UNIVERSAL L^2-TORSION AND TAUT SUTURED DECOMPOSITIONS

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ABSTRACT. Given an admissible 3-manifold M and a first cohomology class ϕ, we show that the universal L^2-torsion of M detects fiberedness of ϕ unless M is a closed graph manifold. As a natural extension in sutured manifold theory, we provide a simple formula that shows how this invariant change under taut sutured decompositions. We show that a taut sutured manifold is a product if and only if its universal L^2-torsion is trivial. Our proof is based on a study of the leading term map on Linnell's skew field.

1. INTRODUCTION

Torsion invariants of a finite cellular complex X contain delicate topological information. To begin with, one has to produce an exact sequence from the cellular chain complex of the universal cover $\widehat{X}$. One way to do this is to "base change" the group ring $\mathbb{Z}[\pi_1(X)]$ to a commutative field. For 3-manifolds, this idea produces the Reidemeister–Franz torsion and the (multi-variable) Alexander polynomial, considering extensions of scalars to the complex number field $\mathbb{C}$ and the field of rational functions $\mathbb{Q}(H_1(X))$, respectively. However, this base change loses the non-commutative information of the fundamental group. The torsion introduced by J.H.C Whitehead does not involve any base change and retain most information; the so-called Whitehead torsion lives in a certain group $\mathrm{Wh}(\pi_1(X))$ consisting of matrices invertible over $\mathbb{Z}[\pi_1(X)]$ up to equivalence. As a tradeoff, the Whitehead torsion only applies to pairs (X, Y) where X deformation retracts to Y. Also, it is too restrictive for a matrix to be invertible over $\mathbb{Z}[\pi_1(X)]$ and conjecturally $\mathrm{Wh}(G)$ is trivial for all torsion-free groups G.

The universal L^2-torsion is potentially a powerful yet applicable torsion invariant at the same time. For G belonging to a large family of torsion-free groups, there is a canonical associative field $\mathcal{D}_G$ which contains $\mathbb{Z}G$ as a subring.

Definition 1.1. Let X be a finite CW-complex with fundamental group G. We call X L^2-acyclic if the $\mathcal{D}_G$-chain complex $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*(\widehat{X})$ is exact; its torsion $\tau_u^{(2)}(X)$ is therefore called the universal L^2-torsion of X, living in the Whitehead group $\mathrm{Wh}(\mathcal{D}_G)$ of the field $\mathcal{D}_G$.

L^2-acyclic spaces abound, including mapping torus, spaces with amenable fundamental groups, most 3-manifolds and all odd-dimensional closed hyperbolic manifolds. The Whitehead group $\mathcal{D}_G$ is never trivial; in fact there are interesting homomorphisms from $\mathrm{Wh}(\mathcal{D}_G)$ such as the polytope map and the Fuglede–Kadison determinant.

We apply the universal L^2-torsion to the fiberedness of 3-manifolds. Let M be a compact oriented 3-manifold. A homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$ is called fibered if it is induced by a fibration of M over the circle. Thurston showed that there is a disjoint union of open cones in $H^1(M; \mathbb{R})$ such that ϕ is fibered if and only if ϕ lies in the cones. Such open cones are hence called the fibered cones of M. We will also call a real class $\phi \in H^1(M; \mathbb{R})$ fibered if it lies in the fibered cones.

It is known that if a class ϕ is fibered, then its Alexander polynomial, twisted Alexander polynomial and L^2-Alexander torsion are all monic. Conversely, the degree and leading term of those Alexander-type invariants are not enough to characterize fiberedness. Friedl and Vidussi showed that the collection of all twisted Alexander polynomial of ϕ do determine the fiberedness of ϕ. Their work indicates that torsion invariants carry enough information for the detection of fiberedness.

Given any real character $\phi : G \to \mathbb{R}$ and a nonzero element $a = \sum_{g\in G} n_g \cdot g \in \mathbb{Z}G$, the ϕ-leading term of a is defined to be the sum of nonzero terms $n_g \cdot g$ where $\phi(g)$ attains minimum. This construction can be generalized to the leading term map

$$L_\phi : \mathcal{D}_G \to \mathcal{D}_G.$$

Furthermore, this naturally induces a map $L_\phi : \mathrm{Wh}(\mathcal{D}_G) \to \mathrm{Wh}(\mathcal{D}_G)$ on the Whitehead group. Our first main result shows that the universal L^2-torsion detects fiberedness of most 3-manifolds:

Theorem 1.2. *Suppose M is an admissible 3-manifold which is not a closed graph manifold. Let G be the fundamental group of M and $\phi \in H^1(M;\mathbb{R})$ be any nonzero character. Then ϕ is fibered if and only if $L_\phi \tau_u^{(2)}(M) = 1 \in \mathrm{Wh}(\mathcal{D}_G)$.*

A 3-manifold is called admissible if it is compact, connected, orientable and irreducible, its boundary is empty or a collection of tori, and the fundamental group is infinite. A sutured manifold (M, R_+, R_-, γ) is a compact oriented 3-manifold with a partition of its boundary into two oriented subsurfaces R_+ and R_- along their common boundary γ. A sutured manifold can be decomposed along a nicely embedded surface S and the resulting manifold is again a sutured manifold. We write $(M, R_+, R_-, \gamma) \overset{S}{\rightsquigarrow} (M', R'_+, R'_-, \gamma')$ for a sutured decomposition. Any taut sutured manifold admits a taut sutured hierarchy. A beautiful result by Herrmann shows that a sutured manifold (M, R_+, R_-, γ) being taut is almost equivalent to the pair (M, R_+) being L^2-acyclic. Hence the universal L^2-torsion of a taut sutured manifold (M, R_+, R_-, γ) is defined to be $\tau_u^{(2)}(M, R_+)$ which lives in $\mathrm{Wh}(\mathcal{D}_{\pi_1(M)})$. The second main result of this paper describes the change of the universal L^2-torsion during taut sutured decompositions.

Theorem 1.3. *Let $(N, R_+, R_-, \gamma) \overset{\Sigma}{\rightsquigarrow} (N', R'_+, R'_-, \gamma')$ be a taut sutured decomposition and let $\phi \in H^1(N;\mathbb{Z})$ be the Poincaré dual of the surface Σ, then*

$$j_* \tau_u^{(2)}(N', R'_+) = L_\phi \tau_u^{(2)}(N, R_+)$$

where $j_* : \mathrm{Wh}(\mathcal{D}_{\pi_1(N')}) \to \mathrm{Wh}(\mathcal{D}_{\pi_1(N)})$ is induced by the inclusion map $j : N' \hookrightarrow N$.

Finally, we can show that the universal L^2-torsion detects product sutured manifolds.

Theorem 1.4. *Let (N, R_+, R_-, γ) be a taut sutured manifold with non-empty sutured annuli γ. Then (N, γ) is a product sutured manifold if and only if $\tau_u^{(2)}(N, R_+) = 1 \in \mathrm{Wh}(\mathcal{D}_{\pi_1(N)})$.*

2. Algebraic preliminaries

2.1. **Hilbert modules and the affiliated algebra.** Let G be a group. Consider the following Hilbert space

$$l^2(G) = \Big\{ \sum_{g\in G} c_g \cdot g \;\Big|\; c_g \in \mathbb{C},\ \sum_{g\in G} |c_g|^2 < \infty \Big\}$$

with inner product

$$\Big\langle \sum_{g\in G} c_g \cdot g, \sum_{g\in G} d_g \cdot g \Big\rangle = \sum_{g\in G} c_g \overline{d_g}.$$

This Hilbert space has a natural left and right isometric G-action by multiplications. The group von Neumann algebra $\mathcal{N}G$ is defined to be the algebra of all bounded linear operators of $l^2(G)$ that commutes with the left G-action. A (finitely generated) Hilbert $\mathcal{N}G$-module is defined to be a closed G-invariant subspace of $l^2(G)^n$. Each Hilbert $\mathcal{N}G$-module V can be assigned the von-Neumann dimension $\dim_{\mathcal{N}G} V$ which takes value in $[0, +\infty)$.

Let $\mathcal{U}G$ be the set of all densely-defined, closed operators (possibly unbounded) on $l^2(G)$ that commutes with the left G-action. The composition and addition of two operators in $\mathcal{U}G$ is well-defined [Lü02, Section 8.1], hence $\mathcal{U}G$ forms a $\mathbb{C}$-algebra and is called the affiliated algebra of G. In particular, we have the following inclusion relations

$$\mathbb{Z}G \subset \mathcal{N}G \subset \mathcal{U}G$$

where the integral group ring $\mathbb{Z}G$ embeds into $\mathcal{N}G$ by the right regular representation.

2.2. **Atiyah conjecture and Linnell's skew-field.**

Definition 2.1 (Atiyah Conjecture)**.** A group is said to satisfy the Atiyah Conjecture if for any matrix $A^{m\times n}$ over $\mathbb{Z}G$ the von Neumann dimension of $\ker(r_A)$ is an integer, where $r_A : l^2(G)^m \to l^2(G)^n$ is given by right multiplication with A.

The Atiyah Conjecture has been verified for a large class of groups. We mark the following class of groups given by Linnell, which is large enough to include all 3-manifold groups.

Theorem 2.2 ([Lin93])**.** *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions by elementary amenable groups. Then any torsion-free group in $\mathcal{C}$ satisfies the Atiyah Conjecture.*

Theorem 2.3 ([KL24, Theorem 1.4])**.** *The fundamental group of any connected 3-manifold lies in $\mathcal{C}$.*

Definition 2.4 (Division closure)**.** Let R be a subring of a ring S. The division closure of R in S is the smallest subring $\mathcal{D}(R \subset S)$ of S containing R such that if an element of R is invertible in S, then it is also invertible in $\mathcal{D}(R \subset S)$. Let G be a group, define $\mathcal{D}_G$ to be the division closure of $\mathbb{Z}G$ in $\mathcal{U}G$.

Theorem 2.5 (Linnell)**.** *A torsion-free group G satisfies the Atiyah Conjecture if and only if $\mathcal{D}_G$ is a skew-field.*

Proposition 2.6 ([Kie20, Proposition 4.6])**.** *Let G be a torsion-free group satisfying the Atiyah Conjecture. Then the following statements hold.*

1. *The involution on $\mathbb{Z}G$ naturally extends to an involution on $\mathcal{D}_G$.*
2. *Every automorphism of the group G extends to an automorphism of $\mathcal{D}_G$.*
3. *If K is a subgroup of G, then K satisfies the Atiyah Conjecture. Moreover, the natural embedding $\mathbb{Z}K \hookrightarrow \mathbb{Z}G$ extends to an embedding $\mathcal{D}_K \hookrightarrow \mathcal{D}_G$.*

2.3. **Ore localization.**

Definition 2.7 (Ore localization)**.** Let R be a ring with unit and let $S \subset R$ be a multiplicatively closed subset. The pair (R, S) satisfies the (right) Ore condition if the following two conditions hold:

(1) for any $(r, s) \in R \times S$ there exists $(r', s') \in R \times S$ such that $rs' = sr'$, and
(2) for any $r \in R$ and $s \in S$ with $sr = 0$, there is $t \in S$ with $rt = 0$.

If (R, S) satisfies the Ore condition, define an equivalence relation on $R \times S$

$$(r, s) \sim (rx, sx) \quad \text{whenever} \quad x \in R,\ sx \in S.$$

The quotient set $R\times S/\sim$ is denoted by RS^{-1}. Define a ring structure on RS^{-1} as follows. Given two representatives $(r, s), (r', s') \in RS^{-1}$, we can find $c \in R,\ d \in S$ with $sc = s'd \in S$ and define

$$(r, s) + (r', s') = (rc + r'd, sc).$$

We can find $e \in R,\ f \in S$ with $se = r'f$ and define

$$(r, s) \cdot (r', s') = (re, s'f).$$

The ring RS^{-1} is called the Ore localization of R at S.

Intuitively, a pair $(r, s) \in RS^{-1}$ is understood as a formal fraction rs^{-1}. The Ore condition (1) can be remembered as whenever there is a left (=wrong way) fraction $s^{-1}r$ then there is a right fraction $r'(s')^{-1}$ such that $rs' = sr'$. Condition (2) is automatically satisfied if S contains no zero divisors. In this paper, we only need to deal with Ore localizations of the following simple form:

Lemma 2.8 ([Coh95, Corollary 1.3.3])**.** *Let R be an integral domain such that $aR \cap bR \neq \{0\}$ for all $a, b \in R^\times$. Then $(R, R^\times)$ satisfies the Ore localization. Moreover, the Ore localization of R at $R^\times$ is a field K and the natural homomorphism $\lambda : R \to K$ is an embedding.*

An integral domain R satisfying the condition in Lemma 2.8 is called an Ore domain. The field K is called the field of fraction of R.

2.4. **Crossed products.** Assume that G is a torsion-free group which satisfies the Atiyah conjecture. Given any short exact sequence of groups

$$1 \to K \to G \xrightarrow{\nu} H \to 1.$$

Then K also satisfies the Atiyah conjecture by Proposition 2.6. Denote by $\mathcal{D}_K$ and $\mathcal{D}_G$ the Linnell's skew fields of K and G, respectively.

Choose a section $s : H \to G$ for the epimorphism ν such that $\nu \circ s$ is the identity. We do not require that s be a group homomorphism. Consider the following subset of $\mathcal{D}_G$:

$$\mathcal{D}_K *_s H := \left\{ \sum_{h \in H} x_h \cdot s(h) \in \mathcal{D}_G \;\middle|\; x_h \in \mathcal{D}_K,\ x_h = 0 \text{ for all but finitely many } h \in H \right\}.$$

This set contains the zero element 0, the identity element $1 = s(1_H)^{-1} \cdot s(1_H)$ and is closed under addition. Moreover, it is also closed under multiplication, since

$$\begin{aligned} \left(\sum_{h \in H} x_h \cdot s(h) \right) \cdot \left(\sum_{h \in H} y_h \cdot s(h) \right) &= \sum_{h_1, h_2 \in H} x_{h_1} \cdot s(h_1) \cdot y_{h_2} \cdot s(h_2) \\ &= \sum_{h_1, h_2 \in H} \underbrace{x_{h_1} s(h_1) y_{h_2} s(h_1)^{-1}}_{\in \mathcal{D}_K} \cdot \underbrace{s(h_1) s(h_2) s(h_1 h_2)^{-1}}_{\in K} \cdot s(h_1 h_2). \end{aligned}$$

Recall that the group automorphism of conjugation by $s(h_1)$ in K extends to an automorphism of $\mathcal{D}_K$ by Proposition 2.6, so $s(h_1) y_{h_2} s(h_1)^{-1} \in \mathcal{D}_K$. It follows that $\mathcal{D}_K *_s H$ is a subring of the field $\mathcal{D}_G$.

Proposition 2.9. *With notations as above, we have the following properties.*

(1) If an element $\sum_{h \in H} x_h \cdot s(h)$ of $\mathcal{D}_K *_s H$ is zero then $x_h = 0$ for all $h \in H$.
(2) Given another section $s' : H \to G$, then

$$\sum_{h \in H} x_h \cdot s(h) = \sum_{h \in H} y_h \cdot s'(h)$$

if and only if $y_h = x_h s(h) s'(h)^{-1}$ for all $h \in H$.

Proof. The first statement is a consequence of [Lüc02, Lemma 10.57]. The second statement follows from the previous one. □

As a corollary, the subring $\mathcal{D}_K *_s H \subset \mathcal{D}_G$ does not depend on the choice of the section. We call this subring the crossed product of $\mathcal{D}_K$ and H, denoted by $\mathcal{D}_K * H$. It is clear that $\mathcal{D}_K * H$ is an integral domain since it embeds in the field $\mathcal{D}_G$. When H is nice, the relation between $\mathcal{D}_G$ and its subring $\mathcal{D}_K * H$ is surprisingly simple.

Proposition 2.10. *Let $1 \to K \to G \to H \to 1$ be an extension of groups.*

(1) *Suppose H is a finite group, then $\mathcal{D}_G = \mathcal{D}_K * H$.*
(2) *Suppose H is a virtually finitely generated abelian group, then the integral domain $\mathcal{D}_K * H$ is an Ore domain whose field of fraction agrees with $\mathcal{D}_G$.*

Proof. These two statements are Lemma 10.59 and Lemma 10.69 of [Lüc02], respectively. □

2.5. **The K_1-group and Dieudonné determinant.** Let R be an associative ring with identity. For any positive integer n let $GL(n,R)$ be the group of invertible $(n\times n)$-matrices over R. Identifying each $M \in GL(n,R)$ with the matrix

$$\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL(n+1,R)$$

we obtain inclusions

$$GL(1,R) \subset GL(2,R) \subset \cdots .$$

The union $GL(R) = \bigcup_{n\geqslant 1} GL(n,R)$ is called the infinite general linear group. Define

$$K_1(R) := GL(R)/[GL(R), GL(R)].$$

It is a classical result of Whitehead that the commutator subgroup $[GL(R), GL(R)]$ is exactly the subgroup generated by all elementary matrices in $GL(R)$.

Example 2.11. If $R = F$ is a commutative field, then the determinant gives an isomorphism

$$\det : K_1(F) \to F^\times, \quad [A] \mapsto \det A.$$

from $K_1(F)$ to the multiplicative group $F^\times$.

When $R = \mathcal{D}$ is an skew-field, the Dieudonné determinant given by [Die43] is the unique map

$$\det : GL(\mathcal{D}) \to \mathcal{D}^\times/[D^\times, D^\times]$$

satisfying following properties (see [Ros95, Theorem 2.2.5]):

(a) The determinant is invariant under (left) elementary row operations. In other words, if $A \in GL(\mathcal{D})$ and A' is obtained from A by adding a (left-)multiple of a row to another row, then $\det A = \det A'$.
(b) If $A \in GL(\mathcal{D})$, and A' is obtained from A by (left-)multiplying one of the rows by $a \in \mathcal{D}$, then $\det A = \bar{a}\cdot \det A'$ where $\bar{a}$ is the image of a in $D^\times/[D^\times, D^\times]$.
(c) The determinant of the identity matrix is 1.

The determinant also has the following additional properties.

(d) If $A, B \in GL(n,\mathcal{D})$, then $\det(AB) = \det A \cdot \det B$.
(e) If $A \in GL(\mathcal{D})$ and A' is obtained from A by interchanging two of its rows, then $\det A' = (-\bar{1})\det A$.
(f) The determinant is invariant under taking transpose.

Since the target group of det is abelian, the Dieudonné determinant factors through the abelianization of $GL(\mathcal{D})$ and induces $\overline{\det} : K_1(\mathcal{D}) \to \mathcal{D}^\times/[\mathcal{D}^\times, \mathcal{D}^\times]$.

Lemma 2.12. *For any skew field $\mathcal{D}$, the Dieudonné determinant induces an group isomorphism*

$$\overline{\det} : K_1(\mathcal{D}) \to \mathcal{D}^\times/[\mathcal{D}^\times, \mathcal{D}^\times].$$

The inverse map is given by viewing an element $a \in \mathcal{D}^\times/[\mathcal{D}^\times, \mathcal{D}^\times]$ as an (1×1)-matrix $[a] \in K_1(\mathcal{D})$.

Suppose G is a finitely generated torsion-free group satisfying the Atiyah Conjecture. We apply the above constructions to Linnell's field $\mathcal{D}_G$. Note that the (1×1)-matrices $[\pm 1]$ and $[\pm G] := \{[\pm g] \mid g \in G\}$ form two subgroups of $K_1(\mathcal{D}_G)$, respectively. Define the reduced K_1-group and the Whitehead group of $\mathcal{D}_G$ as

$$\widetilde{K}_1(\mathcal{D}_G) := K_1(\mathcal{D}_G)/[\pm 1], \quad \mathrm{Wh}(\mathcal{D}_G) := K_1(\mathcal{D}_G)/[\pm G].$$

By Lemma 2.12 we identify the group $K_1(\mathcal{D}_G)$ and the abelianization of $\mathcal{D}_G^\times$ throughout this paper. The operators δ_ϕ and L_ϕ in Definition 3.2 can be defined for the above quotient groups.

Lemma 2.13. *Given any $\phi \in H^1(G;\mathbb{R})$. The homomorphism $\delta_\phi : \mathcal{D}_G^\times \to \mathbb{R}$ induces a well-defined homomorphism $\delta_\phi : \Lambda \to \mathbb{R}$ for $\Lambda = K_1(\mathcal{D}_G)$ and $\widetilde{K}_1(\mathcal{D}_G)$. The homomorphism $L_\phi : \mathcal{D}_G^\times \to \mathcal{D}_G^\times$ induces a well-defined homomorphism $L_\phi : \Lambda \to \Lambda$ for $\Lambda = K_1(\mathcal{D}_G)$, $\widetilde{K}_1(\mathcal{D}_G)$ and $\mathrm{Wh}(\mathcal{D}_G)$.*

$$\begin{tikzcd}
& & GL(\mathcal{D}_G) \arrow[dl, "\det"'] \arrow[d, "\det_r"] \arrow[dr, "\det_w"] & \\
\mathcal{D}_G^\times/[\mathcal{D}_G^\times, \mathcal{D}_G^\times] \arrow[r, equal] & K_1(\mathcal{D}_G) \arrow[r] \arrow[loop below, "{\delta_\phi, L_\phi}"] & \widetilde{K}_1(\mathcal{D}_G) \arrow[r] \arrow[loop below, "{\delta_\phi, L_\phi}"] & \mathrm{Wh}(\mathcal{D}_G) \arrow[loop below, "L_\phi"]
\end{tikzcd}$$

Convention 2.14. For the rest of the paper, whenever the group G and the skew field $\mathcal{D}_G$ is clear in the context, given any square matrix A over $\mathcal{D}_G$ denote by $\det A \in K_1(\mathcal{D}_G)$ the Dieudonné determinant of A; denote by $\det_r A$ and $\det_w A$ the image of $\det A$ in $\widetilde{K}_1(\mathcal{D}_G)$ and $\mathrm{Wh}(\mathcal{D}_G)$, respectively.

We use the same symbols δ_ϕ, L_ϕ for their induced maps on $K_1(\mathcal{D}_G)$, $\widetilde{K}_1(\mathcal{D}_G)$ and $\mathrm{Wh}(\mathcal{D}_G)$ and its domain of definition will be clear from the context.

In order to keep track with our convention of notation in $\mathcal{D}_G$, we will use multiplicative symbol for the group operations in the K_1-groups $K_1(\mathcal{D}_G)$, $\widetilde{K}_1(\mathcal{D}_G)$ and $\mathrm{Wh}(\mathcal{D}_G)$. This coincides with [Ros95, Tur01] but differs from other references [FL17, FL19].

3. Leading term map, restriction map and determinant

As usual, let G be a finitely generated torsion-free group which satisfies the Atiyah Conjecture.

3.1. **The leading term map.** Let $\nu : G \to H_1(G)_f$ be the natural quotient map to the free abelianization group $H_1(G)_f$, then we have the short exact sequence

$$1 \to K \to G \xrightarrow{\nu} H_1(G)_f \to 1.$$

Fix $\phi \in H^1(G;\mathbb{R})$ be any real cohomology class. Given any nonzero element $u \in (\mathcal{D}_K * H_1(G)_f)^\times$, we choose a section s and write

$$u = \sum_{h \in H_1(G)_f} x_h \cdot s(h) \in \mathcal{D}_K * H_1(G)_f.$$

The support of u is defined to be the set $\mathrm{supp}(u) := \{h \in H_1(G)_f \mid x_h \neq 0\}$, this is a finite subset of $H_1(G)_f$ and does not depend on the choice of section by Proposition 2.9. Define $\delta_\phi(u)$ to be the minimal value of $\phi(h)$ for all $h \in \mathrm{supp}(u)$. Define

$$L_\phi(u) := \sum_{\substack{h \in \mathrm{supp}(u), \\ \phi(h) = \delta_\phi(z)}} x_h \cdot s(h).$$

This element is nonzero and lies in $(\mathcal{D}_K * H_1(G)_f)^\times$.

Lemma 3.1. *The definition of $\delta_\phi(u)$ and $L_\phi(u)$ do not depend on the choice of section s. Moreover, we have*

$$\delta_\phi(u_1u_2) = \delta_\phi(u_1) + \delta_\phi(u_2),$$
$$L_\phi(u_1u_2) = L_\phi(u_1) \cdot L_\phi(u_2)$$

for all $u_1, u_2 \in (\mathcal{D}_K * H_1(G)_f)^\times$. Hence

$$\delta_\phi : (\mathcal{D}_K * H_1(G)_f)^\times \to \mathbb{R},$$
$$L_\phi : (\mathcal{D}_K * H_1(G)_f)^\times \to (\mathcal{D}_K * H_1(G)_f)^\times$$

are well-defined group homomorphisms.

Proof. Let $u = \sum_h x_h \cdot s(h)$ and s' be another section, then by Proposition 2.9 $u = \sum_h y_h \cdot s'(h)$ where $y_h = x_h s(h)s'(h)^{-1}$. It follows that $\delta_\phi(u)$ and $L_\phi(u)$ do not depend on the choice of section. The terms of u_1u_2 with minimal ϕ-value exactly comes from the multiplication of that of u_1 and u_2. This explains the homomorphism. ☐

Recall that $\mathcal{D}_G$ is the field of fraction of the subring $\mathcal{D}_K * H_1(G)_f$ by Proposition 2.10.

Definition 3.2. The group homomorphism δ_ϕ and L_ϕ extend to group homomorphisms

$$\delta_\phi : \mathcal{D}_G^\times \to \mathbb{R}, \quad \delta_\phi(uv^{-1}) := \delta_\phi(u) - \delta_\phi(v),$$
$$L_\phi : \mathcal{D}_G^\times \to \mathcal{D}_G^\times, \quad L_\phi(uv^{-1}) := L_\phi(u)L_\phi(v)^{-1}$$

for all $u, v \in (\mathcal{D}_K * H_1(G)_f)^\times$. It is convenient to set $\delta_\phi(0) = +\infty$ and $L_\phi(0) = 0$. Then we have

$$\delta_\phi(z_1z_2) = \delta_\phi(z_1) + \delta_\phi(z_2), \quad L_\phi(z_1z_2) = L_\phi(z_1) \cdot L_\phi(z_2)$$

for all $z_1, z_2 \in \mathcal{D}_G$.

Proof. We prove the well-definedness. Suppose $z \in \mathcal{D}_G^\times$ can be expressed as $z = u_1v_1^{-1} = u_2v_2^{-1}$, then there exists $w_1, w_2 \in (\mathcal{D}_K * H_1(G)_f)^\times$ such that $u_1w_1 = u_2w_2,\ v_1w_1 = v_2w_2$. Hence

$$\begin{aligned} L_\phi(u_1)L_\phi(v_1)^{-1} &= L_\phi(u_1)L_\phi(w_1)L_\phi(w_1)^{-1}L_\phi(v_1)^{-1} \\ &= L_\phi(u_1w_1)L_\phi(v_1w_1)^{-1} \\ &= L_\phi(u_2w_2)L_\phi(v_2w_2)^{-1} \\ &= L_\phi(u_2)L_\phi(v_2)^{-1}. \end{aligned}$$

To verify that L_ϕ is a homomorphism, let $z_1, z_2 \in \mathcal{D}_G^\times$. By the Ore condition, we can arrange that $z_1 = u_1w^{-1}$ and $z_2 = wu_2^{-1}$ for $u_1, u_2, w \in (\mathcal{D}_K * H_1(G)_f)^\times$, so

$$L_\phi(z_1)L_\phi(z_2) = L_\phi(u_1)L_\phi(w)^{-1} \cdot L_\phi(w)L_\phi(u_2)^{-1} = L_\phi(u_1)L_\phi(u_2)^{-1} = L_\phi(z_1z_2).$$

The statements for δ_ϕ can be proved similarly. ☐

Here are some basic facts about the mappings δ_ϕ, L_ϕ, especially about their properties under additions in $\mathcal{D}_G$. Most of the properties clearly holds true in the subring $\mathcal{D}_K * H_1(G)_f$ and it is routine to verify them in its field of fraction $\mathcal{D}_G$.

Proposition 3.3. *Let G be a finitely generated torsion-free group which satisfies the Atiyah Conjecture. Let $\phi \in H^1(G;\mathbb{R})$ be any real cohomology class and $\delta_\phi : \mathcal{D}_G \to \mathbb{R}$, $L_\phi : \mathcal{D}_G \to \mathcal{D}_G$ be as in Definition 3.2. Suppose $z, z_1, \dots, z_n \in \mathcal{D}_G$, then:*

(1) *$\delta_{r\phi}(z) = r \cdot \delta_\phi(z)$, $L_{r\phi}(z) = L_\phi(z)$ for all $r \in R_+$.*
(2) *$\delta_\phi(cz) = \delta_\phi(z)$, $L_\phi(cz) = c \cdot L_\phi(z)$ for all $c \in \mathbb{Q} \setminus \{0\}$.*
(3) *$\delta_\phi(L_\phi(z)) = \delta_\phi(z)$, $L_\phi(L_\phi(z)) = L_\phi(z)$.*
(4) *If $L_\phi(z_1) = L_\phi(z_2)$ nonzero, then $\delta_\phi(z_1) = \delta_\phi(z_2) < \delta_\phi(z_1 - z_2)$.*

(5) $\delta_\phi(z_1+z_2) \geqslant \min\{\delta_\phi(z_1), \delta_\phi(z_2)\}$. *If* $\delta_\phi(z_1) < \delta_\phi(z_2)$, *then*

$$\delta_\phi(z_1+z_2) = \delta_\phi(z_1), \quad L_\phi(z_1+z_2) = L_\phi(z_1).$$

(6) *If* $\delta_\phi(z_1) = \cdots = \delta_\phi(z_n) =: \delta$ *and* $\sum_{k=1}^n L_\phi(z_k) \neq 0$, *then*

$$\delta_\phi\left(\sum_{k=1}^n z_k\right) = \delta, \quad L_\phi\left(\sum_{k=1}^n z_k\right) = \sum_{k=1}^n L_\phi(z_k).$$

(7) *Given* $z \in \mathcal{D}_G$. *For any open neighborhood* $U \subset H^1(G;\mathbb{R})$ *of* ϕ, *there is a rational cohomology class* $\psi \in U$ *such that* $L_\psi(z) = L_\phi(z)$.

(8) *Suppose* $L \subset G$ *is an inclusion of a finitely generated subgroup* L, *and* $\mathcal{D}_L \subset \mathcal{D}_G$ *is the induced inclusion. Denote by* $\phi|_L : L \to \mathbb{R}$ *the restriction of* ϕ *to* L. *Then the mappings*

$$\delta_{\phi|_L} : \mathcal{D}_L \to \mathbb{R}, \quad L_{\phi|_L} : \mathcal{D}_L \to \mathcal{D}_L$$

are exactly the restrictions of δ_ϕ *and* L_ϕ *to* $\mathcal{D}_L$.

Proof. For (1)–(3), the statements hold in $\mathcal{D}_K * H_1(G)_f$ and directly extends to $\mathcal{D}_G$ by definition.

For (4), assume that $z_1 = u_1 w^{-1}$, $z_2 = u_2 w^{-1}$ for $u_1, u_2, w \in \mathcal{D}_K * H_1(G)_f$, then $L_\phi(z_1) = L_\phi(z_2)$ implies that $L_\phi(u_1) = L_\phi(u_2)$. It follows that $\delta_\phi(u_1) = \delta_\phi(u_2) < \delta_\phi(u_1 - u_2)$. Therefore $\delta_\phi(z_1) = \delta_\phi(z_2) < \delta_\phi(z_1 - z_2)$.

For (5), assume that $z_1 = u_1 w^{-1}$, $z_2 = u_2 w^{-1}$ for $u_1, u_2, w \in \mathcal{D}_K * H_1(G)_f$. Since it is clear that $\delta_\phi(u_1+u_2) \geqslant \min\{\delta_\phi(u_1), \delta_\phi(u_2)\}$, then $\delta_\phi(z_1+z_2) = \delta_\phi(u_1+u_2) - \delta_\phi(w) \geqslant \min\{\delta_\phi(u_1), \delta_\phi(u_2)\} - \delta_\phi(w) = \min\{\delta_\phi(z_1), \delta_\phi(z_2)\}$. If $\delta_\phi(z_1) < \delta_\phi(z_2)$, then $\delta_\phi(u_1) < \delta_\phi(u_2)$ and $L_\phi(u_1+u_2) = L_\phi(u_1)$. It follows that $\delta_\phi(z_1+z_2) = \delta_\phi(z_1)$ and $L_\phi(z_1+z_2) = L_\phi(z_1)$.

For (6), assume that $z_i = u_i w^{-1}$ for $u_i, w \in \mathcal{D}_K * H_1(G)_f$, $i = 1, \ldots, n$. By assumption we have $\delta_\phi(u_i) = \delta + \delta_\phi(w)$ and $\sum_{i=1}^k L_\phi(u_i) \neq 0$. It follows that $\delta_\phi(\sum_{i=1}^k u_i) = \delta + \delta_\phi(w)$ and $L_\phi(\sum_{i=1}^k u_i) = \sum_{i=1}^k L_\phi(u_i)$. Hence $\delta_\phi(\sum_{i=1}^k z_i) = \delta$ and $L_\phi(\sum_{i=1}^k z_i) = \sum_{i=1}^k L_\phi(z_i)$.

For (7), assume $z = uv^{-1}$ with $u, v \in \mathcal{D}_K * H_1(G)_f$. Write $u = \sum_{h \in H_1(G)_f} x_h \cdot s(h)$ for a section s. Given another cohomology class $\psi \in H^1(G;\mathbb{R})$, then $L_\phi(z) = L_\psi(z)$ if the following two conditions hold:

- for all $h, h' \in \mathrm{supp}(u) \cup \mathrm{supp}(v)$, $\psi(h-h') < 0$ whenever $\phi(h-h') < 0$;
- for all $h, h' \in \mathrm{supp}(u) \cup \mathrm{supp}(v)$, $\psi(h-h') = 0$ whenever $\phi(h-h') = 0$.

The domain Ω of such ψ is the intersection of finitely many closed hyperplanes and open half spaces of $H^1(G;\mathbb{R})$, each given by an integral linear equation. Since $\phi \in \Omega$, given any open neighborhood $U \ni \phi$ there are rational classes in $U \cap \Omega$.

For (8), consider the short exact sequence $1 \to K' \to L \to H_1(L)_f \to 1$. There is a commutative diagram

$$\begin{tikzcd}
\mathcal{D}_{K'} * H_1(L)_f \arrow[r, hook] \arrow[d, hook] & \mathcal{D}_K * H_1(G)_f \arrow[d, hook] \\
\mathcal{D}_L \arrow[r, hook] & \mathcal{D}_G
\end{tikzcd}$$

and $\mathcal{D}_L$ is the field of fraction of $\mathcal{D}_{K'} * H_1(L)_f$. Choose any nonzero $u \in \mathcal{D}_{K'} * H_1(L)_f$ and write $u = \sum_{h \in H_1(L)_f} x_h \cdot s(h)$, $x_h \in \mathcal{D}_{K'}$ for a section $s : H_1(L)_f \to L$. By definition $\delta_{\phi|_L}(u) = \min\{\phi(h) \mid h \in H_1(L)_f,\ x_h \neq 0\}$. Write $\delta := \delta_{\phi|_L}(u)$. Decompose

$$u = \sum_{\substack{h \in H_1(L)_f, \\ \phi|_L(h) = \delta}} x_h \cdot s(h) + \sum_{\substack{h \in H_1(L)_f, \\ \phi|_L(h) > \delta}} x_h \cdot s(h) =: u_1 + u_2.$$

Then by definition $L_{\phi|_L}(u) = u_1$ is nonzero. We want to show

$$\delta_\phi(u) = \delta, \quad L_\phi(u) = u_1.$$

This does not directly follow from the definition of δ_ϕ and L_ϕ since $s : H_1(L)_f \to L$ is not a section of $G \to H_1(G)_f$. We choose to argue as follows. By (6) applied to u_1 we know that

$$\delta_\phi(u_1) = \delta, \quad L_\phi(u_1) = u_1.$$

By (5) applied to u_2 we know that

$$\delta_\phi(u_2) \geqslant \min\{\delta_\phi(x_h \cdot s(h)) \mid h \in H_1(L)_f,\ \phi|_L(h) > \delta\} > \delta,$$

Again by (5) applied to $u = u_1 + u_2$ we know that $\delta_\phi(u) = \delta_\phi(u_1) = \delta$ and $L_\phi(u) = L_\phi(u_1) = u_1$. Hence $\delta_\phi(u) = \delta_{\phi|_L}(u)$ and $L_\phi(u) = L_{\phi|_L}(u)$ for all nonzero $u \in \mathcal{D}_{K'} * H_1(L)_f$. Passing to the field of fraction we have $\delta_\phi(z) = \delta_{\phi|_L}(z)$ and $L_\phi(z) = L_{\phi|_L}(z)$ for all $z \in \mathcal{D}_L$. ☐

Definition 3.4. An element $z \in \mathcal{D}_G$ is called ϕ-pure if $L_\phi(z) = z$.

Lemma 3.5. *Here are some properties of the ϕ-pure elements.*

1. *Elements of $\mathbb{Z}[\ker\phi] \subset \mathcal{D}_G$ are ϕ-pure; elements of $G \subset \mathcal{D}_G$ are ϕ-pure.*
2. *The product of two ϕ-pure elements is ϕ-pure.*
3. *Given any nonzero $z \in \mathcal{D}_G$, then $L_\phi(z)$ is the unique element $w \in \mathcal{D}_G$ such that w is ϕ-pure and $\delta_\phi(z - w) > \delta_\phi(z)$.*
4. *Suppose $z_1, \dots, z_n$ are ϕ-pure elements with $\delta_\phi(z_1) = \cdots = \delta_\phi(z_1) = \delta$, then the sum $\sum_{i=1}^n z_i$ is also ϕ-pure.*

Proof. The properties (1) and (2) follow from the definition of L_ϕ.

For (3), it follows from Proposition 3.3 (3), (4) that $L_\phi(z)$ is ϕ-pure and $\delta_\phi(z - L_\phi(z)) > \delta_\phi(z)$. On the other hand, if a ϕ-pure element w satisfies $\delta_\phi(z - w) > \delta_\phi(z)$, then by Proposition 3.3 (4)

$$w = L_\phi(w) = L_\phi(z - (z - w)) = L_\phi(z).$$

Finally, (4) is a consequence of Proposition 3.3 (6). ☐

Theorem 3.6. *Let $\phi \in H^1(H; \mathbb{R})$ be any real cohomology class. Suppose P and Q are two square matrices over $\mathcal{D}_G$ of size $n \geqslant 1$, such that the following conditions hold:*

- (i) *P is invertible over $\mathcal{D}_G$;*
- (ii) *there exist real numbers $d_0, d_1, \dots, d_n$ such that if P_{ij} is nonzero, then P_{ij} is ϕ-pure with $\delta_\phi(P_{ij}) = d_0 + d_i - d_j$;*
- (iii) *$\delta_\phi(Q_{ij}) > d_0 + d_i - d_j$ for all i, j.*

Then we have $L_\phi(\det(P + Q)) = \det P \in K_1(\mathcal{D}_G)$.

Proof. We prove by induction on n. When $n = 1$ then $P, Q \in \mathcal{D}_G$, by the conditions we have $\delta_\phi(P) = d_0$, and $\delta_\phi(Q) > d_0$ if Q is nonzero. Then $L_\phi(\det(P + Q)) = \det P$ by definition.

Now assume Lemma 3.6 holds for size n. Assume that P, Q are $(n + 1)$ by $(n + 1)$ matrices

$$P = \begin{pmatrix} A & U \\ X & p \end{pmatrix}, \quad Q = \begin{pmatrix} B & V \\ Y & q \end{pmatrix}$$

where $p, q \in \mathcal{D}_G$, A, B are n by n matrices over $\mathcal{D}_G$ and

$$X = (x_1, \dots, x_n), \quad Y = (y_1, \dots, y_n),$$
$$U = (u_1, \dots, u_n)^T, \quad V = (v_1, \dots, v_n)^T$$

are matrices over $\mathcal{D}_G$ of appropriate size. Without loss of generality we can assume $p \neq 0$, hence also $p + q \neq 0$ by condition (iii). Note that

$$\begin{pmatrix} I & -(U + V)(p + q)^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A + B & U + V \\ X + Y & p + q \end{pmatrix} = \begin{pmatrix} W & 0 \\ X + Y & p + q \end{pmatrix}$$

where

$$W = A + B - (U + V)(p + q)^{-1}(X + Y),$$

is an n by n matrix. So

$$\det(P + Q) = \det W \cdot \det(p + q),$$
$$L_\phi \det(P + Q) = L_\phi \det W \cdot \det p$$

(note that $L_\phi(p + q) = p$). Staring at the expression of W, it is easy to guess that the terms of lowest δ_ϕ form the matrix $A - Up^{-1}X$. We show that this is indeed the case.

Claim. *Define*

$$W' := A - Up^{-1}X, \quad W'' := W - W'.$$

Then W' is invertible over $\mathcal{D}_G$ and $\det P = \det W' \cdot \det p$. Moreover, W' and W'' satisfies the three conditions of Lemma 3.6 for size n and then by the induction hypothesis we have $L_\phi \det(W) = \det(W')$.

Admitting this claim. Then we have

$$L_\phi \det(P + Q) = L_\phi \det W \cdot \det p = \det W' \cdot \det p = \det P$$

and the induction is finished. It remains to prove the Claim.

Proof of Claim. To prove (i), it follows from

$$\begin{pmatrix} I & -Up^{-1} \\ 0 & 1 \end{pmatrix} \cdot P = \begin{pmatrix} W' & 0 \\ X & p \end{pmatrix}$$

that W' is invertible over $\mathcal{D}_G$ and $\det P = \det W' \cdot \det p$.

For (ii), note that

$$W_{ij} = A_{ij} + B_{ij} - (u_i + v_i)(p + q)^{-1}(x_j + y_j),$$
$$W'_{ij} = A_{ij} - u_i p^{-1} x_j,$$
$$W''_{ij} = B_{ij} + u_i p^{-1} x_j - (u_i + v_i)(p + q)^{-1}(x_j + y_j).$$

If $A_{ij} \neq 0$, then A_{ij} is ϕ-pure with $\delta_\phi(A_{ij}) = d_0 + d_i - d_j$; if $u_i p^{-1} x_j \neq 0$, then $u_i p^{-1} x_j$ is also ϕ-pure with

$$\delta_\phi(u_i p^{-1} x_j) = (d_0 + d_i - d_{n+1}) - d_0 + (d_0 + d_{n+1} - d_j) = d_0 + d_i - d_j.$$

In conclusion, if $W'_{ij} \neq 0$ then W'_{ij} is ϕ-pure with $\delta_\phi(W'_{ij}) = d_0 + d_i - d_j$ and this proves (ii).

For (iii), by assumption we have $\delta_\phi(B_{ij}) > d_0 + d_i - d_j$. We also have $\delta_\phi(u_i p^{-1} x_j - (u_i + v_i)(p + q)^{-1}(x_j + y_j)) > d_0 + d_i - d_j$, since if $u_i p^{-1} x_j \neq 0$, then

$$L_\phi((u_i + v_i)(p + q)^{-1}(x_j + y_j)) = u_i p^{-1} x_j, \quad \delta_\phi(u_i p^{-1} x_j) = d_0 + d_i - d_j$$

and we can apply Proposition 3.3 (4); if $u_i p^{-1} x_j = 0$ then $u_i = 0$ or $x_j = 0$ and

$$\begin{aligned} \delta_\phi((u_i + v_i)(p + q)^{-1}(x_j + y_j)) &= \delta_\phi(u_i + v_i) - \delta_\phi(p + q) + \delta_\phi(x_j + y_j) \\ &> \delta_\phi(u_i) - \delta_\phi(p + q) + \delta_\phi(x_j) \\ &= d_0 + d_i - d_j. \end{aligned}$$

In either cases, we have $\delta_\phi(u_i p^{-1} x_j - (u_i + v_i)(p + q)^{-1}(x_j + y_j)) > d_0 + d_i - d_j$. These combines to show that $\delta_\phi(W'') > d_0 + d_i - d_j$ by Proposition 3.3 (5). $\square$

3.2. **The restriction map.** Suppose G is a finitely generated torsion-free group satisfying the Atiyah Conjecture and let $L \triangleleft G$ be a normal subgroup of finite index d. In this section we define the restriction map $\operatorname{res}_L^G : K_1(\mathcal{D}_G) \to K_1(\mathcal{D}_L)$. Recall that $\mathcal{D}_G$ is naturally isomorphic to the crossed product $\mathcal{D}_L * (G/L)$ by Proposition 2.10.

Definition 3.7. Fix a section $s : G/L \to G$ and suppose its image is $s(G/L) = \{g_1, \ldots, g_d\}$. Then $G = g_1 L \sqcup \ldots \sqcup g_d L$. For any element $z \in \mathcal{D}_G$ and for any $k \in \{1, \ldots, d\}$, there is a unique way to express $g_k \cdot z$ as

$$g_k \cdot z = \sum_{j=1}^{d} l_{kj} \cdot g_j, \quad l_{kj} \in \mathcal{D}_L.$$

Define $\Lambda_s(z)$ to be the $(d \times d)$-matrix over $\mathcal{D}_L$ whose (k,j)-entry is l_{kj}. In other words, $\Lambda_s(z)$ is the unique matrix over $\mathcal{D}_L$ such that

$$\begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z = \Lambda_s(z) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}.$$

Lemma 3.8. *With the notations as in Definition 3.7, the following statements hold.*

(1) *For any $z_1, z_2 \in \mathcal{D}_G$, we have $\Lambda_s(z_1 z_2) = \Lambda_s(z_2) \cdot \Lambda_s(z_1)$.*
(2) *If $z \neq 0$, then $\Lambda_s(z)$ is invertible over $\mathcal{D}_L$.*
(3) *If s' is another section, then $\Lambda_{s'}(z) = \Omega \Lambda_s(z) \Omega^{-1}$ for an invertible matrix Ω over $\mathbb{Z}L$ which only depends on s and s'.*
(4) *For any $\phi \in H^1(G;\mathbb{R})$ we have $\delta_\phi(g_k \cdot z) \leqslant \delta_\phi(l_{kj} \cdot g_j)$. If z is ϕ-pure, then l_{kj} is also ϕ-pure; moreover if $l_{kj} \neq 0$ then $\delta_\phi(g_k \cdot z) = \delta_\phi(l_{kj} \cdot g_j)$.*

Proof. For (1), it follows from

$$\begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z_1 z_2 = \Lambda_s(z_1) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z_2 = \Lambda_s(z_1) \cdot \Lambda_s(z_2) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}$$

that $\Lambda_s(z_1 z_2) = \Lambda_s(z_2) \cdot \Lambda_s(z_1)$. Note that (2) is a direct consequence of (1).

For (3), let s' be another section with $s'(L) = \{g_1', \ldots, g_d'\}$ and let Ω be the $(d \times d)$-matrix over $\mathbb{Z}L$ such that

$$\begin{pmatrix} g_1' \\ \vdots \\ g_d' \end{pmatrix} = \Omega \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}.$$

Then

$$\begin{pmatrix} g_1' \\ \vdots \\ g_d' \end{pmatrix} \cdot z = \Omega \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z = \Omega \Lambda_s(z) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} = \Omega \Lambda_s(z) \Omega^{-1} \begin{pmatrix} g_1' \\ \vdots \\ g_d' \end{pmatrix}$$

and therefore $\Lambda_{s'}(z) = \Omega \Lambda_s(z) \Omega^{-1}$.

For (4). First suppose the contrary that there exists $k, j \in \{1, \ldots, d\}$ such that $\delta_\phi(g_k \cdot z) > \delta_\phi(l_{kj} \cdot g_j)$. Fix k and let $\mathcal{J}$ be the collection of indices j such that $\delta_\phi(g_k \cdot z) > \delta_\phi(l_{kj} \cdot g_j)$. Since

$$\delta_\phi(g_k \cdot z) = \delta_\phi\Bigl(\sum_{j=1}^{d} l_{kj} \cdot g_j\Bigr),$$

it follows from Proposition 3.3 (5) that $\sum_{j\in\mathcal{J}} l_{kj}\cdot g_j = 0$. Recall that $g_1,\dots,g_d$ are $\mathcal{D}_L$-independent by Proposition 2.9. This implies that $l_{kj}=0$ for all $j\in\mathcal{J}$ and forces $\mathcal{J}$ to be empty, a contradiction. Hence $\delta_\phi(g_k\cdot z)\leqslant \delta_\phi(l_{kj}\cdot g_j)$ for all k,j. It can be seen from Lemma 3.5 (3) that

$$L_\phi(g_k\cdot z) = L_\phi\left(\sum_{j=1}^d l_{kj}\cdot g_j\right) = \sum_{j=1}^d l'_{kj}\cdot g_j$$

where

$$l'_{kj} = \begin{cases} L_\phi(l_{kj}) & \text{if } \delta_\phi(l_{kj}\cdot g_j) = \delta_\phi(g_k\cdot z) \\ 0 & \text{if } \delta_\phi(l_{kj}\cdot g_j) > \delta_\phi(g_k\cdot z). \end{cases}$$

If in addition z is ϕ-pure, then $g_k\cdot z$ is also ϕ-pure by Lemma 3.5 and therefore $l_{kj}=l'_{kj}$ for all k,j. It follows that l_{kj} is ϕ-pure for all k,j and moreover if $l_{kj}\neq 0$ then $\delta_\phi(g_k\cdot z)=\delta_\phi(l_{kj}\cdot g_j)$. □

Definition 3.9. Let $L\triangleleft G$ be a normal subgroup of finite index d. Choose a section $s: G/L\to G$ with $s(G/L)=\{g_1,\dots,g_d\}$. Then $G = g_1L\sqcup\ldots\sqcup g_dL$. Given any element $z\in\mathcal{D}_G^\times$ and $[z]\in K_1(\mathcal{D}_G)$ is the corresponding (1×1)-matrix. Define

$$\operatorname{res}_L^G : K_1(\mathcal{D}_G)\to K_1(\mathcal{D}_L),\quad [z]\mapsto \det(\Lambda_s(z)).$$

This mapping is a group homomorphism independent of the choice of $g_1,\dots,g_d$.

Remark 3.10. An element $z\in\mathcal{D}_G^\times$ can be associated with $R_z:\mathcal{D}_G\to\mathcal{D}_G$, the operator of right multiplication by z. The choice of coset representatives identifies $\mathcal{D}_G$ with $\bigoplus_{k=1}^d \mathcal{D}_L\cdot g_k$ as $\mathcal{D}_L$-vector spaces and R_z is naturally a $\mathcal{D}_L$-linear automorphism represented by the matrix $\Lambda_s(z)$. By definition $\operatorname{res}_L^G([z]) = \det R_z$. A different choice of coset representatives amounts to a change of basis which preserves the determinant. This shows that $\operatorname{res}_L^G([z])$ does not depend on the choice of $g_1,\dots,g_d$.

Theorem 3.11. *Suppose G is a finitely generated torsion-free group satisfying the Atiyah Conjecture and let $L\triangleleft G$ be a normal subgroup of finite index. Let $\phi\in H^1(G;\mathbb{R})$ and denote by $\phi|_L\in H^1(L;\mathbb{R})$ the restriction of ϕ to L. Then for any $[z]\in K_1(\mathcal{D}_G)$, we have*

$$L_{\phi|_L}(\operatorname{res}_L^G([z])) = \operatorname{res}_L^G(L_\phi([z]))\in K_1(\mathcal{D}_L).$$

Proof. Choose $z\in\mathcal{D}_G^\times$ representing the class $[z]$. Let $z =: L_\phi(z)+z'$. Fix a choice of coset representatives $G=g_1L\sqcup\ldots\sqcup g_dL$. Let P and Q be $(d\times d)$-matrices over $\mathcal{D}_L$ such that

$$g_k\cdot L_\phi(z) = \sum_{j=1}^d P_{kj}\cdot g_j,\quad g_k\cdot z' = \sum_{j=1}^d Q_{kj}\cdot g_j.$$

Then $\operatorname{res}_L^G([z]) = \det(P+Q)$ and $\operatorname{res}_L^G([L_\phi(z)]) = \det P$. Since $L_\phi(z)$ is ϕ-pure, it follows from Lemma 3.7 that P_{kj} is ϕ-pure, and moreover

$$\begin{aligned} &\delta_\phi(P_{kj}) = \delta_\phi(z)+\delta_\phi(g_k)-\delta_\phi(g_j) \quad \text{if } P_{kj}\neq 0,\\ &\delta_\phi(Q_{kj}) \geqslant \delta_\phi(z')+\delta_\phi(g_k)-\delta_\phi(g_j) > \delta_\phi(z)+\delta_\phi(g_k)-\delta_\phi(g_j) \end{aligned}$$

for all k,j. By Proposition 3.3 (8) $\delta_\phi=\delta_{\phi|_L}$ and $L_\phi = L_{\phi|_L}$ in $\mathcal{D}_L$. Applying Theorem 3.6 to P and Q over $\mathcal{D}_L$ we have

$$\det P = L_{\phi|_L}(\det(P+Q))$$

hence $L_{\phi|_L}(\operatorname{res}_L^G([z])) = \operatorname{res}_L^G(L_\phi([z]))$ and the proof is finished. □

4. Universal L^2-torsion

Let G be a finitely generated torsion-free group satisfying the Atiyah conjecture.

4.1. **Universal L^2-torsion of chain complexes.** A chain complex C_* is called a finite based free $\mathbb{Z}G$-chain complex if there exists $n \geqslant 0$ such that

$$C_* = (0 \to C_n \to \cdots \to C_1 \to C_0 \to 0),$$

where each C_k is a finitely generated free (left) $\mathbb{Z}G$-module equipped with a preferred (unordered) free $\mathbb{Z}G$-basis, and the boundary operators are $\mathbb{Z}G$-linear maps.

Definition 4.1. An $(n \times n)$-matrix A over $\mathbb{Z}G$ is called a weak isomorphism if the operator $r_A : l^2(G)^n \to l^2(G)$ given by right multiplication with A is injective and has dense image. A finite based free $\mathbb{Z}G$-chain complex C_* is said to be L^2-acyclic if the chain complex $l^2(G) \otimes_{\mathbb{Z}G} C_*$ is weakly exact as a chain complex of Hilbert modules, i.e. the boundary operators $\partial_k^{(2)} : l^2(G) \otimes_{\mathbb{Z}G} C_k \to l^2(G) \otimes_{\mathbb{Z}G} C_{k-1}$ satisfies $\ker \partial_k^{(2)} = \overline{\operatorname{im} \partial_{k+1}^{(2)}}$ for all k.

We will not work with those analytic-flavored definitions but prefer the more algebraic-flavored ones given by the following Lemma.

Lemma 4.2 ([FL17, Lemma 2.21])**.** *A finite based free $\mathbb{Z}G$-chain complex C_* is L^2-acyclic if and only if the chain complex $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*$ is exact as a chain complex of (left) $\mathcal{D}_G$-modules. A square matrix over $\mathbb{Z}G$ is a weak isomorphism if and only if it is invertible over $\mathcal{D}_G$.*

It is classical to define the torsion of an acyclic chain complex of free modules, see for example [Mil66, Coh73, Tur01]. The situation here is particularly nice because $\mathcal{D}_G$ is a skew-field and any module over a skew field is free. We adopt the definitions in [Tur01, Section 3].

Definition 4.3. Let $\mathcal{D}$ be a skew-field and let V be a finitely generated (left) $\mathcal{D}$-module. Suppose $\dim V = k$ and pick two (unordered) bases $b = \{b_1, \ldots, b_k\}$ and $c = \{c_1, \ldots, c_k\}$. Then

$$b_i = \sum_{j=1}^{k} a_{ij} c_j, \quad k = 1, \ldots, k,$$

where the transition matrix $(a_{ij})_{i,j=1,\ldots,k}$ is a non-degenerate $(k \times k)$-matrix. Define $[b/c] = \det_r(a_{ij}) \in \widetilde{K}_1(\mathcal{D})$.

Definition 4.4 (Universal L^2-torsion of chain complexes)**.** Assume that C_* is a finite based $\mathbb{Z}G$-chain complex of length n which is L^2-acyclic, then the chain complex $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*$ is an exact $\mathcal{D}_G$-chain complex, with c_i the preferred basis of $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_i$. Let ∂ be the boundary homomorphism and pick bases b_i of the free $\mathcal{D}_G$-module $B_i := \operatorname{im} \partial_i$. Combine them to bases $b_i b_{i-1}$ of C_i. Then the universal L^2-torsion of C_* is defined to be

$$\tau_u^{(2)}(C_*) := \prod_{i=0}^{n} [b_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \widetilde{K}_1(\mathcal{D}_G).$$

This definition does not depend on the choice of basis b_i.

An exact sequence $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ of finitely generated based free $\mathbb{Z}G$-modules is called based exact, if $i(b_0) \subset b_1$ and p maps $b_1 \setminus i(b_0)$ bijectively to b_2, where b_i is the preferred basis of M_i, $i = 0, 1, 2$. An exact sequence $0 \to C_* \to D_* \to E_* \to 0$ of finite based free $\mathbb{Z}G$-chain complex is called based exact if

$$0 \to C_k \to D_k \to E_k \to 0$$

is based exact for all k. The following basic property can be found in [Tur01, Theorem 3.4].

Proposition 4.5. *The universal L^2-torsion for chain complexes have the following properties.*

(1) *For any L^2-acyclic finite based free $\mathbb{Z}G$-chain complex*
$$C_* = (0 \to C_1 \xrightarrow{A} C_0 \to 0)$$
where C_0 and C_1 are isomorphic to $\mathbb{Z}G^r$ under the preferred basis and A is a square matrix A over $\mathbb{Z}G$. Then $\tau_u^{(2)}(C_*) = (\det_r A)^{-1} \in \widetilde{K}_1(\mathcal{D}_G)$.

(2) *If C_*, C'_*, C''_* are finite based free $\mathbb{Z}G$-chain complexes and there is a based short exact sequence*
$$0 \to C'_* \to C_* \to C''_* \to 0.$$
If C'_* and C''_* are L^2-acyclic then C_* is L^2-acyclic and
$$\tau_u^{(2)}(C_*) = \tau_u^{(2)}(C'_*) \cdot \tau_u^{(2)}(C''_*) \in \widetilde{K}_1(\mathcal{D}_G).$$

Recall that $\mathbb{Z}G$ is a ring with an involution $x \mapsto \bar{x}$ which sends $\sum n_g \cdot g$ to $\sum n_g \cdot g^{-1}$. Given any left $\mathbb{Z}G$-module A, define the dual module A^* to be $\mathrm{Hom}_{\mathbb{Z}G}(A, \mathbb{Z}G)$, considered as a left $\mathbb{Z}G$-module as follows. For each $x \in \mathbb{Z}G$ and $f : A \to \mathbb{Z}G$ define $xf : A \to \mathbb{Z}G$ by the formula $(xf)(y) = f(y) \cdot \bar{x}$, $\forall y \in A$. If A is a free $\mathbb{Z}G$-module with basis a_i, then A^* is a free $\mathbb{Z}G$-module with basis a_i^*. Suppose $f : A \to B$ is a $\mathbb{Z}G$-linear map between two based free $\mathbb{Z}G$-modules represented by a matrix P under the given bases, then the dual map $f^* : B^* \to A^*$ is represented by the matrix P^*, the involution transpose of P. The following Proposition 4.6 is a classic property of torsion invariants and can be proved as in [Mil62].

Proposition 4.6. *If $C_* = (0 \to C_n \to \cdots \to C_0 \to 0)$ is a finite based free $\mathbb{Z}G$-chain complex which is L^2-acyclic. Then the dual chain complex C^* is L^2-acyclic and*
$$\overline{\tau_u^{(2)}(C^*)} = \tau_u^{(2)}(C_*)^{(-1)^{n+1}} \in \widetilde{K}_1(\mathcal{D}_G).$$

Remark 4.7. The universal L^2-torsion of chain complexes was first defined in [FL17]. They defined the weak K_1-group $K_1^w(\mathbb{Z}G)$ and the reduced weak K_1 group $\widetilde{K}_1^w(\mathbb{Z}G)$ for general groups G where the universal L^2-torsion lives in. The universal property of the universal L^2-torsion is also established in that paper. When G satisfies the Atiyah Conjecture then there is a natural homomorphism $i_G : K_1^w(\mathbb{Z}G) \to K_1(\mathcal{D}_G)$ and therefore $\widetilde{i}_G : \widetilde{K}_1^w(\mathbb{Z}G) \to \widetilde{K}_1(\mathcal{D}_G)$. By universal property, the universal L^2-torsion defined in our paper is the image of theirs under $\tilde{i}$.

Recall that Linnell's class $\mathcal{C}$ is the smallest class of groups which contains all free groups and is closed under directed unions and extensions by elementary amenable groups (see Theorem 2.2). It is proved [LL17, Theorem 0.1] that the i_G becomes an isomorphism if G is a torsion-free group in $\mathcal{C}$. So for any torsion-free group G in Linnell's class $\mathcal{C}$ our definition of the universal L^2-torsion coincides with the original one in [FL17]. In particular, this includes all torsion-free 3-manifold groups by Theorem 2.3.

4.2. **Universal L^2-torsion of CW-complexes.** Let X be a connected finite CW-complex with fundamental group G and let $Y \subset X$ be a subcomplex. Let $p : \widehat{X} \to X$ be the universal covering of X, and let $\widehat{Y} := p^{-1}(Y)$ be the preimage. Then $\widehat{X}$ admits the induced CW-structure and $\widehat{Y}$ is a subcomplex of $\widehat{X}$. The natural left G-action on $\widehat{X}$ gives rise to the left $\mathbb{Z}G$-module structure on the cellular chain complex $C_*(\widehat{X}, \widehat{Y})$. By choosing a lift $\widehat{\sigma}$ for each cell σ in $X \setminus Y$, we find a free $\mathbb{Z}G$-basis for each $\mathbb{Z}G$-module $C_k(\widehat{X}, \widehat{Y})$. Therefore $C_*(\widehat{X}, \widehat{Y})$ becomes a finite based free $\mathbb{Z}G$-chain complex. The following definition does not depend on the choice of the lifting of cells.

Definition 4.8. Let X be a finite connected CW-complex with fundamental group G and let Y be a subcomplex of X. The pair (X, Y) is called L^2-acyclic if the finite based free chain complex $C_*(\widehat{X}, \widehat{Y})$ is L^2-acyclic (c.f. Lemma 4.2). If the pair (X, Y) is L^2-acyclic, then we define its universal L^2-torsion
$$\tau_u^{(2)}(X, Y) \in \mathrm{Wh}(\mathcal{D}_G)$$

to be the image of $\tau_u^{(2)}(C_*(\widehat{X},\widehat{Y}))$ under the projection $\widetilde{K}_1(\mathcal{D}_G) \to \mathrm{Wh}(\mathcal{D}_G)$.

When X is not necessarily connected, we say (X,Y) is L^2-acyclic if for every component $X_i \in \pi_0(X)$ the pair $(X_i, X_i \cap Y)$ is L^2-acyclic. Furthermore, if the fundamental groups $\pi_1(X_i)$ satisfy the Atiyah conjecture, then we define

$$\mathrm{Wh}(\mathcal{D}_{\Pi(X)}) := \bigoplus_{X_i \in \pi_0(X)} \mathrm{Wh}(\mathcal{D}_{\pi_1(X_i)}),$$
$$\tau_u^{(2)}(X,Y) := (\tau_u^{(2)}(X_i, X_i \cap Y))_{X_i \in \pi_0(X)} \in \mathrm{Wh}(\mathcal{D}_{\Pi(X)}).$$

Assume that (X,Y) and (X',Y') are finite CW-pairs. We say a CW-mapping $f:(X',Y') \to (X,Y)$ is π_1-injective, if the restriction of f to each component of X' induces an injection on fundamental groups. In this case, there is a natural homomorphism $\iota_* : \mathrm{Wh}(\mathcal{D}_{\Pi(X')}) \to \mathrm{Wh}(\mathcal{D}_{\Pi(X)})$. Suppose (X,Y) and (X',Y') are L^2-acyclic, then the induced universal L^2-torsion

$$\iota_* \tau_u^{(2)}(X',Y') \in \mathrm{Wh}(\mathcal{D}_{\Pi(X)})$$

is the image of the universal L^2-torsion of (X',Y') under the homomorphism ι_*.

Theorem 4.9. *We record the fundamental properties of the universal L^2-torsion.*

(1) (Simple-homotopy invariance) *Suppose (X,X_0) and (Y,Y_0) are CW-pairs. Let $f:(X,X_0) \to (Y,Y_0)$ be a mapping such that $f: X \to Y$ and $f|_{X_0}: X_0 \to Y_0$ are simple-homotopy equivalences. Then (X,X_0) is L^2-acyclic if and only if (Y,Y_0) is L^2-acyclic. In this case, we have $\tau_u^{(2)}(Y,Y_0) = f_* \tau_u^{(2)}(X,X_0)$.*

(2) (Sum formula) *Let $(U,V) = (X,C) \cup (Y,D)$ where (X,C), (Y,D) and $(X \cap Y, C \cap D)$ are L^2-acyclic sub-pairs that embeds π_1-injectively into (U,V), then*
$$\tau_u^{(2)}(U,V) = (\iota_1)_* \tau_u^{(2)}(X,C) \cdot (\iota_2)_* \tau_u^{(2)}(Y,D) \cdot (\iota_3)_* \tau_u^{(2)}(X \cap Y, C \cap D)^{-1}$$
where $\iota_i,\ i = 1,2,3$ are the embeddings of the corresponding space pairs into (M,N).

(3) (Induction) *Let $f:(X_0,Y_0) \subset (X,Y)$ be a π_1-injective inclusion. Let $\widehat{X}$ be the universal cover of X and let $\widehat{X}_0, \widehat{Y}_0$ be the preimage of X_0, Y_0 in $\widehat{X}_0$. Then the finite based free $\mathbb{Z}[\pi_1(X)]$-chain complex $C_*(\widehat{X}_0, \widehat{Y}_0)$ is L^2-acyclic if and only if (X_0,Y_0) is L^2-acyclic. In this case we have*
$$\tau_u^{(2)}(C_*(\widehat{X}_0,\widehat{Y}_0)) = f_* \tau_u^{(2)}(X_0,Y_0).$$

(4) (Restriction) *Let X be a connected finite CW-complex and let $\overline{X}$ be a connected finite degree covering of X. Suppose that $\pi_1(X) = G$ and $\pi_1(\overline{X}) = H$ and recall the restriction map $\mathrm{res}_H^G : \widetilde{K}_1(\mathcal{D}_G) \to \widetilde{K}_1(\mathcal{D}_H)$ defined in Section 3.2. Let $Y \subset X$ be a subcomplex and let $\overline{Y}$ be its preimage in $\overline{X}$. Then*
$$\tau_u^{(2)}(\overline{X},\overline{Y}) = \mathrm{res}_H^G\, \tau_u^{(2)}(X,Y).$$

Proof. The first one will be proved in Remark 4.14 after we introduced the notion of universal L^2-torsion of mappings. Properties (2)–(4) are natural generalization of [FL17, Theorem 3.5] to CW-pairs and the proof carry over without essential changes to the relative cases. □

4.3. Universal L^2-torsion of an L^2-weak homotopy equivalence.

Definition 4.10. Let X,Y be finite CW-complexes. A cellular map $f: X \to Y$ is an L^2-weak homotopy equivalence if the CW-pair (M_f, X) is L^2-acyclic, where $M_f = (X \times I) \cup_f Y$ is the mapping cylinder with $X = X \times \{1\}$ as a subcomplex. If in addition the fundamental group of the target Y satisfies the Atiyah Conjecture, then define

$$\tau_u^{(2)}(f) := \iota_* \tau_u^{(2)}(M_f, X) \in \mathrm{Wh}(\mathcal{D}_{\Pi(Y)})$$

where $\iota: M_f \to Y$ is the natural deformation retraction.

Proposition 4.11. *Suppose the spaces X, Y, Z are finite CW-complexes whose fundamental groups satisfy the Atiyah Conjecture.*

1. *A CW-pair (X, Y) is L^2-acyclic if and only if the inclusion map $f : Y \to X$ is an L^2-weak homotopy equivalence. In this case $\tau_u^{(2)}(X, Y) = \tau_u^{(2)}(f)$.*
2. *If $f, g : X \to Z$ are homotopic L^2-weak homotopy equivalences then $\tau_u^{(2)}(f) = \tau_u^{(2)}(g)$.*
3. *If $f : X \to Z$ is a simple-homotopy equivalence, then f is a L^2-weak homotopy equivalence and $\tau_u^{(2)}(f) = 1$.*
4. *If $f : X \to Y$ and $g : Y \to Z$ are L^2-weak homotopy equivalences. Suppose that g is π_1-injective, then $g \circ f$ is an L^2-weak homotopy equivalence with $\tau_u^{(2)}(g \circ f) = g_* \tau_u^{(2)}(f) \cdot \tau_u^{(2)}(g)$.*

Proof. Following [Coh73], we write $K \curvearrowright L$ if two finite CW-complexes K and L are related by a finite sequence of elementary collapses or expansions; if there is a common subcomplex K_0 that no cells are removed during the process, then we write $K \curvearrowright L$ rel K_0. In this case, it is clear that $\tau_u^{(2)}(K, K_0) = \iota_* \tau_u^{(2)}(L, K_0)$ where $\iota : L \to K$ is the natural homotopy equivalence.

For (1), there are elementary expansions $M_f \curvearrowright X \times I$ and elementary collapses $X \times I \curvearrowright X \times \{1\}$ relative to $Y = Y \times \{1\}$. Let $\iota : M_f \to X$ be the deformation retract, then $\tau_u^{(2)}(f) = \iota_* \tau_u^{(2)}(M_f, Y) = \tau_u^{(2)}(X, Y)$. (2) follows from the fact that $M_f \curvearrowright M_g$ rel X [Coh73, (5.5)]. For (3), if $f : X \to Z$ is a simple-homotopy equivalence then $M_f \curvearrowright X$ rel X [Coh73, (5.8)].

For the proof of (4) we need the following "L^2-excision" property.

Lemma 4.12 (Excision)**.** *If K, L and M are subcomplexes of the complex $K \cup L$ with $M = K \cap L$. Suppose the inclusion $i : K \hookrightarrow K \cup L$ is π_1-injective. Then $\tau_u^{(2)}(K \cup L, L) = i_* \tau_u^{(2)}(K, M)$.*

Proof. As in the proof of [Coh73, (20.3)], we may assume that L and $K \cup L$ are connected. Let $\widehat{K \cup L}$ be the universal covering of $K \cup L$ and let $\widehat{L}$, $\widehat{K}$ and $\widehat{M}$ be the preimage under the covering. Then there is an isomorphism of chain complexes $C_*(\widehat{K \cup L}, \widehat{L}) = C_*(\widehat{K}, \widehat{M})$ and hence $\tau_u^{(2)}(K \cup L, L) = \tau_u^{(2)}(C_*(\widehat{K}, \widehat{M})) = j_* \tau_u^{(2)}(K, M)$ where the second identity follows from the induction property (see Theorem 4.9). □

The proof of Proposition 4.11(4) proceeds as follows. Let M be the union of M_f and M_g along the identity map on Y. Then $M \curvearrowright M_{g \circ f}$ rel $X \cup Z$ by [Coh73, (5.6)]. There is a commutative diagram

$$\begin{tikzcd} M_f \arrow[r, "\iota_f"] \arrow[d, hook, "i_1"'] & Y \arrow[r, hook, "i"] & M_g \arrow[d, "\iota_g"] \arrow[dll, "i_2"'] \\ M \arrow[rr, "\iota"] & & Z \end{tikzcd}$$

where i_1, i_2 are inclusions, ι, ι_1, ι_2 are natural deformation retracts. Then we have

$$\begin{aligned} \tau_u^{(2)}(g \circ f) &= \iota_* \tau_u^{(2)}(M, X) \\ &= \iota_*(\tau_u^{(2)}(M, M_f) \cdot (i_1)_* \tau_u^{(2)}(M_f, X)), \quad \text{by sum formula} \\ &= \iota_*((i_2)_* \tau_u^{(2)}(M_g, Y) \cdot (i_1)_* \tau_u^{(2)}(M_f, X)), \quad \text{by excision} \\ &= (\iota_g)_* \tau_u^{(2)}(M_g, Y) \cdot g_*(\iota_f)_* \tau_u^{(2)}(M_f, X), \quad \text{note that } \iota_g \circ i = g \\ &= \tau_u^{(2)}(g) \cdot g_* \tau_u^{(2)}(f). \end{aligned}$$

The proof is finished. □

The identity $\tau_u^{(2)}(g \circ f) = g_* \tau_u^{(2)}(f) \cdot \tau_u^{(2)}(g)$ is often called the multiplicativity of the universal L^2-torsion. The conditions that f, g are L^2-weak homotopy equivalences and g is π_1-injective is in general necessary. But when one of the mappings is a simple homotopy equivalence then the conditions can be relaxed as follows.

Lemma 4.13. *Suppose X, Y, Z, W are finite CW-complexes whose fundamental groups satisfy the Atiyah Conjecture. Suppose $f : Y \to Z$ is a simple homotopy equivalence.*

(1) *A mapping $h : X \to Y$ is an L^2-weak homotopy equivalence if and only if $f \circ h$ is an L^2-weak homotopy equivalence. In this case we have $\tau_u^{(2)}(f \circ h) = f_* \tau_u^{(2)}(h)$.*
(2) *A mapping $g : Z \to W$ is an L^2-weak homotopy equivalence if and only if $g \circ f$ is an L^2-weak homotopy equivalence. In this case we have $\tau_u^{(2)}(g \circ f) = \tau_u^{(2)}(g)$.*

Proof. The key observation is that a simple homotopy equivalence f admits an inverse f^{-1} which is also a simple homotopy equivalence. For (1), the forward direction follows from Proposition 4.11(4). The inverse direction is similar, noting that $h = f^{-1} \circ (f \circ h)$. If h is an L^2-weak homotopy equivalence then the identity $\tau_u^{(2)}(f \circ h) = f_* \tau_u^{(2)}(h)$ follows again from Proposition 4.11(4).

For (2), suppose g is an L^2-weak homotopy equivalence. Let M be the union of M_f and M_g along the identity map on Y. Then $M \curvearrowright M_{g \circ f}$ rel $X \cup Z$. Apply the Excision Lemma 4.12 to $M_f \subset M$ and $i : M_g \subset M$ then (M, M_f) is L^2-acyclic and $\tau_u^{(2)}(M, M_f) = i_* \tau_u^{(2)}(M_g, Y)$. Since f is a simple homotopy equivalence we have $M_f \curvearrowright X$ rel X. Let $\iota : M \to Z$ be the deformation retract. Then

$$\tau_u^{(2)}(g \circ f) := \iota_* \tau_u^{(2)}(M, X) = \iota_* \tau_u^{(2)}(M, M_f) = (\iota \circ i)_* \tau_u^{(2)}(M_g, Y) = \tau_u^{(2)}(g).$$

Therefore $g \circ f$ is an L^2-weak homotopy equivalence and the identity $\tau_u^{(2)}(g \circ f) = \tau_u^{(2)}(g)$ holds. The inverse direction is proved the same way, noting that $g = (g \circ f) \circ f^{-1}$. $\square$

Remark 4.14. As a corollary of Lemma 4.13, we prove the simple-homotopy invariance stated in Theorem 4.9. Let $f : (X, X_0) \to (Y, Y_0)$ be a mapping of CW-pairs such that $f : X \to Y$ and $f|_{X_0} : X_0 \to Y_0$ are simple-homotopy equivalences. Then we have the following commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, hook, "i_X"'] & Y_0 \arrow[d, hook, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then: (X, X_0) is L^2-acyclic $\Leftrightarrow$ i_X is an L^2-homotopy equivalence $\Leftrightarrow$ $f \circ i_X$ is an L^2-homotopy equivalence $\Leftrightarrow$ $i_Y \circ f_0$ is an L^2-homotopy equivalence $\Leftrightarrow$ i_Y is an L^2-homotopy equivalence $\Leftrightarrow$ (Y, Y_0) is L^2-acyclic. In this case,

$$f_* \tau_u^{(2)}(X, X_0) = f_* \tau_u^{(2)}(i_X) = \tau_u^{(2)}(f \circ i_X) = \tau_u^{(2)}(i_Y \circ f_0) = \tau_u^{(2)}(i_Y) = \tau_u^{(2)}(Y, Y_0).$$

4.4. **Universal L^2-torsion of manifolds.** We define the universal L^2-torsion for smooth manifold pairs as follows. Recall that a smooth triangulation of a smooth manifold M is a homeomorphism from a simplicial complex to M whose restriction to each simplex is smooth.

Definition 4.15 (Universal L^2-torsion of manifold pairs)**.** Let M be a compact, smooth manifold, possibly with boundary, and let N be a compact, smooth submanifold of M. Suppose that there is a smooth triangulation of M such that N is a subcomplex of M. Then we use the triangulation to identify (M, N) with a CW-pair (X, Y) and define $\tau_u^{(2)}(M, N) := \tau_u^{(2)}(X, Y)$.

For the purpose of this paper, assume that either N is a zero-codimensional submanifold of ∂M, or the embedding $N \hookrightarrow M$ is proper (i.e. $N \cap \partial M = \partial N$). In these cases one can find a smooth triangulation of M such that N is a subcomplex of M (see [Mun66, Chapter 10]). Any two such triangulations have a common subdivision and are simple homotopy equivalent as CW-complexes [Whi40]. Therefore $\tau_u^{(2)}(M,N)$ is well-defined by simple homotopy invariance of the universal L^2-torsion (see Theorem 4.9).

Definition 4.16 (Universal L^2-torsion of mappings between manifolds)**.** Suppose $f : N \to M$ is a continuous mapping between compact smooth manifolds (possibly with boundaries) M, N. Choose any smooth triangulations of M, N and choose a simplicial mapping g homotopic to f. We say f is an L^2-weak homotopy equivalence if g is an L^2-weak homotopy equivalence. In this case, define the universal L^2-torsion of f as

$$\tau_u^{(2)}(f) := \tau_u^{(2)}(g) \in \mathrm{Wh}(\mathcal{D}_{\Pi(M)}).$$

It follows from and Proposition 4.11 and Lemma 4.13 that $\tau_u^{(2)}(f)$ does not depend on the choice of triangulations on M, N or the simplicial approximation g. When $N \subset M$ is a smooth submanifold with f the inclusion map, then $\tau_u^{(2)}(M,N) = \tau_u^{(2)}(f)$.

4.5. **Computation of the universal L^2-torsion.** We state and prove the matrix chain method for computing the universal L^2-torsion of a chain complex which goes back to [Tur01, Theorem 2.2].

Let $C_* = (0 \to C_n \to \cdots \to C_1 \to C_0 \to 0)$ be a finite based free $\mathbb{Z}G$-chain complex and let $\partial_i : C_i \to C_{i-1}$ be the boundary operator. Suppose $d_i := \mathrm{rank}_{\mathbb{Z}G}\, C_i$ is the rank of the free module C_i. Then ∂_i is given by a matrix

$$A_i = (a^i_{jk})_{\substack{j=1,\dots,d_i\\ k=1,\dots,d_{i-1}}}, \quad a^i_{jk} \in \mathbb{Z}G.$$

Definition 4.17. A matrix chain for C_* is a collection of finite sets $\mathcal{A} = \{\mathcal{I}_0, \dots, \mathcal{I}_n\}$ where $\mathcal{I}_i \subset \{1, \dots, d_i\}$ and $\mathcal{I}_n = \emptyset$. Let B_i be the submatrix of A_i formed by the entries a^i_{jk} with $j \notin \mathcal{I}_i$ and $k \in \mathcal{I}_{i-1}$. Then B_i are called the matrices associated to the matrix chain.

A matrix chain is called non-degenerate if each associated matrix is a square matrix and is invertible over $\mathcal{D}_G$.

Theorem 4.18. *A finite based free $\mathbb{Z}G$-chain complex C_* is L^2-acyclic if and only if there exists an non-degenerate matrix chain $\mathcal{A} = \{\mathcal{I}_0, \dots, \mathcal{I}_n\}$ for C_*. If this happens, then*

$$\tau_u^{(2)}(C_*) = \prod_{i=1}^{n} \det{}_r(B_i)^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G)$$

where B_i are the matrices associated to the matrix chain.

The proof is a generalization of the idea of [DFL16, Lemma 3.1] to larger chain complexes.

Proof. Suppose C_* is L^2-acyclic. Since $H_n^{(2)}(C_*) = 0$, we know that A_n is L^2-injective. Then there is a submatrix B_n of A_n, such that B_n is a square matrix of size $d_n \times d_n$ and is a weak isomorphism. We set $\mathcal{I}_{n-1} \subset \{1, \dots, d_{n-1}\}$ to be the set of indices of the columns of B_i. Write $C_{n-1} = C'_{n-1} \oplus C''_{n-1}$ where C'_{n-1} corresponds to the set of indices $\mathcal{I}_{n-1}$ and C''_{n-1} corresponds to the remaining indices. Then $B_n : C_n \to C'_{n-1}$ is a weak isomorphism. Since $H_{n-1}^{(2)}(C_*) = 0$, we know that the restriction of A_{n-1} to C''_{n-1} is injective, whose matrix A'' is exactly the submatrix of A consisting of the rows whose index does not belong to $\mathcal{I}_{n-1}$. So we obtain the following L^2-acyclic chain complex

$$0 \longrightarrow C''_{n-1} \xrightarrow{A''_{n-1}} C_{n-2} \xrightarrow{A_{n-2}} \cdots \longrightarrow C_1 \xrightarrow{A_1} C_0 \longrightarrow 0.$$

Repeat this procedure and in the end we find matrices $B_n, \dots, B_1$ which gives the matrix chain for C_*.

For the backward direction, again write $C_{n-1} = C'_{n-1} \oplus C''_{n-1}$ where C'_{n-1} corresponds to the set of indices $\mathcal{I}_{n-1}$ and C''_{n-1} corresponds to the remaining indices. Then we have the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & C_n \arrow[r, "\cong"] \arrow[d, "B_n"] & C_n \arrow[r] \arrow[d, "A_n"] & 0 \arrow[d] & \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i"] \arrow[d] & C_{n-1} \arrow[r, "p"] \arrow[d, "A_{n-1}"] & C''_{n-1} \arrow[r] \arrow[d, "A''_{n-1}"] & 0 \\
 & 0 \arrow[r] & C_{n-2} \arrow[r, "\cong"] \arrow[d, "A_{n-2}"] & C_{n-2} \arrow[r] \arrow[d, "A_{n-2}"] & 0 \\
 & & \vdots & \vdots &
\end{tikzcd}$$

If we set $C'_* = (0 \to C_n \xrightarrow{B_n} C'_{n-1} \to 0)$ and

$$C''_* = (0 \longrightarrow C''_{n-1} \xrightarrow{A''_{n-1}} C_{n-2} \xrightarrow{A_{n-2}} \cdots \longrightarrow C_1 \xrightarrow{A_1} C_0 \longrightarrow 0).$$

We then have the short exact sequence of based $\mathbb{C}G$-chain complex

$$0 \to C'_* \to C_* \to C''_* \to 0.$$

By [FL17, Lemma 2.9], if C'_* is L^2-acyclic then C_* is also L^2-acyclic and $\tau_u^{(2)}(C_*) = \tau_u^{(2)}(C''_*) \cdot \det(B_n)^{(-1)^n}$. Repeat the above decomposition to C''_* and in the end we know that C_* is L^2-acyclic and

$$\tau_u^{(2)}(C_*) = \prod_{i=1}^{n} \det{}_r(B_i)^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G).$$

□

Proposition 4.19. *We calculate the universal L^2-torsion of some manifold pairs.*

(1) *Let N be a compact smooth manifolds whose fundamental group satisfies the Atiyah Conjecture. then for any $s \in [0,1]$ we have*
$$\tau_u^{(2)}(N \times I, N \times \{s\}) = 1 \in \mathrm{Wh}(\mathcal{D}_{\Pi(N)}).$$
(2) *Let S^1 be the circle with fundamental group $\mathbb{Z} = \langle t \rangle$. Then $\tau_u^{(2)}(S^1) = [(t-1)^{-1}] \in \mathrm{Wh}(\mathcal{D}_{\mathbb{Z}})$.*
(3) *Let T^2 be the torus then*
$$\tau_u^{(2)}(T^2) = 1 \in \mathrm{Wh}(\mathcal{D}_{\mathbb{Z}^2}).$$

Proof. For (1), let $f : N \times \{s\} \to N \times I$ be the inclusion, then f is a simple-homotopy equivalence and $\tau_u^{(2)}(N \times I, N \times \{s\}) = \tau_u^{(2)}(f) = 1$ by Proposition 4.11.

For (2), a CW-structure of S^1 is given by a 0-cell p and an 1-cell e. By choosing appropriate liftings $\hat{p}$ and $\hat{e}$ the cellular chain complex of the universal cover is

$$C_*(\widehat{S}^1) = (0 \to \mathbb{Z}[t^{\pm}] \cdot \langle e \rangle \xrightarrow{(t-1)} \mathbb{Z}[t^{\pm}] \cdot \langle p \rangle \to 0)$$

and hence $\tau_u^{(2)}(S^1) = [(t-1)^{-1}]$.

For (3), consider the CW-structure for T^2 given by identifying pairs of sides of a square. Let p be the 0-cell, e_1, e_2 be the 1-cells and σ be the 2-cell. Then the boundary of σ is a loop

$e_1e_2e_1^{-1}e_2^{-1}$. Suppose the loop e_1, e_2 represents $t_1, t_2 \in \pi_1(T^2)$, respectively. Then by choosing appropriate liftings of the cells the chain complex of the universal cover is

$$C_*(\widehat{T}^2) = (0 \to \mathbb{Z}[t_1^{\pm}, t_2^{\pm}] \cdot \langle \sigma \rangle \xrightarrow{\begin{pmatrix} 1-t_2 & t_1-1 \end{pmatrix}} \mathbb{Z}[t_1^{\pm}, t_2^{\pm}] \cdot \langle e_1, e_2 \rangle \xrightarrow{\begin{pmatrix} t_1-1 \\ t_2-1 \end{pmatrix}} \mathbb{Z}[t_1^{\pm}, t_2^{\pm}] \cdot \langle p \rangle \to 0).$$

A matrix chain can be given by $B_2 = (1-t_2)$ and $B_1 = (t_2 - 1)$, hence $\tau_u^{(2)}(T^2) = \det_w(1-t_2) \cdot \det_w(t_2-1)^{-1} = 1$. □

5. Universal L^2-torsion for taut sutured manifolds

5.1. **Taut surfaces.** Given a compact orientable surface Σ with path-components $\Sigma_1, \ldots, \Sigma_k$ we define its complexity as

$$\chi_-(\Sigma) := \sum_{i=1}^{k} \max\{0, -\chi(\Sigma)\}.$$

Let N be a compact oriented 3-manifold. A properly embedded oriented surface Σ is taut if Σ is incompressible, and has minimal complexity among all properly embedded oriented surfaces representing the homology class $[\Sigma, \partial\Sigma] \in H_2(N, \nu(\partial\Sigma); \mathbb{Z})$.

5.2. **Sutured 3-manifolds.** A sutured manifold (M, R_-, R_+, γ) consists of an oriented 3-manifold M with a decomposition of its boundary into two subsurfaces R_+ and R_- along their common boundary γ. The orientation on $R_\pm$ is defined in the way that the normal vector of R_+ points out of M and the normal vector of R_- points inward of M. The boundary orientations of $R_\pm$ coincide and give the orientation of the simple closed curves γ. If the surfaces $R_\pm$ are not important in the statement we sometimes abbreviate a sutured manifold (M, R_-, R_+, γ) as (M, γ).

A sutured manifold (M, R_-, R_+, γ) is called taut if M is irreducible and $R_\pm$ are both taut surfaces (after pushing slightly into M).

5.3. **Sutured manifold decompositions.** First we introduce some notation. Let M be a compact oriented 3-manifold and let S be a (not necessarily connected) properly embedded surface. Denote by $\nu(S) := S \times (-1, 1)$ a product neighborhood of S in M and denote by $M \backslash\backslash S := M \backslash \nu(S)$ the complement. Let S_+ (resp. S_-) be the components of $S \times \{-1\} \cup S \times \{1\}$ in $M \backslash\backslash S$ whose normal vector points out of (resp. into) M'. We remark that, if the neighborhood $S \times (-1, 1)$ is chosen in the way that the normal direction of $S = S \times \{0\}$ coincides with the positive direction of $(-1, 1)$, then S'_+ (resp. S'_-) is actually the surface $S \times \{-1\}$ (resp. $S \times \{+1\}$).

Let (M, R_-, R_+, γ) be a sutured manifold. A properly embedded oriented surface S is called a decomposition surface if for every component λ of ∂S one of the following holds:

(1) λ is transverse to γ.
(2) λ is a component of γ and the boundary induced orientation on $\lambda = \partial S$ coincides with the orientation on γ.
(3) No component of ∂S bounds a disk in $R_\pm$ and no component of S is a disk D with $\partial D \subset R_\pm$.

Given a decomposition surface S of (M, R_-, R_+, γ), define the sutured manifold decomposition

$$(M, R_-, R_+, \gamma) \overset{S}{\rightsquigarrow} (M', R'_-, R'_+, \gamma')$$

where

$$\begin{aligned} M' &= M \setminus (S \times (-1,1)), \\ R'_+ &= (R_+ \cap M') \cup S_+, \\ R'_- &= (R_- \cap M') \cup S_-, \\ \gamma &= \partial R_+ = \partial R_-. \end{aligned}$$

A sutured manifold decomposition $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ as defined above is called a taut sutured decomposition if (M',γ') is taut. In this case, Gabai [Gab87, Lemma 0.4] proved that (M,γ) is automatically a taut sutured manifold.

We make the following remarks:

(1) The definition of sutured manifold here follows [AD19] and slightly differs from many other sources where the suture are disjoint union of annuli and tori (c.f. [Gab83, Gab87]). The definition of sutured manifold decomposition is modified accordingly.
(2) Suppose $(M,R_+,R_-,\gamma) \overset{S}{\rightsquigarrow} (M',R'_+,R'_-,\gamma')$ is a taut sutured decomposition, then S is incompressible in M. The reason is as follows: R'_+ is the union of the surfaces S_+ and $R_+ \cap M'$ along their common part $S_+ \cap R_+$. This common part consists of some arcs and some boundary circles of S_+. By assumption, no boundary circles of S_+ bound a disk in R_+ or a disk in S_+, so the closed curves of the common part are homotopy nontrivial in S_+ and $R_+ \cap M'$. By Van-Kampen Theorem the surface S_+ is π_1-injective in R'_+, therefore π_1-injective in M' since R'_+ is incompressible in M'. This proves that S can not admit a compressing disk in M.
(3) Given a taut sutured decomposition $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$. Since S is incompressible in M, it follows that for any component of M' the inclusion into M induces monomorphism on fundamental groups.

5.4. **Universal L^2-torsion for taut sutured 3-manifolds.**

Theorem 5.1 ([Her23]). *Let (M,R_-,R_+,γ) be a sutured 3-manifold with infinite fundamental group.*

(1) *Suppose (M,γ) is taut, then the pair (M,R_+) is L^2-acyclic.*
(2) *Suppose M is irreducible and $R_\pm$ are both incompressible. If the pair (M,R_+) is L^2-acyclic, then (M,γ) is taut.*

Proof. Suppose (M,γ) is taut. If a component of $R_\pm$ is a disk or sphere, then M must be the 3-ball, contradicting the infinite fundamental group assumption. Therefore the complexity of $R_\pm$ equals $-\chi(R_\pm)$, and we have $\chi(R_+) = \chi(R_-)$, then we apply [Her23, Theorem 1.1].

Suppose M is irreducible and $R_\pm$ is incompressible. If the pair (M,R_+) is L^2-acyclic, then the Euler characteristic $\chi(M,R_+) = \chi(M) - \chi(R_+)$ is zero. But $\chi(M) = \frac{1}{2}\chi(\partial M) = \frac{1}{2}(\chi(R_+) + \chi(R_-))$, it follows that $\chi(R_+) = \chi(R_-)$, then we apply [Her23, Theorem 1.1]. □

The reader might wonder why we prefer (M,R_+) than (M,R_-). In fact, these two pairs are dual to each other by the following Proposition 5.2, while the pair (M,R_+) gives us a neater statement.

Proposition 5.2. *Let (N,R_+,R_-,γ) be a taut sutured manifold, then the pairs (N,R_+) and (N,R_-) are L^2-acyclic and*

$$\tau_u^{(2)}(N,R_+) = \overline{\tau_u^{(2)}(N,R_-)}.$$

Proof. The proof is a modification of [Mil62, Page 139] and does not use any L^2-theories. Form a triangulation of N such that R_+ and R_- are subcomplexes. Let $\widehat{N}$ be the universal cover and let

$\widehat{R}_\pm$ be the preimage of $R_\pm$. Consider the dual cellular complex $\widehat{N}'$ of $\widehat{N}$. A cell of $\widehat{N}$ is canonically dual to a cell of $\widehat{N}' \setminus \partial\widehat{N}'$. More explicitly, we have the intersection pairing

$$p : C_*(\widehat{N}) \times C_{3-*}(\widehat{N}', \partial\widehat{N}') \to \mathbb{Z}G, \quad p(\sigma, \sigma') = \sum_{g \in G} \langle \sigma, g\sigma' \rangle \cdot g$$

where $\langle \sigma, g\sigma' \rangle$ is the intersection number of σ and $g\sigma'$. It is easy to verify the identities

$$\begin{aligned} p(g\sigma, \sigma') &= g \cdot p(\sigma, \sigma'), \\ p(\sigma, g\sigma') &= p(\sigma, \sigma') \cdot g^{-1}, \\ p(\partial\sigma, \sigma') &= \pm p(\sigma, \partial\sigma'). \end{aligned}$$

The pairing is non-degenerate in the sense that $\sigma \mapsto p(\sigma, *)$ gives an isomorphism of $\mathbb{Z}G$-chain complexes $C_*(\widehat{N}) \cong C^{3-*}(\widehat{N}', \partial N')$. Note that a cell of $\widehat{R}_+$ is canonically dual to a cell of $\widehat{N}' \setminus \partial\widehat{N}'$ which meets $\widehat{R}_+$ at a nonempty set and vise versa. Let K be the union of the cells of N' which is disjoint with R_+ and let $\widehat{K}$ be the preimage in the universal cover, then there is an induced non-degenerate pairing

$$C_*(\widehat{N}, \widehat{R}_+) \times C_{3-*}(\widehat{K}, \partial\widehat{N}') \to \mathbb{Z}G$$

and hence the isomorphism of $\mathbb{Z}G$-chain complexes $C_*(\widehat{N}, \widehat{R}_+) \cong C^{3-*}(\widehat{K}, \partial\widehat{N}')$. By Proposition 4.6 the finite based free $\mathbb{Z}G$-chain complex $C_*(\widehat{K}, \partial\widehat{N}')$ is L^2-acyclic and

$$\tau_u^{(2)}(C_*(\widehat{N}, \widehat{R}_+)) = \overline{\tau_u^{(2)}(C_*(\widehat{K}, \partial\widehat{N}'))},$$

therefore $\tau_u^{(2)}(N, R_+) = \overline{\tau_u^{(2)}(K, \partial N')}$. The pair $(K, \partial N')$ is a deformation retract of the pair (N', R'_-), hence $\tau_u^{(2)}(N, R_+) = \overline{\tau_u^{(2)}(N', R'_-)}$. The proof is finished since the universal L^2-torsion does not depend on the cellular structure. □

Lemma 5.3. *Let (N, γ) be a taut sutured manifold. Suppose there is a decomposition surface S such that*

1. *The sutured manifold decomposition $(N, R_+, R_-, \gamma) \overset{S}{\rightsquigarrow} (N', R'_+, R'_-, \gamma')$ is taut.*
2. *S is separating and N' is a disjoint union of two (possibly non-connected) manifolds N_0, N_1.*
3. *$S_- \subset N_0$, $S_+ \subset N_1$.*

Then $\tau_u^{(2)}(N, R_+) = i_* \tau_u^{(2)}(N', R'_+)$ where $i : N' \hookrightarrow N$ is the inclusion.

Proof. There is a short exact sequence of chain complexes

$$0 \to C_*^{(2)}(N_0, R_+ \cap N_0) \to C_*^{(2)}(N, R_+) \to C_*^{(2)}(N, N_0 \cup R_+) \to 0$$

and a natural isomorphism $C_*^{(2)}(N, N_0 \cup R_+) = C_*^{(2)}(N_1, S_+ \cup (R_+ \cap N_1))$. Note that $R_+ \cap N_0$ is the R_+-part of N_0 and $S_+ \cup (R_+ \cap N_1)$ is the R_+-part of N_1. By assumption N_0 and N_1 are taut sutured manifold. It follows from Theorem 5.1 that the three chain complexes are L^2-acyclic. By additivity of the universal L^2-torsion [FL17, Lemma 2.9] we have

$$\tau_u^{(2)}(N, R_+) = (i_0)_* \tau_u^{(2)}(N_0, R_+ \cap N_0) \cdot (i_1)_* \tau_u^{(2)}(N_1, S_+ \cup (R_+ \cap N_1))$$

where i_0 and i_1 are the inclusion of N_0 and N_1 into N. The right hand side of the above equation is exactly $i_* \tau_u^{(2)}(N', R'_+)$ and the proof is finished. □

A weighted surface $\widehat{S}$ in a compact oriented 3-manifold N is a collection of pairs (S_i, w_i), $i = 1, \ldots, n$, where S_i is a connected properly embedded oriented surface, w_i is a positive integer, and $S_i \cap S_j = \emptyset$ for $i \neq j$. The realization of $\widehat{S}$ is the properly embedded oriented surface

$$\bar{S} := \bigcup_{i=1}^{n} w_i \cdot S_i$$

where $w_i \cdot S_i$ is the union of w_i parallel copies of S_i. The reduction of $\widehat{S}$ is the properly embedded oriented surface

$$S := \bigcup_{i=1}^{n} S_i.$$

A weighted decomposition surface is a weighted surface whose realization is a decomposition surface. If $\widehat{S}$ is a weighted decomposition surface, then $N\backslash\backslash\bar{S}$ is the union of $N\backslash\backslash S$ and some product sutured manifolds. So the sutured decomposition along $\bar{S}$ is taut if and only if the sutured decomposition along S is taut.

Proposition 5.4. *Let (N, R_+, R_-, γ) be a taut sutured manifold and let $(N, \gamma) \overset{\Sigma}{\rightsquigarrow} (N', \gamma')$ be a taut sutured decomposition. Then there is a weighted decomposition surface $\widehat{S}$ in N (with $\bar{S}$ the realization and S the reduction) such that*

1. *$N\backslash\backslash S$ is connected,*
2. *$[\bar{S}] = [\Sigma] \in H_2(N, \partial N; \mathbb{Z})$,*
3. *the sutured decomposition of N along S is taut,*
4. *$i_*\tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+) = j_*\tau_u^{(2)}(N\backslash\backslash\Sigma, \Sigma_+ \cup R_+)$ where i, j are the natural inclusions of $N\backslash\backslash S$ and $N\backslash\backslash\Sigma$ into N.*

Proof. For any weighted surface $\widehat{S}$ in N, define $c(\widehat{S}) := \#\pi_0(N\backslash\backslash S)$. First take $\widehat{S}$ to be the surface Σ with weight 1 assigned to each component, then $\bar{S} = S = \Sigma$ and $\widehat{S}$ satisfies (2)–(4). It suffices to prove that given any weighted surface $\widehat{S}$ with $c(\widehat{S}) > 1$ such that (2)–(4) holds, then there exists a weighted surface $\widehat{T}$ such that (2)–(4) holds and $c(\widehat{T}) < c(\widehat{S})$.

Given such $\widehat{S}$. Since $c(\widehat{S}) > 1$ there is a component $C \subset S$ such that $C_\pm$ lies in different components of $N\backslash\backslash S$. Choose C to be a component with minimal weight w among such components, let M_0 (resp. M_1) be the component of $N\backslash\backslash S$ containing C_- (resp. C_+). Let $C = C_1, C_1, \ldots, C_k$ be the components of S whose normal direction points into M_1 and let $D_1, \ldots, D_l$ be the components of S whose normal direction points out of M_1. It may happen that $C_i = D_j$ for some i, j, in this case the two sides of C_i both belong to M_1. It follows that

$$[C_1] + \cdots + [C_k] = [D_1] + \cdots + [D_l] \in H_2(N, \partial N; \mathbb{Z}).$$

We change the weights of $\widehat{S}$ by increasing the weights of $D_1, \ldots, D_l$ by w, and decreasing the weights of $C_1, \ldots, C_k$ by w. If a component has weight zero, we simply discard this component. This new weighted surface is denoted by $\widehat{T}$. Clearly, we have $[\bar{S}] = [\bar{T}] \in H_2(N, \partial N; \mathbb{Z})$. Since T is a subcollection of S and the decomposition along S is taut, it follows that the decomposition along T is also taut. So (2), (3) holds true for $\widehat{T}$. Moreover, $c(\widehat{T}) < c(\widehat{S})$ since M_0 and M_1 are in the same component of $N\backslash\backslash T$.

Finally, let S_0 be $S \setminus T$. Then $N\backslash\backslash S$ is obtained from the sutured manifold decomposition of $N\backslash\backslash T$ along S_0. By construction, S_0 separates $N\backslash\backslash T$; in particular, it separates M_0 from M_1 and $(S_0)_+ \subset M_1$. Apply Lemma 5.3 with $N_0 := (N\backslash\backslash S) - M_1$ and $N_1 := M_1$, we have

$$i'_*\tau_u^{(2)}(N\backslash\backslash T, T_+ \cup R_+) = i_*\tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+) = j_*\tau_u^{(2)}(N\backslash\backslash\Sigma, \Sigma_+ \cup R_+)$$

where i, i', j' are the inclusions of $N\backslash\backslash S$, $N\backslash\backslash T$ and $N\backslash\backslash\Sigma$ into N, respectively. This verifies (4) for $\widehat{T}$ and finishes the proof. $\square$

Definition 5.5. Let $\phi \in H^1(G;\mathbb{R})$ be an 1-cohomology class. For any non-zero matrix A over $\mathbb{Z}G$, let $\delta_\phi(A)$ be the smallest real number $\delta_\phi(A_{ij})$ among all non-zero entries A_{ij}. Then we can decompose the matrix in a unique way

$$A = L_\phi(A) + (A - L_\phi(A))$$

where any group element g appearing in $L_\phi(A)$ satisfies $\phi(g) = \delta_\phi(A)$, and any group element h appearing in $(A - L_\phi(A))$ satisfies $\phi(h) > \delta_\phi(A)$.

We define $L_\phi(A) = 0$ if A is a zero matrix. Otherwise $L_\phi(A)$ is always nonzero.

For any finite based free $\mathbb{Z}G$-chain complex

$$C_* = (0 \longrightarrow C_n \xrightarrow{A_n} \cdots \xrightarrow{A_2} C_1 \xrightarrow{A_1} C_0 \longrightarrow 0),$$

define

$$L_\phi(C_*) = (0 \longrightarrow C_n \xrightarrow{L_\phi(A_n)} \cdots \xrightarrow{L_\phi(A_2)} C_1 \xrightarrow{L_\phi(A_1)} C_0 \longrightarrow 0).$$

Remark 5.6. It is easy to verify that $L_\phi(A)L_\phi(B) = L_\phi(AB)$ holds for arbitrary matrices A, B. In particular, the chain complex $L_\phi(C_*)$ is well-defined.

Theorem 5.7. *Let $\phi \in H^1(G;\mathbb{R})$ be an 1-cohomology class and let*

$$C_* = (0 \longrightarrow C_n \xrightarrow{A_n} \cdots \xrightarrow{A_2} C_1 \xrightarrow{A_1} C_0 \longrightarrow 0)$$

be a finite based free $\mathbb{C}G$-chain complex. Suppose that $L_\phi(C_*)$ is L^2-acyclic, then C_* is also L^2-acyclic and

$$\tau_u^{(2)}(L_\phi(C_*)) = L_\phi(\tau_u^{(2)}(C_*)).$$

Proof. Suppose that $L_\phi(C_*)$ is L^2-acyclic, then by Theorem 4.18 we can find a non-degenerate matrix chain $\mathcal{A}$ of $L_\phi(C_*)$. Let $B_1, \ldots, B_n$ be the associated submatrices of $L_\phi(A_1), \ldots, L_\phi(A_n)$, then

$$\tau_u^{(2)}(L_\phi(C_*)) = \prod_{i=1}^{n} \det{}_r(B_i)^{(-1)^n}$$

Let $C_1, \ldots, C_n$ be the submatrices of $A_1, \ldots, A_n$ associated to the same matrix chain $\mathcal{A}$. Since B_i must be non-zero, then $L_\phi(C_i) = B_i$. By Theorem 3.6 we have

$$L_\phi \det{}_r(C_i) = \det{}_r B_i.$$

In particular, C_i are weak isomorphisms and therefore $\mathcal{A}$ is a non-degenerate matrix chain of C_*. By Theorem 4.18,

$$\tau_u^{(2)}(C_*) = \prod_{i=1}^{n} \det{}_r(C_i)^{(-1)^n}$$

and in particular $\tau_u^{(2)}(L_\phi(C_*)) = L_\phi(\tau_u^{(2)}(C_*))$. □

Theorem 5.8. *Let (N,γ) be a taut sutured manifold and let $(N,\gamma) \overset{\Sigma}{\rightsquigarrow} (N',\gamma')$ be a taut sutured decomposition. Let $\phi = PD([\Sigma, \partial\Sigma]) \in H^1(N;\mathbb{Z})$ be the Poincaré dual of the surface Σ, then*

$$j_*\tau_u^{(2)}(N', R'_+) = L_\phi(\tau_u^{(2)}(N, R_+))$$

where $j : N' \hookrightarrow N$ is the natural inclusion.

Proof. We find a weighted decomposition surface $\widehat{S}$ in N as provided by Proposition 5.4. Then $N\backslash\backslash S$ is connected, $\phi = PD([\Sigma, \partial\Sigma]) = PD([\bar{S}, \partial\bar{S}]) \in H^1(N;\mathbb{Z})$ and $j_*\tau_u^{(2)}(N', R'_+) = i_*\tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+)$ where $i : N\backslash\backslash S \to N$ is the inclusion. We are left to show that

$$L_\phi(\tau_u^{(2)}(N, R_+)) = i_*\tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+).$$

Find a CW-structure for N such that $S \times I$ and $R_\pm$ are subcomplexes. Fix a base point $p \in N \setminus \nu(S)$. For any cell σ in the CW-structure of N, choose a path γ_σ connecting p and σ such that

- γ_σ is disjoint with S_-, if $\sigma \subset N \setminus S_-$,
- γ_σ is disjoint with S_+, if $\sigma \subset S_-$.

Lift the base point p to $\hat{p}$ in the universal cover $\widehat{N}$ and lift each cell σ to $\hat{\sigma}$ using the path γ_σ. The cells $\hat{\sigma}$ form a basis for the finite based free $\mathbb{Z}[\pi_1(N)]$-chain complex $C_*(\widehat{N})$. Now consider the L^2-cellular chain complex $C_*^{(2)}(N, R_+)$, each chain group admits the following direct sum decomposition

$$C_k^{(2)}(N, R_+) = C_k^{(2)}(N\backslash\backslash S, S_+ \cup R_+) \oplus C_k^{(2)}(S \times I, S_- \cup R_+), \quad k = 0, 1, 2, 3.$$

Accordingly, the boundary homomorphism $\partial_* : C_*^{(2)}(N, R_+) \to C_{*-1}^{(2)}(N, R_+)$ admits the following expression

$$\partial_* = \begin{pmatrix} \partial_*^1 & \partial_*^2 \\ \partial_*^3 & \partial_*^4 \end{pmatrix},$$

$$\partial_*^1 : C_*^{(2)}(N\backslash\backslash S, S_+ \cup R_+) \to C_{*-1}^{(2)}(N\backslash\backslash S, S_+ \cup R_+),$$

$$\partial_*^2 : C_*^{(2)}(N\backslash\backslash S, S_+ \cup R_+) \to C_{*-1}^{(2)}(S \times I, S_- \cup R_+),$$

$$\partial_*^3 : C_*^{(2)}(S \times I, S_- \cup R_+) \to C_{*-1}^{(2)}(N\backslash\backslash S, S_+ \cup R_+).$$

$$\partial_*^4 : C_*^{(2)}(S \times I, S_- \cup R_+) \to C_{*-1}^{(2)}(S \times I, S_- \cup R_+).$$

Note that any group element g appearing in ∂_*^3 satisfies $\phi(g) > 0$, and any group element h appearing in $\partial_*^1, \partial_*^2$ or ∂_*^4 satisfies $\phi(h) = 0$. It follows that

$$L_\phi(\partial_*) = \begin{pmatrix} \partial_*^1 & \partial_*^2 \\ 0 & \partial_*^4 \end{pmatrix}$$

and we hence obtain a short exact sequence of chain complexes

$$0 \to C_*^{(2)}(N\backslash\backslash S, S_+ \cup R_+) \to L_\phi(C_*^{(2)}(N, R_+)) \to C_*^{(2)}(S \times I, S_- \cup R_+) \to 0.$$

Then by [FL17, Lemma 2.9] we have

$$\tau_u^{(2)}(L_\phi(C_*^{(2)}(N, R_+))) = i_* \tau_u^{(2)}(N\backslash\backslash S, S_- \cup R_+) \cdot i'_* \tau_u^{(2)}(S \times I, S_- \cup R_+).$$

The left hand side equals $L_\phi(\tau_u^{(2)}(N, R_+))$ by Theorem 5.7. On the right hand side, since $(S_+ \cup R_+) \cap (S \times I)$ deformation retracts onto S_-, we have $\tau_u^{(2)}(S \times I, S_- \cup R_+) = \tau_u^{(2)}(S \times I, S_-) = 1$. It follows that $L_\phi(\tau_u^{(2)}(N, R_+)) = i'_* \tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+)$ and the proof is finished. $\square$

6. A criterion for fiberedness of 3-manifold

6.1. **Polytopes.** Let H be a finitely generated free abelian group. Note that $H_1(H; \mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Z}} H$ is a finite-dimensional real vector space.

A polytope in $H_1(H; \mathbb{R})$ is a compact set which is the convex hull of a finite subset. We allow the empty set $\emptyset$ to be a polytope.

Definition 6.1 (Face map F_ϕ)**.** Given a polytope P and any character $\phi \in H^1(H; \mathbb{R})$. Set $\delta_\phi(P) := \inf_{x \in P} \phi(x)$, define the face associated to ϕ by

$$F_\phi(P) := \{x \in P \mid \phi(x) = \delta_\phi(P)\}.$$

It is clear that $F_\phi(P)$ is a polytope contained in P. The collection $\{F_\phi(P) \mid \phi \in H^1(H;\mathbb{R})\}$ is the collection of faces of P. A face is called a vertex if it is a single point. Any polytope is the convex hull of all its vertices. A polytope is called integral if all its vertices lie in the integral lattice $H \subset H_1(H;\mathbb{R})$.

Given any two non-empty polytopes P_1, P_2 in $H_1(H;\mathbb{R})$, define their Minkowski sum to be the polytope

$$P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1,\ p_2 \in P_2\}.$$

It is the convex hull of the set $\{v_1 + v_2 \mid v_i \text{ is a vertex of } P_i,\ i = 1, 2\}$. The operator δ_ϕ and the face map L_ϕ is additive under the Minkowski sum:

$$\delta_\phi(P_1 + P_2) = \delta_\phi(P_1) + \delta_\phi(P_2), \quad F_\phi(P_1 + P_2) = F_\phi(P_1) + F_\phi(P_2)$$

for all character ϕ and all polytopes P_1, P_2.

Example 6.2. Let M be an admissible 3-manifold. The Thurston norm ball $B_x(M) := \{\phi \in H^1(M;\mathbb{R}) \mid x_M(\phi) \leqslant 1\}$ is a (perhaps non-compact) polyhedron in $H^1(M;\mathbb{R})$; the dual Thurston norm ball is defined to be $B^*_x(M) := \{z \in H_1(M;\mathbb{R}) \mid \phi(z) \leqslant 1 \text{ for all } \phi \in B_x(M)\}$. Thurston proved that $B^*_x(M)$ is an integral polytope in $H_1(M;\mathbb{R})$ with vertices $\pm v_1, \dots, \pm v_k$, and the Thurston norm ball is given by

$$B_x(M) = \{\phi \in H^1(M;\mathbb{R}) \mid |\phi(v_i)| \leqslant 1,\ i = 1, \dots, k\}.$$

A Thurston cone in $H^1(M;\mathbb{R})$ is either an open cone formed by the origin and a face of $B_x(M)$, or a maximal connected component of $H^1(M;\mathbb{R}) \setminus \{0\}$ on which the Thurston norm x_M vanishes. It follows that $H^1(M;\mathbb{R}) \setminus \{0\}$ is the disjoint union of all Thurston cones of various dimensions. A Thurston cone is called top-dimensional if its dimension equals $\dim H^1(M;\mathbb{R})$. The following Lemma 6.3 is Thurston's result stated differently.

Lemma 6.3. *A nonzero character $\phi \in H^1(M;\mathbb{R})$ lies in a top-dimensional Thurston cone if and only if $F_\phi B^*_x(M)$ is a vertex.*

6.2. **Polytope group $\mathcal{P}_\mathbb{Z}(H)$ and $\mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H)$.** Given the finitely generated abelian group H, define $\mathcal{P}_\mathbb{Z}(H)$ to be the Grothendieck group of integral polytopes in $H_1(H;\mathbb{R})$ under the Minkowski sum. More precisely, $\mathcal{P}_\mathbb{Z}$ is the abelian group with a generating set

$$\{[P] \mid P \text{ is an non-empty integral polytope in } H_1(H;\mathbb{R})\}$$

and a relation $[P] + [Q] = [P + Q]$ for each pair of non-empty integral polytopes P, Q in $H_1(H;\mathbb{R})$. Any element of $\mathcal{P}_\mathbb{Z}(H)$ can be expressed in the formal sum $[P] - [Q]$ for non-empty integral polytopes P, Q in $H_1(H;\mathbb{R})$ and

$$[P_1] - [Q_1] = [P_2] - [Q_2] \Longleftrightarrow [P_1] + [Q_2] = [P_2] + [Q_1].$$

Note that every element $h \in H$ determines an one-vertex polytope $[h]$ in $\mathcal{P}_\mathbb{Z}(H)$ and this defines an embedding of H into $\mathcal{P}_\mathbb{Z}(H)$. Define

$$\mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H) = \mathcal{P}_\mathbb{Z}(H)/H.$$

In other words, two polytopes are identified in $\mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H)$ if and only if they differ by a translation with an element in the lattice H.

The operators δ_ϕ and F_ϕ naturally extends to the polytope groups:

$$\begin{aligned} &\delta_\phi : \mathcal{P}_\mathbb{Z}(H) \to \mathbb{R}, \quad \delta_\phi([P] - [Q]) = \delta_\phi(P) - \delta_\phi(Q), \\ &F_\phi : \mathcal{P}_\mathbb{Z}(H) \to \mathcal{P}_\mathbb{Z}(H), \quad F_\phi([P] - [Q]) = [F_\phi(P)] - [F_\phi(Q)]. \end{aligned}$$

Since the face map F_ϕ preserves the subgroup H, it naturally induces the face map $F_\phi : \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H) \to \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H)$.

6.3. **Polytope homomorphism $\mathbb{P}$.** Given a finitely generated group G satisfying the Atiyah Conjecture. Let H be the free abelianization of G, then we have the short exact sequence

$$1 \to K \to G \to H \to 1.$$

Recall that $\mathcal{D}_K * H$ is the twisted group ring embedded in $\mathcal{D}_G$. If given any section $s : H \to G$ then any element u has a unique expression

$$u = \sum_{h \in H} x_h \cdot s(h).$$

For any nonzero $u \in \mathcal{D}_K * H$, define $\mathbb{P}(u)$ to be the integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H$ spanned by $\{h \in H \mid x_h \neq 0\}$. This polytope does not depend on the choice of the section $s : H \to G$. It is proved in [FL19, Lemma 6.4] that

$$\mathbb{P}(uv) = \mathbb{P}(u) + \mathbb{P}(v)$$

for all nonzero $u, v \in \mathcal{D}_K * H$. Recall that the Linnell's skew field $\mathcal{D}_G$ is the field of fraction of $\mathcal{D}_K * H$, we then define the polytope homomorphism as

$$\mathbb{P} : \mathcal{D}_G^{\times} \to \mathcal{P}_{\mathbb{Z}}(H), \quad \mathbb{P}(uv^{-1}) := [\mathbb{P}(u)] - [\mathbb{P}(v)]$$

for all nonzero $u, v \in \mathcal{D}_K * H$. This homomorphism is well-defined. The following commutative diagram is immediate from the definition.

$$\begin{tikzcd}
\mathcal{D}_G^{\times} \arrow[r, "L_\phi"] \arrow[d, "\mathbb{P}"] & \mathcal{D}_G^{\times} \arrow[d, "\mathbb{P}"] \\
\mathcal{P}_{\mathbb{Z}}(H) \arrow[r, "F_\phi"] & \mathcal{P}_{\mathbb{Z}}(H).
\end{tikzcd}$$

Recall that $K_1(\mathcal{D}_G)$ is the abelianization of $\mathcal{D}_G^{\times}$ and $\mathrm{Wh}(\mathcal{D}_G) = K_1(\mathcal{D}_G)/[\pm G]$, then the polytope homomorphism naturally induces

$$\mathbb{P} : K_1(\mathcal{D}_G) \to \mathcal{P}_{\mathbb{Z}}(H), \quad \mathbb{P} : \mathrm{Wh}(\mathcal{D}_{\mathrm{G}}) \to \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H).$$

To save notation, we still use the same symbol $\mathbb{P}$ for the induced homomorphisms. We have the commutative diagram for the induced homomorphisms

$$\begin{tikzcd}
\mathrm{Wh}(\mathcal{D}_G) \arrow[r, "L_\phi"] \arrow[d, "\mathbb{P}"] & \mathrm{Wh}(\mathcal{D}_G) \arrow[d, "\mathbb{P}"] \\
\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H) \arrow[r, "F_\phi"] & \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H).
\end{tikzcd}$$

We record the following important result which relate the universal L^2-torsion of a 3-manifold with its dual Thurston norm ball. Note that the minus sign come out since the L^2-torsion polytope is defined to be the negative of the image of the universal L^2-torsion under the polytope homomorphism [FL17, Definition 4.21].

Theorem 6.4 ([FL17, Theorem 4.37])**.** *Let M be an admissible 3-manifold which is not homeomorphic to $S^1 \times D^2$. Then*

$$-[B_x^*(M)] = 2 \cdot \mathbb{P}(\tau_u^{(2)}(M)) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(M)_f),$$

where $B_x^* \subset H_1(M;\mathbb{R})$ is the dual Thurston norm ball, and $\tau_u^{(2)}(M) \in \mathrm{Wh}(\mathcal{D}_{\pi_1(M)})$ is the universal L^2-torsion of M.

6.4. **The universal L^2-torsion detects fiberedness.**

Theorem 6.5. *Suppose M is an admissible 3-manifold which is not a closed graph manifold. Let G be the fundamental group of M and $\phi \in H^1(M;\mathbb{R})$ be any nonzero character. Then ϕ is fibered if and only if $L_\phi \tau_u^{(2)}(M) = 1 \in \mathrm{Wh}(\mathcal{D}_G)$.*

Proof. Suppose ϕ is a fibered class. By Proposition 3.3 (7) we can find a rational fibered class $\psi \in H^1(M;\mathbb{Q})$ arbitrarily close to ϕ such that $L_\phi(\tau_u^{(2)}(M)) = L_\psi(\tau_u^{(2)}(M))$. Choose a positive integer n such that $n\psi$ is an integral fibered class, and let S be a Thurston norm-minimizing surface dual to $n\psi$, then $M\backslash\backslash S$ is a product sutured manifold. By Theorem 5.8 we have

$$L_{n\psi}\tau_u^{(2)}(M) = j_*\tau_u^{(2)}(M\backslash\backslash S, S_+) = 1,$$

where $j : M\backslash\backslash S \hookrightarrow M$ is the inclusion. It follows that $L_\phi\tau_u^{(2)}(M) = L_{n\psi}\tau_u^{(2)}(M) = 1$.

Now suppose $L_\phi\tau_u^{(2)}(M) = 1$. If M is homeomorphic to the solid torus then any nonzero class is fibered and the result is direct. Now we assume that M is not the solid torus. Since M is not a closed graph manifold, the virtual fibering theorem asserts that there is a connected regular finite covering $\overline{M} \to M$ such that the pull back class $\bar{\phi}$ lies in the closure of a fibered cone. Let $\pi_1(\overline{M}) = L < G$. Then by the restriction property of Theorem 4.9 (4)

$$\tau_u^{(2)}(\overline{M}) = \mathrm{res}_L^G\, \tau_u^{(2)}(M) \in \mathrm{Wh}(\mathcal{D}_L).$$

By Theorem 3.11 the restriction map commutes with the leading term map,

$$L_{\bar{\phi}}(\tau_u^{(2)}(\overline{M})) = L_{\bar{\phi}}(\mathrm{res}_L^G\, \tau_u^{(2)}(M)) = \mathrm{res}_L^G(L_\phi\tau_u^{(2)}(M)) = 1.$$

Applying the polytope map $\mathbb{P}$ to both sides we get

$$\mathbb{P}(L_{\bar{\phi}}(\tau_u^{(2)}(\overline{M}))) = 0 \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(\overline{M};\mathbb{R})).$$

Note that $\mathbb{P}(\tau_u^{(2)}(\overline{M})) = 2 \cdot [B_x^*(\overline{M})]$ by Theorem 6.4, we have that

$$0 = \mathbb{P}(L_{\bar{\phi}}(\tau_u^{(2)}(\overline{M}))) = F_{\bar{\phi}}\mathbb{P}(\tau_u^{(2)}(\overline{M})) = -2 \cdot F_{\bar{\phi}}[B_x^*(\overline{M})].$$

Since $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(\overline{M};\mathbb{R}))$ is torsion-free by [FL17, Lemma 4.8], we obtain

$$F_{\bar{\phi}}[B_x^*(\overline{M})] = 0.$$

It follows from Lemma 6.3 that $\bar{\phi}$ lies in a top dimensional Thurston cone of $H^1(\overline{M};\mathbb{R})$. By assumption $\bar{\phi}$ lies in the closure of a fibered cone. So $\bar{\phi}$ must lie in the interior of the fibered cone since the boundary of a fibered cone consists of Thurston cones of strictly lower dimensions. therefore $\bar{\phi}$ is a fibered class for $\overline{M}$ and ϕ is a fibered class for M. □

Theorem 6.6. *Let (N, R_+, R_-, γ) be a taut sutured manifold with non-empty sutured annuli γ. Then (N, γ) is a product sutured manifold if and only if $\tau_u^{(2)}(N, R_+) = 1 \in \mathrm{Wh}(\mathcal{D}_{\pi_1(\mathrm{N})})$.*

Proof. If (N, γ) is a product sutured manifold then clearly $\tau_u^{(2)}(N, R_+) = 1$.

Now we suppose $\tau_u^{(2)}(N, R_+) = 1$. Let $(N, \overline{R}_+, \overline{R}_-, \bar{\gamma})$ be the sutured manifold whose underlying oriented manifold is the same as N, but with

$$\bar{\gamma} = -\gamma, \quad \overline{R}_+ = -R_-, \quad \overline{R}_- = -R_+.$$

Let M be the double of N, then $M = (N, \gamma) \cup (N, \bar{\gamma})$ is the union of two sutured manifolds, with R_+ and $\overline{R}_-$ identified together, R_- and $\overline{R}_+$ identified together. Then $\Sigma := R_+ \cup R_-$ is a Thurston norm minimizing surface dual to a cohomology class ϕ. By Theorem 5.8,

$$L_\phi\tau_u^{(2)}(M) = (j_1)_*\tau_u^{(2)}(N, R_+) \cdot (j_2)_*\tau_u^{(2)}(N, \overline{R}_+).$$

But $\tau_u^{(2)}(N,R_+)=\tau_u^{(2)}(N,R_-)=\tau_u^{(2)}(N,\overline{R}_+)=1$ by Proposition 5.2. Therefore $L_\phi\tau_u^{(2)}(M)=1$. Since M has non-empty torus boundary, it follows from Theorem 6.5 that ϕ is a fibered class and hence $M\backslash\backslash\Sigma$ is a product sutured manifold. In particular (N,γ) is a product sutured manifold. □

7. Examples

7.1. **Free group endomorphisms.** Let $F=\langle x_1,\dots,x_n\rangle$ be a free group of finite rank n and let $X=\vee_{i=1}^n S^1$ be the wedge of n circles. The space X is given the usual CW-structure with one 0-cell p and n 1-cells $e_1,\dots,e_n$. Identify the fundamental group $\pi_1(X,p)$ with F in such a way that $x_i=[e_i]$. Then any homomorphism $\phi: F\to F$ determines a continuous mapping $\Phi: X\to X$ up to homotopy. We call Φ the realization of ϕ with respect to the basis $x_1,\dots,x_n$.

Definition 7.1. Let $F=\langle x_1,\dots,x_n\rangle$. We say $\phi: F\to F$ is a weak isomorphism if its realization Φ is an L^2-weak homotopy equivalence. If ϕ is a weak isomorphism, define $\tau_u^{(2)}(\phi):=\tau_u^{(2)}(\Phi)\in \mathrm{Wh}(\mathcal{D}_F)$; otherwise, define $\tau_u^{(2)}(\phi):=0$. We call $\tau_u^{(2)}(\phi)\in\mathrm{Wh}(\mathcal{D}_F)\sqcup\{0\}$ the universal L^2-torsion of ϕ.

Later we will show that the universal L^2-torsion of ϕ does not depend on the choice of basis of F. Before that, let's explicitly calculate $\tau_u^{(2)}(\phi)$ under the given basis of F. Given $\phi: F\to F$ and its topological realization $\Phi: X\to X$. The mapping cylinder M_Φ can be formed as follows. Choose another copy X' with cells p' and $e_1',\dots,e_n'$, form the wedge $X\vee X'$ by identifying $p\in X$ and $p'\in X'$. For any $i=1,\dots,n$, attach a 2-cell σ_i whose boundary is the concatenation of $\Phi(e_i)$ and e_i^{-1}. The resulting cellular complex is the mapping cylinder M_Φ which contains X as a subcomplex. Let $\widehat{M_\Phi}$ be the universal cover of M_Φ and let $\widehat{X}$ be the preimage. Fix a lifting $\hat p\in\widehat{M_\Phi}$ of p and lift the other cells with respect to the base point $\hat p$. Then we have the following $\mathbb{Z}F$-chain complex

$$(\dagger)\qquad C_*(\widehat{M_\Phi},\widehat{X})=(0\to\mathbb{Z}F\langle\sigma_1,\dots,\sigma_n\rangle\xrightarrow{A_\phi}\mathbb{Z}F\langle e_1,\dots,e_n\rangle\to 0\to 0).$$

The square matrix A_ϕ is given by the Fox derivative

$$A_{ij}=\frac{\partial\phi(x_i)}{\partial x_j}\in\mathbb{Z}F,\quad i,j=1,\dots,n.$$

Recall that the Fox derivative $\frac{\partial}{\partial x_i}:\mathbb{Z}F\to\mathbb{Z}F$, $i=1,\dots,n$ are $\mathbb{Z}$-linear maps characterized by the following two properties:

- $\frac{\partial}{\partial x_i}1=0$, $\frac{\partial}{\partial x_i}x_j=\delta_{ij}$.
- $\frac{\partial}{\partial x_i}(uv)=\frac{\partial}{\partial x_i}u+u\cdot\frac{\partial}{\partial x_i}v$ for all $u,v\in F$.

Lemma 7.2. *If A_ϕ is a weak isomorphism then ϕ is injective.*

Proof. Suppose the contrary that ϕ is not injective, then there is a reduced word $w\in F$ such that $\phi(w)=1$. Let $w=x_{i_1}^{\epsilon_1}\cdots x_{i_k}^{\epsilon_k}$ be such a word with shortest length $k\geqslant 1$, where $i_1,\dots,i_k\in\{1,\dots,n\}$ and $\epsilon_1,\dots,\epsilon_k\in\{\pm1\}$. We may assume that $x_{i_1}^{\epsilon_1}=x_1$ and $x_{i_k}^{\epsilon_k}\neq x_1^{-1}$. Then

$$\phi(x_{i_1})^{\epsilon_1}\cdots\phi(x_{i_k})^{\epsilon_k}=1.$$

Denote by $w_s:=x_{i_1}^{\epsilon_1}\cdots x_{i_s}^{\epsilon_s}$, $0\leqslant s\leqslant k$ to be the prefix of w of length s; set $w_0=1$. For any $j=1,\dots,n$, applying $\frac{\partial}{\partial x_j}$ to both sides we have

$$\sum_{s=1}^k u_s\cdot\frac{\partial\phi(x_{i_s})}{\partial x_j}=0,\quad\text{where}\quad u_s=\begin{cases}\phi(w_{s-1}), & \epsilon_s=1,\\ -\phi(w_s), & \epsilon_s=-1.\end{cases}$$

Note that u_s does not depend on j. Rearranging the summation, we have

$$\sum_{i=1}^{n} U_i \cdot \frac{\partial \phi(x_i)}{\partial x_j} = 0, \quad j = 1, \ldots, n$$

where U_i is the sum of all u_s such that $i_s = i$. Therefore

$$(U_1, U_2, \ldots, U_n) \cdot A_\phi = 0.$$

If we could show that the row vector is nonzero then this implies that A_ϕ is not a weak isomorphism, hence a contradiction. We prove that $U_1 \neq 0$. Let $1 = s_1 < s_2 < \cdots < s_r \leqslant k$ be the collection of indices such that $i_{s_1} = \cdots = i_{s_r} = 1$. Then $u_{s_1} = 1$ by assumption and $U_1 = 1 + u_{s_2} + \cdots + u_{s_r}$. Write $U_1 = 1 \pm \phi(w_{s'_2}) \pm \cdots \pm \phi(w_{s'_r})$, where $w_{s'_u} = w_{s_u}$ or $w_{s_u - 1}$, $u = 1, \ldots, r$ depending on the sign of ϵ_{s_u}. Then we have $0 < s'_2 < \cdots < s'_r < k$. This is because there are no segments of $(\cdots x_1^{-1} x_1 \cdots)$ in the reduced word w, and that w does not ends with x_1^{-1}. The group elements $\phi(w_{s'_u})$ are pairwise distinct, otherwise we find a reduced word with length shorter than k which lies in the kernel of ϕ. This shows that U_1 is nonzero in $\mathbb{Z}F$ and finishes the proof. □

Proposition 7.3. *Let F be a finitely generated free group. Choose a free basis $F = \langle x_1, \ldots, x_n \rangle$ and define $\tau_u^{(2)}(\phi) \in \mathrm{Wh}(\mathcal{D}_F) \sqcup \{0\}$ for any homomorphism $\phi : F \to F$ as in Definition 7.1. Then:*

1. *$\phi : F \to F$ is a weak isomorphism if and only if A_ϕ is a weak isomorphism. In this case, we have $\tau_u^{(2)}(\phi) = \det_w(A_\phi) \in \mathrm{Wh}(\mathcal{D}_F)$.*
2. *$\tau_u^{(2)}(\phi) = 0$ if ϕ is not injective; $\tau_u^{(2)}(\phi) = 1$ if ϕ is an isomorphism.*
3. *If $\phi, \psi : F \to F$ are weak isomorphisms then $\tau_u^{(2)}(\phi \circ \psi) = \phi_* \tau_u^{(2)}(\psi) \cdot \tau_u^{(2)}(\phi)$.*
4. *The definition of $\tau_u^{(2)}(\phi)$ does not depend on the choice of free basis of F.*

Proof. Firstly, (1) is immediate from Equation † and the definition.

For (2), if ϕ is not injective then $\tau_u^{(2)}(\phi) = 0$ by Lemma 7.2. If ϕ is an isomorphism, then $\Phi : X \to X$ is a homotopy equivalence. Since the Whitehead group of a free group is trivial [Sta65] then Φ is a simple homotopy equivalence and $\tau_u^{(2)}(\phi) = \tau_u^{(2)}(\Phi) = 1$ by Proposition 4.11 (3).

For (3), ϕ and ψ are injective by (2). Therefore the result follows from Proposition 4.11 (4) since

$$\tau_u^{(2)}(\phi \circ \psi) = \tau_u^{(2)}(\Phi \circ \Psi) = \phi_* \tau_u^{(2)}(\Psi) \cdot \tau_u^{(2)}(\Phi) = \phi_* \tau_u^{(2)}(\psi) \cdot \tau_u^{(2)}(\phi).$$

For (4), change of free basis amounts to replacing ϕ with $\psi \circ \phi \circ \psi^{-1}$ where $\psi : F \to F$ is an isomorphism. If ϕ is a weak isomorphism, then by (3) we have $\tau_u^{(2)}(\psi \circ \phi \circ \psi^{-1}) = \tau_u^{(2)}(\phi)$ is nonzero, hence $\psi \circ \phi \circ \psi^{-1}$ is a weak isomorphism. Similarly, if $\psi \circ \phi \circ \psi^{-1}$ is a weak isomorphism then ϕ is also a weak isomorphism. In conclusion, $\psi \circ \phi \circ \psi^{-1}$ is a weak isomorphism if and only if ϕ is a weak isomorphism. Moreover, we have $\tau_u^{(2)}(\psi \circ \phi \circ \psi^{-1}) = \tau_u^{(2)}(\phi)$ by (3). □

The question of whether a given homomorphism $\phi : F \to F$ is a weak isomorphism is already interesting. We say that a finitely generated subgroup H of F is compressed if for any subgroup L of F containing H we have $\operatorname{rank} H \leqslant \operatorname{rank} L$. The following characterization given by [JZ24] is notable since it does not involve any L^2-theories in its statement.

Theorem 7.4. *Let F be a free group of finite rank. Then a homomorphism $\phi : F \to F$ is a weak isomorphism if and only if ϕ is injective with compressed image $\operatorname{im} \phi \subset F$.*

Proof. If ϕ is not injective then ϕ is never a weak isomorphism. So we assume that ϕ is injective and aim to show that ϕ is a weak isomorphism if and only if $\operatorname{im} \phi$ is compressed.

Recall that $\Phi : X \to X$ is the topological realization of ϕ and M_Φ is the mapping cylinder. The homomorphism $\phi : F \to F$ is a weak isomorphism if and only if the pair (M_Φ, X) is L^2-acyclic. By the homology long exact sequence this is equivalent to

$$H_1(X; \mathcal{D}_F) \to H_1(M_\Phi; \mathcal{D}_F)$$

being injective (note that $\operatorname{rank} H_1(X; \mathcal{D}_F) = \operatorname{rank} H_1(M_\Phi; \mathcal{D}_F) = \operatorname{rank} F - 1$). By [JZ24, Corollary 1.2] this is equivalent to $\pi_1(X) \subset \pi_1(M_\phi)$ being compressed, i.e. $\operatorname{im}\phi \subset F$ is compressed. $\square$

7.2. **Applications to knots and links.** Let $K \subset S^3$ be a knot. A free Seifert surface of K is a Seifert surface whose complement is a handlebody. Any Seifert surface obtained from Seifert's algorithm applied to a knot diagram is a free Seifert surface, so any knot admits abundant free Seifert surfaces.

Given any free Seifert surface Σ of a knot K. Let g be the genus of Σ, then the complement $S^3 \backslash\backslash \Sigma$ is a handlebody of genus $g+1$. Fix an orientation on Σ and view $S^3 \backslash\backslash \Sigma$ as a sutured manifold, so the boundary $\partial(S^3 \backslash\backslash \Sigma)$ is the union of Σ_+ and Σ_-, where Σ_+ (resp. Σ_-) is the copy of Σ whose normal direction points out of (resp. into) $S^3 \backslash\backslash \Sigma$. Note that the fundamental groups $\pi_1(\Sigma)$ and $\pi_1(S^3 \backslash\backslash \Sigma)$ are both free groups of rank $g+1$.

Lemma 7.5. *Let $\iota : \pi_1(\Sigma_+) \to \pi_1(S^3 \backslash\backslash \Sigma)$ be the inclusion-induced map on fundamental groups. Then $S^3 \backslash\backslash \Sigma$ is a taut sutured manifold if and only if ι is a weak isomorphism. In this case, we have*

$$\tau_u^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+) = \tau_u^{(2)}(\iota).$$

Proof. Choose a base point $p \in \Sigma_+$. Let X be the wedge of $(g+1)$ circles with base point x. Let $f : (X, x) \to (\Sigma_+, p)$ be a homotopy equivalence and let $g : (S^3 \backslash\backslash \Sigma, p) \to (X, x)$ be a deformation retract. We use f_*, g_* to identify $\pi_1(\Sigma_+)$ and $\pi_1(S^3 \backslash\backslash \Sigma)$ with $\pi_1(X)$. Let $i : \Sigma_+ \to S^3 \backslash\backslash \Sigma$ be the inclusion. Both f, g are simple-homotopy equivalences since the Whitehead group of free groups are trivial. Then by Lemma 4.13 we have $\tau_u^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+) = \tau_u^{(2)}(i) = \tau_u^{(2)}(g \circ i \circ f)$. Since $g \circ i \circ f : X \to X$ induces ι on fundamental groups, therefore $\tau_u^{(2)}(g \circ i \circ f) = \tau_u^{(2)}(\iota)$. $\square$

7.2.1. *Pretzel knot $K(p,q,r)$.* Let $K(p,q,r)$ be the (p,q,r)-pretzel link, $p,q,r \in \mathbb{Z}$. The (p,q,r)-pretzel link can be viewed as the boundary of the surface obtained from attaching bands to two disks of p,q,r half-twists, respectively. Suppose that p,q,r are both odd numbers. Then $K(p,q,r)$ is a knot and we can form a Seifert surface of genus 1 as in the figure below. Choose x, y generating $\pi_1(S^3 \backslash\backslash K)$ as in the Figure. Pick two loops u, v on Σ generating $\pi_1(\Sigma)$ and push them slightly into $u_+, v_+ \subset \Sigma_+$. It can be checked directly that $[u_+] = x^{\frac{p+1}{2}} y^{\frac{1-q}{2}}$ and $[v_+] = y^{\frac{q+1}{2}} (xy)^{\frac{r-1}{2}}$, hence

$$\operatorname{im}\iota = \langle x^{\frac{p+1}{2}} y^{\frac{1-q}{2}}, y^{\frac{q+1}{2}} (xy)^{\frac{r-1}{2}} \rangle \subset \langle x, y \rangle = \pi_1(S^3 \setminus \Sigma).$$

Set $a := \frac{p+1}{2},\ b := \frac{1-q}{2},\ c := \frac{r-1}{2}$, then $p = 2a-1,\ q = 1-2b,\ r = 2c+1$. Then

$$\operatorname{im}\iota = \langle x^a y^b, y^{1-b}(xy)^c \rangle \subset \langle x, y \rangle.$$

Remark 7.6. The Seifert matrix V is a matrix given by the intersection number of u_+, v_+ and x, y. It is clear that

$$V = \begin{pmatrix} a & b \\ c & 1-b+c \end{pmatrix}$$

and the Alexander polynomial of $K(p,q,r)$ is $A_{K_{p,q,r}}(t) = \det(tV - V^T)$.

Examples 7.7. We compute the universal L^2-torsion of the sutured manifold $S^3 \backslash\backslash \Sigma$ for some p, q, r.

- $a = 2,\ b = 1,\ c = 1$. In this case $K = K(3, -1, 3)$, $\operatorname{im}\iota = \langle x^2y, xy\rangle \subset \langle x, y\rangle$ and
$$A_\iota = \begin{pmatrix} 1+x & x^2 \\ 1 & x \end{pmatrix}.$$
We have $\tau_u^{(2)}(S^3\backslash\backslash\Sigma, \Sigma_+) = \det_w(A_\iota) = [x] = 1 \in \mathrm{Wh}(\mathcal{D}_{\pi_1(S^3\backslash\backslash\Sigma)})$. This shows that $S^3\backslash\backslash\Sigma$ is a product sutured manifold and K is fibered.
- $a = -2,\ b = 5,\ c = 1$. In this case $K = K(-5, -9, 3)$, $\operatorname{im}\iota = \langle x^{-2}y^5, y^{-4}xy\rangle \subset \langle x, y\rangle$ and
$$A_\iota = \begin{pmatrix} -x^{-1} - x^{-2} + x^{-2}y & x^{-2} + x^{-2}yx \\ y^{-4} & -y^{-1} - y^{-2} - y^{-3} - y^{-4} + y^{-4}x \end{pmatrix}.$$
By elementary row operation we have
$$\begin{aligned}\det_w(A_\iota) &= [x^{-2} + x^{-2}yx + (x^{-1} + x^{-2} - x^{-2}y)y^4(-y^{-1} - y^{-2} - y^{-3} - y^{-4} + y^{-4}x)] \\ &= [x^2 + y^4 - x(y + y^2 + y^3)].\end{aligned}$$
The image under the polytope map $\mathbb{P} : \mathrm{Wh}(\mathcal{D}_{\pi_1(S^3\backslash\backslash\Sigma)}) \to \mathcal{P}_{\mathbb{Z}}(H_1(S^3\backslash\backslash\Sigma))$ is a polytope spanned by vertices.
- $a = -4,\ b = 3,\ c = 1$. In this case $K = K(-9, -5, 3)$, $\operatorname{im}\iota = \langle x^{-2}y^5, y^{-4}xy\rangle \subset \langle x, y\rangle$ and
$$A_\iota = \begin{pmatrix} -x^{-1} - x^{-2} - x^{-3} - x^{-4} & x^{-4}(1 + y + y^2) \\ y^{-2} & -y^{-1} - y^{-2} + y^{-2}x \end{pmatrix}.$$
By elementary row operation we have
$$\begin{aligned}\det_w(A_\iota) &= [x^{-4}(1 + y + y^2) + (x^{-1} + x^{-2} + x^{-3} + x^{-4})y^2(-y^{-1} - y^{-2} + y^{-2}x)] \\ &= [y^2 + x^4 - xy - x^2y - x^3y].\end{aligned}$$
The image under the polytope map $\mathbb{P} : \mathrm{Wh}(\mathcal{D}_{\pi_1(S^3\backslash\backslash\Sigma)}) \to \mathcal{P}_{\mathbb{Z}}(H_1(S^3\backslash\backslash\Sigma))$ is a polytope spanned by vertices.

7.2.2. *n-chain link L_n.* Let L_n be the n-chain link with orientation as in the figure. Consider the Seifert surface Σ as in the figure. Then Σ deformation retracts to the wedge of $(n-1)$ circles. The complement of Σ is a handle body of genus $(n-1)$. Choose a free basis as in the figure, then $\pi_1(S^3 \setminus \Sigma) = \langle x_1, \ldots, x_{n-1}\rangle$. Let $\iota : \pi_1(\Sigma_+) \to \pi_1(S^3\backslash\backslash\Sigma)$ be the inclusion-induced map then
$$\operatorname{im}\iota = \langle x_1x_2^{-1}, x_2x_3^{-1}, \ldots, x_{n-2}x_{n-1}^{-1}, x_{n-1}^2x_{n-2}\cdots x_2x_1\rangle \subset \langle x_1, \ldots, x_{n-1}\rangle$$
Then the Fox derivative matrix is
$$A = \begin{pmatrix} 1 & -x_1x_2^{-1} & & & \\ & 1 & -x_2x_3^{-1} & & \\ & & \ddots & & \\ & & & 1 & -x_{n-2}x_{n-1}^{-1} \\ x_{n-1}^2x_{n-2}\cdots x_2 & x_{n-1}^2x_{n-2}\cdots x_3 & \cdots & x_{n-1}^2 & 1 + x_{n-1} \end{pmatrix}$$

Lemma 7.8. *Let*
$$B = \begin{pmatrix} 1 & -s_1 & & & \\ & 1 & -s_2 & & \\ & & \ddots & & \\ & & & 1 & -s_{n-2} \\ f_1 & f_2 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix}$$
be a matrix over a skew field. Then the Dieudonné determinant
$$\det(B) = [f_1s_1s_2\cdots s_{n-2} + f_2s_2s_3\cdots s_{n-2} + \cdots f_{n-2}s_{n-2} + f_{n-1}].$$

Applying this Lemma, note that $s_i s_{i+1} \cdots s_{n-2} = x_i x_{n-1}^{-1}$, it follows that

$$f_i s_i s_{i+1} \cdots s_{n-2} = x_{n-1}^2 x_{n-2} \cdots x_i x_{n-1}^{-1}, \quad i < n-1, \quad f_{n-1} = 1 + x_{n-1}.$$

By conjugation with x_{n-1}, it follows that

$$\det(A) = [x_{n-1}x_{n-2}\cdots x_1 + x_{n-1}x_{n-2}\cdots x_2 + \cdots + x_{n-1}x_{n-2} + x_{n-1} + 1].$$

The Fuglede–Kadison determinant equals

$$\det{}^{\mathrm{r}}_{\mathcal{N}\pi_1(S^3\setminus\Sigma)}(A) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}}.$$

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