Matemática Discreta

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Planar Graphs

Embedding

An embedding of a graph is a collection of points and curves in a plane that satisfies the following conditions:

- Each vertex of the graph is assigned a point in the plane; distinct vertices receive distinct points (i.e., no two vertices share the same point).
- Each edge of the graph is assigned a curve in the plane. If the edge is $e = xy$, then the endpoints of the curve for e are exactly the points assigned to x and y. Furthermore, no other vertex point is on this curve.

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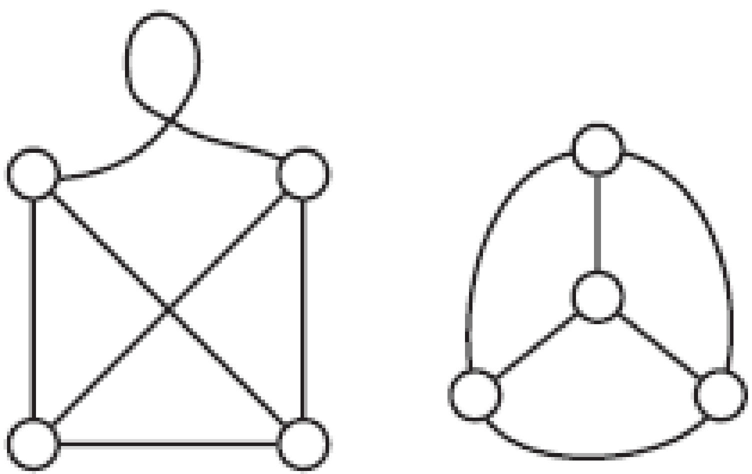
If all the curves are simple (do not cross themselves) and if the curves from two edges do not intersect (except at an endpoint if they both are incident with the same vertex), then we call the embedding *crossing-free*.

Planar Graphs



The figure shows two embeddings of the graph K_4. Note that we greatly exaggerated the points, drawing them as large round dots. The drawing on the right represents a crossing-free embedding on K_4.

Planar Graphs

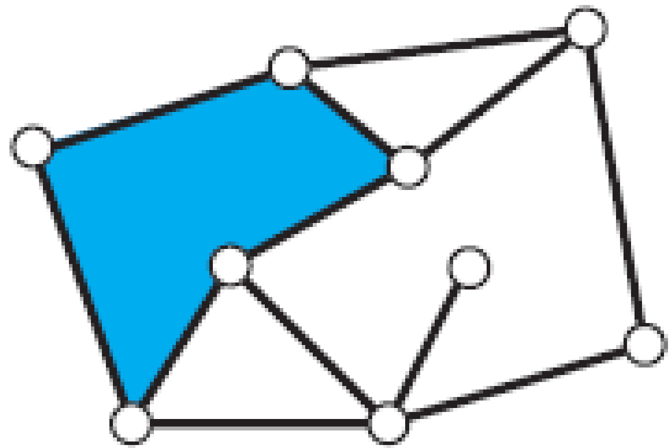
Definition 53.2

(**Planar graph**) A *planar graph* is a graph that has a crossing-free embedding in the plane.

Euler's Formula

Let G be a planar graph and consider a crossing-free embedding of G, as in the figure. In this drawing, we see the points and curves of the embedding. We also see another feature: *faces*. A *face* is a portion of the plane cut off by the drawing. Imagine the graph drawn on a physical piece of paper. If we cut along the curves representing the edges of G, the paper falls apart into various pieces. Each of these pieces is called a *face* (or *region*) of the embedding.

Planar Graphs



The graph in this figure has $n = 9$ vertices, $m = 12$ edges, and $f = 5$ faces.

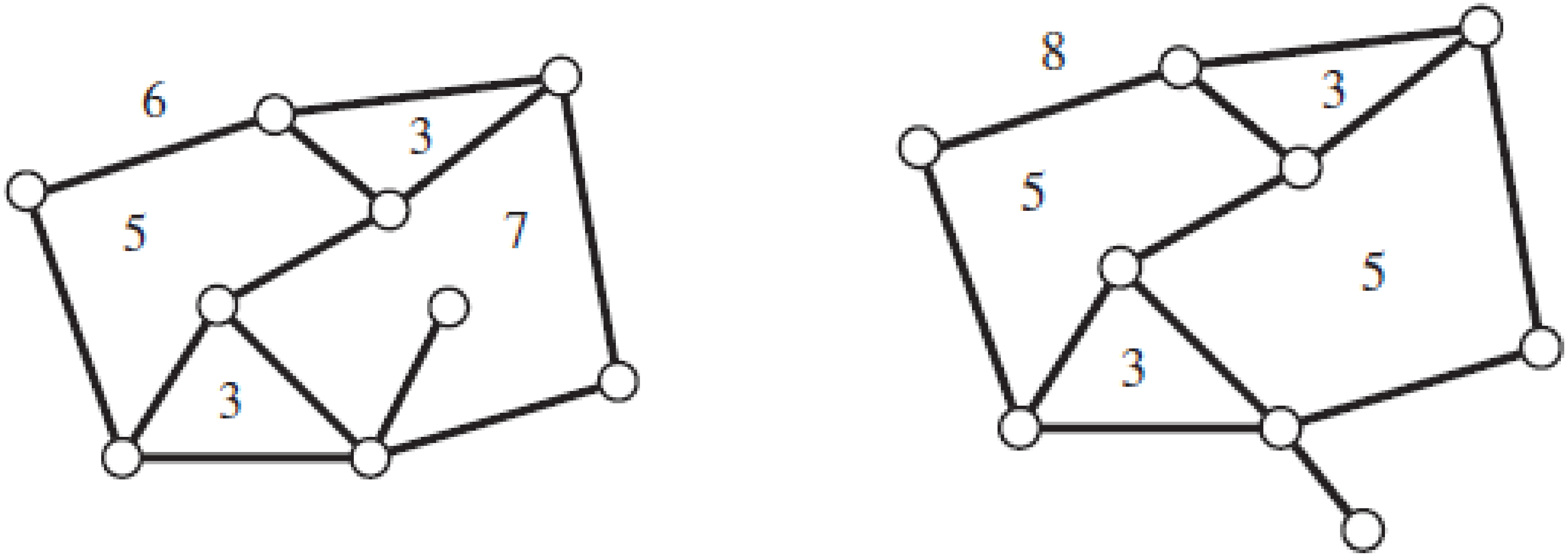
Planar Graphs

Theorem 53.3

(Euler's formula) Let G be a connected planar graph with n vertices and m edges. Choose a crossing-free embedding for G, and let f be the number of faces in the embedding. Then

$$n - m + f = 2.$$

Planar Graphs



For example, consider the two drawings of the graph in the figure. In both cases, the graph has $f = 2 - n + m = 2 - 9 + 12 = 5$ faces.

Planar Graphs

Proposition 53.4

Let G be a planar graph. The sum of the degrees of the faces in a crossing-free embedding of G in the plane equals $2|E(G)|$.

Corollary 53.5

Let G be a planar graph with at least two edges. Then

$$|E(G)| \leq 3|V(G)| - 6.$$

Furthermore, if G does not contain K_3 as a subgraph, then

$$|E(G)| \leq 2|V(G)| - 4.$$

Planar Graphs

Corollary 53.6

Let G be a planar graph with minimum degree δ. Then $\delta \leq 5$.

Nonplanar Graphs

Proposition 53.7 The graph K_5 is nonplanar.

Proposition 53.8 The graph $K_{3,3}$ is nonplanar.

Theorem 53.9

(**Kuratowski**) A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

Coloring Planar Graphs

Theorem 53.10

(Four Color) If G is a planar graph, then $\chi(G) \leq 4$.

Proposition 53.11 **(Six color)** If G is a planar graph, then $\chi(G) \leq 6$.

Theorem 53.12 **(Five color)** If G is a planar graph, then $\chi(G) \leq 5$.