Matemática Discreta

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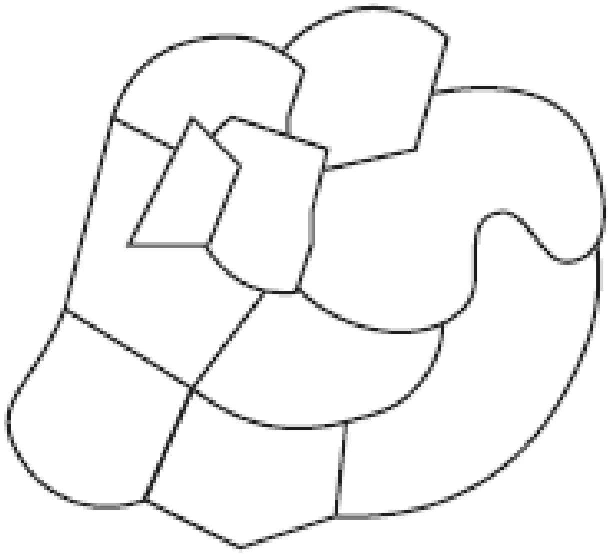
Fundamentals of Graph Theory

Map Coloring



Fundamentals of Graph Theory

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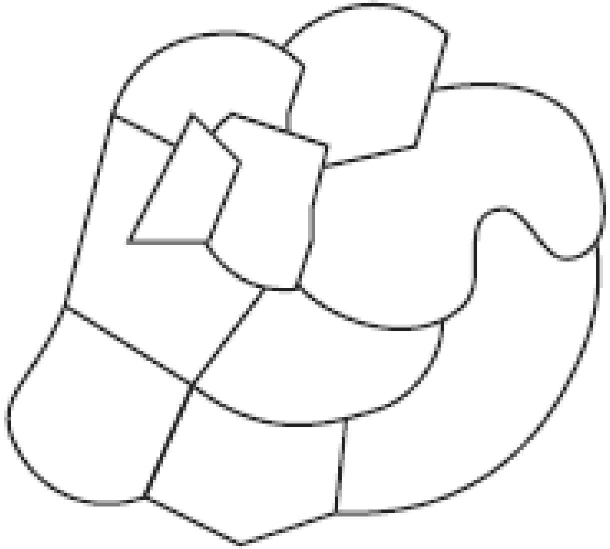
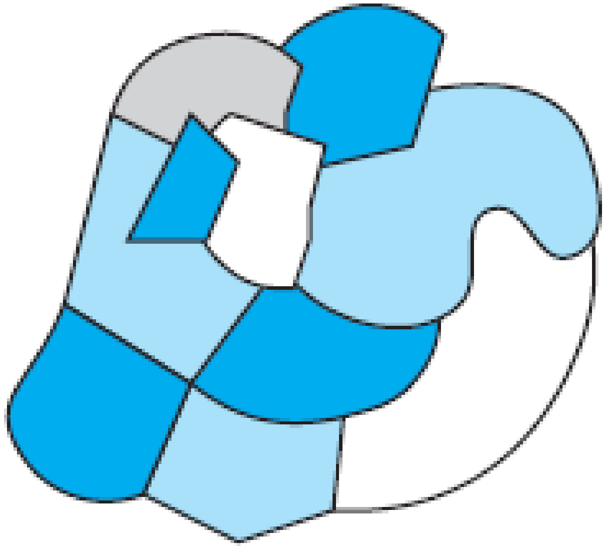


The question is: What is the smallest number of colors you need to color your map?

Fundamentals of Graph Theory

Map Coloring



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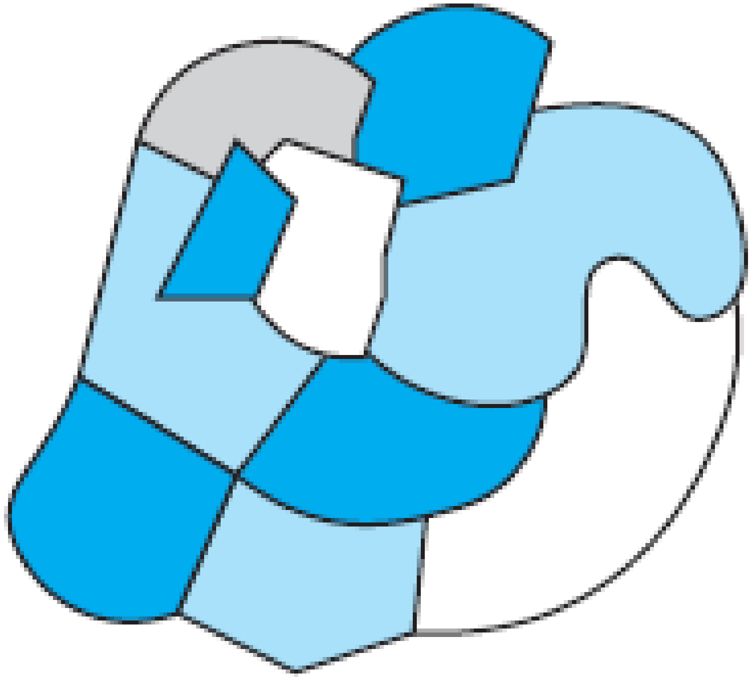
Fundamentals of Graph Theory

Map Coloring

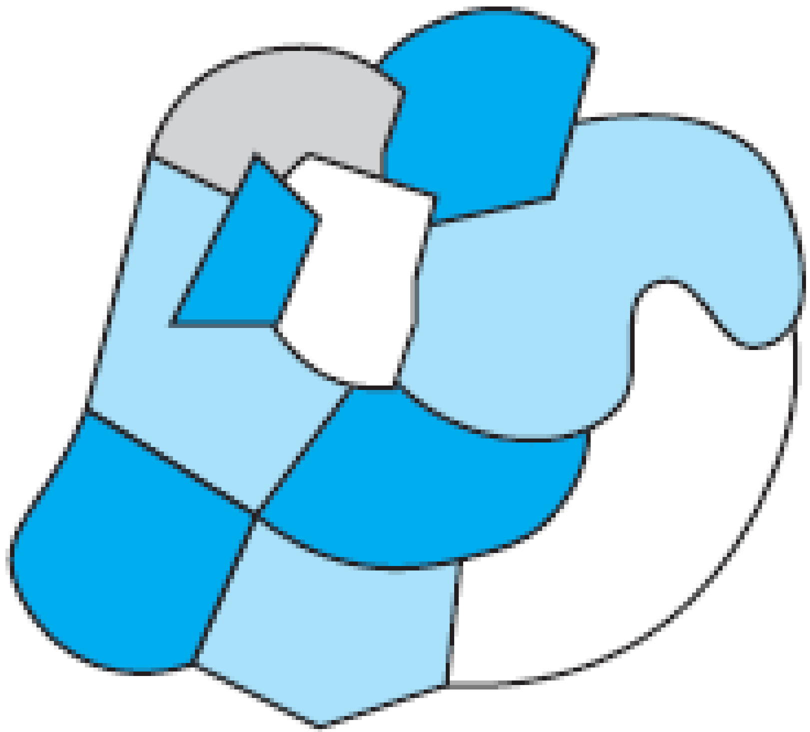


The question is: What is the smallest number of colors you need to color your map?

We can color the map in the figure with just four colors, as shown

Fundamentals of Graph Theory

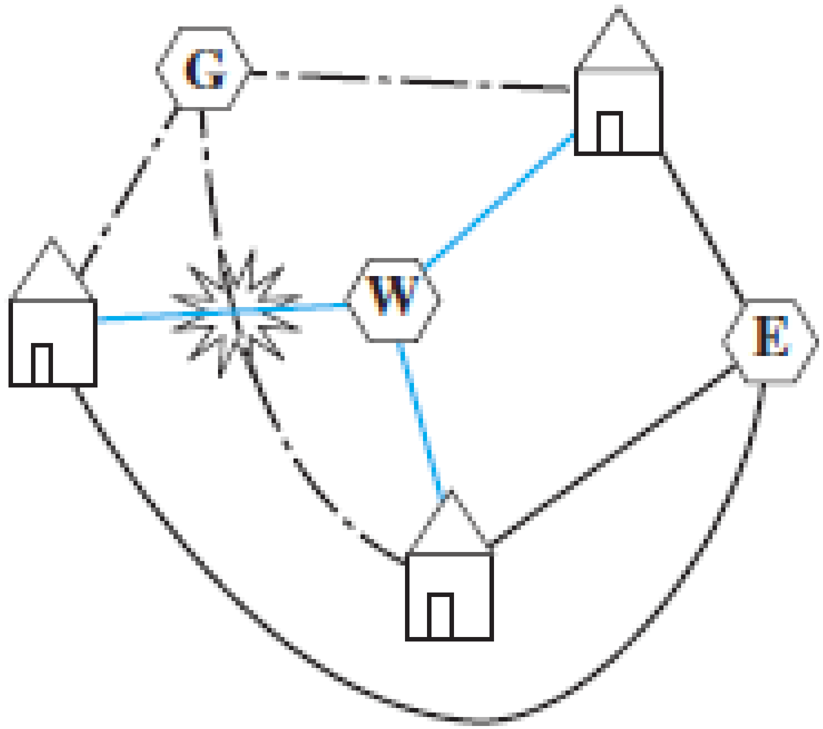
Map Coloring



- Can this map be colored with fewer than four colors? (Notice that we have only one country that is gray; perhaps if we are clever, we can color this map with only three colors.)
- Is there another map that can be colored with fewer than four colors?
- Is there a map that requires more than four colors?

Fundamentals of Graph Theory

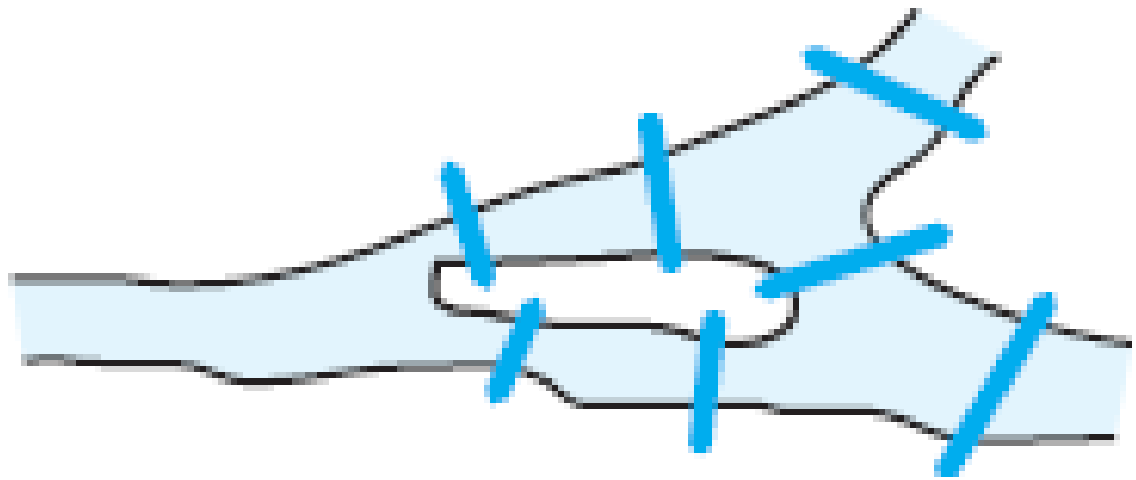
Three Utilities



The following is a classic puzzle. Imagine a "city" containing three houses and three utility plants. The three utilities supply gas, water, and electricity. As an urban planner, your job is to run connections from every utility plant to every home. You need to have three electric wires (from the electric plant to each of the three houses), three water pipes (from the water plant to the houses), and three gas lines (from the gas facility to the houses). You may place the houses and the utility plants anywhere you desire. However, you may not allow two wires/pipes/lines to cross! The diagram shows a failed attempt to construct a suitable layout.

Fundamentals of Graph Theory

Seven Bridges

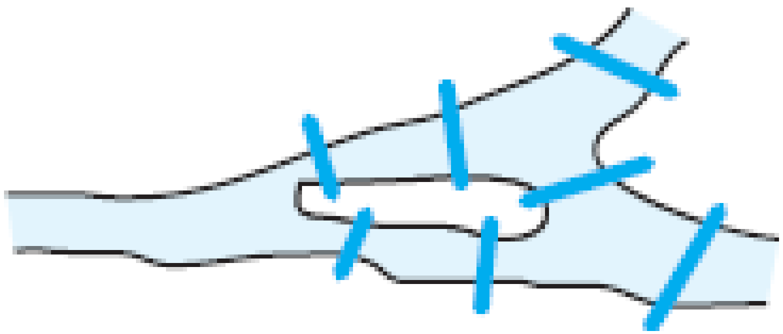


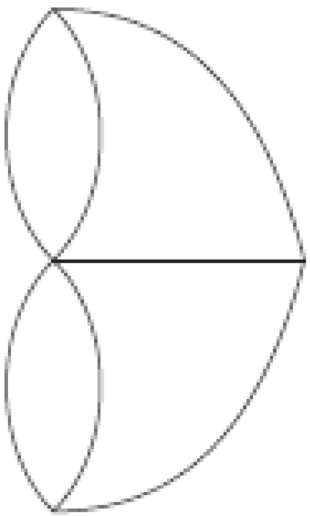
The following is another classic puzzle. In the late 1700s, in the city of Königsburg (now called Kaliningrad) located in the aforementioned disconnected section of Russia, there were seven bridges connecting various parts of the city; these were configured as shown in the figure.

The townspeople enjoyed strolling through their city in the evening. They wondered: Is there a tour we can take through our city so that we cross every bridge exactly once?

Fundamentals of Graph Theory

Seven Bridges





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Fundamentals of Graph Theory

What Is a Graph?

Definition 47.1

(**Graph**) A *graph* is a pair $G = (V, E)$ where V is a nonempty finite set and E is a set of two-element subsets of V.

Example 47.2

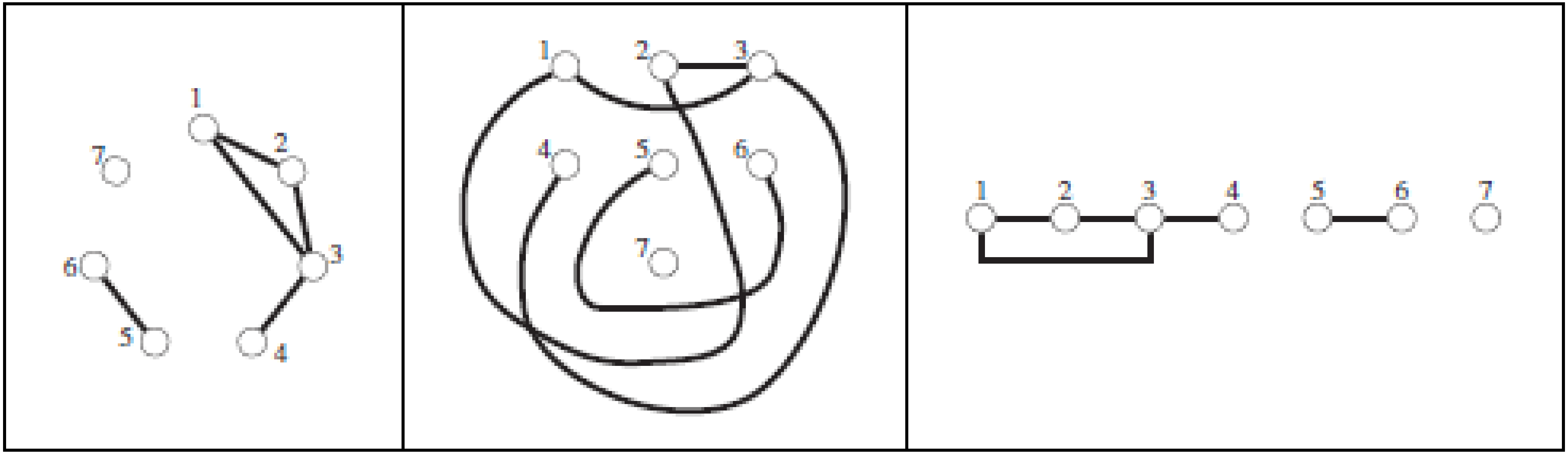
Let

$$G = \left(\{1,2,3,4,5,6,7\}, \{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{5,6\}\}\right).$$

Here V is the finite set $\{1,2,3,4,5,6,7\}$ and E is a set containing 5 two-element subsets of V: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{3,4\}$, and $\{5,6\}$. Therefore $G = (V, E)$ is a graph.

The elements of V are called the *vertices* (singular: *vertex*) of the graph, and the elements of E are called the *edges* of the graph.

Fundamentals of Graph Theory



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Fundamentals of Graph Theory

Adjacency

Definition 47.3

(Adjacent) Let $G = (V, E)$ be a graph and let $u, v \in V$. We say that u is *adjacent* to v provided $\{u, v\} \in E$. The notation $u \sim v$ means that u is adjacent to v.

If $\{u, v\}$ is an edge of G, we call u and v the *endpoints* of the edge.

Suppose v is a vertex and an endpoint of the edge e. We can express this fact as $v \in e$ since e is a two-element set, one of whose elements is v. We also say that v is *incident on* (or *incident with*) e.

Fundamentals of Graph Theory

A Matter of Degree

Let $G = (V, E)$ be a graph and suppose u and v are vertices of G. If u and v are adjacent, we also say that u and v are *neighbors*. The set of all neighbors of a vertex v is called the *neighborhood* of v and is denoted $N(v)$. That is,

$$N(v) = \{u \in V : u \sim v\}.$$

Example 47.2

Definition 47.4

(**Degree**) Let $G = (V, E)$ be a graph and let $v \in V$. The degree of v is the number of edges with which v is incident. The degree of v is denoted $d_G(v)$ or, if there is no risk of confusion, simply $d(v)$.

Fundamentals of Graph Theory

Definition 47.4

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In other words,

$$d(v) = |N(v)|.$$

Example 47.2.

$$\begin{aligned}\sum_{v \in V} d(v) &= d(1) + d(2) + d(3) + d(4) + d(5) + d(6) + d(7) \\ &= 2 + 2 + 3 + 1 + 1 + 1 + 0 = 10\end{aligned}$$

Fundamentals of Graph Theory

Theorem 47.5

Let $G = (V, E)$. The sum of the degrees of the vertices in G is twice the number of edges; that is,

$$\sum_{v \in V} d(v) = 2|E|.$$

Fundamentals of Graph Theory

Theorem 47.5

Proof. Suppose the vertex set is $V = \{v_1, v_2, \ldots, v_n\}$. We can create an $n \times n$ matrix as follows. The entry in row i and column j of this matrix is 1 if $v_i \sim v_j$ and is 0 otherwise.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fundamentals of Graph Theory

Theorem 47.5

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We ask,

> How many 1s are in this matrix?

Fundamentals of Graph Theory

Theorem 47.5

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First answer: Notice that for every edge of G there are exactly two 1s in the matrix. For example, if $v_i v_j \in E$, then there is a 1 in position ij (row i, column j) and a 1 in position ji. Thus the number of 1s in this matrix is exactly $2|E|$.

Second answer: Consider a given row of this matrix—say, the row corresponding to some vertex v_i. There is a 1 in this row exactly for those vertices adjacent to v_i (i.e., there is a 1 in the j^{th} spot of this row exactly when there is an edge from v_i to v_j). Thus, the number of 1s in this row is exactly the degree of the vertex—that is, $d(v_i)$.

Fundamentals of Graph Theory

Further Notation and Vocabulary

- *Maximum and minimum degree.*
  The maximum degree of a vertex in G is denoted $\Delta(G)$. The minimum degree of a vertex in G is denoted $\delta(G)$. The letters Δ and δ are upper- and lowercase Greek deltas, respectively. For the graph in Example 47.2, we have $\Delta(G) = 3$ and $\delta(G) = 0$.

Fundamentals of Graph Theory

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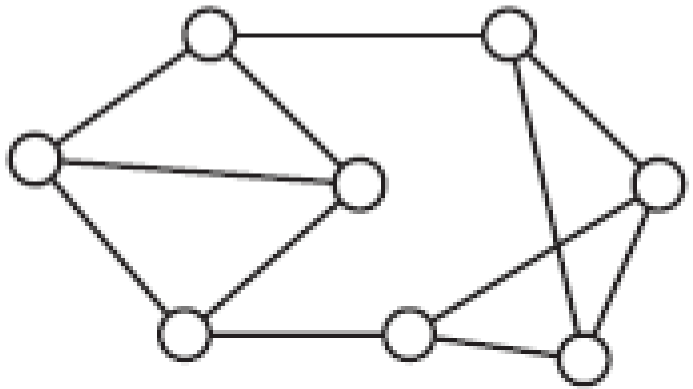
- *Regular graphs.*

  If all vertices in G have the same degree, we call G *regular*. If a graph is regular and all vertices have degree r, we also call the graph r-regular. The graph in the figure is 3-regular.



Fundamentals of Graph Theory

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- *Regular graphs.*

  If all vertices in G have the same degree, we call G *regular*. If a graph is regular and all vertices have degree r, we also call the graph r-regular. The graph in the figure is 3-regular.

- *Vertex and edge sets.*

  Let G be a graph. If we neglect to give a name to the vertex and edge sets of G, we can simply write $V(G)$ and $E(G)$ for the vertex and edge sets, respectively.

Fundamentals of Graph Theory

Further Notation and Vocabulary

- *Order and size.*

  Let $G = (V, E)$ be a graph. The *order* of G is the number of vertices in G—that is, $|V|$. The *size* of G is the number of edges—that is, $|E(G)|$.

  It is customary (but certainly not mandatory) to use the letters n and m to stand for $|V|$ and $|E|$, respectively.

  Various authors invent special symbols to stand for the number of vertices and the number of edges in a graph. Personally, I like the following:

$$\nu(G) = |V(G)| \qquad \text{and} \qquad \varepsilon(G) = |E(G)|.$$

Fundamentals of Graph Theory

Further Notation and Vocabulary

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- *Complete graphs.*

  Let G be a graph. If all pairs of distinct vertices are adjacent in G, we call G *complete*. A complete graph on n vertices is denoted K_n. The graph in the figure is a K_5.

  The opposite extreme is a graph with no edges. We call such graphs *edgeless*.

Fundamentals of Graph Theory

Further Notation and Vocabulary

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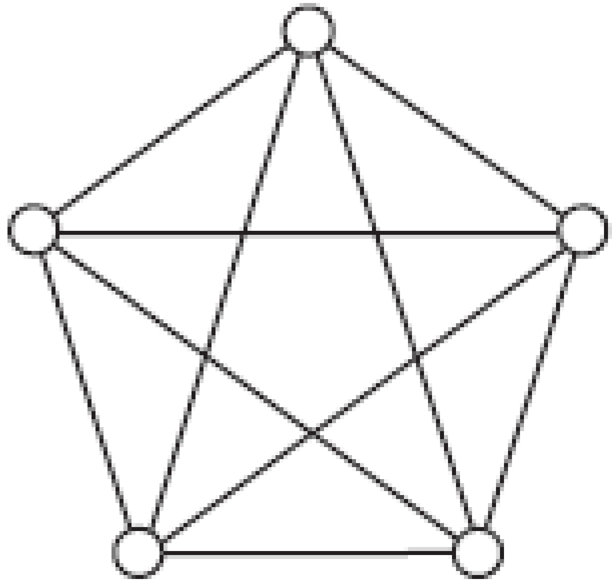
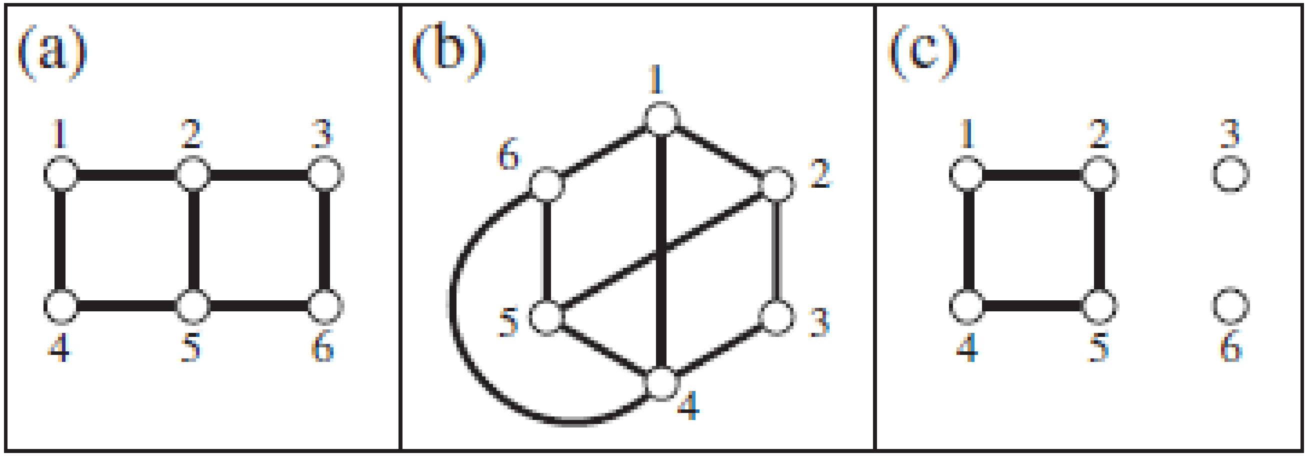
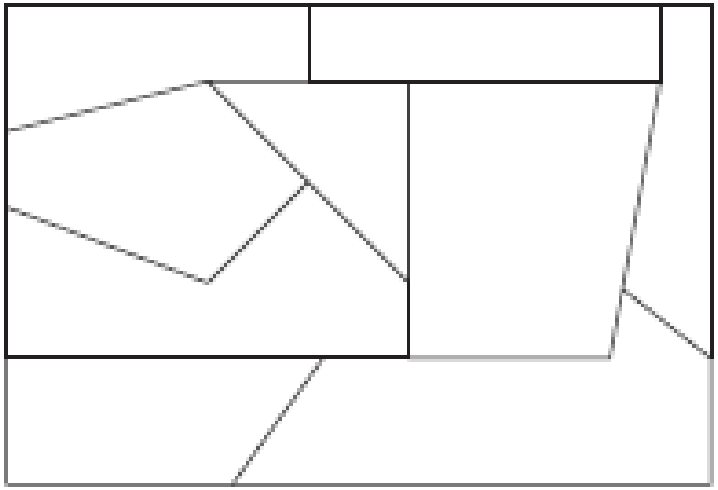


Fundamentals of Graph Theory

47.1. The following pictures represent graphs. Please write each of these graphs as a pair of sets (V, E).



47.3. Color the map in the figure with four colors (so that adjacent countries have different colors) and explain why it is not possible to color this map with only three colors.



Fundamentals of Graph Theory

47.16. Prove that in any graph with two or more vertices, there must be two vertices of the same degree.

47.18. Find all 3-regular graphs on nine vertices.

47.19. How many edges are in K_n, a complete graph on n vertices?

47.20. How many different graphs can be formed with vertex set $V = \{1, 2, 3, \ldots, n\}$?

Fundamentals of Graph Theory

Subgraphs

Definition 48.1

(Subgraph) Let G and H be graphs. We call G a *subgraph* of H provided $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

Fundamentals of Graph Theory

Subgraphs

Example 48.2 Let G and H be the following graphs:

$$V(G) = \{1, 2, 3, 4, 6, 7, 8\} \qquad E(G) = \big\{\{1,2\}, \{2,3\}, \{2,6\}, \{3,6\}, \{4,7\}, \{6,8\}, \{7,8\}\big\}$$

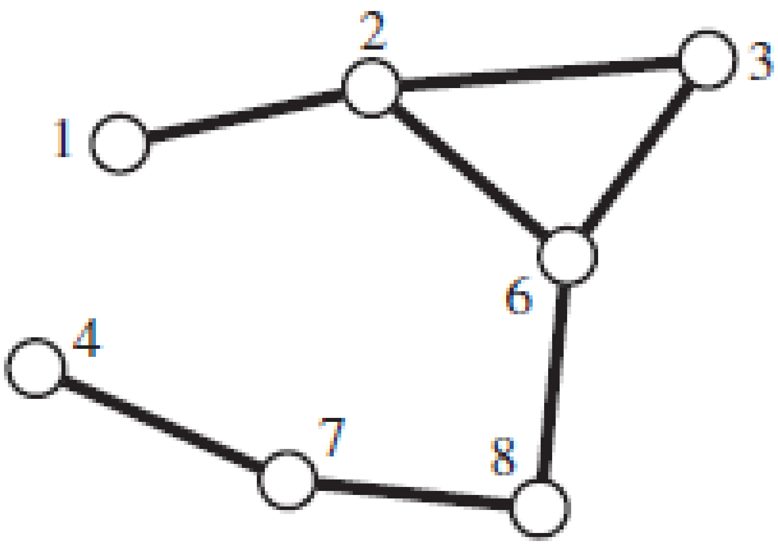
$$V(H) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \qquad E(H) = \big\{\{1,2\}, \{1,4\}, \{2,3\}, \{2,5\}, \{2,6\}, \{3,6\}, \{3,9\}, \{4,7\}, \{5,6\}, \{5,7\}, \{6,8\}, \{6,9\}, \{7,8\}, \{8,9\}\big\}$$

Notice that $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$ and so G is a subgraph of H

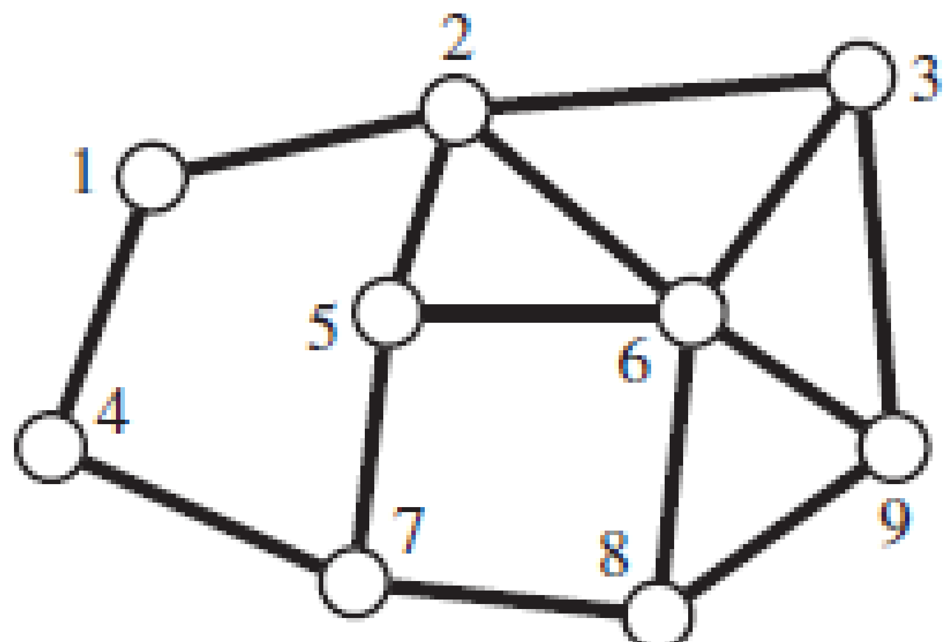
Fundamentals of Graph Theory

Subgraphs

G



H



Fundamentals of Graph Theory

Induced and Spanning Subgraphs

Definition 48.3

(Spanning subgraph) Let G and H be graphs. We call G a *spanning subgraph* of H provided G is a subgraph of H and $V(G) = V(H)$.

When G is a spanning subgraph of H, the definition requires that $V(G) = V(H)$; that is, G and H have all the same vertices. Thus the only allowable deletions from H are edge deletions.

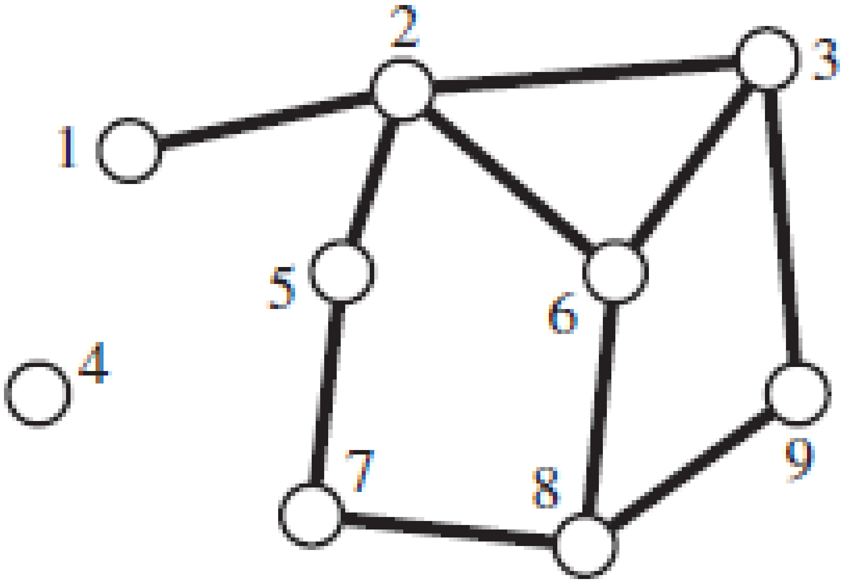
Example 48.4

Fundamentals of Graph Theory

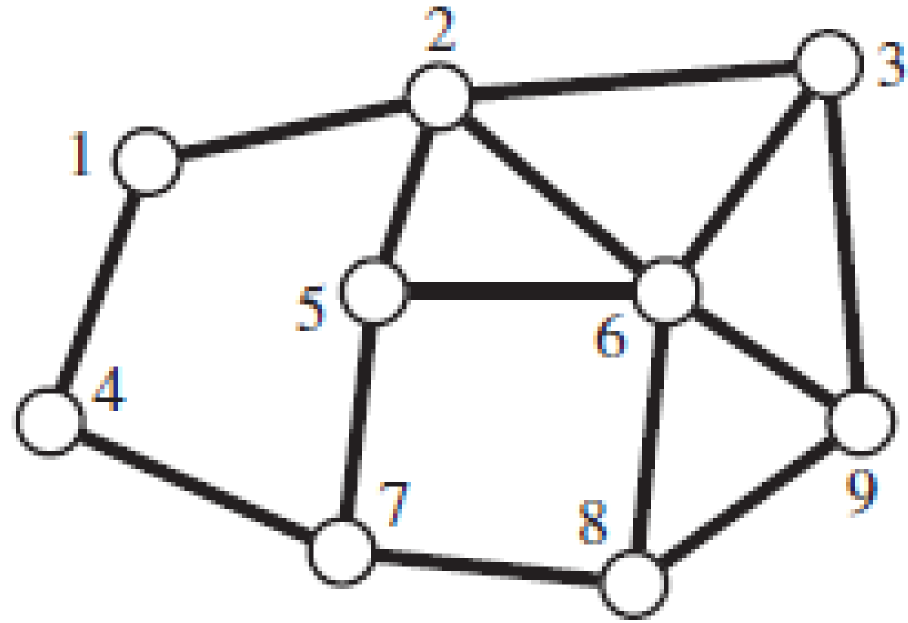
Induced and Spanning Subgraphs

Example 48.4

G



H



Fundamentals of Graph Theory

Induced and Spanning Subgraphs

Definition 48.5

(Induced subgraph) Let H be a graph and let A be a subset of the vertices of H; that is, $A \subseteq V(H)$. The *subgraph of* H *induced on* A is the graph $H[A]$ defined by

$$V(H[A]) = A \quad \text{and}$$
$$E(H[A]) = \{xy \in E(H) : x \in A \text{ and } y \in A\}.$$

When we say that G is an induced subgraph of H, we mean that $G = H[A]$ for some $A \subseteq V(H)$.

Fundamentals of Graph Theory

Induced and Spanning Subgraphs

Example 48.6

$V(G) = \{1, 2, 3, 5, 6, 7, 8\}$ and
$E(G) = \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{5, 6\}, \{5, 7\}, \{6, 8\}, \{7, 8\}\}$.

$$G = H[A] \qquad \text{where} \qquad A = \{1, 2, 3, 5, 6, 7, 8\}.$$

We can also write $G = (H - 4) - 9 = (H - 9) - 4$.

Fundamentals of Graph Theory

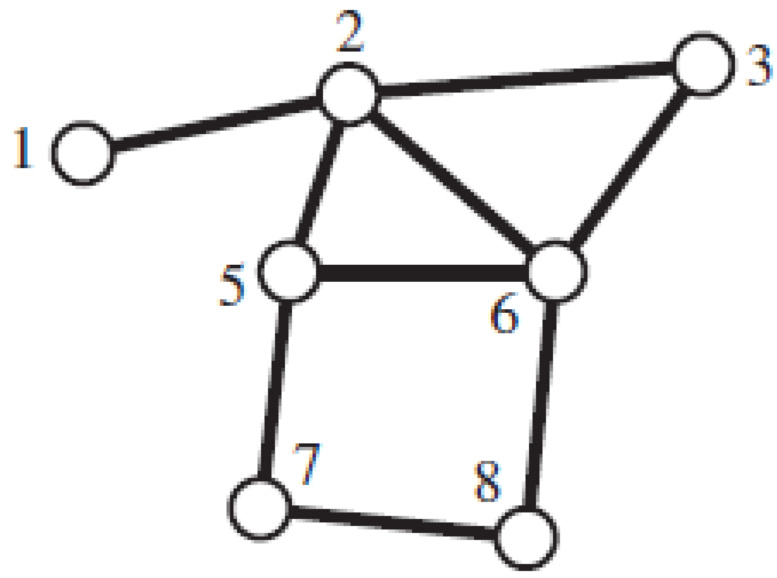
Induced and Spanning Subgraphs

Example 48.6

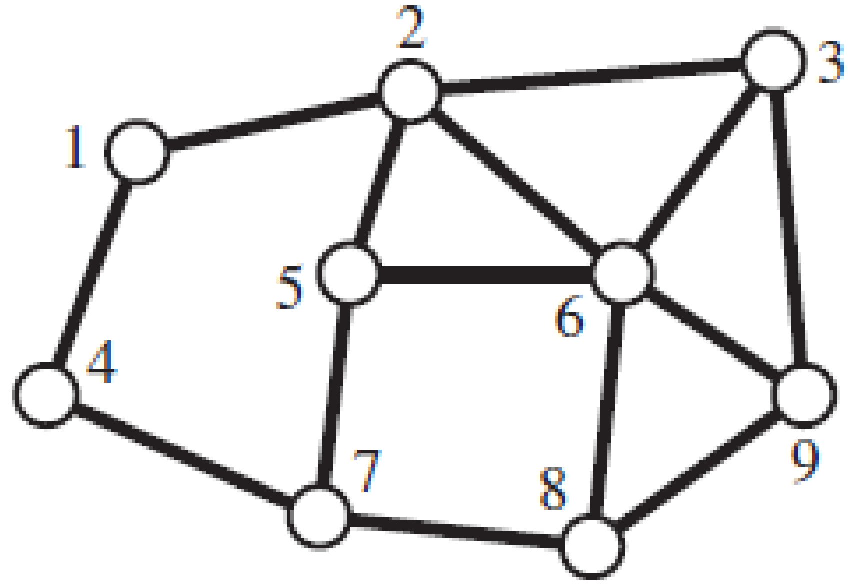
$V(G) = \{1, 2, 3, 5, 6, 7, 8\}$ and

$E(G) = \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{5, 6\}, \{5, 7\}, \{6, 8\}, \{7, 8\}\}$.

G



H



Fundamentals of Graph Theory

Cliques and Independent Sets

Definition 48.7

(Clique, clique number) Let G be a graph. A subset of vertices $S \subseteq V(G)$ is called a *clique* provided any two distinct vertices in S are adjacent.

The *clique number* of G is the size of a largest clique; it is denoted $\omega(G)$.

In other words, a set $S \subseteq V(G)$ is called a clique provided $G[S]$ is a complete graph.

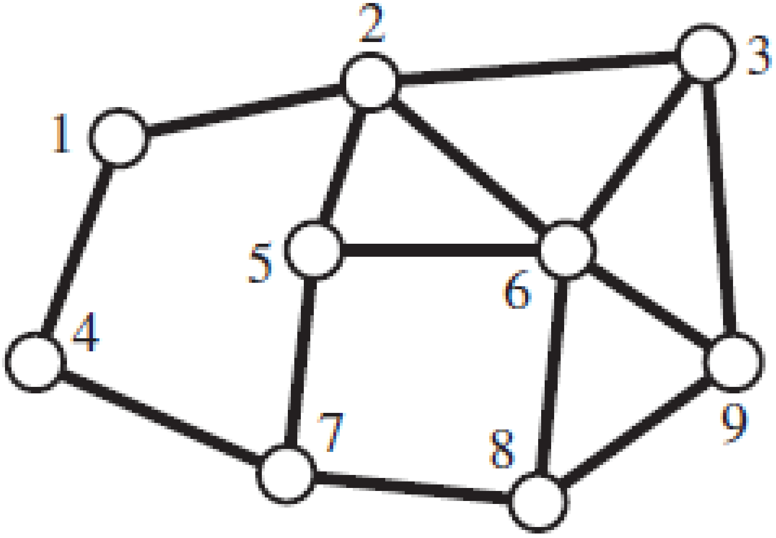
Example 48.8

Fundamentals of Graph Theory

Cliques and Independent Sets

Example 48.8

Let H be the graph from the earlier examples in this section, shown again here.



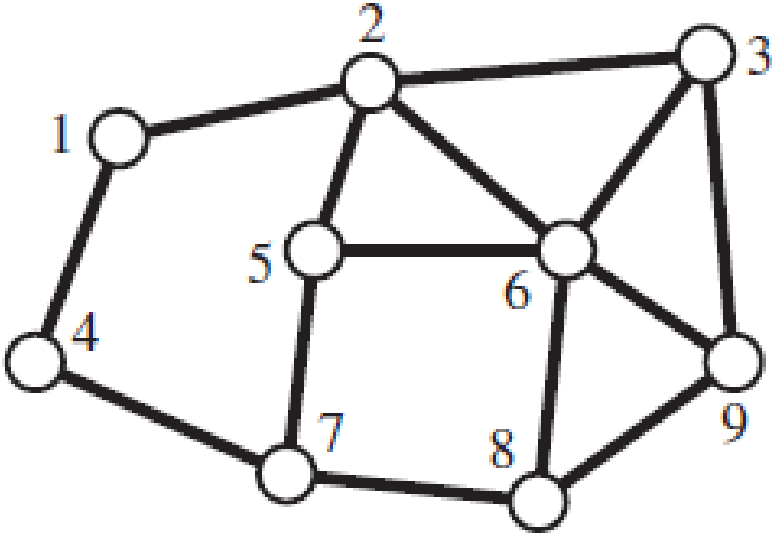
Fundamentals of Graph Theory

Cliques and Independent Sets

Example 48.8

Let H be the graph from the earlier examples in this section, shown again here.

maximal



maximum

$\{1,4\}$ $\{2,5,6\}$ $\{9\}$ $\{2,3,6\}$ $\{6,8,9\}$ $\{4\}$ $\emptyset$.

The largest size of a clique in H is 3, so $\omega(H) = 3$.

Fundamentals of Graph Theory

Cliques and Independent Sets

(Independent set, independence number) Let G be a graph. A subset of vertices $S \subseteq V(G)$ is called an *independent set* provided no two vertices in S are adjacent.

The *independence number* of G is the size of a largest independent set; it is denoted $\alpha(G)$.

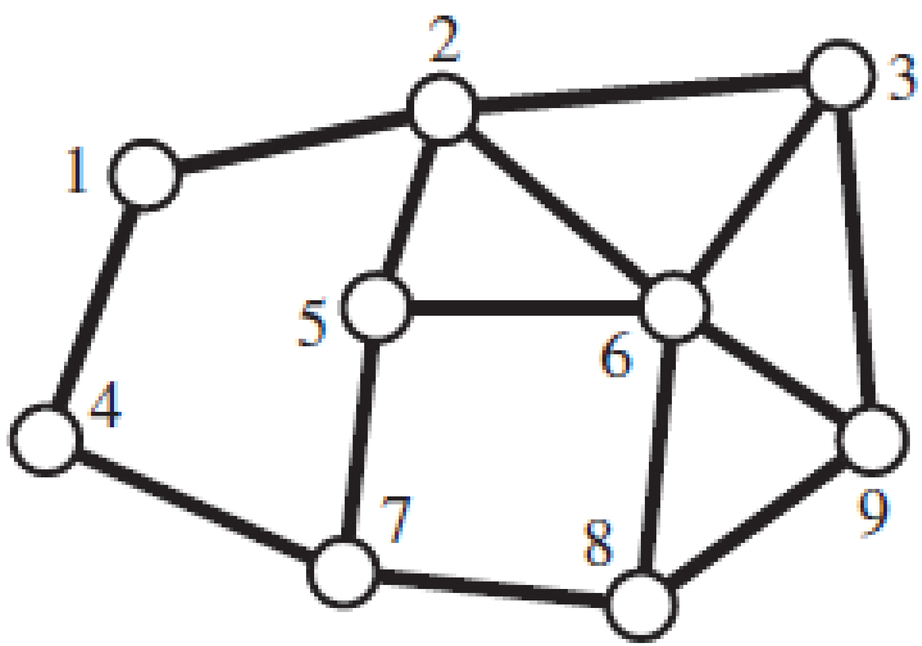
In other words, a set $S \subseteq V(G)$ is independent provided $G[S]$ is an edgeless graph.

Example 48.10

Fundamentals of Graph Theory

Cliques and Independent Sets

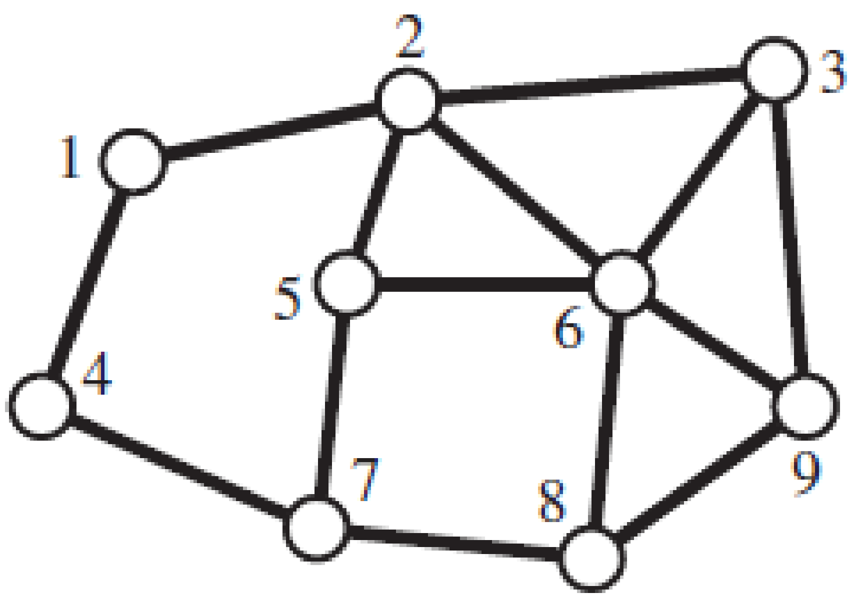
Example 48.10 Let H be the graph from the earlier examples in this section.



Fundamentals of Graph Theory

Cliques and Independent Sets

Example 48.10 Let H be the graph from the earlier examples in this section.



$$\{1,3,5\} \qquad \{1,7,9\} \qquad \{4\} \qquad \{1,3,5,8\} \qquad \{4,6\} \qquad \{1,3,7\} \qquad \emptyset.$$

The largest size of an independent set in H is 4, so $\alpha(H) = 4$.

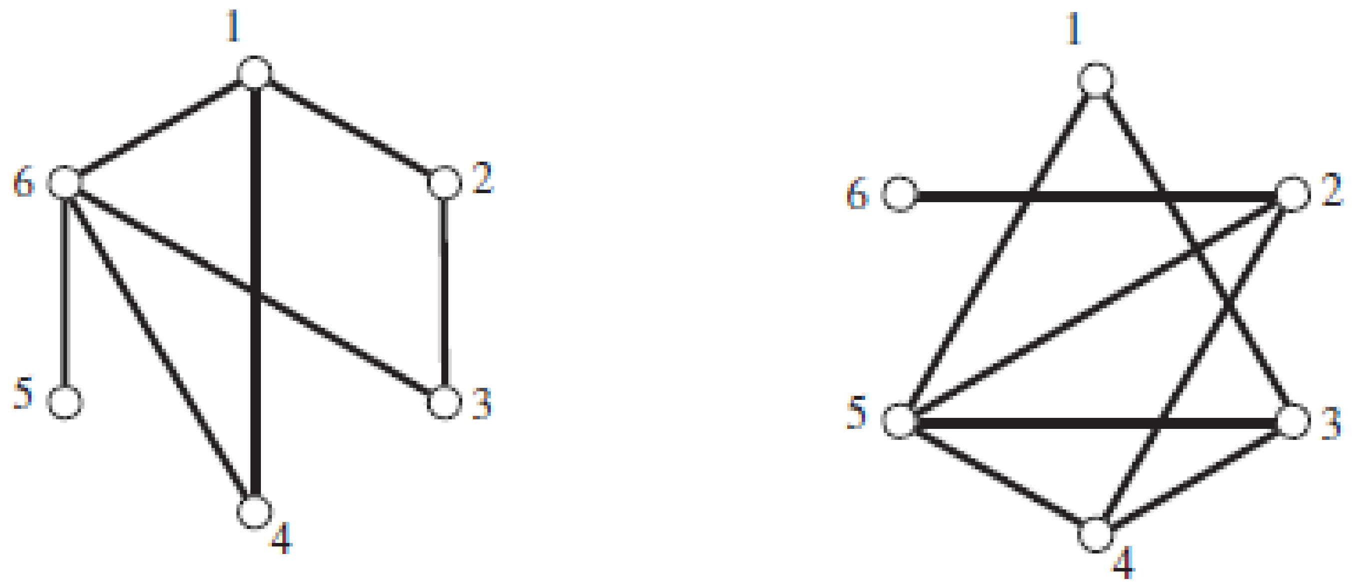
Fundamentals of Graph Theory

Complements

Definition 48.11

(Complement) Let G be a graph. The *complement* of G is the graph denoted $\overline{G}$ defined by

$$V(\overline{G}) = V(G) \quad \text{and}$$
$$E(\overline{G}) = \{xy : x, y \in V(G),\ x \neq y,\ xy \notin E(G)\}.$$



Fundamentals of Graph Theory

Complements

Proposition 48.12

Let G be a graph. A subset of $V(G)$ is a clique of G if and only if it is an independent set of $\overline{G}$. Furthermore,

$$\omega(G) = \alpha(\overline{G}) \qquad \text{and} \qquad \alpha(G) = \omega(\overline{G}).$$

Fundamentals of Graph Theory

Complements

Proposition 48.13

Let G be a graph with at least six vertices. Then $\omega(G) \geq 3$ or $\omega(\overline{G}) \geq 3$.