Calcular o valor, em função de x, das seguintes integrais aplicando o método de integração por substituição trigonométrica:

1) $\quad I = \displaystyle\int \frac{1}{x^3\sqrt{x^2-9}}\, dx$;

Solução

intervalos: $\quad \pi \le \theta$ e $\theta < \dfrac{3\,\pi}{2}$, se x < -3 ; $\qquad 0 \le \theta < \dfrac{\pi}{2}$, se x > 3

considerando $x = 3\sec(\theta) \quad \Rightarrow \quad dx = 3\sec(\theta)\,\mathrm{tg}(\theta)\, d\theta$

temos que: $\quad x^3 = (3\sec(\theta))^3 = 27\sec(\theta)^3 \qquad$ e

$$\sqrt{x^2-9} = \sqrt{(3\sec(\theta))^2-9} = \sqrt{9\sec(\theta)^2-9} = 3\sqrt{\sec(\theta)^2-1} = 3\sqrt{\mathrm{tg}(\theta)^2} = 3\,\mathrm{tg}(\theta)$$

substituindo em I , temos:

$$I = \int \frac{3\sec(\theta)\,\mathrm{tg}(\theta)}{27\sec(\theta)^3\, 3\,\mathrm{tg}(\theta)}\, d\theta = \frac{1}{27}\int \frac{1}{\sec(\theta)^2}\, d\theta = \frac{1}{27}\int \cos(\theta)^2\, d\theta = \frac{1}{27}\int \frac{\cos(2\,\theta)+1}{2}\, d\theta =$$

$$= \frac{1}{54}\int \cos(2\,\theta)\, d\theta + \frac{1}{54}\int 1\, d\theta = \frac{1}{54}\left(\frac{1}{2}\,\mathrm{sen}(2\,\theta)\right) + \frac{1}{54}\,\theta + K = \frac{\theta}{54} + \frac{1}{54}\,\mathrm{sen}(\theta)\cos(\theta) + K$$

temos que: $\quad x = 3\sec(\theta) = \dfrac{3}{\cos(\theta)} \quad \Rightarrow \quad \cos(\theta) = \dfrac{3}{x}$

temos também: $\quad \sqrt{x^2-9} = 3\,\mathrm{tg}(\theta) \quad \Rightarrow \quad \mathrm{tg}(\theta) = \dfrac{\sqrt{x^2-9}}{3} \quad \Rightarrow \quad \theta =$

$$\operatorname{arctg}\left(\frac{\sqrt{x^2-9}}{3}\right)$$

fazendo: $\operatorname{sen}(\theta) = \cos(\theta)\dfrac{\operatorname{sen}(\theta)}{\cos(\theta)} = \cos(\theta)\operatorname{tg}(\theta) \quad => \quad \operatorname{sen}(\theta) = \dfrac{3}{x}\dfrac{\sqrt{x^2-9}}{3} =$
$\dfrac{\sqrt{x^2-9}}{x}$

substituindo estes valores em I , temos:

$$I = \frac{1}{54}\operatorname{arctg}\left(\frac{\sqrt{x^2-9}}{3}\right) + \frac{1}{54}\frac{\sqrt{x^2-9}}{x}\frac{3}{x} + K =$$

$$= \frac{1}{54}\operatorname{arctg}\left(\frac{\sqrt{x^2-9}}{3}\right) + \frac{1}{18x^2}\sqrt{x^2-9} + K$$

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2) $I = \displaystyle\int \frac{12x^3}{\sqrt{2x^2+7}}\,dx$;

Solução

intervalos: $0 \le \theta$ e $\theta < \dfrac{\pi}{2}$, se $0 \le x$; $-\dfrac{\pi}{2} < \theta < 0$, se $x < 0$

$$I = \int \frac{12x^3}{\sqrt{2x^2+7}}\,dx = \int \frac{12x^3}{\sqrt{\dfrac{2x^2}{7}+1}}\,dx$$

considerando $x = \sqrt{\dfrac{7}{2}}\operatorname{tg}(\theta) \quad => \quad dx = \sqrt{\dfrac{7}{2}}\sec(\theta)^2\,d\theta$

temos que: $12x^3 = 12\left(\sqrt{\dfrac{7}{2}}\operatorname{tg}(\theta)\right)^3 = 42\sqrt{\dfrac{7}{2}}\operatorname{tg}(\theta)^3$ e

$$\sqrt{2x^2+7} = \sqrt{2\left(\sqrt{\frac{7}{2}}\,\mathrm{tg}(\theta)\right)^2+7} = \sqrt{7\,\mathrm{tg}(\theta)^2+7} = \sqrt{7\,(\mathrm{tg}(\theta)^2+1)} =$$

$$= \sqrt{7}\,\sqrt{\mathrm{tg}(\theta)^2+1} = \sqrt{7}\,\sqrt{\sec(\theta)^2} = \sqrt{7}\,\sec(\theta)$$

subtituindo em I , temos:

$$I = \int \frac{42\sqrt{\frac{7}{2}}\,\mathrm{tg}(\theta)^3\sqrt{\frac{7}{2}}\,\sec(\theta)^2}{\sqrt{\frac{7}{2}}\,\sec(\theta)}\,d\theta = 42\,\frac{\sqrt{\frac{7}{2}}}{\sqrt{7}}\int \mathrm{tg}(\theta)^3\sec(\theta)\,d\theta = 21\sqrt{7}$$

$$\int \mathrm{tg}(\theta)^3\sec(\theta)\,d\theta =$$

$$= 21\sqrt{7}\int (\sec(\theta)^2-1)\,\mathrm{tg}(\theta)\sec(\theta)\,d\theta = 21\sqrt{7}$$

$$\int \sec(\theta)^2\,\mathrm{tg}(\theta)\sec(\theta) - \mathrm{tg}(\theta)\sec(\theta)\,d\theta =$$

$$= 21\sqrt{7}\int \sec(\theta)^2\sec(\theta)\,\mathrm{tg}(\theta)\,d\theta\ -21\sqrt{7}\int \sec(\theta)\,\mathrm{tg}(\theta)\,d\theta$$

considerando $u = \sec(\theta)$ => $du = \sec(\theta)\,\mathrm{tg}(\theta)\,d\theta$

sutstituindo , temos:

$$I = 21\sqrt{7}\int u^2\,du\ -21\sqrt{7}\int 1\,du = 21\sqrt{7}\,\frac{u^3}{3}\ -21\sqrt{7}\,u + K$$

sustituindo u por seu valor inicial, temos:

$$I = 21\sqrt{7}\,\frac{\sec(\theta)^3}{3}\ -21\sqrt{7}\sec(\theta) + K = 7\sqrt{7}\,\sec(\theta)^3\ -21\sqrt{7}\sec(\theta) + K$$

como temos que $\sqrt{2x^2+7} = \sqrt{7}\,\sec(\theta)$ => $\sec(\theta) = \dfrac{\sqrt{7}\sqrt{2x^2+7}}{7}$

substituindo este valor em I, temos:

$$I = 7\sqrt{7}\left(\frac{\sqrt{7}\sqrt{2x^2+7}}{7}\right)^3 - \frac{21\sqrt{7}\sqrt{7}\sqrt{2x^2+7}}{7} = 7\sqrt{7}\,\frac{\sqrt{7}^3\sqrt{2x^2+7}^3}{7^3}$$
$$-21\sqrt{2x^2+7} =$$

$$= \sqrt{2x^2+7}^3 - 21\sqrt{2x^2+7} + K = (2x^2+7)\sqrt{2x^2+7} - 21\sqrt{2x^2+7} + K =$$

$$= 2x^2\sqrt{2x^2+7} - 14\sqrt{2x^2+7} + K$$

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3) $$I = \int \frac{\sqrt{x^2-2x-3}}{x+1}\,dx \;;$$

Solução

considerando $x + 1 = u$ => $x = u - 1$ => $dx = du$

substituindo em I , temos:

$$I = \int \frac{\sqrt{(u-1)^2 - 2(u-1) - 3}}{u}\,du = \int \frac{\sqrt{u^2 - 2u + 1 - 2u + 2 - 3}}{u}\,du =$$

$$= \int \frac{\sqrt{u^2-4u}}{u}\,du = \int \frac{u^2-4u}{u\sqrt{u^2-4u}}\,du = \int \frac{u-4}{\sqrt{u^2-4u}}\,du = \int \frac{u-4}{\sqrt{(u-2)^2-4}}\,du$$

no intervalo $0 < \theta < \frac{\pi}{2}$

podemos considerar $u - 2 = 2\sec(\theta)$ => $u = 2\sec(\theta) + 2$ => $du = 2\sec(\theta)\,tg(\theta)\,d\theta$

assim, temos: $u - 4 = 2\sec(\theta) - 2$

e $\sqrt{(u-2)^2-4} = \sqrt{(2\sec(\theta)+2-2)^2-4} = \sqrt{4\sec(\theta)^2-4} =$

$$= 2\sqrt{\sec(\theta)^2-1} = 2\sqrt{\operatorname{tg}(\theta)^2} = 2\operatorname{tg}(\theta)$$

subtituindo estes valores em I , temos:

$$I = \int \frac{(2\sec(\theta)-2)\,2\sec(\theta)\operatorname{tg}(\theta)}{2\operatorname{tg}(\theta)}\,d\theta = \int 2\sec(\theta)^2 - 2\sec(\theta)\,d\theta =$$

$$2\int \sec(\theta)^2\,d\theta - 2\int \sec(\theta)\,d\theta =$$

$$= 2\operatorname{tg}(\theta) - 2\ln(\,|\sec(\theta)+\operatorname{tg}(\theta)|\,) + K$$

mas temos que: $u-2 = 2\sec(\theta) \Rightarrow \sec(\theta) = \dfrac{u-2}{2}$

e que: $\sqrt{(u-2)^2-4} = 2\operatorname{tg}(\theta) \Rightarrow \operatorname{tg}(\theta) = \dfrac{\sqrt{(u-2)^2-4}}{2}$

substituindo em I , temos:

$$I = 2\,\frac{\sqrt{(u-2)^2-4}}{2} - 2\ln\left(\,\left|\frac{u-2}{2} + \frac{\sqrt{(u-2)^2-4}}{2}\right|\,\right) + K$$

mas temos que: u = x + 1
substituindo em I , temos:

$$I = \sqrt{(x+1-2)^2-4} - 2\ln\left(\,\left|\frac{x+1-2}{2} + \frac{\sqrt{(x+1-2)^2-4}}{2}\right|\,\right) + K =$$

$$= \sqrt{(x-1)^2-4} - 2\ln\left(\,\left|\frac{x-1}{2} + \frac{\sqrt{(x-1)^2-4}}{2}\right|\,\right) + K =$$

$$= \sqrt{x^2-2x-3} - 2\ln\left(\,\left|\frac{x-1+\sqrt{x^2-2x-3}}{2}\right|\,\right) + K$$

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4) $I = \displaystyle\int \frac{1}{(x^2 - 2x - 3)^{\left(\frac{3}{2}\right)}}\, dx$;

Solução

$$I = \int \frac{1}{(x^2 - 2x - 3)^{\left(\frac{3}{2}\right)}}\, dx = \int \frac{1}{[(x-1)^2 - 4]^{\left(\frac{3}{2}\right)}}\, dx = \int \frac{1}{\sqrt{[(x-1)^2 - 4]^3}}\, dx$$

no intervalo $0 \le \theta$ e $\theta < \dfrac{\pi}{2}$

podemos considerar $x - 1 = 2\sec(\theta)$ => $x = 2\sec(\theta) + 1$ => $dx = 2\sec(\theta)\,\mathrm{tg}(\theta)\, d\theta$

e $\sqrt{[(x-1)^2 - 4]^3} = \sqrt{(2\sec(\theta))^2 - 4}^{\,3} = 2^3 \sqrt{\sec(\theta)^2 - 1}^{\,3} = 8\sqrt{\mathrm{tg}(\theta)^2}^{\,3} = 8\,\mathrm{tg}(\theta)^3$

substituindo estes valores em I, temos:

$$I = \int \frac{2\sec(\theta)\,\mathrm{tg}(\theta)}{8\,\mathrm{tg}(\theta)^3}\, d\theta = \frac{1}{4}\int \frac{\sec(\theta)}{\mathrm{tg}(\theta)^2}\, d\theta = \frac{1}{4}\int \frac{\cos(\theta)}{\mathrm{sen}(\theta)^2}\, d\theta$$

considerando $u = \mathrm{sen}(\theta)$ => $du = \cos(\theta)\, d\theta$

substituindo em I, temos:

$$I = \frac{1}{4}\int \frac{1}{u^2}\, du = -\frac{1}{4u} + K$$

substituindo u por seu valor inicial, temos:

$$I = -\frac{1}{4\,\mathrm{sen}(\theta)} + K$$

como temos que: $x - 1 = 2\sec(\theta) = \dfrac{2}{\cos(\theta)}$ => $\cos(\theta) = \dfrac{2}{x-1}$

e que: $\sqrt{[(x-1)^2 - 4]^3} = 8\,\mathrm{tg}(\theta)^3 = 8\left(\dfrac{\mathrm{sen}(\theta)}{\cos(\theta)}\right)^3$ => $8\,\dfrac{\mathrm{sen}(\theta)}{\cos(\theta)} = \sqrt{(x-1)^2 - 4}$ =>

=> $8\,\mathrm{sen}(\theta) = \cos(\theta)\sqrt{(x-1)^2 - 4} = \dfrac{2}{x-1}\sqrt{(x-1)^2 - 4} = 2\sqrt{\dfrac{(x-1)^2 - 4}{(x-1)^2}} =$

$= 2\sqrt{1 - \left(\dfrac{2}{x-1}\right)^2}$ => $\mathrm{sen}(\theta) = \dfrac{1}{4}\sqrt{1 - \left(\dfrac{2}{x-1}\right)^2}$

logo, $I = \dfrac{1}{16}\sqrt{1 - \left(\dfrac{2}{x-1}\right)^2} + K$

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5) $I = \displaystyle\int \frac{1}{(x+2)\sqrt{x^2 + 4x + 3}}\,dx$;

Solução

$$I = \int \frac{1}{(x+2)\sqrt{x^2 + 4x + 3}}\,dx = \int \frac{1}{(x+2)\sqrt{(x+2)^2 - 1}}\,dx$$

no intervalo $0 < \theta < \dfrac{\pi}{2}$

podemos considerar $x + 2 = \sec(\theta)$ => $x = \sec(\theta) - 2$ => $dx = \sec(\theta)\,\mathrm{tg}(\theta)\,d\theta$

e $\sqrt{(x+2)^2 - 1} = \sqrt{\sec(\theta)^2 - 1} = \sqrt{\mathrm{tg}(\theta)^2} = \mathrm{tg}(\theta)$

substituindo estes valores em I , temos:

$$I = \int \frac{\sec(\theta)\,\mathrm{tg}(\theta)}{\sec(\theta)\,\mathrm{tg}(\theta)}\,d\theta = \int 1\,d\theta = \theta + K$$

como temos que: $x + 2 = \sec(\theta)$ => $\theta = \mathrm{arcsec}(x + 2)$

logo, $I = \mathrm{arcsec}(x + 2) + K$

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