Exercises

1. Let X have the exponential distribution with parameter β. Suppose that we wish to test the hypotheses $H_0 : \beta \geq 1$ versus $H_1 : \beta < 1$. Consider the test procedure δ that rejects H_0 if $X \geq 1$.

a. Determine the power function of the test.

b. Compute the size of the test.

2. Suppose that $X_1, \ldots, X_n$ form a random sample from the uniform distribution on the interval $[0, \theta]$, and that the following hypotheses are to be tested:

$$H_0: \theta \geq 2, \qquad H_1: \theta < 2.$$

Let $Y_n = \max\{X_1, \ldots, X_n\}$, and consider a test procedure such that the critical region contains all the outcomes for which $Y_n \leq 1.5$.

a. Determine the power function of the test.

b. Determine the size of the test.

3. Suppose that the proportion p of defective items in a large population of items is unknown, and that it is desired to test the following hypotheses:

$$H_0: p = 0.2, \qquad H_1: p \neq 0.2.$$

Suppose also that a random sample of 20 items is drawn from the population. Let Y denote the number of defective items in the sample, and consider a test procedure δ such that the critical region contains all the outcomes for which either $Y \geq 7$ or $Y \leq 1$.

a. Determine the value of the power function $\pi(p|\delta)$ at the points $p = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$, and 1; sketch the power function.

b. Determine the size of the test.

4. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with unknown mean μ and known variance 1. Suppose also that μ_0 is a certain specified number, and that the following hypotheses are to be tested:

$$H_0: \mu = \mu_0, \qquad H_1: \mu \neq \mu_0.$$

Finally, suppose that the sample size n is 25, and consider a test procedure such that H_0 is to be rejected if $|\overline{X}_n - \mu_0| \geq c$. Determine the value of c such that the size of the test will be 0.05.

5. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with unknown mean μ and unknown variance σ^2. Classify each of the following hypotheses as either simple or composite:

a. H_0: $\mu = 0$ and $\sigma = 1$

b. H_0: $\mu > 3$ and $\sigma < 1$

c. H_0: $\mu = -2$ and $\sigma^2 < 5$

d. H_0: $\mu = 0$

6. Suppose that a single observation X is to be taken from the uniform distribution on the interval $\left[\theta - \frac{1}{2}, \theta + \frac{1}{2}\right]$, and suppose that the following hypotheses are to be tested:

$$H_0: \theta \leq 3, \qquad H_1: \theta \geq 4.$$

Construct a test procedure δ for which the power function has the following values: $\pi(\theta|\delta) = 0$ for $\theta \leq 3$ and $\pi(\theta|\delta) = 1$ for $\theta \geq 4$.

7. Return to the situation described in Example 9.1.7. Consider a different test δ^* that rejects H_0 if $Y_n \leq 2.9$ or $Y_n \geq 4.5$. Let δ be the test described in Example 9.1.7.

a. Prove that $\pi(\theta|\delta^*) = \pi(\theta|\delta)$ for all $\theta \leq 4$.

b. Prove that $\pi(\theta|\delta^*) < \pi(\theta|\delta)$ for all $\theta > 4$.

c. Which of the two tests seems better for testing the hypotheses (9.1.8)?

8. Assume that $X_1, \ldots, X_n$ are i.i.d. with the normal distribution that has mean μ and variance 1. Suppose that we wish to test the hypotheses

$$H_0: \mu \leq \mu_0, \qquad H_1: \mu > \mu_0.$$

Consider the test that rejects H_0 if $Z \geq c$, where Z is defined in Eq. (9.1.10).

a. Show that $\Pr(Z \geq c|\mu)$ is an increasing function of μ.

b. Find c to make the test have size α_0.

9. Assume that $X_1, \ldots, X_n$ are i.i.d. with the normal distribution that has mean μ and variance 1. Suppose that we wish to test the hypotheses

$$H_0: \mu \geq \mu_0, \qquad H_1: \mu < \mu_0.$$

Find a test statistic T such that, for every c, the test δ_c that rejects H_0 when $T \geq c$ has power function $\pi(\mu|\delta_c)$ that is decreasing in μ.

10. In Exercise 8, assume that $Z = z$ is observed. Find a formula for the p-value.

11. Assume that $X_1, \ldots, X_9$ are i.i.d. having the Bernoulli distribution with parameter p. Suppose that we wish to test the hypotheses

$$H_0: p = 0.4, \qquad H_1: p \neq 0.4.$$

Let $Y = \sum_{i=1}^{9} X_i$.

a. Find c_1 and c_2 such that

$$\Pr(Y \le c_1 | p = 0.4) + \Pr(Y \ge c_2 | p = 0.4)$$

is as close as possible to 0.1 without being larger than 0.1.

b. Let δ be the test that rejects H_0 if either $Y \le c_1$ or $Y \ge c_2$. What is the size of the test δ_c?

c. Draw a graph of the power function of δ_c.

12. Consider a single observation X from a Cauchy distribution centered at θ. That is, the p.d.f. of X is

$$f(x|\theta) = \frac{1}{\pi[1 + (x - \theta)^2]}, \quad \text{for } -\infty < x < \infty.$$

Suppose that we wish to test the hypotheses

$$H_0: \ \theta \le \theta_0,$$
$$H_1: \ \theta > \theta_0.$$

Let δ_c be the test that rejects H_0 if $X \ge c$.

a. Show that $\pi(\theta|\delta_c)$ is an increasing function of θ.

b. Find c to make δ_c have size 0.05.

c. If $X = x$ is observed, find a formula for the p-value.

13. Let X have the Poisson distribution with mean θ. Suppose that we wish to test the hypotheses

$$H_0: \ \theta \le 1.0,$$
$$H_1: \ \theta > 1.0.$$

Let δ_c be the test that rejects H_0 if $X \ge c$. Find c to make the size of δ_c as close as possible to 0.1 without being larger than 0.1.

14. Let $X_1, \ldots, X_n$ be i.i.d. with the exponential distribution with parameter θ. Suppose that we wish to test the hypotheses

$$H_0: \ \theta \ge \theta_0,$$
$$H_1: \ \theta < \theta_0.$$

Let $X = \sum_{i=1}^{n} X_i$. Let δ_c be the test that rejects H_0 if $X \ge c$.

a. Show that $\pi(\theta|\delta_c)$ is a decreasing function of θ.

b. Find c in order to make δ_c have size α_0.

c. Let $\theta_0 = 2$, $n = 1$, and $\alpha_0 = 0.1$. Find the precise form of the test δ_c and sketch its power function.

15. Let X have the uniform distribution on the interval $[0, \theta]$, and suppose that we wish to test the hypotheses

$$H_0: \ \theta \le 1,$$
$$H_1: \ \theta > 1.$$

We shall consider test procedures of the form "reject H_0 if $X \ge c$." For each possible value x of X, find the p-value if $X = x$ is observed.

16. Consider the confidence interval found in Exercise 5 in Sec. 8.5. Find the collection of hypothesis tests that are equivalent to this interval. That is, for each $c > 0$, find a test δ_c of the null hypothesis $H_{0,c} : \sigma^2 = c$ versus some alternative such that δ_c rejects $H_{0,c}$ if and only if c is not in the interval. Write the test in terms of a test statistic $T = r(\boldsymbol{X})$ being in or out of some nonrandom interval that depends on c.

17. Let $X_1, \ldots, X_n$ be i.i.d. with a Bernoulli distribution that has parameter p. Let $Y = \sum_{i=1}^{n} X_i$. We wish to find a coefficient γ confidence interval for p of the form $(q(y), 1)$. Prove that, if $Y = y$ is observed, then $q(y)$ should be chosen to be the smallest value p_0 such that $\Pr(Y \ge y | p = p_0) \ge 1 - \gamma$.

18. Consider the situation described immediately before Eq. (9.1.12). Prove that the expression (9.1.12) equals the smallest α_0 such that we would reject H_0 at level of significance α_0.

19. Return to the situation described in Example 9.1.17. Suppose that we wish to test the hypotheses

$$H_0: \ \mu \ge \mu_0,$$
$$H_1: \ \mu < \mu_0 \tag{9.1.27}$$

at level α_0. It makes sense to reject H_0 if $\overline{X}_n$ is small. Construct a one-sided coefficient $1 - \alpha_0$ confidence interval for μ such that we can reject H_0 if μ_0 is not in the interval. Make sure that the test formed in this way rejects H_0 if $\overline{X}_n$ is small.

20. Prove Theorem 9.1.3.

21. Return to the situations described in Example 9.1.17 and Exercise 19. We wish to compare what might happen if we switch the null and alternative hypotheses. That is, we want to compare the results of testing the hypotheses in (9.1.22) at level α_0 to the results of testing the hypotheses in (9.1.27) at level α_0.

a. Let $\alpha_0 < 0.5$. Prove that there are no possible data sets such that we would reject both of the null hypotheses simultaneously. That is, for every possible $\overline{X}_n$ and σ', we must fail to reject at least one of the two null hypotheses.

b. Let $\alpha_0 < 0.5$. Prove that there are data sets that would lead to failing to reject both null hypotheses. Also prove that there are data sets that would lead to rejecting each of the null hypotheses while failing to reject the other.

c. Let $\alpha_0 > 0.5$. Prove that there are data sets that would lead to rejecting both null hypotheses.