Fuzzy ES

What is fuzzy logic?

- Words (and the thoughts they describe) are often vague and imprecise: How hot is “hot”? How tall is “tall”?
- Even more concrete concepts are not necessarily black and white. What makes a chair a chair?
- Fuzzy logic attempts to capture the idea of *degrees of membership* in a class. Contrast with Boolean logic, in which membership is absolute.
- Class question: What makes a chair a chair?

www.backstreet.demon.co.uk

www.petdiscounters.com

www.greenwoodworking.com

www.moosemaple.com/

www.johnston.k12.ia.us

Which of these is the most “chair-like”? What features make a chair a chair?

Classical (“crisp”) sets

If X is the *universe of discourse* (also called the *finite reference super set)*, with elements x, then the **characteristic function** of crisp set A is:

$$f_A(x): X \rightarrow 0,1$$

Where:

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Fuzzy sets

If X is the universe of discourse, with elements x, then the characteristic function of fuzzy set A is:

$$\mu_A(x): X \rightarrow [0,1]$$

Where:

$\mu_A(x) = 1$	If x is totally in A
$\mu_A(x) = 0$	If x is totally not in A
$0 < \mu_A(x) < 1$	If x is partially in A

Defining fuzzy sets with *fit vectors*

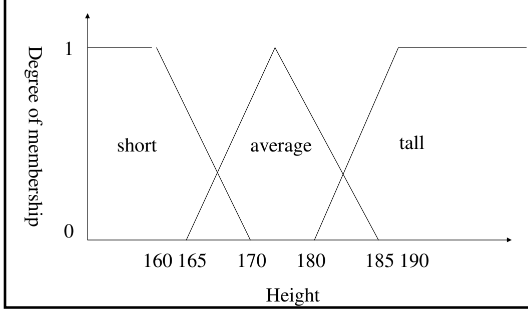
A can be defined as:

$$\mathrm{A} = \{\mu_{\mathrm{A}}(x_1)/x_1\}\ldots\{\mu_A(x_n)/x_n\}$$

Where *linear fit functions* join the points given in the set definition. So, for example:

Tall men = (0/180, 1/190)
Short men=(1/160, 0/170)
Average men=(0/165,1/175,0/185)

Graphical representation of set definition



What if key features don't have a scale?

- Determine list of binary features (either present or absent)
- Determine weight (importance) of each feature
- Sum over weights, and normalize
- This gives your membership function!

Class exercise: develop a membership function for "chair".

Hedges

Hedges are fuzzy set qualifiers, and modify the shape of the set. Can be:

- *Truth values:* e.g. quite true, mostly false
- *Probabilities:* e.g. likely, unlikely
- *Quantifiers:* e.g. most, several, few
- *Possibilities:* e.g. almost impossible, quite possible

Interpretations of hedges

Hedges are given mathematical interpretations, for example, the operation of *concentration* (i.e. making the set smaller):

$$\mu_A^{very}(x) = [\mu_A(x)]^2$$

$$\mu_A^{extremely}(x) = [\mu_A(x)]^3$$

$$\mu_A^{veryvery}(x) = [\mu_A^{very}(x)]^2 = [\mu_A(x)]^4$$

Interpretation of hedges (2)

Also, operations of *dilation* (i.e. make set bigger):

$$\mu_A^{moreorless}(x) = \sqrt{\mu_A(x)}$$

And operations of *intensification* (exaggerates the extremes of the set):

$$\mu_A^{indeed}(x) = \begin{cases} 2[\mu_A(x)]^2 & 0 \le \mu_A(x) \le 0.5 \\ 1 - 2[1-\mu_A(x)]^2 & 0.5 < \mu_A(x) \le 1 \end{cases}$$

Fuzzy Operations (1)

The **complement** of a set is its opposite, given by:

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

Fuzzy operations (2)

Intersection determines gives how much of each element belongs to both sets.

$$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)]$$

Fuzzy operations (3)

Union gives how much of an element is in either set.

$$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)]$$

Properties

Same as for crisp sets:

- Commutativity (e.g. A OR B = B OR A)
- Associativity (e.g. A OR (B OR C) = (A OR B) OR C))
- Distributivity (e.g. A AND (B OR C) = (A AND B) OR (A AND C))
- Idempotency (e.g. A AND A = A)
- Involution (NOT (NOT A) = A)
- Transitivity (if (A → B) AND (B→C) then (A→C))
- De Morgan's Laws:
  NOT (A AND B) = NOT A OR NOT B
  NOT (A OR B) = NOT A AND NOT B

Fuzzy rules

A fuzzy rule is like a classical IF-THEN rule, but uses fuzzy classes.

IF x is [a member of] A THEN y is [a member of] B

Where x and y are linguistic variables and A and B are linguistic values, determined by fuzzy sets on the universes of discourse X and Y.

Fuzzy rules example

IF **height** is *tall* THEN **weight** is *heavy*.

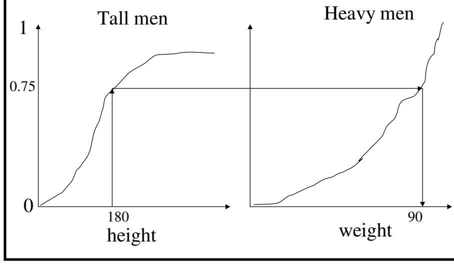
Where **height** and **weight** have a range (i.e. the universe of discourse), and *tall* and *heavy* are fuzzy classes.

Reasoning with fuzzy rules

- In classical systems, rules with true antecedents fire.
- In fuzzy systems, truth (i.e. membership in some class) is relative – so all rules fire (to some extent).
- If the antecedent is true to some degree, the consequent is true to the same degree.

Monotonic selection

Allows estimation of the variable in the consequent. However, it only works for monotonic membership functions – not common!



Antecedents/consequents with multiple parts

- Antecedents: Use unification (OR) or intersection (AND) operations to calculate a membership value for the whole antecedent.
- Consequents: Each consequent is affected equally by the membership in the antecedent class(es).

Fuzzy inference: Mamdani-style (see Negnevitsky)

1. Fuzzify the input variables
2. Evaluate the rules
3. Aggregate the rule outputs
4. Defuzzify the output

Fuzzification

- Using the crisp inputs from the user, turn them into fuzzy memberships for *all* the relevant (i.e. right Universe of Discourse) classes.

Rule evaluation

- Apply the fuzzified inputs to *all* the relevant rules, using union and intersection operations to handle complex antecedents.

Aggregate the results

Build a membership function for each output UoD, by aggregating all the relevant classes.

Defuzzify

Use a center of mass formula to calculate the crisp output value (integration approximated as summation):

$$COG = \frac{\sum_{x=a}^{b} \mu_A(x)x}{\sum_{x=a}^{b} \mu_A(x)}$$

Class exercise

Write 3-5 fuzzy rules that determine heart attack risk, using:

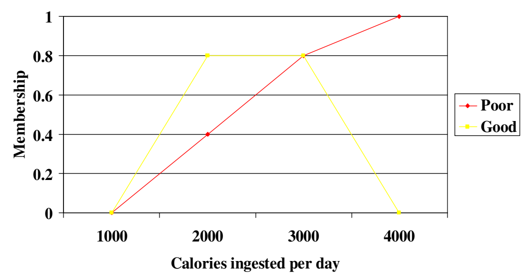
- Three 'universes of discourse' (UoD): diet, exercise, and risk
- 2 or 3 fuzzy classes per UoD, *and* their membership functions (represent graphically)
- Show fuzzy inference for one set of sample data

One solution: Rules

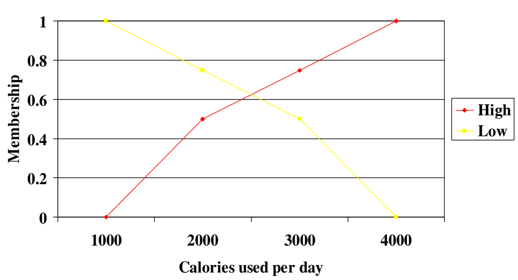
- *diet* is *poor* AND *exercise* is *low* → *risk* is *high*
- *diet* is *good* AND *exercise* is *high* → *risk* is *low*
- *diet* is *good* OR *exercise* is *high* → *risk* is *average*

Your rules and membership functions may vary.

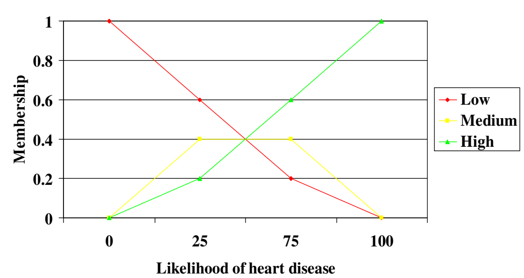
Membership function: Diet



Membership function: Exercise



Membership function: Risk



User data

- Exercise: expends 1000 calories/day
- Diet: eats 2500 calories/day
- What is the risk of heart disease?

Step 1: Fuzzification

Determine the degree of membership of the input in the relevant fuzzy sets.

$$\mu_{poordiet}(2500) = 0.6$$

$$\mu_{gooddiet}(2500) = 0.8$$

$$\mu_{highexer}(1000) = 0$$

$$\mu_{lowexer}(1000) = 1$$

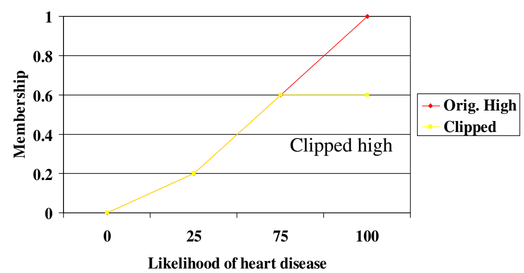
Rule Evaluation: antecedents

- *diet* is *poor* (0.6) AND *exercise* is *low* (1) *risk* is *high*

$$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)]$$

- If the antecedent is true to some degree, the consequent is true to the same degree.
- So, the membership function for the antecedent is 0.6, which is used to *clip* the membership function of the consequent, for example:

Membership function: Risk HIGH



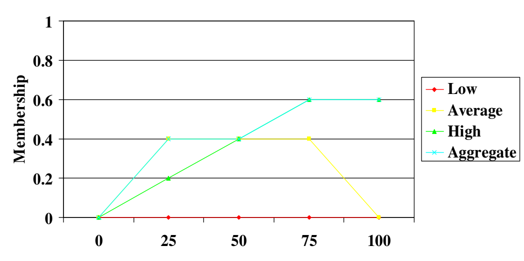
Evaluation (2)

Continue in this way for other rules, giving:

- *diet* is *good (0.8)* AND *exercise* is *high* (0)→ *risk* is *low (0)* [having applied the intersection rule]
- *diet* is *good* (0.8) OR *exercise* is *high* (0)→ *risk* is *average (0.8)* [having applied the union rule, that is:]

$$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)]$$

Aggregation: membership functions for RISK



Note that "low" has been clipped down to zero, and "average" hasn't been clipped, because the derived # was higher than the original membership function. The aggregate is simply the union of the three functions.

Defuzzification

- Use the COG formula to determine the center the aggregate function.

$$COG = \frac{\sum_{x=a}^{b} \mu_A(x)x}{\sum_{x=a}^{b} \mu_A(x)}$$

Defuzzification(2)

- To simplify the calculation, we will sample every 12.5 units (better to do more, or to integrate properly):

$$COG = \frac{12.5*.2 + (25+37.5+50)*.4 + 62.5*.5 + (75+87.5+100)*.6}{.2+3*.4+.5+3*.6} = 70$$

Result

- So, the likelihood of heart disease for our patient is 70%.
- Please keep in mind that I made up these membership functions and rules without any medical knowledge – they are just for the exercise!!