Bonus Θέμα 2018-2019

$D = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ * \end{pmatrix} \right\}$ με $x_1, x_2 > 0$

$p(x_1) = \begin{cases} \frac{1}{\theta_1} \cdot e^{-\frac{x_1}{\theta_1}} & x_1 \geq 0 \\ 0, & \text{αλλιώς} \end{cases} \qquad p(x_2) = \begin{cases} \frac{1}{\theta_2}, & x_2 \in (0, \theta_2) \\ 0, & \text{αλλιώς} \end{cases}$

$\vec{\theta}^{\circ} = \begin{pmatrix} \theta_1 = 2 \\ \theta_2 = 4 \end{pmatrix}$

Ισχύει $D_g = \{ \vec{x}^{(1)}, \vec{x}^{(2)}, x_1^{(3)} \}$, $D_b = \{ x_2^{(3)} \}$

$Q(\vec{\theta}, \vec{\theta}^{\circ}) = \mathbb{E}_{D_b} \{ \ln p(D; \vec{\theta} \mid D_g; \vec{\theta}^{\circ}) \}$

$= \underbrace{\sum_{i=1}^{2} \ln p(\vec{x}^{(i)}; \vec{\theta})}_{A} + \mathbb{E}_{x_2^{(3)}} \{ \ln p(\vec{x}^{(3)}; \vec{\theta} \mid x_1^{(3)}; \vec{\theta}^{\circ}) \}$

$= A + \int_{-\infty}^{+\infty} dx_2^{(3)} \ln p(\vec{x}^{(3)} \mid \vec{\theta}) \cdot p(x_2^{(3)} \mid \vec{\theta}^{\circ}; x_1^{(3)} = 3)$

Αλλά $\ln p(\vec{x}^{(3)} \mid \vec{\theta}) = \left[\ln\left(\frac{1}{\theta_1} \cdot e^{-\frac{x_1^{(3)}}{\theta_1}} \right) + \ln\left(\frac{1}{\theta_2} \right) \right] \mathbb{1}_{0 < x_2^{(3)} < \theta_2}$,

όπου $\mathbb{1}_{0 < x_2^{(3)} < \theta_2} = H(x_2^{(3)}) \cdot H(\theta_2 - x_2^{(3)})$, H - Heaviside step

Αντίστοιχα, $p(x_2^{(3)} \mid \vec{\theta}^{\circ}; x_1^{(3)} = 3) = {}^1/_{\theta_2^{\circ}} \cdot \mathbb{1}_{0 < x_2^{(3)} < \theta_2^{\circ}} =$

$= {}^1/_4 \cdot H(x_2^{(3)}) H(4 - x_2^{(3)})$. Συνεπώς

$\int_{-\infty}^{+\infty} dx_2^{(3)} \ln p(\vec{x}^{(3)} \mid \vec{\theta}) \cdot p(x_2^{(3)} \mid \vec{\theta}^{\circ}; x_1^{(3)} = 3) = -\frac{1}{4} \left[\ln\theta_1 + \frac{3}{\theta_1} + \ln\theta_2 \right] \cdot$

$\cdot \int_{-\infty}^{+\infty} dx_2^{(3)} H(x_2^{(3)}) H(4 - x_2^{(3)}) H(\theta_2 - x_2^{(3)}) =$

$$= -\frac{1}{4}\left[\ln\theta_1 + \ln\theta_2 + \frac{3}{\theta_1}\right]\cdot\int_0^{+\infty} dx_2^{(3)}\, H(4-x_2^{(3)})\, H(\theta_2 - x_2^{(3)}) \quad (1)$$

Εδώ πρέπει να πάρουμε περιπτώσεις, αφού η μόνη πληροφορία που έχουμε για το θ_2 είναι πως $\theta_2 \geq 2$, καθώς υπάρχει δεδομένο με $x_2 = 2$.

- Αν $\theta_2 \geq 4$, τότε:

$$(1) = -\frac{1}{4}\left[\ln\theta_1 + \ln\theta_2 + \frac{3}{\theta_1}\right]\int_0^4 dx_2^{(3)} = -\left(\ln\theta_1 + \ln\theta_2 + \frac{3}{\theta_1}\right)$$

- Αν $\theta_2 < 4$, τότε:

$$(1) = -\frac{1}{4}\left[\ln\theta_1 + \ln\theta_2 + \frac{3}{\theta_1}\right]\int_0^{\theta_2} dx_2^{(3)} = -\frac{\theta_2}{4}\left(\ln\theta_1 + \ln\theta_2 + \frac{3}{\theta_1}\right)$$

Πρόσθετα, $A = -2\ln\theta_1 - 2\ln\theta_2 - \frac{3}{\theta_1}$. Τελικά, έχουμε

$$-Q(\vec{\theta},\vec{\theta}^{\circ}) = 2\ln\theta_1 + 2\ln\theta_2 + \frac{3}{\theta_1} + \left(\ln\theta_1 + \ln\theta_2 + \frac{3}{\theta_1}\right)\cdot\begin{cases} 1, & \theta_2 \geq 4 \\ \frac{\theta_2}{4}, & \theta_2 < 4 \end{cases}$$

Προχωρώντας στο M-step, ισχύει $\vec{\theta} = \underset{\vec{\theta}}{\arg\min}\left[-Q(\vec{\theta},\vec{\theta}^{\circ})\right]$.

Για $\theta_2 \geq 4$ ισχύει πως

$$\underset{\theta_2}{\arg\min}\left[-Q(\vec{\theta},\vec{\theta}^{\circ})\right] = \underset{\theta_2}{\arg\min}\left[2\ln\theta_2 + \ln\theta_2\right] = 4,$$

ενώ για $2 \leq \theta_2 \leq 4$ ισχύει πως

$$\underset{\theta_2}{\arg\min}\left[-Q(\vec{\theta},\vec{\theta}^{\circ})\right] = \underset{\theta_2}{\arg\min}\left[2\ln\theta_2 + \frac{\theta_2}{4}\ln\theta_2\right] = 2$$

Επιπλέον, $\frac{\partial Q}{\partial \theta_1} = 0 \Rightarrow \frac{2}{\theta_1} - \frac{3}{\theta_1^2} + \frac{1}{\theta_1}\begin{cases} 1 \\ \frac{\theta_2}{4} \end{cases} = 0 \Leftrightarrow$

$$2\theta_1 - 3 + \theta_1 \begin{cases} 1 \\ \frac{\theta_2}{4} \end{cases} = 0 \Leftrightarrow$$

$$\begin{cases} 2\theta_1 - 3 + \theta_1 = 0 \\ 2\theta_1 - 3 + \frac{\theta_1}{2} = 0 \end{cases} \Leftrightarrow \begin{matrix} \theta_1 = 1 \quad (\text{και } \theta_2 = 4) \\ \theta_1 = \frac{6}{5} \quad (\text{και } \theta_2 = 2) \end{matrix}$$

Στην πρώτη περίπτωση ισχύει

$$-Q = 6 + 3\ln 4 \simeq 10.159$$

Στη δεύτερη περίπτωση ισχύει

$$-Q = \frac{5}{2}\ln 2 + \frac{5}{2} + \frac{5}{4} + \frac{5}{2}\ln\frac{6}{5} \simeq 5.939$$

Άρα, τελικά, $\vec{\theta}' = \begin{pmatrix} 6/5 \\ 2 \end{pmatrix}$