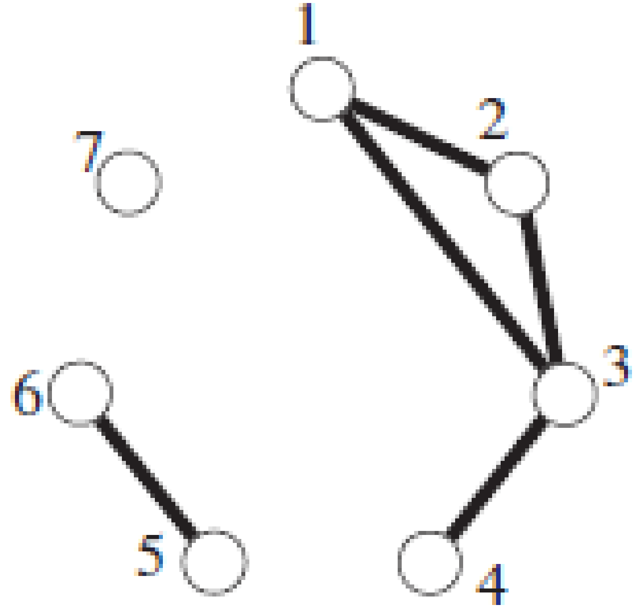
Matemática Discreta

Escola de Artes, Ciências e Humanidades - USP
Profa Dra. Karla Lima
email:ksampaiolima@usp.br

Fundamentals of Graph Theory

Connection



Walks

Definition 49.1

(Walk) Let $G = (V, E)$ be a graph. A *walk* in G is a sequence (or list) of vertices, with each vertex adjacent to the next; that is,

$$W = (v_0, v_1, \ldots, v_\ell) \quad \text{with} \quad v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_\ell.$$

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The *length* of this walk is ℓ. Note that we started the subscripts at zero and that there are $\ell + 1$ vertices on the walk.

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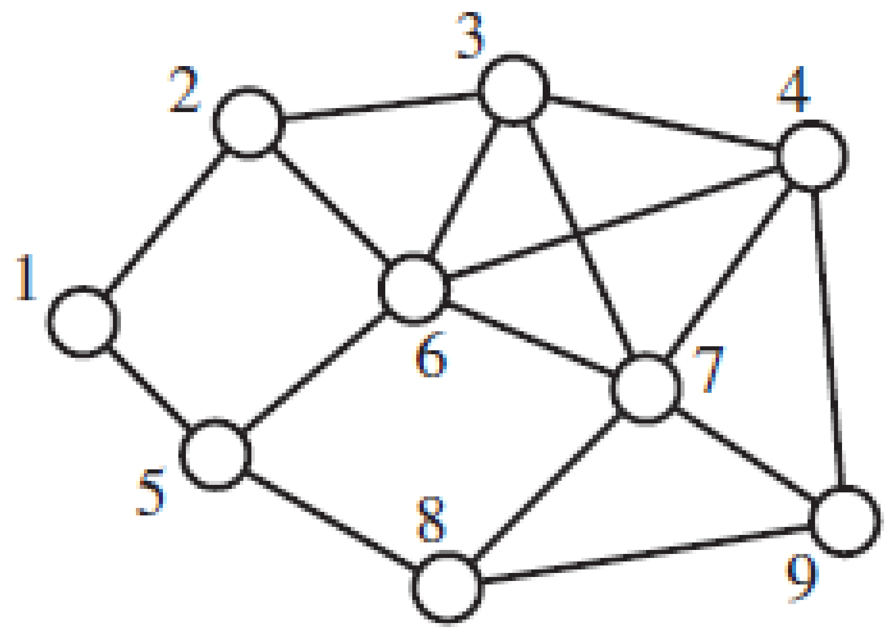
Walks



In general, a (u, v)-*walk* is a walk in a graph whose first vertex is u and whose last vertex is v.

We are permitted to visit a vertex more than once on a walk.

If $W = v_0 \sim v_1 \sim \cdots \sim v_{\ell-1} \sim v_\ell$, then its *reversal* is also a walk (because $\sim$ is symmetric). The reversal of W is $W^{-1} = v_\ell \sim v_{\ell-1} \sim \cdots \sim v_1 \sim v_0$.

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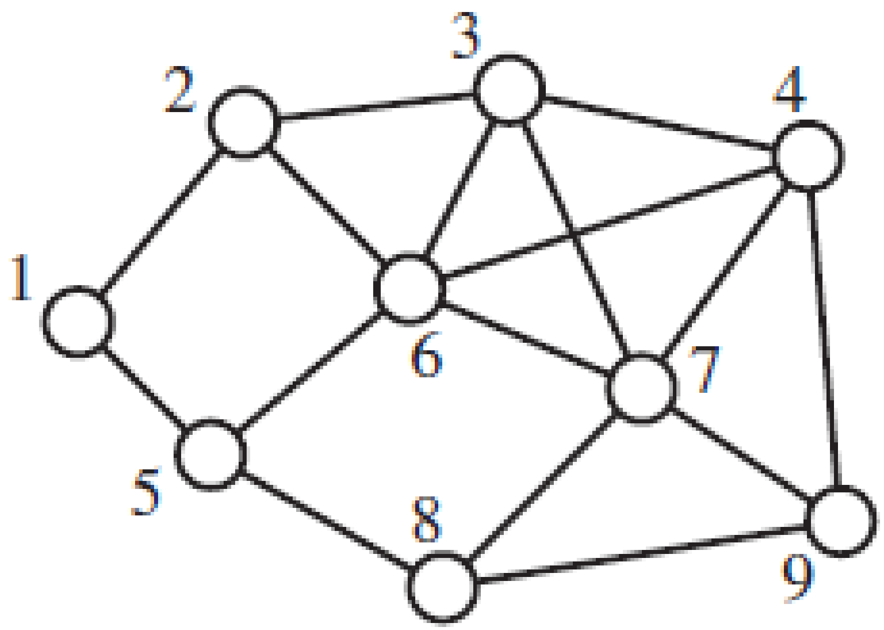
Walks



$1 \sim 2 \sim 3 \sim 4.$

$1 \sim 2 \sim 3 \sim 6 \sim 2 \sim 1 \sim 5.$

$5 \sim 1 \sim 2 \sim 6 \sim 3 \sim 2 \sim 1.$

$1 \sim 5 \sim 1 \sim 5 \sim 1.$

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Definition 49.2

(Concatenation) Let G be a graph. Suppose W_1 and W_2 are the following walks:

$$W_1 = v_0 \sim v_1 \sim \cdots \sim v_\ell$$
$$W_2 = w_0 \sim w_1 \sim \cdots \sim w_k$$

and suppose $v_\ell = w_0$. Their *concatenation*, denoted $W_1 + W_2$, is the walk

$$v_0 \sim v_1 \sim \cdots \sim (v_\ell = w_0) \sim w_1 \sim \cdots \sim w_k.$$

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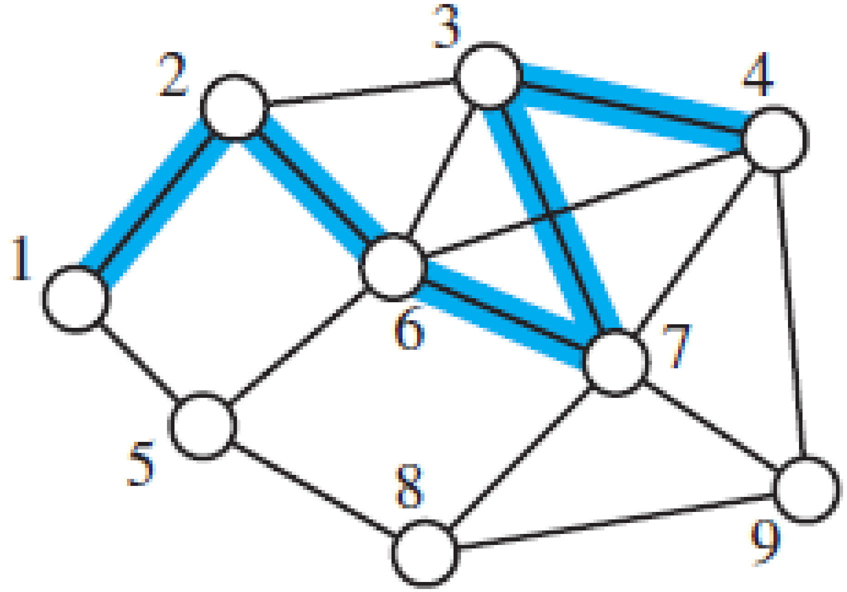
Paths

Definition 49.3

(Path) A *path* in a graph is a walk in which no vertex is repeated.



(u, v)-*path* is a path whose first vertex is u and whose last vertex is v.

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Paths

Proposition 49.4

Let P be a path in a graph G. Then P does not traverse any edge of G more than once.

Proof. Suppose, for the sake of contradiction, that some path P in a graph G traverses the edge $e = uv$ more than once. Without loss of generality, we have

$$P = \cdots \sim u \sim v \sim \cdots \sim u \sim v \sim \cdots \qquad \text{or}$$

$$P = \cdots \sim u \sim v \sim \cdots \sim v \sim u \sim \cdots .$$

$$\vdots$$

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Paths

Thus a path of length k contains exactly $k + 1$ (distinct) vertices and traverses exactly k (distinct) edges.

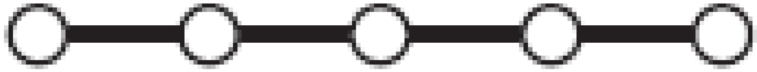
Definition 49.5

(Path graph) A *path* is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set

$$E = \{v_i v_{i+1} : 1 \leq i < n\}.$$

A path on n vertices is denoted P_n.

A P_5 graph:



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Definition 49.6

(Connected to) Let G be a graph and let $u, v \in V(G)$. We say that u *is connected to* v provided there is a (u, v)-path in G (i.e., a path whose first vertex is u and whose last vertex is v).

Lemma 49.7

Let G be a graph and let $x, y \in V(G)$. If there is an (x, y)-walk in G, then there is an (x, y)-path in G.

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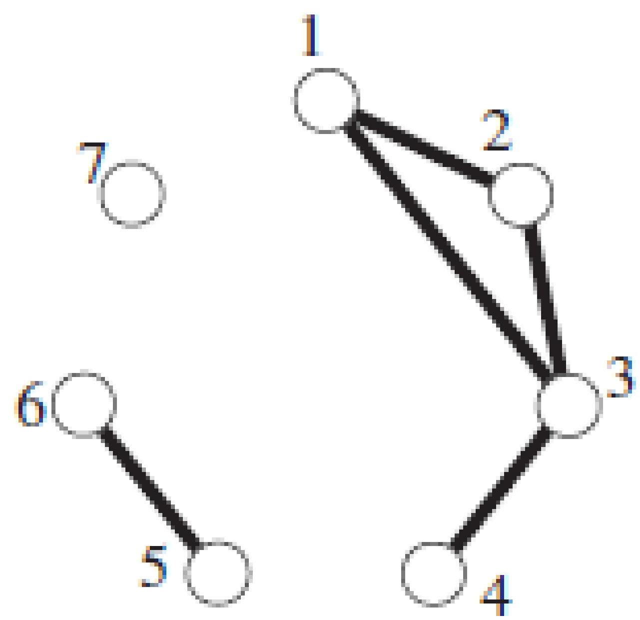
Definition 49.9

(Component) A *component* of G is a subgraph of G induced on an equivalence class of the is-connected-to relation on $V(G)$.



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Definition 49.10

(Connected) A graph is called *connected* provided each pair of vertices in the graph is connected by a path; that is, for all $x, y \in V(G)$, there is an (x, y)-path.

Disconnection

Definition 49.11

(Cut vertex, cut edge) Let G be a graph. A vertex $v \in V(G)$ is called a *cut vertex* of G provided $G - v$ has more components than G.

Similarly, an edge $e \in E(G)$ is called a *cut edge* of G provided $G - e$ has more components than G.

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Disconnection

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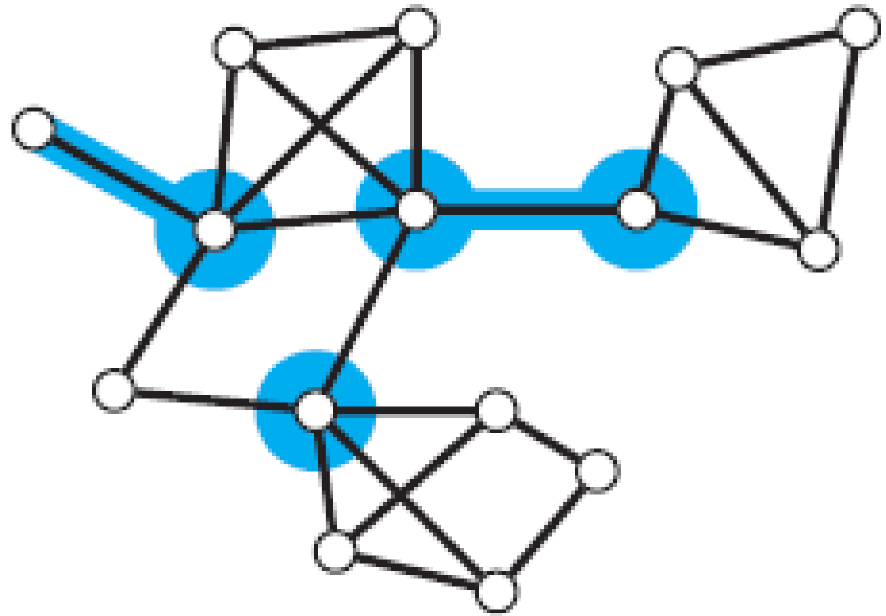


Fundamentals of Graph Theory

Theorem 49.12

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G-e$ has exactly two components.

Proof. Let G be a connected graph and let $e \in E(G)$ be a cut edge. Because G is connected, it has exactly one component. Because e is a cut edge, $G - e$ has more components than G (i.e., $G - e$ has at least two components). Our job is to show that it does not have more than two components.

Suppose, for the sake of contradiction, $G - e$ has three (or more) components. Let a, b, and c be three vertices of $G - e$, each in a separate component. This implies that there is no path joining any pair of them.

Fundamentals of Graph Theory

Theorem 49.12

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G - e$ has exactly two components.

Let P be an (a, b)-path in G. Because there is no (a, b)-path in $G - e$, we know P must traverse the edge e. Suppose x and y are the endpoints of the edge e, and without loss of generality, the path P traverses e in the order x, then y; that is,

$$P = a \sim \cdots \sim x \sim y \sim \cdots \sim b.$$

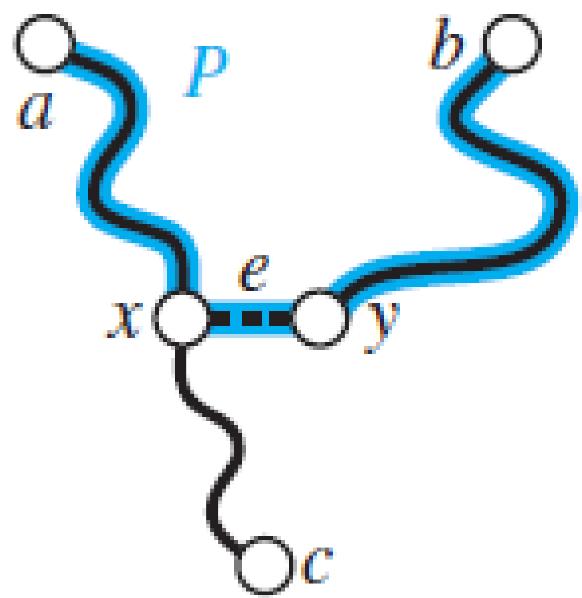
Similarly, since G is connected, there is a path Q from c to a that must use the edge $e = xy$. Which vertex, x or y, appears first on Q as we travel from c to a?

Fundamentals of Graph Theory

Theorem 49.12

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G - e$ has exactly two components.

If x appears before y on the (c, a)-path Q, then notice that we have, in $G - e$, a walk from c to a. Use the (c, x)-portion of Q, concatenated with the (x, a)-portion of P^{-1}. This yields a (c, a)-walk in $G - e$ and hence a (c, a)-path in $G - e$ (by Lemma 49.7). This, however, is a contradiction, because a and c are in separate components of $G - e$.

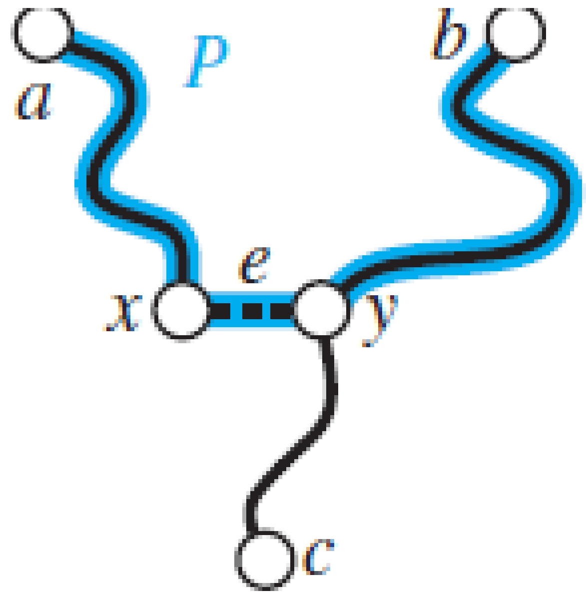


Fundamentals of Graph Theory

Theorem 49.12

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G-e$ has exactly two components.

If y appears before x on the (c,a)-path Q, then notice that we have, in $G-e$, a walk from c to b. Concatenate that (c,y)-section of Q with the (y,b)-section of P. This walk does not use the edge e. Therefore there is a (c,a)-walk in $G-e$ and hence (Lemma 49.7) a (c,a)-walk in $G-e$. This contradicts the fact that in $G-e$ we have c and b in separate components.



Fundamentals of Graph Theory

Theorem 49.12

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G-e$ has exactly two components.

Therefore $G - e$ has at most two components.