We need to prove that $\theta x_1' + (1-\theta) x_2' \in S \Leftrightarrow$

$\theta^2 x_1 y_1 + (1-\theta)^2 x_2 y_2 + \theta(1-\theta)(x_1 y_2 + x_2 y_1) \ge 1$. ~~Because~~ (1)

$0 \le (x_1 y_2 - x_2 y_1)^2 = (x_1 y_2)^2 - 2 x_1 y_1 x_2 y_2 + (x_2 y_1)^2 \Rightarrow$

$4 \le 4 x_1 y_1 x_2 y_2 \le (x_1 y_2)^2 + 2 x_1 y_1 x_2 y_2 + (x_2 y_1)^2 =$

$= (x_1 y_2 + x_2 y_1)^2 \overset{x_1, x_2, y_1, y_2 \ge 0}{\Rightarrow} 2 \le x_1 y_2 + x_2 y_1$. Taking in account

$x_1 y_1 \ge 1$, $x_2 y_2 \ge 1$ and replacing in (1) we get

$\theta^2 x_1 y_1 + (1-\theta)^2 x_2 y_2 + \theta(1-\theta)(x_1 y_2 + x_2 y_1) \geq$

$\theta^2 + (1-\theta)^2 + 2(\theta(1-\theta)) = (\theta + 1 - \theta)^2 = 1.$

So the condition for convexity is satisfied.

3) The set of points closer to a given point than a given set, i.e., $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$, where $S \subseteq \mathbb{R}^n$.

Let's look at our inequality

$\|x - x_0\|_2 \leq \|x - y\|_2 \iff (x - x_0)^T (x - x_0) \leq (x-y)^T (x-y) \iff$
$\iff x^T x - 2x_0^T x + x_0^T x_0 \leq x^T x - 2y^T x + y^T y \iff$
$\iff 2(y - x_0)^T x \leq y^T y - x_0^T x_0$. This expression is half space expression (with division by 2 and transfer everything apart x to the right). It means that we can show our set as $\bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$, which is intersection of halfspaces which are convex.

4) The set of points closer to one set than to another, i.e., $\{x \mid dist(x, S) \leq dist(x, T)\}$, where $S, T \subseteq \mathbb{R}^n$, and $dist(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$

The set seems to be nonconvex, because obviously it can form holes in the space. This example shows it. Suppose $S = \{(x,y) \mid x^2 + y^2 = 4\}$, $T = \{(0,0)\}$, then $\{(x,y) \mid dist((x,y), S) \leq dist((x,y), T)\} =$

$= \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \} = A$. This set is not convex, because for points $(1,0), (-1,0) = x_1, x_2$ $\quad \theta x_1 + (1-\theta) x_2 \notin A$ for $\theta \in (0,1)$

5) The set $\{ x \mid x + S_2 \subseteq S_1 \}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

Suppose we take one $y \in S_2$, then $x + y \in S_1 \Rightarrow$ ~~set~~ set of x can be expressed as $S_1 - y$ for one $y \in S_2$.

$S_1 - y$ is affine, so it is convex. Therefore the initial set can be presented as intersection of convex sets: $\{ x \mid x + S_2 \subseteq S_1 \} = \bigcap_{y \in S_2} (S_1 - y)$.

Exercise 2.

For each of the following functions determine whether it is convex or concave or not.

1) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

The f is twicely differentiable, so using Hessian is possible.

$\nabla^2 f(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \Rightarrow \lambda_1 = 1, \lambda_2 = -1.$

Therefore, the Hessian is not positive semidefinite, so f is not convex. Also, the Hessian is not negative semidefinite, so f is not concave.

2) $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$.

The f is twicely differentiable, lets take the Hessian again

$$\nabla^2 f(x) = \begin{pmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{pmatrix} \quad \begin{vmatrix} \frac{2}{x_1^3 x_2} - \lambda & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} - \lambda \end{vmatrix} =$$

$$= \left(\frac{2}{x_1^3 x_2} - \lambda \right) \left(\frac{2}{x_1 x_2^3} - \lambda \right) - \frac{1}{x_1^4 x_2^4} = 0$$

$$\lambda_1 = \frac{1}{x_1^4 x_2^4} \left(x_1^3 x_2 - \sqrt{x_1^2 x_2^2 (x_1^4 - x_1^2 x_2^2 + x_2^4)} + x_1 x_2^3 \right)$$

$$\lambda_2 = \frac{1}{x_1^4 x_2^4} \left(x_1^3 x_2 + \sqrt{x_1^2 x_2^2 (x_1^4 - x_1^2 x_2^2 + x_2^4)} + x_1 x_2^3 \right)$$

$\lambda_1, \lambda_2 \geq 0$ ~~on~~ $\mathbb{R}^2_{++} \Rightarrow$ function is convex and nonconcave

3) $f(x_1, x_2) = \frac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$.

Again we can take the Hessian. $\nabla^2 f(x) = \begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$

$$\begin{vmatrix} -\lambda & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} - \lambda \end{vmatrix} = \lambda^2 - \frac{2x_1\lambda}{x_2^3} - \frac{1}{x_2^4} = 0 \Rightarrow$$

$$\lambda_{1,2} = \frac{x_1 x_2 \pm x_2\sqrt{x_1^2 + x_2^2}}{x_2^4}$$ if $x_1 = 3$, $x_2 = 4$, then $\lambda_1 = -\frac{1}{32}$, $\lambda_2 = \frac{1}{8} \Rightarrow$

$\Rightarrow f$ is neither convex or concave.

4) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$ on $\mathbb{R}^2_{++}$.

Taking the Hessian again. $\nabla^2 f(x) = \begin{pmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{(1-\alpha)} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & -(1-\alpha)\alpha x_1^\alpha x_2^{-1-\alpha} \end{pmatrix} =$

$= -\alpha(\alpha-1) x_1^\alpha x_2^{1-\alpha} \begin{pmatrix} -\frac{1}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & -\frac{1}{x_2^2} \end{pmatrix}$. The left part is ≥ 0, so we can drop it $(-\frac{1}{x_1^2} - \lambda)(-\frac{1}{x_2^2} - \lambda) - \frac{1}{x_1^2 x_2^2} = 0$ $\lambda_{1,2} = \frac{-x_1^2 - x_2^2}{x_1^2 x_2^2} \leq 0 \Rightarrow$

$\Rightarrow f$ is ~~can~~ not convex, but it's concave.

Exercise 3

Show that following functions are convex.

1) $f(X) = Tr(X^{-1})$ on $\text{dom} f = S^n_{++}$.

Because trace is the sum of eigenvalues lets focus on them. We can express X as $Z + tV$, $Z > 0$, $V \in S^n$

$tr((Z + tV)^{-1}) = tr(Z^{-1}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})^{-1}) = tr(Z^{-1}Q(I + t\Lambda)^{-1}Q^T) =$

(~~...~~ Here eigenvalue decomposition is what we need) $=$

$= \sum_{i=1}^n (Q^T Z^{-1} Q)_{(i\text{-th row} \times i\text{th column})} \cdot \frac{1}{1 + t\lambda_i}$ - this linear combination of convex functions of t.

3) $f(X)=\sum_{i=1}^{n}\sigma_i(X)$ on dom S^n, where $\sigma_1(X),\dots,\sigma_n(X)$ are singular values of a matrix $X\in\mathbb{R}^{n\times n}$.

Let's define the function $f_k(X)=\sum_{i=1}^{k}\sigma_i(X)$ of k largest singular values of matrix X. Let's define $\Phi_k=\{B\in S^n: f_1(B)\le 1, f_n(B)=k\}$. Then according to the Theorem 3.2 of "On Extreme Singular Values of Matrix Valued Functions" $f_k(X)=\max\{\langle X,B\rangle : B\in\Phi_k\}$. Therefore, function $f_k(X)$ is convex, because it is the maximum of linear functions of X.

2) $f(X,y)=y^TX^{-1}y$ on dom $f=S^n_{++}\times\mathbb{R}^n$.