Matemática Discreta

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Fundamentals of Graph Theory

Theorem 49.12

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G - e$ has exactly two components.

Let P be an (a, b)-path in G. Because there is no (a, b)-path in $G - e$, we know P must traverse the edge e. Suppose x and y are the endpoints of the edge e, and without loss of generality, the path P traverses e in the order x, then y; that is,

$$P = a \sim \cdots \sim x \sim y \sim \cdots \sim b.$$

Similarly, since G is connected, there is a path Q from c to a that must use the edge $e = xy$. Which vertex, x or y, appears first on Q as we travel from c to a?

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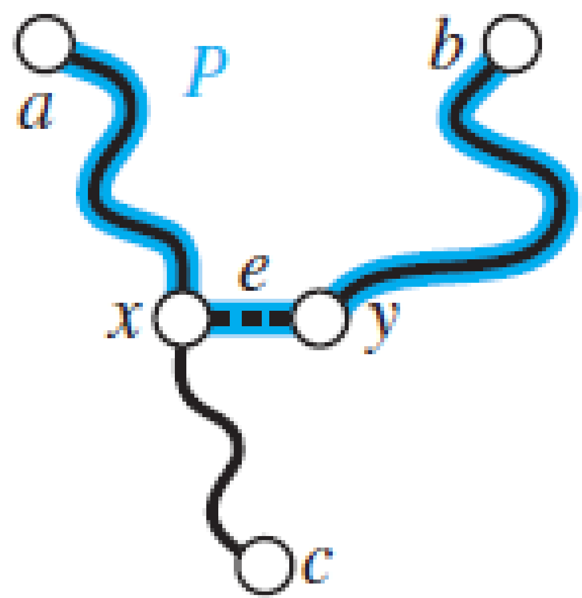
If x appears before y on the (c, a)-path Q, then notice that we have, in $G - e$, a walk from c to a. Use the (c, x)-portion of Q, concatenated with the (x, a)-portion of P^{-1}. This yields a (c, a)-walk in $G - e$ and hence a (c, a)-path in $G - e$ (by Lemma 49.7). This, however, is a contradiction, because a and c are in separate components of $G - e$.



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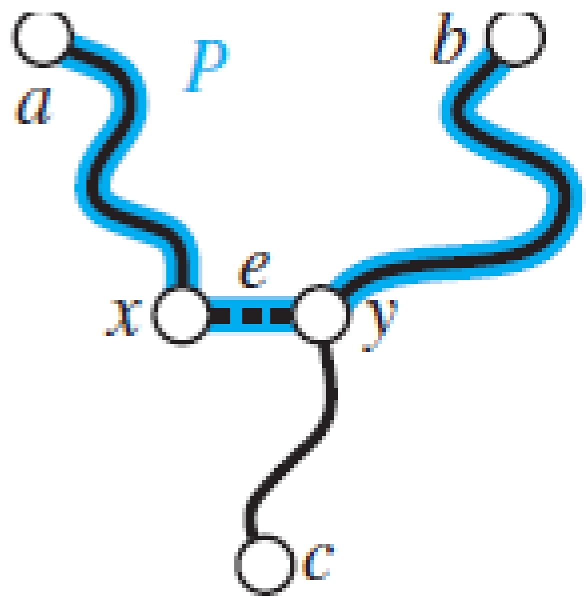
If y appears before x on the (c,a)-path Q, then notice that we have, in $G-e$, a walk from c to b. Concatenate that (c,y)-section of Q with the (y,b)-section of P. This walk does not use the edge e. Therefore there is a (c,a)-walk in $G-e$ and hence (Lemma 49.7) a (c,a)-walk in $G-e$. This contradicts the fact that in $G-e$ we have c and b in separate components.



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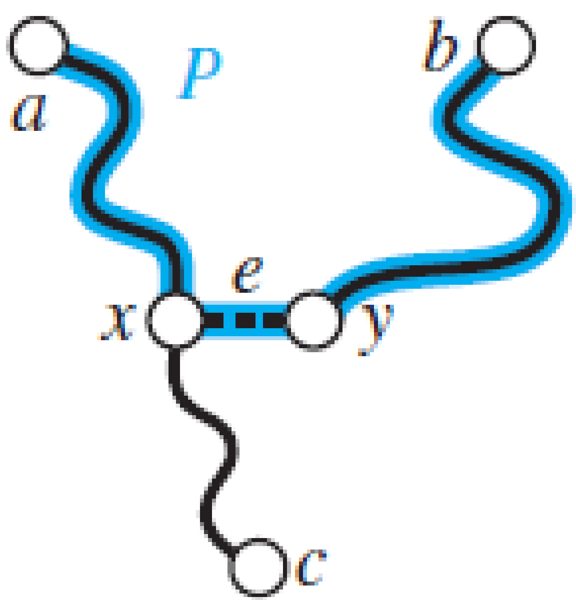
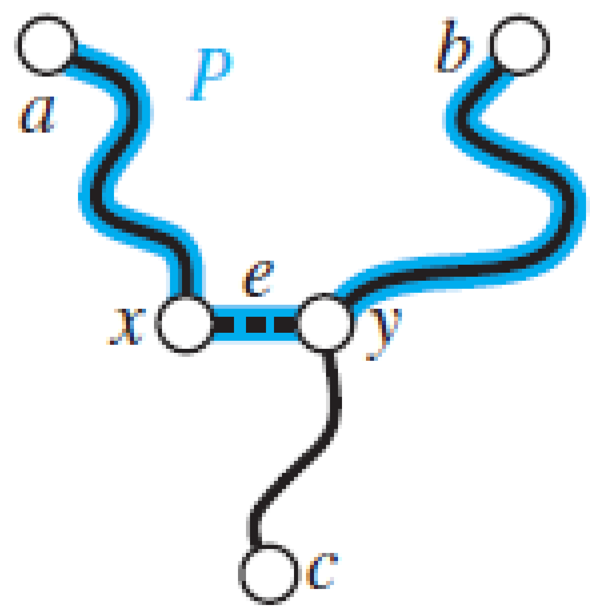
Therefore $G-e$ has at most two components.





Fundamentals of Graph Theory

Definition 50.1

(Cycle) A *cycle* is a walk of length at least three in which the first and last vertex are the same, but no other vertices are repeated.

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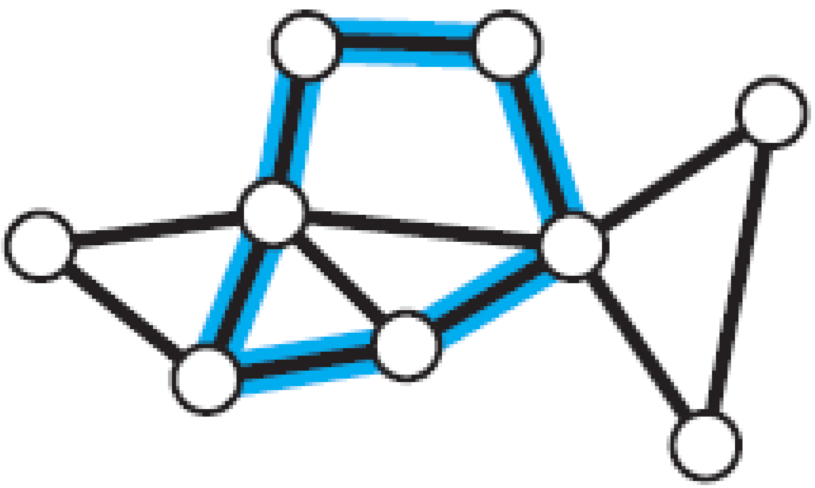
The term *cycle* also refers to a (sub)graph consisting of the vertices and edges of such a walk. In other words, a cycle is a graph of the form $G = (V, E)$ where

$$V = \{v_1, v_2, \ldots, v_n\} \quad \text{and}$$
$$E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}.$$



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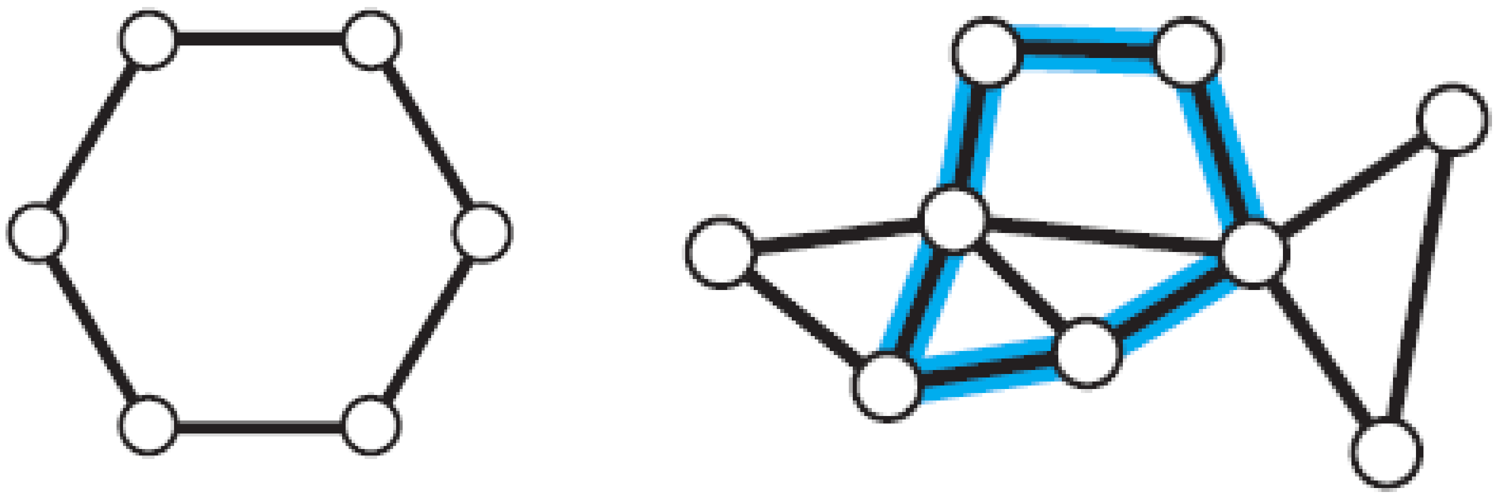


A cycle (graph) on n vertices is denoted C_n.

Fundamentals of Graph Theory

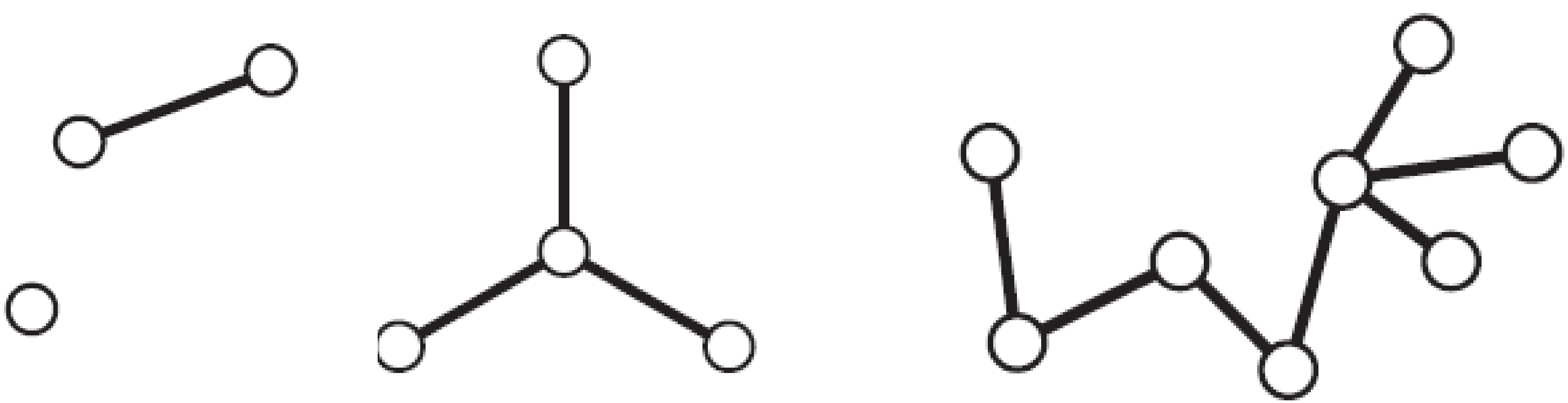
Forests and Trees

Definition 50.2

(Forest) Let G be a graph. If G contains no cycles, then we call G *acyclic*. Alternatively, we call G a *forest*.

Definition 50.3

(Tree) A *tree* is a connected, acyclic graph.



Fundamentals of Graph Theory

Properties of Trees

Theorem 50.4

Let T be a tree. For any two vertices a and b in $V(T)$, there is a unique (a, b)-path.

Conversely, if G is a graph with the property that for any two vertices u, v, there is exactly one (u, v)-path, then G must be a tree.

($\Rightarrow$) Suppose T is a tree and let $a, b \in V(T)$. We need to prove that there is a unique (a, b)-path in T. We have two things to prove:

- *Existence*: The path exists.
- *Uniqueness*: There can be only one such path.

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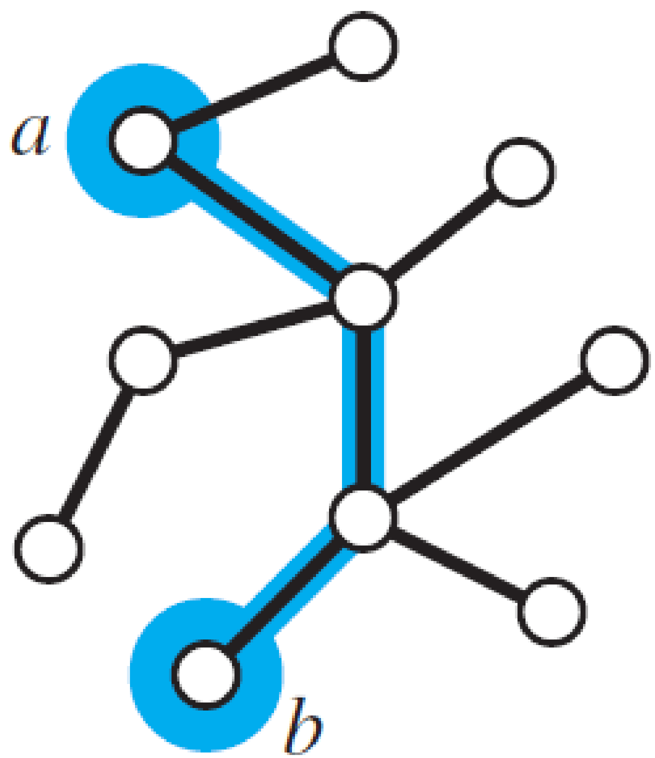
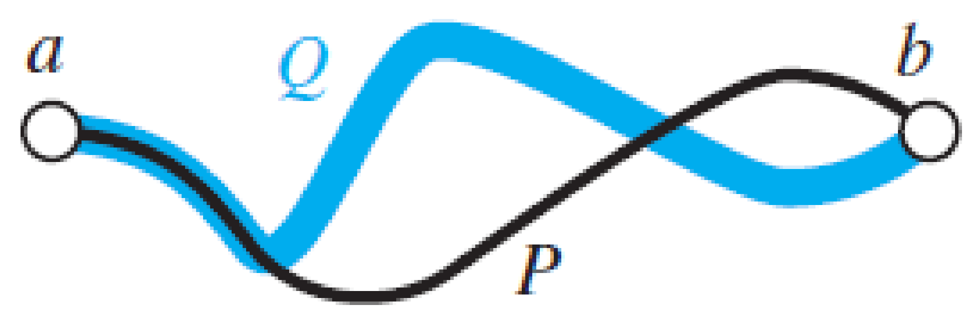


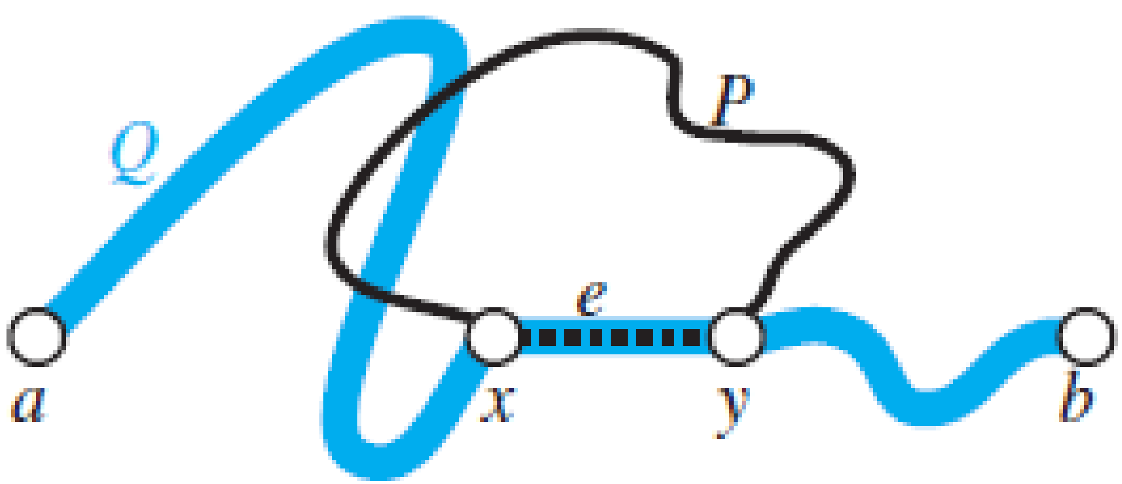


Fundamentals of Graph Theory

Properties of Trees

Theorem 50.5

Let G be a connected graph. Then G is a tree if and only if every edge of G is a cut edge.



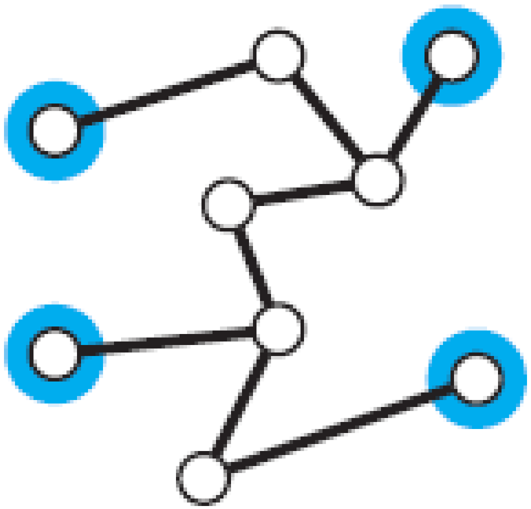
Fundamentals of Graph Theory

Properties of Trees

Leaves

Definition 50.6

(Leaf) A *leaf* of a graph is a vertex of degree 1.



Leaves are also called *end vertices* or *pendant vertices*. The tree in the figure has four leaves (marked).

Fundamentals of Graph Theory

Leaves

Theorem 50.7 Every tree with at least two vertices has a leaf.

Proposition 50.8 Let T be a tree and let v be a leaf of T. Then $T - v$ is a tree.

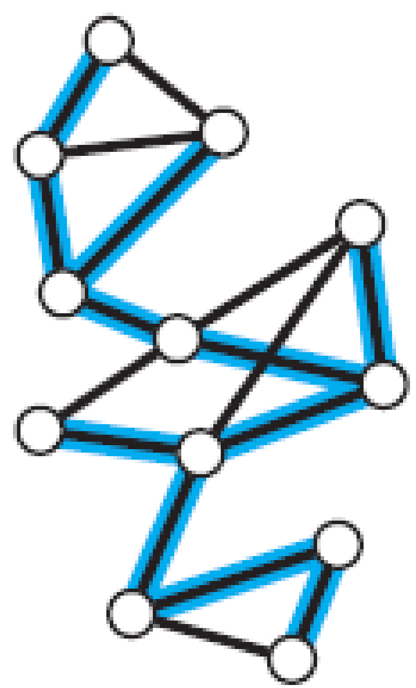
Theorem 50.9 Let T be a tree with $n \geq 1$ vertices. Then T has $n - 1$ edges.

Fundamentals of Graph Theory

Spanning Trees

Definition 50.10

(Spanning tree) Let G be a graph. A *spanning tree* of G is a spanning subgraph of G that is a tree.



Fundamentals of Graph Theory

Spanning Trees

Theorem 50.11

A graph has a spanning tree if and only if it is connected.

Proof. ($\Leftarrow$) Suppose G has a spanning tree T. We want to show that G is connected. Let $u, v \in V(G)$. Since T is spanning, we have $V(T) = V(G)$, and so $u, v \in V(T)$. Since T is connected, there is a (u, v)-path P in T. Since T is a subgraph of G, P is a (u, v)-path of G. Therefore G is connected.
($\Rightarrow$) Suppose G is connected. Let T be a spanning connected subgraph of G with the least number of edges.

We claim that T is a tree. By construction, T is connected. Furthermore, we claim that every edge of T is a cut edge. Otherwise, if $e \in E(T)$ were not a cut edge of T, then $T - e$ would be a smaller spanning connected subgraph of G.$\Rightarrow\Leftarrow$ Therefore every edge of T is a cut edge. Hence (Theorem 50.5) T is a tree, and so G has a spanning tree. ∎

Fundamentals of Graph Theory

Spanning Trees

Theorem 50.12

Let G be a connected graph on $n \geq 1$ vertices. Then G is a tree if and only if G has exactly $n - 1$ edges.

Proof. ($\Rightarrow$) This was shown in Theorem 50.9.

($\Leftarrow$) Suppose G is a connected graph with n vertices and $n - 1$ edges. By Theorem 50.11, we know that G has a spanning tree T; that is, T is a tree, $V(T) = V(G)$, and $E(T) \subseteq E(G)$. Note, however, that

$$|E(T)| = |V(T)| - 1 = |V(G)| - 1 = |E(G)|$$

so we actually have $E(T) = E(G)$. Therefore $G = T$ (i.e., G is a tree). ■