UNIVERSAL L^2-TORSION AND TAUT SUTURED DECOMPOSITIONS

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Abstract. Given an admissible 3-manifold M and a cohomology class $\phi \in H^1(M;\mathbb{R})$, we prove that the universal L^2-torsion of M detects the fiberedness of ϕ, except when M is a closed graph manifold that admits no non-positively curved metric. We further extend this invariant to sutured manifolds, where we derive a decomposition formula for taut sutured decompositions. Moreover, we show that a taut sutured manifold is a product if and only if its universal L^2-torsion is trivial. Our methods are based on a detailed study of the leading term map over Linnell's skew field. As an application, we apply the theory to homomorphisms between finitely generated free groups, which enables explicit computations of the invariant for sutured handlebodies.

1. Introduction

Let M be a compact orientable 3-manifold. A homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$ is said to be fibered if it is induced by a fibration of M over S^1. In his seminal work, Thurston [Thu86] introduced a semi-norm on $H^1(M;\mathbb{R})$, now known as the Thurston norm, and showed that its unit ball $B_x(M)$ is a finite-sided polyhedron. Moreover, he showed that the fibered classes in $H^1(M;\mathbb{R})$ correspond exactly to certain open cones over the top-dimensional faces of $B_x(M)$. Marked by the success of Gabai's sutured manifold theory [Gab83, Gab87] and the confirmation of the Virtual Fibering Conjecture [AGM13], determining the Thurston norm and the fibered structure of a 3-manifold has become a central theme in three-dimensional topology. In this paper we focus on the class of admissible 3-manifolds, defined as follows:

Definition 1.1 (Admissible 3-manifold)**.** A 3-manifold is called *admissible* if it is compact, connected, orientable, and irreducible, its boundary is either empty or a collection of tori, and its fundamental group is infinite.

The universal L^2-torsion introduced by Friedl and Lück [FL17] is defined for a finite CW-complex X with vanishing L^2-Betti numbers. In the case when X is an admissible 3-manifold, this invariant $\tau_u^{(2)}(X)$ takes value in the weak Whitehead group $\mathrm{Wh}^w(\pi_1(X))$ associated to the fundamental group of X. It has been shown that the universal L^2-torsion completly determines the Thurston norm of such a manifold [FL17], highlighting its potential to reflect subtle topological information of 3-manifolds. This leads naturally to the following questions:

- Does the universal L^2-torsion also characterize the fibered structure of an admissible 3-manifold? If so, in what manner?
- Can this invariant be extended to fit into the framework of Gabai's sutured manifold theory? If so, how does it change under sutured manifold decompositions?

This paper is devoted to answering these two questions.

To investigate the first question, we are motivated by the role of the leading coefficient of the Alexander polynomial in detecting fiberedness. For any cohomology class $\phi \in H^1(M;\mathbb{R})$ of an admissible 3-manifold M, we define a natural "leading term map"

$$L_\phi : \mathrm{Wh}^w(\pi_1(M)) \to \mathrm{Wh}^w(\pi_1(M))$$

on the weak Whitehead group of $\pi_1(M)$. We show that, unless M is a closed graph manifold that does not admit a non-positively curved (NPC) metric, the class ϕ is fibered if and only if the image of $\tau_u^{(2)}(M)$ under L_ϕ is trivial.

Theorem 1.2. *Suppose M is an admissible 3-manifold that is not a closed graph manifold without an NPC metric. For any nonzero cohomology class $\phi \in H^1(M;\mathbb{R})$, the class ϕ is fibered if and only if $L_\phi \tau_u^{(2)}(M) = 1 \in \mathrm{Wh}^w(\pi_1(M))$.*

As a consequence, Theorem 1.2 provides a clear description of the marking on the L^2-torsion polytope $\mathcal{P}(M)$ introduced in [Kie20] which determines the fibered structure of M.

A substantial part of this paper is devoted to the second question, namely, extending the theory of universal L^2-torsion to the setting of sutured manifolds. A sutured manifold (N, R_+, R_-, γ) is a compact oriented 3-manifold whose boundary is partitioned into two oriented subsurfaces R_+ and R_-, meeting along their common boundary γ. A sutured manifold can be decomposed along a nicely embedded surface S and the resulting manifold is again a sutured manifold, we write

$$(N, R_+, R_-, \gamma) \overset{S}{\rightsquigarrow} (N', R'_+, R'_-, \gamma')$$

for such a decomposition. A key result of Herrmann [Her23] shows that, roughly speaking, a sutured manifold (N, R_+, R_-, γ) is taut if and only if the pair (N, R_+) has trivial L^2-Betti numbers (see Theorem 5.6). This motivates the definition of the universal L^2-torsion $\tau_u^{(2)}(N, R_+)$ for a taut sutured manifold, which takes values in $\mathrm{Wh}^w(\pi_1(N))$.

The second main result of this paper describes how the universal L^2-torsion behaves under taut sutured decompositions. Specifically, we relate its change to the leading term map introduced earlier.

Theorem 1.3. *Let $(N, R_+, R_-, \gamma) \overset{\Sigma}{\rightsquigarrow} (N', R'_+, R'_-, \gamma')$ be a taut sutured decomposition and let $\phi \in H^1(N;\mathbb{Z})$ be the Poincaré dual of the surface Σ, then*

$$j_* \tau_u^{(2)}(N', R'_+) = L_\phi \tau_u^{(2)}(N, R_+)$$

where $j_* : \mathrm{Wh}^w(\pi_1(N')) \to \mathrm{Wh}^w(\pi_1(N))$ is induced by the inclusion $j : N' \hookrightarrow N$.

Furthermore, we show that the universal L^2-torsion serves as an obstruction to the product structure on sutured manifolds. Indeed, a sutured manifold (N, R_+, R_-, γ) may be viewed as a cobordism between two compact surfaces $R_\pm$. When N is taut, the inclusion map $R_+ \hookrightarrow N$ induces isomorphism on L^2-homology. We prove that this cobordism is a trivial product if and only if its universal L^2-torsion is trivial—a result reminiscent of the celebrated s-cobordism theorem.

Theorem 1.4. *Let (N, R_+, R_-, γ) be a taut sutured manifold such that both R_+ and R_- are non-empty. Then N is diffeomorphic to the product $R_+ \times [0,1]$ if and only if $\tau_u^{(2)}(N, R_+) = 1 \in \mathrm{Wh}^w(\pi_1(N))$.*

We further study the computation of the universal L^2-torsion $\tau_u^{(2)}(X, Y)$ in the case where the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ are finitely generated free groups. In this setting, the universal L^2-torsion is completely determined by the homomorphism $\varphi : \pi_1(Y) \to \pi_1(X)$ induced by the inclusion $Y \hookrightarrow X$. It is natural to define the universal L^2-torsion $\tau_u^{(2)}(\varphi)$ for a group homomorphism between finitely generated free groups. The following result gives an explicit formula (see Proposition 6.2):

Theorem 1.5. *Let $\varphi : F_1 \to F_2$ be a homomorphism between finitely generated free groups. Then the universal L^2-torsion of φ is*

$$\tau_u^{(2)}(\varphi) = [J_\varphi] \in \mathrm{Wh}^w(F_2) \sqcup \{0\}$$

where J_φ is the Fox Jacobian matrix of φ over $\mathbb{Z}F_2$, whose entries are the Fox derivatives

$$(J_\varphi)_{ij} = \frac{\partial \varphi(x_i)}{\partial y_i} \in \mathbb{Z}F_2, \quad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$$

with respect to any chosen bases $F_1 = \langle x_1, \dots, x_n \rangle$, $F_2 = \langle y_1, \dots, y_m \rangle$.

We apply this formula to explicitly compute the universal L^2-torsion of certain sutured handlebodies. A key example is the sutured manifold $S^3 \backslash\backslash \Sigma$ obtained by decomposing the n-chain link complement along a minimal-genus Seifert surface Σ (see Figure 1). We show that

$$\tau_u^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+) = [1 + y_1 + \cdots + y_{n-1}] \in \mathrm{Wh}^w(\pi_1(S^3 \backslash\backslash \Sigma))$$

where $\{y_1, \dots, y_{n-1}\}$ is a free generating set of $\pi_1(S^3 \backslash\backslash \Sigma)$. Combining this with the calculations of [BA22], we obtain the following formula for the classical L^2-torsion:

$$\tau^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}} \sim \sqrt{n/e}, \quad \text{as } n \to +\infty.$$

Here, the left-hand side denotes the classical L^2-torsion of the pair $(S^3 \backslash\backslash \Sigma, \Sigma_+)$, obtained from the universal L^2-torsion via the Fuglede–Kadison determinant. Together with results from [Dua25], this computation yields an infinite family of hyperbolic manifolds for which the leading coefficient of the L^2-Alexander torsion associated to a nonzero class is greater than 1. This confirms the final remaining case of [BAFH22, Conjecture 1.7].

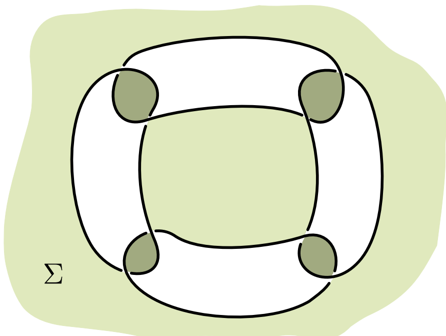

Figure 1. *The n-chain link and the minimal-genus Seifert surface Σ, where $n = 4$.*

1.1. **Motivation.**

1.1.1. *Why universal L^2-torsion.* To define the classical torsion invariants of a finite CW-complex X, one has to produce an exact sequence from the cellular chain complex of the universal cover $\widehat{X}$. One way to do this is to extend the scalars from the group ring $\mathbb{Z}[\pi_1(X)]$ to a *commutative* field. For 3-manifolds, this yields the Reidemeister–Franz torsion and the (multi-variable) Alexander polynomials [Mil62, McM02, FV11a], via scalar extensions to $\mathbb{C}$ and the field of rational functions $\mathbb{Q}(H_1(X))$, respectively. However, this base change loses non-commutative information of the fundamental group.

Whitehead torsion, introduced by J.H.C Whitehead, avoids this issue by retaining the full group ring structure. It takes values in the Whitehead group $\mathrm{Wh}(\pi_1(X))$ consisting of invertible matrices over $\mathbb{Z}[\pi_1(X)]$ modulo elementary relations [Mil66, Coh73]. However, the Whitehead torsion is only defined for pairs (X, Y) where X deformation retracts to Y. Also, it is too restrictive for a matrix to be invertible over $\mathbb{Z}[\pi_1(X)]$. Indeed, it is a well-known conjecture that the Whitehead group of any torsion-free group should vanish.

The universal L^2-torsion emerges as a powerful and widely applicable torsion invariant. Its advantages are threefold:

1. The universal L^2-torsion is defined via the faithful infinite-dimensional regular representation of the fundamental group and captures its deep noncommutative information.
2. It applies to a broader class of spaces whose L^2-Betti numbers vanish, including mapping tori [Lüc94], spaces with infinite amenable fundamental groups [CG86], admissible 3-manifolds [LL95] and all odd-dimensional closed hyperbolic manifolds [HS98] (see [Lüc02] for further examples).
3. The weak Whitehead group $\mathrm{Wh}^w(G)$ of any group G is never trivial (it always contains $\mathbb{Z}$). In fact there are interesting non-trivial homomorphisms from $\mathrm{Wh}^w(G)$ such as the Fuglede–Kadison determinant and the polytope maps [FL17].

The universal L^2-torsion has been used to define group-theoretic Thurston norms for broader classes of groups, including free-by-cyclic groups [FK18, KS24] and coherent right-angled Artin groups [Kud24]. The BNS-invariant, which generalizes the notion of fiberedness from 3-manifold groups to arbitrary finitely generated groups, is deeply connected to the universal L^2-torsion, as evidenced by recent work on two-generator one-relator groups [FST17, FT20, HK20], free-by-cyclic groups [FK18] and more general agrarian groups by [Kie20]. These studies focus on the L^2-torsion polytope derived from the universal L^2-torsion and investigate the existence of a *marking* that determines the structure of the BNS-invariant.

A key novelty of our Theorem 1.2 is to provide a clear and direct description of the fibered structure for 3-manifolds in terms of the universal L^2-torsion itself.

1.1.2. *Fiberedness and the leading term of torsions.* It is well-known that if a class ϕ is fibered, then its Alexander polynomial, twisted Alexander polynomial, and L^2-Alexander torsion all have trivial leading coefficients. While the converse does not hold true in general, Friedl and Vidussi [FV11b] showed that the collection of all twisted Alexander polynomial associated to ϕ does determine whether ϕ is fibered. This suggests that the leading term of torsion invariants in fact carry enough information for the detection of fiberedness.

The present work is largely motivated by the study of L^2-Alexander torsions. For a cohomology class $\phi \in H^1(M;\mathbb{R})$, the L^2-Alexander torsion was introduced by Dubois, Friedl and Lück [DFL16] as an L^2-analogue of the classical Alexander polynomial. Liu [Liu17] showed that a leading coefficient $C(M,\phi) \in [1,+\infty)$ of the L^2-Alexander torsion is well-defined and $C(M,\phi) = 1$ whenever ϕ is a fibered class. Conversely, it is not difficult to find examples that $C(M,\phi) = 1$ while ϕ is non-fibered. For example, take M to be any two-bridge knot complement and ϕ the canonical class; the leading coefficient $C(M,\phi)$ can be interpreted as the relative L^2-torsion of the guts $\Gamma(\phi)$ of ϕ [Dua25], which is a disjoint union of solid tori relative to an annulus on its boundary [AZ22]. The relative L^2-torsion in such cases is trivial because the Fuglede–Kadison determinant

$$\det_{\mathcal{N}\pi_1(M)} : \mathrm{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(t^{\pm})^{\times}/\{\pm t^n \mid n \in \mathbb{Z}\} \to \mathbb{R}_+$$

is not faithful, mapping certain nontrivial elements to 1. This observation suggests the need for a more refined invariant that is defined algebraically and does not rely on the Fuglede–Kadison determinant.

Accordingly, we consider the universal L^2-torsion and introduce the leading term map

$$L_\phi : \mathrm{Wh}^w(\pi_1(M)) \to \mathrm{Wh}^w(\pi_1(M))$$

which serves as an algebraic analogue of "taking the leading coefficient" within the weak Whitehead group. We refer to Section 4 for further details.

1.1.3. *Analogy with sutured Floer homology.* The Heegaard Floer homology, defined by Ozsváth and Szabó [OS04] for closed 3-manifolds, was generalized by Juhász [Juh06] to sutured Floer homology (SFH), an invariant for balanced sutured manifolds. In [Juh08] the fundamental properties of SFH are proved:

(1) If (M, γ) is a taut balanced sutured manifold, then $SFH(M, \gamma)$ is non-trivial.
(2) If $(M, \gamma) \rightsquigarrow (M', \gamma')$ is a sutured decomposition of balanced sutured manifolds, then $SFH(M', \gamma')$ is a direct summand of $SFH(M, \gamma)$.
(3) A taut balanced sutured manifold (M, γ) is a product sutured manifold if and only if $SFH(M, \gamma) = \mathbb{Z}$.

We observe that analogous properties hold in the theory of universal L^2-torsions. Specifically, property (1) corresponds to Herrmann's result (Theorem 5.6), while (2) and (3) are reflected in Theorems 1.3–1.4, respectively.

Question 1.6. *Is there a deeper connection between sutured Floer homology and the universal L^2-torsion of a taut sutured manifold?*

This question is particularly interesting because SFH is defined in terms of holomorphic curves and is not directly related to the fundamental group, whereas the universal L^2-torsion is algebraic and computable from a presentation of the fundamental group.

Furthermore, Juhász [Juh10] showed that the rank of SFH serves as a complexity measure for sutured manifolds that decreases under non-trivial decompositions. This motivates a parallel question in the L^2-context:

Question 1.7. *Can one define a natural complexity function on the weak Whitehead group that decreases under the leading term map L_ϕ?*

1.2. **Proof ingredients.** The remaining part of the paper is divided into five sections. We briefly discuss the contents of each these.

1.2.1. *Algebraic preliminaries.* Section 2 reviews the basic notions of L^2-theory, focusing on three topics that play a central role in the sequel: *Hilbert modules*, the *Atiyah Conjecture* and the *Linnell's skew field*, and K_1*-groups* together with the *Dieudonné determinant.*

1.2.2. *Universal L^2-torsion.* In Section 3, we define the universal L^2-torsion and discuss its computation. Let X be a finite CW-complex whose fundamental group G is torsion-free and satisfies the Atiyah Conjecture, the universal L^2-torsion $\tau_u^{(2)}(X)$ is defined as the torsion of the chain complex $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*(\widehat{X})$ viewed as a complex of modules over Linnell's skew field $\mathcal{D}_G$, and takes values in the Whitehead group $\mathrm{Wh}(\mathcal{D}_G)$. Slightly different from the original definition in [FL17], where the universal L^2-torsion lives in the weak Whitehead group $\mathrm{Wh}^w(G)$, our definition is more restrictive but is better suited to algebraic manipulation. Importantly, the two definitions coincide when X is a 3-manifold, or more generally when G belongs to Linnell's class $\mathcal{C}$; see Section 3.1.1 for a detailed discussion.

We further extend the definition of universal L^2-torsion for CW-pairs and to continuous mappings between finite CW-complexes via mapping cylinders. Several basic properties are established. This generalization brings greater flexibility to the application of the invariant.

Finally, we generalize Turaev's *matrix chain method* [Tur01] for the direct computation of the universal L^2-torsion of a chain complex. Informally, a chain complex is acyclic if and only if we can extract a chain of invertible submatrices from the matrices representing its connecting operators. The torsion is then given by the alternating product of their determinants.

1.2.3. *Leading term map, restriction map and polytope map.* The content of Section 4 is mostly technical, focusing on the study of three important homomorphisms.

Given any real character $\phi : G \to \mathbb{R}$ and a nonzero element $a = \sum_{g\in G} n_g \cdot g \in \mathbb{Z}G$, the ϕ-leading term of a is defined to be the sum of nonzero terms $n_g \cdot g$ for which $\phi(g)$ is minimal. Using the crossed product structure of the skew field $\mathcal{D}_G$, this construction extends to the *leading term homomorphism*

$$L_\phi : \mathcal{D}_G^\times \to \mathcal{D}_G^\times.$$

This homomorphism further induces homomorphisms on the K_1-group and the Whitehead group of $\mathcal{D}_G$. The key result concerning the leading term map (Theorem 4.12) says that, roughly speaking, if A, B are square matrices over $\mathcal{D}_G$ such that the ϕ-values of entries of B are strictly greater than those of A, then the ϕ-leading term of the determinant $\det(A + B)$ depends only on A. This intuitive result is surprisingly useful throughout our paper.

The *restriction map* relates the K_1-group of a group G to that of a finite-index subgroup $L < G$:

$$\mathrm{res}_L^G : K_1(\mathcal{D}_G) \to K_1(\mathcal{D}_L).$$

This is defined by interpreting an element $z \in K_1(\mathcal{D}_G)$ as an operator $r_z : \ell^2(L)^{[G:L]} \to \ell^2(L)^{[G:L]}$ and then taking its determinant. We prove that the leading term map commutes with the restriction map (see Theorem 4.17).

Every element $z \in \mathcal{D}_G^\times$ may be viewed as being "supported" on a certain Newton polytope in the real vector space $H_1(G;\mathbb{R})$. The *polytope map*

$$\mathbb{P} : \mathcal{D}_G^\times \to \mathcal{P}_{\mathbb{Z}}(H)$$

formalizes this intuition, where H is the free abelianization of G and $\mathcal{P}_{\mathbb{Z}}(H)$ is the Grothendieck group of the integral polytopes in $\mathbb{R} \otimes_{\mathbb{Z}} H$ under the Minkowski sum. It is known that $\mathcal{P}_{\mathbb{Z}}(H)$ is a free abelian group possessing an explicit basis [Fun21]. The polytope map is particularly useful for detecting nontrivial elements in $\mathcal{D}_G^\times/[\mathcal{D}_G^\times, \mathcal{D}_G^\times] = K_1(\mathcal{D}_G)$.

At the end of Section 4.3 we prove the "if" direction of Theorem 1.2, namely Theorem 4.24. The idea is as follows: the condition $L_\phi \tau_u^{(2)}(M) = 1$, when passing to finite index coverings, implies that the class ϕ is lifted to a top-dimensional cone in each covering. We then apply the Virtual Fibering Theorem on admissible 3-manifolds non-positively curved metrics (see [AFW15] and references therein) to conclude that ϕ is fibered.

1.2.4. *Universal L^2-torsion for taut sutured manifolds.* In Section 5, we investigate the universal L^2-torsion for sutured 3-manifolds, an object invented by Gabai [Gab83] to describe 3-manifolds via cut-and-paste constructions. To prove the decomposition formula in Theorem 1.3, we adopt the idea of Turaev's algorithm [Tur02, BAFH22] to reduce to the case of non-separating decomposition surfaces. Under an explicit CW-structure, the chain complex of the decomposed manifold is seen to be exactly the "leading term" of the chain complex of the original manifold. This idea rigorously formulated in Theorem 5.13.

Once the decomposition formula is established, the "only if" direction of Theorem 1.2 follows easily, see Theorem 5.14.

A refined doubling trick (Lemma 5.16) enables us to convert a taut sutured manifold into an admissible 3-manifold which is not a closed graph manifold. Combining this trick with the fiberedness criterion for admissible 3-manifolds (Theorem 1.2) and the decomposition formula (Theorem 1.3), we show that the universal L^2-torsion detects product sutured manifolds (Theorem 1.4).

1.2.5. *Applications of the universal L^2-torsion.* Section 6 is devoted to applications and concrete computations of the universal L^2-torsion. We begin by defining this invariant for homomorphisms between finitely generated free groups. An explicit computational formula is established in Proposition 6.2, which in particular proves Theorem 1.5. We then apply this formula to sutured manifolds modeled on 3-dimensional handlebodies, leading to criteria for detecting whether such a sutured manifold is taut or a product (Proposition 6.6). Explicit computations are carried out for the family of sutured manifolds obtained from cutting up the n-chain link complement along a Seifert surface (Example 6.8).

2. Algebraic preliminaries

2.1. **Hilbert spaces associated to groups.** Let G be a group. Consider the following Hilbert space

$$\ell^2(G) = \Big\{ \sum_{g\in G} c_g \cdot g \;\Big|\; c_g \in \mathbb{C},\ \sum_{g\in G} |c_g|^2 < \infty \Big\}$$

with inner product

$$\Big\langle \sum_{g\in G} c_g \cdot g, \sum_{g\in G} d_g \cdot g \Big\rangle = \sum_{g\in G} c_g \overline{d_g}.$$

This Hilbert space has a natural left and right isometric G-action by multiplications. Every G-invariant closed subspace V of $\ell^2(G)^n$ is called a *Hilbert $\mathcal{N}G$-module* and can be assigned the von-Neumann dimension $\dim_{\mathcal{N}G} V$ which takes real value in $[0, n]$.

Let $\mathcal{U}G$ be the set of all densely-defined, closed operators (possibly unbounded) on $\ell^2(G)$ that commutes with the left G-action. The composition and addition of two operators in $\mathcal{U}G$ is well-defined [Lü02, Section 8.1], hence $\mathcal{U}G$ forms a $\mathbb{C}$-algebra and is called the *affiliated algebra of G*. In particular, we have the natural inclusion

$$\mathbb{Z}G \subset \mathcal{U}G$$

where the integral group ring $\mathbb{Z}G$ embeds into $\mathcal{N}G$ by the right regular representation. Moreover, any matrix A over $\mathbb{Z}G$ of size $m \times n$ can be viewed as a G-invariant bounded operator $r_A : \ell^2(G)^m \to \ell^2(G)^n$ by right multiplication.

Definition 2.1 (Weak isomorphism)**.** A G-invariant bounded operator $f : \ell^2(G)^m \to \ell^2(G)^n$ is called a *weak isomorphism* if f is injective with dense image. Note that if f is a weak isomorphism then m and n must be equal. A square matrix A over $\mathbb{Z}G$ is called a *weak isomorphism* if the G-invariant bounded operator $r_A : \ell^2(G)^m \to \ell^2(G)^n$ is a weak isomorphism.

2.2. **Atiyah Conjecture and Linnell's skew field.**

Definition 2.2 (Atiyah Conjecture)**.** A torsion-free group G is said to satisfy the *Atiyah Conjecture* if for any matrix $A^{m\times n}$ over $\mathbb{Z}G$ the von Neumann dimension of $\ker(r_A)$ is an integer.

The Atiyah Conjecture has been verified for a large class of groups (c.f. [Kie20, Theorem 4.2]). We mark the following class of groups given by Linnell, which is large enough to include all 3-manifold groups.

Theorem 2.3 ([Lin93])**.** *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions by elementary amenable groups. Then any torsion-free group in $\mathcal{C}$ satisfies the Atiyah Conjecture.*

Theorem 2.4 ([KL24, Theorem 1.4])**.** *The fundamental group of any connected 3-manifold lies in $\mathcal{C}$.*

An alternative algebraic characterization of the Atiyah Conjecture is given by the affiliated algebra.

Definition 2.5 (Division closure)**.** Let R be a subring of a ring S. The *division closure* of R in S is the smallest subring of S containing R such that if an element of this subring is invertible in S, then it is also invertible in the subring. Let G be a group, define $\mathcal{D}_G$ to be the division closure of $\mathbb{Z}G$ in $\mathcal{U}G$.

Theorem 2.6 ([Lin93])**.** *A torsion-free group G satisfies the Atiyah Conjecture if and only if $\mathcal{D}_G$ is a skew field.*

Therefore, when G is a torsion-free group satisfying the Atiyah Conjecture, we will call $\mathcal{D}_G$ the *Linnell's skew field*. The following Proposition 2.7 gives some useful functorial properties of the Linnell's skew field.

Proposition 2.7 ([Kie20, Proposition 4.6])**.** *Let G be a torsion-free group satisfying the Atiyah Conjecture. Then the following statements hold.*

(1) *Every automorphism of the group G extends to an automorphism of $\mathcal{D}_G$.*
(2) *If K is a subgroup of G, then K satisfies the Atiyah Conjecture. Moreover, the natural embedding $\mathbb{Z}K \hookrightarrow \mathbb{Z}G$ extends to an embedding $\mathcal{D}_K \hookrightarrow \mathcal{D}_G$.*

2.3. **The K_1-group and Dieudonné determinant.** Let $\mathcal{F}$ be an skew field. For any positive integer n let $GL(n,\mathcal{F})$ be the group of invertible $(n\times n)$-matrices over $\mathcal{F}$. Identifying each $M \in GL(n,\mathcal{F})$ with the matrix

$$\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL(n+1,\mathcal{F})$$

we obtain inclusions

$$GL(1,\mathcal{F}) \subset GL(2,\mathcal{F}) \subset \cdots .$$

The union $GL(\mathcal{F}) = \bigcup_{n\geqslant 1} GL(n,R)$ is called the *infinite general linear group*. We remark that a classical result of Whitehead (see for example [Mil66]) says that the commutator subgroup $[GL(\mathcal{F}), GL(\mathcal{F})]$ is exactly the subgroup generated by all elementary matrices in $GL(\mathcal{F})$.

Definition 2.8 (Dieudonné determinant [Die43, Ros95])**.** The Dieudonné determinant is the unique map

$$\det : GL(\mathcal{F}) \to \mathcal{F}^\times/[\mathcal{F}^\times, \mathcal{F}^\times]$$

satisfying following the properties (a)–(c):

(a) The determinant is invariant under (left) elementary row operations. In other words, if $A \in GL(\mathcal{F})$ and A' is obtained from A by adding a (left-)multiple of a row to another row, then $\det A = \det A'$.
(b) If $A \in GL(\mathcal{F})$, and A' is obtained from A by (left-)multiplying one of the rows by $a \in \mathcal{F}$, then $\det A = \bar{a}\cdot \det A'$ where $\bar{a}$ is the image of a in $\mathcal{F}^\times/[\mathcal{F}^\times,\mathcal{F}^\times]$.
(c) The determinant of the identity matrix is 1.

The determinant also has the following additional properties (d)–(f).

(d) If $A, B \in GL(n,\mathcal{F})$, then $\det(AB) = \det A \cdot \det B$.
(e) If $A \in GL(\mathcal{F})$ and A' is obtained from A by interchanging two of its rows, then $\det A' = (-\bar{1}) \det A$.
(f) The determinant is invariant under taking transpose.

The Dieudonné determinant factors through the abelianization of $GL(\mathcal{F})$. The induced homomorphism turns out to be an isomorphism.

Lemma 2.9 ([Ros95, Corollary 2.2.6]). *For any skew field $\mathcal{F}$, the Dieudonné determinant induces a group isomorphism*

$$GL(\mathcal{F})/[GL(\mathcal{F}), GL(\mathcal{F})] \xrightarrow{\cong} \mathcal{F}^\times/[\mathcal{F}^\times, \mathcal{F}^\times].$$

The inverse map is given by viewing $\bar{a} \in \mathcal{F}^\times/[\mathcal{F}^\times, \mathcal{F}^\times]$ as an (1×1)-matrix (a), for all $a \in \mathcal{F}^\times$.

Definition 2.10 ($K_1(\mathcal{F})$ and $\widetilde{K}_1(\mathcal{F})$). The K_1-*group of* $\mathcal{F}$ is defined to be

$$K_1(\mathcal{F}) := F^\times/[\mathcal{F}^\times, \mathcal{F}^\times].$$

The *reduced* K_1-*group* $\widetilde{K}_1(\mathcal{F})$ is the quotient of $K_1(\mathcal{F})$ by the subgroup $\{[\pm 1]\}$.

We now apply these definitions to Linnell's skew field $\mathcal{D}_G$, where G is a torsion-free group satisfying the Atiyah Conjecture. In addition to the K_1-groups $K_1(\mathcal{D}_G)$ and $\widetilde{K}_1(\mathcal{D}_G)$ defined above, we introduce the following Whitehead group:

Definition 2.11 ($\mathrm{Wh}(\mathcal{D}_G)$). The group $\mathrm{Wh}(\mathcal{D}_G)$ is defined to be the quotient of $K_1(\mathcal{D}_G)$ by the subgroup $\{[\pm g] \mid g \in G\}$.

Definition 2.12. In the remainder of this paper we will frequently work with determinants and their image in the quotient groups $\widetilde{K}_1(\mathcal{D}_G), \mathrm{Wh}(\mathcal{D}_G)$. To this end, we introduce the homomorphisms

$$\det_r : GL(\mathcal{D}_G) \to \widetilde{K}_1(\mathcal{D}_G), \quad \det_w : GL(\mathcal{D}_G) \to \mathrm{Wh}(\mathcal{D}_G)$$

so that the following diagram commutes.

$$\begin{tikzcd}
 & & GL(\mathcal{D}_G) \arrow[dl, "\det"'] \arrow[d, "\det_r"] \arrow[dr, "\det_w"] & \\
\mathcal{D}_G^\times/[\mathcal{D}_G^\times, \mathcal{D}_G^\times] \arrow[r, equal] & K_1(\mathcal{D}_G) \arrow[r] & \widetilde{K}_1(\mathcal{D}_G) \arrow[r] & \mathrm{Wh}(\mathcal{D}_G)
\end{tikzcd}$$

3. Universal L^2-torsion

Let G be a torsion-free group satisfying the Atiyah conjecture and let $\mathcal{D}_G$ be the Linnell's skew field. The universal L^2-torsion defined in this section takes values in the abelian groups $\widetilde{K}_1(\mathcal{D}_G)$ for $\mathbb{Z}G$-chain complexes, and in $\mathrm{Wh}(\mathcal{D}_G)$ for finite CW-complexes with fundamental group G. Although our definition is more restrictive than the original one given in [FL17], the two definitions coincide when G is a 3-manifold group. A detailed discussion of the relationship between the two is provided in Section 3.1.1.

3.1. **Universal L^2-torsion of chain complexes.** A chain complex C_* is called a *finite based free $\mathbb{Z}G$-chain complex* if there exists $n \geqslant 0$ such that

$$C_* = (0 \to C_n \to \cdots \to C_1 \to C_0 \to 0),$$

where each C_k is a finitely generated free (left) $\mathbb{Z}G$-module equipped with a preferred (unordered) free $\mathbb{Z}G$-basis, and the boundary operators are $\mathbb{Z}G$-linear maps.

Definition 3.1. A finite based free $\mathbb{Z}G$-chain complex C_* is said to be L^2-*acyclic* if the chain complex $\ell^2(G) \otimes_{\mathbb{Z}G} C_*$ is weakly exact, i.e. the image $\operatorname{im} \partial_{k+1}^{(2)}$ is dense in $\ker \partial_k^{(2)}$ for all k, where $\partial_k^{(2)} : \ell^2(G) \otimes_{\mathbb{Z}G} C_k \to \ell^2(G) \otimes_{\mathbb{Z}G} C_{k-1}$ is the boundary operator.

We will not work with those analytic-flavored definitions but prefer the more algebraic-flavored ones given by the following Lemma 3.2.

Lemma 3.2 ([FL17, Lemma 2.21])**.** *A square matrix over $\mathbb{Z}G$ is a weak isomorphism if and only if it is invertible over $\mathcal{D}_G$. A finite based free $\mathbb{Z}G$-chain complex C_* is L^2-acyclic if and only if the chain complex $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*$ is exact as a chain complex of left $\mathcal{D}_G$-modules.*

It is classical to define the torsion of an exact chain complex of free modules, see for example [Mil66, Coh73, Tur01]. The situation here is particularly nice because $\mathcal{D}_G$ is a skew field and any module over a skew field is free. We adopt the definitions in [Tur01, Section 3].

Definition 3.3. Suppose V is a finitely generated left $\mathcal{D}_G$-module with $\dim V = k$. Pick two (unordered) bases $b = \{b_1, \dots, b_k\}$ and $c = \{c_1, \dots, c_k\}$. Then

$$b_i = \sum_{j=1}^{k} a_{ij} c_j, \quad i = 1, \dots, k,$$

where $(a_{ij})_{i,j=1,\dots,k}$ is a non-degenerate k-by-k matrix over $\mathcal{D}_G$. Define $[b/c]$ to be the determinant $\det_r(a_{ij}) \in \widetilde{K}_1(\mathcal{D}_G)$. This definition does not depend on the ordering of bases b, c.

Definition 3.4 (Universal L^2-torsion of chain complexes)**.** Let C_* be a finite based $\mathbb{Z}G$-chain complex of length n that is L^2-acyclic. Let c_i be the preferred basis of $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_i$ and let ∂ be the boundary homomorphism in degree i. Choose bases b_i for the free $\mathcal{D}_G$-modules $B_i := \operatorname{im} \partial_i$, and combine them to bases $b_i b_{i-1}$ of C_i. The *universal L^2-torsion of C_** is defined to be

$$\tau_u^{(2)}(C_*) := \prod_{i=0}^{n} [b_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \widetilde{K}_1(\mathcal{D}_G).$$

This value does not depend on the choice of the bases b_i.

An exact sequence $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ of finitely generated based free $\mathbb{Z}G$-modules is called *based exact*, if $i(b_0) \subset b_1$ and p maps $b_1 \setminus i(b_0)$ bijectively to b_2, where b_i is the preferred basis of M_i, $i = 0, 1, 2$. Similarly, an exact sequence $0 \to C_* \to D_* \to E_* \to 0$ of finite based free $\mathbb{Z}G$-chain complex is called *based exact* if

$$0 \to C_k \to D_k \to E_k \to 0$$

is based exact for all k. The following basic property can be found in [Tur01, Theorem 3.4].

Proposition 3.5. *The universal L^2-torsion for chain complexes have the following properties.*

1. *For any L^2-acyclic finite based free $\mathbb{Z}G$-chain complex*
$$C_* = (0 \to C_1 \xrightarrow{A} C_0 \to 0)$$
where C_0 and C_1 are isomorphic to $\mathbb{Z}G^r$ under the preferred basis and A is a square matrix A over $\mathbb{Z}G$. Then $\tau_u^{(2)}(C_*) = \det_r(A)^{-1} \in \widetilde{K}_1(\mathcal{D}_G)$.
2. *If C_*, C'_*, C''_* are finite based free $\mathbb{Z}G$-chain complexes and there is a based short exact sequence*
$$0 \to C'_* \to C_* \to C''_* \to 0.$$
If C'_* and C''_* are L^2-acyclic then C_* is L^2-acyclic and
$$\tau_u^{(2)}(C_*) = \tau_u^{(2)}(C'_*) \cdot \tau_u^{(2)}(C''_*) \in \widetilde{K}_1(\mathcal{D}_G).$$

Definition 3.6 (Dual $\mathbb{Z}G$-modules)**.** Recall that $\mathbb{Z}G$ is a ring with an involution $x \mapsto \bar{x}$ which sends $\sum n_g \cdot g$ to $\sum n_g \cdot g^{-1}$. Given any left $\mathbb{Z}G$-module A, the *dual module* A^* is defined to be $\mathrm{Hom}_{\mathbb{Z}G}(A, \mathbb{Z}G)$, considered as a left $\mathbb{Z}G$-module as follows: for each $x \in \mathbb{Z}G$ and $f : A \to \mathbb{Z}G$ define $xf : A \to \mathbb{Z}G$ by the formula $(xf)(y) = f(y) \cdot \bar{x}$, $\forall y \in A$.

If A is a free $\mathbb{Z}G$-module with basis a_i, then A^* is a free $\mathbb{Z}G$-module with basis a_i^*. Suppose $f : A \to B$ is a $\mathbb{Z}G$-linear map between two based free $\mathbb{Z}G$-modules represented by a matrix P under the given bases, then the dual map $f^* : B^* \to A^*$ is represented by the matrix P^*, the involution transpose of P.

The involution on $\mathbb{Z}G$ is compatible with taking adjoint in $\mathcal{U}G$. The Linnell's skew field $\mathcal{D}_G$, as the division closure of $\mathbb{Z}G$ in $\mathcal{U}G$, is closed under taking adjoint (otherwise $\mathcal{D}_G \cap \overline{\mathcal{D}_G}$ would be a smaller subring which is inversion-closed). Hence $\mathcal{D}_G$ admits a canonical involution which naturally extends the involution of $\mathbb{Z}G$. The following Proposition 3.7 is a classic property of torsion invariants and can be proved as in [Mil62].

Proposition 3.7. *If $C_* = (0 \to C_n \to \cdots \to C_0 \to 0)$ is a finite based free $\mathbb{Z}G$-chain complex which is L^2-acyclic. Then the dual chain complex C^* is L^2-acyclic and*

$$\overline{\tau_u^{(2)}(C^*)} = \tau_u^{(2)}(C_*)^{(-1)^{n+1}} \in \widetilde{K}_1(\mathcal{D}_G).$$

3.1.1. *Comparing Friedl–Lück's definition.* Our Definition 3.4 of the universal L^2-torsion slightly differs from that of [FL17]. We first recall the definition of the weak K_1-groups introduced by [FL17], which is an abelian group analogous to the classical K_1-group, but rather than requiring the matrices over $\mathbb{Z}G$ to be invertible, we only require that they are weak isomorphisms (recall Definition 2.1).

Definition 3.8 ($K_1^w(\mathbb{Z}G)$, $\widetilde{K}_1^w(\mathbb{Z}G)$ and $\mathrm{Wh}^w(G)$)**.** Suppose G is any group. The *weak K_1-group* $K_1^w(\mathbb{Z}G)$ is defined in terms of generators and relations as follows. Generators $[A]$ are given by square matrices A over $\mathbb{Z}G$ such that A is a weak isomorphism. There are two sets of relations:

(i) If A, B are square matrices over $\mathbb{Z}G$ of the same size such that A, B are weak isomorphisms, then
$$[AB] = [A] \cdot [B].$$
(ii) If A, B are square matrices over $\mathbb{Z}G$ of size n and m, respectively. Suppose A, B are weak isomorphisms and let C be any matrix over $\mathbb{Z}G$ of size $n \times m$. Then
$$\left[\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\right] = [A] \cdot [B] = [B] \cdot [A].$$

The *reduced weak K_1-group* $\widetilde{K}_1^w(\mathbb{Z}G)$ is defined to be the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup $\{[1], [-1]\}$. The *weak Whitehead group* $\mathrm{Wh}^w(G)$ is defined to be the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup $\{[\pm g] \mid g \in G\}$.

Proposition 3.9. *Suppose G is a torsion-free group which satisfies the Atiyah Conjecture.*

(1) *There are natural homomorphisms*
$$i_G : K_1^w(\mathbb{Z}G) \to K_1(\mathcal{D}_G), \quad \tilde{i}_G : \widetilde{K}_1^w(\mathbb{Z}G) \to \widetilde{K}_1(\mathcal{D}_G), \quad i_G^w : \mathrm{Wh}^w(G) \to \mathrm{Wh}(\mathcal{D}_G).$$
(2) *If G falls into the Linnell's class $\mathcal{C}$ (recall Theorem 2.3), then the three homomorphisms above are all isomorphisms.*

Proof. By Lemma 3.2 below, a square matrix A over $\mathbb{Z}G$ is a weak isomorphism if and only if it is invertible over $\mathcal{D}_G$. The homomorphism in (1) is given by viewing a weak isomorphism A over $\mathbb{Z}G$ as an invertible matrix over $\mathcal{D}_G$. (2) is the content of [LL17]. □

For any group G and any L^2-acyclic finite based $\mathbb{Z}G$-chain complex C_*, the universal L^2-torsion $\rho_u^{(2)}(C_*)$ defined in [FL17] takes values in the reduced weak K_1 group $\widetilde{K}_1^w(\mathbb{Z}G)$. Proposition 3.5 holds true for $\rho_u^{(2)}$ and moreover it characterizes the universal property of $\rho_u^{(2)}$. Therefore when G is torsion-free and satisfies the Atiyah Conjecture, the universal L^2-torsion $\tau_u^{(2)}(C_*)$ defined in the present paper is the image of $\rho_u^{(2)}(C_*)$ under the natural map

$$\tilde{i}_G : K_1^w(\mathbb{Z}G) \to K_1(\mathcal{D}_G).$$

If furthermore G lies in Linnell's class $\mathcal{C}$ then $\tilde{i}_G$ is a canonical isomorphism and the definitions of $\tau_u^{(2)}$ and $\rho_u^{(2)}$ are equivalent by Proposition 3.9. In particular, this includes all 3-manifold groups by Theorem 2.4.

3.2. **Universal L^2-torsion of CW-complexes.** Let X be a connected finite CW-complex whose fundamental group is torsion-free satisfying the Atiyah Conjecture and let $Y \subset X$ be a subcomplex. Let $p : \widehat{X} \to X$ be the universal covering of X, and let $\widehat{Y} := p^{-1}(Y)$ be the preimage. Then $\widehat{X}$ admits the induced CW-structure and $\widehat{Y}$ is a subcomplex of $\widehat{X}$. The natural left $\pi_1(X)$-action on $\widehat{X}$ gives rise to the left $\mathbb{Z}[\pi_1(X)]$-module structure on the cellular chain complex $C_*(\widehat{X}, \widehat{Y})$. By choosing a lift $\widehat{\sigma}$ for each cell σ in $X \setminus Y$, $C_*(\widehat{X}, \widehat{Y})$ becomes a finite based free $\mathbb{Z}[\pi_1(X)]$-chain complex. The following definition does not depend on the choice of the lifting of cells.

Definition 3.10 (Universal L^2-torsion of CW-complexes)**.** Let X be a finite connected CW-complex with fundamental group G and let Y be a subcomplex of X. The pair (X, Y) is called *L^2-acyclic* if the finite based free chain complex $C_*(\widehat{X}, \widehat{Y})$ is L^2-acyclic (c.f. Lemma 3.2). The *universal L^2-torsion*

$$\tau_u^{(2)}(X, Y) \in \mathrm{Wh}(\mathcal{D}_{\pi_1(X)}) \sqcup \{0\}$$

is defined as follows: if (X, Y) is L^2-acyclic, define $\tau_u^{(2)}(X, Y)$ to be the image of $\tau_u^{(2)}(C_*(\widehat{X}, \widehat{Y}))$ under the quotient $\widetilde{K}_1(\mathcal{D}_{\pi_1(X)}) \to \mathrm{Wh}(\mathcal{D}_{\pi_1(X)})$; if (X, Y) is not L^2-acyclic, define $\tau_u^{(2)}(X, Y) := 0$.

When X is not necessarily connected, we say (X, Y) is *L^2-acyclic* if for every component $X_i \in \pi_0(X)$ the pair $(X_i, X_i \cap Y)$ is L^2-acyclic. Furthermore, if the fundamental groups $\pi_1(X_i)$ satisfy the Atiyah conjecture, then we define

$$\mathrm{Wh}(\mathcal{D}_{\Pi(X)}) := \bigoplus_{X_i \in \pi_0(X)} \mathrm{Wh}(\mathcal{D}_{\pi_1(X_i)}),$$
$$\tau_u^{(2)}(X, Y) := (\tau_u^{(2)}(X_i, X_i \cap Y))_{X_i \in \pi_0(X)} \in \mathrm{Wh}(\mathcal{D}_{\Pi(X)}).$$

If $(X_i, X_i \cap Y)$ is not L^2-acyclic for some i, then define $\tau_u^{(2)}(X, Y) := 0$.

Assume that (X, Y) and (X', Y') are finite CW-pairs. We say a CW-mapping $f : (X', Y') \to (X, Y)$ is *π_1-injective*, if the restriction of f to each component of X' induces an injection on fundamental groups. In this case, there is a natural homomorphism

$$\iota_* : \mathrm{Wh}(\mathcal{D}_{\Pi(X')}) \sqcup \{0\} \to \mathrm{Wh}(\mathcal{D}_{\Pi(X)}) \sqcup \{0\}.$$

Define

$$\iota_* \tau_u^{(2)}(X', Y') \in \mathrm{Wh}(\mathcal{D}_{\Pi(X)}) \sqcup \{0\}$$

to be the image of $\tau_u^{(2)}(X, Y)$ under the homomorphism ι_*.

Theorem 3.11. *We record the fundamental properties of the universal L^2-torsion.*

(1) (Simple-homotopy invariance) *Suppose* (X, X_0) *and* (Y, Y_0) *are CW-pairs. Let* $f : (X, X_0) \to (Y, Y_0)$ *be a mapping such that* $f : X \to Y$ *and* $f|_{X_0} : X_0 \to Y_0$ *are simple-homotopy equivalences. Then* $\tau_u^{(2)}(Y, Y_0) = f_* \tau_u^{(2)}(X, X_0)$*. In particular,* (X, X_0) *is* L^2*-acyclic if and only if* (Y, Y_0) *is* L^2*-acyclic.*

(2) (Sum formula) *Let* $(U, V) = (X, C) \cup (Y, D)$ *where* (X, C)*,* (Y, D) *and* $(X \cap Y, C \cap D)$ *are* L^2*-acyclic sub-pairs that embeds* π_1*-injectively into* (U, V)*, then*

$$\tau_u^{(2)}(U, V) = (\iota_1)_* \tau_u^{(2)}(X, C) \cdot (\iota_2)_* \tau_u^{(2)}(Y, D) \cdot (\iota_3)_* \tau_u^{(2)}(X \cap Y, C \cap D)^{-1}$$

where ι_i*,* $i = 1, 2, 3$ *are the embeddings of the corresponding space pairs into* (M, N)*.*

(3) (Induction) *Let* $f : (X_0, Y_0) \subset (X, Y)$ *be a* π_1*-injective inclusion. Let* $\widehat{X}$ *be the universal cover of* X *and let* $\widehat{X_0}, \widehat{Y_0}$ *be the preimage of* X_0, Y_0 *in* $\widehat{X_0}$*. Then the finite based free* $\mathbb{Z}[\pi_1(X)]$*-chain complex* $C_*(\widehat{X_0}, \widehat{Y_0})$ *is* L^2*-acyclic if and only if* (X_0, Y_0) *is* L^2*-acyclic. Moreover, we have*

$$\tau_u^{(2)}(C_*(\widehat{X_0}, \widehat{Y_0})) = f_* \tau_u^{(2)}(X_0, Y_0).$$

(4) (Restriction) *Let* X *be a connected finite CW-complex and let* $\overline{X}$ *be a connected finite degree covering of* X*. Suppose that* $\pi_1(X) = G$ *and* $\pi_1(\overline{X}) = H$ *and denote by* $\operatorname{res}_H^G : \operatorname{Wh}(\mathcal{D}_G) \to \operatorname{Wh}(\mathcal{D}_H)$ *the homomorphism given by restriction (which will be discussed in details in Section 4.4). Let* $Y \subset X$ *be a subcomplex and let* $\overline{Y}$ *be its preimage in* $\overline{X}$*. Then*

$$\tau_u^{(2)}(\overline{X}, \overline{Y}) = \operatorname{res}_H^G \tau_u^{(2)}(X, Y).$$

Proof. The first one will be proved in Remark 3.17 after we introduced the notion of universal L^2-torsion of mappings. Properties (2)–(4) are natural generalization of [FL17, Theorem 3.5] to CW-pairs and the proof carry over without essential changes to the relative cases. □

3.3. Universal L^2-torsion of mappings.

Definition 3.12 (Universal L^2-torsion of mappings)**.** Let X, Y be finite CW-complexes. Given a cellular map $f : Y \to X$, form the mapping cylinder

$$M_f := ((Y \times I) \sqcup X)/\sim, \quad \text{where } (y, 0) \sim f(y) \text{ for all } y \in Y.$$

View $Y = Y \times \{1\}$ as a subcomplex of M_f. If the fundamental group of X is torsion-free satisfying the Atiyah Conjecture, then the *universal* L^2*-torsion of the mapping* f is defined to be

$$\tau_u^{(2)}(f) := \iota_* \tau_u^{(2)}(M_f, Y) \in \operatorname{Wh}(\mathcal{D}_{\Pi(X)}) \sqcup \{0\}$$

where $\iota : M_f \to X$ is the natural deformation retraction.

Definition 3.13 (L^2-homology equivalence)**.** A mapping $f : Y \to X$ is called an L^2-*homology equivalence* if $\tau_u^{(2)}(f) \neq 0$, or equivalently (M_f, Y) is L^2-acyclic.

Proposition 3.14. *Suppose the spaces* X, Y, Z *are finite CW-complexes whose fundamental groups are torsion-free satisfying the Atiyah Conjecture.*

(1) *If* (X, Y) *is a CW-pair with* $f : Y \to X$ *the inclusion map. Then* $\tau_u^{(2)}(X, Y) = \tau_u^{(2)}(f)$*.*
(2) *If* $f, g : X \to Z$ *are homotopic cellular maps. Then* $\tau_u^{(2)}(f) = \tau_u^{(2)}(g)$*.*
(3) *If* $f : X \to Z$ *is a simple-homotopy equivalence, then* $\tau_u^{(2)}(f) = 1$*.*
(4) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are* L^2*-homology equivalences. Suppose that* g *is* π_1*-injective, then* $g \circ f$ *is an* L^2*-homology equivalence with* $\tau_u^{(2)}(g \circ f) = g_* \tau_u^{(2)}(f) \cdot \tau_u^{(2)}(g)$*.*

Proof. Following [Coh73], we write $K \curvearrowright L$ if two finite CW-complexes K and L are related by a finite sequence of elementary collapses or expansions; if there is a common subcomplex K_0 that no cells are removed during the process, then we write $K \curvearrowright L$ rel K_0. In this case, it is clear that $\tau_u^{(2)}(K, K_0) = \iota_* \tau_u^{(2)}(L, K_0)$ where $\iota : L \to K$ is the natural homotopy equivalence.

For (1), there are elementary expansions $M_f \curvearrowright X \times I$ and elementary collapses $X \times I \curvearrowright X \times \{1\}$ relative to $Y = Y \times \{1\}$. Let $\iota : M_f \to X$ be the deformation retract, then $\tau_u^{(2)}(f) = \iota_* \tau_u^{(2)}(M_f, Y) = \tau_u^{(2)}(X, Y)$. (2) follows from the fact that $M_f \curvearrowright M_g$ rel X [Coh73, (5.5)]. For (3), if $f : X \to Z$ is a simple-homotopy equivalence then $M_f \curvearrowright X$ rel X [Coh73, (5.8)].

For the proof of (4) we need the following "L^2-excision" property.

Lemma 3.15 (Excision)**.** *If K, L and M are subcomplexes of the complex $K \cup L$ with $M = K \cap L$. Suppose the inclusion $i : K \hookrightarrow K \cup L$ is π_1-injective. Then $\tau_u^{(2)}(K \cup L, L) = i_* \tau_u^{(2)}(K, M)$.*

Proof. As in the proof of [Coh73, (20.3)], we may assume that L and $K \cup L$ are connected. Let $\widehat{K \cup L}$ be the universal covering of $K \cup L$ and let $\widehat{L}$, $\widehat{K}$ and $\widehat{M}$ be the preimage under the covering. Then there is an isomorphism of chain complexes $C_*(\widehat{K \cup L}, \widehat{L}) = C_*(\widehat{K}, \widehat{M})$ and hence $\tau_u^{(2)}(K \cup L, L) = \tau_u^{(2)}(C_*(\widehat{K}, \widehat{M})) = j_* \tau_u^{(2)}(K, M)$ where the second identity follows from the induction property (see Theorem 3.11). ☐

The proof of Proposition 3.14(4) proceeds as follows. Let M be the union of M_f and M_g along the identity map on Y. Then $M \curvearrowright M_{g \circ f}$ rel $X \cup Z$ by [Coh73, (5.6)]. There is a commutative diagram

$$\begin{tikzcd}
M_f \arrow[r, "\iota_f"] \arrow[d, hook, "i_1"'] & Y \arrow[r, hook, "i"] & M_g \arrow[d, "\iota_g"] \arrow[dll, hook', "i_2"'] \\
M \arrow[rr, "\iota"] & & Z
\end{tikzcd}$$

where i_1, i_2 are inclusions, ι, ι_1, ι_2 are natural deformation retracts. Then we have

$$\begin{aligned}
\tau_u^{(2)}(g \circ f) &= \iota_* \tau_u^{(2)}(M, X) \\
&= \iota_*(\tau_u^{(2)}(M, M_f) \cdot (i_1)_* \tau_u^{(2)}(M_f, X)), \quad \text{by sum formula} \\
&= \iota_*((i_2)_* \tau_u^{(2)}(M_g, Y) \cdot (i_1)_* \tau_u^{(2)}(M_f, X)), \quad \text{by excision} \\
&= (\iota_g)_* \tau_u^{(2)}(M_g, Y) \cdot g_*(\iota_f)_* \tau_u^{(2)}(M_f, X), \quad \text{note that } \iota_g \circ i = g \\
&= \tau_u^{(2)}(g) \cdot g_* \tau_u^{(2)}(f).
\end{aligned}$$

The proof is finished. ☐

The identity $\tau_u^{(2)}(g \circ f) = g_* \tau_u^{(2)}(f) \cdot \tau_u^{(2)}(g)$ is called the *multiplicativity* of the universal L^2-torsion. The conditions that f, g are L^2-homology equivalences and g is π_1-injective is in general necessary. But when one of the mappings is a simple homotopy equivalence then the conditions can be relaxed as follows.

Lemma 3.16. *Let X, Y, Z, W be finite CW-complexes and consider the chain of mappings $X \xrightarrow{h} Y \xrightarrow{f} Z \xrightarrow{g} W$. Suppose $f : Y \to Z$ is a simple homotopy equivalence. Then*

(1) *A mapping $h : X \to Y$ is an L^2-homology equivalence if and only if $f \circ h$ is an L^2-homology equivalence. Moreover $\tau_u^{(2)}(f \circ h) = f_* \tau_u^{(2)}(h)$.*

(2) *A mapping* $g : Z \to W$ *is an* L^2*-homology equivalence if and only if* $g \circ f$ *is an* L^2*-homology equivalence. Moreover* $\tau_u^{(2)}(g \circ f) = \tau_u^{(2)}(g)$.

Proof. The key observation is that a simple homotopy equivalence f admits an inverse f^{-1} which is also a simple homotopy equivalence. For (1), the forward direction follows from Proposition 3.14(4). For the inverse direction note that $h \simeq f^{-1} \circ (f \circ h)$. The identity $\tau_u^{(2)}(f \circ h) = f_* \tau_u^{(2)}(h)$ follows again from Proposition 3.14(4).

For (2), suppose g is an L^2-homology equivalence. Let M be the union of M_f and M_g along the identity map of Y. Then $M \curvearrowright M_{g\circ f}$ rel $X \cup Z$. The Excision Lemma 3.15 applied to $M_f \subset M$ and $i : M_g \subset M$ shows that (M, M_f) is L^2-acyclic and $\tau_u^{(2)}(M, M_f) = i_* \tau_u^{(2)}(M_g, Y)$. Since f is a simple homotopy equivalence we have $M_f \curvearrowright X$ rel X. Let $\iota : M \to Z$ be the deformation retract. Then

$$\tau_u^{(2)}(g \circ f) := \iota_* \tau_u^{(2)}(M, X) = \iota_* \tau_u^{(2)}(M, M_f) = (\iota \circ i)_* \tau_u^{(2)}(M_g, Y) = \tau_u^{(2)}(g).$$

Therefore $g \circ f$ is an L^2-homology equivalence and the identity $\tau_u^{(2)}(g \circ f) = \tau_u^{(2)}(g)$ holds. The inverse direction is proved the same way, noting that $g \simeq (g \circ f) \circ f^{-1}$. $\square$

Remark 3.17. As a corollary of Lemma 3.16, we prove the simple-homotopy invariance stated in Theorem 3.11. Let $f : (X, X_0) \to (Y, Y_0)$ be a mapping of CW-pairs such that $f : X \to Y$ and $f|_{X_0} : X_0 \to Y_0$ are simple-homotopy equivalences. Then we have the following commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, hook, "i_X"'] & Y_0 \arrow[d, hook, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then: (X, X_0) is L^2-acyclic $\Leftrightarrow$ i_X is an L^2-homology equivalence $\Leftrightarrow$ $f \circ i_X$ is an L^2-homology equivalence $\Leftrightarrow$ $i_Y \circ f_0$ is an L^2-homology equivalence $\Leftrightarrow$ i_Y is an L^2-homology equivalence $\Leftrightarrow$ (Y, Y_0) is L^2-acyclic. Moreover, we have

$$f_* \tau_u^{(2)}(X, X_0) = f_* \tau_u^{(2)}(i_X) = \tau_u^{(2)}(f \circ i_X) = \tau_u^{(2)}(i_Y \circ f_0) = \tau_u^{(2)}(i_Y) = \tau_u^{(2)}(Y, Y_0).$$

3.4. **Universal L^2-torsion of manifolds.** We define the universal L^2-torsion for smooth manifold pairs as follows. Recall that a smooth triangulation of a smooth manifold M is a homeomorphism from a simplicial complex to M whose restriction to each simplex is smooth.

Definition 3.18 (Universal L^2-torsion of manifold pairs)**.** Let M be a compact, smooth manifold, possibly with boundary, and let N be a compact, smooth submanifold of M. Suppose that there is a smooth triangulation of M such that N is a subcomplex of M. Then we use the triangulation to identify (M, N) with a CW-pair (X, Y) and define $\tau_u^{(2)}(M, N) := \tau_u^{(2)}(X, Y)$.

For the purpose of this paper, assume that either N is a zero-codimensional submanifold of ∂M, or the embedding $N \hookrightarrow M$ is proper (i.e. $N \cap \partial M = \partial N$). In these cases one can find a smooth triangulation of M such that N is a subcomplex of M (see [Mun66, Chapter 10]). Any two such triangulations have a common subdivision and are simple homotopy equivalent as CW-complexes [Whi40]. Therefore $\tau_u^{(2)}(M, N)$ is well-defined by simple homotopy invariance of the universal L^2-torsion (see Theorem 3.11).

Definition 3.19 (Universal L^2-torsion of mappings between manifolds)**.** Suppose $f : N \to M$ is a continuous mapping between compact smooth manifolds (possibly with boundaries) M, N. Choose any smooth triangulations of M, N and choose a simplicial mapping g homotopic to f.Define the *universal L^2-torsion of f* as

$$\tau_u^{(2)}(f) := \tau_u^{(2)}(g) \in \mathrm{Wh}(\mathcal{D}_{\Pi(M)}) \sqcup \{0\}.$$

It follows from and Proposition 3.14 and Lemma 3.16 that $\tau_u^{(2)}(f)$ does not depend on the choice of triangulations on M, N or the simplicial approximation g. Note that when $N \subset M$ is a smooth submanifold with f the inclusion map, then $\tau_u^{(2)}(f) = \tau_u^{(2)}(M, N)$.

3.5. **Methods for computations.** We state and prove the *matrix chain method* for computing the universal L^2-torsion of a chain complex which goes back to [Tur01, Theorem 2.2].

Let $C_* = (0 \to C_n \to \cdots \to C_1 \to C_0 \to 0)$ be a finite based free $\mathbb{Z}G$-chain complex and let $\partial_i : C_i \to C_{i-1}$ be the boundary operator. Suppose $d_i := \mathrm{rank}_{\mathbb{Z}G} C_i$ is the rank of the free module C_i. Then ∂_i is given by a matrix

$$A_i = (a^i_{jk})_{\substack{j=1,\ldots,d_i \\ k=1,\ldots,d_{i-1}}}, \quad a^i_{jk} \in \mathbb{Z}G.$$

Definition 3.20. A *matrix chain* for C_* is a collection of finite sets $\mathcal{A} = \{\mathcal{I}_0, \ldots, \mathcal{I}_n\}$ where $\mathcal{I}_i \subset \{1, \ldots, d_i\}$ and $\mathcal{I}_n = \emptyset$. Let B_i be the submatrix of A_i formed by the entries a^i_{jk} with $j \notin \mathcal{I}_i$ and $k \in \mathcal{I}_{i-1}$. Then $\{B_i\}$ are called the matrices associated to the matrix chain.

A matrix chain is called *non-degenerate* if each associated matrix is a square matrix and is invertible over $\mathcal{D}_G$.

Theorem 3.21. *A finite based free $\mathbb{Z}G$-chain complex C_* is L^2-acyclic if and only if there exists an non-degenerate matrix chain $\mathcal{A} = \{\mathcal{I}_0, \ldots, \mathcal{I}_n\}$ for C_*. If this happens, then*

$$\tau_u^{(2)}(C_*) = \prod_{i=1}^{n} \det{}_r(B_i)^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G)$$

where B_i are the matrices associated to the matrix chain.

The proof is a generalization of the idea of [DFL16, Lemma 3.1] to larger chain complexes and is well-known to experts. We give an sketched proof here.

Proof Sketch. Suppose C_* is L^2-acyclic. Then A_n is injective over $\mathcal{D}_G$ and there is a $d_n \times d_n$ submatrix B_n of A_n such that B_n is non-singular. We set $\mathcal{I}_{n-1} \subset \{1, \ldots, d_{n-1}\}$ to be the set of indices of the columns of B_i. Write $C_{n-1} = C'_{n-1} \oplus C''_{n-1}$ where C'_{n-1} corresponds to the set of indices $\mathcal{I}_{n-1}$ and C''_{n-1} corresponds to the remaining indices. Then $A_{n-1} : C_{n-1} \to C_{n-2}$ is trivial on C'_{n-1} and we obtain the following commuting diagram where each row and column is exact:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & C_n \arrow[r, "\cong"] \arrow[d, "\cong"', "B_n"] & C_n \arrow[r] \arrow[d, "A_n"] & 0 \arrow[d] & \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i"] \arrow[d] & C_{n-1} \arrow[r, "p"] \arrow[d, "A_{n-1}"] & C''_{n-1} \arrow[r] \arrow[d, "A''_{n-1}"] & 0 \\
& 0 \arrow[r] & C_{n-2} \arrow[r, "\cong"] \arrow[d, "A_{n-2}"] & C_{n-2} \arrow[r] \arrow[d, "A_{n-2}"] & 0 \\
& & \vdots & \vdots &
\end{tikzcd}$$

Repeat this procedure to A''_{n-1} and the vertical sequences on the right and in the end we find matrices $B_n, \ldots, B_1$ which gives the non-degenerate matrix chain for C_*.

For the backward direction, suppose $\mathcal{A} = \{\mathcal{I}_0, \ldots, \mathcal{I}_n\}$ is a non-degenerate matrix chain for C_*. Write $C_{n-1} = C'_{n-1} \oplus C''_{n-1}$ where C'_{n-1} corresponds to the set of indices $\mathcal{I}_{n-1}$ and C''_{n-1} corresponds to the remaining indices. Then we again obtain the commutative diagram as above. If we set $C'_* = (0 \to C_n \xrightarrow{B_n} C'_{n-1} \to 0)$ and

$$C''_* = (0 \longrightarrow C''_{n-1} \xrightarrow{A''_{n-1}} C_{n-2} \xrightarrow{A_{n-2}} \cdots \longrightarrow C_1 \xrightarrow{A_1} C_0 \longrightarrow 0).$$

Then $\{\mathcal{I}_0, \ldots, \mathcal{I}_{n-1}\}$ is a non-degenerate matrix chain for C''_* and we have the short exact sequence of based $\mathbb{Z}G$-chain complex

$$0 \to C'_* \to C_* \to C''_* \to 0.$$

An induction argument on n allows us to assume that C''_* is L^2-acyclic and

$$\tau_u^{(2)}(C''_*) = \prod_{i=1}^{n-1} \det_r(B_i)^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G).$$

Then by Proposition 3.5 we have

$$\tau_u^{(2)}(C_*) = \tau_u^{(2)}(C'_*) \cdot \tau_u^{(2)}(C''_*) = \prod_{i=1}^{n} \det_r(B_i)^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G).$$

□

Proposition 3.22. *We calculate the universal L^2-torsion of some manifold pairs.*

(1) *Let N be a compact smooth manifold whose fundamental group satisfies the Atiyah Conjecture. then for any $s \in [0,1]$ we have*

$$\tau_u^{(2)}(N \times I, N \times \{s\}) = 1 \in \mathrm{Wh}(\mathcal{D}_{\Pi(N)}).$$

(2) *Let S^1 be the circle with fundamental group $\pi_1(S^1) = \langle t \rangle$. Then $\tau_u^{(2)}(S^1) = [t - 1]^{-1} \in \mathrm{Wh}(\mathcal{D}_{\mathbb{Z}})$.*

(3) *Let T^2 be the torus then*

$$\tau_u^{(2)}(T^2) = 1 \in \mathrm{Wh}(\mathcal{D}_{\mathbb{Z}^2}).$$

Proof. For (1), let $f : N \times \{s\} \to N \times I$ be the inclusion, then f is a simple-homotopy equivalence and $\tau_u^{(2)}(N \times I, N \times \{s\}) = \tau_u^{(2)}(f) = 1$ by Proposition 3.14.

For (2), a CW-structure of S^1 is given by a 0-cell p and an 1-cell e. Let t be the generator of $\pi_1(S^1)$. By choosing appropriate liftings $\hat{p}$ and $\hat{e}$ we have

$$C_*(\widehat{S}^1) = (0 \to \mathbb{Z}[t^\pm] \cdot \langle \hat{e} \rangle \xrightarrow{(t-1)} \mathbb{Z}[t^\pm] \cdot \langle \hat{p} \rangle \to 0)$$

and hence $\tau_u^{(2)}(S^1) = [t-1]^{-1}$.

For (3), consider the CW-structure for T^2 given by identifying pairs of sides of a square. Let p be the 0-cell, e_1, e_2 be the 1-cells and σ be the 2-cell. Then the boundary of σ is a loop $e_1 e_2 e_1^{-1} e_2^{-1}$. Suppose the loop e_1, e_2 represents $t_1, t_2 \in \pi_1(T^2)$, respectively. Then by choosing appropriate liftings of the cells we have

$$C_*(\widehat{T}^2) = (0 \to \mathbb{Z}[t_1^\pm, t_2^\pm] \cdot \langle \hat{\sigma} \rangle \xrightarrow{\begin{pmatrix} 1 - t_2 & t_1 - 1 \end{pmatrix}} \mathbb{Z}[t_1^\pm, t_2^\pm] \cdot \langle \hat{e}_1, \hat{e}_2 \rangle \xrightarrow{\begin{pmatrix} t_1 - 1 \\ t_2 - 1 \end{pmatrix}} \mathbb{Z}[t_1^\pm, t_2^\pm] \cdot \langle \hat{p} \rangle \to 0).$$

A matrix chain is given by $B_2 = (1 - t_2)$ and $B_1 = (t_2 - 1)$, hence $\tau_u^{(2)}(T^2) = [1 - t_2] \cdot [t_2 - 1]^{-1} = 1$. □

4. Leading term map, restriction map and polytope map

4.1. Ore localization.

Definition 4.1 (Ore localization)**.** Let R be a ring with unit and let $S \subset R$ be a multiplicatively closed subset. The pair (R, S) satisfies the (right) Ore condition if the following two conditions hold:

(1) for any $(r, s) \in R \times S$ there exists $(r', s') \in R \times S$ such that $rs' = sr'$, and
(2) for any $r \in R$ and $s \in S$ with $sr = 0$, there is $t \in S$ with $rt = 0$.

If (R, S) satisfies the Ore condition, define an equivalence relation on $R \times S$

$$(r, s) \sim (rx, sx) \quad \text{whenever} \quad x \in R,\ sx \in S.$$

The quotient set $R \times S/\sim$ is denoted by RS^{-1}. Define a ring structure on RS^{-1} as follows. Given two representatives $(r, s), (r', s') \in RS^{-1}$, we can find $c \in R,\ d \in S$ with $sc = s'd \in S$ and define

$$(r, s) + (r', s') = (rc + r'd, sc).$$

We can find $e \in R,\ f \in S$ with $se = r'f$ and define

$$(r, s) \cdot (r', s') = (re, s'f).$$

The ring RS^{-1} is called the Ore localization of R at S.

Intuitively, a pair $(r, s) \in RS^{-1}$ is understood as a formal fraction rs^{-1}. The Ore condition (1) can be remembered as whenever there is a left (=wrong way) fraction $s^{-1}r$ then there is a right fraction $r'(s')^{-1}$ such that $rs' = sr'$. Condition (2) is automatically satisfied if S contains no zero divisors. In this paper, we only need to deal with Ore localizations of the following simple form:

Lemma 4.2 ([Coh95, Corollary 1.3.3])**.** *Let R be an integral domain such that $aR \cap bR \neq \{0\}$ for all $a, b \in R^\times$. Then $(R, R^\times)$ satisfies the Ore condition. Moreover, the Ore localization of R at $R^\times$ is a field K and the natural homomorphism $\lambda : R \to K$ is an embedding.*

An integral domain R satisfying the condition in Lemma 4.2 is called an Ore domain. The field K is called the field of fraction of R.

4.2. **Crossed products.** Assume that G is a torsion-free group which satisfies the Atiyah conjecture. Given any short exact sequence of groups

$$1 \to K \to G \xrightarrow{\nu} H \to 1.$$

Then K also satisfies the Atiyah conjecture by Proposition 2.7. Denote by $\mathcal{D}_K$ and $\mathcal{D}_G$ the Linnell's skew fields of K and G, respectively.

Choose a section $s : H \to G$ for the epimorphism ν such that $\nu \circ s$ is the identity. We do not require that s be a group homomorphism. Consider the following subset of $\mathcal{D}_G$:

$$\mathcal{D}_K *_s H := \left\{ \sum_{h \in H} x_h \cdot s(h) \in \mathcal{D}_G \;\middle|\; x_h \in \mathcal{D}_K,\ x_h = 0 \text{ for all but finitely many } h \in H \right\}.$$

This set contains the zero element 0, the identity element $1 = s(1_H)^{-1} \cdot s(1_H)$ and is closed under addition. Moreover, it is also closed under multiplication, since

$$\begin{aligned}\Big(\sum_{h \in H} x_h \cdot s(h)\Big) \cdot \Big(\sum_{h \in H} y_h \cdot s(h)\Big) &= \sum_{h_1, h_2 \in H} x_{h_1} \cdot s(h_1) \cdot y_{h_2} \cdot s(h_2) \\ &= \sum_{h_1, h_2 \in H} \underbrace{x_{h_1} s(h_1) y_{h_2} s(h_1)^{-1}}_{\in \mathcal{D}_K} \cdot \underbrace{s(h_1) s(h_2) s(h_1 h_2)^{-1}}_{\in K} \cdot s(h_1 h_2).\end{aligned}$$

Recall that the group automorphism of conjugation by $s(h_1)$ in K extends to an automorphism of $\mathcal{D}_K$ by Proposition 2.7, so $s(h_1)y_{h_2}s(h_1)^{-1} \in \mathcal{D}_K$. It follows that $\mathcal{D}_K *_s H$ is a subring of the field $\mathcal{D}_G$.

Proposition 4.3. *With notations as above, we have the following properties.*

(1) *An element $\sum_{h\in H} x_h \cdot s(h)$ of $\mathcal{D}_K *_s H$ is zero if and only if $x_h = 0$ for all $h \in H$.*
(2) *Given another section $s' : H \to G$, then*
$$\sum_{h\in H} x_h \cdot s(h) = \sum_{h\in H} y_h \cdot s'(h)$$
if and only if $y_h = x_h s(h)s'(h)^{-1}$ for all $h \in H$.

Proof. The first statement is a consequence of [Lüc02, Lemma 10.57]. The second statement follows from the previous one. □

As a corollary, the subring $\mathcal{D}_K *_s H \subset \mathcal{D}_G$ does not depend on the choice of the section. We call this subring the *crossed product* of $\mathcal{D}_K$ and H, denoted by $\mathcal{D}_K * H$. It is clear that $\mathcal{D}_K * H$ is an integral domain since it embeds in the field $\mathcal{D}_G$. In some cases, the relation between $\mathcal{D}_G$ and its subring $\mathcal{D}_K * H$ is surprisingly simple.

Proposition 4.4. *Let $1 \to K \to G \to H \to 1$ be an extension of groups.*

(1) *Suppose H is a finite group, then $\mathcal{D}_G = \mathcal{D}_K * H$.*
(2) *Suppose H is virtually a finitely generated abelian group, then the integral domain $\mathcal{D}_K * H$ is an Ore domain whose field of fraction agrees with $\mathcal{D}_G$.*

Proof. These two statements are Lemma 10.59 and Lemma 10.69 of [Lüc02], respectively. □

4.3. **The leading term map.** Let G be a finitely generated torsion-free group which satisfies the Atiyah Conjecture. Let $\nu : G \to H_1(G)_f$ be the natural quotient map to the free abelianization group $H_1(G)_f$, then we have the short exact sequence
$$1 \to K \to G \xrightarrow{\nu} H_1(G)_f \to 1.$$
Fix a section $s : H_1(G)_f \to G$ and let $\phi \in H^1(G;\mathbb{R})$ be a real cohomology class.

Definition 4.5. Given any nonzero element $u \in (\mathcal{D}_K * H_1(G)_f)^\times$, we write
$$u = \sum_{h\in H_1(G)_f} x_h \cdot s(h) \in \mathcal{D}_K * H_1(G)_f.$$
The *support of* u is defined to be the set $\operatorname{supp}(u) := \{h \in H_1(G)_f \mid x_h \neq 0\}$, this is a finite subset of $H_1(G)_f$ and does not depend on the choice of section by Proposition 4.3. Define $\delta_\phi(u)$ to be the minimal value of $\phi(h)$ for all $h \in \operatorname{supp}(u)$. Define
$$L_\phi(u) := \sum_{\substack{h\in \operatorname{supp}(u),\\ \phi(h)=\delta_\phi(z)}} x_h \cdot s(h).$$
This element is nonzero and lies in $(\mathcal{D}_K * H_1(G)_f)^\times$.

Lemma 4.6. *The definition of $\delta_\phi(u)$ and $L_\phi(u)$ do not depend on the choice of section s. Moreover, we have*
$$\delta_\phi(u_1u_2) = \delta_\phi(u_1) + \delta_\phi(u_2),$$
$$L_\phi(u_1u_2) = L_\phi(u_1) \cdot L_\phi(u_2)$$
for all $u_1, u_2 \in (\mathcal{D}_K * H_1(G)_f)^\times$. Hence
$$\delta_\phi : (\mathcal{D}_K * H_1(G)_f)^\times \to \mathbb{R},$$
$$L_\phi : (\mathcal{D}_K * H_1(G)_f)^\times \to (\mathcal{D}_K * H_1(G)_f)^\times$$

are well-defined group homomorphisms.

Proof. Let $u = \sum_h x_h \cdot s(h)$ and s' be another section, then by Proposition 4.3 $u = \sum_h y_h \cdot s'(h)$ where $y_h = x_h s(h)s'(h)^{-1}$. It follows that $\delta_\phi(u)$ and $L_\phi(u)$ do not depend on the choice of section. The terms of u_1u_2 with minimal ϕ-value exactly comes from the multiplication of that of u_1 and u_2. This explains the homomorphism. $\square$

Recall that G is finitely generated and $\mathcal{D}_G$ is the field of fraction of the subring $\mathcal{D}_K * H_1(G)_f$ by Proposition 4.4.

Definition 4.7 (Leading term map L_ϕ)**.** The group homomorphisms δ_ϕ and L_ϕ extend to group homomorphisms

$$\delta_\phi : \mathcal{D}_G^\times \to \mathbb{R}, \quad \delta_\phi(uv^{-1}) := \delta_\phi(u) - \delta_\phi(v),$$
$$L_\phi : \mathcal{D}_G^\times \to \mathcal{D}_G^\times, \quad L_\phi(uv^{-1}) := L_\phi(u)L_\phi(v)^{-1}$$

for all $u, v \in (\mathcal{D}_K * H_1(G)_f)^\times$. It is convenient to set $\delta_\phi(0) = +\infty$ and $L_\phi(0) = 0$. Then we have

$$\delta_\phi(z_1z_2) = \delta_\phi(z_1) + \delta_\phi(z_2), \quad L_\phi(z_1z_2) = L_\phi(z_1) \cdot L_\phi(z_2)$$

for all $z_1, z_2 \in \mathcal{D}_G$.

Proof. Well-definedness needs proving. Suppose $z \in \mathcal{D}_G^\times$ can be expressed as $z = u_1v_1^{-1} = u_2v_2^{-1}$, then there exists $w_1, w_2 \in (\mathcal{D}_K * H_1(G)_f)^\times$ such that $u_1w_1 = u_2w_2$, $v_1w_1 = v_2w_2$. Hence

$$\begin{aligned} L_\phi(u_1)L_\phi(v_1)^{-1} &= L_\phi(u_1)L_\phi(w_1)L_\phi(w_1)^{-1}L_\phi(v_1)^{-1} \\ &= L_\phi(u_1w_1)L_\phi(v_1w_1)^{-1} \\ &= L_\phi(u_2w_2)L_\phi(v_2w_2)^{-1} \\ &= L_\phi(u_2)L_\phi(v_2)^{-1}. \end{aligned}$$

To verify that L_ϕ is a homomorphism, let $z_1, z_2 \in \mathcal{D}_G^\times$. By the Ore condition, we can arrange that $z_1 = u_1w^{-1}$ and $z_2 = wu_2^{-1}$ for $u_1, u_2, w \in (\mathcal{D}_K * H_1(G)_f)^\times$, so

$$L_\phi(z_1)L_\phi(z_2) = L_\phi(u_1)L_\phi(w)^{-1} \cdot L_\phi(w)L_\phi(u_2)^{-1} = L_\phi(u_1)L_\phi(u_2)^{-1} = L_\phi(z_1z_2).$$

The statements for δ_ϕ can be proved similarly. $\square$

Here are some basic facts about the mappings δ_ϕ, L_ϕ, especially about their properties under additions in $\mathcal{D}_G$. Most of the properties clearly holds true in the subring $\mathcal{D}_K * H_1(G)_f$ and it is routine to verify them in its field of fraction $\mathcal{D}_G$.

Proposition 4.8. *Let G be a finitely generated torsion-free group which satisfies the Atiyah Conjecture. Let $\phi \in H^1(G;\mathbb{R})$ be any real cohomology class and $\delta_\phi : \mathcal{D}_G \to \mathbb{R}$, $L_\phi : \mathcal{D}_G \to \mathcal{D}_G$ be as in Definition 4.7. Suppose $z, z_1, \ldots, z_n \in \mathcal{D}_G$, then:*

(1) $\delta_{r\phi}(z) = r \cdot \delta_\phi(z)$, $L_{r\phi}(z) = L_\phi(z)$ *for all* $r \in \mathbb{R}_+$.
(2) $\delta_\phi(cz) = \delta_\phi(z)$, $L_\phi(cz) = c \cdot L_\phi(z)$ *for all* $c \in \mathbb{Q} \setminus \{0\}$.
(3) $\delta_\phi(L_\phi(z)) = \delta_\phi(z)$, $L_\phi(L_\phi(z)) = L_\phi(z)$.
(4) *If* $L_\phi(z_1) = L_\phi(z_2)$ *nonzero, then* $\delta_\phi(z_1) = \delta_\phi(z_2) < \delta_\phi(z_1 - z_2)$.
(5) $\delta_\phi(z_1 + z_2) \geqslant \min\{\delta_\phi(z_1), \delta_\phi(z_2)\}$. *If* $\delta_\phi(z_1) < \delta_\phi(z_2)$, *then*

$$\delta_\phi(z_1 + z_2) = \delta_\phi(z_1), \quad L_\phi(z_1 + z_2) = L_\phi(z_1).$$

(6) *If* $\delta_\phi(z_1) = \cdots = \delta_\phi(z_n) =: \delta$ *and* $\sum_{k=1}^n L_\phi(z_k) \neq 0$, *then*

$$\delta_\phi\Big(\sum_{k=1}^n z_k\Big) = \delta, \quad L_\phi\Big(\sum_{k=1}^n z_k\Big) = \sum_{k=1}^n L_\phi(z_k).$$

(7) *Given $z \in \mathcal{D}_G$. For any open neighborhood $U \subset H^1(G;\mathbb{R})$ of ϕ, there is a rational cohomology class $\psi \in U$ such that $L_\psi(z) = L_\phi(z)$.*

(8) *Suppose $L \subset G$ is a finitely generated subgroup and $\mathcal{D}_L \subset \mathcal{D}_G$ is the induced inclusion. Denote by $\phi|_L : L \to \mathbb{R}$ the restriction of ϕ to L. Then the mappings*

$$\delta_{\phi|_L} : \mathcal{D}_L \to \mathbb{R}, \quad L_{\phi|_L} : \mathcal{D}_L \to \mathcal{D}_L$$

are exactly the restrictions of δ_ϕ and L_ϕ to $\mathcal{D}_L$.

Proof. For (1)–(3), the statements hold in $\mathcal{D}_K * H_1(G)_f$ and directly extends to $\mathcal{D}_G$ by definition.

For (4), assume that $z_1 = u_1 w^{-1}$, $z_2 = u_2 w^{-1}$ for $u_1, u_2, w \in \mathcal{D}_K * H_1(G)_f$, then $L_\phi(z_1) = L_\phi(z_2)$ implies that $L_\phi(u_1) = L_\phi(u_2)$. It follows that $\delta_\phi(u_1) = \delta_\phi(u_2) < \delta_\phi(u_1 - u_2)$. Therefore $\delta_\phi(z_1) = \delta_\phi(z_2) < \delta_\phi(z_1 - z_2)$.

For (5), assume that $z_1 = u_1 w^{-1}$, $z_2 = u_2 w^{-1}$ for $u_1, u_2, w \in \mathcal{D}_K * H_1(G)_f$. Since it is clear that $\delta_\phi(u_1 + u_2) \geqslant \min\{\delta_\phi(u_1), \delta_\phi(u_2)\}$, then $\delta_\phi(z_1 + z_2) = \delta_\phi(u_1 + u_2) - \delta_\phi(w) \geqslant \min\{\delta_\phi(u_1), \delta_\phi(u_2)\} - \delta_\phi(w) = \min\{\delta_\phi(z_1), \delta_\phi(z_2)\}$. If $\delta_\phi(z_1) < \delta_\phi(z_2)$, then $\delta_\phi(u_1) < \delta_\phi(u_2)$ and $L_\phi(u_1 + u_2) = L_\phi(u_1)$. It follows that $\delta_\phi(z_1 + z_2) = \delta_\phi(z_1)$ and $L_\phi(z_1 + z_2) = L_\phi(z_1)$.

For (6), assume that $z_i = u_i w^{-1}$ for $u_i, w \in \mathcal{D}_K * H_1(G)_f$, $i = 1, \ldots, n$. By assumption we have $\delta_\phi(u_i) = \delta + \delta_\phi(w)$ and $\sum_{i=1}^k L_\phi(u_i) \neq 0$. It follows that $\delta_\phi(\sum_{i=1}^k u_i) = \delta + \delta_\phi(w)$ and $L_\phi(\sum_{i=1}^k u_i) = \sum_{i=1}^k L_\phi(u_i)$. Hence $\delta_\phi(\sum_{i=1}^k z_i) = \delta$ and $L_\phi(\sum_{i=1}^k z_i) = \sum_{i=1}^k L_\phi(z_i)$.

For (7), assume $z = uv^{-1}$ with $u, v \in \mathcal{D}_K * H_1(G)_f$. Write $u = \sum_{h \in H_1(G)_f} x_h \cdot s(h)$ for a section s. Given another cohomology class $\psi \in H^1(G;\mathbb{R})$, then $L_\phi(z) = L_\psi(z)$ if the following two conditions hold:

- for all $h, h' \in \mathrm{supp}(u) \cup \mathrm{supp}(v)$, $\psi(h - h') < 0$ whenever $\phi(h - h') < 0$;
- for all $h, h' \in \mathrm{supp}(u) \cup \mathrm{supp}(v)$, $\psi(h - h') = 0$ whenever $\phi(h - h') = 0$.

The domain Ω of such ψ is the intersection of finitely many closed hyperplanes and open half spaces of $H^1(G;\mathbb{R})$, each given by an integral linear equation. Since $\phi \in \Omega$, given any open neighborhood $U \ni \phi$ there are rational classes in $U \cap \Omega$.

For (8), consider the short exact sequence $1 \to K' \to L \to H_1(L)_f \to 1$. There is a commutative diagram

$$\begin{tikzcd} \mathcal{D}_{K'} * H_1(L)_f \arrow[r, hook] \arrow[d, hook] & \mathcal{D}_K * H_1(G)_f \arrow[d, hook] \\ \mathcal{D}_L \arrow[r, hook] & \mathcal{D}_G \end{tikzcd}$$

and $\mathcal{D}_L$ is the field of fraction of $\mathcal{D}_{K'} * H_1(L)_f$. Choose any nonzero $u \in \mathcal{D}_{K'} * H_1(L)_f$ and write $u = \sum_{h \in H_1(L)_f} x_h \cdot s(h)$, $x_h \in \mathcal{D}_{K'}$ for a section $s : H_1(L)_f \to L$. By definition $\delta_{\phi|_L}(u) = \min\{\phi(h) \mid h \in H_1(L)_f,\ x_h \neq 0\}$. Write $\delta := \delta_{\phi|_L}(u)$. Decompose

$$u = \sum_{\substack{h \in H_1(L)_f, \\ \phi|_L(h) = \delta}} x_h \cdot s(h) + \sum_{\substack{h \in H_1(L)_f, \\ \phi|_L(h) > \delta}} x_h \cdot s(h) =: u_1 + u_2.$$

Then by definition $L_{\phi|_L}(u) = u_1$ is nonzero. We want to show

$$\delta_\phi(u) = \delta, \quad L_\phi(u) = u_1.$$

This does not directly follow from the definition of δ_ϕ and L_ϕ since $s : H_1(L)_f \to L$ is not a section of $G \to H_1(G)_f$. We choose to argue as follows. By (6) applied to u_1 we know that

$$\delta_\phi(u_1) = \delta, \quad L_\phi(u_1) = u_1.$$

By (5) applied to u_2 we know that

$$\delta_\phi(u_2) \geqslant \min\{\delta_\phi(x_h \cdot s(h)) \mid h \in H_1(L)_f,\ \phi|_L(h) > \delta\} > \delta,$$

Again by (5) applied to $u = u_1 + u_2$ we know that $\delta_\phi(u) = \delta_\phi(u_1) = \delta$ and $L_\phi(u) = L_\phi(u_1) = u_1$. Hence $\delta_\phi(u) = \delta_{\phi|_L}(u)$ and $L_\phi(u) = L_{\phi|_L}(u)$ for all nonzero $u \in \mathcal{D}_{K'} * H_1(L)_f$. Passing to the field of fraction we have $\delta_\phi(z) = \delta_{\phi|_L}(z)$ and $L_\phi(z) = L_{\phi|_L}(z)$ for all $z \in \mathcal{D}_L$. $\square$

Definition 4.9. An element $z \in \mathcal{D}_G$ is called *ϕ-pure* if $L_\phi(z) = z$.

Lemma 4.10. *Here are some properties of the ϕ-pure elements.*

1. *Elements of $\mathbb{Z}[\ker\phi] \subset \mathcal{D}_G$ are ϕ-pure; elements of $G \subset \mathcal{D}_G$ are ϕ-pure.*
2. *The product of two ϕ-pure elements is ϕ-pure.*
3. *Given any nonzero $z \in \mathcal{D}_G$, then $L_\phi(z)$ is the unique element $w \in \mathcal{D}_G$ such that w is ϕ-pure and $\delta_\phi(z - w) > \delta_\phi(z)$.*
4. *Suppose $z_1, \dots, z_n$ are ϕ-pure elements with $\delta_\phi(z_1) = \dots = \delta_\phi(z_1) = \delta$, then the sum $\sum_{i=1}^n z_i$ is also ϕ-pure.*

Proof. The properties (1) and (2) follow from the definition of L_ϕ.

For (3), it follows from Proposition 4.8(3)–(4) that $L_\phi(z)$ is ϕ-pure and $\delta_\phi(z - L_\phi(z)) > \delta_\phi(z)$. On the other hand, if a ϕ-pure element w satisfies $\delta_\phi(z - w) > \delta_\phi(z)$, then by Proposition 4.8(4)

$$w = L_\phi(w) = L_\phi(z - (z - w)) = L_\phi(z).$$

Finally, (4) is a consequence of Proposition 4.8(6). $\square$

Remark 4.11. Given any $\phi \in H^1(G;\mathbb{R})$. The homomorphism $\delta_\phi : \mathcal{D}_G^\times \to \mathbb{R}$ induces a well-defined homomorphism $\delta_\phi : \Lambda \to \mathbb{R}$ for $\Lambda = K_1(\mathcal{D}_G)$ and $\widetilde{K}_1(\mathcal{D}_G)$. The homomorphism $L_\phi : \mathcal{D}_G^\times \to \mathcal{D}_G^\times$ induces a well-defined homomorphism $L_\phi : \Lambda \to \Lambda$ for $\Lambda = K_1(\mathcal{D}_G)$, $\widetilde{K}_1(\mathcal{D}_G)$ and $\mathrm{Wh}(\mathcal{D}_G)$.

We use the same symbols δ_ϕ, L_ϕ for their induced maps on $K_1(\mathcal{D}_G)$, $\widetilde{K}_1(\mathcal{D}_G)$ and $\mathrm{Wh}(\mathcal{D}_G)$ and its domain of definition will be clear from the context.

$$\begin{tikzcd}
 & & GL(\mathcal{D}_G) \arrow[dl, "\det"'] \arrow[d, "\det_r"] \arrow[dr, "\det_w"] & \\
\mathcal{D}_G^\times/[\mathcal{D}_G^\times, \mathcal{D}_G^\times] \arrow[r, equal] & K_1(\mathcal{D}_G) \arrow[r] \arrow[loop below, "L_\phi"] & \widetilde{K}_1(\mathcal{D}_G) \arrow[r] \arrow[loop below, "L_\phi"] & \mathrm{Wh}(\mathcal{D}_G) \arrow[loop below, "L_\phi"]
\end{tikzcd}$$

Theorem 4.12. *Let $\phi \in H^1(H;\mathbb{R})$ be any real cohomology class. Suppose P and Q are two square matrices over $\mathcal{D}_G$ of size $n \times n$, such that the following conditions hold:*

- (i) *P is invertible over $\mathcal{D}_G$;*
- (ii) *there exist real numbers $d_0, d_1, \dots, d_n$ such that if P_{ij} is nonzero, then P_{ij} is ϕ-pure with $\delta_\phi(P_{ij}) = d_0 + d_i - d_j$;*
- (iii) *$\delta_\phi(Q_{ij}) > d_0 + d_i - d_j$ for all i, j.*

Then $P + Q$ is invertible and moreover $L_\phi(\det(P + Q)) = \det P \in K_1(\mathcal{D}_G)$.

Proof. We prove by induction on n. When $n = 1$ then $P, Q \in \mathcal{D}_G$, by the conditions we have $\delta_\phi(P) = d_0$, and $\delta_\phi(Q) > d_0$ if Q is nonzero. Then $L_\phi(\det(P + Q)) = \det P$ by definition.

Now assume Lemma 4.12 holds for size n. Assume that P, Q are $(n+1)$ by $(n+1)$ matrices

$$P = \begin{pmatrix} A & U \\ X & p \end{pmatrix}, \quad Q = \begin{pmatrix} B & V \\ Y & q \end{pmatrix}$$

where $p, q \in \mathcal{D}_G$, A, B are n by n matrices over $\mathcal{D}_G$ and

$$X = (x_1, \ldots, x_n), \quad Y = (y_1, \ldots, y_n),$$
$$U = (u_1, \ldots, u_n)^T, \quad V = (v_1, \ldots, v_n)^T$$

are matrices over $\mathcal{D}_G$ of appropriate size. Without loss of generality we can assume $p \neq 0$, hence also $p + q \neq 0$ by condition (iii). Note that

$$\begin{pmatrix} I & -(U+V)(p+q)^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A+B & U+V \\ X+Y & p+q \end{pmatrix} = \begin{pmatrix} W & 0 \\ X+Y & p+q \end{pmatrix}$$

where

$$W = A + B - (U+V)(p+q)^{-1}(X+Y),$$

is an n by n matrix. So

$$\det(P+Q) = \det W \cdot \det(p+q),$$
$$L_\phi \det(P+Q) = L_\phi \det W \cdot \det p$$

(note that $L_\phi(p+q) = p$). Staring at the expression of W, it is easy to guess that the terms of lowest δ_ϕ form the matrix $A - Up^{-1}X$. We show that this is indeed the case.

Claim. *Define*

$$W' := A - Up^{-1}X, \quad W'' := W - W'.$$

Then W' is invertible over $\mathcal{D}_G$ and $\det P = \det W' \cdot \det p$. Moreover, W' and W'' satisfies the three conditions of Lemma 4.12 for size n and then by the induction hypothesis we have $L_\phi \det(W) = \det(W')$.

Admitting this claim. Then we have

$$L_\phi \det(P+Q) = L_\phi \det W \cdot \det p = \det W' \cdot \det p = \det P$$

and the induction is finished. It remains to prove the Claim.
Proof of Claim. To prove (i), it follows from

$$\begin{pmatrix} I & -Up^{-1} \\ 0 & 1 \end{pmatrix} \cdot P = \begin{pmatrix} W' & 0 \\ X & p \end{pmatrix}$$

that W' is invertible over $\mathcal{D}_G$ and $\det P = \det W' \cdot \det p$.

For (ii), note that

$$W_{ij} = A_{ij} + B_{ij} - (u_i + v_i)(p+q)^{-1}(x_j + y_j),$$
$$W'_{ij} = A_{ij} - u_i p^{-1} x_j,$$
$$W''_{ij} = B_{ij} + u_i p^{-1} x_j - (u_i + v_i)(p+q)^{-1}(x_j + y_j).$$

If $A_{ij} \neq 0$, then A_{ij} is ϕ-pure with $\delta_\phi(A_{ij}) = d_0 + d_i - d_j$; if $u_i p^{-1} x_j \neq 0$, then $u_i p^{-1} x_j$ is also ϕ-pure with

$$\delta_\phi(u_i p^{-1} x_j) = (d_0 + d_i - d_{n+1}) - d_0 + (d_0 + d_{n+1} - d_j) = d_0 + d_i - d_j.$$

In conclusion, if $W'_{ij} \neq 0$ then W'_{ij} is ϕ-pure with $\delta_\phi(W'_{ij}) = d_0 + d_i - d_j$ and this proves (ii).

For (iii), by assumption we have $\delta_\phi(B_{ij}) > d_0 + d_i - d_j$. We also have $\delta_\phi(u_i p^{-1} x_j - (u_i + v_i)(p+q)^{-1}(x_j + y_j)) > d_0 + d_i - d_j$, since if $u_i p^{-1} x_j \neq 0$, then

$$L_\phi((u_i + v_i)(p+q)^{-1}(x_j + y_j)) = u_i p^{-1} x_j, \quad \delta_\phi(u_i p^{-1} x_j) = d_0 + d_i - d_j$$

and we can apply Proposition 4.8 (4); if $u_i p^{-1} x_j = 0$ then $u_i = 0$ or $x_j = 0$ and

$$\begin{aligned}\delta_\phi((u_i+v_i)(p+q)^{-1}(x_j+y_j)) &= \delta_\phi(u_i+v_i) - \delta_\phi(p+q) + \delta_\phi(x_j+y_j)\\ &> \delta_\phi(u_i) - \delta_\phi(p+q) + \delta_\phi(x_j)\\ &= d_0 + d_i - d_j.\end{aligned}$$

In either cases, we have $\delta_\phi(u_i p^{-1} x_j - (u_i+v_i)(p+q)^{-1}(x_j+y_j)) > d_0 + d_i - d_j$. These combines to show that $\delta_\phi(W'') > d_0 + d_i - d_j$ by Proposition 4.8 (5). □

4.4. **The restriction map.** Suppose G is a finitely generated torsion-free group satisfying the Atiyah Conjecture and let $L \lhd G$ be a normal subgroup of finite index d. In this section we define the restriction map $\mathrm{res}_L^G : K_1(\mathcal{D}_G) \to K_1(\mathcal{D}_L)$. Recall that $\mathcal{D}_G$ is naturally isomorphic to the crossed product $\mathcal{D}_L * (G/L)$ by Proposition 4.4.

Definition 4.13. Fix a section $s : G/L \to G$ and suppose its image is $s(G/L) = \{g_1, \ldots, g_d\}$. Then $G = g_1 L \sqcup \ldots \sqcup g_d L$. For any element $z \in \mathcal{D}_G$ and for any $k \in \{1, \ldots, d\}$, there is a unique way to express $g_k \cdot z$ as

$$g_k \cdot z = \sum_{j=1}^{d} l_{kj} \cdot g_j, \quad l_{kj} \in \mathcal{D}_L.$$

Define $\Lambda_s(z)$ to be the $(d \times d)$-matrix over $\mathcal{D}_L$ whose (k,j)-entry is l_{kj}. In other words, $\Lambda_s(z)$ is the unique matrix over $\mathcal{D}_L$ such that

$$\begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z = \Lambda_s(z) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}.$$

Lemma 4.14. *With the notations as in Definition 4.13, the following statements hold.*

(1) *For any $z_1, z_2 \in \mathcal{D}_G$, we have $\Lambda_s(z_1 z_2) = \Lambda_s(z_2) \cdot \Lambda_s(z_1)$.*
(2) *If $z \neq 0$, then $\Lambda_s(z)$ is invertible over $\mathcal{D}_L$.*
(3) *If s' is another section, then $\Lambda_{s'}(z) = \Omega \Lambda_s(z) \Omega^{-1}$ for an invertible matrix Ω over $\mathbb{Z}L$ which only depends on s and s'.*
(4) *For any $\phi \in H^1(G; \mathbb{R})$ and any j, k we have $\delta_\phi(g_k \cdot z) \leqslant \delta_\phi(l_{kj} \cdot g_j)$. If z is ϕ-pure, then l_{kj} is also ϕ-pure; moreover if $l_{kj} \neq 0$ then $\delta_\phi(g_k \cdot z) = \delta_\phi(l_{kj} \cdot g_j)$.*

Proof. For (1), it follows from

$$\begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z_1 z_2 = \Lambda_s(z_1) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z_2 = \Lambda_s(z_1) \cdot \Lambda_s(z_2) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}$$

that $\Lambda_s(z_1 z_2) = \Lambda_s(z_2) \cdot \Lambda_s(z_1)$. Note that (2) is a direct consequence of (1).

For (3), let s' be another section with $s'(L) = \{g_1', \ldots, g_d'\}$ and let Ω be the $(d \times d)$-matrix over $\mathbb{Z}L$ such that

$$\begin{pmatrix} g_1' \\ \vdots \\ g_d' \end{pmatrix} = \Omega \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}.$$

Then

$$\begin{pmatrix} g_1' \\ \vdots \\ g_d' \end{pmatrix} \cdot z = \Omega \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z = \Omega \Lambda_s(z) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} = \Omega \Lambda_s(z) \Omega^{-1} \begin{pmatrix} g_1' \\ \vdots \\ g_d' \end{pmatrix}$$

and therefore $\Lambda_{s'}(z) = \Omega \Lambda_s(z) \Omega^{-1}$.

For (4). First suppose the contrary that there exists $k, j \in \{1, \dots, d\}$ such that $\delta_\phi(g_k \cdot z) > \delta_\phi(l_{kj} \cdot g_j)$. Fix k and let $\mathcal{J}$ be the collection of indices j such that $\delta_\phi(g_k \cdot z) > \delta_\phi(l_{kj} \cdot g_j)$. Since

$$\delta_\phi(g_k \cdot z) = \delta_\phi\Big(\sum_{j=1}^{d} l_{kj} \cdot g_j\Big),$$

it follows from Proposition 4.8 (5) that $\sum_{j \in \mathcal{J}} l_{kj} \cdot g_j = 0$. Recall that $g_1, \dots, g_d$ are $\mathcal{D}_L$-independent by Proposition 4.3. This implies that $l_{kj} = 0$ for all $j \in \mathcal{J}$ and forces $\mathcal{J}$ to be empty, a contradiction. Hence $\delta_\phi(g_k \cdot z) \leqslant \delta_\phi(l_{kj} \cdot g_j)$ for all k, j. It can be seen from Lemma 4.10 (3) that

$$L_\phi(g_k \cdot z) = L_\phi\Big(\sum_{j=1}^{d} l_{kj} \cdot g_j\Big) = \sum_{j=1}^{d} l'_{kj} \cdot g_j$$

where

$$l'_{kj} = \begin{cases} L_\phi(l_{kj}) & \text{if } \delta_\phi(l_{kj} \cdot g_j) = \delta_\phi(g_k \cdot z) \\ 0 & \text{if } \delta_\phi(l_{kj} \cdot g_j) > \delta_\phi(g_k \cdot z). \end{cases}$$

If in addition z is ϕ-pure, then $g_k \cdot z$ is also ϕ-pure by Lemma 4.10 and therefore $l_{kj} = l'_{kj}$ for all k, j. It follows that l_{kj} is ϕ-pure for all k, j and moreover if $l_{kj} \neq 0$ then $\delta_\phi(g_k \cdot z) = \delta_\phi(l_{kj} \cdot g_j)$. ☐

Definition 4.15 (Restriction map)**.** Let $L \triangleleft G$ be a normal subgroup of finite index. Choose a section $s : G/L \to G$. The mapping Given any element $z \in \mathcal{D}_G^\times$, define

$$\mathrm{res}_H^G : \mathcal{D}_G^\times \to K_1(\mathcal{D}_L), \quad z \mapsto \det(\Lambda_s(z)) \in K_1(\mathcal{D}_L)$$

is a group homomorphism independent of the section by Lemma 4.14. We use the same symbol to denote the induced homomorphism on the K_1-group

$$\mathrm{res}_L^G : K_1(\mathcal{D}_G) \to K_1(\mathcal{D}_L).$$

Remark 4.16. (1) An element $z \in \mathcal{D}_G^\times$ can be associated with $R_z : \mathcal{D}_G \to \mathcal{D}_G$, the operator of right multiplication by z. A choice of section s identifies $\mathcal{D}_G$ with $\bigoplus_{k=1}^d \mathcal{D}_L \cdot g_k$ as $\mathcal{D}_L$-vector spaces and R_z is naturally a $\mathcal{D}_L$-linear automorphism represented by the matrix $\Lambda_s(z)$ which is invertible (with inverse matrix $\Lambda_s(z^{-1})$). By definition $\mathrm{res}_L^G(z) = \det R_z$. A different choice of coset representatives amounts to a change of basis which preserves the determinant.

(2) Given $g \in G \subset \mathcal{D}_G^\times$, the matrix $\Lambda_s(g)$ is a permutation matrix whose nonzero entries are $l_i \in L$. It follows that

$$\mathrm{res}_L^G(g) = \det(\Lambda_s(g)) = \pm \prod_i l_i$$

is represented by a element of L. Therefore the restriction map naturally induces a homomorphism on the Whitehead groups

$$\mathrm{res}_L^G : \mathrm{Wh}(\mathcal{D}_G) \to \mathrm{Wh}(\mathcal{D}_L).$$

Theorem 4.17. *Suppose G is a finitely generated torsion-free group satisfying the Atiyah Conjecture and let $L \triangleleft G$ be a normal subgroup of finite index. Let $\phi \in H^1(G; \mathbb{R})$ and denote by $\phi|_L \in H^1(L; \mathbb{R})$ the restriction of ϕ to L. Then for any $z \in \mathcal{D}_G^\times$, we have*

$$L_{\phi|_L}(\mathrm{res}_L^G(z)) = \mathrm{res}_L^G(L_\phi(z)) \in K_1(\mathcal{D}_L).$$

Proof. Write $z = L_\phi(z) + z'$. Fix a choice of coset representatives $G = g_1 L \sqcup \ldots \sqcup g_d L$. Let P and Q be $(d \times d)$-matrices over $\mathcal{D}_L$ such that

$$g_k \cdot L_\phi(z) = \sum_{j=1}^{d} P_{kj} \cdot g_j, \quad g_k \cdot z' = \sum_{j=1}^{d} Q_{kj} \cdot g_j.$$

Then $\mathrm{res}_L^G(z) = \det(P+Q)$ and $\mathrm{res}_L^G(L_\phi(z)) = \det P$. Since $L_\phi(z)$ is ϕ-pure, it follows from Lemma 4.13 that P_{kj} is ϕ-pure, and moreover

$$\begin{aligned} &\delta_\phi(P_{kj}) = \delta_\phi(z) + \delta_\phi(g_k) - \delta_\phi(g_j) \quad \text{if } P_{kj} \neq 0, \\ &\delta_\phi(Q_{kj}) \geqslant \delta_\phi(z') + \delta_\phi(g_k) - \delta_\phi(g_j) > \delta_\phi(z) + \delta_\phi(g_k) - \delta_\phi(g_j) \end{aligned}$$

for all k, j. By Proposition 4.8(8) $\delta_\phi = \delta_{\phi|_L}$ and $L_\phi = L_{\phi|_L}$ in $\mathcal{D}_L$. Applying Theorem 4.12 to P and Q over $\mathcal{D}_L$ we have

$$\det P = L_{\phi|_L}(\det(P+Q)) \in K_1(\mathcal{D}_L),$$

hence $L_{\phi|_L}(\mathrm{res}_L^G(z)) = \mathrm{res}_L^G(L_\phi(z))$ and the proof is finished. □

4.5. **The polytope map.** Let H be a finitely generated free abelian group. Note that $H_1(H;\mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Z}} H$ is a finite-dimensional real vector space. A *polytope* in $H_1(H;\mathbb{R})$ is a compact set which is the convex hull of a finite subset. We allow the empty set $\emptyset$ to be a polytope.

Definition 4.18 (Faces of polytopes)**.** Given a polytope P and any character $\phi \in H^1(H;\mathbb{R})$. Set $\delta_\phi(P) := \inf_{x\in P} \phi(x)$, define the *face associated to* ϕ by

$$F_\phi(P) := \{x \in P \mid \phi(x) = \delta_\phi(P)\}.$$

It is clear that $F_\phi(P)$ is a polytope contained in P and $\{F_\phi(P) \mid \phi \in H^1(H;\mathbb{R})\}$ is the collection of faces of P. A face is called a *vertex* if it is a single point. Any polytope is the convex hull of all its vertices. A polytope is called *integral* if all its vertices lie in the integral lattice $H \subset H_1(H;\mathbb{R})$.

Given any two non-empty polytopes P_1, P_2 in $H_1(H;\mathbb{R})$, their *Minkowski sum* is defined to be the polytope

$$P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1,\ p_2 \in P_2\}.$$

It is the convex hull of the set $\{v_1 + v_2 \mid v_i \text{ is a vertex of } P_i,\ i = 1, 2\}$. The operator δ_ϕ and the face map F_ϕ are additive under the Minkowski sum, namely

$$\delta_\phi(P_1 + P_2) = \delta_\phi(P_1) + \delta_\phi(P_2), \quad F_\phi(P_1 + P_2) = F_\phi(P_1) + F_\phi(P_2)$$

for all character ϕ and all polytopes P_1, P_2.

Example 4.19. Let M be an admissible 3-manifold. The *Thurston norm ball* $B_x(M) := \{\phi \in H^1(M;\mathbb{R}) \mid x_M(\phi) \leqslant 1\}$ is a (perhaps non-compact) polyhedron in $H^1(M;\mathbb{R})$; the *dual Thurston norm ball* is defined to be $B_x^*(M) := \{z \in H_1(M;\mathbb{R}) \mid \phi(z) \leqslant 1 \text{ for all } \phi \in B_x(M)\}$. Thurston proved that $B_x^*(M)$ is an integral polytope in $H_1(M;\mathbb{R})$ with vertices $\pm v_1, \ldots, \pm v_k$, and the Thurston norm ball is determined by these vertices:

$$B_x(M) = \{\phi \in H^1(M;\mathbb{R}) \mid |\phi(v_i)| \leqslant 1,\ i = 1, \ldots, k\}.$$

A *Thurston cone* in $H^1(M;\mathbb{R})$ is either an open cone formed by the origin and a face of $B_x(M)$, or a maximal connected component of $H^1(M;\mathbb{R}) \setminus \{0\}$ on which the Thurston norm x_M vanishes. It follows that $H^1(M;\mathbb{R}) \setminus \{0\}$ is the disjoint union of all Thurston cones of various dimensions. A Thurston cone is called top-dimensional if its dimension equals $\dim H^1(M;\mathbb{R})$. The following Lemma 4.20 is Thurston's result stated differently.

Lemma 4.20. *A nonzero character $\phi \in H^1(M;\mathbb{R})$ lies in a top-dimensional Thurston cone if and only if $F_\phi B_x^*(M)$ is a vertex.*

Definition 4.21 ($\mathcal{P}_{\mathbb{Z}}(H)$ and $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$)**.** The *(integral) polytope group* $\mathcal{P}_{\mathbb{Z}}(H)$ is defined to be the Grothendieck group of integral polytopes in $H_1(H;\mathbb{R})$ under the Minkowski sum. Namely, $\mathcal{P}_{\mathbb{Z}}$ is the abelian group with generating set

$$\{[P] \mid P \text{ is an non-empty integral polytope in } H_1(H;\mathbb{R})\}$$

and a relation $[P]+[Q]=[P+Q]$ for each pair of non-empty integral polytopes P,Q.

Every element $h\in H$ determines an one-point polytope $[h]$ in $\mathcal{P}_{\mathbb{Z}}(H)$ and this defines an embedding of H into $\mathcal{P}_{\mathbb{Z}}(H)$. Define

$$\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)=\mathcal{P}_{\mathbb{Z}}(H)/H.$$

In other words, two polytopes are identified in $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ if and only if they differ by a translation with an element in the lattice H.

We make the following remarks:

(1) Any element of $\mathcal{P}_{\mathbb{Z}}(H)$ can be expressed as a formal sum $[P]-[Q]$ for non-empty integral polytopes P,Q in $H_1(H;\mathbb{R})$. Two expressions $[P_1]-[Q_1]$ and $[P_2]-[Q_2]$ represents the same element if and only if $P_1+Q_2=P_2+Q_1$. Another useful criterion says that two elements $x,y\in\mathcal{P}_{\mathbb{Z}}(H)$ are different if and only if there exists $\phi: H\to\mathbb{Z}$ such that they projects to different elements $\phi(x),\phi(y)\in\mathcal{P}_{\mathbb{Z}}(\mathbb{Z})$. This observation can be used to show that $\mathcal{P}_{\mathbb{Z}}(H)$ and $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ are both free abelian groups [FL17, Lemma 4.8].

(2) The face map F_ϕ naturally extends to a homomorphism of the polytope groups:

$$F_\phi:\mathcal{P}_{\mathbb{Z}}(H)\to\mathcal{P}_{\mathbb{Z}}(H),\quad F_\phi([P]-[Q])=[F_\phi(P)]-[F_\phi(Q)].$$

Since F_ϕ preserves the subgroup H, it naturally induces the homomorphism (by abuse of notation)

$$F_\phi:\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)\to\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H).$$

4.5.1. *Polytope homomorphism* $\mathbb{P}$. Given a finitely generated torsion-free group G satisfying the Atiyah Conjecture. Let H be the free abelianization of G, then we have the short exact sequence

$$1\to K\to G\to H\to 1$$

and $\mathcal{D}_G$ is the field of fraction of the subring $\mathcal{D}_K * H$ by Proposition 4.4. Recall from Definition 4.5 that given any section $s: H\to G$ then every $u\in\mathcal{D}_K*H$ has a unique expression

$$u=\sum_{h\in H}x_h\cdot s(h),\quad x_h\in\mathcal{D}_K.$$

The support of u is the set $\mathrm{supp}(u):=\{h\in H\mid x_h\neq 0\}$ which does not depend on the choice of the section s.

Definition 4.22 (Polytope map)**.** For any $u\in(\mathcal{D}_K*H)^\times$, define $\mathbb{P}(u)$ to be the integral polytope in $\mathbb{R}\otimes_{\mathbb{Z}}H$ which is the convex hull of $\mathrm{supp}(u)$. It is proved in [FL19, Lemma 6.4] that

$$\mathbb{P}(uv)=\mathbb{P}(u)+\mathbb{P}(v)$$

for all $u,v\in(\mathcal{D}_K*H)^\times$. The *polytope homomorphism* is defined to be

$$\mathbb{P}:\mathcal{D}_G^\times\to\mathcal{P}_{\mathbb{Z}}(H),\quad \mathbb{P}(uv^{-1}):=[\mathbb{P}(u)]-[\mathbb{P}(v)]$$

for all $u,v\in(\mathcal{D}_K*H)^\times$.

The following commutative diagram is immediate from the definition.

$$\begin{tikzcd} \mathcal{D}_G^\times \arrow[r, "L_\phi"] \arrow[d, "\mathbb{P}"] & \mathcal{D}_G^\times \arrow[d, "\mathbb{P}"] \\ \mathcal{P}_\mathbb{Z}(H) \arrow[r, "F_\phi"] & \mathcal{P}_\mathbb{Z}(H). \end{tikzcd}$$

Recall that $K_1(\mathcal{D}_G)$ is the abelianization of $\mathcal{D}_G^\times$ and $\mathrm{Wh}(\mathcal{D}_G) = K_1(\mathcal{D}_G)/[\pm G]$, then the polytope homomorphism naturally induces (by abuse of notation)

$$\mathbb{P} : \mathrm{Wh}(\mathcal{D}_\mathrm{G}) \to \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H).$$

We have the commutative diagram for the induced homomorphisms

$$\begin{tikzcd} \mathrm{Wh}(\mathcal{D}_G) \arrow[r, "L_\phi"] \arrow[d, "\mathbb{P}"] & \mathrm{Wh}(\mathcal{D}_G) \arrow[d, "\mathbb{P}"] \\ \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H) \arrow[r, "F_\phi"] & \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H). \end{tikzcd}$$

We record the following important result which relate the universal L^2-torsion of an admissible 3-manifold with its dual Thurston norm ball. Unlike the original source [FL17], the minus sign come out because the L^2-torsion polytope defined there is the negative of $\mathbb{P}(\tau_u^{(2)}(M))$.

Theorem 4.23 ([FL17, Theorem 4.37])**.** *Let M be an admissible 3-manifold which is not homeomorphic to $S^1 \times D^2$. Then*

$$-[B_x^*(M)] = 2 \cdot \mathbb{P}(\tau_u^{(2)}(M)) \in \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H_1(M)_f),$$

where $B_x^* \subset H_1(M;\mathbb{R})$ is the dual Thurston norm ball, and $\tau_u^{(2)}(M) \in \mathrm{Wh}(\mathcal{D}_{\pi_1(M)})$ is the universal L^2-torsion of M.

4.6. **Half of Theorem 1.2.** We are ready to prove the "if" part of Theorem 1.2. A stronger version of the "only if" part will be stated and proved in Theorem 5.14 after establishing Theorem 1.3.

Theorem 4.24 (The "if" part of Theorem 1.2)**.** *Suppose M is an admissible 3-manifold such that M is not a closed graph manifold which admits no NPC metric. Let $\phi \in H^1(M;\mathbb{R})$ be a nonzero class such that $L_\phi \tau_u^{(2)}(M) = 1 \in \mathrm{Wh}(\mathcal{D}_{\pi_1(M)})$, then ϕ is fibered.*

Proof. If M is homeomorphic to the solid torus then any nonzero class is fibered. Let's assume that M is not the solid torus. The assumption posed on M implies that $\pi_1(M)$ is virtually special by a combination of work (c.f. [AFW15, Section 4.7] and references therein). In particular it follows from Agol's criterion for virtual fibering [Ago08] that there is a regular finite covering $\overline{M} \to M$ such that the pull back class $\bar{\phi}$ lies in the closure of a fibered cone. Let G be the fundamental group of M and $\pi_1(\overline{M}) =: L < G$. By the restriction property of Theorem 3.11(4),

$$\tau_u^{(2)}(\overline{M}) = \mathrm{res}_L^G \, \tau_u^{(2)}(M) \in \mathrm{Wh}(\mathcal{D}_L).$$

Applying $L_{\bar{\phi}}$ to both sides, we have

$$\begin{aligned} L_{\bar{\phi}}(\tau_u^{(2)}(\overline{M})) &= L_{\bar{\phi}}(\mathrm{res}_L^G \, \tau_u^{(2)}(M)) \\ &= \mathrm{res}_L^G(L_\phi \tau_u^{(2)}(M)) \quad \text{by Theorem 4.17} \\ &= 1 \in \mathrm{Wh}(\mathcal{D}_L). \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= \mathbb{P}(L_{\bar\phi}(\tau_u^{(2)}(\overline{M}))) \\ &= F_{\bar\phi}\mathbb{P}(\tau_u^{(2)}(\overline{M})) \quad \text{by commutativity} \\ &= -2 \cdot F_{\bar\phi}[B_x^*(\overline{M})] \quad \text{by Theorem 4.23.} \end{aligned}$$

Since $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(\overline{M};\mathbb{R}))$ is torsion-free by [FL17, Lemma 4.8], we obtain that

$$F_{\bar\phi}[B_x^*(\overline{M})] = 0.$$

It follows from Lemma 4.20 that $\bar\phi$ lies in a top dimensional Thurston cone of $H^1(\overline{M};\mathbb{R})$. Because we have chosen $\overline{M}$ so that $\bar\phi$ lies in the closure of a fibered cone, $\bar\phi$ must lie in a fibered cone since the boundary of a fibered cone consists of Thurston cones of strictly lower dimensions. Therefore $\bar\phi$ is a fibered class for $\overline{M}$ and it follows that ϕ is a fibered class for M. □

5. Universal L^2-torsion for taut sutured manifolds

In this section, we first briefly recall the terminologies of the sutured manifold theory, then we discuss the universal L^2-torsion of a taut sutured manifold and prove the decomposition formula Theorem 1.3, which is then used to finish the proof of Theorem 1.2. After that, we use a doubling trick to prove Theorem 1.4.

5.1. Background on sutured manifold theory.

Throughout this section N will be an arbitrary compact, oriented 3-manifold.

Definition 5.1 (Taut surfaces)**.** Given a compact orientable surface Σ with path-components $\Sigma_1, \ldots, \Sigma_k$ we define its *complexity* as

$$\chi_-(\Sigma) := \sum_{i=1}^{k} \max\{0, -\chi(\Sigma_i)\}.$$

A properly embedded oriented surface Σ in N is *taut* if Σ is incompressible, and has minimal complexity among all properly embedded oriented surfaces representing the homology class $[\Sigma, \partial\Sigma] \in H_2(N, \nu(\partial\Sigma); \mathbb{Z})$, where $\nu(\partial\Sigma)$ is a regular neighborhood of $\partial\Sigma$ in ∂N.

Definition 5.2 (Sutured manifold)**.** A *sutured manifold* (N, R_-, R_+, γ) is a compact oriented 3-manifold N with a decomposition of its boundary into two subsurfaces R_+ and R_- along their common boundary γ which we call the *suture*. The orientation on $R_\pm$ is defined in the way that the normal vector of R_+ points out of N and the normal vector of R_- points inward of N. The boundary orientations of $R_\pm$ on γ coincide and induce an orientation of the suture γ. We sometimes abbreviate a sutured manifold (N, R_-, R_+, γ) as (N, γ) where no confusions caused.

A sutured manifold (N, R_-, R_+, γ) is called *taut* if N is irreducible and $R_\pm$ are both taut surfaces (viewed as properly embedded surfaces after pushing slightly into N).

Before we introduce the important notion of sutured manifold decompositions, let's fix some notations. Let S be a (not necessarily connected) properly embedded oriented surface in N. Choose a product neighborhood $S \times (-1, 1)$ of S in N and denote by $N\backslash\backslash S := N \setminus S \times (-1, 1)$ the complement. Let S_+ (resp. S_-) be the components of $S \times \{-1\} \cup S \times \{1\}$ in $N\backslash\backslash S$ whose normal vector points out of (resp. into) N'. This convention of notation has the advantage that $(N\backslash\backslash\Sigma, \Sigma_+, \Sigma_-, \emptyset)$ is a sutured manifold when N is closed.

Definition 5.3 (Decomposition surface)**.** Let (N, R_-, R_+, γ) be a sutured manifold and S a properly embedded surface in N such that ∂S is transverse to γ on ∂N. Furthermore, no component of ∂S bounds a disk in $R_\pm$ and no component of S is a disk D with $\partial D \subset R_\pm$. Then we call S a *decomposition surface* for (N, γ).

Definition 5.4 (Sutured manifold decomposition)**.** Given an oriented decomposition surface S for (N, R_-, R_+, γ), define the *sutured manifold decomposition*

$$(N, R_-, R_+, \gamma) \overset{S}{\rightsquigarrow} (N', R'_-, R'_+, \gamma')$$

where

$$\begin{aligned} N' &= N \backslash\backslash S, \\ R'_+ &= (R_+ \cup S_+) \cap N', \\ R'_- &= (R_- \cup S_-) \cap N', \\ \gamma' &= \partial R'_+ = \partial R'_-. \end{aligned}$$

See Figure 3–4 for illustrations. A sutured manifold decomposition $(N, \gamma) \overset{S}{\rightsquigarrow} (N', \gamma')$ is called a *taut sutured decomposition* if (N', γ') is taut. Gabai [Gab87, Lemma 0.4] proved that if $(N, \gamma) \overset{S}{\rightsquigarrow} (N', \gamma')$ is a taut sutured decomposition then (N, γ) is taut.

We make the following remarks:

(1) Like [AD19], we view sutures as simple closed curves and do not allow torus sutures, which differs from many other sources where the suture are disjoint union of annuli and tori. In fact, if given a sutured manifold (N, γ) which contains toral sutures, we can arbitrarily absorb the toral sutures into $R_\pm$ without changing the universal L^2-torsion. Namely, we have the following:

Lemma 5.5. *If (N, γ) is a taut sutured manifold and $T \subset \gamma$ is a torus component. Then $\tau_u^{(2)}(N, R_+) = \tau_u^{(2)}(N, R_+ \cup T)$.*

Proof. This follows from the Sum Formula 3.11 and the fact that $\tau_u^{(2)}(T) = 1$. □

The definition of sutured manifold decomposition is modified accordingly. In the classical definition of decomposition surface S (c.f. [Gab83, Gab87]), a component of S can be the core of a sutured annulus. If so, S can be slightly isotoped into a decomposition surface S' such that $\partial S'$ lies completely in R_+ or R_-, and decompositions along S and S' produce isomorphic sutured manifolds.

(2) Suppose $(N, R_+, R_-, \gamma) \overset{S}{\rightsquigarrow} (N', R'_+, R'_-, \gamma')$ is a taut sutured decomposition, then S is incompressible in N. The reason is as follows: R'_+ is the union of the surfaces S_+ and $R_+ \cap N'$ along their common part $S_+ \cap R_+$. This common part consists of some arcs and some boundary circles of S_+. By assumption, no boundary circles of S_+ bound a disk in R_+ or a disk in S_+, so the closed curves of $S_+ \cap R_+$ are homotopy nontrivial in both S_+ and $R_+ \cap N'$. By Van Kampen Theorem the surface S_+ is π_1-injective in R'_+, therefore π_1-injective in N' since R'_+ is incompressible in N'. This proves that S can not admit a compressing disk in N by .

(3) Given a taut sutured decomposition $(N, \gamma) \overset{S}{\rightsquigarrow} (N', \gamma')$. Since S is incompressible in N, it follows that for any component of N' the inclusion into N induces monomorphism on fundamental groups. The pair $(N', R'_+) = (N \backslash\backslash S, (S_+ \cup R_+) \cap (N \backslash\backslash S))$ will be abbreviated as $(N \backslash\backslash S, S_+ \cup R_+)$, since a CW-pair (X, Y) always means $(X, X \cap Y)$ in order to make sense.

5.2. **Universal L^2-torsion for taut sutured 3-manifolds.**

Theorem 5.6 ([Her23]). *Let (N,γ) be a sutured 3-manifold with infinite fundamental group. Suppose that N is irreducible and $R_\pm$ are both incompressible. Then (N,γ) is taut if and only if the pair (N,R_+) is L^2-acyclic.*

Proof. Suppose (N,γ) is taut. No components of $R_\pm$ are disks or spheres, otherwise N must be the 3-ball whose fundamental group is trivial. It follows that the complexity of $R_\pm$ equals $-\chi(R_\pm)$, and we have $\chi(R_+)=\chi(R_-)$, then we apply [Her23, Theorem 1.1] to conclude that (N,R_+) is L^2-acyclic.

Suppose on the other hand that the pair (N,R_+) is L^2-acyclic, then the Euler characteristic $\chi(N,R_+)=\chi(N)-\chi(R_+)$ is zero. But $\chi(N)=\frac{1}{2}\chi(\partial N)=\frac{1}{2}(\chi(R_+)+\chi(R_-))$, it follows that $\chi(R_+)=\chi(R_-)$, then we apply [Her23, Theorem 1.1]. $\square$

The reader might wonder why we prefer (N,R_+) than (N,R_-). In fact, these two pairs are dual to each other by the following Proposition 5.7. We prefer the pair (N,R_+) since it suits our convention of orientations better.

Proposition 5.7. *Let (N,R_+,R_-,γ) be a sutured manifold. Suppose the pair (N,R_+) is L^2-acyclic. Then (N,R_-) is also L^2-acyclic. Moreover*

$$\tau_u^{(2)}(N,R_+)=\overline{\tau_u^{(2)}(N,R_-)}.$$

Proof. The proof is a modification of [Mil62, Page 139] and does not use any L^2-theories. Form a smooth triangulation of N such that R_+ and R_- are subcomplexes. Let $\widehat{N}$ be the universal cover and let $\widehat{R}_\pm$ be the preimage of $R_\pm$. Consider the dual cellular complex $\widehat{N}'$ of $\widehat{N}$. A cell of $\widehat{N}$ is canonically dual to a cell of $\widehat{N}'\setminus\partial\widehat{N}'$, see Figure 2 below.

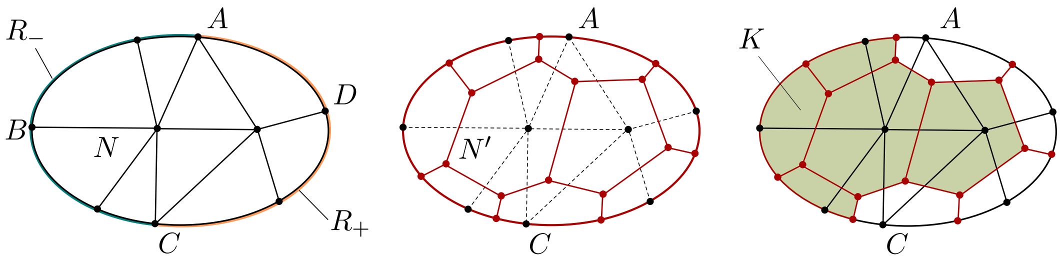

Figure 2. *An illustration of the one-dimension-lower case. The figure on the left shows a simplicial complex N whose boundary is a union of a subcomplex R_- (the arc $\widehat{ABC}$) and a subcomplex R_+ (the arc $\widehat{ADC}$). The figure in the middle shows the dual cellular complex N' (in red). The last figure shows the subcomplex $K\subset N'$ which is the union of cells disjoint from R_+. The complex N' deformation retracts to K along a product neighborhood of R_+.*

More explicitly, we have the intersection pairing

$$p: C_*(\widehat{N})\times C_{3-*}(\widehat{N}',\partial\widehat{N}')\to\mathbb{Z}G,\quad p(\sigma,\sigma')=\sum_{g\in G}\langle\sigma,g\sigma'\rangle\cdot g$$

where $\langle\sigma,g\sigma'\rangle$ is the intersection number of σ and $g\sigma'$. It is easy to verify the identities

$$\begin{aligned}p(g\sigma,\sigma')&=g\cdot p(\sigma,\sigma'),\\ p(\sigma,g\sigma')&=p(\sigma,\sigma')\cdot g^{-1},\\ p(\partial\sigma,\sigma')&=\pm p(\sigma,\partial\sigma').\end{aligned}$$

The pairing is non-degenerate in the sense that $\sigma\mapsto p(\sigma,*)$ gives an isomorphism of $\mathbb{Z}G$-chain complexes $C_*(\widehat{N})\cong C^{3-*}(\widehat{N}',\partial N')$. Note that a cell of $\widehat{R}_+$ is canonically dual to a cell of $\widehat{N}' \setminus \partial \widehat{N}'$ which intersects $\widehat{R}_+$ at a non-empty set and vise versa. Let K be the union of the cells of N' which is disjoint with R_+ and let $\widehat{K}$ be the preimage in the universal cover, then there is an induced non-degenerate pairing

$$C_*(\widehat{N}, \widehat{R}_+) \times C_{3-*}(\widehat{K}, \partial \widehat{N}') \to \mathbb{Z}G$$

and hence the isomorphism of $\mathbb{Z}G$-chain complexes $C_*(\widehat{N}, \widehat{R}_+) \cong C^{3-*}(\widehat{K}, \partial \widehat{N}')$. By Proposition 3.7 the finite based free $\mathbb{Z}G$-chain complex $C_*(\widehat{K}, \partial \widehat{N}')$ is L^2-acyclic and

$$\tau_u^{(2)}(C_*(\widehat{N}, \widehat{R}_+)) = \overline{\tau_u^{(2)}(C_*(\widehat{K}, \partial \widehat{N}'))},$$

therefore $\tau_u^{(2)}(N, R_+) = \overline{\tau_u^{(2)}(K, \partial N')}$. The pair $(K, \partial N')$ is a deformation retract of the pair (N', R'_-) (barycentric-subdivide N, if necessary), hence $\tau_u^{(2)}(N, R_+) = \overline{\tau_u^{(2)}(N', R'_-)}$. This shows that the pair (N, R_-) is L^2-acyclic with $\tau_u^{(2)}(N, R_+) = \overline{\tau_u^{(2)}(N, R_-)}$. ☐

5.3. Turaev's algorithm.

Lemma 5.8. *Let (N, γ) be a taut sutured manifold. Suppose there is a decomposition surface S such that*

(1) *The sutured manifold decomposition $(N, R_+, R_-, \gamma) \overset{S}{\rightsquigarrow} (N', R'_+, R'_-, \gamma')$ is taut.*
(2) *S is separating and N' is a disjoint union of two (possibly non-connected) manifolds N_0, N_1.*
(3) *$S_- \subset N_0$, $S_+ \subset N_1$.*

Then $\tau_u^{(2)}(N, R_+) = i_* \tau_u^{(2)}(N', R'_+)$ where $i : N' \hookrightarrow N$ is the inclusion.

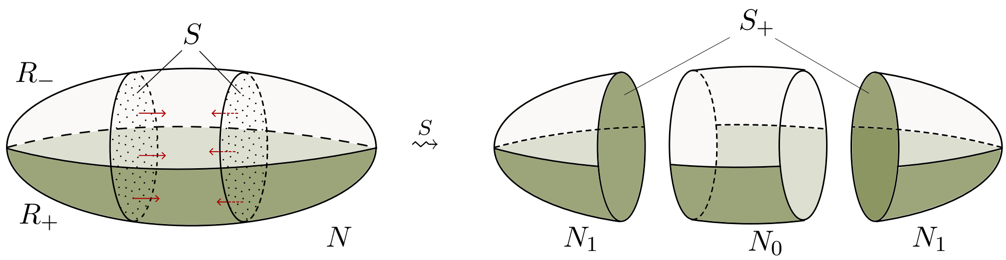

Figure 3. *Decomposing N along a separating decomposition surface S (the dotted region in the left figure whose normal direction as indicated by the red arrows). The R_+-part of the sutured manifolds are in green.*

Proof. See Figure 3 for an illustration of the decomposition. There is a short exact sequence of chain complexes

$$0 \to C_*^{(2)}(N_0, N_0 \cap R_+) \to C_*^{(2)}(N, R_+) \to C_*^{(2)}(N, N_0 \cup R_+) \to 0$$

and a natural isomorphism $C_*^{(2)}(N, N_0 \cup R_+) = C_*^{(2)}(N_1, (S_+ \cup R_+) \cap N_1)$. Note that $N_0 \cap R_+$ is the R_+-part of N_0 and $(S_+ \cup R_+) \cap N_1$ is the R_+-part of N_1. By assumption N_0 and N_1 are taut sutured manifold. It follows from Theorem 5.6 that the three chain complexes are L^2-acyclic and

$$\tau_u^{(2)}(N, R_+) = (i_0)_* \tau_u^{(2)}(N_0, N_0 \cap R_+) \cdot (i_1)_* \tau_u^{(2)}(N_1, (S_+ \cup R_+) \cap N_1)$$

where i_0 and i_1 are the inclusion of N_0 and N_1 into N. The right hand side of the above equation is exactly $i_* \tau_u^{(2)}(N', R'_+)$ and the proof is finished. ☐

Definition 5.9 (Weighted surface)**.** A *weighted surface* $\widehat{S}$ in a compact oriented 3-manifold N is a collection of pairs (S_i, w_i), $i = 1, \dots, n$, where S_i is a connected properly embedded oriented surface, w_i is a positive integer, and $S_i \cap S_j = \emptyset$ for $i \neq j$. The *realization* of $\widehat{S}$ is the properly embedded oriented surface

$$\bar{S} := \bigcup_{i=1}^{n} w_i \cdot S_i$$

where $w_i \cdot S_i$ is the union of w_i parallel copies of S_i. The *reduction* of $\widehat{S}$ is the properly embedded oriented surface

$$S := \bigcup_{i=1}^{n} S_i.$$

A *weighted decomposition surface* is a weighted surface whose realization is a decomposition surface.

If $\widehat{S}$ is a weighted decomposition surface, then $N\backslash\backslash\bar{S}$ is a disjoint union of $N\backslash\backslash S$ and some copies of $S \times I$. So the sutured decomposition along $\bar{S}$ is taut if and only if the sutured decomposition along S is taut.

The following Proposition 5.10 is a generalization of [BAFH22, Proposition 3.3]. The idea goes back to Turaev [Tur02].

Proposition 5.10. *Let (N, R_+, R_-, γ) be a taut sutured manifold and let $(N, \gamma) \overset{\Sigma}{\rightsquigarrow} (N', \gamma')$ be a taut sutured decomposition. Then there is a weighted decomposition surface $\widehat{S}$ in N (with $\bar{S}$ the realization and S the reduction) such that*

1. *$N\backslash\backslash S$ is connected,*
2. *$[\bar{S}] = [\Sigma] \in H_2(N, \partial N; \mathbb{Z})$,*
3. *the sutured decomposition of N along S is taut,*
4. *$i_*\tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+) = j_*\tau_u^{(2)}(N\backslash\backslash\Sigma, \Sigma_+ \cup R_+)$ where i, j are the natural inclusions of $N\backslash\backslash S$ and $N\backslash\backslash\Sigma$ into N.*

Proof. For any weighted surface $\widehat{S}$ in N, define $c(\widehat{S}) := \#\pi_0(N\backslash\backslash S)$. First take $\widehat{S}$ to be the surface Σ with weight 1 assigned to each component, then $\bar{S} = S = \Sigma$ and $\widehat{S}$ satisfies (2)–(4). It suffices to prove that given any weighted surface $\widehat{S}$ with $c(\widehat{S}) > 1$ such that (2)–(4) holds, then there exists a weighted surface $\widehat{T}$ such that (2)–(4) holds and $c(\widehat{T}) < c(\widehat{S})$.

Given such $\widehat{S}$. Since $c(\widehat{S}) > 1$ there is a component $C \subset S$ such that $C_\pm$ lies in different components of $N\backslash\backslash S$. Choose C to be a component with minimal weight w among such components, let M_0 (resp. M_1) be the component of $N\backslash\backslash S$ containing C_- (resp. C_+). Let $C_1 = C, C_2, \dots, C_k$ be the components of S whose normal direction points into M_1 and let $D_1, \dots, D_l$ be the components of S whose normal direction points out of M_1. It may happen that $C_i = D_j$ for some i, j, in this case the two sides of C_i both belong to M_1. It follows that

$$[C_1] + \cdots + [C_k] = [D_1] + \cdots + [D_l] \in H_2(N, \partial N; \mathbb{Z}).$$

We change the weights of $\widehat{S}$ by increasing the weights of $D_1, \dots, D_l$ by w, and decreasing the weights of $C_1, \dots, C_k$ by w. If a component has weight zero, we simply discard this component. This new weighted surface is denoted by $\widehat{T}$. Clearly, we have $[\bar{S}] = [\bar{T}] \in H_2(N, \partial N; \mathbb{Z})$. Since T is a subcollection of S and the decomposition along S is taut, it follows that the decomposition along T is also taut. So (2), (3) holds true for $\widehat{T}$. Moreover, $c(\widehat{T}) < c(\widehat{S})$ since M_0 and M_1 are in the same component of $N\backslash\backslash T$.

Finally, let S_0 be $S \setminus T$. Then $N \backslash\backslash S$ is obtained from the sutured manifold decomposition of $N \backslash\backslash T$ along S_0. By construction, S_0 separates $N \backslash\backslash T$; in particular, it separates M_0 from M_1 and $(S_0)_+ \subset M_1$. Apply Lemma 5.8 to $N_0 := (N \backslash\backslash S) - M_1$ and $N_1 := M_1$, we have

$$i'_* \tau_u^{(2)}(N \backslash\backslash T, T_+ \cup R_+) = i_* \tau_u^{(2)}(N \backslash\backslash S, S_+ \cup R_+) = j_* \tau_u^{(2)}(N \backslash\backslash \Sigma, \Sigma_+ \cup R_+)$$

where i, i', j' are the inclusions of $N \backslash\backslash S$, $N \backslash\backslash T$ and $N \backslash\backslash \Sigma$ into N, respectively. This verifies (4) for $\widehat{T}$ and finishes the proof. □

5.4. **Decomposition formula for universal L^2-torsion.**

Definition 5.11. Let G be a group and $\phi \in H^1(G; \mathbb{R})$ an 1-cohomology class. There are natural homomorphisms

$$\delta_\phi : (\mathbb{Z}G)^\times \to \mathbb{R}, \quad L_\phi : (\mathbb{Z}G)^\times \to (\mathbb{Z}G)^\times$$

defined by considering the terms with minimal ϕ-value (c.f. Section 4.3). For any nonzero matrix A over $\mathbb{Z}G$, let $\delta_\phi(A)$ be the smallest real number $\delta_\phi(A_{ij})$ among all nonzero entries A_{ij}. Then we can decompose the matrix in a unique way

$$A = L_\phi(A) + (A - L_\phi(A))$$

where any group element g appearing in $L_\phi(A)$ satisfies $\phi(g) = \delta_\phi(A)$, and any group element h appearing in $(A - L_\phi(A))$ satisfies $\phi(h) > \delta_\phi(A)$.

We define $L_\phi(A) = 0$ when A is a zero matrix. Otherwise $L_\phi(A)$ is always nonzero.

For any finite based free $\mathbb{Z}G$-chain complex

$$C_* = (0 \longrightarrow C_n \xrightarrow{A_n} \cdots \xrightarrow{A_2} C_1 \xrightarrow{A_1} C_0 \longrightarrow 0),$$

define

$$L_\phi(C_*) = (0 \longrightarrow C_n \xrightarrow{L_\phi(A_n)} \cdots \xrightarrow{L_\phi(A_2)} C_1 \xrightarrow{L_\phi(A_1)} C_0 \longrightarrow 0).$$

It is easy to verify that $L_\phi(A)L_\phi(B) = L_\phi(AB)$ holds for arbitrary matrices A, B. In particular $L_\phi(C_*)$ is a well-defined finite based free $\mathbb{Z}G$-chain complex.

Lemma 5.12. *Suppose G is a torsion-free group satisfying the Atiyah Conjecture. Let $\phi \in H^1(G; \mathbb{R})$ be an 1-cohomology class and let*

$$C_* = (0 \longrightarrow C_n \xrightarrow{A_n} \cdots \xrightarrow{A_2} C_1 \xrightarrow{A_1} C_0 \longrightarrow 0)$$

be a finite based free $\mathbb{Z}G$-chain complex. Suppose that $L_\phi(C_*)$ is L^2-acyclic, then C_* is also L^2-acyclic and

$$\tau_u^{(2)}(L_\phi(C_*)) = L_\phi(\tau_u^{(2)}(C_*)).$$

Proof. Suppose that $L_\phi(C_*)$ is L^2-acyclic, then by Theorem 3.21 we can find a non-degenerate matrix chain $\mathcal{A}$ of $L_\phi(C_*)$. Let $B_1, \ldots, B_n$ be the associated submatrices of $L_\phi(A_1), \ldots, L_\phi(A_n)$, then B_i are weak isomorphisms and

$$\tau_u^{(2)}(L_\phi(C_*)) = \prod_{i=1}^{n} \det{}_r(B_i)^{(-1)^n} \in \widetilde{K}_1(\mathcal{D}_G)$$

Let $C_1, \ldots, C_n$ be the submatrices of $A_1, \ldots, A_n$ associated to the same matrix chain $\mathcal{A}$. Since B_i are nonzero, then $L_\phi(C_i) = B_i$. By Theorem 4.12 we have

$$L_\phi \det{}_r(C_i) = \det{}_r(B_i) \in \widetilde{K}_1(\mathcal{D}_G).$$

In particular, C_i are weak isomorphisms and therefore $\mathcal{A}$ is a non-degenerate matrix chain of C_*. By Theorem 3.21,

$$\tau_u^{(2)}(C_*) = \prod_{i=1}^{n} \det{}_r(C_i)^{(-1)^n} \in \widetilde{K}_1(\mathcal{D}_G)$$

and in particular $\tau_u^{(2)}(L_\phi(C_*)) = L_\phi(\tau_u^{(2)}(C_*))$. □

Theorem 5.13 (Theorem 1.3). *Let (N,γ) be a taut sutured manifold and let $(N,\gamma) \overset{\Sigma}{\rightsquigarrow} (N',\gamma')$ be a taut sutured decomposition. Let $\phi = PD([\Sigma,\partial\Sigma]) \in H^1(N;\mathbb{Z})$ be the Poincaré dual of the surface Σ, then*

$$j_*\tau_u^{(2)}(N',R'_+) = L_\phi(\tau_u^{(2)}(N,R_+))$$

where $j : N' \hookrightarrow N$ is the natural inclusion.

Proof. We find a weighted decomposition surface $\widehat{S}$ in N as provided by Proposition 5.10. Then $N\backslash\backslash S$ is connected, $\phi = PD([\Sigma,\partial\Sigma]) = PD([\bar{S},\partial\bar{S}]) \in H^1(N;\mathbb{Z})$ and $j_*\tau_u^{(2)}(N',R'_+) = i_*\tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+)$ where $i : N\backslash\backslash S \to N$ is the inclusion. We are left to show that

$$L_\phi(\tau_u^{(2)}(N,R_+)) = i_*\tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+).$$

Find a CW-structure for N such that $S \times I$ and $R_\pm$ are subcomplexes. Fix a base point

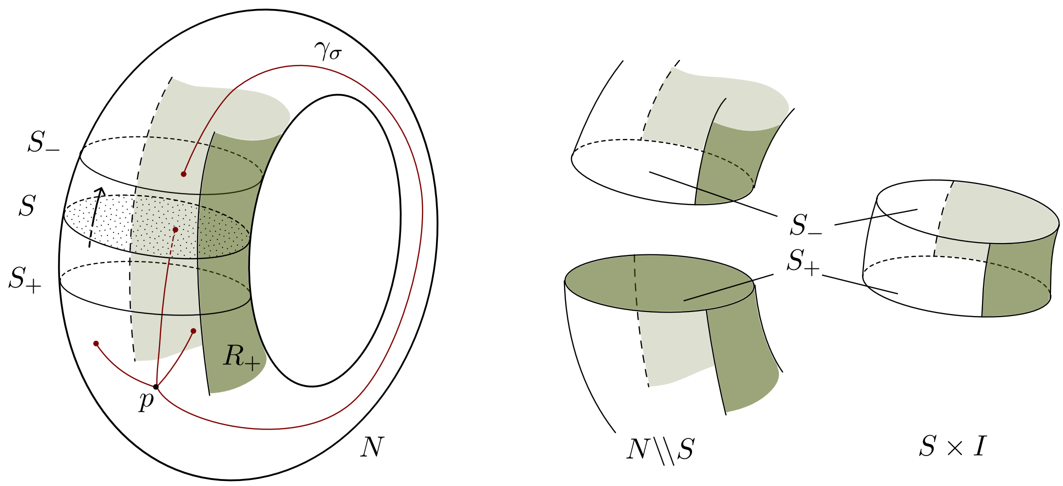

Figure 4. *Consider N as the union of $N\backslash\backslash S$ and $S \times I$.*

$p \in N \setminus (S \times I)$. For any cell σ in the CW-structure of N, choose a path γ_σ (the paths in red in Figure 4) connecting p and σ such that

- γ_σ is disjoint with S_-, if $\sigma \subset N \setminus S_-$,
- γ_σ is disjoint with S_+, if $\sigma \subset S_-$.

Lift the base point p to $\hat{p}$ in the universal cover $\widehat{N}$ and lift each cell σ to $\hat{\sigma}$ using the path γ_σ. The cells $\hat{\sigma}$ form a basis for the finite based free $\mathbb{Z}[\pi_1(N)]$-chain complex $C_*(\widehat{N})$. Now consider the L^2-cellular chain complex $C_*^{(2)}(N,R_+)$, for each $k = 0,1,2,3$ the chain group admits the following direct sum decomposition

$$C_k^{(2)}(N,R_+) = C_k^{(2)}(N\backslash\backslash S, S_+ \cup R_+) \oplus C_k^{(2)}(S \times I, S_- \cup R_+).$$

Accordingly, the boundary homomorphism $\partial_* : C_*^{(2)}(N,R_+) \to C_{*-1}^{(2)}(N,R_+)$ admits the following decomposition

$$
\begin{aligned}
&\partial_* = \begin{pmatrix} \partial_*^1 & \partial_*^2 \\ \partial_*^3 & \partial_*^4 \end{pmatrix},\\
&\partial_*^1 : C_*^{(2)}(N\backslash\backslash S, S_+ \cup R_+) \to C_{*-1}^{(2)}(N\backslash\backslash S, S_+ \cup R_+),\\
&\partial_*^2 : C_*^{(2)}(N\backslash\backslash S, S_+ \cup R_+) \to C_{*-1}^{(2)}(S\times I, S_- \cup R_+),\\
&\partial_*^3 : C_*^{(2)}(S\times I, S_- \cup R_+) \to C_{*-1}^{(2)}(N\backslash\backslash S, S_+ \cup R_+),\\
&\partial_*^4 : C_*^{(2)}(S\times I, S_- \cup R_+) \to C_{*-1}^{(2)}(S\times I, S_- \cup R_+).
\end{aligned}
$$

In particular, if $\hat{\sigma}$ is a basis element of $C_*^{(2)}(S\times I, S_-\cup R_+)$ then there are basis elements τ_i (allowing repetitions) of $C_{*-1}^{(2)}(N\backslash\backslash S, S_+ \cup R_+)$ and $g_i \in \pi_1(N)$ such that

$$\partial_*^3(\hat{\sigma}) = \sum_i g_i \hat{\tau}_i.$$

Then the cells $\hat{\tau}_i$ must lie in $\widehat{S}_-$. By our choice of basis, each g_i satisfies $\phi(g_i) > 0$. A similar argument shows that any group element h appearing in $\partial_*^1, \partial_*^2$ or ∂_*^4 satisfies $\phi(h) = 0$. It follows that

$$L_\phi(\partial_*) = \begin{pmatrix} \partial_*^1 & \partial_*^2 \\ 0 & \partial_*^4 \end{pmatrix}$$

and we hence obtain a short exact sequence of chain complexes

$$0 \to C_*^{(2)}(N\backslash\backslash S, S_+ \cup R_+) \to L_\phi(C_*^{(2)}(N,R_+)) \to C_*^{(2)}(S\times I, S_- \cup R_+) \to 0.$$

Then by the product formula 3.5 and the induction property Theorem 3.11 we have

$$\tau_u^{(2)}(L_\phi(C_*^{(2)}(N,R_+))) = i_*\tau_u^{(2)}(N\backslash\backslash S, S_- \cup R_+) \cdot i'_*\tau_u^{(2)}(S\times I, S_- \cup R_+).$$

The left hand side equals $L_\phi(\tau_u^{(2)}(N,R_+))$ by Lemma 5.12. On the right hand side, since $(S_-\cup R_+)\cap(S\times I)$ deformation retracts onto S_-, we have $\tau_u^{(2)}(S\times I, S_-\cup R_+) = \tau_u^{(2)}(S\times I, S_-) = 1$. It follows that $L_\phi(\tau_u^{(2)}(N,R_+)) = i'_*\tau_u^{(2)}(N\backslash\backslash S, S_+\cup R_+)$ and the proof is finished. $\square$

The (stronger) "only if" part of Theorem 1.2 follows easily.

Theorem 5.14 (The "only if" part of Theorem 1.2)**.** *Suppose M is an admissible 3-manifold and $\phi \in H^1(M;\mathbb{R})$ is a fibered class. Then $L_\phi\tau_u^{(2)}(M) = 1 \in \mathrm{Wh}(\mathcal{D}_G)$.*

Proof. By Proposition 4.8(7) we can find a rational fibered class $\psi \in H^1(M;\mathbb{Q})$ arbitrarily close to ϕ such that $L_\phi(\tau_u^{(2)}(M)) = L_\psi(\tau_u^{(2)}(M))$. Choose a positive integer n such that $n\psi$ is an integral fibered class, and let S be a fiber surface dual to $n\psi$, then $M\backslash\backslash S = S\times[0,1]$ is a product. By Theorem 1.3 we have

$$L_{n\psi}\tau_u^{(2)}(M) = j_*\tau_u^{(2)}(M\backslash\backslash S, S_+) = 1,$$

where $j : M\backslash\backslash S \hookrightarrow M$ is the inclusion. It follows that $L_\phi\tau_u^{(2)}(M) = L_{n\psi}\tau_u^{(2)}(M) = 1$. $\square$

Combined with Theorem 4.24 the proof of Theorem 1.2 is now complete.

5.5. **Proof of Theorem 1.4.** We use a refined doubling trick to show that universal L^2-torsion detects product sutured manifolds.

Definition 5.15 (Double of taut sutured manifolds)**.** Let (N, R_+, R_-, γ) be a taut sutured manifold and let $f : R_+ \to R_+$ be an orientation-preserving homeomorphism. Let $(N, \overline{R}_+, \overline{R}_-, \bar{\gamma})$ be the sutured manifold whose underlying oriented manifold is the same as N, but with R_+, R_- interchanged (see Figure 5 below). Namely,

$$\bar{\gamma} = -\gamma, \quad \overline{R}_+ = -R_-, \quad \overline{R}_- = -R_+.$$

The *double of N with monodromy f* is defined to be the admissible 3-manifold

$$DN_f = (N, \gamma) \cup (N, \bar{\gamma})/\sim$$

formed by gluing together the two sutured manifolds with the gluing relation $\sim$ as follows: we identify R_- and $\overline{R}_+$ via the identity map and identify R_+ and $\overline{R}_-$ via the homeomorphism f.

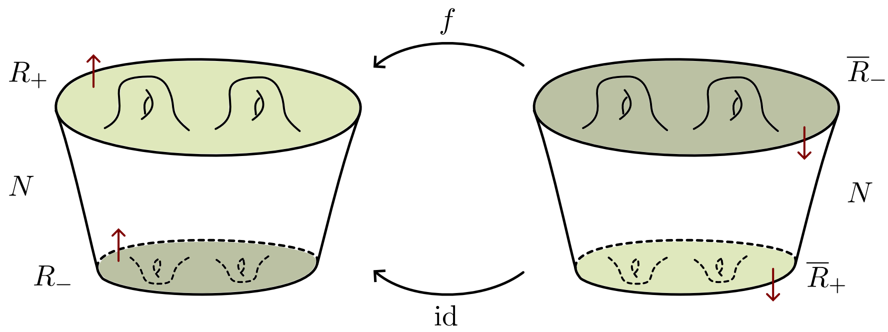

Figure 5. *Double of a sutured manifold N with monodromy f*

Lemma 5.16. *If (N, R_+, R_-, γ) is a taut sutured manifold with R_+ and R_- both non-empty. Suppose $R_+ \cup R_-$ is not a union of tori, then there exists $f : R_+ \to R_+$ such that DN_f is not a closed graph manifold.*

Proof. If N contains non-empty sutured annuli, then DN_f has non-empty torus boundary for any f by construction. If $\pi_1(N)$ is finite then N is the product 3-ball and has non-empty sutured annuli.

If N has infinite fundamental group and does not contain sutured annuli, then each component of $R_\pm$ is a closed surfaces with positive genus. By assumption, $R_+ \cup R_-$ contains a component of genus greater than 1. Since $\chi(R_+) = \chi(R_-)$, we know that R_+ contains a component of genus greater than 1. Denote this component by Σ. Write

$$f = (\phi : \Sigma \to \Sigma,\ \mathrm{id} : R_+ \setminus \Sigma \to R_+ \setminus \Sigma).$$

Let M be the manifold obtained from gluing together (N, γ) and $(N, \bar{\gamma})$ via $\mathrm{id} : R_- \to \overline{R}_+$ and $\mathrm{id} : R_+ \setminus \Sigma \to \overline{R}_- \setminus (-\Sigma)$. Then M is a 3-manifold whose boundary are two copies of Σ left unglued. It is clear that DN_f is obtained from gluing $\Sigma_1, \Sigma_2 \subset \partial M$ together via $\phi : \Sigma_1 \to \Sigma_2$. We are left to prove the following:

Claim. *Let M be a compact, orientable, irreducible 3-manifold. Suppose $\partial M = \Sigma_1 \sqcup \Sigma_2$ is the union of two connected incompressible surfaces of the same genus greater than 1. Then there is a homeomorphism $\phi : \Sigma_1 \to \Sigma_2$ such that the closed manifold M_ϕ formed by gluing together the two boundary surfaces of M via ϕ is not a graph manifold.*

Proof of Claim. If $M = \Sigma_1 \times I$ is a product, then a pseudo-Anosov homeomorphism ϕ suffices. Now suppose M is not a product. The Characteristic Submanifold Theorem asserts that there is a submanifold $X \subset M$ which is a disjoint union of Seifert fibered spaces and I-bundle over surfaces, such that any incompressible torus and annulus of M can be homotoped into X [JS79, Chapter V]. The intersections $X \cap \Sigma_1$ and $X \cap \Sigma_2$ are incompressible proper subsurfaces of Σ_1 and Σ_2 respectively; otherwise let's suppose $\Sigma_1 \subset X$, then Σ_1 is contained in an I-bundle component P of X. Since $\partial P \subset \partial M$, P must be the entire M, contradicting our assumption. Let C_i be a component of $\partial(X \cap \Sigma_i)$ which is an essential simple closed curve on Σ_i, $i = 1, 2$. Choose a homeomorphism $\phi : \Sigma_1 \to \Sigma_2$ such that the two simple closed curves $\phi(C_1)$ and C_2 fill Σ_2. This can be done by assigning $\phi = \psi^n \circ \phi_0$ for any homeomorphism $\phi_0 : \Sigma_1 \to \Sigma_2$, any pseudo-Anosov homeomorphism $\psi : \Sigma_2 \to \Sigma_2$ and a sufficiently large integer n. In this case the distance between $[\phi(C_1)]$ and $[C_2]$ on the curve complex of Σ_2 will be arbitrarily large [MM99, Proposition 4.6]. In particular $\phi(C_1)$ fills with C_2.

We now show that the glued-up manifold M_f is not a closed graph manifold. Suppose $T \subset M_f$ is an essential torus. Up to isotopy we assume that T is transverse to $\Sigma = \Sigma_1 = \Sigma_2$ in M_f, and the intersections $T \cap \Sigma$ (if non-empty) are essential simple closed curves on Σ. If $T \cap \Sigma$ is indeed non-empty, choose any component $C \subset T \cap \Sigma$, then $T \setminus C$ is an essential annulus in M, therefore the image of C on $\Sigma_2 \subset \partial M$ can be homotoped into both $X \cap \Sigma_2$ and $\phi(X \cap \Sigma_1)$, hence the geometric intersection numbers $i(C, \phi(C_1))$ and $i(C, C_2)$ are both zero. This contradicts the fact that $\phi(C_1)$ and C_2 fill Σ_2. Therefore any essential torus T in M_f can be isotoped to be disjoint with Σ. Let Y be the JSJ-piece of M_f containing Σ. Note that Y must not be a Seifert fibered space, for otherwise Σ would be a horizontal surface in Y since Σ is an essential surface with genus greater than 1. The circle fiber over any essential simple closed curves of Σ becomes an essential torus in Y which intrinsically intersects Σ, a contradiction. Therefore Y is a hyperbolic piece and M_f is not a closed graph manifold. □

Theorem 5.17 (Theorem 1.4)**.** *Let (N, R_+, R_-, γ) be a taut sutured manifold with R_+ and R_- both non-empty. Then (N, γ) is a product sutured manifold if and only if $\tau_u^{(2)}(N, R_+) = 1 \in \mathrm{Wh}(\mathcal{D}_{\pi_1(N)})$.*

Proof. If (N, γ) is a product sutured manifold then clearly $\tau_u^{(2)}(N, R_+) = 1$.

Now we suppose $\tau_u^{(2)}(N, R_+) = 1$ and wish to prove that N is a product.

Case 1: $R_+ \cup R_-$ is a disjoint union of tori. In this case N is an admissible 3-manifold and $\tau_u^{(2)}(N) = \tau_u^{(2)}(N, R_+) = 1$. By Theorem 1.2 we know that any nonzero class is fibered, and whose Thurston norm vanishes by Theorem 4.23. This shows that N is homeomorphic to one of the following: the solid torus $D^2 \times S^1$, the thickened torus $T^2 \times I$, or the twisted I-bundle over Klein bottle $K \widetilde{\times} I$. Since R_+ and R_- are both non-empty, it follows that $(N, R_+, R_-, \gamma) = (T^2 \times I, T^2 \times \{1\}, T^2 \times \{0\}, \emptyset)$ and is indeed a product sutured manifold.

Case 2: $R_+ \cup R_-$ is not a union of tori. Choose a homeomorphism $f : R_+ \to R_+$ as in Lemma 5.16 such that the doubling DN_f is not a closed graph manifold. Then $R_+ \cup R_-$ is a Thurston norm minimizing surface dual to a cohomology class ϕ. By Theorem 1.3,

$$L_\phi \tau_u^{(2)}(DN_f) = (j_1)_* \tau_u^{(2)}(N, R_+) \cdot (j_2)_* \tau_u^{(2)}(N, \overline{R}_+)$$

where j_1, j_2 are the natural inclusions. Since $\tau_u^{(2)}(N, R_+) = \tau_u^{(2)}(N, R_-) = \tau_u^{(2)}(N, \overline{R}_+) = 1$ by Proposition 5.7, we have $L_\phi \tau_u^{(2)}(DN_f) = 1$. It follows from Theorem 1.2 that ϕ is a fibered class and hence $R_+ \cup R_-$ is a fiber surface. Therefore (N, γ) is a product sutured manifold. □

6. Applications of the universal L^2-torsions

6.1. **Universal L^2-torsion for groups homomorphisms.** Let G_1, G_2 be torsion-free groups satisfying the Atiyah Conjecture. Assume that their Whitehead groups $\mathrm{Wh}(G_i)$ vanish and they have finite classifying spaces X_i, $i = 1, 2$. Then any homomorphism $\varphi : G_1 \to G_2$ determines a homotopy class of mapping $\Phi : X_1 \to X_2$ which we call a *realization of* φ.

Definition 6.1 (Universal L^2-torsion for group homomorphisms)**.** Let $\varphi : G_1 \to G_2$ be a homomorphism and let $\Phi : X_1 \to X_2$ be its realization. The *universal L^2-torsion of* φ is defined to be $\tau_u^{(2)}(\varphi) := \tau_u^{(2)}(\Phi) \in \mathrm{Wh}(\mathcal{D}_{X_2}) \sqcup \{0\}$. We say φ is an *L^2-homology equivalence* if $\tau_u^{(2)}(\varphi) \neq 0$.

If X_1', X_2' are different choice of classifying spaces then we have the following commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi} & X_2 \\ {\scriptstyle\psi_1}\downarrow & & \downarrow{\scriptstyle\psi_2} \\ X_1' & \xrightarrow{\Phi'} & X_2' \end{array}$$

where $\psi_i : X_i \to X_i'$ are homotopy equivalences (hence simple-homotopy equivalences) $i = 1, 2$ and Φ' is the realization of φ under the classifying spaces X_1', X_2'. By Lemma 3.16 we know that

$$\tau_u^{(2)}(\Phi') = (\psi_2)_* \tau_u^{(2)}(\Phi),$$

hence the definition of $\tau_u^{(2)}(\varphi)$ does not depend on the choice of classifying spaces X_i.

6.1.1. *Homomorphism between free groups.* Let $F_1 = \langle x_1, \dots, x_n \rangle$ and $F_2 = \langle y_1, \dots, y_m \rangle$ be finitely generated free groups. Let us explicitly calculate $\tau_u^{(2)}(\varphi)$ for a homomorphism $\varphi : F_1 \to F_2$ under the given basis.

Let $X_1 = \vee_{i=1}^n S^1$ and $X_2 = \vee_{i=1}^m S^1$ be the wedge of circles. The space X_1 is given the usual CW-structure with one 0-cell p and n 1-cells $e_1, \dots, e_n$. Identify the fundamental group $\pi_1(X_1, p)$ with F_1 in such a way that $x_i = [e_i]$. Similarly, X_2 is given the CW-structure with one 0-cell q and n 1-cells $f_1, \dots, f_n$ and $y_i = [f_i]$. Denote by $\Phi : X_1 \to X_2$ the realization of φ. Form the wedge space $X_1 \vee X_2$ by identifying $p \in X$ and $q \in X_2$; then for any $i = 1, \dots, n$, attach a 2-cell σ_i whose boundary is the concatenation of $\Phi(e_i)$ and e_i^{-1}. The resulting CW-complex M_Φ is simple-homotopy equivalent to the mapping cylinder of Φ. Let $\widehat{M_\Phi}$ be the universal cover of M_Φ and let $\widehat{X_1}$ be the preimage of $X_1 \subset M_\Phi$. Fix a lifting $\hat{p} \in \widehat{M_\Phi}$ of p and lift the other cells with respect to the base point $\hat{p}$. Then we have the following $\mathbb{Z}F_2$-chain complex

$$(\dagger) \qquad C_*(\widehat{M_\Phi}, \widehat{X}) = (0 \to \mathbb{Z}F_2\langle \hat{\sigma}_1, \dots, \hat{\sigma}_n \rangle \xrightarrow{J_\varphi} \mathbb{Z}F_2\langle \hat{f}_1, \dots, \hat{f}_m \rangle \to 0 \to 0).$$

The square matrix J_φ is called the *Jacobian of φ with respect to the basis $y_1, \dots, y_m$*. Recall that the Fox derivative $\frac{\partial}{\partial y_i} : \mathbb{Z}F_2 \to \mathbb{Z}F_2$, $i = 1, \dots, m$ are $\mathbb{Z}$-linear maps characterized by the following two properties:

- $\frac{\partial}{\partial y_i} 1 = 0$, $\frac{\partial}{\partial y_i} y_j = \delta_{ij}$.
- $\frac{\partial}{\partial y_i}(uv) = \frac{\partial}{\partial y_i} u + u \cdot \frac{\partial}{\partial x_i} v$ for all $u, v \in F_2$.

The entries of J_φ are then given by the Fox derivative

$$J_{ij} = \frac{\partial \varphi(x_i)}{\partial y_j} \in \mathbb{Z}F_2, \quad 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m.$$

Proposition 6.2. *Let $\varphi : F_1 \to F_2$ be a homomorphism between finitely generated free groups $F_1 = \langle x_1, \dots, x_n \rangle$ and $F_2 = \langle y_1, \dots, y_m \rangle$. Then:*

(1) $\tau_u^{(2)}(\varphi)=\det_w(J_\varphi)$ *where* J_φ *is the Jacobian of* φ *with respect to the basis* $\{x_i\},\{y_j\}$. *In particular* φ *is an* L^2*-homology equivalence if and only if* J_φ *is a weak isomorphism.*
(2) $\tau_u^{(2)}(\varphi)=1$ *if* φ *is an isomorphism;* $\tau_u^{(2)}(\varphi)=0$ *if* $m\neq n$ *or* φ *is not injective.*
(3) *If* $F_3=\langle z_1,\dots,z_k\rangle$ *is another free group and* $\psi: F_2\to F_3$ *is a homomorphism. Suppose* φ,ψ *are* L^2*-homology equivalences then* $\tau_u^{(2)}(\psi\circ\varphi)=\psi_*\tau_u^{(2)}(\varphi)\cdot\tau_u^{(2)}(\psi)$.

Proof. Firstly, (1) is immediate from Equation (†) and Definition 6.1.

For (2), if φ is an isomorphism, then $\Phi: X_1\to X_2$ is a homotopy equivalence. Since the Whitehead group of a free group is trivial [Sta65] then Φ is a simple homotopy equivalence and $\tau_u^{(2)}(\varphi)=\tau_u^{(2)}(\Phi)=1$ by Proposition 3.14(3). If $m\neq n$, then J_φ is not a square matrix and clearly $\tau_u^{(2)}(\varphi)=0$. The proof of (2) is finished once we established the following Lemma 6.3.

Lemma 6.3. *If* $n=m$ *and* φ *is not injective, then the Jacobian* J_φ *is not a weak isomorphism.*

Proof of Lemma 6.3. Suppose the contrary that φ is not injective, then there is a reduced word $w\in F_1$ such that $\varphi(w)=1$. Let $w=x_{i_1}^{\epsilon_1}\cdots x_{i_k}^{\epsilon_k}$ be such a word with shortest length $k\geqslant 1$, where $i_1,\dots,i_k\in\{1,\dots,n\}$ and $\epsilon_1,\dots,\epsilon_k\in\{\pm1\}$. We may assume that $x_{i_1}^{\epsilon_1}=x_1$ and $x_{i_k}^{\epsilon_k}\neq x_1^{-1}$. Denote by $w_s:=x_{i_1}^{\epsilon_1}\cdots x_{i_s}^{\epsilon_s}$, $s=1,\dots,k$ to be the prefix of w of length s; set $w_0=1$. For any $j=1,\dots,n$, apply $\frac{\partial}{\partial y_j}$ to both sides of the identity $\varphi(x_{i_1})^{\epsilon_1}\cdots\varphi(x_{i_k})^{\epsilon_k}=1$, we have

$$\sum_{s=1}^k u_s\cdot\frac{\partial\varphi(x_{i_s})}{\partial y_j}=0,\quad\text{where}\quad u_s=\begin{cases}\varphi(w_{s-1}), & \epsilon_s=1,\\ -\varphi(w_s), & \epsilon_s=-1.\end{cases}$$

Note that u_s does not depend on j. Rearranging the identities, we have

$$\sum_{i=1}^n U_i\cdot\frac{\partial\varphi(x_i)}{\partial y_j}=0,\quad j=1,\dots,n$$

where U_i is the summation of all u_s such that $i_s=i$. Therefore

$$(U_1,U_2,\dots,U_n)\cdot J_\varphi=0.$$

If we could show that $U_i\neq 0$ for some i then this implies that J_φ is not a weak isomorphism, hence a contradiction. We prove that $U_1\neq 0$. Let $1\leqslant s_1<s_2<\cdots<s_r\leqslant k$ be the collection of indices such that $i_{s_1}=\cdots=i_{s_r}=1$. Then $s_1=1$ by assumption and $U_1=1+u_{s_2}+\cdots+u_{s_r}$. Write $U_1=1\pm\varphi(w_{s'_2})\pm\cdots\pm\varphi(w_{s'_r})$, where $w_{s'_j}=w_{s_j}$ or w_{s_j-1}, $j=1,\dots,r$ depending on the sign of ϵ_{s_j}. Then we have $0<s'_2<\cdots<s'_r<k$. This is because there are no segments of $(\cdots x_1^{-1}x_1\cdots)$ in the reduced word w, and that w does not ends with x_1^{-1}. We claim that $\varphi(w_{s'_j})\in F$ are pairwise distinct, otherwise there are two distinct prefixes of w with the same image under φ and we find a reduced word whose length shorter than k that lies in $\ker\varphi$, a contradiction. This shows that $U_1\in\mathbb{Z}F$ is nonzero in $\mathbb{Z}F$ and finishes the proof. □

It remains to prove Proposition 6.2(3). Note that φ and ψ are injective by (2) and the result follows from Proposition 3.14(4). □

It is an interesting question to characterize when a homomorphism $\varphi: F_1\to F_2$ is an L^2-homology equivalence. A finitely generated subgroup H of a free group F is called *compressed* if for any subgroup L of F containing H we have $\operatorname{rank}H\leqslant\operatorname{rank}L$. The following characterization given by [JZ24] is notable since it does not involve any L^2-theories in its statement.

Theorem 6.4. *Let $\varphi : F_1 \to F_2$ be a homomorphism between finitely generated free groups of the same rank. Then φ is an L^2-homology equivalence if and only if φ is injective with compressed image* $\operatorname{im}\varphi \subset F_2$.

Proof. If φ is not injective then $\tau_u^{(2)}(\varphi) = 0$ by Proposition 6.2(2). Now we assume that φ is injective and aim to show that $\tau_u^{(2)}(\varphi) \neq 0$ if and only if $\operatorname{im}\varphi$ is compressed in F_2.

Let $\Phi : X_1 \to X_2$ be the topological realization of φ and let M_Φ be the mapping cylinder. Then $\tau_u^{(2)}(\varphi) \neq 0$ if and only if the pair (M_Φ, X_1) is L^2-acyclic. By the homology long exact sequence this is equivalent to

$$H_1(X_1; \mathcal{D}_{F_2}) \to H_1(M_\Phi; \mathcal{D}_{F_2})$$

being injective (note that $\operatorname{rank} H_1(X_1; \mathcal{D}_{F_2}) = \operatorname{rank} H_1(M_\Phi; \mathcal{D}_{F_2}) = \operatorname{rank} F_2 - 1$). By [JZ24, Corollary 1.2] this is equivalent to $\pi_1(X_1) \subset \pi_1(M_\varphi)$ being compressed, i.e. $\operatorname{im}\varphi \subset F_2$ is compressed. □

Conjecture 6.5. *A homomorphism φ between finitely generated free groups is an isomorphism if and only if $\tau_u^{(2)}(\varphi) = 1$.*

6.2. **3-dimensional handlebodies.** In the remaining part of this paper, a "handlebody" refers to a compact connected orientable 3-manifold obtained from attaching 1-handles to a 3-ball. The boundary of a handlebody is a connected closed surface. The homeomorphism type of a handlebody is determined by the genus of its boundary. A genus-g handlebody H_g refers to a handlebody whose boundary is a genus $g \geqslant 0$ surface. Note that H_g deformation retracts to the one-point union of g circles.

Fix $g \geqslant 1$. Consider the sutured manifold (H_g, R_+, R_-, γ) such that R_+ and R_- are both connected and $\chi(R_+) = \chi(R_-)$. It is clear that $\chi(R_+) = \chi(R_-) = 1 - g$, and the fundamental group of $R_\pm$ are both isomorphic to a free group of rank g.

Proposition 6.6. *Suppose (H_g, R_+, R_-, γ) is a sutured manifold such that R_+, R_- are both connected and $\chi(R_+) = \chi(R_-)$ (we do not assume that $R_\pm$ are incompressible). Then:*

1. *$\tau_u^{(2)}(H_g, R_+) = \tau_u^{(2)}(\varphi)$ where $\varphi : \pi_1(R_+) \to \pi_1(H_g)$ is the inclusion-induced homomorphism.*
2. *(H_g, R_+, R_-, γ) is a taut sutured manifold if and only if $\tau_u^{(2)}(H_g, R_+) \neq 0$.*
3. *(H_g, R_+, R_-, γ) is a product sutured manifold if and only if $\tau_u^{(2)}(H_g, R_+) = 1$.*

Proof. For (1), Let $\Phi : R_+ \to H_g$ be the inclusion map. Note that R_+ and H_g are classifying spaces for $\pi_1(R_+)$ and $\pi_1(H_g)$, respectively. Therefore

$$\tau_u^{(2)}(H_g, R_+) = \tau_u^{(2)}(\Phi) = \tau_u^{(2)}(\varphi).$$

For (2), the forward direction follows from Theorem 5.6(1). Now suppose $\tau_u^{(2)}(H_g, R_+)$ is nonzero. By Proposition 5.7 we know that $\tau_u^{(2)}(H_g, R_-)$ is also nonzero. Then it follows from (1) and Proposition 6.2(2) that $R_\pm \subset H_g$ are both incompressible surfaces. Therefore (H_g, γ) is taut by Theorem 5.6(2).

Finally, (3) is a direct corollary of (2) and Theorem 1.4. □

Example 6.7. A sutured manifold M as in the left of Figure 6 is a 3-ball with 2 arcs removed. There are 3 sutured annulus separating ∂M into two pairs of pants R_+ and R_-. The manifold M is homeomorphic to a genus-2 handlebody whose fundamental group is generated by the loops x and y. The fundamental group of R_+ is generated by the loops x and u where $u = yxyx^{-1}y^{-1}$ in $\pi_1(M)$. Let $\varphi : \pi_1(R_+) \to \pi_1(M)$ be the inclusion-induced homomorphism, then under the basis $\pi_1(R_+) = \langle x, u\rangle$, $\pi_1(M) = \langle x, y\rangle$, we

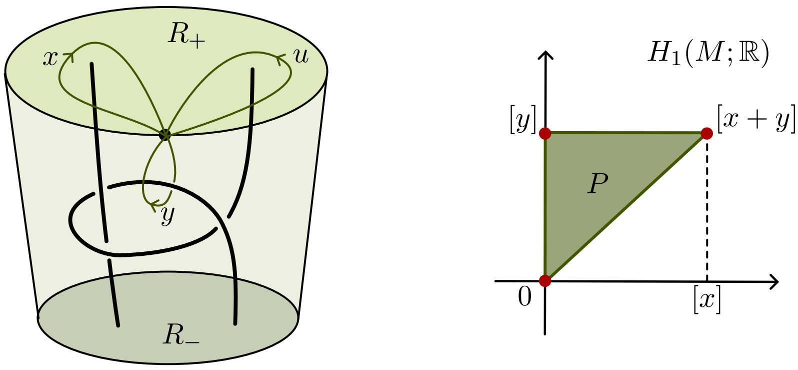

Figure 6. *A sutured manifold M and a representative P of its L^2-polytope in $H_1(M;\mathbb{R})$*

have

$$J_\varphi = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y - yxyx^{-1} & 1 + yx - u \end{pmatrix}.$$

Therefore by Proposition 6.6,

$$\tau_u^{(2)}(M, R_+) = \det_w(J_\varphi) = [1 + yx - u].$$

The polytope map $\mathbb{P} : \mathrm{Wh}(\mathcal{D}_{\pi_1(M)}) \to \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(M;\mathbb{Z}))$ sends $\tau_u^{(2)}(M, R_+)$ to a polytope $[P]$ where P is the convex hull of $\{0, [u] = [y], [x+y]\}$. Proposition 6.6 implies that M is a taut sutured manifold but not a product sutured manifold.

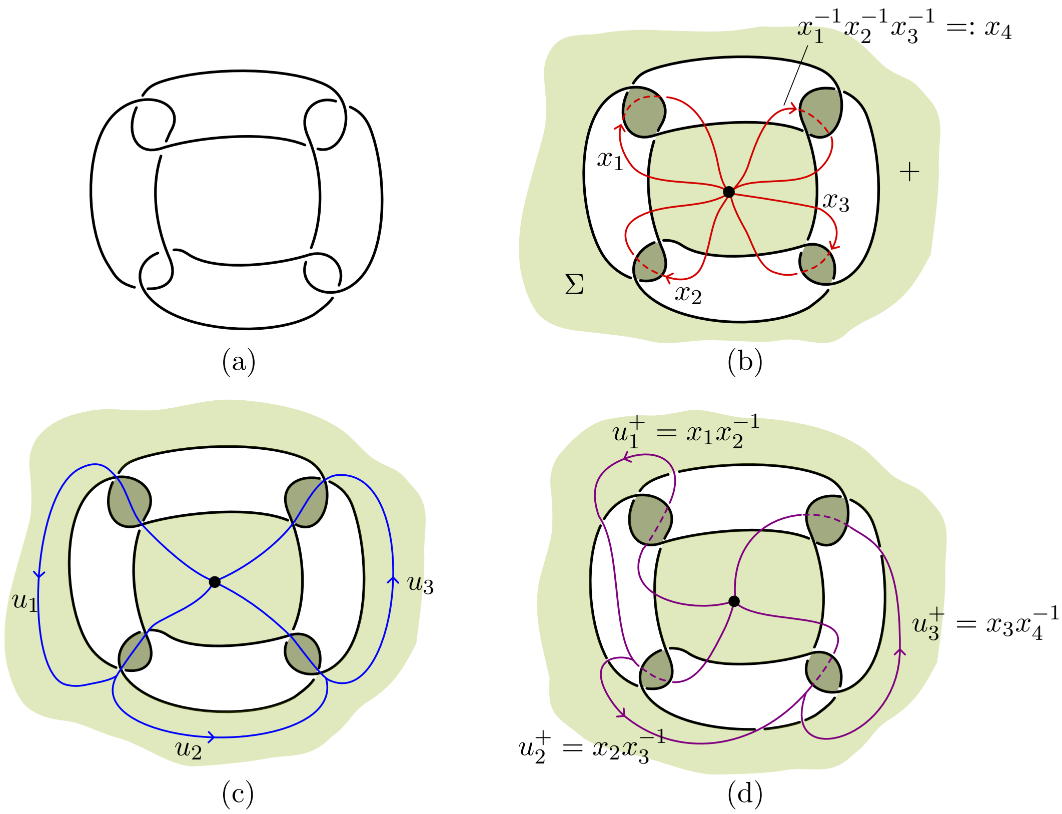

Figure 7. *The n-chain link example, where $n = 4$*

Example 6.8 (n-chain link)**.** For each $n \geqslant 3$, the n-chain link L_n is an alternating link obtained from linking together n unknots in a cyclic way. A diagram of a 4-chain link is illustrated in Figure 7(a). Consider the natural Seifert surface Σ as is shown in Figure 7(b) (where the positive-side of Σ is in light green, while the negative-side of Σ is in dark green). Then Σ is obtained from two copies of disks D^2 by attaching n twisted-bands, and Σ deformation retracts to the wedge of $(n-1)$ circles. The complement $S^3 \backslash\backslash \Sigma$ is a handle body of genus $(n-1)$, whose boundary is a union of two copies of Σ, namely Σ_+ and Σ_-. Choose a free basis for the fundamental group $\pi_1(S^3\backslash\backslash\Sigma) = \langle x_1, \ldots, x_{n-1}\rangle$ where x_i are represented by the red loops in Figure 7(b); a free basis for $\pi_1(\Sigma_+)$ is represented by the blue loops $u_1, \ldots, u_{n-1}$ in Figure 7(c). Pushing u_i slightly into the positive direction, we obtain its image u_i^+ under the inclusion $\Sigma_+ \hookrightarrow S^3\backslash\backslash\Sigma$, which is represented by $x_i x_{i+1}^{-1}$ (we assume that $x_n := x_1^{-1}\cdots x_{n-1}^{-1}$, see Figure 7(b)). Let $\varphi : \pi_1(\Sigma_+) \to \pi_1(S^3\backslash\backslash\Sigma)$ be the inclusion-induced map, then

$$\operatorname{im}\varphi = \langle x_1x_2^{-1}, x_2x_3^{-1}, \ldots, x_{n-2}x_{n-1}^{-1}, x_{n-1}^2 x_{n-2}\cdots x_2x_1\rangle \subset \langle x_1, \ldots, x_{n-1}\rangle$$

and the Jacobian of φ is

$$J_\varphi = \begin{pmatrix} 1 & -x_1x_2^{-1} & & & \\ & 1 & -x_2x_3^{-1} & & \\ & & \ddots & & \\ & & & 1 & -x_{n-2}x_{n-1}^{-1} \\ x_{n-1}^2x_{n-2}\cdots x_2 & x_{n-1}^2x_{n-2}\cdots x_3 & \cdots & x_{n-1}^2 & 1+x_{n-1} \end{pmatrix}.$$

Lemma 6.9. *For $n \geqslant 3$, let*

$$B_n = \begin{pmatrix} 1 & -s_1 & & & \\ & 1 & -s_2 & & \\ & & \ddots & & \\ & & & 1 & -s_{n-2} \\ f_1 & f_2 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix}$$

be a matrix over a skew field. Then its Dieudonné determinant

$$\det(B_n) = [f_1s_1s_2\cdots s_{n-2} + f_2s_2s_3\cdots s_{n-2} + \cdots f_{n-2}s_{n-2} + f_{n-1}].$$

Proof. When $n = 3$, the identity

$$\det(B_3) = \det\begin{pmatrix} 1 & -s_1 \\ f_1 & f_2 \end{pmatrix} = [f_1s_1 + f_2]$$

holds true. For the general case, left-multiply $(-f_1)$ to the first row of B_n and add it to the last row, we eliminate the bottom-left entry and hence

$$\det(B_n) = \det\begin{pmatrix} 1 & -s_2 & & \\ & \ddots & & \\ & & 1 & -s_{n-2} \\ f_1s_1+f_2 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix}.$$

The conclusion follows easily from an induction argument with respect to n. □

Applying Lemma 6.9 to J_φ, note that $s_is_{i+1}\cdots s_{n-2} = x_ix_{n-1}^{-1}$, it follows that

$$f_is_is_{i+1}\cdots s_{n-2} = x_{n-1}^2x_{n-2}\cdots x_ix_{n-1}^{-1}, \quad 1 \leqslant i < n-1$$

and $f_{n-1} = 1+x_{n-1}$. Denote by $y_i := x_{n-1}x_{n-2}\cdots x_i$, $i = 1,\ldots,n-1$, then $\{y_1,\ldots,y_{n-1}\}$ is another free basis of the fundamental group $\pi_1(S^3\backslash\backslash\Sigma)$. By Proposition 6.6

$$\begin{aligned}\tau_u^{(2)}(S^3\backslash\backslash\Sigma,\Sigma_+) &= \det_w(J_\varphi)\\ &= [x_{n-1}\cdot(y_1+y_2+\cdots+y_{n-1}+1)\cdot x_{n-1}^{-1}]\\ &= [y_1+y_2+\cdots+y_{n-1}+1].\end{aligned}$$

The polytope map $\mathbb{P}$ sends the universal L^2-torsion $\tau_u^{(2)}(S^3\backslash\backslash\Sigma,\Sigma_+)$ to an $(n-1)$-simplex of $H_1(S^3\backslash\backslash\Sigma;\mathbb{R})$ spanned by n vertices $\{0,[y_1],\ldots,[y_{n-1}]\}$. By Proposition 6.6 $S^3\backslash\backslash\Sigma$ is a taut sutured manifold and Σ is a norm-minimizing Seifert surface for L. Moreover, $\tau_u^{(2)}(S^3\backslash\backslash\Sigma,\Sigma_+)\neq 1$ and L is not fibered as an oriented link.

Remark 6.10. Suppose G is group satisfying the Determinant Conjecture (see [Lüc02, Section 13] for a detailed discussion), then the Fuglede–Kadison determinant is a well-defined homomorphism $\det_{\mathcal{N}G} : \mathrm{Wh}(\mathcal{D}_G)\to\mathbb{R}_+$. For any L^2-acyclic finite CW-complex X with $\pi_1(X) = G$, the Fuglede–Kadison determinant brings the universal L^2-torsion $\tau_u^{(2)}(X)$ to the L^2-torsion $\tau^{(2)}(X)$. Let G be the fundamental group $\pi_1(S^3\backslash\backslash\Sigma)$. It follows from [BA22] that

$$\det_{\mathcal{N}G}([1+y_1+\cdots+y_{n-1}]) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}}.$$

Therefore the L^2-torsion $\tau^{(2)}(S^3\backslash\backslash\Sigma,\Sigma_+) = (n-1)^{\frac{n-1}{2}}/n^{\frac{n-2}{2}}$.

For an admissible 3-manifold N and a cohomology class $\phi\in H^1(N;\mathbb{Z})$, the L^2-Alexander torsion is a function $\tau^{(2)}(N,\phi):\mathbb{R}_+\to[0,+\infty)$ [DFL16]. This function has a well-defined "degree" which equals to the Thurston norm of ϕ [FL19, Liu17], and a "leading coefficient" $C(N,\phi)\geqslant 1$ [Liu17]. It is proved in [Dua25] that $C(N,\phi)$ equals the L^2-torsion of the pair $(N\backslash\backslash\Sigma,\Sigma_+)$ where Σ is a norm-minimizing surface dual to ϕ. Let X_n be the n-chain link complement and let $\phi\in H^1(X_n)$ be the Poincaré dual of Σ. It follows that

$$C(X_n,\phi) = \tau^{(2)}(X_n,\Sigma_+) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}}\sim\sqrt{n/e},\quad \text{as } n\to+\infty.$$

Therefore the n-chain link complements X_n give an infinite family of hyperbolic manifolds such that $C(X_n,\phi)>1$ for a nonzero class $\phi\in H^1(X_n;\mathbb{Z})\setminus\{0\}$, answering a question of [BAFH22, Conjecture 1.7].

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