Universal L^2-torsion detects fibered 3-manifold

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Background

The **Thurston norm** and **fibered structure** are central in 3-manifold theory.

	Thurston norm detection	**Fibered detection**
Δ	Lower bound	$\Longrightarrow$ monic
Δ_α	Exact (for some α)	$\Longleftrightarrow$ monic for all α
HF	Exact	$\Longleftrightarrow$ monic
$\tau_{\mathbf{u}}^{(\mathbf{2})}$	**Exact**	$\Longleftrightarrow$ **monic**

The **universal** L^2**-torsion** $\tau_u^{(2)}(M)$ is the Reidemeister torsion of M over certain skew field [Friedl–Lück, 2017].

Two technical requirements for the space M:

- L^2-acyclic compact CW-complex.
- $G := \pi_1(M)$ satisfies the Atiyah Conjecture and torsion-free.

e.g. irreducible 3-manifolds with $\chi = 0$ and $|\pi_1| = \infty$.

Outcome:

- $\mathbb{Z}G$ embeds in a canonical skew field $\mathcal{D}_G$ [Linnell, 1993]. (a field whose multiplication is non-commutative)
- $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*(\widehat{M})$ is exact as a $\mathcal{D}_G$-chain complex.

Definition (Friedl–Lück, 2017)

The universal L^2-torsion of M, denoted by $\tau_u^{(2)}(M)$, is the Reidemeister torsion of $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*(\widehat{M})$.

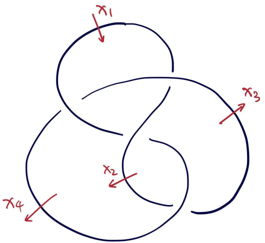
Reidemeister torsion of $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*(\widehat{M})$:

$$
\begin{array}{ccccc}
A_{n+1} & & A_n & & A_{n-1} \\
\oplus & \overset{f_{n+1}}{\underset{\cong}{\searrow}} & \oplus & \overset{f_n}{\underset{\cong}{\searrow}} & \oplus \\
\ker \partial_{n+1} & & \ker \partial_n & & \ker \partial_{n-1} \\
\| & & \| & & \| \\
\mathcal{D}_G \otimes_{\mathbb{Z}G} C_{n+1}(\widehat{M}) & \xrightarrow{\partial_{n+1}} & \mathcal{D}_G \otimes_{\mathbb{Z}G} C_n(\widehat{M}) & \xrightarrow{\partial_n} & \mathcal{D}_G \otimes_{\mathbb{Z}G} C_{n-1}(\widehat{M})
\end{array}
$$

Then $\tau_u^{(2)}(M) := \prod_i \det(f_i)^{(-1)^n}$.

- Dieudonné determinant $\det : \mathrm{GL}(\mathcal{D}_G) \to K_1(\mathcal{D}_G)$.
- A classical fact: $K_1(\mathcal{D}_G) = (\mathcal{D}_G^\times)_{\mathsf{abel}}$.
- The invariant is well-defined in the Whitehead group

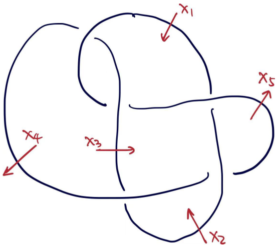
$$\mathrm{Wh}(\mathcal{D}_G) := K_1(\mathcal{D}_G)/\pm G.$$



M_1 is 4_1-knot complement,

$$\Delta(M_1) = (1 - 3t + t^2),$$

$$\tau_u^{(2)}(M_1) = [1 - (x_2 + x_3 + x_4) + x_4 x_3] \cdot [1 - x_4]^{-1}.$$

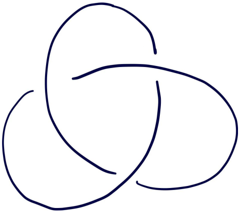


M_2 is 5_2-knot complement,

$$\Delta(M_2) = (2 - 3t + 2t^2),$$

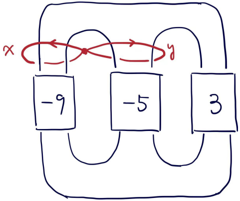
$$\tau_u^{(2)}(M_2) = \left[(1 + x_4^{-1} x_3) - (x_1 + x_5 + x_2) + (x_5 x_1 + x_2 x_1)\right] \cdot [1 - x_5]^{-1}.$$

Trefoil Knot 3_1



- Fibered
- $\Delta(t) = t^2 - t + 1$
- Leading term of $\tau_u^{(2)}$: 1

Pretzel Knot $P(-9, -5, 3)$



- Non-fibered
- $\Delta(t) = t^2 - t + 1$
- Leading term of $\tau_u^{(2)}$: $[(x + x^2 + x^3)y - x^4 - y^2]$

Main result

Let M be a compact, aspherical 3-manifold with $\chi(M) = 0$. Let $\phi \in H^1(M; \mathbb{R}) \setminus \{0\}$.

Theorem (D., 2025)

Suppose M is not a closed graph manifold. Then:

$$\phi \text{ is fibered} \iff \phi\text{-leading term of } \tau_u^{(2)}(M) \text{ equals } 1.$$

- Fills the missing corner of the Table.
- "$\implies$" is easy to believe.
- "$\impliedby$", let's prove it:

Suppose $L < G$ is a finite index normal subgroup. There is a restriction map

$$\mathrm{res}_L^G : \mathrm{Wh}(\mathcal{D}_G) \to \mathrm{Wh}(\mathcal{D}_L).$$

Theorem (Friedl–Lück '17)

Suppose $\overline{M} \to M$ is a regular finite covering with $\pi_1(\overline{M}) = L$ and $\pi_1(M) = G$. Then

$$\tau_u^{(2)}(\overline{M}) = \mathrm{res}_L^G(\tau_u^{(2)}(M)).$$

Theorem (D., 2025)

The leading term map commutes with the restriction map:

$$\begin{tikzcd} \mathrm{Wh}(\mathcal{D}_G) \arrow[r, "\mathrm{res}_L^G"] \arrow[d, "L_\phi"'] & \mathrm{Wh}(\mathcal{D}_L) \arrow[d, "L_{\phi|_L}"] \\ \mathrm{Wh}(\mathcal{D}_G) \arrow[r, "\mathrm{res}_L^G"] & \mathrm{Wh}(\mathcal{D}_L) \end{tikzcd}$$

where $\phi \in H^1(G;\mathbb{R})$ and $\phi|_L \in H^1(L;\mathbb{R})$ is the restriction.

Proof of "$\implies$": Suppose $L_\phi \tau_u^{(2)}(M) = 1$.

- By Agol's Virtual Fibering Theorem, $\exists L \overset{f.i.}{\lhd} G$ such that $\phi|_L$ is quasi-fibered. Let $\overline{M}$ be the corresponding finite covering.
- $L_{\phi|_L} \tau_u^{(2)}(\overline{M}) = 1$ by the commutative diagram.
- $\phi|_L$ is in the top dimensional Thurston cone of $H^1(\overline{M};\mathbb{R})$.
- $\phi|_L$ is fibered, therefore ϕ is fibered.

Thank you for listening!