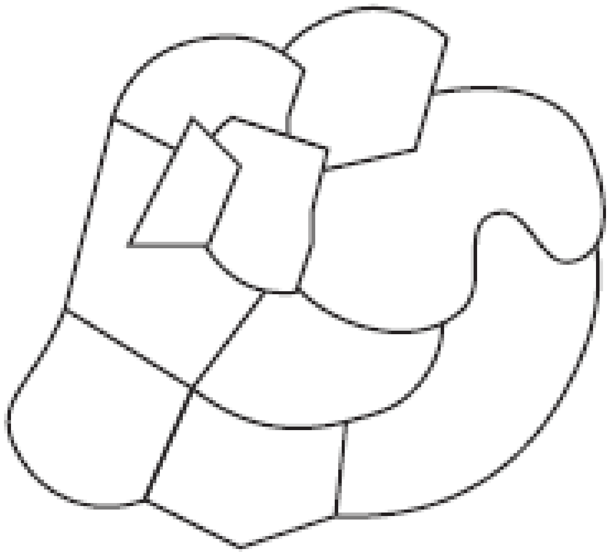
Matemática Discreta

Escola de Artes, Ciências e Humanidades - USP
Profa Dra. Karla Lima
email:ksampaiolima@usp.br

Fundamentals of Graph Theory

Map Coloring



Fundamentals of Graph Theory

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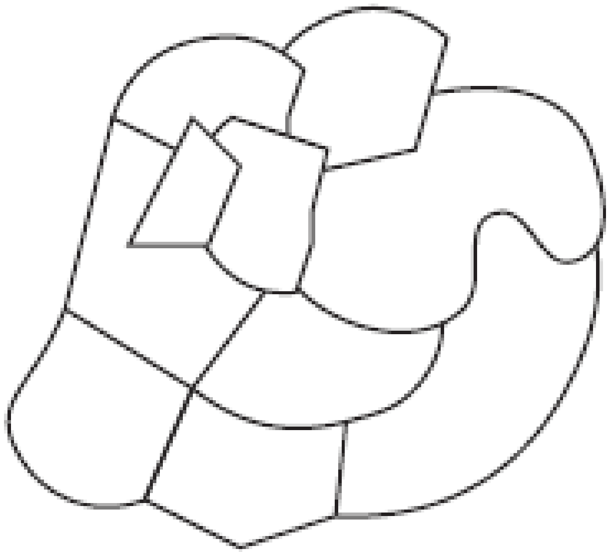

The question is: What is the smallest number of colors you need to color your map?

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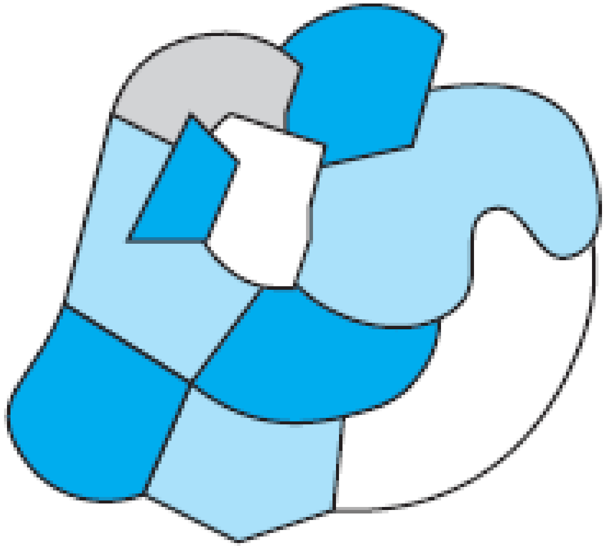
Map Coloring



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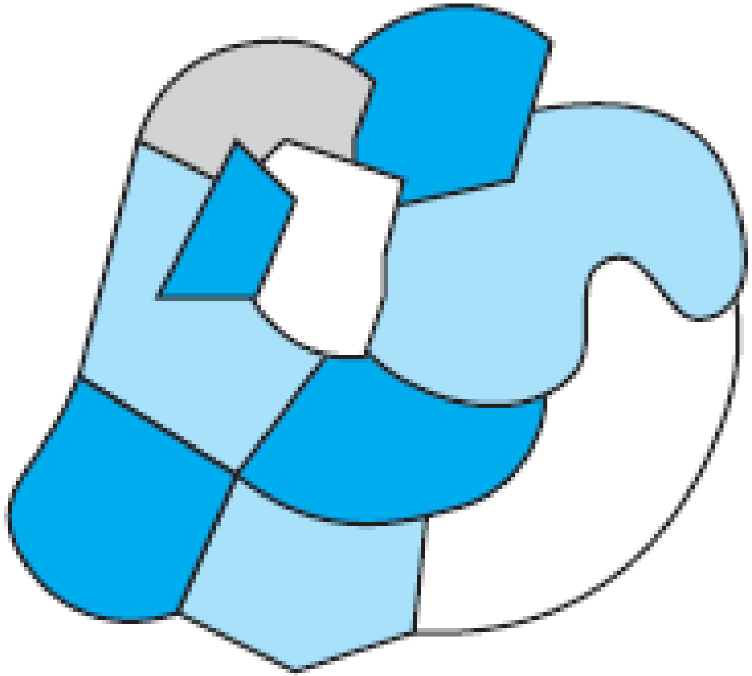

The question is: What is the smallest number of colors you need to color your map?

We can color the map in the figure with just four colors, as shown

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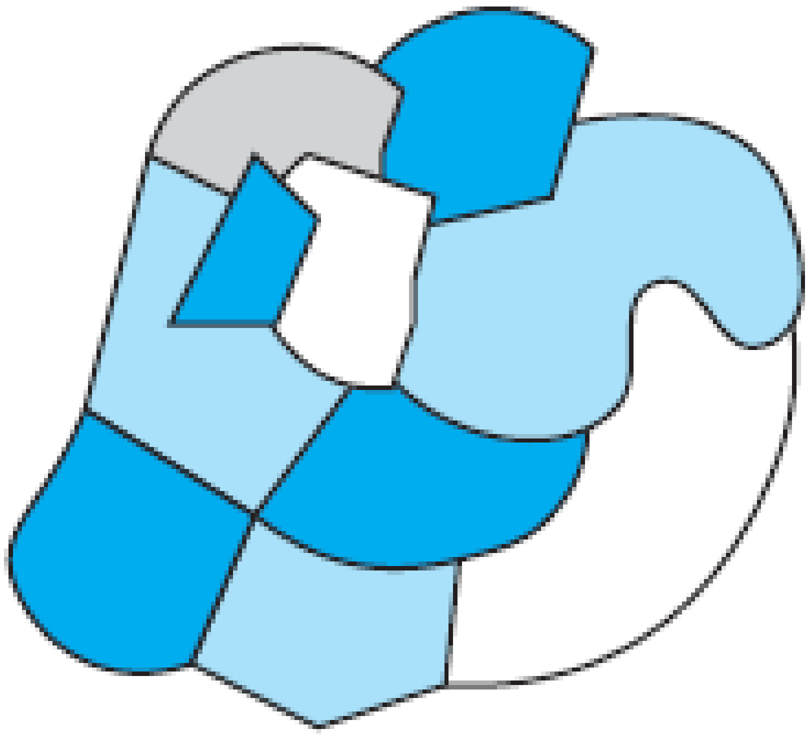
Map Coloring



- Can this map be colored with fewer than four colors? (Notice that we have only one country that is gray; perhaps if we are clever, we can color this map with only three colors.)
- Is there another map that can be colored with fewer than four colors?
- Is there a map that requires more than four colors?

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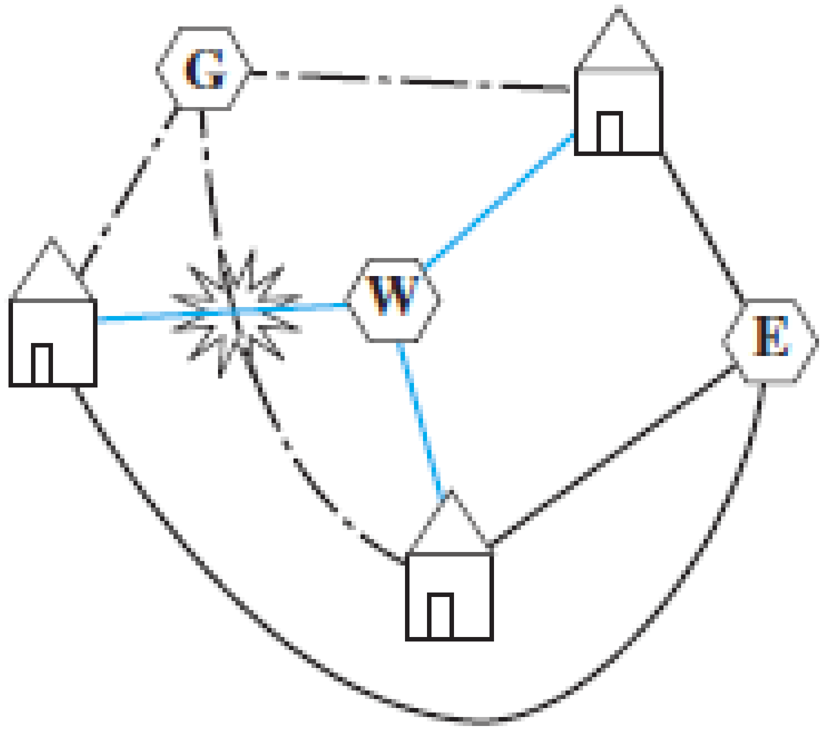
Three Utilities



The following is a classic puzzle. Imagine a "city" containing three houses and three utility plants. The three utilities supply gas, water, and electricity. As an urban planner, your job is to run connections from every utility plant to every home. You need to have three electric wires (from the electric plant to each of the three houses), three water pipes (from the water plant to the houses), and three gas lines (from the gas facility to the houses). You may place the houses and the utility plants anywhere you desire. However, you may not allow two wires/pipes/lines to cross! The diagram shows a failed attempt to construct a suitable layout.

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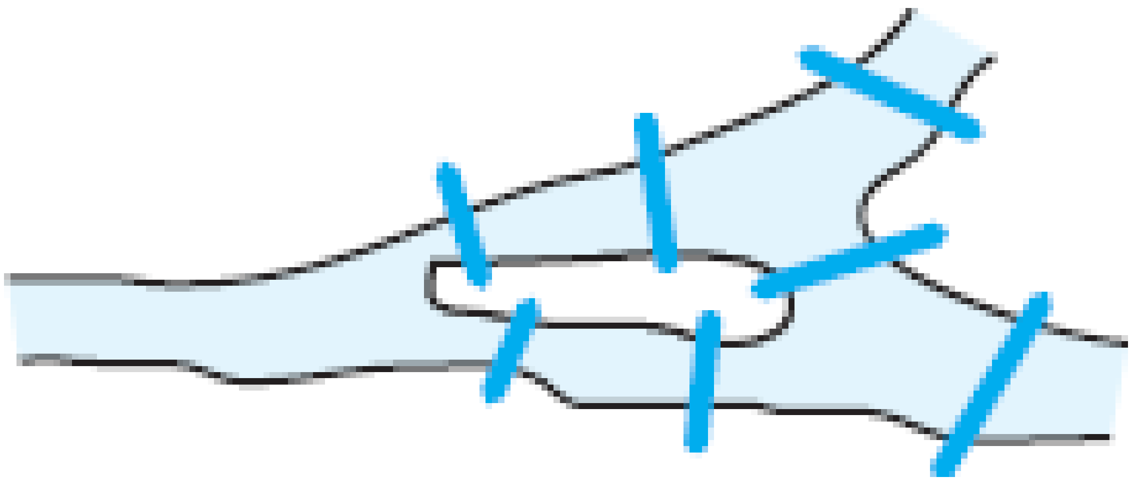
Seven Bridges



The following is another classic puzzle. In the late 1700s, in the city of Königsburg (now called Kaliningrad) located in the aforementioned disconnected section of Russia, there were seven bridges connecting various parts of the city; these were configured as shown in the figure.

The townspeople enjoyed strolling through their city in the evening. They wondered: Is there a tour we can take through our city so that we cross every bridge exactly once?

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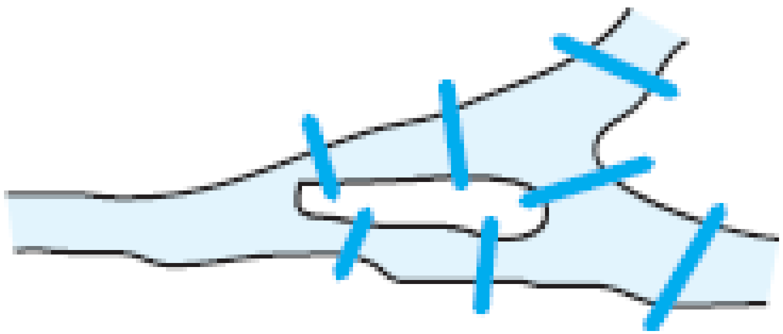
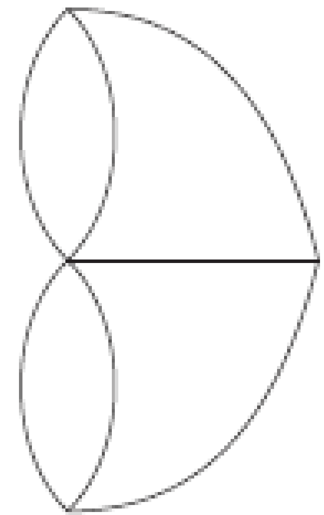
Seven Bridges





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Fundamentals of Graph Theory

What Is a Graph?

Definition 47.1

(Graph) A *graph* is a pair $G = (V, E)$ where V is a nonempty finite set and E is a set of two-element subsets of V.

Example 47.2

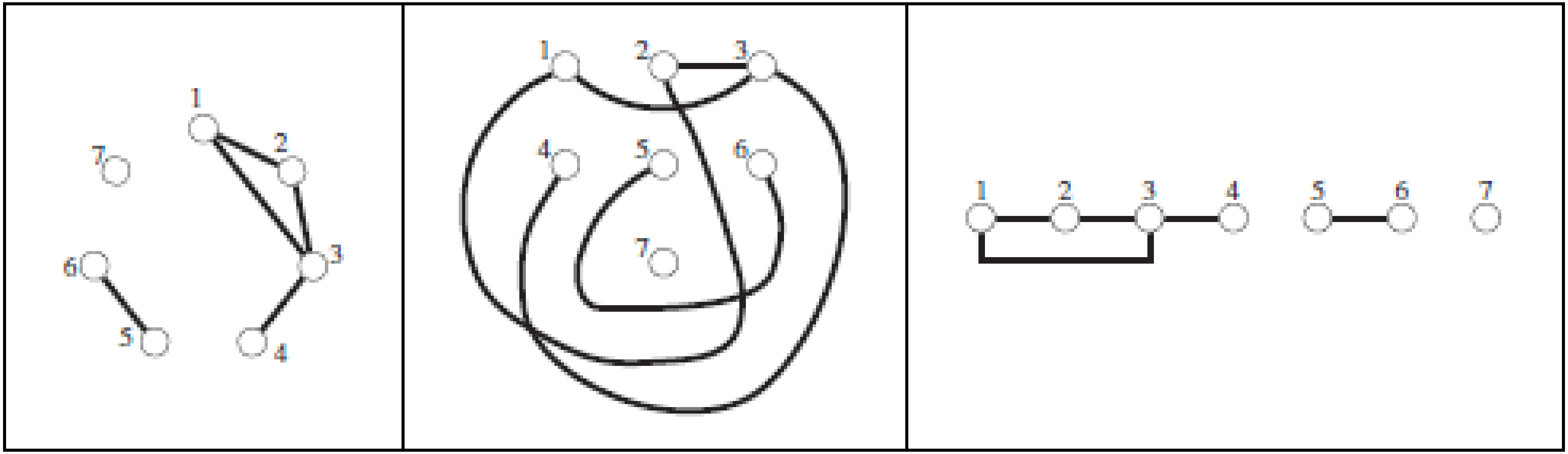
Let

$$G = \Big(\{1,2,3,4,5,6,7\}, \{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{5,6\}\}\Big).$$

Here V is the finite set $\{1,2,3,4,5,6,7\}$ and E is a set containing 5 two-element subsets of V: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{3,4\}$, and $\{5,6\}$. Therefore $G = (V, E)$ is a graph.

The elements of V are called the *vertices* (singular: *vertex*) of the graph, and the elements of E are called the *edges* of the graph.

Fundamentals of Graph Theory



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Fundamentals of Graph Theory

Adjacency

Definition 47.3

(Adjacent) Let $G = (V, E)$ be a graph and let $u, v \in V$. We say that u is *adjacent* to v provided $\{u, v\} \in E$. The notation $u \sim v$ means that u is adjacent to v.

If $\{u, v\}$ is an edge of G, we call u and v the *endpoints* of the edge.

Suppose v is a vertex and an endpoint of the edge e. We can express this fact as $v \in e$ since e is a two-element set, one of whose elements is v. We also say that v is *incident on* (or *incident with*) e.

Fundamentals of Graph Theory

A Matter of Degree

Let $G = (V, E)$ be a graph and suppose u and v are vertices of G. If u and v are adjacent, we also say that u and v are *neighbors*. The set of all neighbors of a vertex v is called the *neighborhood* of v and is denoted $N(v)$. That is,

$$N(v) = \{u \in V : u \sim v\}.$$

Example 47.2.

Definition 47.4

(**Degree**) Let $G = (V, E)$ be a graph and let $v \in V$. The degree of v is the number of edges with which v is incident. The degree of v is denoted $d_G(v)$ or, if there is no risk of confusion, simply $d(v)$.

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Definition 47.4

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In other words,

$$d(v) = |N(v)|.$$

Example 47.2.

$$\begin{aligned}\sum_{v\in V} d(v) &= d(1) + d(2) + d(3) + d(4) + d(5) + d(6) + d(7) \\ &= 2 + 2 + 3 + 1 + 1 + 1 + 0 = 10\end{aligned}$$

Fundamentals of Graph Theory

Theorem 47.5

Let $G = (V, E)$. The sum of the degrees of the vertices in G is twice the number of edges; that is,

$$\sum_{v \in V} d(v) = 2|E|.$$

Fundamentals of Graph Theory

Theorem 47.5

Proof. Suppose the vertex set is $V = \{v_1, v_2, \ldots, v_n\}$. We can create an $n \times n$ matrix as follows. The entry in row i and column j of this matrix is 1 if $v_i \sim v_j$ and is 0 otherwise.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fundamentals of Graph Theory

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We ask,

How many 1s are in this matrix?

Fundamentals of Graph Theory

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First answer: Notice that for every edge of G there are exactly two 1s in the matrix. For example, if $v_i v_j \in E$, then there is a 1 in position ij (row i, column j) and a 1 in position ji. Thus the number of 1s in this matrix is exactly $2|E|$.

Second answer: Consider a given row of this matrix—say, the row corresponding to some vertex v_i. There is a 1 in this row exactly for those vertices adjacent to v_i (i.e., there is a 1 in the j^{th} spot of this row exactly when there is an edge from v_i to v_j). Thus, the number of 1s in this row is exactly the degree of the vertex—that is, $d(v_i)$.

Fundamentals of Graph Theory

Further Notation and Vocabulary

- *Maximum and minimum degree.*
  The maximum degree of a vertex in G is denoted $\Delta(G)$. The minimum degree of a vertex in G is denoted $\delta(G)$. The letters Δ and δ are upper- and lowercase Greek deltas, respectively. For the graph in Example 47.2, we have $\Delta(G) = 3$ and $\delta(G) = 0$.

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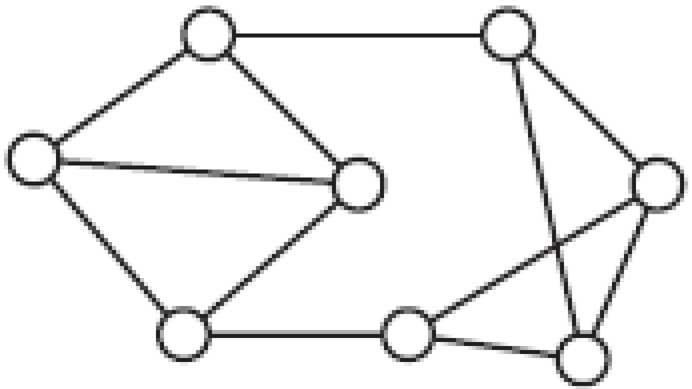
- *Regular graphs.*
  If all vertices in G have the same degree, we call G *regular*. If a graph is regular and all vertices have degree r, we also call the graph r-regular. The graph in the figure is 3-regular.



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- *Regular graphs.*

  If all vertices in G have the same degree, we call G *regular*. If a graph is regular and all vertices have degree r, we also call the graph r-regular. The graph in the figure is 3-regular.

- *Vertex and edge sets.*

  Let G be a graph. If we neglect to give a name to the vertex and edge sets of G, we can simply write $V(G)$ and $E(G)$ for the vertex and edge sets, respectively.

Fundamentals of Graph Theory

Further Notation and Vocabulary

- *Order and size.*

  Let $G = (V, E)$ be a graph. The *order* of G is the number of vertices in G—that is, $|V|$. The *size* of G is the number of edges—that is, $|E(G)|$.

  It is customary (but certainly not mandatory) to use the letters n and m to stand for $|V|$ and $|E|$, respectively.

  Various authors invent special symbols to stand for the number of vertices and the number of edges in a graph. Personally, I like the following:

$$\nu(G) = |V(G)| \qquad \text{and} \qquad \varepsilon(G) = |E(G)|.$$

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- *Complete graphs.*

  Let G be a graph. If all pairs of distinct vertices are adjacent in G, we call G *complete*. A complete graph on n vertices is denoted K_n. The graph in the figure is a K_5.

  The opposite extreme is a graph with no edges. We call such graphs *edgeless*.

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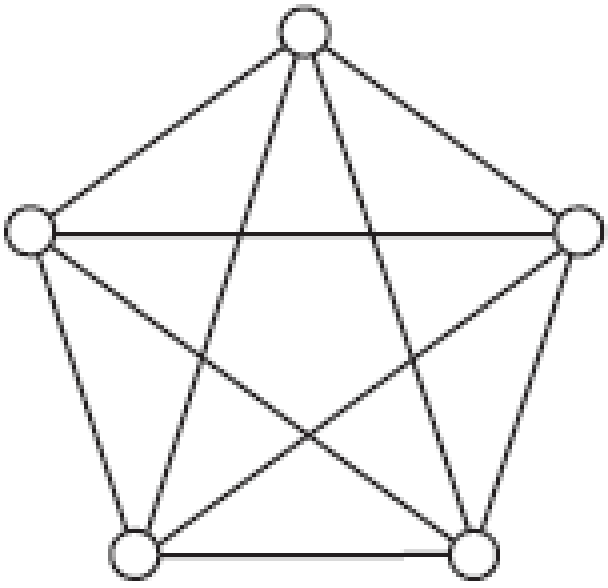
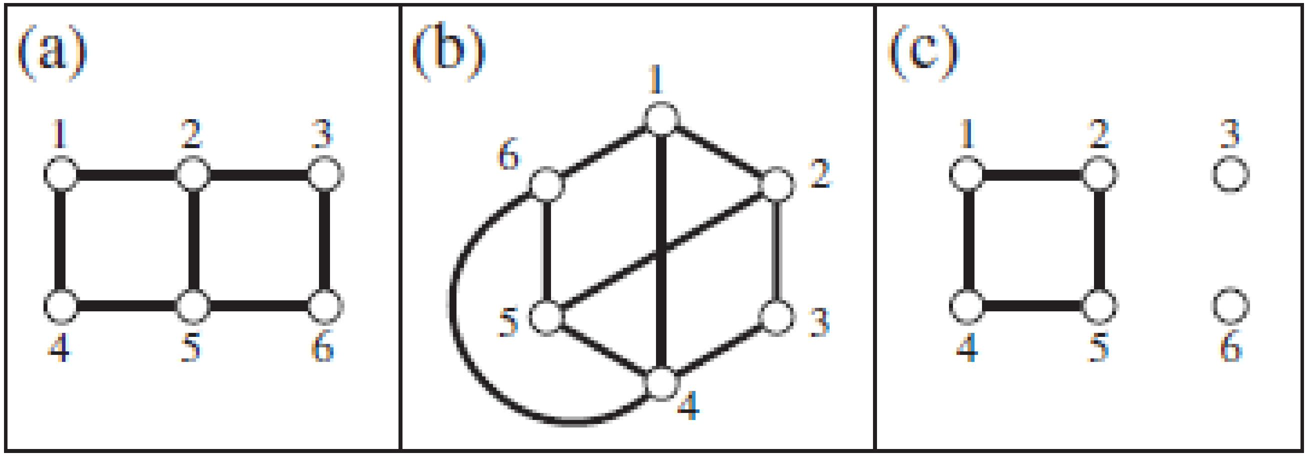
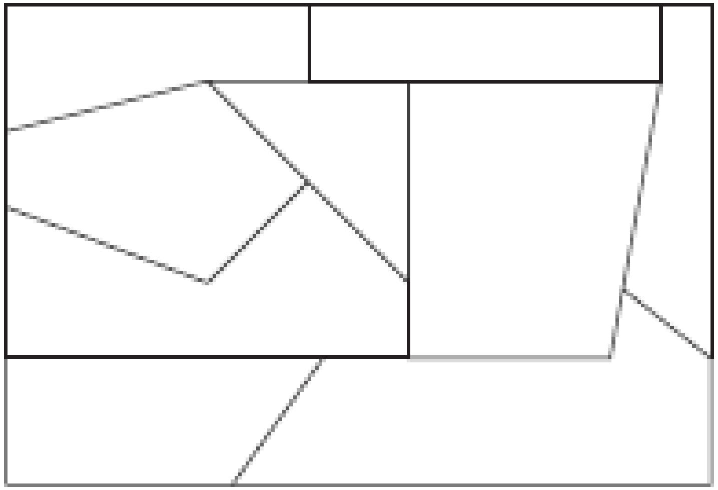


Fundamentals of Graph Theory

47.1. The following pictures represent graphs. Please write each of these graphs as a pair of sets (V, E).



47.3. Color the map in the figure with four colors (so that adjacent countries have different colors) and explain why it is not possible to color this map with only three colors.



Fundamentals of Graph Theory

47.16. Prove that in any graph with two or more vertices, there must be two vertices of the same degree.

47.18. Find all 3-regular graphs on nine vertices.

47.19. How many edges are in K_n, a complete graph on n vertices?

47.20. How many different graphs can be formed with vertex set $V = \{1, 2, 3, \ldots, n\}$?