Matemática Discreta

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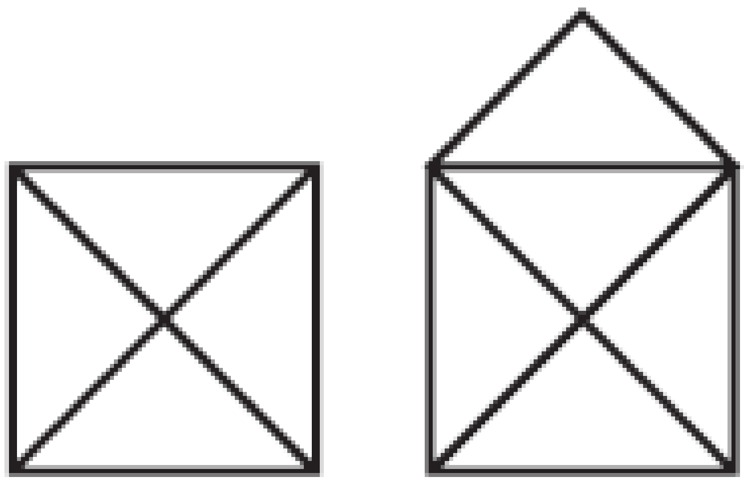
Eulerian Graphs



Definition 51.1

(Eulerian trail, tour) Let G be a graph. A walk in G that traverses every edge exactly once is called an *Eulerian trail*. If, in addition, the trail begins and ends at the same vertex, we call the walk an *Eulerian tour*. Finally, if G has an Eulerian tour, we call G *Eulerian*.

Eulerian Graphs

The problems we consider are the following: Which graphs have Eulerian trails? Which graphs have Eulerian tours (i.e., are Eulerian)? In this section, we give a complete answer.

Necessary Conditions

Let us call a component of a graph *trivial* if it contains only one vertex. Otherwise we call the component *nontrivial*. Thus the first necessary condition for the existence of an Eulerian trail is the following:

- If G is Eulerian, then G has at most one nontrivial component.

Eulerian Graphs

Necessary Conditions

$$W = \text{first} \sim \cdots \sim ? \sim v \sim ? \sim \cdots \sim ? \sim v \sim ? \sim \cdots \sim ? \sim v \sim ? \sim \cdots \sim \text{last}.$$

Since every edge of the graph is traversed exactly once, and since for every edge entering v on this tour there is another edge exiting v, it must be the case that $d(v)$ is even.

We therefore have the following:

- If G has an Eulerian trail, then it has at most two vertices of odd degree.

- If G has an Eulerian trail that begins at a vertex a and ends a vertex b (with $a \neq b$), then vertices a and b have odd degree.

Eulerian Graphs

Necessary Conditions

$$W = \text{first} \sim \cdots \sim ? \sim v \sim ? \sim \cdots \sim ? \sim v \sim ? \sim \cdots \sim ? \sim v \sim ? \sim \cdots \sim \text{last}.$$

Since every edge of the graph is traversed exactly once, and since for every edge entering v on this tour there is another edge exiting v, it must be the case that $d(v)$ is even.

We therefore have the following:

- If G has an Eulerian trail, then it has at most two vertices of odd degree.

- If G has an Eulerian trail that begins at a vertex a and ends a vertex b (with $a \neq b$), then vertices a and b have odd degree.

- If G has an Eulerian tour (i.e., if G is Eulerian), then all vertices in G have even degree.

Eulerian Graphs

Necessary Conditions

Suppose we have an Eulerian tour in a connected graph that begins and ends at a vertex a, and suppose b is the second vertex on this tour:

$$W = a \sim b \sim \cdots\cdots \sim a.$$

We can, instead, begin the tour at b, follow the original tour until we get to the last visit to a, and finish at b; that is,

$$W' = b \sim \cdots\cdots \sim a \sim b$$

- If G is a connected Eulerian graph, then G has an Euler tour that begins/ends at any vertex.

Eulerian Graphs

Lemma 51.4

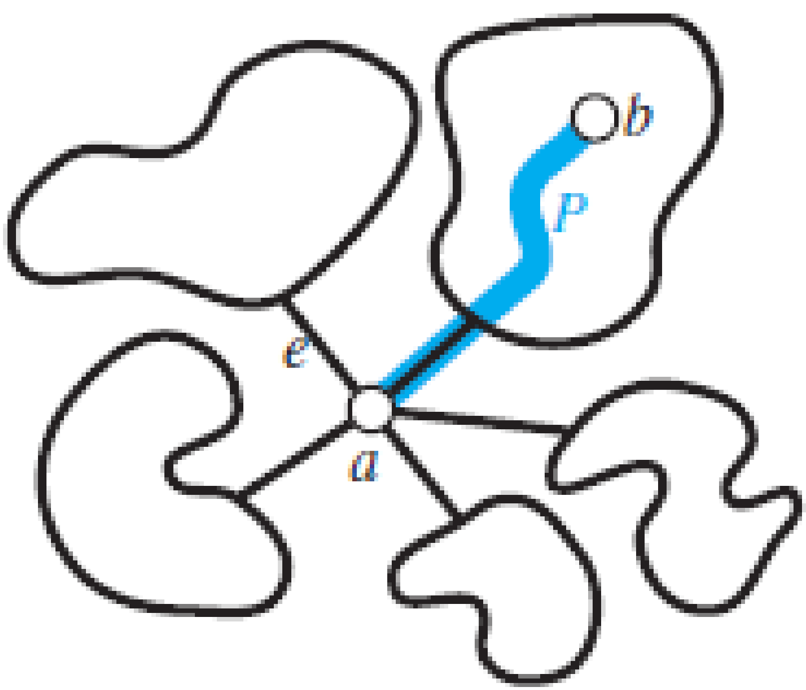
Let G be a graph all of whose vertices have even degree. Then no edge of G is a cut edge.

Proof. Suppose, for the sake of contradiction, $e = xy$ is a cut edge of such a graph. Notice that $G - e$ has exactly two components (by Theorem 49.12), and each of these components contains exactly one vertex of odd degree, contradicting Exercise 47.15. ■

Eulerian Graphs

Lemma 51.5

Let G be a connected graph with exactly two vertices of odd degree. Let a be a vertex of odd degree and suppose $d(a) \neq 1$. Then at least one of the edges incident with a is not a cut edge.



Eulerian Graphs

Lemma 51.5

Let G be a connected graph with exactly two vertices of odd degree. Let a be a vertex of odd degree and suppose $d(a) \neq 1$. Then at least one of the edges incident with a is not a cut edge.

Proof. Suppose, for the sake of contradiction, that all edges incident at a are cut edges. Let b be the other vertex of odd degree in G.

Since G is connected, there is an (a, b)-path P in G. Exactly one edge incident at a is traversed by P. Let e be any other edge incident at a.

Now consider the graph $G' = G - e$. This graph has exactly two components (Theorem 49.12). Since the path P does not use the edge e, vertices a and b are in the same component. Notice also that, in G', vertex a has even degree, and all other vertices in its component have not changed degree. This means that, in G', the component containing vertex a has exactly one vertex of odd degree, contradicting Exercise 47.15. ■

Eulerian Graphs

Theorem 51.2

Let G be a connected graph all of whose vertices have even degree. For every vertex $v \in V(G)$, there is an Eulerian tour that begins and ends at v.

Theorem 51.3

Let G be a connected graph with exactly two vertices of odd degree: a and b. Then G has an Eulerian trail that begins at a and ends at b.