# Homework 2

## 1D Boundary Value Problem

## Introduction

The first homework assignment is concerned with solving 1D boundary value problems using the finite element method. The main purpose it to introduce the "mFEM" library that will be used throughout this course and to which will be expaned upon for each assignment. The mFEM library is an object-oriented set of code designed for this course. If you are not familiar with object-oriented programming in general or with MATLAB refer to the folowing online documentation.

www.mathworks.com/discovery/object-oriented-programming.html

## mFEM Installation

Before the mFEM library is ready to use, it must be installed, this is accomplished by following these simple steps.

1. Change the current folder in MATLAB to the folder containing this file, i.e. the folder to which you unzipped the contents of the assignment.
2. Run the install script:

```
install;
```

## mFEM Documentation

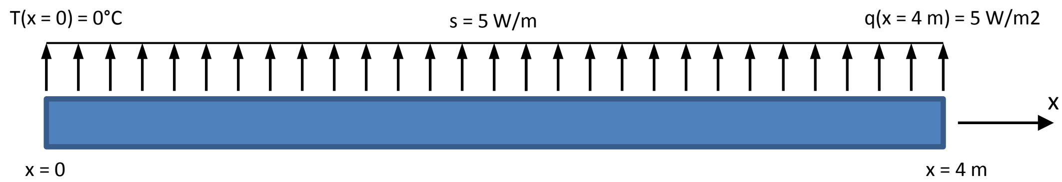
The mFEM documentation will be available from the MATLAB helpbrowser after the `install` function has been executed. To access the mFEM documenation:

1. Open the help browser by typing `doc` into the command line or selecting the help icon.
2. Then search for "mFEM," this will open the main documentation page for the library.

## Tutorial Problem

Before completing this homework, review the tutorial program (`tutorial.m`) (a link to the detailed documentation is available in the mFEM documentation), which solves the following problem.

Consider a bar with a uniformly distributed heat source ($s$) of 5 W/m. The bar has a uniform cross sectional area ($A$) of 0.1 m$^2$ and thermal conductivity ($k$) of 2 W/(°Cm). The length of the bar is 4 m. The boundary conditions are $T(0) = 0$ °C and $q(4) = 5$ W/m$^2$ as shown in the figure below.




## Problem 1

The strong form of for the 1D heat conduction problem is given as follows, where $T$ is temperature, $k$ is thermal conductivity, $A$ is cross-sectional area, $s$ is the heat source, $q$ is heat flux. The domain is denoted as $\Omega$. The boundaries ares denoted as $\Gamma_T$ where the temperature is prescribed, $\Gamma_q$ where the flux is prescribed, and $\Gamma_h$ on the flux is known in terms of the convective heat transfer coefficient ($h$) and ambient temperature ($T_\infty$).

$$\begin{aligned}
&\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + s = 0 \quad \text{on} \quad \Omega, \\
&\vec{q}\cdot\hat{n} = \bar{q} \quad \text{on} \quad \Gamma_q, \\
&\vec{q}\cdot\hat{n} = h(T - T_\infty) \quad \text{on} \quad \Gamma_h, \\
&T = \bar{T} \quad \text{on} \quad \Gamma_T.
\end{aligned} \tag{1.1}$$

Derive the weak form of Equation (1.1), the finite element stiffness matrix, and the finite element force vector.

## Solution

First, multiple the governing equation by an arbitrary weight function and integrate over the domain.

$$\int_\Omega w\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) d\Omega + \int_\Omega ws\, d\Omega = 0 \quad \forall w \tag{1.2}$$

Integrate the left-hand term by parts as follows.

$$\int_\Omega w\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) d\Omega = wAk\frac{dT}{dx}\cdot\hat{n}\bigg|_\Gamma - \int_\Omega \frac{dw}{dx}Ak\frac{dT}{dx}\, d\Omega, \tag{1.3}$$

where $\hat{n}$ is -1 on the left boundary and +1 on the right boundary.

Noting that $q\cdot\hat{n} = -k\frac{dT}{dx}\cdot\hat{n}$ and that on the essential boundaries ($T = \bar{T}$) $w$ is zero, Equations (1.2) and (1.3) may combined to form the weak form of the equation:

$$\int_\Omega \frac{dw}{dx}Ak\frac{dT}{dx}d\Omega + \int_\Omega ws\, d\Omega + wA\bar{q}\bigg|_{\Gamma_q} + wAhT\bigg|_{\Gamma_h} - wAhT_\infty\bigg|_{\Gamma_h} = 0\,\forall w \text{ with } w = 0 \text{ on } \Gamma_T. \tag{1.4}$$

Equation (1.4) may be written as a summation over the elements as:

$$\sum_e \Bigg[\int_{\Omega^e}\left(\frac{dw^e}{dx}\right)^T A^e k^e \frac{dT^e}{dx}d\Omega + \int_{\Omega^e}(w^e)^T s^e\, d\Omega + (w^e)^T A^e \bar{q}^e\bigg|_{\Gamma_q} + (w^e)^T A^e h^e T^e\bigg|_{\Gamma_h} - (w^e)^T A^e h^e T^e_\infty\bigg|_{\Gamma_h}\Bigg] = 0\,\forall w \text{ with } w = 0 \text{ on } \Gamma_T. \tag{1.5}$$

Note, the all terms are scalar, as such the use of the transpose is exploited so that when the vector-base approximations are introduced the vector multiplications will be stated correctly.

Next, it is assumed that that $T^e$ and $w^e$ as well as their derivatives may be approximated by a set of basis functions ($N$) multiplied by the nodal values ($\{T^e\}$ and $\{w^e\}$ respectively), where $B = \frac{dN}{ddx}$.

$$w^T = \{w^e\}^T[N^e]^T \qquad T^e = [N^e]\{T^e\}$$

$$\left(\frac{dw^e}{dx}\right)^T = \{w^e\}^T[B^e]^T \qquad \frac{dT^e}{dx} = [B^e]\{T^e\}$$

It is also assumed that there exists an assembly matrix $[L^e]$ such that it maps the element values to the global values, e.g., $T = [L^e]\{T^e\}$.

Given the approximations Equation (1.5) may be written as:

$$\{w\}^T \sum_e \Big[[L^e]^T[K^e][L^e]\Big]\{T\} - \{w\}^T \sum_e \Big[[L^e]^T[f^e]\Big]\{T\} = 0, \tag{1.6}$$

where $[K^e]$ is the element stiffness matrix and $[f^e]$ is the element force vector:

$$[K^e] = \int_{\Omega_e} [B^e]^T A^e k^e [B] d\Omega + \int_{\Gamma_h} [N^e]^T h^e [N^e] d\Gamma, \tag{1.7}$$

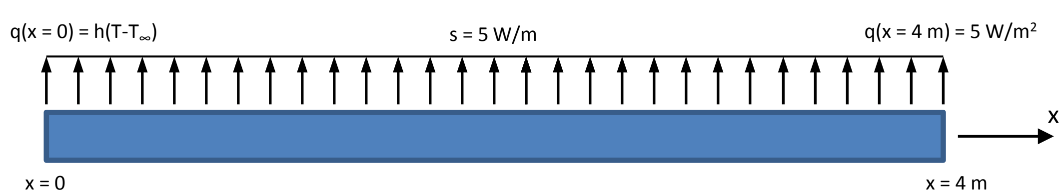
$$[f^e] = \int_{\Omega_e} [N^e]^T s^e \, d\Omega + \int_{\Gamma_q} [N^e]^T \bar{q} \, d\Gamma - \int_{\Gamma_h} [N^e]^T h^e T_\infty \, d\Gamma. \tag{1.8}$$

## Problem 2

Modify the tutorial program to solve the above problem but change the left-hand boundary condition ($x = 0$) to a convective boundary:

$$q(0) = h(T - T_\infty),$$

where $h$ is heat transfer coefficient with a value of 100 W/(°C$\text{m}^2$) and the ambient temperature $T_\infty$ = 10 °C.




## Solution

The additional constants $h$ and $T_inf$ must be added to the System class:

```
sys.add_constant('k',2,'A',0.1,'b',5,'q_bar',5,'h',100,'T_inf',10);
```

The convective term contributes to both the stiffness matrix and force vector, thus an additional matrix and vector must be added to the System.

```
sys.add_matrix('K_h', 'N''*h*N');
sys.add_vector('f_h', '-T_inf*N''',1);
```

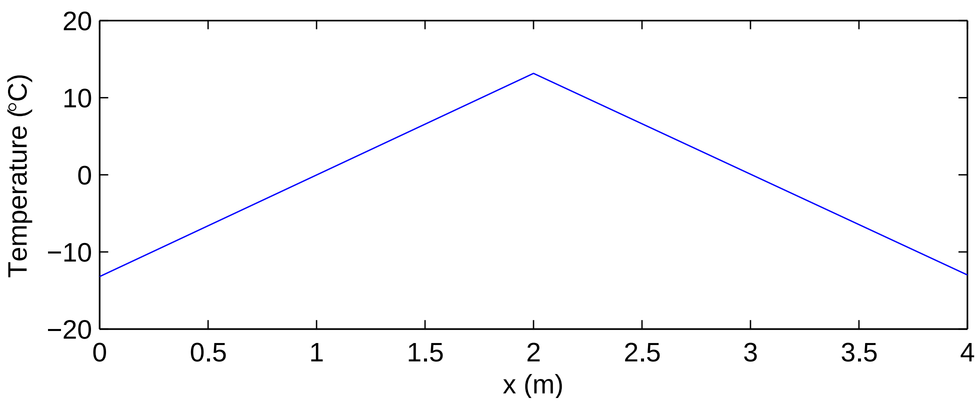
These newly created terms must be added to the assembled stiffness matrix and force vector.

```
K = sys.assemble('K') + sys.assemble('K_h');
f = sys.assemble('f_s') + sys.assemble('f_q') + sys.assemble('f_h');
```

Since there are no explicitly defined essential boundary conditions, only the mixed convective boundary, the solution for temperatures is done using the complete stiffness matrix and force vector.

```
T = K\f;
```

The resulting temperatures are -13.1961, 13.1487, and -13.0064 °C for the left, middle, and right of the bar.




## Problem 3

Solve for the temperature gradient at the Gauss points for both the Tutorial Problem and Problem 2.

**No solution exists for this question, to add solution create a file named: "HW2/HW2-3/soln.tex"**

## Problem 4

Using the center of the rod as the reference point, perform a convergence test for Problem 2 on the temperature at this point to demonstrate how many elements are needed to provide an accurate solution.

**No solution exists for this question, to add solution create a file named:**
**"HW2/HW2-4/soln.tex"**

## Problem 5

Using the `Line2.m` class as a guide and the incomplete `Line3.m` file provided, build a 1D, 3-node quadratic finite element and solve the Tutorial and Problem 2 above using this element. Compare the results between the two elements.

**No solution exists for this question, to add solution create a file named:**
**"HW2/HW2-5/soln.tex"**