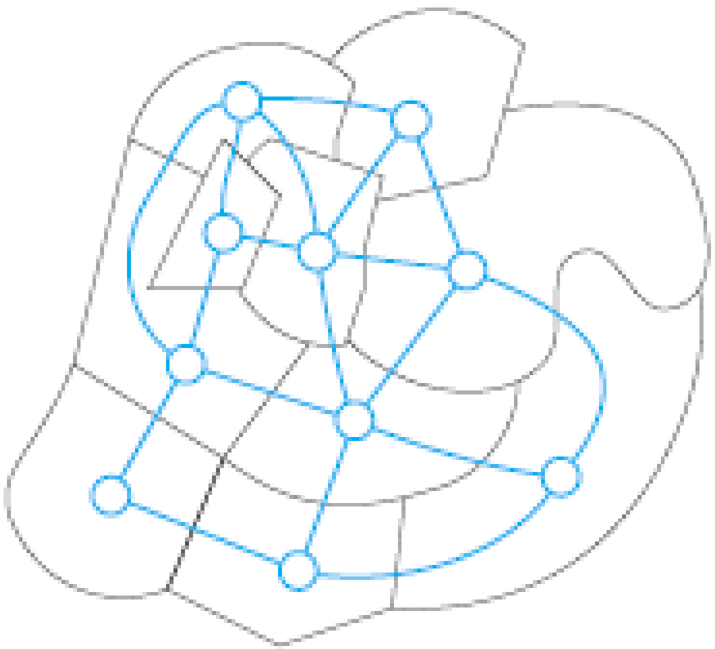
Matemática Discreta

Escola de Artes, Ciências e Humanidades - USP
Profa Dra. Karla Lima
email:ksampaiolima@usp.br

Coloring



The general problem is as follows: Let G be a graph. To each vertex of G, we wish to assign a color such that adjacent vertices receive different colors. Of course, we could give every vertex its own color, but this is not terribly interesting and not relevant to applications. The objective is to use as few colors as possible.

Coloring

Definition 52.1

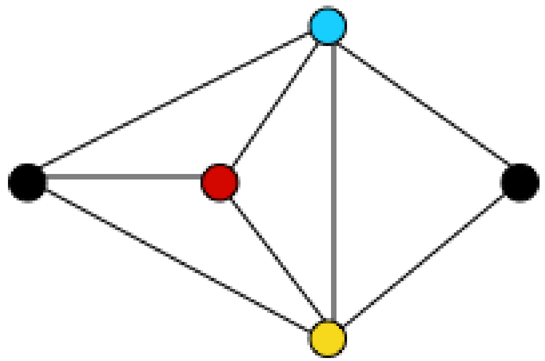
(Graph coloring) Let G be a graph and let k be a positive integer. A *k-coloring* of G is a function

$$f : V(G) \to \{1, 2, \ldots, k\}.$$

We call this coloring *proper* provided

$$\forall xy \in E(G),\ f(x) \neq f(y).$$

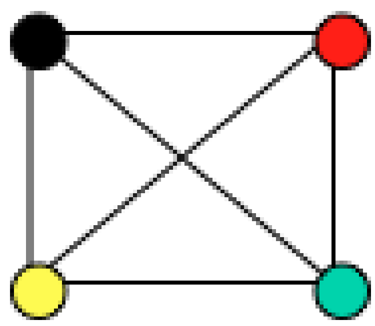
If a graph has a proper k-coloring, we call it *k-colorable*.

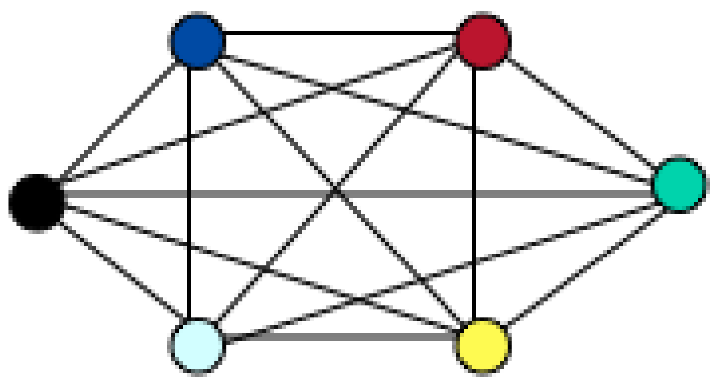


Coloring

Definition 52.2

(Chromatic number) Let G be a graph. The smallest positive integer k for which G is k-colorable is called the *chromatic number* of G. The chromatic number of G is denoted $\chi(G)$.





Coloring

Proposition 52.4

Let G be a subgraph of H. Then $\chi(G) \le \chi(H)$.

Proof. Given a proper coloring of H, we can simply copy those colors to the vertices of G to achieve a proper coloring of G. So if we used only $\chi(H)$ colors to color the vertices of H, we have used at most $\chi(H)$ colors in a proper coloring of G. ■

Coloring

Proposition 52.5

Let G be a graph with maximum degree Δ. Then $\chi(G) \le \Delta + 1$.

Proof. Suppose the vertices of G are $\{v_1, v_2, \cdots, v_n\}$ and we have a palette of $\Delta + 1$ colors. We color the vertices of G as follows:

To begin, no vertex in G is assigned a color. Assign any color from the palette to vertex v_1. Next we color vertex v_2. We take any color we wish from the palette, as long as the coloring is proper. In other words, if $v_1 v_2$ is an edge, we may not assign the same color to v_2 that we gave to v_1. We continue in exactly this fashion through all the vertices. That is, when we come to vertex v_j, we assign to vertex v_j any color from the palette we wish, just making certain that the color on vertex v_j is not the same as any of its already-colored neighbors.

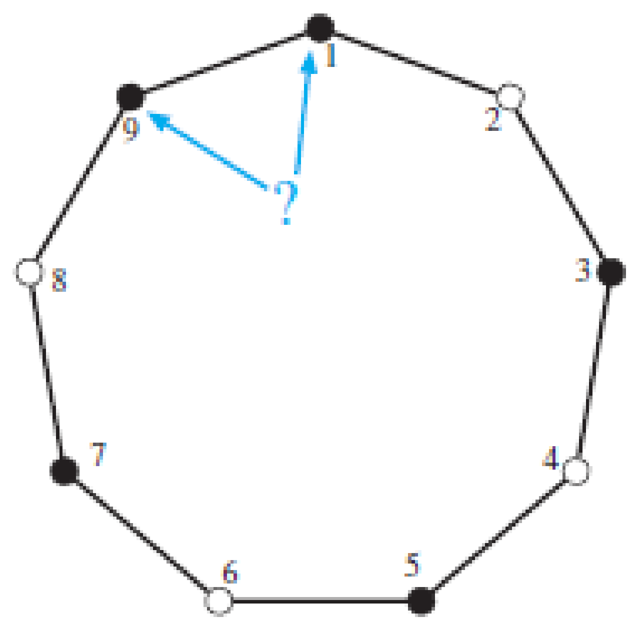
The issue is whether there are sufficiently many colors in the palette so that this procedure never gets stuck (i.e., we never reach a vertex where there is no legal color left to choose). Since every vertex has at most Δ neighbors and since there are $\Delta + 1$ colors in the palette, we can never get stuck. Thus this procedure produces a proper $\Delta + 1$-coloring of the graph. Hence $\chi(G) \le \Delta + 1$. ■

Coloring

Example 52.6

What is the chromatic number of the cycle C_n? If n is even, then we can alternate colors (black, white, black, white, etc.) around the cycle. When n is even, this yields a valid coloring. However, if n is odd, then vertex 1 and vertex n would both be black if we alternated colors around the cycle. See the figure. Thus, for n-odd, C_n is not two-colorable. It is, however, three-colorable. We can alternately color vertices 1 through $n-1$ with black and white and then color vertex n with, say, blue. This gives a proper three-coloring of C_n. [Also, by Proposition 52.5, we have $\chi(C_n) \le \Delta(C_n) + 1 = 2 + 1 = 3$.] Thus

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even and} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$



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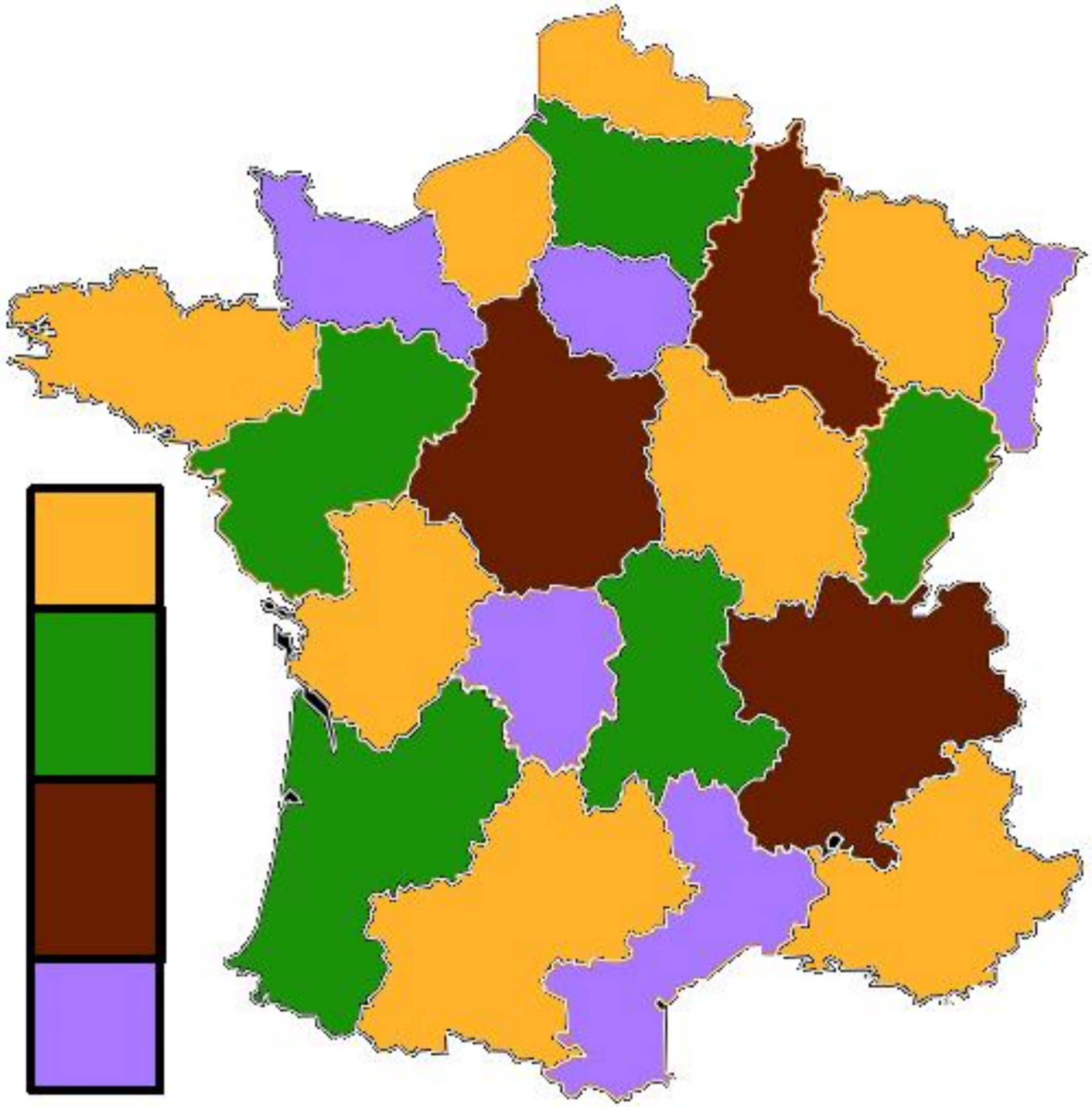

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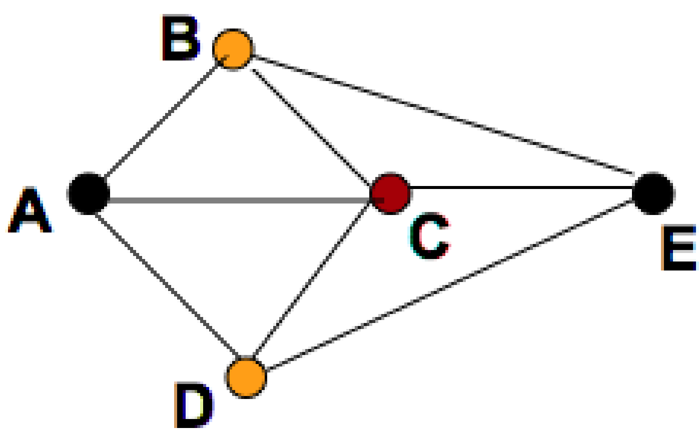
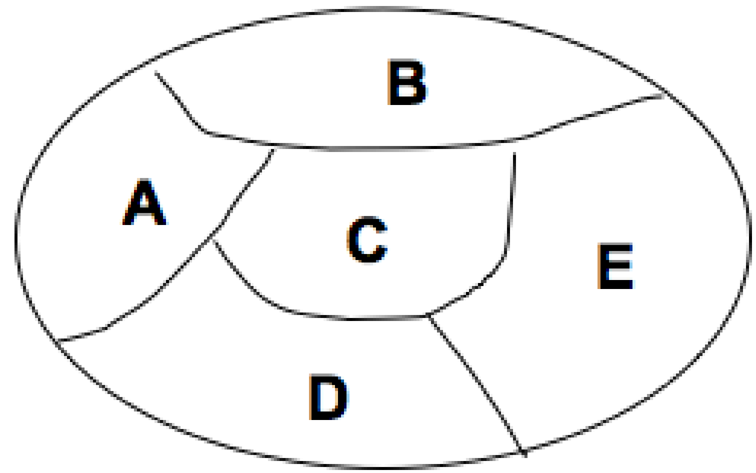
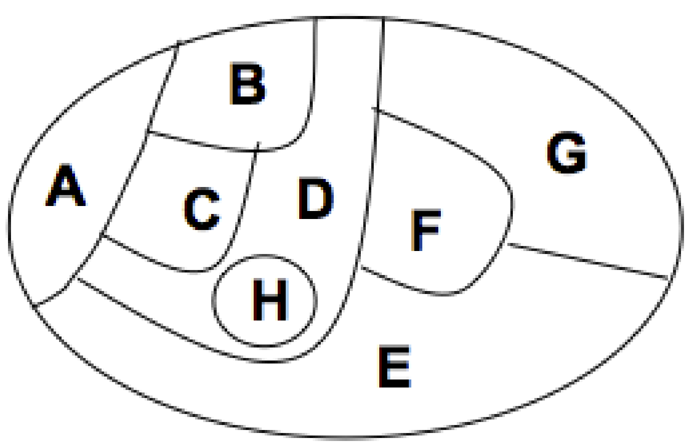
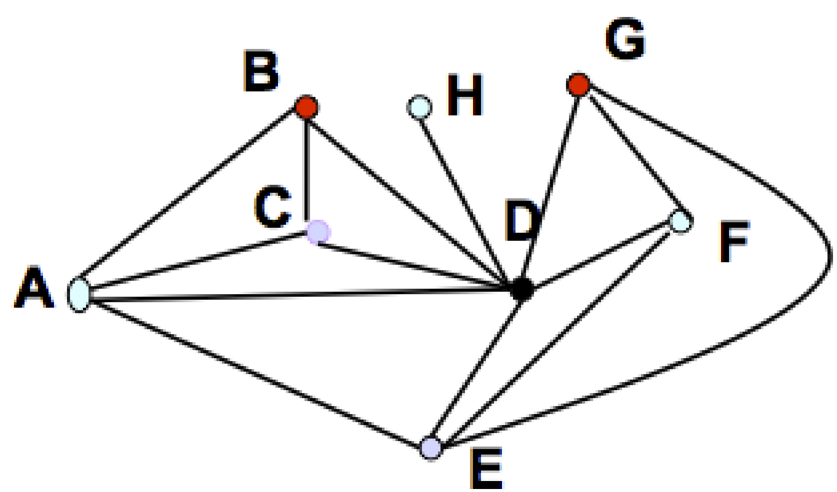




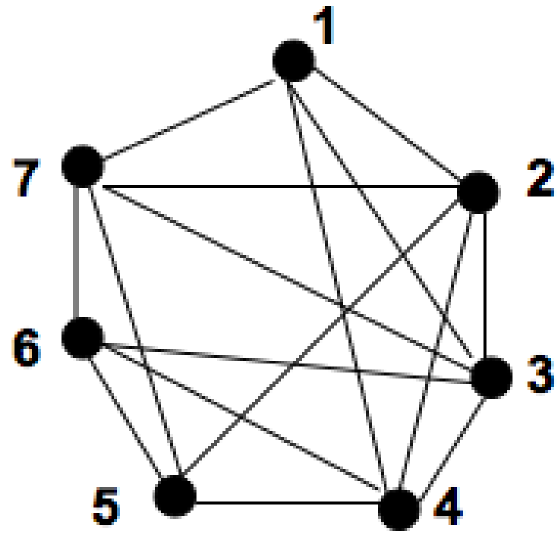




Coloring

- Exemplo: existem 7 dsciplinas. A seguinte tabela mostra a existência de alunos em comum: onde há * na célula ij, existe um aluno matriculado na disciplina I e na disciplina j.

	1	2	3	4	5	6	7
1	-	*	*	*	-	-	*
2		-	*	*	*	-	*
3			-	*	-	*	*
4				-	*	*	-
5					-	*	*
6						-	*
7							-



Bipartite Graphs

Which graphs are one-colorable? That is, can we describe the class of all graphs G for which $\chi(G) = 1$?

Proposition 52.7

A graph G is one-colorable if and only if it is edgeless.

That was easy! Let's move on to characterizing two-colorable graphs—that is, graphs G for which $\chi(G) \le 2$. These graphs have a special name.

Definition 52.8

A graph G is called *bipartite* provided it is 2-colorable.

Bipartite Graphs

Proposition 52.9 Trees are bipartite.

Proof. The proof is by induction on the number of vertices in the tree.

Basis case: Clearly a tree with only one vertex is bipartite. Indeed, $\chi(K_1) = 1 \leq 2$.

Induction hypothesis: Every tree with n vertices is bipartite.

Let T be a tree with $n + 1$ vertices. Let v be a leaf of T and let $T' = T - v$. Since T is a tree with n vertices, by induction T' is bipartite. Properly color T' using the two colors black and white.

Now consider v's neighbor—call it w. Whatever color w has, we can give v the other color (e.g., if w is white, we color v black).

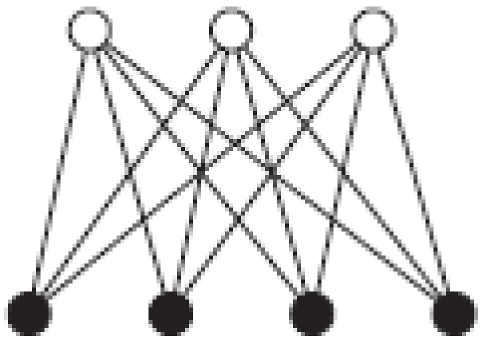
Since v has only one neighbor, this gives a proper two-coloring of T. ■

Bipartite Graphs

Definition 52.10

(Complete bipartite graphs) Let n, m be positive integers. The *complete bipartite graph* $K_{n,m}$ is a graph whose vertices can be partitioned $V = X \cup Y$ such that

- $|X| = n$,
- $|Y| = m$,
- for all $x \in X$ and for all $y \in Y$, xy is an edge, and
- no edge has both its endpoints in X or both its endpoints in Y.



Bipartite Graphs

Theorem 52.11

A graph is bipartite if and only if it does not contain an odd cycle.