

UNIVERSAL L^2 -TORSION AND TAUT SUTURED DECOMPOSITIONS

JIANRU DUAN

ABSTRACT. Given an admissible 3-manifold M and a first cohomology class ϕ , we show that the universal L^2 -torsion of M detects the fiberedness of ϕ , except when M is a closed graph manifold. As a natural extension in sutured manifold theory, we derive a simple formula describing how this invariant changes under taut sutured decompositions. Furthermore, we establish that a taut sutured manifold is a product if and only if its universal L^2 -torsion is trivial. Our approach relies on a detailed study of the leading term map over Linnell's skew field. As applications, we initiate the investigation of the universal L^2 -torsion for free group homomorphisms, which enables explicit computations in the case of sutured handlebodies.

1. INTRODUCTION

Let M be a compact orientable 3-manifold. A homomorphism $\phi : \pi_1(M) \rightarrow \mathbb{Z}$ is called fibered if it is induced by a fibration of M over the circle. Thurston [Thu86] defined a semi-norm on the real cohomology group $H^1(M; \mathbb{R})$ and showed that its unit ball $B_x(M)$ is a finite polygon. Moreover, Thurston showed that there is a disjoint union of cones over some of the open faces of $B_x(M)$ such that ϕ is fibered if and only if ϕ lies in the cones. Marked by the success of Gabai's sutured manifold theory [Gab83, Gab87] and the confirmation of the Virtual Fiber Conjecture [AGM13], it has been one of the most important topics in 3-dimensional topology to determine the Thurston norm and the fibered structure of a 3-manifold. In this paper we pay attention to the case of admissible 3-manifolds:

Definition 1.1 (Admissible 3-manifold). A 3-manifold is called admissible if it is compact, connected, orientable and irreducible, its boundary is empty or a collection of tori, and its fundamental group is infinite.

The universal L^2 -torsion introduced by Friedl and Lück [FL17] is defined for a finite CW-complex X whose L^2 -Betti numbers vanish. When X is an admissible 3-manifold, then this invariant $\tau_u^{(2)}(X)$ takes value in the weak Whitehead group $\text{Wh}^w(\pi_1(X))$ associated to the fundamental group $\mathbb{Z}\pi_1(X)$. While the universal L^2 -torsion fully characterizes the Thurston norm of an admissible 3-manifold [FL17], it is natural to ask whether it provides a necessary and sufficient condition for the fiberedness of an admissible 3-manifold. To this end, given any cohomology class $\phi \in H^1(M; \mathbb{R})$ of an admissible 3-manifold M , we define a natural “leading term map”

$$L_\phi : \text{Wh}^w(\pi_1(M)) \rightarrow \text{Wh}^w(\pi_1(M))$$

on the weak Whitehead group of $\pi_1(M)$. Unless M is a closed graph manifold which does not admit a non-positively-curved (or NPC) metric, we show that ϕ is fibered if and only if the image of $\tau_u^{(2)}(M)$ under L_ϕ is trivial.

Theorem 1.2. *Suppose M is an admissible 3-manifold such that M is not a closed graph manifold which admits no NPC metric. Let $\phi \in H^1(M; \mathbb{R})$ be any nonzero character. Then ϕ is fibered if and only if $L_\phi \tau_u^{(2)}(M) = 1 \in \text{Wh}^w(\pi_1(M))$.*

It follows that Theorem 1.2 gives a clean description of the marking on the L^2 -torsion polytope $\mathcal{P}(M)$ given by [Kie20] which determines the fibered structure of M .

A large part of this paper focuses on examining the universal L^2 -torsion of a sutured manifold. A sutured manifold (N, R_+, R_-, γ) is a compact oriented 3-manifold with a partition of its boundary into two oriented subsurfaces R_+ and R_- along their common boundary γ . A sutured manifold can be decomposed along a nicely embedded surface S and the resulting manifold is again a sutured manifold. We write

$$(N, R_+, R_-, \gamma) \xrightarrow{S} (N', R'_+, R'_-, \gamma')$$

for a sutured decomposition. A beautiful result by Herrmann [Her23] shows that, roughly speaking, a sutured manifold (N, R_+, R_-, γ) is taut if and only if the pair (N, R_+) has trivial L^2 -Betti numbers (see Theorem 5.6). Hence we define the universal L^2 -torsion of a taut sutured manifold (N, R_+, R_-, γ) as $\tau_u^{(2)}(N, R_+)$ which lives in $\text{Wh}(\mathcal{D}_{\pi_1(N)})$. The second main result of this paper describes the change of the universal L^2 -torsion during taut sutured decompositions via the leading term map.

Theorem 1.3. *Let $(N, R_+, R_-, \gamma) \xrightarrow{\Sigma} (N', R'_+, R'_-, \gamma')$ be a taut sutured decomposition and let $\phi \in H^1(N; \mathbb{Z})$ be the Poincaré dual of the surface Σ , then*

$$j_* \tau_u^{(2)}(N', R'_+) = L_\phi \tau_u^{(2)}(N, R_+)$$

where $j_* : \text{Wh}^w(\pi_1(N')) \rightarrow \text{Wh}^w(\pi_1(N))$ is induced by the inclusion map $j : N' \hookrightarrow N$.

Moreover, we can show that the universal L^2 -torsion detects product sutured manifolds. One can view a sutured manifold (N, R_+, R_-, γ) as a cobordism between two compact surfaces R_\pm . If N is taut then the inclusion map $R_+ \hookrightarrow N$ induces isomorphism on L^2 -homology. We show that this cobordism is a trivial product if and only if its universal L^2 -torsion is trivial, a reminiscent of the celebrated s -cobordism theorem.

Theorem 1.4. *Let (N, R_+, R_-, γ) be a taut sutured manifold with R_+ and R_- both non-empty. Then N is diffeomorphic to $R_+ \times [0, 1]$ if and only if $\tau_u^{(2)}(N, R_+) = 1 \in \text{Wh}^w(\pi_1(N))$.*

We initiate the computation of the universal L^2 -torsion $\tau_u^{(2)}(X, Y)$ where the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ are finitely generated free groups. In this case, the universal L^2 -torsion is completely determined by the inclusion-induced homomorphism $\phi : \pi_1(Y) \rightarrow \pi_1(X)$. It is then natural to define the universal L^2 -torsion $\tau_u^{(2)}(\phi)$ for a group homomorphism between finitely generated free groups and we derive the following formula:

Theorem 1.5. *Let $\phi : F_1 \rightarrow F_2$ be a homomorphism between finitely generated free groups $F_1 = \langle x_1, \dots, x_n \rangle$, $F_2 = \langle y_1, \dots, y_m \rangle$. Then the universal L^2 -torsion of ϕ is*

$$\tau_u^{(2)}(\phi) = [J_\phi] \in \text{Wh}^w(F_2) \sqcup \{0\}$$

where J_ϕ is a matrix over $\mathbb{Z}F_2$ called the Jacobian of ϕ , whose entries are given by Fox derivative

$$(J_\phi)_{ij} = \frac{\partial \phi(x_i)}{\partial y_j} \in \mathbb{Z}F, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m.$$

It is understood that $[J_\phi] = 0$ if and only if J_ϕ is not a weak isomorphism. Moreover, the universal L^2 -torsion of ϕ does not depend on the choice of the free basis.

We apply this formula to explicitly compute the universal L^2 -torsion of some sutured handlebodies, particularly the sutured manifold $S^3 \setminus \Sigma$ obtained from decomposing the n -chain link complement along a minimal-genus Seifert surface Σ (see Figure 1). We find that

$$\tau_u^{(2)}(S^3 \setminus \Sigma, \Sigma_+) = [1 + y_1 + \dots + y_{n-1}] \in \text{Wh}^w(\pi_1(S^3 \setminus \Sigma))$$

where $\{y_1, \dots, y_{n-1}\}$ freely generates $\pi_1(S^3 \setminus \Sigma)$. Combining the calculations of [BA22], we obtain the following formula:

$$\tau^{(2)}(S^3 \setminus \Sigma, \Sigma_+) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}} \sim \sqrt{n/e}, \quad \text{as } n \rightarrow +\infty.$$

The left-hand-side is the classical L^2 -torsion of the pair $(S^3 \setminus \Sigma, \Sigma_+)$, obtained from the universal L^2 -torsion by taking the Fuglede–Kadison determinant. This gives an infinite family of hyperbolic manifolds whose L^2 -Alexander torsion associated to a nonzero class has leading coefficient greater than 1, confirming the last missing piece of [BAFH22, Conjecture 1.7].

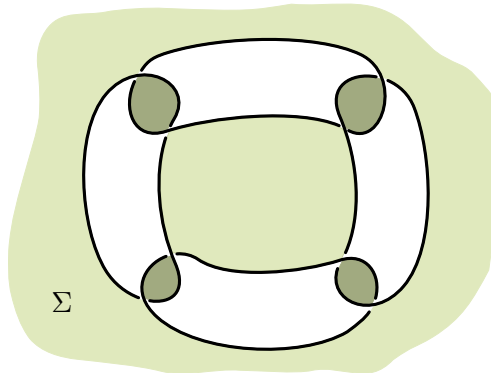


Figure 1. The n -chain link and the minimal-genus Seifert surface Σ , where $n = 4$.

1.1. Motivation.

1.1.1. *Why universal L^2 -torsion.* To define the classical torsion invariants of a finite CW-complex X , one has to produce an exact sequence from the cellular chain complex of the universal cover \widehat{X} . One way to do this is to extend the scalars from the group ring $\mathbb{Z}[\pi_1(X)]$ to a commutative field. For 3-manifolds, this idea produces the Reidemeister–Franz torsion and the (multi-variable) Alexander polynomials, considering extensions of scalars to the complex number field \mathbb{C} and the field of rational functions $\mathbb{Q}(H_1(X))$, respectively. However, this base change loses the non-commutative information of the fundamental group. The torsion introduced by J.H.C Whitehead does not involve any base change and retain most information; the so-called Whitehead torsion lives in a certain group $\text{Wh}(\pi_1(X))$ consisting of matrices invertible over $\mathbb{Z}[\pi_1(X)]$ up to equivalence. As a tradeoff, the Whitehead torsion only applies to pairs (X, Y) where X deformation retracts to Y . Also, it is too restrictive for a matrix to be invertible over $\mathbb{Z}[\pi_1(X)]$ and conjecturally $\text{Wh}(G)$ is trivial for all torsion-free groups G .

The universal L^2 -torsion is potentially a powerful yet applicable torsion invariant at the same time. The reasons are three-fold: 1) The universal L^2 -torsion is defined via the universal covering and extract information from the entire fundamental group. 2) There are a large number of spaces whose L^2 -Betti numbers vanish, such as mapping torus [Lüc94], spaces with infinite amenable fundamental groups [CG86], most 3-manifolds [LL95] and all odd-dimensional closed hyperbolic manifolds [HS98] (see [Lüc02] for more examples). 3) The weak Whitehead group $\text{Wh}^w(G)$ of any group G is never trivial (it always contains \mathbb{Z}). In fact there are interesting non-trivial homomorphisms from $\text{Wh}^w(G)$ such as the Fuglede–Kadison determinant and the polytope maps [FL17].

The universal L^2 -torsion can be used to define a group-theoretic Thurston norm for more general groups, such as for free-by-cyclic groups [FK18, KS24] and coherent right-angled Artin groups (or RAAGs) [Kud24]. The BNS-invariant generalizes the fibered structure of 3-manifold groups

to arbitrary finitely-generated groups, which is intimately related to the universal L^2 -torsion of a group as shown by many recent work, for examples [FST17, FT20, HK20] for two-generator one-relator groups, [FK18] for free-by-cyclic groups, and for more agrarian groups by [Kie20]. These findings concentrate on the L^2 -polytope given by the universal L^2 -torsion and shows the existence of a *marking* which determines the shape of the BNS-invariant. One of the novelty of our Theorem 1.2 is to offer a clean description of the fibered structure for 3-manifolds directly in terms of the universal L^2 -torsion.

1.1.2. Fiberedness and the leading term of torsions. It is known that if a class ϕ is fibered, then its Alexander polynomial, twisted Alexander polynomial and L^2 -Alexander torsion have trivial leading coefficients. Conversely, the leading term of those torsion invariants are not enough to characterize fiberedness. Friedl and Vidussi [FV11] showed that the collection of all twisted Alexander polynomial of ϕ do determine the fiberedness of ϕ . Their work indicates that torsion invariants carry enough information for the detection of fiberedness.

Given a cohomology class $\phi \in H^1(M; \mathbb{R})$, the L^2 -Alexander torsion $\tau^{(2)}(M, \phi) : \mathbb{R}_+ \rightarrow [0, +\infty)$ is a real function introduced by Dubois, Friedl and Lück [DFL16] as an L^2 -analogue of the classical Alexander polynomial. Liu [Liu17] proved that a leading coefficient $C(M, \phi) \in [1, +\infty)$ of the L^2 -Alexander torsion is well-defined and $C(M, \phi) = 1$ if ϕ is a fibered class. Conversely, it is easy to find examples that $C(M, \phi) = 1$ while ϕ is non-fibered. For example, take M to be any two-bridge-knot complement and ϕ the canonical class. The leading coefficient $C(M, \phi)$ can be interpreted as the relative L^2 -torsion of the guts $\Gamma(\phi)$ of ϕ [Dua25], which is a disjoint union of solid tori relative to an annulus on its boundary. The relative L^2 -torsion is trivial because the Fuglede–Kadison determinant

$$\det_{\mathcal{N}\pi_1(M)} : \text{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(t^\pm)^\times / \{\pm t^n \mid n \in \mathbb{Z}\} \rightarrow \mathbb{R}_+$$

is not a faithful map. This indicates that one should investigate the universal L^2 -torsion of a sutured 3-manifold which is defined algebraically and need not taking Fuglede–Kadison determinants. Accordingly, the leading term map $L_\phi : \text{Wh}^w(\pi_1(M)) \rightarrow \text{Wh}^w(\pi_1(M))$ is an algebraic description of “taking leading coefficients” in the weak Whitehead group $\text{Wh}^w(\pi_1(M))$.

1.1.3. Analogy with sutured Floer homology. Heegaard Floer homology, defined by Ozsváth and Szabó [OS04] for closed 3-manifolds, was generalized by Juhász [Juh06] to sutured Floer homology (SFH), an invariant for balanced sutured manifolds. In [Juh08] the fundamental properties of SFH are proved:

- (1) If (M, γ) is a taut balanced sutured manifold, then $SFH(M, \gamma)$ is non-trivial.
- (2) If $(M, \gamma) \rightsquigarrow (M', \gamma')$ is a sutured decomposition of balanced sutured manifolds, then $SFH(M', \gamma')$ is a direct summand of $SFH(M, \gamma)$.
- (3) A taut balanced sutured manifold (M, γ) is a product sutured manifold if and only if $SFH(M, \gamma) = \mathbb{Z}$.

We find counterparts of these three properties in the theory of universal L^2 -torsions. Namely, property (1) corresponds to Herrmann’s result (see Theorem 5.6), and (2)–(3) corresponds to Theorems 1.3–1.4.

Question 1.6. *Are there connections between sutured Floer homology and the universal L^2 -torsion of a taut sutured manifold?*

This question is particularly interesting because SFH is defined holomorphically and is not directly related to the fundamental group, whereas the universal L^2 -torsion is inherently algebraic and computable from a presentation of the fundamental group.

Furthermore, Juhász [Juh10] showed that the rank of SFH serves as a complexity measure for sutured manifolds that decreases under non-trivial decompositions. This motivates a parallel question in the L^2 -setting:

Question 1.7. *Can one define a natural complexity function on the weak Whitehead group that decreases under the leading term map L_ϕ ?*

1.2. Proof ingredients. The remaining part of the paper is divided into five sections. We briefly discuss the contents of each these.

1.2.1. Algebraic preliminaries. In Section 2, we recall the basic notions of L^2 -theory, focusing on three topics which will be of central importance throughout the paper: the Hilbert modules, the Atiyah Conjecture and the Linnell's skew field, K_1 -groups and the Dieudonné determinant.

1.2.2. Universal L^2 -torsion. In Section 3, we give the definition of the universal L^2 -torsion and discuss its computation. For any finite CW-complex X whose fundamental group G is torsion-free satisfying the Atiyah Conjecture, the universal L^2 -torsion $\tau_u^{(2)}(X)$ as living in the Whitehead group $\text{Wh}(\mathcal{D}_G)$, is defined to be the torsion of $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*(\widehat{X})$, a chain complex of vector spaces over Linnell's skew field \mathcal{D}_G . Slightly different from the original definition in [FL17], where the universal L^2 -torsion lives in the intricate weak K_1 -group, our definition is better adapted to algebraic manipulations yet more restrictive. A good news is that when X is a 3-manifold, or more generally when G lies in a large family of groups (Linnell's class \mathcal{C}) then two definitions coincide, see Section 3.1.1 for a discussion.

We define universal L^2 -torsion for CW-pairs and furthermore for continuous mappings between finite CW spaces by looking at its mapping cylinder. Several basic properties are established. This generalization brings greater flexibility to the application of the invariant.

Finally, we generalize Turaev's matrix chain method [Tur01] of directly computing the universal L^2 -torsion of a chain complex. Roughly speaking, a chain complex is acyclic if and only if we can find a chain of invertible submatrices from the matrices of the connecting operators. Moreover the torsion is given by the alternating product of their determinants.

1.2.3. Leading term map, restriction map and polytope map. The content of Section 4 is mostly technical where three important homomorphisms are studied. Given any real character $\phi : G \rightarrow \mathbb{R}$ and a nonzero element $a = \sum_{g \in G} n_g \cdot g \in \mathbb{Z}G$, the ϕ -leading term of a is defined to be the sum of nonzero terms $n_g \cdot g$ where $\phi(g)$ attains minimum. Provided with a knowledge of the crossed product structure of the field \mathcal{D}_G , this construction can be generalized to the leading term homomorphism

$$L_\phi : \mathcal{D}_G^\times \rightarrow \mathcal{D}_G^\times.$$

Furthermore, this homomorphism naturally induces homomorphisms on the K_1 -group and the Whitehead group of \mathcal{D}_G . The key result concerning the leading term map (Theorem 4.11) says that, roughly speaking, if given square matrices A, B over \mathcal{D}_G such that the ϕ -values of entries of B are strictly greater than that of A , then the ϕ -leading term of the determinant $\det(A + B)$ only depends on A . This intuitive result is surprisingly useful throughout our paper.

The restriction map connects the K_1 -group of a group G and its finite index subgroup $L < G$:

$$\text{res}_L^G : K_1(\mathcal{D}_G) \rightarrow K_1(\mathcal{D}_L).$$

This is done by viewing an element $z \in K_1(\mathcal{D}_G)$ as a operator $r_z : \ell^2(L)^{[G:L]} \rightarrow \ell^2(L)^{[G:L]}$ and then taking the determinant. It is proved in Theorem 4.16 using the aforementioned Theorem 4.11 that the leading term map commutes with the restriction map.

Every element $z \in \mathcal{D}_G^\times$ can be viewed as being “supported” on a certain Newton polytope in the real vector space $H_1(G; \mathbb{R})$. The polytope map

$$\mathbb{P} : \mathcal{D}_G^\times \rightarrow \mathcal{P}_{\mathbb{Z}}(H)$$

formalizes this intuition where H is the free abelianization of G and $\mathcal{P}_{\mathbb{Z}}(H)$ is the Grothendieck group of the integral polytopes in $\mathbb{R} \otimes_{\mathbb{Z}} H$ under the Minkowski sum. It is known that $\mathcal{P}_{\mathbb{Z}}(H)$ is a free abelian group possessing an explicit basis [Fun21]. The polytope map is useful when detecting nontrivial elements of $\mathcal{D}_G^\times / [\mathcal{D}_G^\times, \mathcal{D}_G^\times] = K_1(\mathcal{D}_G)$.

At the end of Section 4.3 we prove the “if” part of Theorem 1.2, namely Theorem 4.21. Here is the idea: the condition $L_\phi \tau_u^{(2)}(M) = 1$, when passing to finite index coverings, implies that the class ϕ is lifted to a top-dimensional cone no matter which covering we choose. We then make essential use of the Virtual Fibering Theorem on admissible 3-manifolds with NPC-metric (see [AFW15] and references therein) to conclude that ϕ itself is fibered.

1.2.4. Universal L^2 -torsion for taut sutured manifolds. In Section 5, we study the universal L^2 -torsion for sutured 3-manifolds, an object invented by Gabai [Gab83] to describe 3-manifolds in a cut-and-paste process. In order to prove the decomposition formula Theorem 1.3, we adopt the idea of Turaev’s algorithm [Tur02, BAFH22] to reduce to the case of non-separating decomposition surfaces. Under an explicit CW-structure, the chain complex of the cut up manifold is seen to be exactly the “leading term” of the chain complex of the original manifold. This idea is made rigorous by Theorem 5.13. After establishing the decomposition formula, the “only if” part of Theorem 1.2 follows easily, see Theorem 5.14.

A refined doubling trick (Lemma 5.16) allows one to convert a taut sutured manifold into an admissible 3-manifold which is not a closed graph manifold. A combination of the doubling trick, the criterion for fiberedness of admissible 3-manifolds (Theorem 1.2) and the decomposition formula (Theorem 1.3) shows that the universal L^2 -torsion detects product sutured manifolds (Theorem 1.4).

1.2.5. Applications of the universal L^2 -torsion. In Section 6 we define the universal L^2 -torsions for homomorphisms between finitely generated free groups. Moreover, an explicit formula is given in Proposition 6.2. We then apply this formula to sutured manifolds modeled on 3-dimensional handlebodies, obtaining criteria for tautness and product for such sutured manifolds (Proposition 6.5). Explicit computations are carried out for the n -chain link complement cut up along a Seifert surface (Example 6.7).

2. ALGEBRAIC PRELIMINARIES

2.1. Hilbert spaces associated to groups. Let G be a group. Consider the following Hilbert space

$$\ell^2(G) = \left\{ \sum_{g \in G} c_g \cdot g \mid c_g \in \mathbb{C}, \sum_{g \in G} |c_g|^2 < \infty \right\}$$

with inner product

$$\left\langle \sum_{g \in G} c_g \cdot g, \sum_{g \in G} d_g \cdot g \right\rangle = \sum_{g \in G} c_g \overline{d_g}.$$

This Hilbert space has a natural left and right isometric G -action by multiplications. Every G -invariant closed subspace V of $\ell^2(G)^n$ can be assigned the von-Neumann dimension $\dim_{\mathcal{N}G} V$ which takes real value in $[0, n]$.

Let \mathcal{UG} be the set of all densely-defined, closed operators (possibly unbounded) on $\ell^2(G)$ that commutes with the left G -action. The composition and addition of two operators in \mathcal{UG} is well-defined [Lü02, Section 8.1], hence \mathcal{UG} forms a \mathbb{C} -algebra and is called the affiliated algebra of G .

In particular, we have the following inclusion relations

$$\mathbb{Z}G \subset \mathcal{U}G$$

where the integral group ring $\mathbb{Z}G$ embeds into $\mathcal{N}G$ by the right regular representation. Moreover, any matrix A over $\mathbb{Z}G$ of size $m \times n$ can be viewed as an G -invariant operator $r_A : \ell^2(G)^m \rightarrow \ell^2(G)^n$ by right multiplication.

Definition 2.1 (Weak isomorphism). A G -invariant operator $f : \ell^2(G)^m \rightarrow \ell^2(G)^n$ is called a weak isomorphism if f is injective and whose image is dense. Note that if f is a weak isomorphism then m and n must be equal. A square matrix A over $\mathbb{Z}G$ is called a weak isomorphism if the G -invariant operator $r_A : \ell^2(G)^m \rightarrow \ell^2(G)^n$ is a weak isomorphism.

2.2. Atiyah Conjecture and Linnell's skew-field.

Definition 2.2 (Atiyah Conjecture). A group is said to satisfy the Atiyah Conjecture if for any matrix $A^{m \times n}$ over $\mathbb{Z}G$ the von Neumann dimension of $\ker(r_A)$ is an integer.

The Atiyah Conjecture has been verified for a large class of groups. We mark the following class of groups given by Linnell, which is large enough to include all 3-manifold groups.

Theorem 2.3 ([Lin93]). *Let \mathcal{C} be the smallest class of groups which contains all free groups and is closed under directed unions and extensions by elementary amenable groups. Then any torsion-free group in \mathcal{C} satisfies the Atiyah Conjecture.*

Theorem 2.4 ([KL24, Theorem 1.4]). *The fundamental group of any connected 3-manifold lies in \mathcal{C} .*

Definition 2.5 (Division closure). Let R be a subring of a ring S . The division closure of R in S is the smallest subring $\mathcal{D}(R \subset S)$ of S containing R such that if an element of R is invertible in S , then it is also invertible in $\mathcal{D}(R \subset S)$. Let G be a group, define \mathcal{D}_G to be the division closure of $\mathbb{Z}G$ in $\mathcal{U}G$.

Theorem 2.6 (Linnell). *A torsion-free group G satisfies the Atiyah Conjecture if and only if \mathcal{D}_G is a skew-field.*

Proposition 2.7 ([Kie20, Proposition 4.6]). *Let G be a torsion-free group satisfying the Atiyah Conjecture. Then the following statements hold.*

- (1) *The involution on $\mathbb{Z}G$ naturally extends to an involution on \mathcal{D}_G .*
- (2) *Every automorphism of the group G extends to an automorphism of \mathcal{D}_G .*
- (3) *If K is a subgroup of G , then K satisfies the Atiyah Conjecture. Moreover, the natural embedding $\mathbb{Z}K \hookrightarrow \mathbb{Z}G$ extends to an embedding $\mathcal{D}_K \hookrightarrow \mathcal{D}_G$.*

2.3. The K_1 -group and Dieudonné determinant. Let R be an associative ring with identity. For any positive integer n let $GL(n, R)$ be the group of invertible $(n \times n)$ -matrices over R . Identifying each $M \in GL(n, R)$ with the matrix

$$\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL(n+1, R)$$

we obtain inclusions

$$GL(1, R) \subset GL(2, R) \subset \dots$$

The union $GL(R) = \bigcup_{n \geq 1} GL(n, R)$ is called the *infinite general linear group*. The K_1 -group of R is defined to be the abelianization of the group $GL(R)$. We remark that a classical result of Whitehead says that the commutator subgroup $[GL(R), GL(R)]$ is exactly the subgroup generated by all elementary matrices in $GL(R)$, see for example [Mil66].

The *reduced K_1 -group* $\widetilde{K}_1(R)$ is the quotient of $K_1(R)$ by the subgroup $\{[\pm 1]\}$. The *Whitehead group* $\text{Wh}(G)$ of any group G is defined to be

$$\text{Wh}(G) := K_1(\mathbb{Z}G) / \{[\pm g] \mid g \in G\}.$$

When $R = \mathcal{D}$ is a skew-field, the Dieudonné determinant given by [Die43] is the unique map

$$\det : GL(\mathcal{D}) \rightarrow \mathcal{D}^\times / [D^\times, D^\times]$$

satisfying following properties (see [Ros95, Theorem 2.2.5]):

- (a) The determinant is invariant under (left) elementary row operations. In other words, if $A \in GL(\mathcal{D})$ and A' is obtained from A by adding a (left-)multiple of a row to another row, then $\det A = \det A'$.
- (b) If $A \in GL(\mathcal{D})$, and A' is obtained from A by (left-)multiplying one of the rows by $a \in \mathcal{D}$, then $\det A = \bar{a} \cdot \det A'$ where \bar{a} is the image of a in $D^\times / [D^\times, D^\times]$.
- (c) The determinant of the identity matrix is 1.

The determinant also has the following additional properties.

- (d) If $A, B \in GL(n, \mathcal{D})$, then $\det(AB) = \det A \cdot \det B$.
- (e) If $A \in GL(\mathcal{D})$ and A' is obtained from A by interchanging two of its rows, then $\det A' = (-\bar{1}) \det A$.
- (f) The determinant is invariant under taking transpose.

The Dieudonné determinant factors through the abelianization of $GL(\mathcal{D})$ and induces $\overline{\det} : K_1(\mathcal{D}) \rightarrow \mathcal{D}^\times / [D^\times, D^\times]$.

Lemma 2.8 ([Ros95, Corollary 2.2.6]). *For any skew field \mathcal{D} , the Dieudonné determinant $\overline{\det}$ induces an group isomorphism*

$$K_1(\mathcal{D}) \xrightarrow{\cong} \mathcal{D}^\times / [D^\times, D^\times].$$

The inverse map is given by viewing an element $a \in \mathcal{D}^\times / [D^\times, D^\times]$ as an (1×1) -matrix $[a] \in K_1(\mathcal{D})$.

3. UNIVERSAL L^2 -TORSION

Let G be a finitely generated torsion-free group satisfying the Atiyah conjecture and let \mathcal{D}_G be the Linnell's skew field. The universal L^2 -torsion defined in this section will be living in the abelian groups $\widetilde{K}_1(\mathcal{D}_G)$ (for $\mathbb{Z}G$ -chain complexes) or $\text{Wh}(\mathcal{D}_G)$ (for finite CW-complexes with fundamental group G). While the original universal L^2 -torsion defined in [FL17] lives in $\widetilde{K}_1(\mathbb{Z}G)$ and $\text{Wh}^w(G)$, we will see that our definition is compatible with theirs by the natural homomorphisms

$$\tilde{i}_G : \widetilde{K}_1^w(\mathbb{Z}G) \rightarrow \widetilde{K}_1(\mathcal{D}_G), \quad i_G^w : \text{Wh}^w(G) \rightarrow \text{Wh}(\mathcal{D}_G)$$

described in the above Proposition 3.8. In particular, when G lies in Linnell's class \mathcal{C} , then our definition of the universal L^2 -torsions is identical with that in [FL17], see Section 3.1.1 for a detailed discussion.

3.1. Universal L^2 -torsion of chain complexes. A chain complex C_* is called a finite based free $\mathbb{Z}G$ -chain complex if there exists $n \geq 0$ such that

$$C_* = (0 \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0),$$

where each C_k is a finitely generated free (left) $\mathbb{Z}G$ -module equipped with a preferred (unordered) free $\mathbb{Z}G$ -basis, and the boundary operators are $\mathbb{Z}G$ -linear maps.

Definition 3.1. A finite based free $\mathbb{Z}G$ -chain complex C_* is said to be L^2 -acyclic if the chain complex $\ell^2(G) \otimes_{\mathbb{Z}G} C_*$ is weakly exact, i.e. the image $\text{im } \partial_{k+1}^{(2)}$ is dense in $\ker \partial_k^{(2)}$ for all k , where $\partial_k^{(2)} : \ell^2(G) \otimes_{\mathbb{Z}G} C_k \rightarrow \ell^2(G) \otimes_{\mathbb{Z}G} C_{k-1}$ is the boundary operator.

We will not work with those analytic-flavored definitions but prefer the more algebraic-flavored ones given by the following Lemma.

Lemma 3.2 ([FL17, Lemma 2.21]). *A square matrix over $\mathbb{Z}G$ is a weak isomorphism if and only if it is invertible over \mathcal{D}_G . A finite based free $\mathbb{Z}G$ -chain complex C_* is L^2 -acyclic if and only if the chain complex $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*$ is exact as a chain complex of left \mathcal{D}_G -modules.*

It is classical to define the torsion of an exact chain complex of free modules, see for example [Mil66, Coh73, Tur01]. The situation here is particularly nice because \mathcal{D}_G is a skew-field and any module over a skew field is free. We adopt the definitions in [Tur01, Section 3].

Definition 3.3. Let \mathcal{D} be a skew-field and let V be a finitely generated left \mathcal{D} -module. Suppose $\dim V = k$ and pick two (unordered) bases $b = \{b_1, \dots, b_k\}$ and $c = \{c_1, \dots, c_k\}$. Then

$$b_i = \sum_{j=1}^k a_{ij} c_j, \quad i = 1, \dots, k,$$

where $(a_{ij})_{i,j=1,\dots,k}$ is a non-degenerate k -by- k matrix. Define $[b/c] \in \widetilde{K}_1(\mathcal{D})$ to be the element represented by the matrix (a_{ij}) . This definition does not depend on the ordering of bases b, c .

Definition 3.4 (Universal L^2 -torsion of chain complexes). Assume that C_* is a finite based $\mathbb{Z}G$ -chain complex of length n which is L^2 -acyclic, then the chain complex $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*$ is an exact \mathcal{D}_G -chain complex, with c_i the preferred basis of $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_i$. Let ∂ be the boundary homomorphism and pick bases b_i of the free \mathcal{D}_G -module $B_i := \text{im } \partial_i$. Combine them to bases $b_i b_{i-1}$ of C_i . Then the universal L^2 -torsion of C_* is defined to be

$$\tau_u^{(2)}(C_*) := \prod_{i=0}^n [b_i b_{i-1} / c_i]^{(-1)^{i+1}} \in \widetilde{K}_1(\mathcal{D}_G).$$

This definition does not depend on the choice of basis b_i .

An exact sequence $0 \rightarrow M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \rightarrow 0$ of finitely generated based free $\mathbb{Z}G$ -modules is called based exact, if $i(b_0) \subset b_1$ and p maps $b_1 \setminus i(b_0)$ bijectively to b_2 , where b_i is the preferred basis of M_i , $i = 0, 1, 2$. An exact sequence $0 \rightarrow C_* \rightarrow D_* \rightarrow E_* \rightarrow 0$ of finite based free $\mathbb{Z}G$ -chain complex is called based exact if

$$0 \rightarrow C_k \rightarrow D_k \rightarrow E_k \rightarrow 0$$

is based exact for all k . The following basic property can be found in [Tur01, Theorem 3.4].

Proposition 3.5. *The universal L^2 -torsion for chain complexes have the following properties.*

- (1) *For any L^2 -acyclic finite based free $\mathbb{Z}G$ -chain complex*

$$C_* = (0 \rightarrow C_1 \xrightarrow{A} C_0 \rightarrow 0)$$

where C_0 and C_1 are isomorphic to $\mathbb{Z}G^r$ under the preferred basis and A is a square matrix A over $\mathbb{Z}G$. Then $\tau_u^{(2)}(C_) = [A]^{-1} \in \widetilde{K}_1(\mathcal{D}_G)$.*

- (2) *If C_*, C'_*, C''_* are finite based free $\mathbb{Z}G$ -chain complexes and there is a based short exact sequence*

$$0 \rightarrow C'_* \rightarrow C_* \rightarrow C''_* \rightarrow 0.$$

If C'_ and C''_* are L^2 -acyclic then C_* is L^2 -acyclic and*

$$\tau_u^{(2)}(C_*) = \tau_u^{(2)}(C'_*) \cdot \tau_u^{(2)}(C''_*) \in \widetilde{K}_1(\mathcal{D}_G).$$

Recall that $\mathbb{Z}G$ is a ring with an involution $x \mapsto \bar{x}$ which sends $\sum n_g \cdot g$ to $\sum n_g \cdot g^{-1}$. Given any left $\mathbb{Z}G$ -module A , define the dual module A^* to be $\text{Hom}_{\mathbb{Z}G}(A, \mathbb{Z}G)$, considered as a left $\mathbb{Z}G$ -module as follows. For each $x \in \mathbb{Z}G$ and $f : A \rightarrow \mathbb{Z}G$ define $xf : A \rightarrow \mathbb{Z}G$ by the formula $(xf)(y) = f(y) \cdot \bar{x}$, $\forall y \in A$. If A is a free $\mathbb{Z}G$ -module with basis a_i , then A^* is a free $\mathbb{Z}G$ -module with basis a_i^* . Suppose $f : A \rightarrow B$ is a $\mathbb{Z}G$ -linear map between two based free $\mathbb{Z}G$ -modules represented by a matrix P under the given bases, then the dual map $f^* : B^* \rightarrow A^*$ is represented by the matrix P^* , the involution transpose of P . The following Proposition 3.6 is a classic property of torsion invariants and can be proved as in [Mil62].

Proposition 3.6. *If $C_* = (0 \rightarrow C_n \rightarrow \cdots \rightarrow C_0 \rightarrow 0)$ is a finite based free $\mathbb{Z}G$ -chain complex which is L^2 -acyclic. Then the dual chain complex C^* is L^2 -acyclic and*

$$\overline{\tau_u^{(2)}(C^*)} = \tau_u^{(2)}(C_*)^{(-1)^{n+1}} \in \widetilde{K}_1(\mathcal{D}_G).$$

3.1.1. *Connection with Friedl–Lück’s definition.* Definition 3.4 of the universal L^2 -torsion slightly differs from that of [FL17]. We first recall the definition of the weak K_1 -groups introduced by [FL17], which is an abelian group analogous to the classical K_1 -group, but rather than requiring the matrices over $\mathbb{Z}G$ to be invertible, we only require that they are weak isomorphisms (recall Definition 2.1).

Definition 3.7 ($K_1^w(\mathbb{Z}G)$, $\widetilde{K}_1^w(\mathbb{Z}G)$ and $\text{Wh}^w(G)$). Suppose G is any group. The weak K_1 -group $K_1^w(\mathbb{Z}G)$ is defined in terms of generators and relations as follows. Generators $[A]$ are given by square matrices A over $\mathbb{Z}G$ such that A is a weak isomorphism. There are two sets of relations:

- (i) If A, B are square matrices over $\mathbb{Z}G$ of the same size such that A, B are weak isomorphisms, then

$$[AB] = [A] \cdot [B].$$

- (ii) If A, B are square matrices over $\mathbb{Z}G$ of size n and m , respectively. Suppose A, B are weak isomorphisms and let C be any matrix over $\mathbb{Z}G$ of size $n \times m$. Then

$$\left[\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right] = [A] \cdot [B] = [B] \cdot [A].$$

The reduced weak K_1 -group $\widetilde{K}_1^w(\mathbb{Z}G)$ is defined to be the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup $\{[1], [-1]\}$. The weak Whitehead group $\text{Wh}^w(G)$ is defined to be the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup $\{[\pm g] \mid g \in G\}$.

Proposition 3.8. *Suppose G is a torsion-free group which satisfies the Atiyah Conjecture.*

- (1) *There are natural homomorphisms*

$$i_G : K_1^w(\mathbb{Z}G) \rightarrow K_1(\mathcal{D}_G), \quad \tilde{i}_G : \widetilde{K}_1^w(\mathbb{Z}G) \rightarrow \widetilde{K}_1(\mathcal{D}_G), \quad i_G^w : \text{Wh}^w(G) \rightarrow \text{Wh}(\mathcal{D}_G).$$

- (2) *If G falls into the Linnell’s class \mathcal{C} (which in particular contains all 3-manifold groups), then the three homomorphisms above are all isomorphisms.*

Proof. By Lemma 3.2 below, a square matrix A over $\mathbb{Z}G$ is a weak isomorphism if and only if it is invertible over \mathcal{D}_G . The homomorphism in (1) is given by viewing a weak isomorphism A over $\mathbb{Z}G$ as a matrix over \mathcal{D}_G . (2) is a highly non-trivial fact proven in [LL17]. \square

For any group G and any L^2 -acyclic finite based $\mathbb{Z}G$ -chain complex C_* , the universal L^2 -torsion $\rho_u^{(2)}(C_*)$ defined in [FL17] lives in the reduced weak K_1 group $\widetilde{K}_1^w(\mathbb{Z}G)$. Proposition 3.5 holds true for $\rho_u^{(2)}$ and moreover characterizes the universal property of $\rho_u^{(2)}$. Therefore when G is torsion-free and satisfies the Atiyah Conjecture, then $\tau_u^{(2)}(C_*)$ defined in the present paper is the image of $\rho_u^{(2)}(C_*)$ under the natural map

$$\tilde{i}_G : \widetilde{K}_1^w(\mathbb{Z}G) \rightarrow \widetilde{K}_1(\mathcal{D}_G).$$

If furthermore G lies in Linnell's class \mathcal{C} then \tilde{i}_G is a canonical isomorphism and the definitions of $\tau_u^{(2)}$ and $\rho_u^{(2)}$ are equivalent by Proposition 3.8.

3.2. Universal L^2 -torsion of CW-complexes. Let X be a connected finite CW-complex with fundamental group G and let $Y \subset X$ be a subcomplex. Let $p : \widehat{X} \rightarrow X$ be the universal covering of X , and let $\widehat{Y} := p^{-1}(Y)$ be the preimage. Then \widehat{X} admits the induced CW-structure and \widehat{Y} is a subcomplex of \widehat{X} . The natural left G -action on \widehat{X} gives rise to the left $\mathbb{Z}G$ -module structure on the cellular chain complex $C_*(\widehat{X}, \widehat{Y})$. By choosing a lift $\hat{\sigma}$ for each cell σ in $X \setminus Y$, we find a free $\mathbb{Z}G$ -basis for each $\mathbb{Z}G$ -module $C_k(\widehat{X}, \widehat{Y})$. Therefore $C_*(\widehat{X}, \widehat{Y})$ becomes a finite based free $\mathbb{Z}G$ -chain complex. The following definition does not depend on the choice of the lifting of cells.

Definition 3.9. Let X be a finite connected CW-complex with fundamental group G and let Y be a subcomplex of X . The pair (X, Y) is called L^2 -acyclic if the finite based free chain complex $C_*(\widehat{X}, \widehat{Y})$ is L^2 -acyclic (c.f. Lemma 3.2). The universal L^2 -torsion

$$\tau_u^{(2)}(X, Y) \in \text{Wh}(\mathcal{D}_G) \sqcup \{0\}$$

is defined as follows: if (X, Y) is L^2 -acyclic, define $\tau_u^{(2)}(X, Y)$ to be the image of $\tau_u^{(2)}(C_*(\widehat{X}, \widehat{Y}))$ under the projection $\widetilde{K}_1(\mathcal{D}_G) \rightarrow \text{Wh}(\mathcal{D}_G)$; if (X, Y) is not L^2 -acyclic, define $\tau_u^{(2)}(X, Y) := 0$.

When X is not necessarily connected, we say (X, Y) is L^2 -acyclic if for every component $X_i \in \pi_0(X)$ the pair $(X_i, X_i \cap Y)$ is L^2 -acyclic. Furthermore, if the fundamental groups $\pi_1(X_i)$ satisfy the Atiyah conjecture, then we define

$$\begin{aligned} \text{Wh}(\mathcal{D}_{\Pi(X)}) &:= \bigoplus_{X_i \in \pi_0(X)} \text{Wh}(\mathcal{D}_{\pi_1(X_i)}), \\ \tau_u^{(2)}(X, Y) &:= (\tau_u^{(2)}(X_i, X_i \cap Y))_{X_i \in \pi_0(X)} \in \text{Wh}(\mathcal{D}_{\Pi(X)}). \end{aligned}$$

If $(X_i, X_i \cap Y)$ is not L^2 -acyclic for some i , then define $\tau_u^{(2)}(X, Y) := 0$.

Assume that (X, Y) and (X', Y') are finite CW-pairs. We say a CW-mapping $f : (X', Y') \rightarrow (X, Y)$ is π_1 -injective, if the restriction of f to each component of X' induces an injection on fundamental groups. In this case, there is a natural homomorphism

$$\iota_* : \text{Wh}(\mathcal{D}_{\Pi(X')}) \sqcup \{0\} \rightarrow \text{Wh}(\mathcal{D}_{\Pi(X)}) \sqcup \{0\}.$$

Define

$$\iota_* \tau_u^{(2)}(X', Y') \in \text{Wh}(\mathcal{D}_{\Pi(X)}) \sqcup \{0\}$$

to be the image of $\tau_u^{(2)}(X, Y)$ under the homomorphism ι_* .

Theorem 3.10. *We record the fundamental properties of the universal L^2 -torsion.*

- (1) (Simple-homotopy invariance) *Suppose (X, X_0) and (Y, Y_0) are CW-pairs. Let $f : (X, X_0) \rightarrow (Y, Y_0)$ be a mapping such that $f : X \rightarrow Y$ and $f|_{X_0} : X_0 \rightarrow Y_0$ are simple-homotopy equivalences. Then $\tau_u^{(2)}(Y, Y_0) = f_* \tau_u^{(2)}(X, X_0)$. In particular, (X, X_0) is L^2 -acyclic if and only if (Y, Y_0) is L^2 -acyclic.*
- (2) (Sum formula) *Let $(U, V) = (X, C) \cup (Y, D)$ where (X, C) , (Y, D) and $(X \cap Y, C \cap D)$ are L^2 -acyclic sub-pairs that embeds π_1 -injectively into (U, V) , then*

$$\tau_u^{(2)}(U, V) = (\iota_1)_* \tau_u^{(2)}(X, C) \cdot (\iota_2)_* \tau_u^{(2)}(Y, D) \cdot (\iota_3)_* \tau_u^{(2)}(X \cap Y, C \cap D)^{-1}$$

where ι_i , $i = 1, 2, 3$ are the embeddings of the corresponding space pairs into (U, V) .

- (3) (Induction) Let $f : (X_0, Y_0) \subset (X, Y)$ be a π_1 -injective inclusion. Let \widehat{X} be the universal cover of X and let $\widehat{X}_0, \widehat{Y}_0$ be the preimage of X_0, Y_0 in \widehat{X} . Then the finite based free $\mathbb{Z}[\pi_1(X)]$ -chain complex $C_*(\widehat{X}_0, \widehat{Y}_0)$ is L^2 -acyclic if and only if (X_0, Y_0) is L^2 -acyclic. Moreover, we have

$$\tau_u^{(2)}(C_*(\widehat{X}_0, \widehat{Y}_0)) = f_* \tau_u^{(2)}(X_0, Y_0).$$

- (4) (Restriction) Let X be a connected finite CW-complex and let \overline{X} be a connected finite degree covering of X . Suppose that $\pi_1(X) = G$ and $\pi_1(\overline{X}) = H$ and recall the restriction map $\text{res}_H^G : \text{Wh}(\mathcal{D}_G) \rightarrow \text{Wh}(\mathcal{D}_H)$ defined in Section 4.4. Let $Y \subset X$ be a subcomplex and let \overline{Y} be its preimage in \overline{X} . Then

$$\tau_u^{(2)}(\overline{X}, \overline{Y}) = \text{res}_H^G \tau_u^{(2)}(X, Y).$$

Proof. The first one will be proved in Remark 3.15 after we introduced the notion of universal L^2 -torsion of mappings. Properties (2)–(4) are natural generalization of [FL17, Theorem 3.5] to CW-pairs and the proof carry over without essential changes to the relative cases. \square

3.3. Universal L^2 -torsion of mappings.

Definition 3.11 (Universal L^2 -torsion of mappings). Let X, Y be finite CW-complexes. Given a cellular map $f : Y \rightarrow X$, form the mapping cylinder

$$M_f := ((Y \times I) \sqcup X) / \sim, \quad \text{where } (y, 0) \sim f(y) \text{ for all } y \in Y.$$

View $Y = Y \times \{1\}$ as a subcomplex of M_f . If the fundamental group of X satisfies the Atiyah Conjecture, then the *universal L^2 -torsion of the mapping f* is defined to be

$$\tau_u^{(2)}(f) := \iota_* \tau_u^{(2)}(M_f, Y) \in \text{Wh}(\mathcal{D}_{\Pi(X)}) \sqcup \{0\}$$

where $\iota : M_f \rightarrow X$ is the natural deformation retraction. The mapping f is called an *L^2 -weak homotopy equivalence* if (M_f, Y) is L^2 -acyclic, or equivalently, if $\tau_u^{(2)}(f) \neq 0$.

Proposition 3.12. Suppose the spaces X, Y, Z are finite CW-complexes whose fundamental groups satisfy the Atiyah Conjecture.

- (1) If (X, Y) is a CW-pair with $f : Y \rightarrow X$ the inclusion map. Then $\tau_u^{(2)}(X, Y) = \tau_u^{(2)}(f)$.
- (2) If $f, g : X \rightarrow Z$ are homotopic cellular maps. Then $\tau_u^{(2)}(f) = \tau_u^{(2)}(g)$.
- (3) If $f : X \rightarrow Z$ is a simple-homotopy equivalence, then $\tau_u^{(2)}(f) = 1$.
- (4) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are L^2 -weak homotopy equivalences. Suppose that g is π_1 -injective, then $g \circ f$ is an L^2 -weak homotopy equivalence with $\tau_u^{(2)}(g \circ f) = g_* \tau_u^{(2)}(f) \cdot \tau_u^{(2)}(g)$.

Proof. Following [Coh73], we write $K \curvearrowright L$ if two finite CW-complexes K and L are related by a finite sequence of elementary collapses or expansions; if there is a common subcomplex K_0 that no cells are removed during the process, then we write $K \curvearrowright L \text{ rel } K_0$. In this case, it is clear that $\tau_u^{(2)}(K, K_0) = \iota_* \tau_u^{(2)}(L, K_0)$ where $\iota : L \rightarrow K$ is the natural homotopy equivalence.

For (1), there are elementary expansions $M_f \curvearrowright X \times I$ and elementary collapses $X \times I \curvearrowright X \times \{1\}$ relative to $Y = Y \times \{1\}$. Let $\iota : M_f \rightarrow X$ be the deformation retract, then $\tau_u^{(2)}(f) = \iota_* \tau_u^{(2)}(M_f, Y) = \tau_u^{(2)}(X, Y)$. (2) follows from the fact that $M_f \curvearrowright M_g \text{ rel } X$ [Coh73, (5.5)]. For (3), if $f : X \rightarrow Z$ is a simple-homotopy equivalence then $M_f \curvearrowright X \text{ rel } X$ [Coh73, (5.8)].

For the proof of (4) we need the following “ L^2 -excision” property.

Lemma 3.13 (Excision). If K, L and M are subcomplexes of the complex $K \cup L$ with $M = K \cap L$. Suppose the inclusion $i : K \hookrightarrow K \cup L$ is π_1 -injective. Then $\tau_u^{(2)}(K \cup L, L) = i_* \tau_u^{(2)}(K, M)$.

Proof. As in the proof of [Coh73, (20.3)], we may assume that L and $K \cup L$ are connected. Let $\widehat{K \cup L}$ be the universal covering of $K \cup L$ and let \widehat{L} , \widehat{K} and \widehat{M} be the preimage under the covering. Then there is an isomorphism of chain complexes $C_*(\widehat{K \cup L}, \widehat{L}) = C_*(\widehat{K}, \widehat{M})$ and hence $\tau_u^{(2)}(K \cup L, L) = \tau_u^{(2)}(C_*(\widehat{K}, \widehat{M})) = j_* \tau_u^{(2)}(K, M)$ where the second identity follows from the induction property (see Theorem 3.10). \square

The proof of Proposition 3.12(4) proceeds as follows. Let M be the union of M_f and M_g along the identity map on Y . Then $M \curvearrowright M_{g \circ f} \text{ rel } X \cup Z$ by [Coh73, (5.6)]. There is a commutative diagram

$$\begin{array}{ccccc} M_f & \xrightarrow{\iota_f} & Y & \xhookrightarrow{i} & M_g \\ \downarrow i_1 & & & \nearrow i_2 & \downarrow \iota_g \\ M & \xleftarrow{\iota} & & & Z \end{array}$$

where i_1, i_2 are inclusions, ι, ι_1, ι_2 are natural deformation retracts. Then we have

$$\begin{aligned} \tau_u^{(2)}(g \circ f) &= \iota_* \tau_u^{(2)}(M, X) \\ &= \iota_*(\tau_u^{(2)}(M, M_f) \cdot (i_1)_* \tau_u^{(2)}(M_f, X)), \quad \text{by sum formula} \\ &= \iota_*((i_2)_* \tau_u^{(2)}(M_g, Y) \cdot (i_1)_* \tau_u^{(2)}(M_f, X)), \quad \text{by excision} \\ &= (\iota_g)_* \tau_u^{(2)}(M_g, Y) \cdot g_*(\iota_f)_* \tau_u^{(2)}(M_f, X), \quad \text{note that } \iota_g \circ i = g \\ &= \tau_u^{(2)}(g) \cdot g_* \tau_u^{(2)}(f). \end{aligned}$$

The proof is finished. \square

The identity $\tau_u^{(2)}(g \circ f) = g_* \tau_u^{(2)}(f) \cdot \tau_u^{(2)}(g)$ is called the multiplicativity of the universal L^2 -torsion. The conditions that f, g are L^2 -weak homotopy equivalences and g is π_1 -injective is in general necessary. But when one of the mappings is a simple homotopy equivalence then the conditions can be relaxed as follows.

Lemma 3.14. *Suppose X, Y, Z, W are finite CW-complexes whose fundamental groups satisfy the Atiyah Conjecture. Consider the chain of mappings $X \xrightarrow{h} Y \xrightarrow{f} Z \xrightarrow{g} W$. Suppose $f : Y \rightarrow Z$ is a simple homotopy equivalence. Then*

- (1) *A mapping $h : X \rightarrow Y$ is an L^2 -weak homotopy equivalence if and only if $f \circ h$ is an L^2 -weak homotopy equivalence. Moreover $\tau_u^{(2)}(f \circ h) = f_* \tau_u^{(2)}(h)$.*
- (2) *A mapping $g : Z \rightarrow W$ is an L^2 -weak homotopy equivalence if and only if $g \circ f$ is an L^2 -weak homotopy equivalence. Moreover $\tau_u^{(2)}(g \circ f) = \tau_u^{(2)}(g)$.*

Proof. The key observation is that a simple homotopy equivalence f admits an inverse f^{-1} which is also a simple homotopy equivalence. For (1), the forward direction follows from Proposition 3.12(4). For the inverse direction note that $h \simeq f^{-1} \circ (f \circ h)$. The identity $\tau_u^{(2)}(f \circ h) = f_* \tau_u^{(2)}(h)$ follows again from Proposition 3.12(4).

For (2), suppose g is an L^2 -weak homotopy equivalence. Let M be the union of M_f and M_g along the identity map of Y . Then $M \curvearrowright M_{g \circ f} \text{ rel } X \cup Z$. The Excision Lemma 3.13 applied to $M_f \subset M$ and $i : M_g \subset M$ shows that (M, M_f) is L^2 -acyclic and $\tau_u^{(2)}(M, M_f) = i_* \tau_u^{(2)}(M_g, Y)$. Since f is a simple homotopy equivalence we have $M_f \curvearrowright X \text{ rel } X$. Let $\iota : M \rightarrow Z$ be the deformation retract. Then

$$\tau_u^{(2)}(g \circ f) := \iota_* \tau_u^{(2)}(M, X) = \iota_* \tau_u^{(2)}(M, M_f) = (\iota \circ i)_* \tau_u^{(2)}(M_g, Y) = \tau_u^{(2)}(g).$$

Therefore $g \circ f$ is an L^2 -weak homotopy equivalence and the identity $\tau_u^{(2)}(g \circ f) = \tau_u^{(2)}(g)$ holds. The inverse direction is proved the same way, noting that $g \simeq (g \circ f) \circ f^{-1}$. \square

Remark 3.15. As a corollary of Lemma 3.14, we prove the simple-homotopy invariance stated in Theorem 3.10. Let $f : (X, X_0) \rightarrow (Y, Y_0)$ be a mapping of CW-pairs such that $f : X \rightarrow Y$ and $f|_{X_0} : X_0 \rightarrow Y_0$ are simple-homotopy equivalences. Then we have the following commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ i_X \downarrow & & \downarrow i_Y \\ X & \xrightarrow{f} & Y \end{array}$$

Then: (X, X_0) is L^2 -acyclic $\Leftrightarrow i_X$ is an L^2 -homotopy equivalence $\Leftrightarrow f \circ i_X$ is an L^2 -homotopy equivalence $\Leftrightarrow i_Y \circ f_0$ is an L^2 -homotopy equivalence $\Leftrightarrow i_Y$ is an L^2 -homotopy equivalence $\Leftrightarrow (Y, Y_0)$ is L^2 -acyclic. Moreover, we have

$$f_* \tau_u^{(2)}(X, X_0) = f_* \tau_u^{(2)}(i_X) = \tau_u^{(2)}(f \circ i_X) = \tau_u^{(2)}(i_Y \circ f_0) = \tau_u^{(2)}(i_Y) = \tau_u^{(2)}(Y, Y_0).$$

3.4. Universal L^2 -torsion of manifolds. We define the universal L^2 -torsion for smooth manifold pairs as follows. Recall that a smooth triangulation of a smooth manifold M is a homeomorphism from a simplicial complex to M whose restriction to each simplex is smooth.

Definition 3.16 (Universal L^2 -torsion of manifold pairs). Let M be a compact, smooth manifold, possibly with boundary, and let N be a compact, smooth submanifold of M . Suppose that there is a smooth triangulation of M such that N is a subcomplex of M . Then we use the triangulation to identify (M, N) with a CW-pair (X, Y) and define $\tau_u^{(2)}(M, N) := \tau_u^{(2)}(X, Y)$.

For the purpose of this paper, assume that either N is a zero-codimensional submanifold of ∂M , or the embedding $N \hookrightarrow M$ is proper (i.e. $N \cap \partial M = \partial N$). In these cases one can find a smooth triangulation of M such that N is a subcomplex of M (see [Mun66, Chapter 10]). Any two such triangulations have a common subdivision and are simple homotopy equivalent as CW-complexes [Whi40]. Therefore $\tau_u^{(2)}(M, N)$ is well-defined by simple homotopy invariance of the universal L^2 -torsion (see Theorem 3.10).

Definition 3.17 (Universal L^2 -torsion of mappings between manifolds). Suppose $f : N \rightarrow M$ is a continuous mapping between compact smooth manifolds (possibly with boundaries) M, N . Choose any smooth triangulations of M, N and choose a simplicial mapping g homotopic to f . We say f is an L^2 -weak homotopy equivalence if g is an L^2 -weak homotopy equivalence. In this case, define the universal L^2 -torsion of f as

$$\tau_u^{(2)}(f) := \tau_u^{(2)}(g) \in \text{Wh}(\mathcal{D}_{\Pi(M)}) \sqcup \{0\}.$$

It follows from and Proposition 3.12 and Lemma 3.14 that $\tau_u^{(2)}(f)$ does not depend on the choice of triangulations on M, N or the simplicial approximation g . When $N \subset M$ is a smooth submanifold with f the inclusion map, then $\tau_u^{(2)}(f) = \tau_u^{(2)}(M, N)$.

3.5. Some Computations. We state and prove the matrix chain method for computing the universal L^2 -torsion of a chain complex which goes back to [Tur01, Theorem 2.2].

Let $C_* = (0 \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0)$ be a finite based free $\mathbb{Z}G$ -chain complex and let $\partial_i : C_i \rightarrow C_{i-1}$ be the boundary operator. Suppose $d_i := \text{rank}_{\mathbb{Z}G} C_i$ is the rank of the free module C_i . Then ∂_i is given by a matrix

$$A_i = (a_{jk}^i)_{\substack{j=1, \dots, d_i \\ k=1, \dots, d_{i-1}}}, \quad a_{jk}^i \in \mathbb{Z}G.$$

Definition 3.18. A matrix chain for C_* is a collection of finite sets $\mathcal{A} = \{\mathcal{I}_0, \dots, \mathcal{I}_n\}$ where $\mathcal{I}_i \subset \{1, \dots, d_i\}$ and $\mathcal{I}_n = \emptyset$. Let B_i be the submatrix of A_i formed by the entries a_{jk}^i with $j \notin \mathcal{I}_i$ and $k \in \mathcal{I}_{i-1}$. Then B_i are called the matrices associated to the matrix chain.

A matrix chain is called non-degenerate if each associated matrix is a square matrix and is invertible over \mathcal{D}_G .

Theorem 3.19. *A finite based free $\mathbb{Z}G$ -chain complex C_* is L^2 -acyclic if and only if there exists an non-degenerate matrix chain $\mathcal{A} = \{\mathcal{I}_0, \dots, \mathcal{I}_n\}$ for C_* . If this happens, then*

$$\tau_u^{(2)}(C_*) = \prod_{i=1}^n [B_i]^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G)$$

where B_i are the matrices associated to the matrix chain.

The proof is a generalization of the idea of [DFL16, Lemma 3.1] to larger chain complexes.

Proof. Suppose C_* is L^2 -acyclic. Since $H_n^{(2)}(C_*) = 0$, we know that A_n is L^2 -injective. Then there is a submatrix B_n of A_n , such that B_n is a square matrix of size $d_n \times d_n$ and is a weak isomorphism. We set $\mathcal{I}_{n-1} \subset \{1, \dots, d_{n-1}\}$ to be the set of indices of the columns of B_i . Write $C_{n-1} = C'_{n-1} \oplus C''_{n-1}$ where C'_{n-1} corresponds to the set of indices \mathcal{I}_{n-1} and C''_{n-1} corresponds to the remaining indices. Then $B_n : C_n \rightarrow C'_{n-1}$ is a weak isomorphism. Since $H_{n-1}^{(2)}(C_*) = 0$, we know that the restriction of A_{n-1} to C''_{n-1} is injective, whose matrix A'' is exactly the submatrix of A consisting of the rows whose index does not belong to \mathcal{I}_{n-1} . So we obtain the following L^2 -acyclic chain complex

$$0 \longrightarrow C''_{n-1} \xrightarrow{A''_{n-1}} C_{n-2} \xrightarrow{A_{n-2}} \dots \longrightarrow C_1 \xrightarrow{A_1} C_0 \longrightarrow 0.$$

Repeat this procedure and in the end we find matrices B_n, \dots, B_1 which gives the matrix chain for C_* .

For the backward direction, again write $C_{n-1} = C'_{n-1} \oplus C''_{n-1}$ where C'_{n-1} corresponds to the set of indices \mathcal{I}_{n-1} and C''_{n-1} corresponds to the remaining indices. Then we have the following commutative diagram.

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_n & \xrightarrow{\cong} & C_n & \longrightarrow & 0 \\ & & \downarrow B_n & & \downarrow A_n & & \downarrow \\ 0 & \longrightarrow & C'_{n-1} & \xrightarrow{i} & C_{n-1} & \xrightarrow{p} & C''_{n-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow A_{n-1} & & \downarrow A''_{n-1} \\ & & 0 & \longrightarrow & C_{n-2} & \xrightarrow{\cong} & C_{n-2} \longrightarrow 0 \\ & & & & \downarrow A_{n-2} & & \downarrow A_{n-2} \\ & & & & \vdots & & \vdots \end{array}$$

If we set $C'_* = (0 \rightarrow C_n \xrightarrow{B_n} C'_{n-1} \rightarrow 0)$ and

$$C''_* = (0 \longrightarrow C''_{n-1} \xrightarrow{A''_{n-1}} C_{n-2} \xrightarrow{A_{n-2}} \dots \longrightarrow C_1 \xrightarrow{A_1} C_0 \longrightarrow 0).$$

We then have the short exact sequence of based $\mathbb{C}G$ -chain complex

$$0 \rightarrow C'_* \rightarrow C_* \rightarrow C''_* \rightarrow 0.$$

By Proposition 3.5, if C'_* is L^2 -acyclic then C_* is also L^2 -acyclic and $\tau_u^{(2)}(C_*) = \tau_u^{(2)}(C'_*) \cdot [B_n]^{(-1)^n}$. Repeat the above decomposition to C''_* and in the end we know that C_* is L^2 -acyclic and

$$\tau_u^{(2)}(C_*) = \prod_{i=1}^n [B_i]^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G).$$

□

Proposition 3.20. *We calculate the universal L^2 -torsion of some manifold pairs.*

- (1) *Let N be a compact smooth manifold whose fundamental group satisfies the Atiyah Conjecture. then for any $s \in [0, 1]$ we have*

$$\tau_u^{(2)}(N \times I, N \times \{s\}) = 1 \in \text{Wh}(\mathcal{D}_{\Pi(N)}).$$

- (2) *Let S^1 be the circle with fundamental group $\mathbb{Z} = \langle t \rangle$. Then $\tau_u^{(2)}(S^1) = [t - 1]^{-1} \in \text{Wh}(\mathcal{D}_{\mathbb{Z}})$.*
 (3) *Let T^2 be the torus then*

$$\tau_u^{(2)}(T^2) = 1 \in \text{Wh}(\mathcal{D}_{\mathbb{Z}^2}).$$

Proof. For (1), let $f : N \times \{s\} \rightarrow N \times I$ be the inclusion, then f is a simple-homotopy equivalence and $\tau_u^{(2)}(N \times I, N \times \{s\}) = \tau_u^{(2)}(f) = 1$ by Proposition 3.12.

For (2), a CW-structure of S^1 is given by a 0-cell p and an 1-cell e . By choosing appropriate liftings \hat{p} and \hat{e} the cellular chain complex of the universal cover is

$$C_*(\widehat{S}^1) = (0 \rightarrow \mathbb{Z}[t^{\pm}] \cdot \langle e \rangle \xrightarrow{(t-1)} \mathbb{Z}[t^{\pm}] \cdot \langle p \rangle \rightarrow 0)$$

and hence $\tau_u^{(2)}(S^1) = [t - 1]^{-1}$.

For (3), consider the CW-structure for T^2 given by identifying pairs of sides of a square. Let p be the 0-cell, e_1, e_2 be the 1-cells and σ be the 2-cell. Then the boundary of σ is a loop $e_1 e_2 e_1^{-1} e_2^{-1}$. Suppose the loop e_1, e_2 represents $t_1, t_2 \in \pi_1(T^2)$, respectively. Then by choosing appropriate liftings of the cells the chain complex of the universal cover is

$$C_*(\widehat{T}^2) = (0 \rightarrow \mathbb{Z}[t_1^{\pm}, t_2^{\pm}] \cdot \langle \sigma \rangle \xrightarrow{\begin{pmatrix} 1-t_2 & t_1-1 \end{pmatrix}} \mathbb{Z}[t_1^{\pm}, t_2^{\pm}] \cdot \langle e_1, e_2 \rangle \xrightarrow{\begin{pmatrix} t_1-1 \\ t_2-1 \end{pmatrix}} \mathbb{Z}[t_1^{\pm}, t_2^{\pm}] \cdot \langle p \rangle \rightarrow 0).$$

A matrix chain can be given by $B_2 = (1-t_2)$ and $B_1 = (t_2-1)$, hence $\tau_u^{(2)}(T^2) = [1-t_2] \cdot [t_2-1]^{-1} = 1$. □

4. LEADING TERM MAP, RESTRICTION MAP AND POLYTOPE MAP

4.1. Ore localization.

Definition 4.1 (Ore localization). Let R be a ring with unit and let $S \subset R$ be a multiplicatively closed subset. The pair (R, S) satisfies the (right) Ore condition if the following two conditions hold:

- (1) for any $(r, s) \in R \times S$ there exists $(r', s') \in R \times S$ such that $rs' = sr'$, and
- (2) for any $r \in R$ and $s \in S$ with $sr = 0$, there is $t \in S$ with $rt = 0$.

If (R, S) satisfies the Ore condition, define an equivalence relation on $R \times S$

$$(r, s) \sim (rx, sx) \quad \text{whenever} \quad x \in R, \quad sx \in S.$$

The quotient set $R \times S / \sim$ is denoted by RS^{-1} . Define a ring structure on RS^{-1} as follows. Given two representatives $(r, s), (r', s') \in RS^{-1}$, we can find $c \in R, d \in S$ with $sc = s'd \in S$ and define

$$(r, s) + (r', s') = (rc + r'd, sc).$$

We can find $e \in R$, $f \in S$ with $se = r'f$ and define

$$(r, s) \cdot (r', s') = (re, s'f).$$

The ring RS^{-1} is called the Ore localization of R at S .

Intuitively, a pair $(r, s) \in RS^{-1}$ is understood as a formal fraction rs^{-1} . The Ore condition (1) can be remembered as whenever there is a left (=wrong way) fraction $s^{-1}r$ then there is a right fraction $r'(s')^{-1}$ such that $rs' = sr'$. Condition (2) is automatically satisfied if S contains no zero divisors. In this paper, we only need to deal with Ore localizations of the following simple form:

Lemma 4.2 ([Coh95, Corollary 1.3.3]). *Let R be an integral domain such that $aR \cap bR \neq \{0\}$ for all $a, b \in R^\times$. Then (R, R^\times) satisfies the Ore condition. Moreover, the Ore localization of R at R^\times is a field K and the natural homomorphism $\lambda : R \rightarrow K$ is an embedding.*

An integral domain R satisfying the condition in Lemma 4.2 is called an Ore domain. The field K is called the field of fraction of R .

4.2. Crossed products. Assume that G is a torsion-free group which satisfies the Atiyah conjecture. Given any short exact sequence of groups

$$1 \rightarrow K \rightarrow G \xrightarrow{\nu} H \rightarrow 1.$$

Then K also satisfies the Atiyah conjecture by Proposition 2.7. Denote by \mathcal{D}_K and \mathcal{D}_G the Linnell's skew fields of K and G , respectively.

Choose a section $s : H \rightarrow G$ for the epimorphism ν such that $\nu \circ s$ is the identity. We do not require that s be a group homomorphism. Consider the following subset of \mathcal{D}_G :

$$\mathcal{D}_K *_s H := \left\{ \sum_{h \in H} x_h \cdot s(h) \in \mathcal{D}_G \mid x_h \in \mathcal{D}_K, x_h = 0 \text{ for all but finitely many } h \in H \right\}.$$

This set contains the zero element 0, the identity element $1 = s(1_H)^{-1} \cdot s(1_H)$ and is closed under addition. Moreover, it is also closed under multiplication, since

$$\begin{aligned} \left(\sum_{h \in H} x_h \cdot s(h) \right) \cdot \left(\sum_{h \in H} y_h \cdot s(h) \right) &= \sum_{h_1, h_2 \in H} x_{h_1} \cdot s(h_1) \cdot y_{h_2} \cdot s(h_2) \\ &= \sum_{h_1, h_2 \in H} \underbrace{x_{h_1} s(h_1) y_{h_2} s(h_1)^{-1}}_{\in \mathcal{D}_K} \cdot \underbrace{s(h_1) s(h_2) s(h_1 h_2)^{-1}}_{\in K} \cdot s(h_1 h_2). \end{aligned}$$

Recall that the group automorphism of conjugation by $s(h_1)$ in K extends to an automorphism of \mathcal{D}_K by Proposition 2.7, so $s(h_1) y_{h_2} s(h_1)^{-1} \in \mathcal{D}_K$. It follows that $\mathcal{D}_K *_s H$ is a subring of the field \mathcal{D}_G .

Proposition 4.3. *With notations as above, we have the following properties.*

- (1) *If an element $\sum_{h \in H} x_h \cdot s(h)$ of $\mathcal{D}_K *_s H$ is zero then $x_h = 0$ for all $h \in H$.*
- (2) *Given another section $s' : H \rightarrow G$, then*

$$\sum_{h \in H} x_h \cdot s(h) = \sum_{h \in H} y_h \cdot s'(h)$$

if and only if $y_h = x_h s(h) s'(h)^{-1}$ for all $h \in H$.

Proof. The first statement is a consequence of [Lüc02, Lemma 10.57]. The second statement follows from the previous one. \square

As a corollary, the subring $\mathcal{D}_K *_s H \subset \mathcal{D}_G$ does not depend on the choice of the section. We call this subring the crossed product of \mathcal{D}_K and H , denoted by $\mathcal{D}_K * H$. It is clear that $\mathcal{D}_K * H$ is an integral domain since it embeds in the field \mathcal{D}_G . When H is nice, the relation between \mathcal{D}_G and its subring $\mathcal{D}_K * H$ is surprisingly simple.

Proposition 4.4. *Let $1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$ be an extension of groups.*

- (1) *Suppose H is a finite group, then $\mathcal{D}_G = \mathcal{D}_K * H$.*
- (2) *Suppose H is virtually a finitely generated abelian group, then the integral domain $\mathcal{D}_K * H$ is an Ore domain whose field of fraction agrees with \mathcal{D}_G .*

Proof. These two statements are Lemma 10.59 and Lemma 10.69 of [Lüc02], respectively. \square

As usual, let G be a finitely generated torsion-free group which satisfies the Atiyah Conjecture.

4.3. The leading term map. Let $\nu : G \rightarrow H_1(G)_f$ be the natural quotient map to the free abelianization group $H_1(G)_f$, then we have the short exact sequence

$$1 \rightarrow K \rightarrow G \xrightarrow{\nu} H_1(G)_f \rightarrow 1.$$

Fix $\phi \in H^1(G; \mathbb{R})$ be any real cohomology class. Given any nonzero element $u \in (\mathcal{D}_K * H_1(G)_f)^\times$, we choose a section s and write

$$u = \sum_{h \in H_1(G)_f} x_h \cdot s(h) \in \mathcal{D}_K * H_1(G)_f.$$

The support of u is defined to be the set $\text{supp}(u) := \{h \in H_1(G)_f \mid x_h \neq 0\}$, this is a finite subset of $H_1(G)_f$ and does not depend on the choice of section by Proposition 4.3. Define $\delta_\phi(u)$ to be the minimal value of $\phi(h)$ for all $h \in \text{supp}(u)$. Define

$$L_\phi(u) := \sum_{\substack{h \in \text{supp}(u), \\ \phi(h) = \delta_\phi(u)}} x_h \cdot s(h).$$

This element is nonzero and lies in $(\mathcal{D}_K * H_1(G)_f)^\times$.

Lemma 4.5. *The definition of $\delta_\phi(u)$ and $L_\phi(u)$ do not depend on the choice of section s . Moreover, we have*

$$\begin{aligned} \delta_\phi(u_1 u_2) &= \delta_\phi(u_1) + \delta_\phi(u_2), \\ L_\phi(u_1 u_2) &= L_\phi(u_1) \cdot L_\phi(u_2) \end{aligned}$$

for all $u_1, u_2 \in (\mathcal{D}_K * H_1(G)_f)^\times$. Hence

$$\begin{aligned} \delta_\phi : (\mathcal{D}_K * H_1(G)_f)^\times &\rightarrow \mathbb{R}, \\ L_\phi : (\mathcal{D}_K * H_1(G)_f)^\times &\rightarrow (\mathcal{D}_K * H_1(G)_f)^\times \end{aligned}$$

are well-defined group homomorphisms.

Proof. Let $u = \sum_h x_h \cdot s(h)$ and s' be another section, then by Proposition 4.3 $u = \sum_h y_h \cdot s'(h)$ where $y_h = x_h s(h) s'(h)^{-1}$. It follows that $\delta_\phi(u)$ and $L_\phi(u)$ do not depend on the choice of section. The terms of $u_1 u_2$ with minimal ϕ -value exactly comes from the multiplication of that of u_1 and u_2 . This explains the homomorphism. \square

Recall that \mathcal{D}_G is the field of fraction of the subring $\mathcal{D}_K * H_1(G)_f$ by Proposition 4.4.

Definition 4.6. The group homomorphism δ_ϕ and L_ϕ extend to group homomorphisms

$$\begin{aligned} \delta_\phi : \mathcal{D}_G^\times &\rightarrow \mathbb{R}, \quad \delta_\phi(uv^{-1}) := \delta_\phi(u) - \delta_\phi(v), \\ L_\phi : \mathcal{D}_G^\times &\rightarrow \mathcal{D}_G^\times, \quad L_\phi(uv^{-1}) := L_\phi(u) L_\phi(v)^{-1} \end{aligned}$$

for all $u, v \in (\mathcal{D}_K * H_1(G)_f)^\times$. It is convenient to set $\delta_\phi(0) = +\infty$ and $L_\phi(0) = 0$. Then we have

$$\delta_\phi(z_1 z_2) = \delta_\phi(z_1) + \delta_\phi(z_2), \quad L_\phi(z_1 z_2) = L_\phi(z_1) \cdot L_\phi(z_2)$$

for all $z_1, z_2 \in \mathcal{D}_G$.

Proof. We prove the well-definedness. Suppose $z \in \mathcal{D}_G^\times$ can be expressed as $z = u_1 v_1^{-1} = u_2 v_2^{-1}$, then there exists $w_1, w_2 \in (\mathcal{D}_K * H_1(G)_f)^\times$ such that $u_1 w_1 = u_2 w_2$, $v_1 w_1 = v_2 w_2$. Hence

$$\begin{aligned} L_\phi(u_1) L_\phi(v_1)^{-1} &= L_\phi(u_1) L_\phi(w_1) L_\phi(w_1)^{-1} L_\phi(v_1)^{-1} \\ &= L_\phi(u_1 w_1) L_\phi(v_1 w_1)^{-1} \\ &= L_\phi(u_2 w_2) L_\phi(v_2 w_2)^{-1} \\ &= L_\phi(u_2) L_\phi(v_2)^{-1}. \end{aligned}$$

To verify that L_ϕ is a homomorphism, let $z_1, z_2 \in \mathcal{D}_G^\times$. By the Ore condition, we can arrange that $z_1 = u_1 w^{-1}$ and $z_2 = w u_2^{-1}$ for $u_1, u_2, w \in (\mathcal{D}_K * H_1(G)_f)^\times$, so

$$L_\phi(z_1) L_\phi(z_2) = L_\phi(u_1) L_\phi(w)^{-1} \cdot L_\phi(w) L_\phi(u_2)^{-1} = L_\phi(u_1) L_\phi(u_2)^{-1} = L_\phi(z_1 z_2).$$

The statements for δ_ϕ can be proved similarly. \square

Here are some basic facts about the mappings δ_ϕ, L_ϕ , especially about their properties under additions in \mathcal{D}_G . Most of the properties clearly holds true in the subring $\mathcal{D}_K * H_1(G)_f$ and it is routine to verify them in its field of fraction \mathcal{D}_G .

Proposition 4.7. *Let G be a finitely generated torsion-free group which satisfies the Atiyah Conjecture. Let $\phi \in H^1(G; \mathbb{R})$ be any real cohomology class and $\delta_\phi : \mathcal{D}_G \rightarrow \mathbb{R}$, $L_\phi : \mathcal{D}_G \rightarrow \mathcal{D}_G$ be as in Definition 4.6. Suppose $z, z_1, \dots, z_n \in \mathcal{D}_G$, then:*

- (1) $\delta_{r\phi}(z) = r \cdot \delta_\phi(z)$, $L_{r\phi}(z) = L_\phi(z)$ for all $r \in \mathbb{R}_+$.
- (2) $\delta_\phi(cz) = \delta_\phi(z)$, $L_\phi(cz) = c \cdot L_\phi(z)$ for all $c \in \mathbb{Q} \setminus \{0\}$.
- (3) $\delta_\phi(L_\phi(z)) = \delta_\phi(z)$, $L_\phi(L_\phi(z)) = L_\phi(z)$.
- (4) If $L_\phi(z_1) = L_\phi(z_2)$ nonzero, then $\delta_\phi(z_1) = \delta_\phi(z_2) < \delta_\phi(z_1 - z_2)$.
- (5) $\delta_\phi(z_1 + z_2) \geq \min\{\delta_\phi(z_1), \delta_\phi(z_2)\}$. If $\delta_\phi(z_1) < \delta_\phi(z_2)$, then

$$\delta_\phi(z_1 + z_2) = \delta_\phi(z_1), \quad L_\phi(z_1 + z_2) = L_\phi(z_1).$$

- (6) If $\delta_\phi(z_1) = \dots = \delta_\phi(z_n) =: \delta$ and $\sum_{k=1}^n L_\phi(z_k) \neq 0$, then

$$\delta_\phi\left(\sum_{k=1}^n z_k\right) = \delta, \quad L_\phi\left(\sum_{k=1}^n z_k\right) = \sum_{k=1}^n L_\phi(z_k).$$

- (7) Given $z \in \mathcal{D}_G$. For any open neighborhood $U \subset H^1(G; \mathbb{R})$ of ϕ , there is a rational cohomology class $\psi \in U$ such that $L_\psi(z) = L_\phi(z)$.
- (8) Suppose $L \subset G$ is an inclusion of a finitely generated subgroup L , and $\mathcal{D}_L \subset \mathcal{D}_G$ is the induced inclusion. Denote by $\phi|_L : L \rightarrow \mathbb{R}$ the restriction of ϕ to L . Then the mappings

$$\delta_{\phi|_L} : \mathcal{D}_L \rightarrow \mathbb{R}, \quad L_{\phi|_L} : \mathcal{D}_L \rightarrow \mathcal{D}_L$$

are exactly the restrictions of δ_ϕ and L_ϕ to \mathcal{D}_L .

Proof. For (1)–(3), the statements hold in $\mathcal{D}_K * H_1(G)_f$ and directly extends to \mathcal{D}_G by definition.

For (4), assume that $z_1 = u_1 w^{-1}$, $z_2 = u_2 w^{-1}$ for $u_1, u_2, w \in \mathcal{D}_K * H_1(G)_f$, then $L_\phi(z_1) = L_\phi(z_2)$ implies that $L_\phi(u_1) = L_\phi(u_2)$. It follows that $\delta_\phi(u_1) = \delta_\phi(u_2) < \delta_\phi(u_1 - u_2)$. Therefore $\delta_\phi(z_1) = \delta_\phi(z_2) < \delta_\phi(z_1 - z_2)$.

For (5), assume that $z_1 = u_1 w^{-1}$, $z_2 = u_2 w^{-1}$ for $u_1, u_2, w \in \mathcal{D}_K * H_1(G)_f$. Since it is clear that $\delta_\phi(u_1 + u_2) \geq \min\{\delta_\phi(u_1), \delta_\phi(u_2)\}$, then $\delta_\phi(z_1 + z_2) = \delta_\phi(u_1 + u_2) - \delta_\phi(w) \geq \min\{\delta_\phi(u_1), \delta_\phi(u_2)\} -$

$\delta_\phi(w) = \min\{\delta_\phi(z_1), \delta_\phi(z_2)\}$. If $\delta_\phi(z_1) < \delta_\phi(z_2)$, then $\delta_\phi(u_1) < \delta_\phi(u_2)$ and $L_\phi(u_1 + u_2) = L_\phi(u_1)$. It follows that $\delta_\phi(z_1 + z_2) = \delta_\phi(z_1)$ and $L_\phi(z_1 + z_2) = L_\phi(z_1)$.

For (6), assume that $z_i = u_i w^{-1}$ for $u_i, w \in \mathcal{D}_K * H_1(G)_f$, $i = 1, \dots, n$. By assumption we have $\delta_\phi(u_i) = \delta + \delta_\phi(w)$ and $\sum_{i=1}^k L_\phi(u_i) \neq 0$. It follows that $\delta_\phi(\sum_{i=1}^k u_i) = \delta + \delta_\phi(w)$ and $L_\phi(\sum_{i=1}^k u_i) = \sum_{i=1}^k L_\phi(u_i)$. Hence $\delta_\phi(\sum_{i=1}^k z_i) = \delta$ and $L_\phi(\sum_{i=1}^k z_i) = \sum_{i=1}^k L_\phi(z_i)$.

For (7), assume $z = uv^{-1}$ with $u, v \in \mathcal{D}_K * H_1(G)_f$. Write $u = \sum_{h \in H_1(G)_f} x_h \cdot s(h)$ for a section s . Given another cohomology class $\psi \in H^1(G; \mathbb{R})$, then $L_\phi(z) = L_\psi(z)$ if the following two conditions hold:

- for all $h, h' \in \text{supp}(u) \cup \text{supp}(v)$, $\psi(h - h') < 0$ whenever $\phi(h - h') < 0$;
- for all $h, h' \in \text{supp}(u) \cup \text{supp}(v)$, $\psi(h - h') = 0$ whenever $\phi(h - h') = 0$.

The domain Ω of such ψ is the intersection of finitely many closed hyperplanes and open half spaces of $H^1(G; \mathbb{R})$, each given by an integral linear equation. Since $\phi \in \Omega$, given any open neighborhood $U \ni \phi$ there are rational classes in $U \cap \Omega$.

For (8), consider the short exact sequence $1 \rightarrow K' \rightarrow L \rightarrow H_1(L)_f \rightarrow 1$. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{K'} * H_1(L)_f & \hookrightarrow & \mathcal{D}_K * H_1(G)_f \\ \downarrow & & \downarrow \\ \mathcal{D}_L & \hookrightarrow & \mathcal{D}_G \end{array}$$

and \mathcal{D}_L is the field of fraction of $\mathcal{D}_{K'} * H_1(L)_f$. Choose any nonzero $u \in \mathcal{D}_{K'} * H_1(L)_f$ and write $u = \sum_{h \in H_1(L)_f} x_h \cdot s(h)$, $x_h \in \mathcal{D}_{K'}$ for a section $s : H_1(L)_f \rightarrow L$. By definition $\delta_{\phi|_L}(u) = \min\{\phi(h) \mid h \in H_1(L)_f, x_h \neq 0\}$. Write $\delta := \delta_{\phi|_L}(u)$. Decompose

$$u = \sum_{\substack{h \in H_1(L)_f, \\ \phi|_L(h) = \delta}} x_h \cdot s(h) + \sum_{\substack{h \in H_1(L)_f, \\ \phi|_L(h) > \delta}} x_h \cdot s(h) =: u_1 + u_2.$$

Then by definition $L_{\phi|_L}(u) = u_1$ is nonzero. We want to show

$$\delta_\phi(u) = \delta, \quad L_\phi(u) = u_1.$$

This does not directly follow from the definition of δ_ϕ and L_ϕ since $s : H_1(L)_f \rightarrow L$ is not a section of $G \rightarrow H_1(G)_f$. We choose to argue as follows. By (6) applied to u_1 we know that

$$\delta_\phi(u_1) = \delta, \quad L_\phi(u_1) = u_1.$$

By (5) applied to u_2 we know that

$$\delta_\phi(u_2) \geq \min\{\delta_\phi(x_h \cdot s(h)) \mid h \in H_1(L)_f, \phi|_L(h) > \delta\} > \delta,$$

Again by (5) applied to $u = u_1 + u_2$ we know that $\delta_\phi(u) = \delta_\phi(u_1) = \delta$ and $L_\phi(u) = L_\phi(u_1) = u_1$. Hence $\delta_\phi(u) = \delta_{\phi|_L}(u)$ and $L_\phi(u) = L_{\phi|_L}(u)$ for all nonzero $u \in \mathcal{D}_{K'} * H_1(L)_f$. Passing to the field of fraction we have $\delta_\phi(z) = \delta_{\phi|_L}(z)$ and $L_\phi(z) = L_{\phi|_L}(z)$ for all $z \in \mathcal{D}_L$. \square

Definition 4.8. An element $z \in \mathcal{D}_G$ is called ϕ -pure if $L_\phi(z) = z$.

Lemma 4.9. Here are some properties of the ϕ -pure elements.

- (1) Elements of $\mathbb{Z}[\ker \phi] \subset \mathcal{D}_G$ are ϕ -pure; elements of $G \subset \mathcal{D}_G$ are ϕ -pure.
- (2) The product of two ϕ -pure elements is ϕ -pure.
- (3) Given any nonzero $z \in \mathcal{D}_G$, then $L_\phi(z)$ is the unique element $w \in \mathcal{D}_G$ such that w is ϕ -pure and $\delta_\phi(z - w) > \delta_\phi(z)$.
- (4) Suppose z_1, \dots, z_n are ϕ -pure elements with $\delta_\phi(z_1) = \dots = \delta_\phi(z_n) = \delta$, then the sum $\sum_{i=1}^n z_i$ is also ϕ -pure.

Proof. The properties (1) and (2) follow from the definition of L_ϕ .

For (3), it follows from Proposition 4.7 (3), (4) that $L_\phi(z)$ is ϕ -pure and $\delta_\phi(z - L_\phi(z)) > \delta_\phi(z)$. On the other hand, if a ϕ -pure element w satisfies $\delta_\phi(z - w) > \delta_\phi(z)$, then by Proposition 4.7 (4)

$$w = L_\phi(w) = L_\phi(z - (z - w)) = L_\phi(z).$$

Finally, (4) is a consequence of Proposition 4.7 (6). \square

Lemma 4.10. *Given any $\phi \in H^1(G; \mathbb{R})$. The homomorphism $\delta_\phi : \mathcal{D}_G^\times \rightarrow \mathbb{R}$ induces a well-defined homomorphism $\delta_\phi : \Lambda \rightarrow \mathbb{R}$ for $\Lambda = K_1(\mathcal{D}_G)$ and $\widetilde{K}_1(\mathcal{D}_G)$. The homomorphism $L_\phi : \mathcal{D}_G^\times \rightarrow \mathcal{D}_G^\times$ induces a well-defined homomorphism $L_\phi : \Lambda \rightarrow \Lambda$ for $\Lambda = K_1(\mathcal{D}_G)$, $\widetilde{K}_1(\mathcal{D}_G)$ and $\text{Wh}(\mathcal{D}_G)$.*

We use the same symbols δ_ϕ, L_ϕ for their induced maps on $K_1(\mathcal{D}_G)$, $\widetilde{K}_1(\mathcal{D}_G)$ and $\text{Wh}(\mathcal{D}_G)$ and its domain of definition will be clear from the context.

$$\begin{array}{ccccc} & & GL(\mathcal{D}_G) & & \\ & \swarrow \det & \downarrow \det_r & \searrow \det_w & \\ \mathcal{D}_G^\times / [\mathcal{D}_G^\times, \mathcal{D}_G^\times] & = & K_1(\mathcal{D}_G) & \longrightarrow & \widetilde{K}_1(\mathcal{D}_G) & \longrightarrow & \text{Wh}(\mathcal{D}_G) \\ & & \downarrow \delta_\phi, L_\phi & & \downarrow \delta_\phi, L_\phi & & \downarrow L_\phi \\ & & \text{curved arrow} & & \text{curved arrow} & & \text{curved arrow} \end{array}$$

Theorem 4.11. *Let $\phi \in H^1(H; \mathbb{R})$ be any real cohomology class. Suppose P and Q are two square matrices over \mathcal{D}_G of size $n \geq 1$, such that the following conditions hold:*

- (i) P is invertible over \mathcal{D}_G ;
- (ii) there exist real numbers d_0, d_1, \dots, d_n such that if P_{ij} is nonzero, then P_{ij} is ϕ -pure with $\delta_\phi(P_{ij}) = d_0 + d_i - d_j$;
- (iii) $\delta_\phi(Q_{ij}) > d_0 + d_i - d_j$ for all i, j .

Then we have $L_\phi(\det(P + Q)) = \det P \in K_1(\mathcal{D}_G)$.

Proof. We prove by induction on n . When $n = 1$ then $P, Q \in \mathcal{D}_G$, by the conditions we have $\delta_\phi(P) = d_0$, and $\delta_\phi(Q) > d_0$ if Q is nonzero. Then $L_\phi(\det(P + Q)) = \det P$ by definition.

Now assume Lemma 4.11 holds for size n . Assume that P, Q are $(n + 1)$ by $(n + 1)$ matrices

$$P = \begin{pmatrix} A & U \\ X & p \end{pmatrix}, \quad Q = \begin{pmatrix} B & V \\ Y & q \end{pmatrix}$$

where $p, q \in \mathcal{D}_G$, A, B are n by n matrices over \mathcal{D}_G and

$$\begin{aligned} X &= (x_1, \dots, x_n), & Y &= (y_1, \dots, y_n), \\ U &= (u_1, \dots, u_n)^T, & V &= (v_1, \dots, v_n)^T \end{aligned}$$

are matrices over \mathcal{D}_G of appropriate size. Without loss of generality we can assume $p \neq 0$, hence also $p + q \neq 0$ by condition (iii). Note that

$$\begin{pmatrix} I & -(U + V)(p + q)^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A + B & U + V \\ X + Y & p + q \end{pmatrix} = \begin{pmatrix} W & 0 \\ X + Y & p + q \end{pmatrix}$$

where

$$W = A + B - (U + V)(p + q)^{-1}(X + Y),$$

is an n by n matrix. So

$$\begin{aligned} \det(P + Q) &= \det W \cdot \det(p + q), \\ L_\phi \det(P + Q) &= L_\phi \det W \cdot \det p \end{aligned}$$

(note that $L_\phi(p + q) = p$). Staring at the expression of W , it is easy to guess that the terms of lowest δ_ϕ form the matrix $A - Up^{-1}X$. We show that this is indeed the case.

Claim. *Define*

$$W' := A - Up^{-1}X, \quad W'' := W - W'.$$

Then W' is invertible over \mathcal{D}_G and $\det P = \det W' \cdot \det p$. Moreover, W' and W'' satisfies the three conditions of Lemma 4.11 for size n and then by the induction hypothesis we have $L_\phi \det(W) = \det(W')$.

Admitting this claim. Then we have

$$L_\phi \det(P + Q) = L_\phi \det W \cdot \det p = \det W' \cdot \det p = \det P$$

and the induction is finished. It remains to prove the Claim.

Proof of Claim. To prove (i), it follows from

$$\begin{pmatrix} I & -Up^{-1} \\ 0 & 1 \end{pmatrix} \cdot P = \begin{pmatrix} W' & 0 \\ X & p \end{pmatrix}$$

that W' is invertible over \mathcal{D}_G and $\det P = \det W' \cdot \det p$.

For (ii), note that

$$\begin{aligned} W_{ij} &= A_{ij} + B_{ij} - (u_i + v_i)(p + q)^{-1}(x_j + y_j), \\ W'_{ij} &= A_{ij} - u_i p^{-1} x_j, \\ W''_{ij} &= B_{ij} + u_i p^{-1} x_j - (u_i + v_i)(p + q)^{-1}(x_j + y_j). \end{aligned}$$

If $A_{ij} \neq 0$, then A_{ij} is ϕ -pure with $\delta_\phi(A_{ij}) = d_0 + d_i - d_j$; if $u_i p^{-1} x_j \neq 0$, then $u_i p^{-1} x_j$ is also ϕ -pure with

$$\delta_\phi(u_i p^{-1} x_j) = (d_0 + d_i - d_{n+1}) - d_0 + (d_0 + d_{n+1} - d_j) = d_0 + d_i - d_j.$$

In conclusion, if $W'_{ij} \neq 0$ then W'_{ij} is ϕ -pure with $\delta_\phi(W'_{ij}) = d_0 + d_i - d_j$ and this proves (ii).

For (iii), by assumption we have $\delta_\phi(B_{ij}) > d_0 + d_i - d_j$. We also have $\delta_\phi(u_i p^{-1} x_j - (u_i + v_i)(p + q)^{-1}(x_j + y_j)) > d_0 + d_i - d_j$, since if $u_i p^{-1} x_j \neq 0$, then

$$L_\phi((u_i + v_i)(p + q)^{-1}(x_j + y_j)) = u_i p^{-1} x_j, \quad \delta_\phi(u_i p^{-1} x_j) = d_0 + d_i - d_j$$

and we can apply Proposition 4.7 (4); if $u_i p^{-1} x_j = 0$ then $u_i = 0$ or $x_j = 0$ and

$$\begin{aligned} \delta_\phi((u_i + v_i)(p + q)^{-1}(x_j + y_j)) &= \delta_\phi(u_i + v_i) - \delta_\phi(p + q) + \delta_\phi(x_j + y_j) \\ &> \delta_\phi(u_i) - \delta_\phi(p + q) + \delta_\phi(x_j) \\ &= d_0 + d_i - d_j. \end{aligned}$$

In either cases, we have $\delta_\phi(u_i p^{-1} x_j - (u_i + v_i)(p + q)^{-1}(x_j + y_j)) > d_0 + d_i - d_j$. These combines to show that $\delta_\phi(W'') > d_0 + d_i - d_j$ by Proposition 4.7 (5). \square

4.4. The restriction map. Suppose G is a finitely generated torsion-free group satisfying the Atiyah Conjecture and let $L \triangleleft G$ be a normal subgroup of finite index d . In this section we define the restriction map $\text{res}_L^G : K_1(\mathcal{D}_G) \rightarrow K_1(\mathcal{D}_L)$. Recall that \mathcal{D}_G is naturally isomorphic to the crossed product $\mathcal{D}_L * (G/L)$ by Proposition 4.4.

Definition 4.12. Fix a section $s : G/L \rightarrow G$ and suppose its image is $s(G/L) = \{g_1, \dots, g_d\}$. Then $G = g_1 L \sqcup \dots \sqcup g_d L$. For any element $z \in \mathcal{D}_G$ and for any $k \in \{1, \dots, d\}$, there is a unique way to express $g_k \cdot z$ as

$$g_k \cdot z = \sum_{j=1}^d l_{kj} \cdot g_j, \quad l_{kj} \in \mathcal{D}_L.$$

Define $\Lambda_s(z)$ to be the $(d \times d)$ -matrix over \mathcal{D}_L whose (k, j) -entry is l_{kj} . In other words, $\Lambda_s(z)$ is the unique matrix over \mathcal{D}_L such that

$$\begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z = \Lambda_s(z) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}.$$

Lemma 4.13. *With the notations as in Definition 4.12, the following statements hold.*

- (1) *For any $z_1, z_2 \in \mathcal{D}_G$, we have $\Lambda_s(z_1 z_2) = \Lambda_s(z_2) \cdot \Lambda_s(z_1)$.*
- (2) *If $z \neq 0$, then $\Lambda_s(z)$ is invertible over \mathcal{D}_L .*
- (3) *If s' is another section, then $\Lambda_{s'}(z) = \Omega \Lambda_s(z) \Omega^{-1}$ for an invertible matrix Ω over $\mathbb{Z}L$ which only depends on s and s' .*
- (4) *For any $\phi \in H^1(G; \mathbb{R})$ we have $\delta_\phi(g_k \cdot z) \leq \delta_\phi(l_{kj} \cdot g_j)$. If z is ϕ -pure, then l_{kj} is also ϕ -pure; moreover if $l_{kj} \neq 0$ then $\delta_\phi(g_k \cdot z) = \delta_\phi(l_{kj} \cdot g_j)$.*

Proof. For (1), it follows from

$$\begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z_1 z_2 = \Lambda_s(z_1) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z_2 = \Lambda_s(z_1) \cdot \Lambda_s(z_2) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}$$

that $\Lambda_s(z_1 z_2) = \Lambda_s(z_2) \cdot \Lambda_s(z_1)$. Note that (2) is a direct consequence of (1).

For (3), let s' be another section with $s'(L) = \{g'_1, \dots, g'_d\}$ and let Ω be the $(d \times d)$ -matrix over $\mathbb{Z}L$ such that

$$\begin{pmatrix} g'_1 \\ \vdots \\ g'_d \end{pmatrix} = \Omega \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}.$$

Then

$$\begin{pmatrix} g'_1 \\ \vdots \\ g'_d \end{pmatrix} \cdot z = \Omega \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z = \Omega \Lambda_s(z) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} = \Omega \Lambda_s(z) \Omega^{-1} \begin{pmatrix} g'_1 \\ \vdots \\ g'_d \end{pmatrix}$$

and therefore $\Lambda_{s'}(z) = \Omega \Lambda_s(z) \Omega^{-1}$.

For (4). First suppose the contrary that there exists $k, j \in \{1, \dots, d\}$ such that $\delta_\phi(g_k \cdot z) > \delta_\phi(l_{kj} \cdot g_j)$. Fix k and let \mathcal{J} be the collection of indices j such that $\delta_\phi(g_k \cdot z) > \delta_\phi(l_{kj} \cdot g_j)$. Since

$$\delta_\phi(g_k \cdot z) = \delta_\phi\left(\sum_{j=1}^d l_{kj} \cdot g_j\right),$$

it follows from Proposition 4.7 (5) that $\sum_{j \in \mathcal{J}} l_{kj} \cdot g_j = 0$. Recall that g_1, \dots, g_d are \mathcal{D}_L -independent by Proposition 4.3. This implies that $l_{kj} = 0$ for all $j \in \mathcal{J}$ and forces \mathcal{J} to be empty, a contradiction. Hence $\delta_\phi(g_k \cdot z) \leq \delta_\phi(l_{kj} \cdot g_j)$ for all k, j . It can be seen from Lemma 4.9 (3) that

$$L_\phi(g_k \cdot z) = L_\phi\left(\sum_{j=1}^d l_{kj} \cdot g_j\right) = \sum_{j=1}^d l'_{kj} \cdot g_j$$

where

$$l'_{kj} = \begin{cases} L_\phi(l_{kj}) & \text{if } \delta_\phi(l_{kj} \cdot g_j) = \delta_\phi(g_k \cdot z) \\ 0 & \text{if } \delta_\phi(l_{kj} \cdot g_j) > \delta_\phi(g_k \cdot z). \end{cases}$$

If in addition z is ϕ -pure, then $g_k \cdot z$ is also ϕ -pure by Lemma 4.9 and therefore $l_{kj} = l'_{kj}$ for all k, j . It follows that l_{kj} is ϕ -pure for all k, j and moreover if $l_{kj} \neq 0$ then $\delta_\phi(g_k \cdot z) = \delta_\phi(l_{kj} \cdot g_j)$. \square

Definition 4.14. Let $L \triangleleft G$ be a normal subgroup of finite index d . Choose a section $s : G/L \rightarrow G$ with $s(G/L) = \{g_1, \dots, g_d\}$. Then $G = g_1L \sqcup \dots \sqcup g_dL$. Given any element $z \in \mathcal{D}_G^\times$ and $[z] \in K_1(\mathcal{D}_G)$ is the corresponding (1×1) -matrix. Define

$$\text{res}_L^G : K_1(\mathcal{D}_G) \rightarrow K_1(\mathcal{D}_L), \quad [z] \mapsto [\Lambda_s(z)].$$

This mapping is a group homomorphism independent of the choice of g_1, \dots, g_d .

Remark 4.15. An element $z \in \mathcal{D}_G^\times$ can be associated with $R_z : \mathcal{D}_G \rightarrow \mathcal{D}_G$, the operator of right multiplication by z . The choice of coset representatives identifies \mathcal{D}_G with $\bigoplus_{k=1}^d \mathcal{D}_L \cdot g_k$ as \mathcal{D}_L -vector spaces and R_z is naturally a \mathcal{D}_L -linear automorphism represented by the matrix $\Lambda_s(z)$. By definition $\text{res}_L^G([z]) = \det R_z$. A different choice of coset representatives amounts to a change of basis which preserves the determinant. This shows that $\text{res}_L^G([z])$ does not depend on the choice of g_1, \dots, g_d .

Theorem 4.16. Suppose G is a finitely generated torsion-free group satisfying the Atiyah Conjecture and let $L \triangleleft G$ be a normal subgroup of finite index. Let $\phi \in H^1(G; \mathbb{R})$ and denote by $\phi|_L \in H^1(L; \mathbb{R})$ the restriction of ϕ to L . Then for any $[z] \in K_1(\mathcal{D}_G)$, we have

$$L_{\phi|_L}(\text{res}_L^G([z])) = \text{res}_L^G(L_\phi([z])) \in K_1(\mathcal{D}_L).$$

Proof. Choose $z \in \mathcal{D}_G^\times$ representing the class $[z]$. Let $z =: L_\phi(z) + z'$. Fix a choice of coset representatives $G = g_1L \sqcup \dots \sqcup g_dL$. Let P and Q be $(d \times d)$ -matrices over \mathcal{D}_L such that

$$g_k \cdot L_\phi(z) = \sum_{j=1}^d P_{kj} \cdot g_j, \quad g_k \cdot z' = \sum_{j=1}^d Q_{kj} \cdot g_j.$$

Then $\text{res}_L^G([z]) = \det(P + Q)$ and $\text{res}_L^G([L_\phi(z)]) = \det P$. Since $L_\phi(z)$ is ϕ -pure, it follows from Lemma 4.12 that P_{kj} is ϕ -pure, and moreover

$$\begin{aligned} \delta_\phi(P_{kj}) &= \delta_\phi(z) + \delta_\phi(g_k) - \delta_\phi(g_j) \quad \text{if } P_{kj} \neq 0, \\ \delta_\phi(Q_{kj}) &\geq \delta_\phi(z') + \delta_\phi(g_k) - \delta_\phi(g_j) > \delta_\phi(z) + \delta_\phi(g_k) - \delta_\phi(g_j) \end{aligned}$$

for all k, j . By Proposition 4.7 (8) $\delta_\phi = \delta_{\phi|_L}$ and $L_\phi = L_{\phi|_L}$ in \mathcal{D}_L . Applying Theorem 4.11 to P and Q over \mathcal{D}_L we have

$$\det P = L_{\phi|_L}(\det(P + Q))$$

hence $L_{\phi|_L}(\text{res}_L^G([z])) = \text{res}_L^G(L_\phi([z]))$ and the proof is finished. \square

4.5. The polytope map. Let H be a finitely generated free abelian group. Note that $H_1(H; \mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Z}} H$ is a finite-dimensional real vector space.

A polytope in $H_1(H; \mathbb{R})$ is a compact set which is the convex hull of a finite subset. We allow the empty set \emptyset to be a polytope.

Definition 4.17 (Face map F_ϕ). Given a polytope P and any character $\phi \in H^1(H; \mathbb{R})$. Set $\delta_\phi(P) := \inf_{x \in P} \phi(x)$, define the face associated to ϕ by

$$F_\phi(P) := \{x \in P \mid \phi(x) = \delta_\phi(P)\}.$$

It is clear that $F_\phi(P)$ is a polytope contained in P . The collection $\{F_\phi(P) \mid \phi \in H^1(H; \mathbb{R})\}$ is the collection of faces of P . A face is called a vertex if it is a single point. Any polytope is the convex hull of all its vertices. A polytope is called integral if all its vertices lie in the integral lattice $H \subset H_1(H; \mathbb{R})$.

Given any two non-empty polytopes P_1, P_2 in $H_1(H; \mathbb{R})$, define their Minkowski sum to be the polytope

$$P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1, p_2 \in P_2\}.$$

It is the convex hull of the set $\{v_1 + v_2 \mid v_i \text{ is a vertex of } P_i, i = 1, 2\}$. The operator δ_ϕ and the face map L_ϕ is additive under the Minkowski sum:

$$\delta_\phi(P_1 + P_2) = \delta_\phi(P_1) + \delta_\phi(P_2), \quad F_\phi(P_1 + P_2) = F_\phi(P_1) + F_\phi(P_2)$$

for all character ϕ and all polytopes P_1, P_2 .

Example 4.18. Let M be an admissible 3-manifold. The Thurston norm ball $B_x(M) := \{\phi \in H^1(M; \mathbb{R}) \mid x_M(\phi) \leq 1\}$ is a (perhaps non-compact) polyhedron in $H^1(M; \mathbb{R})$; the dual Thurston norm ball is defined to be $B_x^*(M) := \{z \in H_1(M; \mathbb{R}) \mid \phi(z) \leq 1 \text{ for all } \phi \in B_x(M)\}$. Thurston proved that $B_x^*(M)$ is an integral polytope in $H_1(M; \mathbb{R})$ with vertices $\pm v_1, \dots, \pm v_k$, and the Thurston norm ball is given by

$$B_x(M) = \{\phi \in H^1(M; \mathbb{R}) \mid |\phi(v_i)| \leq 1, i = 1, \dots, k\}.$$

A Thurston cone in $H^1(M; \mathbb{R})$ is either an open cone formed by the origin and a face of $B_x(M)$, or a maximal connected component of $H^1(M; \mathbb{R}) \setminus \{0\}$ on which the Thurston norm x_M vanishes. It follows that $H^1(M; \mathbb{R}) \setminus \{0\}$ is the disjoint union of all Thurston cones of various dimensions. A Thurston cone is called top-dimensional if its dimension equals $\dim H^1(M; \mathbb{R})$. The following Lemma 4.19 is Thurston's result stated differently.

Lemma 4.19. *A nonzero character $\phi \in H^1(M; \mathbb{R})$ lies in a top-dimensional Thurston cone if and only if $F_\phi B_x^*(M)$ is a vertex.*

4.5.1. *Polytope group $\mathcal{P}_{\mathbb{Z}}(H)$ and $\mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H)$.* Given the finitely generated abelian group H , define $\mathcal{P}_{\mathbb{Z}}(H)$ to be the Grothendieck group of integral polytopes in $H_1(H; \mathbb{R})$ under the Minkowski sum. More precisely, $\mathcal{P}_{\mathbb{Z}}$ is the abelian group with a generating set

$$\{[P] \mid P \text{ is an non-empty integral polytope in } H_1(H; \mathbb{R})\}$$

and a relation $[P] + [Q] = [P + Q]$ for each pair of non-empty integral polytopes P, Q in $H_1(H; \mathbb{R})$. Any element of $\mathcal{P}_{\mathbb{Z}}(H)$ can be expressed in the formal sum $[P] - [Q]$ for non-empty integral polytopes P, Q in $H_1(H; \mathbb{R})$ and

$$[P_1] - [Q_1] = [P_2] - [Q_2] \iff [P_1] + [Q_2] = [P_2] + [Q_1].$$

Note that every element $h \in H$ determines an one-vertex polytope $[h]$ in $\mathcal{P}_{\mathbb{Z}}(H)$ and this defines an embedding of H into $\mathcal{P}_{\mathbb{Z}}(H)$. Define

$$\mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H) = \mathcal{P}_{\mathbb{Z}}(H)/H.$$

In other words, two polytopes are identified in $\mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H)$ if and only if they differ by a translation with an element in the lattice H .

The operators δ_ϕ and F_ϕ naturally extends to the polytope groups:

$$\begin{aligned} \delta_\phi : \mathcal{P}_{\mathbb{Z}}(H) &\rightarrow \mathbb{R}, & \delta_\phi([P] - [Q]) &= \delta_\phi(P) - \delta_\phi(Q), \\ F_\phi : \mathcal{P}_{\mathbb{Z}}(H) &\rightarrow \mathcal{P}_{\mathbb{Z}}(H), & F_\phi([P] - [Q]) &= [F_\phi(P)] - [F_\phi(Q)]. \end{aligned}$$

Since the face map F_ϕ preserves the subgroup H , it naturally induces the face map $F_\phi : \mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H) \rightarrow \mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H)$.

4.5.2. *Polytope homomorphism \mathbb{P} .* Given a finitely generated group G satisfying the Atiyah Conjecture. Let H be the free abelianization of G , then we have the short exact sequence

$$1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1.$$

Recall that $\mathcal{D}_K * H$ is the twisted group ring embedded in \mathcal{D}_G . If given any section $s : H \rightarrow G$ then any element u has a unique expression

$$u = \sum_{h \in H} x_h \cdot s(h).$$

For any nonzero $u \in \mathcal{D}_K * H$, define $\mathbb{P}(u)$ to be the integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H$ spanned by $\{h \in H \mid x_h \neq 0\}$. This polytope does not depend on the choice of the section $s : H \rightarrow G$. It is proved in [FL19, Lemma 6.4] that

$$\mathbb{P}(uv) = \mathbb{P}(u) + \mathbb{P}(v)$$

for all nonzero $u, v \in \mathcal{D}_K * H$. Recall that the Linnell's skew field \mathcal{D}_G is the field of fraction of $\mathcal{D}_K * H$, we then define the polytope homomorphism as

$$\mathbb{P} : \mathcal{D}_G^\times \rightarrow \mathcal{P}_{\mathbb{Z}}(H), \quad \mathbb{P}(uv^{-1}) := [\mathbb{P}(u)] - [\mathbb{P}(v)]$$

for all nonzero $u, v \in \mathcal{D}_K * H$. This homomorphism is well-defined. The following commutative diagram is immediate from the definition.

$$\begin{array}{ccc} \mathcal{D}_G^\times & \xrightarrow{L_\phi} & \mathcal{D}_G^\times \\ \downarrow \mathbb{P} & & \downarrow \mathbb{P} \\ \mathcal{P}_{\mathbb{Z}}(H) & \xrightarrow{F_\phi} & \mathcal{P}_{\mathbb{Z}}(H). \end{array}$$

Recall that $K_1(\mathcal{D}_G)$ is the abelianization of \mathcal{D}_G^\times and $\text{Wh}(\mathcal{D}_G) = K_1(\mathcal{D}_G)/[\pm G]$, then the polytope homomorphism naturally induces

$$\mathbb{P} : K_1(\mathcal{D}_G) \rightarrow \mathcal{P}_{\mathbb{Z}}(H), \quad \mathbb{P} : \text{Wh}(\mathcal{D}_G) \rightarrow \mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H).$$

To save notation, we still use the same symbol \mathbb{P} for the induced homomorphisms. We have the commutative diagram for the induced homomorphisms

$$\begin{array}{ccc} \text{Wh}(\mathcal{D}_G) & \xrightarrow{L_\phi} & \text{Wh}(\mathcal{D}_G) \\ \downarrow \mathbb{P} & & \downarrow \mathbb{P} \\ \mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H) & \xrightarrow{F_\phi} & \mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H). \end{array}$$

We record the following important result which relate the universal L^2 -torsion of a 3-manifold with its dual Thurston norm ball. Note that the minus sign come out since the L^2 -torsion polytope is defined to be the negative of the image of the universal L^2 -torsion under the polytope homomorphism [FL17, Definition 4.21].

Theorem 4.20 ([FL17, Theorem 4.37]). *Let M be an admissible 3-manifold which is not homeomorphic to $S^1 \times D^2$. Then*

$$-[B_x^*(M)] = 2 \cdot \mathbb{P}(\tau_u^{(2)}(M)) \in \mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H_1(M)_f),$$

where $B_x^* \subset H_1(M; \mathbb{R})$ is the dual Thurston norm ball, and $\tau_u^{(2)}(M) \in \text{Wh}(\mathcal{D}_{\pi_1(M)})$ is the universal L^2 -torsion of M .

4.6. Half of Theorem 1.2. We are ready to prove the “if” part of Theorem 1.2. A stronger version of the “only if” part will be stated and proved in Theorem 5.14 after establishing Theorem 1.3.

Theorem 4.21 (The “if” part of Theorem 1.2). *Suppose M is an admissible 3-manifold such that M is not a closed graph manifold which admits no NPC metric. Let $\phi \in H^1(M; \mathbb{R})$ be a nonzero class such that $L_\phi \tau_u^{(2)}(M) = 1 \in \text{Wh}(\mathcal{D}_{\pi_1(M)})$, then ϕ is fibered.*

Proof. If M is homeomorphic to the solid torus then any nonzero class is fibered. Let's assume that M is not the solid torus. The assumption posed on M implies that $\pi_1(M)$ is virtually special by a combination of work [AGM13, Liu13, PW18] c.f. [AFW15, Section 4.7], in particular it follows from Agol's criterion for virtual fibering [Ago08] that there is a regular finite covering

$\overline{M} \rightarrow M$ such that the pull back class $\bar{\phi}$ lies in the closure of a fibered cone. Let G be the fundamental group of M and $\pi_1(\overline{M}) =: L < G$. Then by the restriction property of Theorem 3.10(4)

$$\tau_u^{(2)}(\overline{M}) = \text{res}_L^G \tau_u^{(2)}(M) \in \text{Wh}(\mathcal{D}_L).$$

By Theorem 4.16 the restriction map commutes with the leading term map,

$$L_{\bar{\phi}}(\tau_u^{(2)}(\overline{M})) = L_{\bar{\phi}}(\text{res}_L^G \tau_u^{(2)}(M)) = \text{res}_L^G(L_{\phi}\tau_u^{(2)}(M)) = 1.$$

Applying the polytope map \mathbb{P} to both sides we get

$$\mathbb{P}(L_{\bar{\phi}}(\tau_u^{(2)}(\overline{M}))) = 0 \in \mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H_1(\overline{M}; \mathbb{R})).$$

Note that $\mathbb{P}(\tau_u^{(2)}(\overline{M})) = 2 \cdot [B_x^*(\overline{M})]$ by Theorem 4.20, we have that

$$0 = \mathbb{P}(L_{\bar{\phi}}(\tau_u^{(2)}(\overline{M}))) = F_{\bar{\phi}}\mathbb{P}(\tau_u^{(2)}(\overline{M})) = -2 \cdot F_{\bar{\phi}}[B_x^*(\overline{M})].$$

Since $\mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H_1(\overline{M}; \mathbb{R}))$ is torsion-free by [FL17, Lemma 4.8], we obtain

$$F_{\bar{\phi}}[B_x^*(\overline{M})] = 0.$$

It follows from Lemma 4.19 that $\bar{\phi}$ lies in a top dimensional Thurston cone of $H^1(\overline{M}; \mathbb{R})$. So $\bar{\phi}$ must lie in the interior of the fibered cone since the boundary of a fibered cone consists of Thurston cones of strictly lower dimensions. Therefore $\bar{\phi}$ is a fibered class for \overline{M} and it follows that ϕ is a fibered class for M . \square

5. UNIVERSAL L^2 -TORSION FOR TAUT SUTURED MANIFOLDS

In this section, we first briefly recall the terminologies of the sutured manifold theory, then we discuss the universal L^2 -torsion of a taut sutured manifold and prove the decomposition formula Theorem 1.3, which is then used to finish the proof of Theorem 1.2. After that, we use a doubling trick to prove Theorem 1.4.

5.1. Background on sutured manifold theory. Throughout this section N will be an arbitrary compact, oriented 3-manifold.

Definition 5.1 (Taut surfaces). Given a compact orientable surface Σ with path-components $\Sigma_1, \dots, \Sigma_k$ we define its *complexity* as

$$\chi_-(\Sigma) := \sum_{i=1}^k \max\{0, -\chi(\Sigma_i)\}.$$

A properly embedded oriented surface Σ in N is *taut* if Σ is incompressible, and has minimal complexity among all properly embedded oriented surfaces representing the homology class $[\Sigma, \partial\Sigma] \in H_2(N, \nu(\partial\Sigma); \mathbb{Z})$, where $\nu(\partial\Sigma)$ is a regular neighborhood of $\partial\Sigma$ in ∂N .

Definition 5.2 (Sutured manifold). A *sutured manifold* (N, R_-, R_+, γ) is a compact oriented 3-manifold N with a decomposition of its boundary into two subsurfaces R_+ and R_- along their common boundary γ which we call the *suture*. The orientation on R_{\pm} is defined in the way that the normal vector of R_+ points out of N and the normal vector of R_- points inward of N . The boundary orientations of R_{\pm} on γ coincide and induce an orientation of the suture γ . We sometimes abbreviate a sutured manifold (N, R_-, R_+, γ) as (N, γ) where no confusions caused.

A sutured manifold (N, R_-, R_+, γ) is called *taut* if N is irreducible and R_{\pm} are both taut surfaces (viewed as properly embedded surfaces after pushing slightly into N).

Before we introduce the important notion of sutured manifold decompositions, let's fix some notations. Let S be a (not necessarily connected) properly embedded oriented surface in N . Choose a product neighborhood $S \times (-1, 1)$ of S in N and denote by $N \setminus\!\!\setminus S := N \setminus S \times (-1, 1)$ the complement. Let S_+ (resp. S_-) be the components of $S \times \{-1\} \cup S \times \{1\}$ in $N \setminus\!\!\setminus S$ whose normal vector points out of (resp. into) N' . This convention of notation has the advantage that $(N \setminus\!\!\setminus \Sigma, \Sigma_+, \Sigma_-, \emptyset)$ is a sutured manifold when N is closed.

Definition 5.3 (Decomposition surface). Let (N, R_-, R_+, γ) be a sutured manifold and S a properly embedded surface in N such that ∂S is transverse to γ on ∂N . Furthermore, no component of ∂S bounds a disk in R_\pm and no component of S is a disk D with $\partial D \subset R_\pm$. Then we call S a *decomposition surface* for (N, γ) .

Definition 5.4 (Sutured manifold decomposition). Given an oriented decomposition surface S for (N, R_-, R_+, γ) , define the *sutured manifold decomposition*

$$(N, R_-, R_+, \gamma) \xrightarrow{S} (N', R'_-, R'_+, \gamma')$$

where

$$\begin{aligned} N' &= N \setminus\!\!\setminus S, \\ R'_+ &= (R_+ \cup S_+) \cap N', \\ R'_- &= (R_- \cup S_-) \cap N', \\ \gamma' &= \partial R'_+ = \partial R'_-. \end{aligned}$$

See Figure 3–4 for illustrations. A sutured manifold decomposition $(N, \gamma) \xrightarrow{S} (N', \gamma')$ is called a *taut sutured decomposition* if (N', γ') is taut. Gabai [Gab87, Lemma 0.4] proved that if $(N, \gamma) \xrightarrow{S} (N', \gamma')$ is a taut sutured decomposition then (N, γ) is taut.

We make the following remarks:

- (1) Like [AD19], we view sutures as simple closed curves and do not allow torus sutures, which differs from many other sources where the suture are disjoint union of annuli and tori. In fact, if given a sutured manifold (N, γ) which contains toral sutures, we can arbitrarily absorb the toral sutures into R_\pm without changing the universal L^2 -torsion. Namely, we have the following:

Lemma 5.5. *If (N, γ) is a taut sutured manifold and $T \subset \gamma$ is a torus component. Then $\tau_u^{(2)}(N, R_+) = \tau_u^{(2)}(N, R_+ \cup T)$.*

Proof. This follows from the Sum Formula 3.10 and the fact that $\tau_u^{(2)}(T) = 1$. \square

The definition of sutured manifold decomposition is modified accordingly. In the classical definition of decomposition surface S (c.f. [Gab83, Gab87]), a component of S can be the core of a sutured annulus. If so, S can be slightly isotoped into a decomposition surface S' such that $\partial S'$ lies completely in R_+ or R_- , and decompositions along S and S' produce isomorphic sutured manifolds.

- (2) Suppose $(N, R_+, R_-, \gamma) \xrightarrow{S} (N', R'_+, R'_-, \gamma')$ is a taut sutured decomposition, then S is incompressible in N . The reason is as follows: R'_+ is the union of the surfaces S_+ and $R_+ \cap N'$ along their common part $S_+ \cap R_+$. This common part consists of some arcs and some boundary circles of S_+ . By assumption, no boundary circles of S_+ bound a disk in R_+ or a disk in S_+ , so the closed curves of $S_+ \cap R_+$ are homotopy nontrivial in both S_+ and $R_+ \cap N'$. By Van Kampen Theorem the surface S_+ is π_1 -injective in R'_+ , therefore π_1 -injective in N' since R'_+ is incompressible in N' . This proves that S can not admit a compressing disk in N by .

- (3) Given a taut sutured decomposition $(N, \gamma) \xrightarrow{S} (N', \gamma')$. Since S is incompressible in N , it follows that for any component of N' the inclusion into N induces monomorphism on fundamental groups. The pair $(N', R'_+) = (N \setminus S, (S_+ \cup R_+) \cap (N \setminus S))$ will be abbreviated as $(N \setminus S, S_+ \cup R_+)$, since a CW-pair (X, Y) always means $(X, X \cap Y)$ in order to make sense.

5.2. Universal L^2 -torsion for taut sutured 3-manifolds.

Theorem 5.6 ([Her23]). *Let (N, γ) be a sutured 3-manifold with $\pi_1(N)$ infinite. Suppose that N is irreducible and R_\pm are both incompressible. Then (N, γ) is taut if and only if the pair (N, R_+) is L^2 -acyclic.*

Proof. Suppose (N, γ) is taut. No components of R_\pm are disks or spheres, otherwise N must be the 3-ball whose fundamental group is trivial. It follows that the complexity of R_\pm equals $-\chi(R_\pm)$, and we have $\chi(R_+) = \chi(R_-)$, then we apply [Her23, Theorem 1.1] to conclude that (N, R_+) is L^2 -acyclic.

Suppose on the other hand that the pair (N, R_+) is L^2 -acyclic, then the Euler characteristic $\chi(N, R_+) = \chi(N) - \chi(R_+)$ is zero. But $\chi(N) = \frac{1}{2}\chi(\partial N) = \frac{1}{2}(\chi(R_+) + \chi(R_-))$, it follows that $\chi(R_+) = \chi(R_-)$, then we apply [Her23, Theorem 1.1]. \square

The reader might wonder why we prefer (N, R_+) than (N, R_-) . In fact, these two pairs are dual to each other by the following Proposition 5.7. We prefer the pair (N, R_+) since it suits our convention of orientations better.

Proposition 5.7. *Let (N, R_+, R_-, γ) be a sutured manifold. Suppose the pair (N, R_+) is L^2 -acyclic. Then (N, R_-) is also L^2 -acyclic. Moreover*

$$\tau_u^{(2)}(N, R_+) = \overline{\tau_u^{(2)}(N, R_-)}.$$

Proof. The proof is a modification of [Mil62, Page 139] and does not use any L^2 -theories. Form a smooth triangulation of N such that R_+ and R_- are subcomplexes. Let \widehat{N} be the universal cover and let \widehat{R}_\pm be the preimage of R_\pm . Consider the dual cellular complex \widehat{N}' of \widehat{N} . A cell of \widehat{N} is canonically dual to a cell of $\widehat{N}' \setminus \partial \widehat{N}'$, see Figure 2 below.

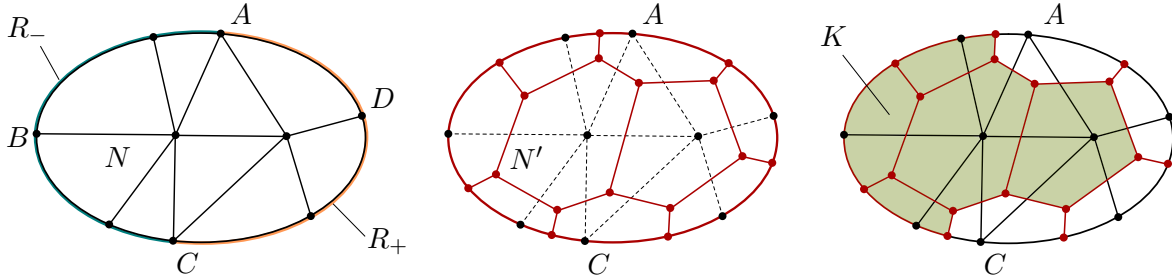


Figure 2. *An illustration of the one-dimension-lower case. The figure on the left shows a simplicial complex N whose boundary is a union of a subcomplex R_- (the arc \widehat{ABC}) and a subcomplex R_+ (the arc \widehat{ADC}). The figure in the middle shows the dual cellular complex N' (in red). The last figure shows the subcomplex $K \subset N'$ which is the union of cells disjoint from R_+ . The complex N' deformation retracts to K along a product neighborhood of R_+ .*

More explicitly, we have the intersection pairing

$$p : C_*(\widehat{N}) \times C_{3-*}(\widehat{N}', \partial \widehat{N}') \rightarrow \mathbb{Z}G, \quad p(\sigma, \sigma') = \sum_{g \in G} \langle \sigma, g\sigma' \rangle \cdot g$$

where $\langle \sigma, g\sigma' \rangle$ is the intersection number of σ and $g\sigma'$. It is easy to verify the identities

$$\begin{aligned} p(g\sigma, \sigma') &= g \cdot p(\sigma, \sigma'), \\ p(\sigma, g\sigma') &= p(\sigma, \sigma') \cdot g^{-1}, \\ p(\partial\sigma, \sigma') &= \pm p(\sigma, \partial\sigma'). \end{aligned}$$

The pairing is non-degenerate in the sense that $\sigma \mapsto p(\sigma, *)$ gives an isomorphism of $\mathbb{Z}G$ -chain complexes $C_*(\widehat{N}) \cong C^{3-*}(\widehat{N}', \partial\widehat{N}')$. Note that a cell of \widehat{R}_+ is canonically dual to a cell of $\widehat{N}' \setminus \partial\widehat{N}'$ which intersects \widehat{R}_+ at a non-empty set and vice versa. Let K be the union of the cells of N' which is disjoint with R_+ and let \widehat{K} be the preimage in the universal cover, then there is an induced non-degenerate pairing

$$C_*(\widehat{N}, \widehat{R}_+) \times C_{3-*}(\widehat{K}, \partial\widehat{N}') \rightarrow \mathbb{Z}G$$

and hence the isomorphism of $\mathbb{Z}G$ -chain complexes $C_*(\widehat{N}, \widehat{R}_+) \cong C^{3-*}(\widehat{K}, \partial\widehat{N}')$. By Proposition 3.6 the finite based free $\mathbb{Z}G$ -chain complex $C_*(\widehat{K}, \partial\widehat{N}')$ is L^2 -acyclic and

$$\tau_u^{(2)}(C_*(\widehat{N}, \widehat{R}_+)) = \overline{\tau_u^{(2)}(C_*(\widehat{K}, \partial\widehat{N}'))},$$

therefore $\tau_u^{(2)}(N, R_+) = \overline{\tau_u^{(2)}(K, \partial N')}$. The pair $(K, \partial N')$ is a deformation retract of the pair (N', R'_-) (barycentric-subdivide N , if necessary), hence $\tau_u^{(2)}(N, R_+) = \overline{\tau_u^{(2)}(N', R'_-)}$. This shows that the pair (N, R_-) is L^2 -acyclic with $\tau_u^{(2)}(N, R_+) = \tau_u^{(2)}(N, R_-)$. \square

5.3. Turaev's algorithm.

Lemma 5.8. *Let (N, γ) be a taut sutured manifold. Suppose there is a decomposition surface S such that*

- (1) *The sutured manifold decomposition $(N, R_+, R_-, \gamma) \xrightarrow{S} (N', R'_+, R'_-, \gamma')$ is taut.*
- (2) *S is separating and N' is a disjoint union of two (possibly non-connected) manifolds N_0, N_1 .*
- (3) *$S_- \subset N_0$, $S_+ \subset N_1$.*

Then $\tau_u^{(2)}(N, R_+) = i_\tau_u^{(2)}(N', R'_+)$ where $i : N' \hookrightarrow N$ is the inclusion.*

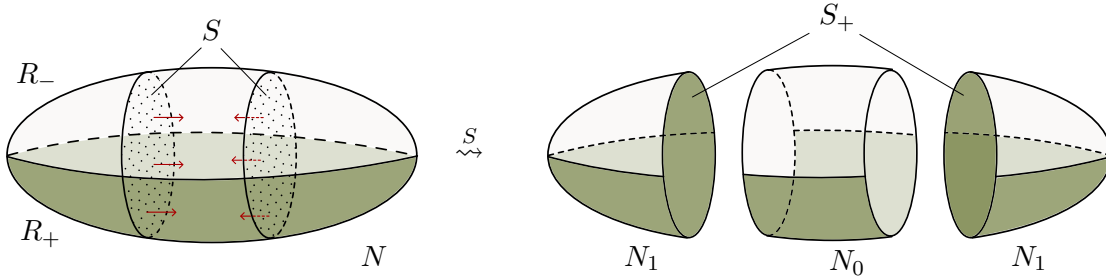


Figure 3. *Decomposing N along a separating decomposition surface S (the dotted region in the left figure whose normal direction as indicated by the red arrows). The R_+ -part of the sutured manifolds are in green.*

Proof. See Figure 3 for an illustration of the decomposition. There is a short exact sequence of chain complexes

$$0 \rightarrow C_*^{(2)}(N_0, N_0 \cap R_+) \rightarrow C_*^{(2)}(N, R_+) \rightarrow C_*^{(2)}(N, N_0 \cup R_+) \rightarrow 0$$

and a natural isomorphism $C_*^{(2)}(N, N_0 \cup R_+) = C_*^{(2)}(N_1, (S_+ \cup R_+) \cap N_1)$. Note that $N_0 \cap R_+$ is the R_+ -part of N_0 and $(S_+ \cup R_+) \cap N_1$ is the R_+ -part of N_1 . By assumption N_0 and N_1 are taut sutured manifold. It follows from Theorem 5.6 that the three chain complexes are L^2 -acyclic and

$$\tau_u^{(2)}(N, R_+) = (i_0)_* \tau_u^{(2)}(N_0, N_0 \cap R_+) \cdot (i_1)_* \tau_u^{(2)}(N_1, (S_+ \cup R_+) \cap N_1)$$

where i_0 and i_1 are the inclusion of N_0 and N_1 into N . The right hand side of the above equation is exactly $i_* \tau_u^{(2)}(N', R'_+)$ and the proof is finished. \square

Definition 5.9 (Weighted surface). A *weighted surface* \widehat{S} in a compact oriented 3-manifold N is a collection of pairs (S_i, w_i) , $i = 1, \dots, n$, where S_i is a connected properly embedded oriented surface, w_i is a positive integer, and $S_i \cap S_j = \emptyset$ for $i \neq j$. The *realization* of \widehat{S} is the properly embedded oriented surface

$$\bar{S} := \bigcup_{i=1}^n w_i \cdot S_i$$

where $w_i \cdot S_i$ is the union of w_i parallel copies of S_i . The *reduction* of \widehat{S} is the properly embedded oriented surface

$$S := \bigcup_{i=1}^n S_i.$$

A *weighted decomposition surface* is a weighted surface whose realization is a decomposition surface.

If \widehat{S} is a weighted decomposition surface, then $N \setminus \bar{S}$ is a disjoint union of $N \setminus S$ and some copies of $S \times I$. So the sutured decomposition along \bar{S} is taut if and only if the sutured decomposition along S is taut.

The following Proposition 5.10 is a generalization of [BAFH22, Proposition 3.3]. The idea goes back to Turaev [Tur02].

Proposition 5.10. *Let (N, R_+, R_-, γ) be a taut sutured manifold and let $(N, \gamma) \xrightarrow{\Sigma} (N', \gamma')$ be a taut sutured decomposition. Then there is a weighted decomposition surface \widehat{S} in N (with \bar{S} the realization and S the reduction) such that*

- (1) $N \setminus S$ is connected,
- (2) $[\bar{S}] = [\Sigma] \in H_2(N, \partial N; \mathbb{Z})$,
- (3) the sutured decomposition of N along S is taut,
- (4) $i_* \tau_u^{(2)}(N \setminus S, S_+ \cup R_+) = j_* \tau_u^{(2)}(N \setminus \Sigma, \Sigma_+ \cup R_+)$ where i, j are the natural inclusions of $N \setminus S$ and $N \setminus \Sigma$ into N .

Proof. For any weighted surface \widehat{S} in N , define $c(\widehat{S}) := \#\pi_0(N \setminus S)$. First take \widehat{S} to be the surface Σ with weight 1 assigned to each component, then $\bar{S} = S = \Sigma$ and \widehat{S} satisfies (2)–(4). It suffices to prove that given any weighted surface \widehat{S} with $c(\widehat{S}) > 1$ such that (2)–(4) holds, then there exists a weighted surface \widehat{T} such that (2)–(4) holds and $c(\widehat{T}) < c(\widehat{S})$.

Given such \widehat{S} . Since $c(\widehat{S}) > 1$ there is a component $C \subset S$ such that C_{\pm} lies in different components of $N \setminus S$. Choose C to be a component with minimal weight w among such components, let M_0 (resp. M_1) be the component of $N \setminus S$ containing C_- (resp. C_+). Let $C_1 = C, C_2, \dots, C_k$ be the components of S whose normal direction points into M_1 and let D_1, \dots, D_l be the components of S whose normal direction points out of M_1 . It may happen that $C_i = D_j$ for some i, j , in this case the two sides of C_i both belong to M_1 . It follows that

$$[C_1] + \dots + [C_k] = [D_1] + \dots + [D_l] \in H_2(N, \partial N; \mathbb{Z}).$$

We change the weights of \widehat{S} by increasing the weights of D_1, \dots, D_l by w , and decreasing the weights of C_1, \dots, C_k by w . If a component has weight zero, we simply discard this component. This new weighted surface is denoted by \widehat{T} . Clearly, we have $[\widehat{S}] = [\widehat{T}] \in H_2(N, \partial N; \mathbb{Z})$. Since T is a subcollection of S and the decomposition along S is taut, it follows that the decomposition along T is also taut. So (2), (3) holds true for \widehat{T} . Moreover, $c(\widehat{T}) < c(\widehat{S})$ since M_0 and M_1 are in the same component of $N \setminus T$.

Finally, let S_0 be $S \setminus T$. Then $N \setminus S$ is obtained from the sutured manifold decomposition of $N \setminus T$ along S_0 . By construction, S_0 separates $N \setminus T$; in particular, it separates M_0 from M_1 and $(S_0)_+ \subset M_1$. Apply Lemma 5.8 to $N_0 := (N \setminus S) - M_1$ and $N_1 := M_1$, we have

$$i'_* \tau_u^{(2)}(N \setminus T, T_+ \cup R_+) = i_* \tau_u^{(2)}(N \setminus S, S_+ \cup R_+) = j_* \tau_u^{(2)}(N \setminus \Sigma, \Sigma_+ \cup R_+)$$

where i, i', j' are the inclusions of $N \setminus S$, $N \setminus T$ and $N \setminus \Sigma$ into N , respectively. This verifies (4) for \widehat{T} and finishes the proof. \square

5.4. Decomposition formula for universal L^2 -torsion.

Definition 5.11. Let G be a group and $\phi \in H^1(G; \mathbb{R})$ an 1-cohomology class. There are natural homomorphisms

$$\delta_\phi : (\mathbb{Z}G)^\times \rightarrow \mathbb{R}, \quad L_\phi : (\mathbb{Z}G)^\times \rightarrow (\mathbb{Z}G)^\times$$

defined by considering the terms with minimal ϕ -value, c.f. Section 4.3. For any nonzero matrix A over $\mathbb{Z}G$, let $\delta_\phi(A)$ be the smallest real number $\delta_\phi(A_{ij})$ among all nonzero entries A_{ij} . Then we can decompose the matrix in a unique way

$$A = L_\phi(A) + (A - L_\phi(A))$$

where any group element g appearing in $L_\phi(A)$ satisfies $\phi(g) = \delta_\phi(A)$, and any group element h appearing in $(A - L_\phi(A))$ satisfies $\phi(h) > \delta_\phi(A)$.

We define $L_\phi(A) = 0$ when A is a zero matrix. Otherwise $L_\phi(A)$ is always nonzero.

For any finite based free $\mathbb{Z}G$ -chain complex

$$C_* = (0 \longrightarrow C_n \xrightarrow{A_n} \dots \xrightarrow{A_2} C_1 \xrightarrow{A_1} C_0 \longrightarrow 0),$$

define

$$L_\phi(C_*) = (0 \longrightarrow C_n \xrightarrow{L_\phi(A_n)} \dots \xrightarrow{L_\phi(A_2)} C_1 \xrightarrow{L_\phi(A_1)} C_0 \longrightarrow 0).$$

It is easy to verify that $L_\phi(A)L_\phi(B) = L_\phi(AB)$ holds for arbitrary matrices A, B . In particular $L_\phi(C_*)$ is a well-defined finite based free $\mathbb{Z}G$ -chain complex.

Lemma 5.12. Suppose G is a torsion-free group satisfying the Atiyah Conjecture. Let $\phi \in H^1(G; \mathbb{R})$ be an 1-cohomology class and let

$$C_* = (0 \longrightarrow C_n \xrightarrow{A_n} \dots \xrightarrow{A_2} C_1 \xrightarrow{A_1} C_0 \longrightarrow 0)$$

be a finite based free $\mathbb{Z}G$ -chain complex. Suppose that $L_\phi(C_*)$ is L^2 -acyclic, then C_* is also L^2 -acyclic and

$$\tau_u^{(2)}(L_\phi(C_*)) = L_\phi(\tau_u^{(2)}(C_*)).$$

Proof. Suppose that $L_\phi(C_*)$ is L^2 -acyclic, then by Theorem 3.19 we can find a non-degenerate matrix chain \mathcal{A} of $L_\phi(C_*)$. Let B_1, \dots, B_n be the associated submatrices of $L_\phi(A_1), \dots, L_\phi(A_n)$, then B_i are weak isomorphisms and

$$\tau_u^{(2)}(L_\phi(C_*)) = \prod_{i=1}^n [B_i]^{(-1)^n} \in \widetilde{K}_1(\mathcal{D}_G)$$

Let C_1, \dots, C_n be the submatrices of A_1, \dots, A_n associated to the same matrix chain \mathcal{A} . Since B_i are nonzero, then $L_\phi(C_i) = B_i$. By Theorem 4.11 we have

$$L_\phi[C_i] = [B_i] \in \widetilde{K}_1(\mathcal{D}_G).$$

In particular, C_i are weak isomorphisms and therefore \mathcal{A} is a non-degenerate matrix chain of C_* . By Theorem 3.19,

$$\tau_u^{(2)}(C_*) = \prod_{i=1}^n [C_i]^{(-1)^n} \in \widetilde{K}_1(\mathcal{D}_G)$$

and in particular $\tau_u^{(2)}(L_\phi(C_*)) = L_\phi(\tau_u^{(2)}(C_*))$. \square

Theorem 5.13 (Theorem 1.3). *Let (N, γ) be a taut sutured manifold and let $(N, \gamma) \xrightarrow{\Sigma} (N', \gamma')$ be a taut sutured decomposition. Let $\phi = PD([\Sigma, \partial\Sigma]) \in H^1(N; \mathbb{Z})$ be the Poincaré dual of the surface Σ , then*

$$j_*\tau_u^{(2)}(N', R'_+) = L_\phi(\tau_u^{(2)}(N, R_+))$$

where $j : N' \hookrightarrow N$ is the natural inclusion.

Proof. We find a weighted decomposition surface \widehat{S} in N as provided by Proposition 5.10. Then $N \setminus S$ is connected, $\phi = PD([\Sigma, \partial\Sigma]) = PD([\widehat{S}, \partial\widehat{S}]) \in H^1(N; \mathbb{Z})$ and $j_*\tau_u^{(2)}(N', R'_+) = i_*\tau_u^{(2)}(N \setminus S, S_+ \cup R_+)$ where $i : N \setminus S \rightarrow N$ is the inclusion. We are left to show that

$$L_\phi(\tau_u^{(2)}(N, R_+)) = i_*\tau_u^{(2)}(N \setminus S, S_+ \cup R_+).$$

Find a CW-structure for N such that $S \times I$ and R_\pm are subcomplexes. Fix a base point

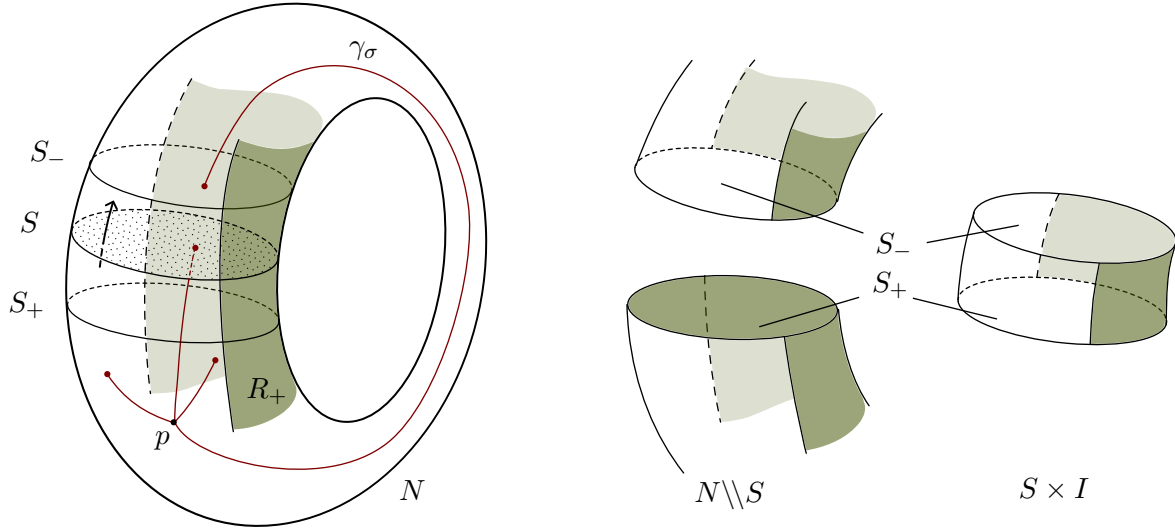


Figure 4. Consider N as the union of $N \setminus S$ and $S \times I$.

$p \in N \setminus (S \times I)$. For any cell σ in the CW-structure of N , choose a path γ_σ (the paths in red in Figure 4) connecting p and σ such that

- γ_σ is disjoint with S_- , if $\sigma \subset N \setminus S_-$,
- γ_σ is disjoint with S_+ , if $\sigma \subset S_-$.

Lift the base point p to \widehat{p} in the universal cover \widehat{N} and lift each cell σ to $\widehat{\sigma}$ using the path γ_σ . The cells $\widehat{\sigma}$ form a basis for the finite based free $\mathbb{Z}[\pi_1(N)]$ -chain complex $C_*(\widehat{N})$. Now consider the

L^2 -cellular chain complex $C_*^{(2)}(N, R_+)$, for each $k = 0, 1, 2, 3$ the chain group admits the following direct sum decomposition

$$C_k^{(2)}(N, R_+) = C_k^{(2)}(N \setminus S, S_+ \cup R_+) \oplus C_k^{(2)}(S \times I, S_- \cup R_+).$$

Accordingly, the boundary homomorphism $\partial_* : C_*^{(2)}(N, R_+) \rightarrow C_{*-1}^{(2)}(N, R_+)$ admits the following decomposition

$$\begin{aligned} \partial_* &= \begin{pmatrix} \partial_*^1 & \partial_*^2 \\ \partial_*^3 & \partial_*^4 \end{pmatrix}, \\ \partial_*^1 &: C_*^{(2)}(N \setminus S, S_+ \cup R_+) \rightarrow C_{*-1}^{(2)}(N \setminus S, S_+ \cup R_+), \\ \partial_*^2 &: C_*^{(2)}(N \setminus S, S_+ \cup R_+) \rightarrow C_{*-1}^{(2)}(S \times I, S_- \cup R_+), \\ \partial_*^3 &: C_*^{(2)}(S \times I, S_- \cup R_+) \rightarrow C_{*-1}^{(2)}(N \setminus S, S_+ \cup R_+), \\ \partial_*^4 &: C_*^{(2)}(S \times I, S_- \cup R_+) \rightarrow C_{*-1}^{(2)}(S \times I, S_- \cup R_+). \end{aligned}$$

In particular, if $\hat{\sigma}$ is a basis element of $C_*^{(2)}(S \times I, S_- \cup R_+)$ then there are basis elements τ_i (allowing repetitions) of $C_{*-1}^{(2)}(N \setminus S, S_+ \cup R_+)$ and $g_i \in \pi_1(N)$ such that

$$\partial_*^3(\hat{\sigma}) = \sum_i g_i \hat{\tau}_i.$$

Then the cells $\hat{\tau}_i$ must lie in \hat{S}_- . By our choice of basis, each g_i satisfies $\phi(g_i) > 0$. A similar argument shows that any group element h appearing in $\partial_*^1, \partial_*^2$ or ∂_*^4 satisfies $\phi(h) = 0$. It follows that

$$L_\phi(\partial_*) = \begin{pmatrix} \partial_*^1 & \partial_*^2 \\ 0 & \partial_*^4 \end{pmatrix}$$

and we hence obtain a short exact sequence of chain complexes

$$0 \rightarrow C_*^{(2)}(N \setminus S, S_+ \cup R_+) \rightarrow L_\phi(C_*^{(2)}(N, R_+)) \rightarrow C_*^{(2)}(S \times I, S_- \cup R_+) \rightarrow 0.$$

Then by the product formula 3.5 and the induction property Theorem 3.10 we have

$$\tau_u^{(2)}(L_\phi(C_*^{(2)}(N, R_+))) = i_* \tau_u^{(2)}(N \setminus S, S_- \cup R_+) \cdot i'_* \tau_u^{(2)}(S \times I, S_- \cup R_+).$$

The left hand side equals $L_\phi(\tau_u^{(2)}(N, R_+))$ by Lemma 5.12. On the right hand side, since $(S_- \cup R_+) \cap (S \times I)$ deformation retracts onto S_- , we have $\tau_u^{(2)}(S \times I, S_- \cup R_+) = \tau_u^{(2)}(S \times I, S_-) = 1$. It follows that $L_\phi(\tau_u^{(2)}(N, R_+)) = i'_* \tau_u^{(2)}(N \setminus S, S_+ \cup R_+)$ and the proof is finished. \square

The (stronger) “only if” part of Theorem 1.2 follows easily.

Theorem 5.14 (The “only if” part of Theorem 1.2). *Suppose M is an admissible 3-manifold and $\phi \in H^1(M; \mathbb{R})$ is a fibered class. Then $L_\phi \tau_u^{(2)}(M) = 1 \in \text{Wh}(\mathcal{D}_G)$.*

Proof. By Proposition 4.7(7) we can find a rational fibered class $\psi \in H^1(M; \mathbb{Q})$ arbitrarily close to ϕ such that $L_\phi(\tau_u^{(2)}(M)) = L_\psi(\tau_u^{(2)}(M))$. Choose a positive integer n such that $n\psi$ is an integral fibered class, and let S be a fiber surface dual to $n\psi$, then $M \setminus S = S \times [0, 1]$ is a product. By Theorem 1.3 we have

$$L_{n\psi} \tau_u^{(2)}(M) = j_* \tau_u^{(2)}(M \setminus S, S_+) = 1,$$

where $j : M \setminus S \hookrightarrow M$ is the inclusion. It follows that $L_\phi \tau_u^{(2)}(M) = L_{n\psi} \tau_u^{(2)}(M) = 1$. \square

Combined with Theorem 4.21 the proof of Theorem 1.2 is now complete.

5.5. Proof of Theorem 1.4. We use a refined doubling trick to show that universal L^2 -torsion detects product sutured manifolds.

Definition 5.15 (Double of taut sutured manifolds). Let (N, R_+, R_-, γ) be a taut sutured manifold and let $f : R_+ \rightarrow R_+$ be an orientation-preserving homeomorphism. Let $(N, \bar{R}_+, \bar{R}_-, \bar{\gamma})$ be the sutured manifold whose underlying oriented manifold is the same as N , but with R_+, R_- interchanged (see Figure 5 below). Namely,

$$\bar{\gamma} = -\gamma, \quad \bar{R}_+ = -R_-, \quad \bar{R}_- = -R_+.$$

The *double of N with monodromy f* is defined to be the admissible 3-manifold

$$DN_f = (N, \gamma) \cup (N, \bar{\gamma}) / \sim$$

formed by gluing together the two sutured manifolds with the gluing relation \sim as follows: we identify R_- and \bar{R}_+ via the identity map and identify R_+ and \bar{R}_- via the homeomorphism f .

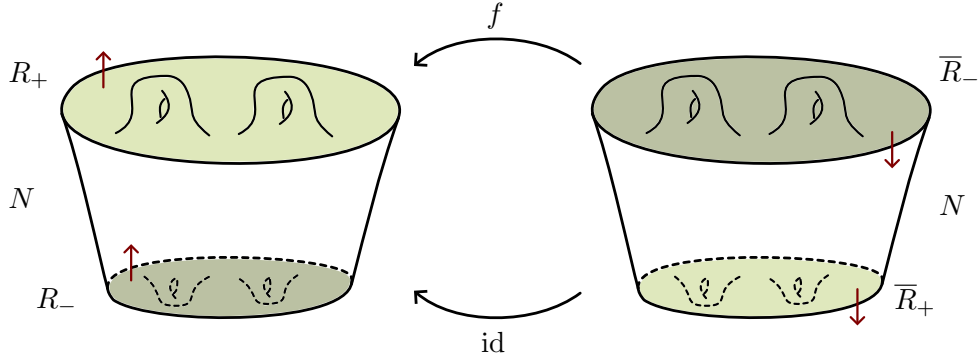


Figure 5. Double of a sutured manifold N with monodromy f

Lemma 5.16. If (N, R_+, R_-, γ) is a taut sutured manifold with R_+ and R_- both non-empty. Suppose $R_+ \cup R_-$ is not a union of tori, then there exists $f : R_+ \rightarrow R_+$ such that DN_f is not a closed graph manifold.

Proof. If N contains non-empty sutured annuli, then DN_f has non-empty torus boundary for any f by construction. If $\pi_1(N)$ is finite then N is the product 3-ball and has non-empty sutured annuli.

If N has infinite fundamental group and does not contain sutured annuli, then each component of R_\pm is a closed surfaces with positive genus. By assumption, $R_+ \cup R_-$ contains a component of genus greater than 1. Since $\chi(R_+) = \chi(R_-)$, we know that R_+ contains a component of genus greater than 1. Denote this component by Σ . Write

$$f = (\phi : \Sigma \rightarrow \Sigma, \text{id} : R_+ \setminus \Sigma \rightarrow R_+ \setminus \Sigma).$$

Let M be the manifold obtained from gluing together (N, γ) and $(N, \bar{\gamma})$ via $\text{id} : R_- \rightarrow \bar{R}_+$ and $\text{id} : R_+ \setminus \Sigma \rightarrow \bar{R}_- \setminus (-\Sigma)$. Then M is a 3-manifold whose boundary are two copies of Σ left unglued. It is clear that DN_f is obtained from gluing $\Sigma_1, \Sigma_2 \subset \partial M$ together via $\phi : \Sigma_1 \rightarrow \Sigma_2$. We are left to prove the following:

Claim. Let M be a compact, orientable, irreducible 3-manifold. Suppose $\partial M = \Sigma_1 \sqcup \Sigma_2$ is the union of two connected incompressible surfaces of the same genus greater than 1. Then there is a homeomorphism $\phi : \Sigma_1 \rightarrow \Sigma_2$ such that the closed manifold M_ϕ formed by gluing together the two boundary surfaces of M via ϕ is not a graph manifold.

Proof of Claim. If $M = \Sigma_1 \times I$ is a product, then a pseudo-Anosov homeomorphism ϕ suffices. Now suppose M is not a product. The Characteristic Submanifold Theorem asserts that there is a submanifold $X \subset M$ which is a disjoint union of Seifert fibered spaces and I -bundle over surfaces, such that any incompressible torus and annulus of M can be homotoped into X [JS79, Chapter V]. The intersections $X \cap \Sigma_1$ and $X \cap \Sigma_2$ are incompressible proper subsurfaces of Σ_1 and Σ_2 respectively; otherwise let's suppose $\Sigma_1 \subset X$, then Σ_1 is contained in an I -bundle component P of X . Since $\partial P \subset \partial M$, P must be the entire M , contradicting our assumption. Let C_i be a component of $\partial(X \cap \Sigma_i)$ which is an essential simple closed curve on Σ_i , $i = 1, 2$. Choose a homeomorphism $\phi : \Sigma_1 \rightarrow \Sigma_2$ such that the two simple closed curves $\phi(C_1)$ and C_2 fill Σ_2 . This can be done by assigning $\phi = \psi^n \circ \phi_0$ for any homeomorphism $\phi_0 : \Sigma_1 \rightarrow \Sigma_2$, any pseudo-Anosov homeomorphism $\psi : \Sigma_2 \rightarrow \Sigma_2$ and a sufficiently large integer n . In this case the distance between $[\phi(C_1)]$ and $[C_2]$ on the curve complex of Σ_2 will be arbitrarily large [MM99, Proposition 4.6]. In particular $\phi(C_1)$ fills with C_2 .

We now show that the glued-up manifold M_f is not a closed graph manifold. Suppose $T \subset M_f$ is an essential torus. Up to isotopy we assume that T is transverse to $\Sigma = \Sigma_1 = \Sigma_2$ in M_f , and the intersections $T \cap \Sigma$ (if non-empty) are essential simple closed curves on Σ . If $T \cap \Sigma$ is indeed non-empty, choose any component $C \subset T \cap \Sigma$, then $T \setminus C$ is an essential annulus in M , therefore the image of C on $\Sigma_2 \subset \partial M$ can be homotoped into both $X \cap \Sigma_2$ and $\phi(X \cap \Sigma_1)$, hence the geometric intersection numbers $i(C, \phi(C_1))$ and $i(C, C_2)$ are both zero. This contradicts the fact that $\phi(C_1)$ and C_2 fill Σ_2 . Therefore any essential torus T in M_f can be isotoped to be disjoint with Σ . Let Y be the JSJ-piece of M_f containing Σ . Note that Y must not be a Seifert fibered space, for otherwise Σ would be a horizontal surface in Y since Σ is an essential surface with genus greater than 1. The circle fiber over any essential simple closed curves of Σ becomes an essential torus in Y which intrinsically intersects Σ , a contradiction. Therefore Y is a hyperbolic piece and M_f is not a closed graph manifold. \square

Theorem 5.17 (Theorem 1.4). *Let (N, R_+, R_-, γ) be a taut sutured manifold with R_+ and R_- both non-empty. Then (N, γ) is a product sutured manifold if and only if $\tau_u^{(2)}(N, R_+) = 1 \in \text{Wh}(\mathcal{D}_{\pi_1(N)})$.*

Proof. If (N, γ) is a product sutured manifold then clearly $\tau_u^{(2)}(N, R_+) = 1$.

Now we suppose $\tau_u^{(2)}(N, R_+) = 1$ and wish to prove that N is a product.

Case 1: $R_+ \cup R_-$ is a disjoint union of tori. In this case N is an admissible 3-manifold and $\tau_u^{(2)}(N) = \tau_u^{(2)}(N, R_+) = 1$. By Theorem 1.2 we know that any nonzero class is fibered, and whose Thurston norm vanishes by Theorem 4.20. This shows that N is homeomorphic to one of the following: the solid torus $D^2 \times S^1$, the thickened torus $T^2 \times I$, or the twisted I -bundle over Klein bottle $K \widetilde{\times} I$. Since R_+ and R_- are both non-empty, it follows that $(N, R_+, R_-, \gamma) = (T^2 \times I, T^2 \times \{1\}, T^2 \times \{0\}, \emptyset)$ and is indeed a product sutured manifold.

Case 2: $R_+ \cup R_-$ is a union of tori. Choose a homeomorphism $f : R_+ \rightarrow R_+$ as in Lemma 5.16 such that the doubling DN_f is not a closed graph manifold. Then $R_+ \cup R_-$ is a Thurston norm minimizing surface dual to a cohomology class ϕ . By Theorem 1.3,

$$L_\phi \tau_u^{(2)}(DN_f) = (j_1)_* \tau_u^{(2)}(N, R_+) \cdot (j_2)_* \tau_u^{(2)}(N, \overline{R}_+)$$

where j_1, j_2 are the natural inclusions. Since $\tau_u^{(2)}(N, R_+) = \tau_u^{(2)}(N, R_-) = \tau_u^{(2)}(N, \overline{R}_+) = 1$ by Proposition 5.7, we have $L_\phi \tau_u^{(2)}(DN_f) = 1$. Note that DN_f is not a closed graph manifold, it follows from Theorem 1.2 that ϕ is a fibered class and hence $R_+ \cup R_-$ is a fiber surface. Therefore (N, γ) is a product sutured manifold. \square

6. APPLICATIONS OF THE UNIVERSAL L^2 -TORSIONS

6.1. Homomorphism between free groups. Let $F_1 = \langle x_1, \dots, x_n \rangle$ and $F_2 = \langle y_1, \dots, y_m \rangle$ be free groups of finite rank. Let $X_1 = \bigvee_{i=1}^n S^1$ and $X_2 = \bigvee_{i=1}^m S^1$ be the wedge of circles. The space X_1 is given the usual CW-structure with one 0-cell p and n 1-cells e_1, \dots, e_n . Identify the fundamental group $\pi_1(X_1, p)$ with F_1 in such a way that $x_i = [e_i]$. Similarly, X_2 is given the usual CW-structure with one 0-cell q and m 1-cells f_1, \dots, f_m . Identify $\pi_1(X_2, q)$ with F_2 accordingly. Then any homomorphism $\phi : F_1 \rightarrow F_2$ determines a continuous mapping $\Phi : X_1 \rightarrow X_2$ up to homotopy. We call Φ the *realization of ϕ with respect to the basis $\{x_i\}$ and $\{y_j\}$* .

Definition 6.1. Let $\phi : F_1 \rightarrow F_2$ be a homomorphism and let Φ be its realization. The universal L^2 -torsion of ϕ is defined to be $\tau_u^{(2)}(\phi) := \tau_u^{(2)}(\Phi) \in \text{Wh}(\mathcal{D}_{F_2}) \sqcup \{0\}$.

Later we will show that the universal L^2 -torsion of ϕ does not depend on the choice of basis $\{x_i\}, \{y_j\}$. Before that, let's explicitly calculate $\tau_u^{(2)}(\phi)$ under the given basis. Denote by $\Phi : X_1 \rightarrow X_2$ the topological realization of ϕ . Form the wedge space $X_1 \vee X_2$ by identifying $p \in X_1$ and $q \in X_2$; then for any $i = 1, \dots, n$, attach a 2-cell σ_i whose boundary is the concatenation of $\Phi(e_i)$ and e_i^{-1} . The resulting cellular complex M_Φ is simple-homotopy equivalent to the mapping cylinder of Φ . Let \widehat{M}_Φ be the universal cover of M_Φ and let \widehat{X}_1 be the preimage of $X_1 \subset M_\Phi$. Fix a lifting $\hat{p} \in \widehat{M}_\Phi$ of p and lift the other cells with respect to the base point \hat{p} . Then we have the following $\mathbb{Z}F_2$ -chain complex

$$(\dagger) \quad C_*(\widehat{M}_\Phi, \widehat{X}) = (0 \rightarrow \mathbb{Z}F_2\langle \hat{\sigma}_1, \dots, \hat{\sigma}_n \rangle \xrightarrow{A_\phi} \mathbb{Z}F_2\langle \hat{f}_1, \dots, \hat{f}_m \rangle \rightarrow 0 \rightarrow 0).$$

The square matrix A_ϕ is called the *Jacobian of ϕ with respect to the basis y_1, \dots, y_m* . Recall that the Fox derivative $\frac{\partial}{\partial y_i} : \mathbb{Z}F_2 \rightarrow \mathbb{Z}F_2$, $i = 1, \dots, m$ are \mathbb{Z} -linear maps characterized by the following two properties:

- $\frac{\partial}{\partial y_i} 1 = 0$, $\frac{\partial}{\partial y_i} y_j = \delta_{ij}$.
- $\frac{\partial}{\partial y_i}(uv) = \frac{\partial}{\partial y_i} u + u \cdot \frac{\partial}{\partial y_i} v$ for all $u, v \in F_2$.

The entries of A_ϕ are then given by the Fox derivative

$$A_{ij} = \frac{\partial \phi(x_i)}{\partial y_j} \in \mathbb{Z}F_2, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m.$$

Proposition 6.2. Let $\phi : F_1 \rightarrow F_2$ be a homomorphism between finitely generated free groups $F_1 = \langle x_1, \dots, x_n \rangle$ and $F_2 = \langle y_1, \dots, y_m \rangle$. Then:

- (1) $\tau_u^{(2)}(\phi) = \det_w(A_\phi)$ where A_ϕ is the Jacobian of ϕ with respect to the basis $\{x_i\}, \{y_j\}$. In particular if A_ϕ is not a weak isomorphism then $\tau_u^{(2)}(\phi) = 0$.
- (2) $\tau_u^{(2)}(\phi) = 1$ if ϕ is an isomorphism; $\tau_u^{(2)}(\phi) = 0$ if $m \neq n$ or ϕ is not injective.
- (3) If $F_3 = \langle z_1, \dots, z_k \rangle$ is another free group and $\psi : F_2 \rightarrow F_3$ is a homomorphism. Suppose ϕ, ψ have nonzero universal L^2 -torsions then $\tau_u^{(2)}(\psi \circ \phi) = \psi_* \tau_u^{(2)}(\phi) \cdot \tau_u^{(2)}(\psi)$.
- (4) The definition of $\tau_u^{(2)}(\phi)$ does not depend on the choice of free basis of $\{x_i\}, \{y_j\}$.

Proof. Firstly, (1) is immediate from Equation (\dagger) and Definition 6.1.

For (2), if ϕ is an isomorphism, then $\Phi : X_1 \rightarrow X_2$ is a homotopy equivalence. Since the Whitehead group of a free group is trivial [Sta65] then Φ is a simple homotopy equivalence and $\tau_u^{(2)}(\phi) = \tau_u^{(2)}(\Phi) = 1$ by Proposition 3.12(3). If $m \neq n$, then A_ϕ is not a square matrix and clearly $\tau_u^{(2)}(\phi) = 0$. The proof of (2) is finished once we established the following Lemma 6.3.

Lemma 6.3. If $n = m$ and ϕ is not injective, then the Jacobian A_ϕ is not a weak isomorphism.

Proof of Lemma 6.3. Suppose the contrary that ϕ is not injective, then there is a reduced word $w \in F_1$ such that $\phi(w) = 1$. Let $w = x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$ be such a word with shortest length $k \geq 1$, where $i_1, \dots, i_k \in \{1, \dots, n\}$ and $\epsilon_1, \dots, \epsilon_k \in \{\pm 1\}$. We may assume that $x_{i_1}^{\epsilon_1} = x_1$ and $x_{i_k}^{\epsilon_k} \neq x_1^{-1}$. Denote by $w_s := x_{i_1}^{\epsilon_1} \cdots x_{i_s}^{\epsilon_s}$, $s = 1, \dots, k$ to be the prefix of w of length s ; set $w_0 = 1$. For any $j = 1, \dots, n$, apply $\frac{\partial}{\partial y_j}$ to both sides of the identity $\phi(x_{i_1})^{\epsilon_1} \cdots \phi(x_{i_k})^{\epsilon_k} = 1$, we have

$$\sum_{s=1}^k u_s \cdot \frac{\partial \phi(x_{i_s})}{\partial y_j} = 0, \quad \text{where} \quad u_s = \begin{cases} \phi(w_{s-1}), & \epsilon_s = 1, \\ -\phi(w_s), & \epsilon_s = -1. \end{cases}$$

Note that u_s does not depend on j . Rearranging the identities, we have

$$\sum_{i=1}^n U_i \cdot \frac{\partial \phi(x_i)}{\partial y_j} = 0, \quad j = 1, \dots, n$$

where U_i is the summation of all u_s such that $i_s = i$. Therefore

$$(U_1, U_2, \dots, U_n) \cdot A_\phi = 0.$$

If we could show that $U_i \neq 0$ for some i then this implies that A_ϕ is not a weak isomorphism, hence a contradiction. We prove that $U_1 \neq 0$. Let $1 \leq s_1 < s_2 < \cdots < s_r \leq k$ be the collection of indices such that $i_{s_1} = \cdots = i_{s_r} = 1$. Then $s_1 = 1$ by assumption and $U_1 = 1 + u_{s_2} + \cdots + u_{s_r}$. Write $U_1 = 1 \pm \phi(w_{s'_2}) \pm \cdots \pm \phi(w_{s'_r})$, where $w_{s'_j} = w_{s_j}$ or w_{s_j-1} , $j = 1, \dots, r$ depending on the sign of ϵ_{s_j} . Then we have $0 < s'_2 < \cdots < s'_r < k$. This is because there are no segments of $(\cdots x_1^{-1} x_1 \cdots)$ in the reduced word w , and that w does not ends with x_1^{-1} . We claim that $\phi(w_{s'_j}) \in F$ are pairwise distinct, otherwise there are two distinct prefixes of w with the same image under ϕ and we find a reduced word whose length shorter than k that lies in $\ker \phi$, a contradiction. This shows that $U_1 \in \mathbb{Z}F$ is nonzero in $\mathbb{Z}F$ and finishes the proof. \square

Let's continue the proof of Proposition 6.2. For (3), ϕ and ψ are injective by (2). Therefore the result follows from Proposition 3.12(4) since

$$\tau_u^{(2)}(\phi \circ \psi) = \tau_u^{(2)}(\Phi \circ \Psi) = \phi_* \tau_u^{(2)}(\Psi) \cdot \tau_u^{(2)}(\Phi) = \phi_* \tau_u^{(2)}(\psi) \cdot \tau_u^{(2)}(\phi).$$

For (4), let F'_i be another copy of F_i but with a different choice of basis and let $\psi_i : F_i \rightarrow F'_i$, $i = 1, 2$ be the isomorphism given by the base change. Suppose $\tau_u^{(2)}(\phi) \neq 0$, then by (3) we have

$$\tau_u^{(2)}(\phi \circ \psi_1) = \tau_u^{(2)}(\phi) \quad \text{and} \quad \tau_u^{(2)}(\psi_2 \circ \phi) = (\psi_2)_* \tau_u^{(2)}(\phi).$$

The first identity says that changing the basis of F_1 does not affect $\tau_u^{(2)}(\phi)$; the second identity says that $\tau_u^{(2)}(\psi_2 \circ \phi)$ is the push-forward of $\tau_u^{(2)}(\phi)$ via the base change ψ_2 , hence $\tau_u^{(2)}(\phi)$ does not depend on the basis of F_2 .

If $\tau_u^{(2)}(\phi) = 0$ then $\tau_u^{(2)}(\phi \circ \psi_1)$ and $\tau_u^{(2)}(\psi_2 \circ \phi)$ are both zero. This can be seen from the identities $\phi = (\phi \circ \psi_1) \circ \psi_1^{-1}$ and $\phi = \psi_2^{-1} \circ (\psi_2 \circ \phi)$; if either one of $\tau_u^{(2)}(\phi \circ \psi_1)$ and $\tau_u^{(2)}(\psi_2 \circ \phi)$ are nonzero, then $\tau_u^{(2)}(\phi)$ is also nonzero by (3), a contradiction. \square

It is an interesting question to characterize when a homomorphism $\phi : F_1 \rightarrow F_2$ to have nonzero universal L^2 -torsion. A finitely generated subgroup H of a free group F is called *compressed* if for any subgroup L of F containing H we have $\text{rank } H \leq \text{rank } L$. The following characterization given by [JZ24] is notable since it does not involve any L^2 -theories in its statement.

Theorem 6.4. *Let F_1, F_2 be finitely generated free groups of same rank and let $\phi : F_1 \rightarrow F_2$ be a homomorphism. Then $\tau_u^{(2)}(\phi) \neq 0$ if and only if ϕ is injective with compressed image $\text{im } \phi \subset F_2$.*

Proof. If ϕ is not injective then $\tau_u^{(2)}(\phi) = 0$ by Proposition 6.2(2). Now we assume that ϕ is injective and aim to show that $\tau_u^{(2)}(\phi) \neq 0$ if and only if $\text{im } \phi$ is compressed in F_2 .

Let $\Phi : X_1 \rightarrow X_2$ be the topological realization of ϕ and let M_Φ be the mapping cylinder. Then $\tau_u^{(2)}(\phi) \neq 0$ if and only if the pair (M_Φ, X_1) is L^2 -acyclic. By the homology long exact sequence this is equivalent to

$$H_1(X_1; \mathcal{D}_{F_2}) \rightarrow H_1(M_\Phi; \mathcal{D}_{F_2})$$

being injective (note that $\text{rank } H_1(X_1; \mathcal{D}_{F_2}) = \text{rank } H_1(M_\Phi; \mathcal{D}_{F_2}) = \text{rank } F_2 - 1$). By [JZ24, Corollary 1.2] this is equivalent to $\pi_1(X_1) \subset \pi_1(M_\Phi)$ being compressed, i.e. $\text{im } \phi \subset F_2$ is compressed. \square

6.2. 3-dimensional handlebodies. In the remaining part of this paper, a “handlebody” refers to a compact connected orientable 3-manifold obtained from attaching 1-handles to a 3-ball. The boundary of a handlebody is a connected closed surface. The homeomorphism type of a handlebody is determined by the genus of its boundary. A genus- g handlebody H_g refers to a handlebody whose boundary is a genus $g \geq 0$ surface. Note that H_g deformation retracts to the one-point union of g circles.

Consider a sutured structure (H_g, R_+, R_-, γ) on a genus- g handlebody, with the assumption that R_+ and R_- are both connected and $\chi(R_+) = \chi(R_-)$. Then it is clear that $\chi(R_+) = \chi(R_-) = 1 - g$, and the fundamental group of R_\pm are both isomorphic to a free group of rank g . The following Proposition 6.5 shows that the tautness is encoded in the group homomorphism $\pi_1(R_+) \rightarrow \pi_1(H_g)$ via the universal L^2 -torsion.

Proposition 6.5. *Suppose (H_g, R_+, R_-, γ) is a sutured manifold such that R_+, R_- are both connected and $\chi(R_+) = \chi(R_-)$ (we do not assume that R_\pm are incompressible). Then:*

- (1) $\tau_u^{(2)}(H_g, R_+) = \tau_u^{(2)}(\phi)$ where $\phi : \pi_1(R_+) \rightarrow \pi_1(H_g)$ is the inclusion-induced homomorphism.
- (2) (H_g, R_+, R_-, γ) is a taut sutured manifold if and only if $\tau_u^{(2)}(H_g, R_+) \neq 0$.
- (3) (H_g, R_+, R_-, γ) is a product sutured manifold if and only if $\tau_u^{(2)}(H_g, R_+) = 1$.

Proof. For (1), let $\Phi : R_+ \rightarrow H_g$ be the inclusion map and Let X be the one-point union of g circles. Since the Whitehead group of a free group is trivial, there are simple-homotopy equivalences $f_1 : \vee^g S^1 \rightarrow R_+$ and $f_2 : H_g \rightarrow \vee^g S^1$, then by Lemma 3.14

$$\tau_u^{(2)}(f_2 \circ \Phi \circ f_1) = (f_2)_* \tau_u^{(2)}(\Phi).$$

By definition, $f_2 \circ \iota \circ f_1$ is the topological realization of the homomorphism $(f_2)_* \circ \phi \circ (f_1)_*$ and therefore $\tau_u^{(2)}(f_2 \circ \Phi \circ f_1) = \tau_u^{(2)}((f_2)_* \circ \phi \circ (f_1)_*)$. By Proposition 6.2 this further equals $(f_2)_* \tau_u^{(2)}(\phi)$. So $(f_2)_* \tau_u^{(2)}(\Phi) = (f_2)_* \tau_u^{(2)}(\phi)$ and hence

$$\tau_u^{(2)}(H_g, R_+) = \tau_u^{(2)}(\Phi) = \tau_u^{(2)}(\phi).$$

For (2), the forward direction follows from Theorem 5.6(1). Now suppose $\tau_u^{(2)}(H_g, R_+) \neq 0$. By Proposition 5.7 we know that $\tau_u^{(2)}(H_g, R_-)$ is also nonzero. Then it follows from (1) and Proposition 6.2(2) that $R_\pm \subset H_g$ are both incompressible surfaces. Therefore (H_g, γ) is taut by Theorem 5.6(2).

Finally, (3) is a direct corollary of (2) and Theorem 1.4. \square

Example 6.6. A sutured manifold M as in the left of Figure 6 is a 3-ball with 2 arcs removed. There are 3 sutured annulus separating ∂M into two pairs of pants R_+ and R_- . The manifold M is homeomorphic to a genus-2 handlebody whose fundamental group is generated by the loops x and y . The fundamental group of R_+ is generated by the loops x and u where $u = yxyx^{-1}y^{-1}$ in

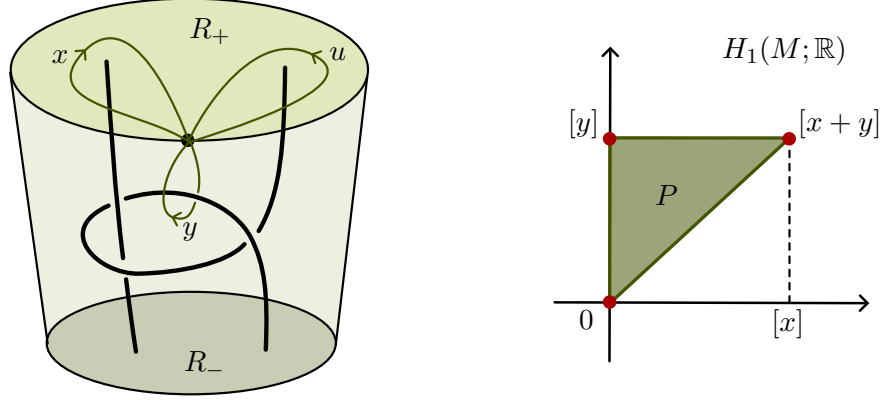


Figure 6. A sutured manifold M and a representative P of its L^2 -polytope in $H_1(M; \mathbb{R})$

$\pi_1(M)$. Let $\phi : \pi_1(R_+) \rightarrow \pi_1(M)$ be the inclusion-induced homomorphism, then under the basis $\pi_1(R_+) = \langle x, u \rangle$, $\pi_1(M) = \langle x, y \rangle$, we have

$$A_\phi = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y - yxyx^{-1} & 1 + yx - u \end{pmatrix}.$$

Therefore by Proposition 6.5,

$$\tau_u^{(2)}(M, R_+) = \det_w(A_\phi) = [1 + yx - u].$$

The polytope map $\mathbb{P} : \text{Wh}(\mathcal{D}_{\pi_1(M)}) \rightarrow \mathcal{P}_{\mathbb{Z}}^{\text{Wh}}(H_1(M; \mathbb{Z}))$ sends $\tau_u^{(2)}(M, R_+)$ to a polytope $[P]$ where P is the convex hull of $\{0, [u] = [y], [x + y]\}$. Proposition 6.5 implies that M is a taut sutured manifold but not a product sutured manifold.

Example 6.7 (n -chain link). For each $n \geq 3$, the n -chain link L_n is an alternating link obtained from linking together n unknots in a cyclic way. A diagram of a 4-chain link is illustrated in Figure 7(a). Consider the natural Seifert surface Σ as is shown in Figure 7(b) (where the positive-side of Σ is in light green, while the negative-side of Σ is in dark green). Then Σ is obtained from two copies of disks D^2 by attaching n twisted-bands, and Σ deformation retracts to the wedge of $(n - 1)$ circles. The complement $S^3 \setminus \Sigma$ is a handle body of genus $(n - 1)$, whose boundary is a union of two copies of Σ , namely Σ_+ and Σ_- . Choose a free basis for the fundamental group $\pi_1(S^3 \setminus \Sigma) = \langle x_1, \dots, x_{n-1} \rangle$ where x_i are represented by the red loops in Figure 7(b); a free basis for $\pi_1(\Sigma_+)$ is represented by the blue loops u_1, \dots, u_{n-1} in Figure 7(c). Pushing u_i slightly into the positive direction, we obtain its image u_i^+ under the inclusion $\Sigma_+ \hookrightarrow S^3 \setminus \Sigma$, which is represented by $x_i x_{i+1}^{-1}$ (we assume that $x_n := x_1^{-1} \cdots x_{n-1}^{-1}$, see Figure 7(b)). Let $\phi : \pi_1(\Sigma_+) \rightarrow \pi_1(S^3 \setminus \Sigma)$ be the inclusion-induced map, then

$$\text{im } \phi = \langle x_1 x_2^{-1}, x_2 x_3^{-1}, \dots, x_{n-2} x_{n-1}^{-1}, x_{n-1}^2 x_{n-2} \cdots x_2 x_1 \rangle \subset \langle x_1, \dots, x_{n-1} \rangle$$

and the Jacobian of ϕ is

$$A_\phi = \begin{pmatrix} 1 & -x_1 x_2^{-1} & & & \\ & 1 & -x_2 x_3^{-1} & & \\ & & \ddots & & \\ & & & 1 & -x_{n-2} x_{n-1}^{-1} \\ x_{n-1}^2 x_{n-2} \cdots x_2 & x_{n-1}^2 x_{n-2} \cdots x_3 & \cdots & x_{n-1}^2 & 1 + x_{n-1} \end{pmatrix}.$$

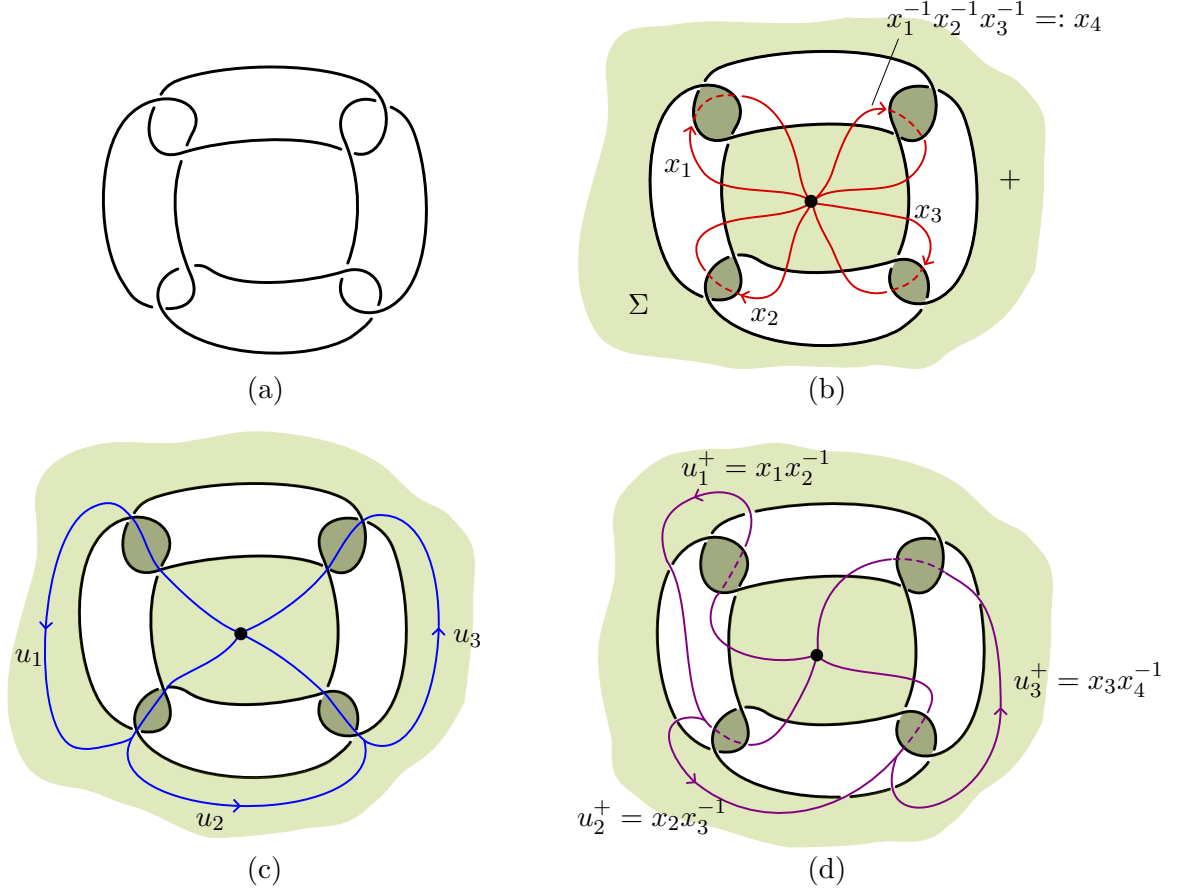


Figure 7. The n -chain link example, where $n = 4$

Lemma 6.8. For $n \geq 3$, let

$$B_n = \begin{pmatrix} 1 & -s_1 & & & \\ & 1 & -s_2 & & \\ & & \ddots & & \\ & & & 1 & -s_{n-2} \\ f_1 & f_2 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix}$$

be a matrix over a skew field. Then its Dieudonné determinant

$$\det(B_n) = [f_1 s_1 s_2 \cdots s_{n-2} + f_2 s_2 s_3 \cdots s_{n-2} + \cdots f_{n-2} s_{n-2} + f_{n-1}].$$

Proof. When $n = 3$, the identity

$$\det(B_3) = \det \begin{pmatrix} 1 & -s_1 \\ f_1 & f_2 \end{pmatrix} = [f_1 s_1 + f_2]$$

holds true. For the general case, left-multiply $(-f_1)$ to the first row of B_n and add it to the last row, we eliminate the bottom-left entry and hence

$$\det(B_n) = \det \begin{pmatrix} 1 & -s_1 & & & \\ & 1 & -s_2 & & \\ & & \ddots & & \\ & & & 1 & -s_{n-2} \\ 0 & f_1 s_1 + f_2 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix} = \det \begin{pmatrix} 1 & & & & \\ & -s_2 & & & \\ & & \ddots & & \\ & & & 1 & -s_{n-2} \\ f_1 s_1 + f_2 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix}.$$

The conclusion follows easily from an induction argument with respect to n . \square

Applying Lemma 6.8 to A_ϕ , note that $s_i s_{i+1} \cdots s_{n-2} = x_i x_{n-1}^{-1}$, it follows that

$$f_i s_i s_{i+1} \cdots s_{n-2} = x_{n-1}^2 x_{n-2} \cdots x_i x_{n-1}^{-1}, \quad 1 \leq i < n-1$$

and $f_{n-1} = 1 + x_{n-1}$. Denote by $y_i := x_{n-1} x_{n-2} \cdots x_i$, $i = 1, \dots, n-1$, then $\{y_1, \dots, y_{n-1}\}$ is another free basis of the fundamental group $\pi_1(S^3 \setminus \Sigma)$. By Proposition 6.5

$$\begin{aligned} \tau_u^{(2)}(S^3 \setminus \Sigma, \Sigma_+) &= \det_w(A_\phi) \\ &= [x_{n-1} \cdot (y_1 + y_2 + \cdots + y_{n-1} + 1) \cdot x_{n-1}^{-1}] \\ &= [y_1 + y_2 + \cdots + y_{n-1} + 1]. \end{aligned}$$

The L^2 -polytope of the sutured manifold $S^3 \setminus \Sigma$ is an $(n-1)$ -simplex of $H_1(S^3 \setminus \Sigma; \mathbb{R})$ spanned by n vertices $\{0, [y_1], \dots, [y_{n-1}]\}$. By Proposition 6.5 $S^3 \setminus \Sigma$ is a taut sutured manifold and Σ is a norm-minimizing Seifert surface for L . Moreover, L is not a fibered link.

Remark 6.9. Suppose G is a residually finite group. The Fuglede–Kadison determinant is a homomorphism $\det_{\mathcal{N}G} : \text{Wh}(\mathcal{D}_G) \rightarrow \mathbb{R}_+$. It has the property that for any L^2 -acyclic finite CW-complex X with $\pi_1(X) = G$, the Fuglede–Kadison determinant brings the universal L^2 -torsion $\tau_u^{(2)}(X)$ to the L^2 -torsion $\tau^{(2)}(X)$. Let G be the fundamental group $\pi_1(S^3 \setminus \Sigma)$. Then it follows from [BA22] that

$$\det_{\mathcal{N}G}([1 + y_1 + \cdots + y_{n-1}]) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}}.$$

Therefore the L^2 -torsion $\tau^{(2)}(S^3 \setminus \Sigma, \Sigma_+) = (n-1)^{\frac{n-1}{2}} / n^{\frac{n-2}{2}}$.

For an admissible 3-manifold N and a cohomology class $\phi \in H^1(N; \mathbb{Z})$, the L^2 -Alexander torsion is a function $\tau^{(2)}(N, \phi) : \mathbb{R}_+ \rightarrow [0, +\infty)$ [DFL16]. This function has a well-defined “degree” which equals to the Thurston norm of ϕ [FL19, Liu17], and a “leading coefficient” $C(N, \phi) \geq 1$ [Liu17]. It is proved in [?] that $C(N, \phi)$ equals the L^2 -torsion of the pair $(N \setminus \Sigma, \Sigma_+)$ where Σ is a norm-minimizing surface dual to ϕ . Let X_n be the n -chain link complement and let $\phi \in H^1(X_n)$ be the Poincaré dual of Σ . It follows that

$$C(X_n, \phi) = \tau^{(2)}(X_n, \Sigma_+) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}} \sim \sqrt{n/e}, \quad \text{as } n \rightarrow +\infty.$$

Therefore the n -chain link complements X_n give an infinite family of hyperbolic manifolds such that $C(X_n, \phi) > 1$ for a nonzero class $\phi \in H^1(X_n; \mathbb{Z}) \setminus \{0\}$, answering a question of [BAFH22, Conjecture 1.7].

REFERENCES

- [AD19] Ian Agol and Nathan M. Dunfield, *Certifying the thurston norm via $sl(2, c)$ -twisted homology*, pp. 1–20, Princeton University Press, Princeton, 2019.
- [AFW15] M. Aschenbrenner, S. Friedl, and H. Wilton, *3-manifold groups*, EMS series of lectures in mathematics, European Mathematical Society, 2015.
- [AGM13] Ian Agol, Daniel Groves, and Jason Manning, *The virtual Haken conjecture*, Doc. Math **18** (2013), no. 1, 1045–1087.
- [Ago08] Ian Agol, *Criteria for virtual fibering*, Journal of Topology **1** (2008), no. 2, 269–284.
- [BA22] Fathi Ben Aribi, *Fuglede-kadison determinants over free groups and lehmer’s constants*, Confluentes Mathematici **14** (2022), no. 1, 3–22.
- [BAFH22] Fathi Ben Aribi, Stefan Friedl, and Gerrit Herrmann, *The leading coefficient of the L^2 -alexander torsion*, Annales de l’Institut Fourier, vol. 72, 2022, pp. 1993–2035.
- [CG86] Jeff Cheeger and Mikhael Gromov, *L_2 -cohomology and group cohomology*, Topology **25** (1986), no. 2, 189–215.

- [Coh73] Marshall M Cohen, *A course in simple-homotopy theory*, vol. 10, Springer Science & Business Media, 1973.
- [Coh95] Paul Moritz Cohn, *Skew fields*, Further algebra and applications, Springer, 1995, pp. 343–370.
- [DFL16] Jérôme Dubois, Stefan Friedl, and Wolfgang Lück, *The L^2 -Alexander torsion of 3-manifolds*, J. Topol. **9** (2016), no. 3, 889–926. MR 3551842
- [Die43] Jean Dieudonné, *Les déterminants sur un corps non commutatif*, Bulletin de la Société Mathématique de France **71** (1943), 27–45.
- [Dua25] Jianru Duan, *Guts determine the leading coefficients of L^2 -alexander torsions*, Transactions of the American Mathematical Society **378** (2025), no. 05, 3699–3720.
- [FK18] Florian Funke and Dawid Kielak, *Alexander and thurston norms, and the bieri–neumann–strebel invariants for free-by-cyclic groups*, Geometry & Topology **22** (2018), no. 5, 2647–2696.
- [FL17] Stefan Friedl and Wolfgang Lück, *Universal l^2 -torsion, polytopes and applications to 3-manifolds*, Proceedings of the London Mathematical Society **114** (2017), no. 6, 1114–1151.
- [FL19] ———, *The L^2 -torsion function and the Thurston norm of 3-manifolds*, Comment. Math. Helv **94** (2019), no. 1, 21–52.
- [FST17] Stefan Friedl, Kevin Schreve, and Stephan Tillmann, *Thurston norm via fox calculus*, Geometry & Topology **21** (2017), no. 6, 3759–3784.
- [FT20] Stefan Friedl and Stephan Tillmann, *Two-generator one-relator groups and marked polytopes*, Annales de l’Institut Fourier, vol. 70, 2020, pp. 831–879.
- [Fun21] Florian Funke, *The integral polytope group*, Advances in Geometry **21** (2021), no. 1, 45–62.
- [FV11] Stefan Friedl and Stefano Vidussi, *Twisted alexander polynomials detect fibered 3-manifolds*, Annals of mathematics (2011), 1587–1643.
- [Gab83] David Gabai, *Foliations and the topology of 3-manifolds*, Journal of Differential Geometry **18** (1983), no. 3, 445.
- [Gab87] ———, *Foliations and the topology of 3-manifolds. iii*, Journal of Differential Geometry **26** (1987), no. 3, 479–536.
- [Her23] Gerrit Herrmann, *Sutured manifolds and l^2 -beti numbers*, The Quarterly Journal of Mathematics **74** (2023), no. 4, 1435–1455.
- [HK20] Fabian Henneke and Dawid Kielak, *The agrarian polytope of two-generator one-relator groups*, Journal of the London Mathematical Society **102** (2020), no. 2, 722–748.
- [HS98] Eckehard Hess and Thomas Schick, *L^2 -torsion of hyperbolic manifolds*, manuscripta mathematica **97** (1998), no. 3, 329–334.
- [JS79] William Jaco and Peter B. Shalen, *Seifert fibered spaces in 3-manifolds*, pp. 91–99, Academic Press, 1979.
- [Juh06] András Juhász, *Holomorphic discs and sutured manifolds*, Algebraic & Geometric Topology **6** (2006), no. 3, 1429–1457.
- [Juh08] ———, *Floer homology and surface decompositions*, Geometry & Topology **12** (2008), no. 1, 299–350.
- [Juh10] ———, *The sutured floer homology polytope*, Geometry & Topology **14** (2010), no. 3, 1303–1354.
- [JZ24] Andrei Jaikin-Zapirain, *Free groups are l^2 -subgroup rigid*, arXiv preprint arXiv:2403.09515 (2024).
- [Kie20] Dawid Kielak, *The bieri–neumann–strebel invariants via newton polytopes*, Inventiones mathematicae **219** (2020), no. 3, 1009–1068.
- [KL24] Dawid Kielak and Marco Linton, *Group rings of three-manifold groups*, Proceedings of the American Mathematical Society **152** (2024), no. 05, 1939–1946.
- [KS24] Dawid Kielak and Bin Sun, *Agrarian and l^2 -beti numbers of locally indicable groups, with a twist*, Mathematische Annalen **390** (2024), no. 3, 3567–3619.
- [Kud24] Monika Kudłinska, *Thurston norm for coherent right-angled artin groups via L^2 -invariants*, arXiv preprint arXiv:2411.02516 (2024).
- [Lin93] Peter A Linnell, *Division rings and group von neumann algebras*.
- [Liu13] Yi Liu, *Virtual cubulation of nonpositively curved graph manifolds*, Journal of Topology **6** (2013), no. 4, 793–822.
- [Liu17] ———, *Degree of L^2 -Alexander torsion for 3-manifolds*, Inventiones mathematicae **207** (2017), no. 3, 981–1030.
- [LL95] John Lott and Wolfgang Lück, *L^2 -topological invariants of 3-manifolds*, Inventiones mathematicae **120** (1995), no. 1, 15–60.
- [LL17] Wolfgang Lück and Peter Linnell, *Localization, whitehead groups and the atiyah conjecture*, Annals of K-Theory **3** (2017), no. 1, 33–53.
- [Lüc94] Wolfgang Lück, *L^2 -beti numbers of mapping tori and groups*, Topology **33** (1994), no. 2, 203–214.
- [Lüc02] ———, *L^2 -invariants: theory and applications to geometry and K-theory*, vol. 44, Springer, 2002.

- [Lü02] Wolfgang Lück, *L2-invariants: Theory and applications to geometry and k-theory*, vol. 3, 2002.
- [Mil62] John Milnor, *A duality theorem for reidemeister torsion*, *Annals of Mathematics* **76** (1962), no. 1, 137–147.
- [Mil66] ———, *Whitehead torsion*, *Bulletin of the American Mathematical Society* **72** (1966), no. 3, 358–426.
- [MM99] Howard A Masur and Yair N Minsky, *Geometry of the complex of curves i: Hyperbolicity*, *Inventiones mathematicae* **138** (1999), no. 1, 103–149.
- [Mun66] James R. Munkres, *Elementary differential topology*, revised ed., *Annals of Mathematics Studies*, vol. No. 54, Princeton University Press, Princeton, NJ, 1966, Lectures given at Massachusetts Institute of Technology, Fall, 1961. MR 198479
- [OS04] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, *Annals of Mathematics* (2004), 1027–1158.
- [PW18] Piotr Przytycki and Daniel T Wise, *Mixed 3-manifolds are virtually special*, *Journal of the American Mathematical Society* **31** (2018), no. 2, 319–347.
- [Ros95] Jonathan Rosenberg, *Algebraic k-theory and its applications*, vol. 147, Springer Science & Business Media, 1995.
- [Sta65] John Stallings, *Whitehead torsion of free products*, *Annals of Mathematics* **82** (1965), no. 2, 354–363.
- [Thu86] William P Thurston, *A norm for the homology of 3-manifolds*, *Mem. Amer. Math. Soc.* **59** (1986), 99–130.
- [Tur01] Vladimir Turaev, *Introduction to combinatorial torsions*, Springer Science & Business Media, 2001.
- [Tur02] ———, *A homological estimate for the thurston norm*, arXiv preprint math/0207267 (2002).
- [Whi40] J. H. C. Whitehead, *On C^1 -complexes*, *Ann. of Math. (2)* **41** (1940), 809–824. MR 2545

BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, NO. 5 YIHEYUAN ROAD, HAIDIAN DISTRICT, BEIJING 100871, CHINA

Email address: duanjr@stu.pku.edu.cn