UNIVERSAL L^2-TORSION AND TAUT SUTURED DECOMPOSITIONS

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Abstract. Given an admissible 3-manifold M and a cohomology class $\phi \in H^1(M;\mathbb{R})$, we prove that the universal L^2-torsion of M detects the fiberedness of ϕ, except when M is a closed graph manifold that admits no non-positively curved metric. We further extend this invariant to sutured manifolds, where we derive a decomposition formula for taut sutured decompositions. Moreover, we show that a taut sutured manifold is a product if and only if its universal L^2-torsion is trivial. Our methods are based on a detailed study of the leading term map over Linnell's skew field. As an application, we apply the theory to homomorphisms between finitely generated free groups, which enables explicit computations of the invariant for sutured handlebodies.

1. Introduction

Let M be a compact orientable 3-manifold. A homomorphism $\phi\colon \pi_1(M) \to \mathbb{Z}$ is said to be fibered if it is induced by a fibration of M over S^1. In his seminal work, Thurston [Thu86] introduced a semi-norm on $H^1(M;\mathbb{R})$, now known as the Thurston norm, and showed that its unit ball $B_x(M)$ is a finite-sided polyhedron. Moreover, he showed that the fibered classes in $H^1(M;\mathbb{R})$ correspond exactly to certain open cones over the top-dimensional faces of $B_x(M)$. Marked by the success of Gabai's sutured manifold theory [Gab83, Gab87] and the confirmation of the Virtual Fibering Conjecture [AGM13], determining the Thurston norm and the fibered structure of a 3-manifold has become a central theme in three-dimensional topology. In this paper we focus on the class of admissible 3-manifolds, defined as follows:

Definition 1.1 (Admissible 3-manifold)**.** A 3-manifold is called *admissible* if it is compact, connected, orientable, and irreducible, its boundary is either empty or a collection of tori, and its fundamental group is infinite.

The universal L^2-torsion introduced by Friedl and Lück [FL17] is defined for a finite CW-complex X with vanishing L^2-Betti numbers. In the case when X is an admissible 3-manifold, this invariant $\tau_u^{(2)}(X)$ takes value in the weak Whitehead group $\mathrm{Wh}^w(\pi_1(X))$ associated to the fundamental group of X. It has been shown that the universal L^2-torsion completly determines the Thurston norm of such a manifold [FL17], highlighting its potential to reflect subtle topological information of 3-manifolds. This leads naturally to the following questions:

- Does the universal L^2-torsion also characterize the fibered structure of an admissible 3-manifold? If so, in what manner?
- Can this invariant be extended to fit into the framework of Gabai's sutured manifold theory? If so, how does it change under sutured manifold decompositions?

This paper is devoted to answering these two questions.

To investigate the first question, we are motivated by the role of the leading coefficient of the Alexander polynomial in detecting fiberedness. For any cohomology class $\phi \in$

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$H^1(M;\mathbb{R})$ of an admissible 3-manifold M, we define a natural "leading term map"

$$L_\phi\colon \operatorname{Wh}^w(\pi_1(M)) \to \operatorname{Wh}^w(\pi_1(M))$$

on the weak Whitehead group of $\pi_1(M)$. We show that, unless M is a closed graph manifold that does not admit a non-positively curved (NPC) metric, the class ϕ is fibered if and only if the image of $\tau_u^{(2)}(M)$ under L_ϕ is trivial.

Theorem 1.2. *Suppose M is an admissible 3-manifold that is not a closed graph manifold without an NPC metric. For any nonzero cohomology class $\phi \in H^1(M;\mathbb{R})$, the class ϕ is fibered if and only if $L_\phi \tau_u^{(2)}(M) = 1 \in \operatorname{Wh}^w(\pi_1(M))$.*

As a consequence, Theorem 1.2 provides a clear description of the marking on the L^2-torsion polytope $\mathcal{P}(M)$ introduced in [Kie20] which determines the fibered structure of M.

A substantial part of this paper is devoted to the second question, namely, extending the theory of universal L^2-torsion to the setting of sutured manifolds. A sutured manifold (N, R_+, R_-, γ) is a compact oriented 3-manifold whose boundary is partitioned into two oriented subsurfaces R_+ and R_-, meeting along their common boundary γ. A sutured manifold can be decomposed along a nicely embedded surface S and the resulting manifold is again a sutured manifold, we write

$$(N, R_+, R_-, \gamma) \overset{S}{\rightsquigarrow} (N', R'_+, R'_-, \gamma')$$

for such a decomposition. A key result of Herrmann [Her23] shows that, roughly speaking, a sutured manifold (N, R_+, R_-, γ) is taut if and only if the pair (N, R_+) has trivial L^2-Betti numbers (see Theorem 5.6). This motivates the definition of the universal L^2-torsion $\tau_u^{(2)}(N, R_+)$ for a taut sutured manifold, which takes values in $\operatorname{Wh}^w(\pi_1(N))$.

The second main result of this paper describes how the universal L^2-torsion behaves under taut sutured decompositions. Specifically, its change is determined by the leading term map introduced earlier.

Theorem 1.3. *Let $(N, R_+, R_-, \gamma) \overset{\Sigma}{\rightsquigarrow} (N', R'_+, R'_-, \gamma')$ be a taut sutured decomposition and let $\phi \in H^1(N;\mathbb{Z})$ be the Poincaré dual of the surface Σ, then*

$$j_*\tau_u^{(2)}(N', R'_+) = L_\phi \tau_u^{(2)}(N, R_+)$$

where $j_*\colon \operatorname{Wh}^w(\pi_1(N')) \to \operatorname{Wh}^w(\pi_1(N))$ is induced by the inclusion $j\colon N' \hookrightarrow N$.

Furthermore, we show that the universal L^2-torsion serves as an obstruction to the product structure on sutured manifolds. Indeed, a sutured manifold (N, R_+, R_-, γ) may be viewed as a cobordism between two compact surfaces $R_\pm$. When N is taut, the inclusion map $R_+ \hookrightarrow N$ induces isomorphism on L^2-homology. We prove that this cobordism is a trivial product if and only if its universal L^2-torsion is trivial—a result reminiscent of the celebrated s-cobordism theorem.

Theorem 1.4. *Let (N, R_+, R_-, γ) be a taut sutured manifold such that both R_+ and R_- are non-empty. Then N is diffeomorphic to the product $R_+ \times [0,1]$ if and only if $\tau_u^{(2)}(N, R_+) = 1 \in \operatorname{Wh}^w(\pi_1(N))$.*

We further study the computation of the universal L^2-torsion $\tau_u^{(2)}(X, Y)$ in the case where the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ are finitely generated free groups. In this setting, the universal L^2-torsion is completely determined by the homomorphism $\varphi\colon \pi_1(Y) \to \pi_1(X)$ induced by the inclusion $Y \hookrightarrow X$. It is natural to define the universal L^2-torsion $\tau_u^{(2)}(\varphi)$ for a group homomorphism between finitely generated free groups. The following result gives an explicit formula (see Proposition 6.2):

Theorem 1.5. *Let $\varphi\colon F_1 \to F_2$ be a homomorphism between finitely generated free groups. Then the universal L^2-torsion of φ is*

$$\tau_u^{(2)}(\varphi) = [J_\varphi] \in \mathrm{Wh}^w(F_2) \sqcup \{0\}$$

where J_φ is the Fox Jacobian matrix of φ over $\mathbb{Z}F_2$, whose entries are the Fox derivatives

$$(J_\varphi)_{ij} = \frac{\partial \varphi(x_i)}{\partial y_j} \in \mathbb{Z}F_2, \quad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$$

with respect to any chosen bases $F_1 = \langle x_1, \dots, x_n \rangle$, $F_2 = \langle y_1, \dots, y_m \rangle$.

We apply this formula to explicitly compute the universal L^2-torsion of certain sutured handlebodies. A key example is the sutured manifold $S^3 \backslash\backslash \Sigma$ obtained by decomposing the n-chain link complement along a minimal-genus Seifert surface Σ (see Figure 1). We show that

$$\tau_u^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+) = [1 + y_1 + \cdots + y_{n-1}] \in \mathrm{Wh}^w(\pi_1(S^3 \backslash\backslash \Sigma))$$

where $\{y_1, \dots, y_{n-1}\}$ is a free generating set of $\pi_1(S^3 \backslash\backslash \Sigma)$. Combining this with the calculations of [BA22], we obtain the following formula for the classical L^2-torsion:

$$\tau^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}} \sim \sqrt{n/e}, \quad \text{as } n \to +\infty.$$

Here, the left-hand side denotes the classical L^2-torsion of the pair $(S^3 \backslash\backslash \Sigma, \Sigma_+)$, obtained from the universal L^2-torsion via the Fuglede–Kadison determinant. Together with results from [Dua25], this computation yields an infinite family of hyperbolic manifolds for which the leading coefficient of the L^2-Alexander torsion associated to a nonzero class is greater than 1. This confirms the final remaining case of [BAFH22, Conjecture 1.7].

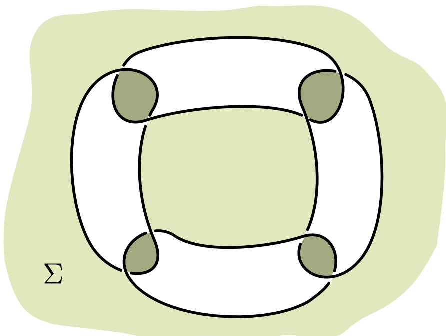

Figure 1. *The n-chain link and the minimal-genus Seifert surface Σ, where $n = 4$.*

1.1. Motivation.

1.1.1. *Why universal L^2-torsion.* To define the classical torsion invariants of a finite CW-complex X, one has to produce an exact sequence from the cellular chain complex of the universal cover $\widehat{X}$. One way to do this is to extend the scalars from the group ring $\mathbb{Z}[\pi_1(X)]$ to a *commutative* field. For 3-manifolds, this yields the Reidemeister–Franz torsion and the (multi-variable) Alexander polynomials [Mil62, McM02, FV11a], via scalar extensions to $\mathbb{C}$ and the field of rational functions $\mathbb{Q}(H_1(X))$, respectively. However, this base change loses non-commutative information of the fundamental group.

Whitehead torsion, introduced by J.H.C Whitehead, avoids this issue by retaining the full group ring structure. It takes values in the Whitehead group $\mathrm{Wh}(\pi_1(X))$ consisting of invertible matrices over $\mathbb{Z}[\pi_1(X)]$ modulo elementary relations [Mil66, Coh73]. However, the Whitehead torsion is only defined for pairs (X, Y) where X deformation retracts to Y. Also, it is too restrictive for a matrix to be invertible over $\mathbb{Z}[\pi_1(X)]$. Indeed, it is a well-known conjecture that the Whitehead group of any torsion-free group should vanish.

The universal L^2-torsion (see Section 3 for definitions) emerges as a powerful and widely applicable torsion invariant. Its advantages are threefold:

1. The universal L^2-torsion is defined via the faithful infinite-dimensional regular representation of the fundamental group and captures its deep noncommutative information.
2. It applies to a broader class of spaces whose L^2-Betti numbers vanish, including mapping tori [Lüc94], spaces with infinite amenable fundamental groups [CG86], admissible 3-manifolds [LL95] and all odd-dimensional closed hyperbolic manifolds [HS98] (see [Lüc02] for further examples).
3. The weak Whitehead group $\mathrm{Wh}^w(G)$ of any group G is never trivial (it contains a $\mathbb{Z}$-summand generated by the identity). In fact there are interesting non-trivial homomorphisms from $\mathrm{Wh}^w(G)$ such as the Fuglede–Kadison determinant and the polytope maps [FL17].

The universal L^2-torsion has been used to define group-theoretic Thurston norms for broader classes of groups, including free-by-cyclic groups [FK18, KS24] and coherent right-angled Artin groups [Kud24]. The BNS-invariant, which generalizes the notion of fiberedness from 3-manifold groups to arbitrary finitely generated groups, is deeply connected to the universal L^2-torsion, as evidenced by recent work on two-generator one-relator groups [FST17, FT20, HK20], free-by-cyclic groups [FK18] and more general agrarian groups by [Kie20]. These studies focus on the L^2-torsion polytope derived from the universal L^2-torsion and investigate the existence of a *marking* that determines the structure of the BNS-invariant.

A key novelty of our Theorem 1.2 is to provide a clear and direct description of the fibered structure for 3-manifolds in terms of the universal L^2-torsion itself. We expect that our technique can be modified to study more groups as mentioned above.

1.1.2. *Fiberedness and the leading term of torsions.* It is well-known that if a class ϕ is fibered, then its Alexander polynomial and twisted Alexander polynomial have trivial leading coefficients [McM02, GKM05, FK06]. While the converse does not hold true in general, Friedl and Vidussi [FV11b] showed that the collection of all twisted Alexander polynomial associated to ϕ does determine whether ϕ is fibered. This suggests that the leading term of torsion invariants in fact carry enough information for the detection of fiberedness.

The present work is largely motivated by the study of L^2-Alexander torsions. For a cohomology class $\phi \in H^1(M; \mathbb{R})$, the L^2-Alexander torsion was introduced by Dubois, Friedl and Lück [DFL16] as an L^2-analogue of the classical Alexander polynomial. Liu [Liu17] showed that a leading coefficient $C(M, \phi) \in [1, +\infty)$ of the L^2-Alexander torsion is well-defined and $C(M, \phi) = 1$ whenever ϕ is a fibered class. Conversely, it is not difficult to find examples that $C(M, \phi) = 1$ while ϕ is non-fibered. For example, take M to be any two-bridge knot complement and ϕ the canonical class; the leading coefficient $C(M, \phi)$ can be interpreted as the relative L^2-torsion of the guts $\Gamma(\phi)$ of ϕ [Dua25], which is a disjoint union of solid tori relative to an annulus on its boundary [AZ22]. The relative L^2-torsion in such cases is trivial because the Fuglede–Kadison determinant

$$\det{}_{\mathcal{N}\pi_1(M)} \colon \mathrm{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(t^{\pm})^{\times}/\{\pm t^n \mid n \in \mathbb{Z}\} \to \mathbb{R}_+$$

is not faithful, mapping certain nontrivial elements to 1. This observation suggests the need for a more refined invariant that is defined algebraically and does not rely on the Fuglede–Kadison determinant.

Accordingly, we consider the universal L^2-torsion and introduce the leading term map

$$L_\phi \colon \operatorname{Wh}^w(\pi_1(M)) \to \operatorname{Wh}^w(\pi_1(M))$$

which serves as an algebraic analogue of "taking the leading coefficient" within the weak Whitehead group. We refer to Section 4 for further details.

1.1.3. *Analogy with sutured Floer homology.* The Heegaard Floer homology, defined by Ozsváth and Szabó [OS04] for closed 3-manifolds, was generalized by Juhász [Juh06] to sutured Floer homology (SFH), an invariant for balanced sutured manifolds. In [Juh08] the fundamental properties of SFH are proved:

(1) If (M, γ) is a taut balanced sutured manifold, then $SFH(M, \gamma)$ is non-trivial.
(2) If $(M, \gamma) \rightsquigarrow (M', \gamma')$ is a sutured decomposition of balanced sutured manifolds, then $SFH(M', \gamma')$ is a direct summand of $SFH(M, \gamma)$.
(3) A taut balanced sutured manifold (M, γ) is a product sutured manifold if and only if $SFH(M, \gamma) = \mathbb{Z}$.

We observe that analogous properties hold in the theory of universal L^2-torsions. Specifically, property (1) corresponds to Herrmann's result (Theorem 5.6), while (2) and (3) are reflected in Theorems 1.3–1.4, respectively.

Question 1.6. *Is there a deeper connection between sutured Floer homology and the universal L^2-torsion of a taut sutured manifold?*

This question is particularly interesting because SFH is defined in terms of holomorphic curves and is not directly related to the fundamental group, whereas the universal L^2-torsion is algebraic and computable from a presentation of the fundamental group.

Furthermore, Juhász [Juh10] showed that the rank of SFH serves as a complexity measure for sutured manifolds that decreases under non-trivial decompositions. This motivates a parallel question in the L^2-context:

Question 1.7. *Can one define a natural complexity function on the weak Whitehead group that decreases under the leading term map L_ϕ?*

1.2. **Proof ingredients.** The remaining part of the paper is divided into five sections. We briefly discuss the contents of each these.

1.2.1. *Algebraic preliminaries.* Section 2 reviews the basic notions of L^2-theory, focusing on three topics that play a central role in the sequel: *Hilbert modules*, the *Atiyah Conjecture* and the *Linnell's skew field*, and *K_1-groups* together with the *Dieudonné determinant.*

1.2.2. *Universal L^2-torsion.* In Section 3, we define the universal L^2-torsion and discuss its computation. Let X be a finite CW-complex whose fundamental group G is torsion-free and satisfies the Atiyah Conjecture, the universal L^2-torsion $\tau_u^{(2)}(X)$ is defined as the torsion of the chain complex $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*(\widehat{X})$ viewed as a complex of modules over Linnell's skew field $\mathcal{D}_G$, and takes values in the Whitehead group $\operatorname{Wh}(\mathcal{D}_G)$. Slightly different from the original definition in [FL17], where the universal L^2-torsion lives in the weak Whitehead group $\operatorname{Wh}^w(G)$, our definition is more restrictive but is better suited to algebraic manipulation. Importantly, the two definitions coincide when X is a 3-manifold, or more generally when G belongs to Linnell's class $\mathcal{C}$; see Section 3.1.1 for a detailed discussion.

We further extend the definition of universal L^2-torsion for CW-pairs and to continuous mappings between finite CW-complexes via mapping cylinders. Several basic properties are established. This generalization brings greater flexibility to the application of the invariant.

Finally, we generalize Turaev's *matrix chain method* [Tur01] for the direct computation of the universal L^2-torsion of a chain complex. Informally, a chain complex is acyclic if and only if we can extract a chain of invertible submatrices from the matrices representing its connecting operators. The torsion is then given by the alternating product of their determinants.

1.2.3. *Leading term map, restriction map and polytope map.* The content of Section 4 is mostly technical, focusing on the study of three important homomorphisms.

Given any real character $\phi\colon G \to \mathbb{R}$ and a nonzero element $a = \sum_{g\in G} n_g \cdot g \in \mathbb{Z}G$, the ϕ-leading term of a is defined to be the sum of nonzero terms $n_g \cdot g$ for which $\phi(g)$ is minimal. Using the crossed product structure of the skew field $\mathcal{D}_G$, this construction extends to the *leading term homomorphism*

$$L_\phi\colon \mathcal{D}_G^\times \to \mathcal{D}_G^\times.$$

This homomorphism further induces homomorphisms on the K_1-group and the Whitehead group of $\mathcal{D}_G$. The key result concerning the leading term map (Theorem 4.13) says that, roughly speaking, if A, B are square matrices over $\mathcal{D}_G$ such that the ϕ-values of entries of B are strictly greater than those of A, then the ϕ-leading term of the determinant $\det(A + B)$ depends only on A. This intuitive result is surprisingly useful throughout our paper.

The *restriction map* relates the K_1-group of a group G to that of a finite-index subgroup $L < G$:

$$\mathrm{res}_L^G\colon K_1(\mathcal{D}_G) \to K_1(\mathcal{D}_L).$$

This is defined by interpreting an element $z \in K_1(\mathcal{D}_G)$ as an operator $r_z\colon \ell^2(L)^{[G\colon L]} \to \ell^2(L)^{[G\colon L]}$ and then taking its determinant. We prove that the leading term map commutes with the restriction map (see Theorem 4.18).

Every element $z \in \mathcal{D}_G^\times$ may be viewed as being "supported" on a certain Newton polytope in the real vector space $H_1(G;\mathbb{R})$. The *polytope map*

$$\mathbb{P}\colon \mathcal{D}_G^\times \to \mathcal{P}_{\mathbb{Z}}(H)$$

formalizes this intuition, where H is the free abelianization of G and $\mathcal{P}_{\mathbb{Z}}(H)$ is the Grothendieck group of the integral polytopes in $\mathbb{R} \otimes_{\mathbb{Z}} H$ under the Minkowski sum. It is known that $\mathcal{P}_{\mathbb{Z}}(H)$ is a free abelian group possessing an explicit basis [Fun21]. The polytope map is particularly useful for detecting nontrivial elements in $\mathcal{D}_G^\times/[\mathcal{D}_G^\times, \mathcal{D}_G^\times] = K_1(\mathcal{D}_G)$.

At the end of Section 4.3 we prove the "if" direction of Theorem 1.2, namely Theorem 4.25. The idea is as follows: the condition $L_\phi \tau_u^{(2)}(M) = 1$, when passing to finite index coverings, implies that the class ϕ is lifted to a top-dimensional cone in each covering. We then apply the Virtual Fibering Theorem on admissible 3-manifolds non-positively curved metrics (see [AFW15] and references therein) to conclude that ϕ is fibered.

1.2.4. *Universal L^2-torsion for taut sutured manifolds.* In Section 5, we investigate the universal L^2-torsion for sutured 3-manifolds, an object invented by Gabai [Gab83] to describe 3-manifolds via cut-and-paste constructions. To prove the decomposition formula in Theorem 1.3, we adopt the idea of Turaev's algorithm [Tur02, BAFH22] to reduce to the case of non-separating decomposition surfaces. Under an explicit CW-structure, the chain complex of the decomposed manifold is seen to be exactly the "leading term" of the chain complex of the original manifold. This idea rigorously formulated in Theorem 5.15.

Once the decomposition formula is established, the “only if” direction of Theorem 1.2 follows easily, see Theorem 5.16.

A refined doubling trick (Lemma 5.18) enables us to convert a taut sutured manifold into an admissible 3-manifold which is not a closed graph manifold. Combining this trick with the fiberedness criterion for admissible 3-manifolds (Theorem 1.2) and the decomposition formula (Theorem 1.3), we show that the universal L^2-torsion detects product sutured manifolds (Theorem 1.4).

1.2.5. *Applications of the universal L^2-torsion.* Section 6 is devoted to applications and concrete computations of the universal L^2-torsion. We begin by defining this invariant for homomorphisms between finitely generated free groups. An explicit computational formula is established in Proposition 6.2, which in particular proves Theorem 1.5. We then apply this formula to sutured manifolds modeled on 3-dimensional handlebodies, leading to criteria for detecting whether such a sutured manifold is taut or a product (Proposition 6.6). Explicit computations are carried out for the family of sutured manifolds obtained from cutting up the n-chain link complement along a Seifert surface (Example 6.8).

2. Algebraic preliminaries

Throughout this paper we adopt the following conventions: all groups are discrete, rings are unital but not necessarily commutative, and modules are left-modules. Fields with noncommutative multiplication are termed skew fields.

2.1. **Hilbert modules.** Let G be a group. Consider the following Hilbert space

$$\ell^2(G) = \Big\{ \sum_{g\in G} c_g \cdot g \;\Big|\; c_g \in \mathbb{C},\ \sum_{g\in G} |c_g|^2 < \infty \Big\}$$

with inner product

$$\Big\langle \sum_{g\in G} c_g \cdot g, \sum_{g\in G} d_g \cdot g \Big\rangle = \sum_{g\in G} c_g \overline{d_g}.$$

This Hilbert space admits natural left and right isometric G-actions by multiplication. The *group von Neumann algebra* $\mathcal{N}G$ is defined as the $\mathbb{C}$-algebra of all bounded linear operators of $\ell^2(G)$ that commutes with the left G-action. Every G-invariant closed subspace V of $\ell^2(G)^n$ is called a *Hilbert $\mathcal{N}G$-module* and can be assigned the *von-Neumann dimension* $\dim_{\mathcal{N}G} V$ which is a real number in $[0, n]$.

Let $\mathcal{U}G$ be the set of all densely-defined, closed operators (possibly unbounded) on $\ell^2(G)$ that commute with the left G-action. The composition and addition of two operators in $\mathcal{U}G$ are well-defined [Lü02, Section 8.1], making $\mathcal{U}G$ a $\mathbb{C}$-algebra and is called the *algebra of operators affiliated to $\mathcal{N}G$*. In particular, there is a natural inclusion

$$\mathbb{Z}G \subset \mathcal{U}G$$

where the integral group ring $\mathbb{Z}G$ embeds into $\mathcal{N}G$ by the right regular representation on $\ell^2(G)$. Moreover, any $m \times n$ matrix A over $\mathbb{Z}G$ can be viewed as a G-invariant bounded operator $r_A\colon \ell^2(G)^m \to \ell^2(G)^n$ defined by right multiplication.

Definition 2.1 (Weak isomorphism)**.** A G-invariant bounded operator $f\colon \ell^2(G)^m \to \ell^2(G)^n$ is called a *weak isomorphism* if f is injective with dense image. Note that if f is a weak isomorphism then m and n must be equal. A square matrix A over $\mathbb{Z}G$ is called a *weak isomorphism* if the G-invariant bounded operator $r_A\colon \ell^2(G)^m \to \ell^2(G)^n$ is a weak isomorphism.

2.2. **Atiyah Conjecture and Linnell's skew field.**

Definition 2.2 (Atiyah Conjecture)**.** A torsion-free group G is said to satisfy the *Atiyah Conjecture* if for any matrix A over $\mathbb{Z}G$ the von Neumann dimension of $\ker(r_A)$ is an integer.

The Atiyah Conjecture has been verified for a large class of groups (c.f. [Kie20, Theorem 4.2]). We mark the following important class of groups given by Linnell, which is large enough to include all 3-manifold groups.

Theorem 2.3 ([Lin93])**.** *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions by elementary amenable groups. Then any torsion-free group in $\mathcal{C}$ satisfies the Atiyah Conjecture.*

Theorem 2.4 ([KL24, Theorem 1.4])**.** *The fundamental group of any connected 3-manifold lies in $\mathcal{C}$.*

An alternative algebraic characterization of the Atiyah Conjecture is given by the division closure of $\mathbb{Z}G$ in $\mathcal{U}G$.

Definition 2.5 (Division closure)**.** Let R be a subring of a ring S. The *division closure* of R in S is the smallest subring of S containing R that is closed under taking inverses in S; that is, any element of this subring which is invertible in S has its inverse also in the subring. For a group G, we denote by $\mathcal{D}_G$ the division closure of $\mathbb{Z}G$ in $\mathcal{U}G$.

Theorem 2.6 ([Lin93])**.** *A torsion-free group G satisfies the Atiyah Conjecture if and only if $\mathcal{D}_G$ is a skew field.*

Therefore, for a torsion-free group G satisfying the Atiyah Conjecture, the ring $\mathcal{D}_G$ is a skew field called the *Linnell's skew field*. The following Proposition 2.7 summarizes useful functorial properties of this construction.

Proposition 2.7 ([Kie20, Proposition 4.6])**.** *Let G be a torsion-free group satisfying the Atiyah Conjecture. Then the following hold.*

(1) *Every automorphism of the group G extends to an automorphism of $\mathcal{D}_G$.*
(2) *If K is a subgroup of G, then K also satisfies the Atiyah Conjecture. Moreover, the natural embedding $\mathbb{Z}K \hookrightarrow \mathbb{Z}G$ extends to an embedding $\mathcal{D}_K \hookrightarrow \mathcal{D}_G$.*

2.3. **The K_1-group and Dieudonné determinant.** Let $\mathcal{F}$ be a skew field. For any positive integer n let $\mathrm{GL}(n, \mathcal{F})$ be the group of invertible $(n \times n)$-matrices over $\mathcal{F}$. By identifying each $M \in \mathrm{GL}(n, \mathcal{F})$ with the block matrix

$$\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1, \mathcal{F}),$$

we obtain a natural chain of inclusions

$$\mathrm{GL}(1, \mathcal{F}) \subset \mathrm{GL}(2, \mathcal{F}) \subset \cdots .$$

The direct union $\mathrm{GL}(\mathcal{F}) = \bigcup_{n \geqslant 1} \mathrm{GL}(n, R)$ is called the *infinite general linear group* over $\mathcal{F}$. A classical result of Whitehead (see e.g. [Mil66]) states that the commutator subgroup $[\mathrm{GL}(\mathcal{F}), \mathrm{GL}(\mathcal{F})]$ coincides with the subgroup generated by all elementary matrices in $\mathrm{GL}(\mathcal{F})$.

Definition 2.8 ([Die43, Ros95])**.** The *Dieudonné determinant* is the unique map

$$\det\colon \mathrm{GL}(\mathcal{F}) \to \mathcal{F}^{\times}/[\mathcal{F}^{\times}, \mathcal{F}^{\times}]$$

satisfying following the properties (a)–(c):

(a) The determinant is invariant under left elementary row operations. That is, if $A \in \mathrm{GL}(\mathcal{F})$ and A' is obtained from A by adding a left multiple of a row to another row, then $\det A = \det A'$.
(b) If $A \in \mathrm{GL}(\mathcal{F})$, and A' is obtained from A by left multiplying one of the rows by $a \in \mathcal{F}^\times$, then $\det A = \bar{a} \cdot \det A'$ where $\bar{a}$ is the image of a in $\mathcal{F}^\times/[\mathcal{F}^\times, \mathcal{F}^\times]$.
(c) The determinant of the identity matrix is $\bar{1}$.

The determinant also has the following additional properties (d)–(e).

(d) If $A, B \in \mathrm{GL}(n, \mathcal{F})$, then $\det(AB) = \det A \cdot \det B$.
(e) If $A \in \mathrm{GL}(\mathcal{F})$ and A' is obtained from A by interchanging two of its rows, then $\det A' = (-\bar{1}) \det A$.
(f) Suppose $\mathcal{F}$ is equipped with an involution. Then the determinant is compatible with the conjugate transpose, that is $\det(A^*) = (\det A)^*$.

Remark 2.9. The Dieudonné determinant is not invariant under the usual matrix transpose. For example, consider Linnell's skew field $\mathcal{D}_F$ of the free group $F = \langle x, y \rangle$. Then the matrix

$$A = \begin{pmatrix} x & 1 \\ xy & y \end{pmatrix}$$

has Dieudonné determinant $\det(A) = [yx - xy]$, while $A^T = \begin{pmatrix} x & xy \\ 1 & y \end{pmatrix}$ is not even invertible.

The Dieudonné determinant factors through the abelianization of $\mathrm{GL}(\mathcal{F})$, and the induced homomorphism is in fact an isomorphism.

Lemma 2.10 ([Ros95, Corollary 2.2.6])**.** *For any skew field $\mathcal{F}$, the Dieudonné determinant induces a group isomorphism*

$$\mathrm{GL}(\mathcal{F})/[\mathrm{GL}(\mathcal{F}), \mathrm{GL}(\mathcal{F})] \xrightarrow{\cong} \mathcal{F}^\times/[\mathcal{F}^\times, \mathcal{F}^\times].$$

The inverse map sends the class of $a \in \mathcal{F}^\times$ to the class of the (1×1)-matrix (a).

Definition 2.11 ($K_1(\mathcal{F})$ and $\widetilde{K}_1(\mathcal{F})$)**.** The *$K_1$-group* of $\mathcal{F}$ is defined as

$$K_1(\mathcal{F}) := F^\times/[\mathcal{F}^\times, \mathcal{F}^\times].$$

The *reduced K_1-group* $\widetilde{K}_1(\mathcal{F})$ is the quotient of $K_1(\mathcal{F})$ by the subgroup $\{[\pm 1]\}$.

We now specialize these definitions to the case of Linnell's skew field $\mathcal{D}_G$, where G is a torsion-free group satisfying the Atiyah Conjecture. In this context, we introduce the following Whitehead group:

Definition 2.12 ($\mathrm{Wh}(\mathcal{D}_G)$)**.** The *Whitehead group* of $\mathcal{D}_G$ is the quotient of $K_1(\mathcal{D}_G)$ by the subgroup generated by the classes $[\pm g]$ for $g \in G$:

$$\mathrm{Wh}(\mathcal{D}_G) := K_1(\mathcal{D}_G)/\langle [\pm g] \mid g \in G \rangle.$$

Definition 2.13. In the following we will frequently consider determinants and their images in the quotient groups $\widetilde{K}_1(\mathcal{D}_G)$ and $\mathrm{Wh}(\mathcal{D}_G)$. We therefore define the homomorphisms

$$\det_r\colon \mathrm{GL}(\mathcal{D}_G) \to \widetilde{K}_1(\mathcal{D}_G), \quad \det_w\colon \mathrm{GL}(\mathcal{D}_G) \to \mathrm{Wh}(\mathcal{D}_G)$$

to be the compositions of the Dieudonné determinant with the respective quotient maps, so that the following diagram commutes.

$$\begin{tikzcd}
 & & \mathrm{GL}(\mathcal{D}_G) \arrow[dl, "\det"'] \arrow[d, "\det_r"] \arrow[dr, "\det_w"] & \\
\mathcal{D}_G^\times/[\mathcal{D}_G^\times, \mathcal{D}_G^\times] \arrow[r, equal] & K_1(\mathcal{D}_G) \arrow[r] & \widetilde{K}_1(\mathcal{D}_G) \arrow[r] & \mathrm{Wh}(\mathcal{D}_G)
\end{tikzcd}$$

3. Universal L^2-torsion

Let G be a torsion-free group satisfying the Atiyah conjecture and let $\mathcal{D}_G$ be the Linnell's skew field. The universal L^2-torsion defined in this section takes values in the abelian groups $\widetilde{K}_1(\mathcal{D}_G)$ for $\mathbb{Z}G$-chain complexes, and in $\mathrm{Wh}(\mathcal{D}_G)$ for finite CW-complexes with fundamental group G. Although our definition is more restrictive than the original one given in [FL17], the two definitions coincide when G is a 3-manifold group. A detailed discussion of the relationship between the two is provided in Section 3.1.1.

3.1. Universal L^2-torsion of chain complexes.

A chain complex C_* is called a *finite based free $\mathbb{Z}G$-chain complex* if there exists $n \geqslant 0$ such that

$$C_* = (0 \to C_n \to \cdots \to C_1 \to C_0 \to 0),$$

where each C_k is a finitely generated free (left) $\mathbb{Z}G$-module equipped with a preferred unordered free $\mathbb{Z}G$-basis, and the boundary operators are $\mathbb{Z}G$-linear maps.

Definition 3.1. A finite based free $\mathbb{Z}G$-chain complex C_* is said to be *L^2-acyclic* if the chain complex $\ell^2(G) \otimes_{\mathbb{Z}G} C_*$ is weakly exact, i.e. the image $\operatorname{im} \partial_{k+1}^{(2)}$ is dense in $\ker \partial_k^{(2)}$ for all k, where $\partial_k^{(2)} \colon \ell^2(G) \otimes_{\mathbb{Z}G} C_k \to \ell^2(G) \otimes_{\mathbb{Z}G} C_{k-1}$ is the boundary operator.

We will not work with those analytic-flavored definitions but prefer the more algebraic-flavored ones given by the following Lemma 3.2.

Lemma 3.2 ([FL17, Lemma 2.21])**.** *A square matrix over $\mathbb{Z}G$ is a weak isomorphism if and only if it is invertible over $\mathcal{D}_G$. A finite based free $\mathbb{Z}G$-chain complex C_* is L^2-acyclic if and only if the chain complex $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_*$ is exact as a chain complex of left $\mathcal{D}_G$-modules.*

It is a classical topic to define the torsion of an exact chain complex of free modules, see for example [Mil66, Coh73, Tur01]. Our situation is particularly nice since $\mathcal{D}_G$ is a skew field and any module over a skew field is free. We follow the definitions given in [Tur01, Section 3].

Definition 3.3. Suppose V is a finitely generated $\mathcal{D}_G$-module with $\dim V = k$. For any two (unordered) bases $b = \{b_1, \dots, b_k\}$ and $c = \{c_1, \dots, c_k\}$, we have

$$b_i = \sum_{j=1}^{k} a_{ij} c_j, \quad i = 1, \dots, k,$$

where the transition matrix $(a_{ij})_{i,j=1,\dots,k}$ is a non-degenerate $(k \times k)$-matrix over $\mathcal{D}_G$. Define $[b/c]$ to be the determinant $\det_r(a_{ij}) \in \widetilde{K}_1(\mathcal{D}_G)$. This definition is independent of the ordering of the elements in b and c.

Definition 3.4 (Universal L^2-torsion of chain complexes)**.** Let C_* be a finite based $\mathbb{Z}G$-chain complex of length n that is L^2-acyclic. For each i, let c_i be the preferred basis of the free $\mathcal{D}_G$-module $\mathcal{D}_G \otimes_{\mathbb{Z}G} C_i$ and let ∂_i be the boundary homomorphism. Choose a basis b_i for the free $\mathcal{D}_G$-module $B_i := \operatorname{im} \partial_i$. Then b_i and b_{i-1} combines to form a basis $b_i b_{i-1}$ of C_i. The *universal L^2-torsion of C_** is defined to be

$$\tau_u^{(2)}(C_*) := \prod_{i=0}^{n} [b_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \widetilde{K}_1(\mathcal{D}_G).$$

This value is independent of the choice of the bases b_i.

An exact sequence $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ of finitely generated based free $\mathbb{Z}G$-modules is called *based exact*, if $i(b_0) \subset b_1$ and p maps $b_1 \setminus i(b_0)$ bijectively to b_2, where b_i is the preferred basis of M_i, $i = 0, 1, 2$. Similarly, an exact sequence $0 \to C_* \to D_* \to E_* \to 0$ of finite based free $\mathbb{Z}G$-chain complex is called *based exact* if in each degree k the sequence

$$0 \to C_k \to D_k \to E_k \to 0$$

is based exact. The following basic property can be found in [Tur01, Theorem 3.4].

Proposition 3.5. *The universal L^2-torsion satisfies the following properties.*

(1) *For any L^2-acyclic finite based free $\mathbb{Z}G$-chain complex*

$$C_* = (0 \to C_1 \xrightarrow{A} C_0 \to 0)$$

where A is a square matrix over $\mathbb{Z}G$, we have $\tau_u^{(2)}(C_*) = \det_r(A)^{-1} \in \widetilde{K}_1(\mathcal{D}_G)$.

(2) *Suppose $0 \to C'_* \to C_* \to C''_* \to 0$ is a based exact sequence of finite based free $\mathbb{Z}G$-chain complexes. If C'_* and C''_* are L^2-acyclic then so is C_*, and*

$$\tau_u^{(2)}(C_*) = \tau_u^{(2)}(C'_*) \cdot \tau_u^{(2)}(C''_*) \in \widetilde{K}_1(\mathcal{D}_G).$$

Definition 3.6 (Dual $\mathbb{Z}G$-modules)**.** Recall that $\mathbb{Z}G$ is a ring with an involution $x \mapsto x^*$ which sends $\sum n_g \cdot g$ to $\sum n_g \cdot g^{-1}$. For any left $\mathbb{Z}G$-module A, its *dual module* A^* is defined to be $\mathrm{Hom}_{\mathbb{Z}G}(A, \mathbb{Z}G)$, considered as a left $\mathbb{Z}G$-module as follows: for each $x \in \mathbb{Z}G$ and $f \in A^*$, the map $xf\colon A \to \mathbb{Z}G$ is given by $(xf)(y) = f(y) \cdot x^*$, $\forall y \in A$.

If A is a free $\mathbb{Z}G$-module with basis a_i, then A^* is also free and admits a dual basis a_i^*. Moreover, if $f\colon A \to B$ is a $\mathbb{Z}G$-linear map between based free $\mathbb{Z}G$-modules represented by a matrix P with respect to the given bases, then the dual map $f^*\colon B^* \to A^*$ is represented by the matrix P^*, the involution transpose of P.

The involution on $\mathbb{Z}G$ is compatible with taking adjoint in $\mathcal{U}G$. Since $\mathcal{D}_G$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}G$, and the set $\mathcal{D}_G \cap (\mathcal{D}_G)^*$ forms an inversion-closed subring containing $\mathbb{Z}G$, it follows that The Linnell's skew field $\mathcal{D}_G$ is itself closed under taking adjoint. Therefore $\mathcal{D}_G$ admits a canonical involution extending that of $\mathbb{Z}G$. The following Proposition 3.7 is a classic property of torsion invariants and can be proved as in [Mil62].

Proposition 3.7. *If $C_* = (0 \to C_n \to \cdots \to C_0 \to 0)$ is a finite based free $\mathbb{Z}G$-chain complex which is L^2-acyclic. Then the dual chain complex C^* is L^2-acyclic and*

$$(\tau_u^{(2)}(C^*))^* = \tau_u^{(2)}(C_*)^{(-1)^{n+1}} \in \widetilde{K}_1(\mathcal{D}_G).$$

3.1.1. *Comparing Friedl–Lück's definition.* Our Definition 3.4 of the universal L^2-torsion slightly differs from that of [FL17]. We first recall the definition of the weak K_1-groups introduced by [FL17], which is an abelian group analogous to the classical K_1-group, but rather than requiring the matrices over $\mathbb{Z}G$ to be invertible, we only require that they are weak isomorphisms (recall Definition 2.1).

Definition 3.8 ($K_1^w(\mathbb{Z}G)$, $\widetilde{K}_1^w(\mathbb{Z}G)$ and $\mathrm{Wh}^w(G)$)**.** Suppose G is any group. The *weak K_1-group* $K_1^w(\mathbb{Z}G)$ is defined in terms of generators and relations as follows. Generators $[A]$ are given by square matrices A over $\mathbb{Z}G$ such that A is a weak isomorphism. There are two sets of relations:

(i) If A, B are square matrices over $\mathbb{Z}G$ of the same size such that A, B are weak isomorphisms, then

$$[AB] = [A] \cdot [B].$$

(ii) If A, B are square matrices over $\mathbb{Z}G$ of size n and m, respectively. Suppose A, B are weak isomorphisms and let C be any matrix over $\mathbb{Z}G$ of size $n \times m$. Then

$$\left[\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right] = [A] \cdot [B] = [B] \cdot [A].$$

The *reduced weak K_1-group* $\widetilde{K}_1^w(\mathbb{Z}G)$ is defined to be the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup $\{[1], [-1]\}$. The *weak Whitehead group* $\mathrm{Wh}^w(G)$ is defined to be the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup $\{[\pm g] \mid g \in G\}$.

Proposition 3.9. *Suppose G is a torsion-free group which satisfies the Atiyah Conjecture.*

(1) *There are natural homomorphisms*

$$i_G \colon K_1^w(\mathbb{Z}G) \to K_1(\mathcal{D}_G), \quad \tilde{i}_G \colon \widetilde{K}_1^w(\mathbb{Z}G) \to \widetilde{K}_1(\mathcal{D}_G), \quad i_G^w \colon \mathrm{Wh}^w(G) \to \mathrm{Wh}(\mathcal{D}_G).$$

(2) *If G falls into the Linnell's class $\mathcal{C}$ (recall Theorem 2.3), then the three homomorphisms above are all isomorphisms.*

Proof. By Lemma 3.2 below, a square matrix A over $\mathbb{Z}G$ is a weak isomorphism if and only if it is invertible over $\mathcal{D}_G$. The homomorphism in (1) is given by viewing a weak isomorphism A over $\mathbb{Z}G$ as an invertible matrix over $\mathcal{D}_G$. (2) is the content of [LL17]. □

For any group G and any L^2-acyclic finite based $\mathbb{Z}G$-chain complex C_*, the universal L^2-torsion $\rho_u^{(2)}(C_*)$ defined in [FL17] takes values in the reduced weak K_1 group $\widetilde{K}_1^w(\mathbb{Z}G)$. Proposition 3.5 holds true for $\rho_u^{(2)}$ and moreover it characterizes the universal property of $\rho_u^{(2)}$ (in fact, in [FL17] the torsion $\rho_u^{(2)}(0 \to C_1 \xrightarrow{A} C_0 \to 0)$ is required to be $[A] \in \widetilde{K}_1^w(\mathbb{Z}G)$, rather than the inverse $[A^{-1}] = -[A]$). Therefore when G is torsion-free and satisfies the Atiyah Conjecture, the universal L^2-torsion $\tau_u^{(2)}(C_*)$ defined in the present paper is the image of $(-\rho_u^{(2)}(C_*))$ under the natural map

$$\tilde{i}_G \colon K_1^w(\mathbb{Z}G) \to K_1(\mathcal{D}_G).$$

If furthermore G lies in Linnell's class $\mathcal{C}$ then $\tilde{i}_G$ is a canonical isomorphism and the definitions of $\tau_u^{(2)}$ and $\rho_u^{(2)}$ are equivalent by Proposition 3.9. In particular, this includes all 3-manifold groups by Theorem 2.4.

Remark 3.10. In the literature, the torsion of a chain complex is defined under two major conventions. Consider the based exact chain complex

$$C_* = (0 \to C_1 \xrightarrow{A} C_0 \to 0).$$

The first convention ρ, commonly used for analytic torsions, typically defines $\rho(C_*)$ as $\det A$, viewing the torsion as an element of an *additive* group [Mil66, RS71, Coh73, BKMF96, Lüc02, FL17, Lüc18]. The second convention τ, common in topological contexts such as the theory of Alexander polynomials, defines $\tau(C_*)$ to be $(\det A)^{-1}$ and treats the torsion as an element of a *multiplicative* group [Mil62, Kit96, Tur01, FV11a, DFL16, Liu17]. Our paper follows the latter convention.

3.2. **Universal L^2-torsion of CW-complexes.** Let X be a connected finite CW-complex with torsion-free fundamental group G satisfying the Atiyah Conjecture and let $Y \subset X$ be a subcomplex. Let $p \colon \widehat{X} \to X$ be the universal covering of X, and set $\widehat{Y} := p^{-1}(Y)$ be the preimage. Then $\widehat{X}$ admits the induced CW-structure and $\widehat{Y}$ is a subcomplex of $\widehat{X}$.

The natural left G-action on $\widehat{X}$ gives rise to the left $\mathbb{Z}G$-module structure on the cellular chain complex $C_*(\widehat{X}, \widehat{Y})$. By choosing a lift $\widehat{\sigma}$ for each cell σ in $X \setminus Y$, $C_*(\widehat{X}, \widehat{Y})$

becomes a finite based free $\mathbb{Z}G$-chain complex. The following definition is independent of the choice of lifts.

Definition 3.11 (Universal L^2-torsion of CW-complexes)**.** Let X be a finite connected CW-complex with fundamental group G and let Y be a subcomplex of X. The pair (X, Y) is called *L^2-acyclic* if the finite based free chain complex $C_*(\widehat{X}, \widehat{Y})$ is L^2-acyclic (c.f. Lemma 3.2). The *universal L^2-torsion*

$$\tau_u^{(2)}(X, Y) \in \mathrm{Wh}(\mathcal{D}_G) \sqcup \{0\}$$

is defined as follows: if (X, Y) is L^2-acyclic, define $\tau_u^{(2)}(X, Y)$ to be the image of $\tau_u^{(2)}(C_*(\widehat{X}, \widehat{Y}))$ under the quotient map $\widetilde{K}_1(\mathcal{D}_G) \to \mathrm{Wh}(\mathcal{D}_G)$. Otherwise, define $\tau_u^{(2)}(X, Y) := 0$.

When X is not necessarily connected, we say the pair (X, Y) is *L^2-acyclic* if for every component $X_i \in \pi_0(X)$ the pair $(X_i, X_i \cap Y)$ is L^2-acyclic. Moreover, if the fundamental group $\pi_1(X_i)$ of each component satisfies the Atiyah Conjecture, we define

$$\mathrm{Wh}(\mathcal{D}_{\Pi(X)}) := \bigoplus_{X_i \in \pi_0(X)} \mathrm{Wh}(\mathcal{D}_{\pi_1(X_i)}),$$

and set

$$\tau_u^{(2)}(X, Y) := (\tau_u^{(2)}(X_i, X_i \cap Y))_{X_i \in \pi_0(X)} \in \mathrm{Wh}(\mathcal{D}_{\Pi(X)}).$$

If $(X_i, X_i \cap Y)$ is not L^2-acyclic for some i, we define $\tau_u^{(2)}(X, Y) := 0$.

Now let (X, Y) and (X', Y') be finite CW-pairs. A CW-mapping $f \colon (X, Y) \to (X', Y')$ is called *π_1-injective* if the restriction of f to each component of X induces an injection on fundamental groups. In this case, there is an induced homomorphism

$$\iota_* \colon \mathrm{Wh}(\mathcal{D}_{\Pi(X)}) \sqcup \{0\} \to \mathrm{Wh}(\mathcal{D}_{\Pi(X')}) \sqcup \{0\}.$$

We define the pushforward of the universal L^2-torsion as

$$\iota_* \tau_u^{(2)}(X, Y) \in \mathrm{Wh}(\mathcal{D}_{\Pi(X')}) \sqcup \{0\}$$

which the image of $\tau_u^{(2)}(X, Y)$ under ι_*. Note that by definition $\iota_* \tau_u^{(2)}(X, Y) = 0$ if and only if $\tau_u^{(2)}(X, Y) = 0$.

Theorem 3.12. *We record the fundamental properties of the universal L^2-torsion.*

(1) (Simple-homotopy invariance) *If $f \colon (X, X_0) \to (Y, Y_0)$ is a mapping of finite CW-pairs such that $f \colon X \to Y$ and $f|_{X_0} \colon X_0 \to Y_0$ are both simple-homotopy equivalences. Then*
$$\tau_u^{(2)}(Y, Y_0) = f_* \tau_u^{(2)}(X, X_0).$$
(2) (Sum formula) *Let $(U, V) = (X, C) \cup (Y, D)$ where (X, C), (Y, D) and $(X \cap Y, C \cap D)$ are L^2-acyclic sub-pairs that embeds π_1-injectively into (U, V), then*
$$\tau_u^{(2)}(U, V) = (\iota_1)_* \tau_u^{(2)}(X, C) \cdot (\iota_2)_* \tau_u^{(2)}(Y, D) \cdot (\iota_3)_* \tau_u^{(2)}(X \cap Y, C \cap D)^{-1}$$
where $\iota_1, \iota_2, \iota_3$ are the inclusions of (X, C), (Y, D), and $(X \cap Y, C \cap D)$ into (U, V), respectively.

(3) (Induction) *Let $f \colon (X_0, Y_0) \subset (X, Y)$ be a π_1-injective inclusion. Let $\widehat{X}$ be the universal cover of X and let $\widehat{X}_0, \widehat{Y}_0$ be the preimage of X_0, Y_0 in $\widehat{X}_0$. Then*
$$\tau_u^{(2)}(C_*(\widehat{X}_0, \widehat{Y}_0)) = f_* \tau_u^{(2)}(X_0, Y_0).$$

(4) (Restriction) *Let X be a connected finite CW-complex with $\pi_1(X) = G$, and let $\overline{X}$ be a connected finite degree covering of X with $\pi_1(\overline{X}) = H < G$. Denote by $\operatorname{res}_H^G \colon \operatorname{Wh}(\mathcal{D}_G) \sqcup \{0\} \to \operatorname{Wh}(\mathcal{D}_H) \sqcup \{0\}$ the restriction homomorphism (to be discussed in details in Section 4.4). For a subcomplex $Y \subset X$ with preimage $\overline{Y} \subset \overline{X}$, we have*
$$\tau_u^{(2)}(\overline{X}, \overline{Y}) = \operatorname{res}_H^G \tau_u^{(2)}(X, Y).$$

Proof. Property (1) will be proved in Remark 3.18 after introducing the universal L^2-torsion for mappings. Properties (2)–(4) are natural generalization of [FL17, Theorem 3.5] to CW-pairs and the proof carry over without essential changes to the relative cases. □

3.3. Universal L^2-torsion of mappings.

Definition 3.13 (Universal L^2-torsion of mappings)**.** Let $f \colon Y \to X$ be a cellular map between finite CW-complexes. The mapping cylinder of f is a finite CW-complex
$$M_f := ((Y \times I) \sqcup X)/\sim, \quad \text{where } (y, 0) \sim f(y) \text{ for all } y \in Y.$$
We identify Y with the subcomplex $Y \times \{1\}$ of M_f. If the fundamental group of X is torsion-free satisfying the Atiyah Conjecture, then the *universal L^2-torsion of the mapping f* is defined as
$$\tau_u^{(2)}(f) := \iota_* \tau_u^{(2)}(M_f, Y) \in \operatorname{Wh}(\mathcal{D}_{\Pi(X)}) \sqcup \{0\}$$
where $\iota \colon M_f \to X$ is the canonical deformation retraction.

Definition 3.14 (L^2-homology equivalence)**.** A map $f \colon Y \to X$ is called an *L^2-homology equivalence* if $\tau_u^{(2)}(f) \neq 0$, or equivalently (M_f, Y) is L^2-acyclic.

Proposition 3.15. *Suppose the spaces X, Y, Z are finite CW-complexes whose fundamental groups are torsion-free satisfying the Atiyah Conjecture.*

(1) *If (X, Y) is a CW-pair and $f \colon Y \to X$ the inclusion map. Then $\tau_u^{(2)}(X, Y) = \tau_u^{(2)}(f)$.*
(2) *If $f, g \colon X \to Z$ are homotopic cellular maps. Then $\tau_u^{(2)}(f) = \tau_u^{(2)}(g)$.*
(3) *If $f \colon X \to Z$ is a simple-homotopy equivalence, then $\tau_u^{(2)}(f) = 1$.*
(4) *If $f \colon X \to Y$ and $g \colon Y \to Z$ are L^2-homology equivalences. Suppose that g is π_1-injective, then $g \circ f$ is an L^2-homology equivalence with*
$$\tau_u^{(2)}(g \circ f) = g_* \tau_u^{(2)}(f) \cdot \tau_u^{(2)}(g).$$

Proof. Following [Coh73], we write $K \curvearrowright L$ if two finite CW-complexes K and L are related by a finite sequence of elementary collapses or expansions. If there is a common subcomplex K_0 that remains unchanged during the process, we write $K \curvearrowright L$ rel K_0. In this case, we have $\tau_u^{(2)}(K, K_0) = \iota_* \tau_u^{(2)}(L, K_0)$ where $\iota \colon L \to K$ is the natural homotopy equivalence.

We now prove the proposition part by part:

For (1), there are elementary expansions $M_f \curvearrowright X \times I$ and elementary collapses $X \times I \curvearrowright X \times \{1\}$, both relative to $Y = Y \times \{1\}$. Let $\iota \colon M_f \to X$ be the deformation retract, then $\tau_u^{(2)}(f) = \iota_* \tau_u^{(2)}(M_f, Y) = \tau_u^{(2)}(X, Y)$.

For (2), if $f, g \colon X \to Z$ are homotopic cellular maps, then $M_f \curvearrowright M_g$ rel X [Coh73, (5.5)], which implies $\tau_u^{(2)}(f) = \tau_u^{(2)}(g)$.

For (3), if $f \colon X \to Z$ is a simple-homotopy equivalence then $M_f \curvearrowright X$ rel X [Coh73, (5.8)].

For the proof of (4) we need the following "L^2-excision" property:

Lemma 3.16 (Excision)**.** *Let K, L be subcomplexes of the complex $K \cup L$ with $M = K \cap L$. If the inclusion $i\colon K \hookrightarrow K \cup L$ is π_1-injective. Then $\tau_u^{(2)}(K \cup L, L) = i_* \tau_u^{(2)}(K, M)$.*

Proof. As in [Coh73, (20.3)], we may assume that L and $K \cup L$ are connected. Let $\widehat{K \cup L}$ be the universal covering of $K \cup L$ and let $\widehat{L}$, $\widehat{K}$ and $\widehat{M}$ be the preimages of L, K and M under the covering, respectively. There is an isomorphism of chain complexes $C_*(\widehat{K \cup L}, \widehat{L}) = C_*(\widehat{K}, \widehat{M})$, so $\tau_u^{(2)}(K \cup L, L) = \tau_u^{(2)}(C_*(\widehat{K}, \widehat{M})) = j_* \tau_u^{(2)}(K, M)$ where the second equality follows from the induction property (Theorem 3.12). $\square$

We now complete the proof of Proposition 3.15(4). Let M be the union of M_f and M_g along the identity map on Y. Then $M \frown M_{g \circ f}$ rel $X \cup Z$ by [Coh73, (5.6)]. There is a diagram commutative up to homotopy

$$\begin{tikzcd}
M_f \arrow[r, "\iota_f"] \arrow[d, hook, "i_1"'] & Y \arrow[r, hook, "i"] & M_g \arrow[d, "\iota_g"] \arrow[dll, hook', "i_2"'] \\
M \arrow[rr, "\iota"] & & Z
\end{tikzcd}$$

where i, i_1, i_2 are inclusions and ι, ι_f, ι_g are the canonical deformation retracts. Then we have

$$\begin{aligned}
\tau_u^{(2)}(g \circ f) &= \iota_* \tau_u^{(2)}(M, X) \\
&= \iota_*(\tau_u^{(2)}(M, M_f) \cdot (i_1)_* \tau_u^{(2)}(M_f, X)), \quad \text{by sum formula} \\
&= \iota_*((i_2)_* \tau_u^{(2)}(M_g, Y) \cdot (i_1)_* \tau_u^{(2)}(M_f, X)), \quad \text{by excision} \\
&= (\iota_g)_* \tau_u^{(2)}(M_g, Y) \cdot g_* (\iota_f)_* \tau_u^{(2)}(M_f, X), \quad \text{since } \iota_g \circ i = g \\
&= \tau_u^{(2)}(g) \cdot g_* \tau_u^{(2)}(f).
\end{aligned}$$

This completes the proof. $\square$

The identity $\tau_u^{(2)}(g \circ f) = g_* \tau_u^{(2)}(f) \cdot \tau_u^{(2)}(g)$ is called the *multiplicativity* of the universal L^2-torsion. The conditions that f, g are L^2-homology equivalences and g is π_1-injective is in general necessary. But when one of the maps is a simple homotopy equivalence, the conditions can be relaxed as follows.

Lemma 3.17. *Let X, Y, Z, W be finite CW-complexes and consider the chain of maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$. Suppose $g\colon Y \to Z$ is a simple homotopy equivalence. Then*

(1) $\tau_u^{(2)}(g \circ f) = g_* \tau_u^{(2)}(f)$, *and*
(2) $\tau_u^{(2)}(h \circ g) = \tau_u^{(2)}(h)$.

In particular, f is an L^2-homology equivalence if and only if $g \circ f$ is, and similarly h is an L^2-homology equivalence if and only if $h \circ g$ is.

Proof. Note that a simple homotopy equivalence g admits a homotopy inverse g^{-1} which is also a simple homotopy equivalence.

For (1), if $\tau_u^{(2)}(f) \neq 0$ then the result follows directly from Proposition 3.15(4). If $\tau_u^{(2)}(f) = 0$ then we must have $\tau_u^{(2)}(g \circ f) = 0$, since otherwise applying the inverse would give the contradiction $\tau_u^{(2)}(f) = (g^{-1})_* \tau_u^{(2)}(g \circ f) \neq 0$.

For (2), suppose first that $\tau_u^{(2)}(h) \neq 0$. Let M be the union of M_g and M_h along the identity map on Z. Then $M \frown M_{h \circ g}$ rel $Y \cup W$. The Excision Lemma 3.16 applied to $i\colon M_h \subset M$ shows that (M, M_g) is L^2-acyclic with $\tau_u^{(2)}(M, M_g) = i_* \tau_u^{(2)}(M_h, Z)$. Since

g is a simple homotopy equivalence we have $M_g \curvearrowright Y$ rel Z. Let $\iota\colon M \to W$ be the deformation retract. Then

$$\tau_u^{(2)}(h \circ g) = \iota_* \tau_u^{(2)}(M, Y) = \iota_* \tau_u^{(2)}(M, M_g) = (\iota \circ i)_* \tau_u^{(2)}(M_h, Z) = \tau_u^{(2)}(h).$$

Now if $\tau_u^{(2)}(h) = 0$, then $\tau_u^{(2)}(h \circ g)$ must also be zero, for otherwise applying the result to the composition $(h \circ g) \circ g^{-1}$ would yield the contradiction $\tau_u^{(2)}(h) = \tau_u^{(2)}((h \circ g) \circ g^{-1}) = \tau_u^{(2)}(h \circ g) \neq 0$. ☐

Remark 3.18. As a corollary of Lemma 3.17, we prove the simple-homotopy invariance stated in Theorem 3.12. Let $f\colon (X, X_0) \to (Y, Y_0)$ be a mapping of CW-pairs such that $f\colon X \to Y$ and $f|_{X_0}\colon X_0 \to Y_0$ are simple-homotopy equivalences. Then we have the following commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, hook, "i_X"'] & Y_0 \arrow[d, hook, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and

$$f_* \tau_u^{(2)}(X, X_0) = f_* \tau_u^{(2)}(i_X) = \tau_u^{(2)}(f \circ i_X) = \tau_u^{(2)}(i_Y \circ f_0) = \tau_u^{(2)}(i_Y) = \tau_u^{(2)}(Y, Y_0).$$

3.4. **Universal L^2-torsion of manifolds.** We now define the universal L^2-torsion for smooth manifold pairs. Recall that a *smooth triangulation* of a smooth manifold M is a homeomorphism from a simplicial complex to M that is smooth on each simplex.

Definition 3.19 (Universal L^2-torsion of manifold pairs)**.** Let M be a compact, smooth manifold, possibly with boundary, and let N be a compact, smooth submanifold of M. Suppose M admits a smooth triangulation X in which N is a subcomplex Y. Then we define $\tau_u^{(2)}(M, N) := \tau_u^{(2)}(X, Y)$.

For the purposes of this paper, we assume that either N is a zero-codimensional submanifold of ∂M, or the embedding $N \hookrightarrow M$ is proper (i.e. $N \cap \partial M = \partial N$). In these cases one can find a smooth triangulation of M such that N is a subcomplex of M ([Mun66, Chapter 10]). Any two such triangulations have a common subdivision and are simple homotopy equivalent as CW-complexes [Whi40]. Therefore $\tau_u^{(2)}(M, N)$ is well-defined by simple homotopy invariance of the universal L^2-torsion (Theorem 3.12).

Definition 3.20 (Universal L^2-torsion of mappings between manifolds)**.** Let $f\colon N \to M$ be a continuous mapping between compact smooth manifolds (possibly with boundaries). Choose smooth triangulations of M, N and let g be a simplicial approximation of f. The *universal L^2-torsion of f* is defined as

$$\tau_u^{(2)}(f) := \tau_u^{(2)}(g) \in \mathrm{Wh}(\mathcal{D}_{\Pi(M)}) \sqcup \{0\}.$$

It follows from Proposition 3.15 and Lemma 3.17 that $\tau_u^{(2)}(f)$ is independent of the choice of triangulations on M, N and the simplicial approximation g. In particular, if $N \subset M$ is a smooth submanifold with f the inclusion map, then $\tau_u^{(2)}(f) = \tau_u^{(2)}(M, N)$.

3.5. **Methods for computations.** We now present the *matrix chain method* for computing the universal L^2-torsion of a chain complex which originates from [Tur01, Theorem 2.2].

Let $C_* = (0 \to C_n \to \cdots \to C_1 \to C_0 \to 0)$ be a finite based free $\mathbb{Z}G$-chain complex and let $\partial_i\colon C_i \to C_{i-1}$ be the boundary operator. Suppose $d_i := \operatorname{rank}_{\mathbb{Z}G} C_i$ is the rank of the free module C_i. Then ∂_i is given by a matrix

$$A_i = (a^i_{jk})_{\substack{j=1,\dots,d_i \\ k=1,\dots,d_{i-1}}}, \quad a^i_{jk} \in \mathbb{Z}G.$$

Definition 3.21. A *matrix chain* for C_* is a collection of finite sets $\mathcal{A} = \{\mathcal{I}_0, \dots, \mathcal{I}_n\}$ where $\mathcal{I}_i \subset \{1, \dots, d_i\}$ and $\mathcal{I}_n = \emptyset$. Let B_i be the submatrix of A_i formed by the entries a^i_{jk} with $j \notin \mathcal{I}_i$ and $k \in \mathcal{I}_{i-1}$. Then $\{B_i\}$ are called the *matrices associated to the matrix chain.*

A matrix chain is called *non-degenerate* if each associated matrix is a square matrix and is invertible over $\mathcal{D}_G$.

Theorem 3.22. *A finite based free $\mathbb{Z}G$-chain complex C_* is L^2-acyclic if and only if there exists an non-degenerate matrix chain $\mathcal{A} = \{\mathcal{I}_0, \dots, \mathcal{I}_n\}$ for C_*. If this happens, then*

$$\tau_u^{(2)}(C_*) = \prod_{i=1}^{n} \det{}_r(B_i)^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G)$$

where B_i are the matrices associated to the matrix chain.

The proof is a generalization of the idea of [DFL16, Lemma 3.1] to larger chain complexes and is well-known to experts. We give an sketched proof here.

Proof Sketch. Suppose C_* is L^2-acyclic. Then A_n is injective over $\mathcal{D}_G$ and there is a $(d_n \times d_n)$-submatrix B_n of A_n such that B_n is invertible over $\mathcal{D}_G$. Let $\mathcal{I}_{n-1} \subset \{1, \dots, d_{n-1}\}$ be the set of indices of the columns of B_i. Write $C_{n-1} = C'_{n-1} \oplus C''_{n-1}$ where C'_{n-1} corresponds to $\mathcal{I}_{n-1}$ and C''_{n-1} corresponds to the remaining indices. The boundary map $A_{n-1} \colon C_{n-1} \to C_{n-2}$ vanishes on C'_{n-1}, yielding the following commuting diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & C_n \arrow[r, "="] \arrow[d, "B_n", "\cong"'] & C_n \arrow[r] \arrow[d, "A_n"] & 0 \arrow[d] & \\
0 \arrow[r] & C'_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d, "A_{n-1}"] & C''_{n-1} \arrow[r] \arrow[d, "A''_{n-1}"] & 0 \\
 & 0 \arrow[r] & C_{n-2} \arrow[r, "="] \arrow[d, "A_{n-2}"] & C_{n-2} \arrow[r] \arrow[d, "A_{n-2}"] & 0 \\
 & & \vdots & \vdots &
\end{tikzcd}$$

Repeating this procedure for A''_{n-1} and the vertical sequences on the right eventually yields matrices $B_n, \dots, B_1$ that form a non-degenerate matrix chain for C_*.

Conversely, suppose $\mathcal{A} = \{\mathcal{I}_0, \dots, \mathcal{I}_n\}$ is a non-degenerate matrix chain for C_*. Decompose $C_{n-1} = C'_{n-1} \oplus C''_{n-1}$ as before and consider the the commutative diagram above. Define the chain complexes:

$$C'_* = (0 \to C_n \xrightarrow{B_n} C'_{n-1} \to 0),$$

$$C''_* = (0 \longrightarrow C''_{n-1} \xrightarrow{A''_{n-1}} C_{n-2} \xrightarrow{A_{n-2}} \cdots \longrightarrow C_1 \xrightarrow{A_1} C_0 \longrightarrow 0).$$

Then $\{\mathcal{I}_0, \dots, \mathcal{I}_{n-1}\}$ is a non-degenerate matrix chain for C''_* and we have the short exact sequence of based $\mathbb{Z}G$-chain complex

$$0 \to C'_* \to C_* \to C''_* \to 0.$$

By induction on n we may assume that C''_* is L^2-acyclic with

$$\tau_u^{(2)}(C''_*) = \prod_{i=1}^{n-1} \det_r(B_i)^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G).$$

Then by Proposition 3.5,

$$\tau_u^{(2)}(C_*) = \tau_u^{(2)}(C'_*) \cdot \tau_u^{(2)}(C''_*) = \prod_{i=1}^{n} \det_r(B_i)^{(-1)^i} \in \widetilde{K}_1(\mathcal{D}_G).$$

□

Proposition 3.23. *We compute the universal L^2-torsion for the following manifold pairs.*

(1) *Let N be a compact smooth manifold whose fundamental group is torsion-free satisfying the Atiyah Conjecture. Then for any $s \in [0,1]$,*
$$\tau_u^{(2)}(N \times I, N \times \{s\}) = 1 \in \mathrm{Wh}(\mathcal{D}_{\Pi(N)}).$$
(2) *Let S^1 be the circle with fundamental group $\pi_1(S^1) = \langle t \rangle$. Then*
$$\tau_u^{(2)}(S^1) = [t-1]^{-1} \in \mathrm{Wh}(\mathcal{D}_{\mathbb{Z}}).$$
(3) *Let T^2 be the torus. Then*
$$\tau_u^{(2)}(T^2) = 1 \in \mathrm{Wh}(\mathcal{D}_{\mathbb{Z}^2}).$$

Proof. For (1), let $f\colon N \times \{s\} \to N \times I$ be the inclusion map, then f is a simple-homotopy equivalence and $\tau_u^{(2)}(N \times I, N \times \{s\}) = \tau_u^{(2)}(f) = 1$ by Proposition 3.15.

For (2), consider the standard CW-structure on S^1 with one 0-cell p and one 1-cell e. After choosing appropriate lifts $\hat{p}$ and $\hat{e}$, the cellular chain complex of the universal cover is

$$C_*(\widehat{S}^1) = (0 \to \mathbb{Z}[t^{\pm}] \cdot \langle \hat{e} \rangle \xrightarrow{(t-1)} \mathbb{Z}[t^{\pm}] \cdot \langle \hat{p} \rangle \to 0)$$

and hence $\tau_u^{(2)}(S^1) = [t-1]^{-1}$.

For (3), consider the standard CW-structure for T^2 given by identifying opposite sides of a square. Let p be the 0-cell, e_1, e_2 the 1-cells and σ the 2-cell. The boundary of σ is the loop $e_1 e_2 e_1^{-1} e_2^{-1}$. If e_1, e_2 represents generators $t_1, t_2 \in \pi_1(T^2)$ respectively, then with appropriate lifts we obtain the chain complex $C_*(\widehat{T}^2)$:

$$0 \to \mathbb{Z}[t_1^{\pm}, t_2^{\pm}] \cdot \langle \hat{\sigma} \rangle \xrightarrow{\begin{pmatrix} 1-t_2 & t_1 - 1 \end{pmatrix}} \mathbb{Z}[t_1^{\pm}, t_2^{\pm}] \cdot \langle \hat{e}_1, \hat{e}_2 \rangle \xrightarrow{\begin{pmatrix} t_1 - 1 \\ t_2 - 1 \end{pmatrix}} \mathbb{Z}[t_1^{\pm}, t_2^{\pm}] \cdot \langle \hat{p} \rangle \to 0.$$

A matrix chain is given by $B_2 = (1 - t_2)$ and $B_1 = (t_2 - 1)$, hence $\tau_u^{(2)}(T^2) = [1 - t_2] \cdot [t_2 - 1]^{-1} = 1$. □

4. Leading term map, restriction map and polytope map

Let G be a torsion-free group satisfying the Atiyah Conjecture. We will define the leading term map $L_\phi : \mathcal{D}_G \to \mathcal{D}_G$ associated to a character $\phi \in H^1(G;\mathbb{R})$. The main objectives of this section are to establish two key results: Theorem 4.13, which relates the leading term map to the Dieudonné determinant, and Theorem 4.18, which shows that the leading term map commutes with the restriction map in appropriate sense. We conclude the section by proving the "if" direction of Theorem 1.2 using the polytope maps (stated as Theorem 4.25).

4.1. **Ore localization.**

Definition 4.1 (Ore localization)**.** Let R be a ring and let $S \subset R$ be a multiplicatively closed subset. The pair (R, S) satisfies the *(right) Ore condition* if the following two conditions hold:

(1) for any $(r, s) \in R \times S$ there exists $(r', s') \in R \times S$ such that $rs' = sr'$, and
(2) for any $(r, s) \in R \times S$ with $sr = 0$, there is $t \in S$ with $rt = 0$.

If (R, S) satisfies the Ore condition, define an equivalence relation on $R \times S$ by

$$(r, s) \sim (rx, sx) \quad \text{whenever} \quad x \in R \text{ and } sx \in S.$$

The quotient set $R \times S/\sim$ is denoted by RS^{-1}. Define a ring structure on RS^{-1} as follows. Given two representatives $(r, s), (r', s') \in RS^{-1}$, choose $c \in R,\ d \in S$ such that $sc = s'd \in S$ and define addition by

$$(r, s) + (r', s') = (rc + r'd, sc).$$

Similarly, choose $e \in R,\ f \in S$ with $se = r'f$ and define

$$(r, s) \cdot (r', s') = (re, s'f).$$

The resulting ring RS^{-1} is called the *Ore localization* of R at S.

Intuitively, a pair $(r, s) \in RS^{-1}$ is viewed as a formal fraction rs^{-1}. The first Ore condition can be understood as whenever there is a "left fraction" $s^{-1}r$ then there is a "right fraction" $r'(s')^{-1}$ such that $rs' = sr'$. The second condition is automatically satisfied if S contains no zero divisors. In this paper, we will only need Ore localizations of this simple type:

Lemma 4.2 ([Coh95, Corollary 1.3.3])**.** *Let R be an integral domain and $R^\times$ the set of nonzero elements. Then $(R, R^\times)$ satisfies the Ore condition if and only if $aR \cap bR \neq \{0\}$ for all nonzero $a, b \in R$. In this case, the Ore localization of R at $R^\times$ is a skew field K and the natural homomorphism $\lambda\colon R \to K$ is an embedding.*

Definition 4.3. An integral domain R satisfying the equivalent conditions in Lemma 4.2 is called an *Ore domain*. The skew field K is called the *field of fractions* of R.

4.2. **Crossed products.** Assume that G is a torsion-free group which satisfies the Atiyah conjecture. Consider a short exact sequence of groups

$$1 \to K \to G \xrightarrow{\nu} H \to 1.$$

Then K is also torsion-free and satisfies the Atiyah Conjecture by Proposition 2.7. Denote by $\mathcal{D}_K$ and $\mathcal{D}_G$ the Linnell's skew fields of K and G, respectively.

Choose a set-theoretic section $s\colon H \to G$ for the epimorphism ν such that $\nu \circ s = \mathrm{id}_H$. Consider the following subset of $\mathcal{D}_G$:

$$\mathcal{D}_K *_s H := \left\{ \sum_{h \in H} x_h \cdot s(h) \in \mathcal{D}_G \;\middle|\; x_h \in \mathcal{D}_K,\ x_h = 0 \text{ for all but finitely many } h \in H \right\}.$$

This set contains the zero element 0, the identity element $1 = s(1_H)^{-1} \cdot s(1_H)$, and is closed under addition. Moreover, it is closed under multiplication, since

$$\begin{aligned} &\Big(\sum_{h \in H} x_h \cdot s(h)\Big) \cdot \Big(\sum_{h \in H} y_h \cdot s(h)\Big) \\ &= \sum_{h_1, h_2 \in H} x_{h_1} \cdot s(h_1) \cdot y_{h_2} \cdot s(h_2) \\ &= \sum_{h_1, h_2 \in H} \underbrace{x_{h_1} s(h_1) y_{h_2} s(h_1)^{-1}}_{\in \mathcal{D}_K} \cdot \underbrace{s(h_1) s(h_2) s(h_1 h_2)^{-1}}_{\in K} \cdot s(h_1 h_2). \end{aligned}$$

Recall that the group automorphism of K given by conjugation by $s(h_1)$ extends to an automorphism of $\mathcal{D}_K$ by Proposition 2.7. It follows that $\mathcal{D}_K *_s H$ is a subring of the field $\mathcal{D}_G$.

Proposition 4.4. *With notations as above, we have the following properties.*

1. *An element $\sum_{h\in H} x_h \cdot s(h)$ of $\mathcal{D}_K *_s H$ is zero if and only if $x_h = 0$ for all $h \in H$.*
2. *Given another section $s'\colon H \to G$, then*
$$\sum_{h\in H} x_h \cdot s(h) = \sum_{h\in H} y_h \cdot s'(h)$$
if and only if $y_h = x_h s(h)s'(h)^{-1}$ for all $h \in H$.

Proof. The first statement is a consequence of [Lüc02, Lemma 10.57]. The second statement follows from the previous one. □

As a corollary, the subring $\mathcal{D}_K *_s H \subset \mathcal{D}_G$ is independent of the choice of the section s. We call this subring the *crossed product* of $\mathcal{D}_K$ and H, denoted by $\mathcal{D}_K * H$. Clearly $\mathcal{D}_K * H$ is an integral domain since it embeds into the skew field $\mathcal{D}_G$. In some cases, the relation between $\mathcal{D}_G$ and its subring $\mathcal{D}_K * H$ is particularly simple.

Proposition 4.5. *Let $1 \to K \to G \to H \to 1$ be a group extension.*

1. *If H is finite, then $\mathcal{D}_G = \mathcal{D}_K * H$.*
2. *If H is virtually a finitely generated abelian group, then the integral domain $\mathcal{D}_K * H$ is an Ore domain whose field of fractions is $\mathcal{D}_G$.*

Proof. These statements are Lemmas 10.59 and 10.69 of [Lüc02], respectively. □

4.3. **The leading term map.** Assume further that G is finitely generated and consider the short exact sequence
$$1 \to K \to G \xrightarrow{\nu} H_1(G)_f \to 1.$$
where $\nu\colon G \to H_1(G)_f$ is the natural quotient map to the free abelian part of the first homology group $H_1(G)_f$. Fix a set-theoretic section $s\colon H_1(G)_f \to G$ and let $\phi \in H^1(G;\mathbb{R})$ be a real cohomology class.

Definition 4.6. For any nonzero element $u \in (\mathcal{D}_K * H_1(G)_f)^\times$, write
$$u = \sum_{h\in H_1(G)_f} x_h \cdot s(h) \in \mathcal{D}_K * H_1(G)_f.$$
The *support of* u is a finite set $\operatorname{supp}(u) := \{h \in H_1(G)_f \mid x_h \neq 0\}$, which is independent of the choice of section by Proposition 4.4. Define $\delta_\phi(u)$ to be the minimal value of $\phi(h)$ for all $h \in \operatorname{supp}(u)$. Define
$$L_\phi(u) := \sum_{\substack{h\in \operatorname{supp}(u),\\ \phi(h)=\delta_\phi(z)}} x_h \cdot s(h).$$
This element is nonzero and belongs to $(\mathcal{D}_K * H_1(G)_f)^\times$.

Lemma 4.7. *The definitions of $\delta_\phi(u)$ and $L_\phi(u)$ is independent of the choice of section. Moreover, for all $u_1, u_2 \in (\mathcal{D}_K * H_1(G)_f)^\times$,*
$$\delta_\phi(u_1u_2) = \delta_\phi(u_1) + \delta_\phi(u_2),$$
$$L_\phi(u_1u_2) = L_\phi(u_1) \cdot L_\phi(u_2).$$
Hence,
$$\delta_\phi\colon (\mathcal{D}_K * H_1(G)_f)^\times \to \mathbb{R},$$
$$L_\phi\colon (\mathcal{D}_K * H_1(G)_f)^\times \to (\mathcal{D}_K * H_1(G)_f)^\times$$

are well-defined group homomorphisms.

Proof. Let $u = \sum_h x_h \cdot s(h)$ and let s' be another section. By Proposition 4.4 $u = \sum_h y_h \cdot s'(h)$ where $y_h = x_h s(h) s'(h)^{-1}$. It follows that $\delta_\phi(u)$ and $L_\phi(u)$ do not depend on the choice of section. The terms of $u_1 u_2$ with minimal ϕ-value are exactly from the products of that of u_1 and u_2. This implies the homomorphism properties. ☐

Recall that G is finitely generated and by Proposition 4.5 the Linnell skew field $\mathcal{D}_G$ is the field of fractions of the subring $\mathcal{D}_K * H_1(G)_f$.

Definition 4.8 (Leading term map L_ϕ)**.** The group homomorphisms δ_ϕ and L_ϕ extend to group homomorphisms

$$\begin{aligned} \delta_\phi \colon \mathcal{D}_G^\times &\to \mathbb{R}, & \delta_\phi(uv^{-1}) &:= \delta_\phi(u) - \delta_\phi(v), \\ L_\phi \colon \mathcal{D}_G^\times &\to \mathcal{D}_G^\times, & L_\phi(uv^{-1}) &:= L_\phi(u) L_\phi(v)^{-1} \end{aligned}$$

for all $u, v \in (\mathcal{D}_K * H_1(G)_f)^\times$.

Furthermore, we set $\delta_\phi(0) = +\infty$ and $L_\phi(0) = 0$. The extended maps satisfy

$$\begin{aligned} \delta_\phi \colon \mathcal{D}_G &\to \mathbb{R} \cup \{+\infty\}, & \delta_\phi(z_1 z_2) &= \delta_\phi(z_1) + \delta_\phi(z_2), \\ L_\phi \colon \mathcal{D}_G &\to \mathcal{D}_G, & L_\phi(z_1 z_2) &= L_\phi(z_1) \cdot L_\phi(z_2) \end{aligned}$$

for all $z_1, z_2 \in \mathcal{D}_G$.

Proof. We first verify well-definedness. Suppose $z \in \mathcal{D}_G^\times$ admits two representations $z = u_1 v_1^{-1} = u_2 v_2^{-1}$, then there exists $w_1, w_2 \in (\mathcal{D}_K * H_1(G)_f)^\times$ such that $u_1 w_1 = u_2 w_2$ and $v_1 w_1 = v_2 w_2$. Hence

$$\begin{aligned} L_\phi(u_1) L_\phi(v_1)^{-1} &= L_\phi(u_1) L_\phi(w_1) L_\phi(w_1)^{-1} L_\phi(v_1)^{-1} \\ &= L_\phi(u_1 w_1) L_\phi(v_1 w_1)^{-1} \\ &= L_\phi(u_2 w_2) L_\phi(v_2 w_2)^{-1} \\ &= L_\phi(u_2) L_\phi(v_2)^{-1}. \end{aligned}$$

To show that L_ϕ is a homomorphism, let $z_1, z_2 \in \mathcal{D}_G^\times$. By the Ore condition, we may write $z_1 = u_1 w^{-1}$ and $z_2 = w u_2^{-1}$ for some $u_1, u_2, w \in (\mathcal{D}_K * H_1(G)_f)^\times$. Then

$$L_\phi(z_1) L_\phi(z_2) = L_\phi(u_1) L_\phi(w)^{-1} \cdot L_\phi(w) L_\phi(u_2)^{-1} = L_\phi(u_1) L_\phi(u_2)^{-1} = L_\phi(z_1 z_2).$$

The corresponding properties for δ_ϕ can be proved similarly. The properties of the extended maps directly follows. ☐

Here are some basic facts about the mappings δ_ϕ, L_ϕ, particularly about their properties under addition in $\mathcal{D}_G$. Most properties clearly hold in the subring $\mathcal{D}_K * H_1(G)_f$ and it is routine to verify them in its field of fractions $\mathcal{D}_G$.

Proposition 4.9. *Let G be a finitely generated torsion-free group which satisfies the Atiyah Conjecture. Let $\phi \in H^1(G; \mathbb{R})$ be a real cohomology class and $\delta_\phi \colon \mathcal{D}_G \to \mathbb{R}$, $L_\phi \colon \mathcal{D}_G \to \mathcal{D}_G$ be as in Definition 4.8. For any $z, z_1, \ldots, z_n \in \mathcal{D}_G$, the following hold:*

1. *$\delta_{r\phi}(z) = r \cdot \delta_\phi(z)$ and $L_{r\phi}(z) = L_\phi(z)$ for all $r \in \mathbb{R}_+$.*
2. *$\delta_\phi(cz) = \delta_\phi(z)$ and $L_\phi(cz) = c \cdot L_\phi(z)$ for all $c \in \mathbb{Q} \setminus \{0\}$.*
3. *$\delta_\phi(L_\phi(z)) = \delta_\phi(z)$ and $L_\phi(L_\phi(z)) = L_\phi(z)$.*
4. *If $L_\phi(z_1) = L_\phi(z_2) \neq 0$, then $\delta_\phi(z_1) = \delta_\phi(z_2) < \delta_\phi(z_1 - z_2)$.*
5. *$\delta_\phi(z_1 + z_2) \geqslant \min\{\delta_\phi(z_1), \delta_\phi(z_2)\}$. If $\delta_\phi(z_1) < \delta_\phi(z_2)$, then*

$$\delta_\phi(z_1 + z_2) = \delta_\phi(z_1), \quad L_\phi(z_1 + z_2) = L_\phi(z_1).$$

(6) *If* $\delta_\phi(z_1) = \cdots = \delta_\phi(z_n) =: \delta$ *and* $\sum_{k=1}^n L_\phi(z_k) \neq 0$*, then*

$$\delta_\phi\Big(\sum_{k=1}^n z_k\Big) = \delta, \quad L_\phi\Big(\sum_{k=1}^n z_k\Big) = \sum_{k=1}^n L_\phi(z_k).$$

(7) *For any open neighborhood* $U \subset H^1(G;\mathbb{R})$ *of* ϕ*, there is a rational cohomology class* $\psi \in U$ *such that* $L_\psi(z) = L_\phi(z)$*.*

(8) *Let* $L \subset G$ *be a finitely generated subgroup with induced inclusion* $\mathcal{D}_L \subset \mathcal{D}_G$*, and let* $\phi|_L \colon L \to \mathbb{R}$ *be the restriction of* ϕ *to* L*. Then the mappings*

$$\delta_{\phi|_L} \colon \mathcal{D}_L \to \mathbb{R} \cup \{+\infty\}, \quad L_{\phi|_L} \colon \mathcal{D}_L \to \mathcal{D}_L$$

are precisely the restrictions of δ_ϕ *and* L_ϕ *to* $\mathcal{D}_L$*.*

Proof. For (1)–(3), the statements hold in $\mathcal{D}_K * H_1(G)_f$ and directly extends to $\mathcal{D}_G$ by definition.

For (4), write $z_1 = u_1 w^{-1},\ z_2 = u_2 w^{-1}$ with $u_1, u_2, w \in \mathcal{D}_K * H_1(G)_f$, then $L_\phi(z_1) = L_\phi(z_2)$ implies that $L_\phi(u_1) = L_\phi(u_2)$. It follows that $\delta_\phi(u_1) = \delta_\phi(u_2) < \delta_\phi(u_1 - u_2)$. Therefore $\delta_\phi(z_1) = \delta_\phi(z_2) < \delta_\phi(z_1 - z_2)$.

For (5), write $z_1 = u_1 w^{-1},\ z_2 = u_2 w^{-1}$ with $u_1, u_2, w \in \mathcal{D}_K * H_1(G)_f$. Since $\delta_\phi(u_1 + u_2) \geqslant \min\{\delta_\phi(u_1), \delta_\phi(u_2)\}$, we have

$$\begin{aligned}\delta_\phi(z_1 + z_2) &= \delta_\phi(u_1 + u_2) - \delta_\phi(w)\\ &\geqslant \min\{\delta_\phi(u_1), \delta_\phi(u_2)\} - \delta_\phi(w) = \min\{\delta_\phi(z_1), \delta_\phi(z_2)\}.\end{aligned}$$

If $\delta_\phi(z_1) < \delta_\phi(z_2)$, then $\delta_\phi(u_1) < \delta_\phi(u_2)$ and $L_\phi(u_1 + u_2) = L_\phi(u_1)$. Hence $\delta_\phi(z_1 + z_2) = \delta_\phi(z_1)$ and $L_\phi(z_1 + z_2) = L_\phi(z_1)$.

For (6), write $z_i = u_i w^{-1}$ with $u_i, w \in \mathcal{D}_K * H_1(G)_f$ for $i = 1, \ldots, n$. By assumption we have $\delta_\phi(u_i) = \delta + \delta_\phi(w)$ and $\sum_{i=1}^k L_\phi(u_i) \neq 0$. It follows that $\delta_\phi(\sum_{i=1}^k u_i) = \delta + \delta_\phi(w)$ and $L_\phi(\sum_{i=1}^k u_i) = \sum_{i=1}^k L_\phi(u_i)$. Hence $\delta_\phi(\sum_{i=1}^k z_i) = \delta$ and $L_\phi(\sum_{i=1}^k z_i) = \sum_{i=1}^k L_\phi(z_i)$.

For (7), write $z = uv^{-1}$ with $u, v \in \mathcal{D}_K * H_1(G)_f$. For another cohomology class $\psi \in H^1(G;\mathbb{R})$, we have $L_\phi(z) = L_\psi(z)$ if ϕ and ψ induce the same ordering on $\mathrm{supp}(u) \cup \mathrm{supp}(v)$, that is:

- for all $h, h' \in \mathrm{supp}(u) \cup \mathrm{supp}(v)$, $\psi(h - h') < 0$ whenever $\phi(h - h') < 0$;
- for all $h, h' \in \mathrm{supp}(u) \cup \mathrm{supp}(v)$, $\psi(h - h') = 0$ whenever $\phi(h - h') = 0$.

The domain Ω of such ψ is the intersection of finitely many closed hyperplanes and open half-spaces of $H^1(G;\mathbb{R})$, each defined by an integral linear equation. Since $\phi \in \Omega$, any open neighborhood $U \ni \phi$ contains rational classes in $U \cap \Omega$.

For (8), consider the short exact sequence $1 \to K' \to L \to H_1(L)_f \to 1$. Recall that $\mathcal{D}_L$ is the field of fractions of $\mathcal{D}_{K'} * H_1(L)_f$ by Proposition 4.5. For any nonzero $u \in \mathcal{D}_{K'} * H_1(L)_f$, write $u = \sum_{h \in H_1(L)_f} x_h \cdot s(h)$ with $x_h \in \mathcal{D}_{K'}$ for a section $s \colon H_1(L)_f \to L$. Let $\delta := \delta_{\phi|_L}(u)$ and decompose

$$u = \sum_{\substack{h \in H_1(L)_f,\\ \phi|_L(h) = \delta}} x_h \cdot s(h) + \sum_{\substack{h \in H_1(L)_f,\\ \phi|_L(h) > \delta}} x_h \cdot s(h) =: u_1 + u_2.$$

Then $L_{\phi|_L}(u) = u_1 \neq 0$. Now identify $\mathcal{D}_L$ with its image in $\mathcal{D}_G$, we want to show

$$\delta_\phi(u) = \delta, \quad L_\phi(u) = u_1.$$

Since $\delta_\phi(x_h \cdot s(h)) = \phi(h)$ for all $h \in L$, applying (6) to u_1 we have $\delta_\phi(u_1) = \delta$ and $L_\phi(u_1) = u_1$. By (5) applied to u_2 we know that

$$\delta_\phi(u_2) \geqslant \min\{\delta_\phi(x_h \cdot s(h)) \mid h \in H_1(L)_f,\ \phi|_L(h) > \delta\} > \delta,$$

Again by (5) applied to $u = u_1 + u_2$ we have $\delta_\phi(u) = \delta_\phi(u_1) = \delta$ and $L_\phi(u) = L_\phi(u_1) = u_1$. Hence

$$\delta_\phi(u) = \delta_{\phi|_L}(u) \quad \text{and} \quad L_\phi(u) = L_{\phi|_L}(u)$$

for all nonzero $u \in \mathcal{D}_{K'} * H_1(L)_f$. Passing to the field of fractions we conclude that $\delta_\phi(z) = \delta_{\phi|_L}(z)$ and $L_\phi(z) = L_{\phi|_L}(z)$ for all $z \in \mathcal{D}_L$. $\square$

Definition 4.10. An element $z \in \mathcal{D}_G$ is called *ϕ-pure* if $L_\phi(z) = z$.

Lemma 4.11. *Here are some properties of the ϕ-pure elements.*

(1) *Elements of $\mathbb{Z}[\ker\phi] \subset \mathcal{D}_G$ are ϕ-pure; elements of $G \subset \mathcal{D}_G$ are ϕ-pure.*
(2) *The product of two ϕ-pure elements is ϕ-pure.*
(3) *For any nonzero $z \in \mathcal{D}_G$, the element $L_\phi(z)$ is the unique ϕ-pure element $w \in \mathcal{D}_G$ satisfying $\delta_\phi(z - w) > \delta_\phi(z)$.*

Proof. The properties (1) and (2) follow from the definition of L_ϕ.

For (3), Proposition 4.9(3)–(4) implies that $L_\phi(z)$ is ϕ-pure and $\delta_\phi(z - L_\phi(z)) > \delta_\phi(z)$. To prove uniqueness, if a ϕ-pure element w satisfies $\delta_\phi(z - w) > \delta_\phi(z)$, then by Proposition 4.9(4),

$$w = L_\phi(w) = L_\phi(z - (z - w)) = L_\phi(z).$$

$\square$

Remark 4.12. For any $\phi \in H^1(G;\mathbb{R})$, the homomorphism $\delta_\phi \colon \mathcal{D}_G^\times \to \mathbb{R}$ induces well-defined homomorphisms $\delta_\phi \colon \Lambda \to \mathbb{R}$ for $\Lambda = K_1(\mathcal{D}_G)$ and $\widetilde{K}_1(\mathcal{D}_G)$. Similarly, the homomorphism $L_\phi \colon \mathcal{D}_G^\times \to \mathcal{D}_G^\times$ induces well-defined homomorphisms $L_\phi \colon \Lambda \to \Lambda$ for $\Lambda = K_1(\mathcal{D}_G)$, $\widetilde{K}_1(\mathcal{D}_G)$ and $\mathrm{Wh}(\mathcal{D}_G)$.

We use the same symbols δ_ϕ, L_ϕ for their induced maps on $K_1(\mathcal{D}_G)$, $\widetilde{K}_1(\mathcal{D}_G)$ and $\mathrm{Wh}(\mathcal{D}_G)$; the domain of definition will be clear from the context. The following commutative diagram illustrates the relevant mappings (compare Definition 2.13).

$$\begin{tikzcd}
 & & \mathrm{GL}(\mathcal{D}_G) \arrow[dl, "\det"'] \arrow[d, "\det_r"] \arrow[dr, "\det_w"] & \\
\mathcal{D}_G^\times/[\mathcal{D}_G^\times,\mathcal{D}_G^\times] \arrow[r, equal] & K_1(\mathcal{D}_G) \arrow[r] \arrow[d, "\delta_\phi"'] \arrow[loop above, "L_\phi"] & \widetilde{K}_1(\mathcal{D}_G) \arrow[r] \arrow[dl, "\delta_\phi"'] \arrow[loop below, "L_\phi"] & \mathrm{Wh}(\mathcal{D}_G) \arrow[loop below, "L_\phi"] \\
 & \mathbb{R} & &
\end{tikzcd}$$

The following Theorem 4.13 is of central importance in this paper, relating the leading term map and the Dieudonné determinant.

Theorem 4.13. *Let $\phi \in H^1(G;\mathbb{R})$ be a real cohomology class. Suppose P and Q are $n \times n$ matrices over $\mathcal{D}_G$ satisfying the following conditions:*

(i) *P is invertible over $\mathcal{D}_G$.*
(ii) *There exist real numbers $d_1, \ldots, d_n$ and $d'_1, \ldots, d'_n$ such that every nonzero entry P_{ij} is ϕ-pure with $\delta_\phi(P_{ij}) = d_i - d'_j$.*
(iii) *$\delta_\phi(Q_{ij}) > d_i - d'_j$ for all i, j.*

Then $P + Q$ is invertible over $\mathcal{D}_G$ and

$$L_\phi(\det(P + Q)) = \det P \in K_1(\mathcal{D}_G).$$

Proof. We prove by induction on n. For the case $n = 1$ we have $P, Q \in \mathcal{D}_G$. By assumption $P \neq 0$ and $\delta_\phi(Q) > \delta_\phi(P)$. Then $L_\phi(\det(P + Q)) = \det P$ follows directly from Proposition 4.9(5).

Now assume the theorem holds for matrices of size n. Let P, Q be $(n+1)\times(n+1)$ matrices

$$P = \begin{pmatrix} A & U \\ X & p \end{pmatrix}, \quad Q = \begin{pmatrix} B & V \\ Y & q \end{pmatrix}$$

where P is invertible, $p, q \in \mathcal{D}_G$, A, B are $n \times n$ matrices over $\mathcal{D}_G$ and

$$\begin{aligned} X &= (x_1, \dots, x_n), \quad Y = (y_1, \dots, y_n), \\ U &= (u_1, \dots, u_n)^T, \quad V = (v_1, \dots, v_n)^T. \end{aligned}$$

Without loss of generality, assume $p \neq 0$; then $\delta_\phi(q) > \delta_\phi(p)$ and $p+q \neq 0$ by condition (iii). Note that

$$\begin{pmatrix} I & -(U+V)(p+q)^{-1} \\ 0 & 1 \end{pmatrix} \cdot (P+Q) = \begin{pmatrix} W & 0 \\ X+Y & p+q \end{pmatrix}$$

where $W = A+B-(U+V)(p+q)^{-1}(X+Y)$ is an $n\times n$ matrix. Examining the expression of W, we observe that the terms of lowest δ_ϕ-value form the matrix $A - Up^{-1}X$. We verify this rigorously.

Claim. *Define*

$$W' := A - Up^{-1}X, \quad W'' := W - W'.$$

Then W' and W'' satisfy the three conditions of Theorem 4.13 for size n. In particular by the induction hypothesis, W' is invertible over $\mathcal{D}_G$ with $\det P = \det W' \cdot \det p$, *and* $L_\phi \det(W) = \det(W')$.

Admitting this Claim. We conclude that W is invertible and

$$\begin{aligned} \det(P+Q) &= \det W \cdot \det(p+q), \\ L_\phi \det(P+Q) &= L_\phi(\det W) \cdot \det p = \det W' \cdot \det p = \det P, \end{aligned}$$

completing the induction. It remains to prove the Claim.

Proof of Claim. To verify condition (i), note that

$$\begin{pmatrix} I & -Up^{-1} \\ 0 & 1 \end{pmatrix} \cdot P = \begin{pmatrix} W' & 0 \\ X & p \end{pmatrix},$$

so W' is invertible over $\mathcal{D}_G$ and $\det P = \det W' \cdot \det p$.

For condition (ii), note that

$$\begin{aligned} W_{ij} &= A_{ij} + B_{ij} - (u_i + v_i)(p+q)^{-1}(x_j + y_j), \\ W'_{ij} &= A_{ij} - u_i p^{-1} x_j, \\ W''_{ij} &= B_{ij} + u_i p^{-1} x_j - (u_i + v_i)(p+q)^{-1}(x_j + y_j). \end{aligned}$$

If $A_{ij} \neq 0$, then A_{ij} is ϕ-pure with $\delta_\phi(A_{ij}) = d_i - d'_j$. If $u_i p^{-1} x_j \neq 0$, then $u_i p^{-1} x_j$ is also ϕ-pure with

$$\delta_\phi(u_i p^{-1} x_j) = (d_i - d'_{n+1}) - (d_{n+1} - d'_{n+1}) + (d_{n+1} - d'_j) = d_i - d'_j.$$

Hence, if $W'_{ij} \neq 0$ then W'_{ij} is ϕ-pure with $\delta_\phi(W'_{ij}) = d_i - d'_j$ by Proposition 4.9(6), proving condition (ii).

For condition (iii), we show

$$\delta_\phi(u_i p^{-1} x_j - (u_i + v_i)(p+q)^{-1}(x_j + y_j)) > d_i - d'_j.$$

If $u_i p^{-1} x_j \neq 0$, then u_i and x_j are both nonzero, and

$$L_\phi((u_i + v_i)(p+q)^{-1}(x_j + y_j)) = u_i p^{-1} x_j, \quad \delta_\phi(u_i p^{-1} x_j) = d_i - d'_j$$

and the inequality follows from Proposition 4.9(4). If $u_i p^{-1} x_j = 0$ then $u_i = 0$ or $x_j = 0$, and

$$\begin{aligned}\delta_\phi((u_i+v_i)(p+q)^{-1}(x_j+y_j)) &= \delta_\phi(u_i+v_i) - \delta_\phi(p+q) + \delta_\phi(x_j+y_j) \\ &> (d_i - d'_{n+1}) - \delta_\phi(p+q) + (d_{n+1} - d'_j) \\ &= d_i - d'_j.\end{aligned}$$

In both cases, we obtained the desired inequality. Since by assumption $\delta_\phi(B_{ij}) > d_i - d'_j$, we conclude that $\delta_\phi(W''_{ij}) > d_i - d'_j$ by Proposition 4.9(5), proving condition (iii). $\square$

4.4. **The restriction map.** Suppose G is a finitely generated torsion-free group satisfying the Atiyah Conjecture, and let $L \triangleleft G$ be a normal subgroup of finite index d. In this section we define the restriction map $\operatorname{res}^G_L \colon K_1(\mathcal{D}_G) \to K_1(\mathcal{D}_L)$. Recall that $\mathcal{D}_G$ is naturally isomorphic to the crossed product $\mathcal{D}_L * (G/L)$ by Proposition 4.5.

Definition 4.14. Fix a section $s \colon G/L \to G$ and let its image be $s(G/L) = \{g_1, \ldots, g_d\}$. Then $G = Lg_1 \sqcup \ldots \sqcup Lg_d$. For any element $z \in \mathcal{D}_G$ and any $k \in \{1, \ldots, d\}$, there is a unique way to express $g_k \cdot z$ as

$$g_k \cdot z = \sum_{j=1}^{d} l_{kj} \cdot g_j, \quad l_{kj} \in \mathcal{D}_L.$$

Define $\Lambda_s(z)$ to be the $d \times d$ matrix over $\mathcal{D}_L$ whose (k,j)-entry is l_{kj}. Equivalently, $\Lambda_s(z)$ is the unique matrix over $\mathcal{D}_L$ such that

$$\begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z = \Lambda_s(z) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}.$$

Lemma 4.15. *With the notation of Definition 4.14, the following hold:*

1. *For any $z_1, z_2 \in \mathcal{D}_G$, we have $\Lambda_s(z_1 z_2) = \Lambda_s(z_2) \cdot \Lambda_s(z_1)$.*
2. *If $z \neq 0$, then $\Lambda_s(z)$ is invertible over $\mathcal{D}_L$.*
3. *If s' is another section, then $\Lambda_{s'}(z) = \Omega \Lambda_s(z) \Omega^{-1}$ for some invertible matrix Ω over $\mathbb{Z}L$ depending only on s and s'.*
4. *For any k and any $\phi \in H^1(G;\mathbb{R})$, we have*

$$L_\phi(g_k \cdot z) = \sum_{j \in \mathcal{J}} L_\phi(l_{kj} \cdot g_j)$$

where $\mathcal{J} = \{j \in \{1, \ldots, d\} \mid \delta_\phi(l_{kj} g_j) = \min_{1 \leqslant j \leqslant d} \delta_\phi(l_{kj} g_j)\}$. In particular

$$\delta_\phi(g_k \cdot z) = \min_{1 \leqslant j \leqslant d} \{\delta_\phi(l_{kj} \cdot g_j)\}.$$

If in addition z is ϕ-pure, then for each j, either $l_{kj} = 0$ or l_{kj} is ϕ-pure with $\delta_\phi(g_k \cdot z) = \delta_\phi(l_{kj} \cdot g_j)$.

Proof. For (1), observe that

$$\begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z_1 z_2 = \Lambda_s(z_1) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z_2 = \Lambda_s(z_1) \cdot \Lambda_s(z_2) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}$$

which implies $\Lambda_s(z_1 z_2) = \Lambda_s(z_2) \cdot \Lambda_s(z_1)$. Note that (2) follows directly from (1).

For (3), let s' be another section with $s'(L) = \{g'_1, \dots, g'_d\}$ and let Ω be the $d \times d$ matrix over $\mathbb{Z}L$ such that

$$\begin{pmatrix} g'_1 \\ \vdots \\ g'_d \end{pmatrix} = \Omega \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}.$$

Then

$$\begin{pmatrix} g'_1 \\ \vdots \\ g'_d \end{pmatrix} \cdot z = \Omega \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \cdot z = \Omega \Lambda_s(z) \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} = \Omega \Lambda_s(z) \Omega^{-1} \begin{pmatrix} g'_1 \\ \vdots \\ g'_d \end{pmatrix},$$

so $\Lambda_{s'}(z) = \Omega \Lambda_s(z) \Omega^{-1}$.

For (4), we may assume that $z \neq 0$, so $l_{kj} \neq 0$ for any $j \in \mathcal{J}$. Note that each $g_j \in G$ is ϕ-pure and by Proposition 4.9(8) $L_\phi(l_{kj}) \in \mathcal{D}_L$. Therefore

$$L_\phi(l_{kj} g_j) = L_\phi(l_{kj}) g_j \in \mathcal{D}_L \cdot g_j.$$

Let $w_1 = \sum_{j \in \mathcal{J}} L_\phi(l_{kj}) g_j$. Since $\mathcal{D}_G = \oplus_j \mathcal{D}_L \cdot g_j$ and $L_\phi(l_{kj}) \neq 0$ for $j \in \mathcal{J}$, it follows that $w_1 \neq 0$. By Proposition 4.9(6) w_1 is ϕ-pure. Now write

$$\begin{aligned} L_\phi(g_k \cdot z) &= \sum_{j \in \mathcal{J}} L_\phi(l_{kj}) g_j + \sum_{j \in \mathcal{J}} (l_{kj} - L_\phi(l_{kj})) g_j + \sum_{j \notin \mathcal{J}} l_{kj} g_j \\ &=: w_1 + w_2 + w_3. \end{aligned}$$

Clearly $\delta_\phi(w_2) > \delta_\phi(w_1)$ by Proposition 4.9(4), and $\delta_\phi(w_3) > \delta_\phi(w_1)$ by the choice of $\mathcal{J}$. We conclude that $L_\phi(g_k \cdot z) = w_1$ by Lemma 4.11(3). In particular,

$$\delta_\phi(g_k \cdot z) = \min\{\delta_\phi(l_{kj} \cdot g_j) \mid j = 1, \dots, n\}.$$

Now suppose additionally that z is ϕ-pure, then $L_\phi(g_k \cdot z) = g_k \cdot z$, that is

$$\sum_{j \in \mathcal{J}} L_\phi(l_{kj}) g_j = \sum_{j=1}^{n} l_{kj} g_j.$$

By the direct sum decomposition $\mathcal{D}_G = \oplus_j \mathcal{D}_L \cdot g_j$, it follows that l_{kj} is pure with $\delta_\phi(l_{kj} g_j) = \delta_\phi(g_k \cdot z)$ for any $j \in \mathcal{J}$, and $l_{kj} = 0$ for any $j \notin \mathcal{J}$. □

Definition 4.16 (Restriction map)**.** Let $L \lhd G$ be a normal subgroup of finite index. Choose a section $s \colon G/L \to G$. Define the *restriction map*

$$\mathrm{res}^G_L \colon \mathcal{D}_G^\times \to K_1(\mathcal{D}_L), \quad z \mapsto \det(\Lambda_s(z)) \in K_1(\mathcal{D}_L).$$

By Lemma 4.15, this is a group homomorphism independent of the choice of section s. We use the same notation for the induced homomorphism on the K_1-group

$$\mathrm{res}^G_L \colon K_1(\mathcal{D}_G) \to K_1(\mathcal{D}_L).$$

Remark 4.17. For $g \in G \subset \mathcal{D}_G^\times$, the matrix $\Lambda_s(g)$ is a permutation matrix whose nonzero entries are elements $l_i \in L$. It follows that

$$\mathrm{res}^G_L(g) = \det(\Lambda_s(g)) = \pm \prod_i l_i \in K_1(\mathcal{D}_L)$$

is represented by a element of $\pm L$. Therefore the restriction map naturally induces a homomorphism on Whitehead groups

$$\mathrm{res}^G_L \colon \mathrm{Wh}(\mathcal{D}_G) \to \mathrm{Wh}(\mathcal{D}_L).$$

The following important result establishes that the restriction map commutes with the leading term map in appropriate sense.

Theorem 4.18. *Let G be a finitely generated torsion-free group satisfying the Atiyah Conjecture, and let $L \triangleleft G$ be a normal subgroup of finite index. Let $\phi \in H^1(G;\mathbb{R})$ and denote by $\phi|_L \in H^1(L;\mathbb{R})$ its to L. Then for any $z \in \mathcal{D}_G^\times$, we have*

$$L_{\phi|_L}(\mathrm{res}_L^G(z)) = \mathrm{res}_L^G(L_\phi(z)) \in K_1(\mathcal{D}_L).$$

Proof. Write $z = L_\phi(z) + z'$. Fix section $s : G/L \to G$ and write $G = Lg_1 \sqcup \ldots \sqcup Lg_d$. Consider $d \times d$ matrices $P := \Lambda_s(L_\phi(z))$ and $Q := \Lambda_s(z')$ over $\mathcal{D}_L$. Explicitly,

$$g_k \cdot L_\phi(z) = \sum_{j=1}^{d} P_{kj} \cdot g_j, \quad g_k \cdot z' = \sum_{j=1}^{d} Q_{kj} \cdot g_j.$$

Then $\mathrm{res}_L^G(z) = \det(P+Q)$ and $\mathrm{res}_L^G(L_\phi(z)) = \det P$. Since $L_\phi(z)$ is ϕ-pure, Lemma 4.15(4) implies that P_{kj} is ϕ-pure, and that

$$\begin{aligned} &\delta_\phi(P_{kj}) = \delta_\phi(z) + \delta_\phi(g_k) - \delta_\phi(g_j) \quad \text{for } P_{kj} \neq 0, \\ &\delta_\phi(Q_{kj}) \geqslant \delta_\phi(z') + \delta_\phi(g_k) - \delta_\phi(g_j) > \delta_\phi(z) + \delta_\phi(g_k) - \delta_\phi(g_j). \end{aligned}$$

By Proposition 4.9(8) the maps δ_ϕ and L_ϕ restrict to $\delta_{\phi|_L}$ and $L_{\phi|_L}$ on $\mathcal{D}_L$. We now apply Theorem 4.13 to P and Q over $\mathcal{D}_L$, taking

$$d_i = \delta_\phi(z) + \delta_\phi(g_i) \quad \text{and} \quad d_i' = \delta_\phi(g_i) \quad \text{for } i = 1, \ldots, d.$$

The three conditions of the Theorem are satisfied, and we conclude

$$L_{\phi|_L}(\det(P+Q)) = \det P \in K_1(\mathcal{D}_L),$$

that is $L_{\phi|_L}(\mathrm{res}_L^G(z)) = \mathrm{res}_L^G(L_\phi(z))$, completing the proof. □

4.5. **The polytope map.** Let H be a finitely generated free abelian group. Note that $H_1(H;\mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Z}} H$ is a finite-dimensional real vector space. A *polytope* in $H_1(H;\mathbb{R})$ is a compact set obtained as the convex hull of a finite subset of points. We allow the empty set $\emptyset$ to be a polytope.

Definition 4.19 (Faces of polytopes)**.** Given a polytope P and a character $\phi \in H^1(H;\mathbb{R})$. Define $\delta_\phi(P) := \inf_{x \in P} \phi(x)$ and the *face associated to* ϕ by

$$F_\phi(P) := \{x \in P \mid \phi(x) = \delta_\phi(P)\}.$$

Clearly $F_\phi(P)$ is a polytope contained in P, and the collection $\{F_\phi(P) \mid \phi \in H^1(H;\mathbb{R})\}$ consists of all faces of P. A face is called a *vertex* if it is a single point. Any polytope is the convex hull of its vertices. A polytope is called *integral* if all its vertices lie in the integral lattice $H \subset H_1(H;\mathbb{R})$.

Given any two non-empty polytopes P_1, P_2 in $H_1(H;\mathbb{R})$, their *Minkowski sum* is defined to be the polytope

$$P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1,\ p_2 \in P_2\}.$$

This is the convex hull of the set $\{v_1 + v_2 \mid v_i \text{ is a vertex of } P_i,\ i = 1, 2\}$. The operator δ_ϕ and the face map F_ϕ are additive under the Minkowski sum:

$$\delta_\phi(P_1 + P_2) = \delta_\phi(P_1) + \delta_\phi(P_2), \quad F_\phi(P_1 + P_2) = F_\phi(P_1) + F_\phi(P_2)$$

for every character ϕ and all polytopes P_1, P_2.

Example 4.20. Let M be an admissible 3-manifold. The *Thurston norm ball*

$$B_x(M) := \{\phi \in H^1(M;\mathbb{R}) \mid x_M(\phi) \leqslant 1\}$$

is a (possibly non-compact) polyhedron in $H^1(M;\mathbb{R})$; the *dual Thurston norm ball* is defined as

$$B_x^*(M) := \{z \in H_1(M;\mathbb{R}) \mid \phi(z) \leqslant 1 \text{ for all } \phi \in B_x(M)\}.$$

Thurston showed that $B_x^*(M)$ is an integral polytope in $H_1(M;\mathbb{R})$ with vertices $\pm v_1,\dots,\pm v_k$, and that the Thurston norm ball is determined by these vertices:

$$B_x(M) = \{\phi \in H^1(M;\mathbb{R}) \mid |\phi(v_i)| \leqslant 1,\ i=1,\dots,k\}.$$

A *Thurston cone* in $H^1(M;\mathbb{R})$ is either an open cone formed by the origin and a face of $B_x(M)$, or a maximal connected component of $H^1(M;\mathbb{R})\setminus\{0\}$ on which the Thurston norm x_M vanishes. It follows that $H^1(M;\mathbb{R})\setminus\{0\}$ is the disjoint union of all Thurston cones of various dimensions. A Thurston cone is called *top-dimensional* if its dimension equals $\dim H^1(M;\mathbb{R})$. The following Lemma 4.21 is a reformulation of Thurston's theorem.

Lemma 4.21. *A nonzero character $\phi \in H^1(M;\mathbb{R})$ lies in a top-dimensional Thurston cone if and only if $F_\phi B_x^*(M)$ is a vertex.*

Definition 4.22 ($\mathcal{P}_{\mathbb{Z}}(H)$ and $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$)**.** The *(integral) polytope group* $\mathcal{P}_{\mathbb{Z}}(H)$ is defined as the Grothendieck group of integral polytopes in $H_1(H;\mathbb{R})$ under the Minkowski sum. More precisely, $\mathcal{P}_{\mathbb{Z}}$ is the abelian group generated by symbols $[P]$ where P is a non-empty integral polytope in $H_1(H;\mathbb{R})$, subject to the relation $[P]+[Q]=[P+Q]$ for each pair of non-empty integral polytopes P,Q.

Every element $h\in H$ determines an one-point polytope $[h]$ in $\mathcal{P}_{\mathbb{Z}}(H)$, defining an embedding of H into $\mathcal{P}_{\mathbb{Z}}(H)$. The *Whitehead polytope group* is defined as the quotient

$$\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H) = \mathcal{P}_{\mathbb{Z}}(H)/H.$$

In other words, two polytopes are identified in $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ if and only if they differ by translation by an element in the lattice H.

We make the following remarks:

(1) Any element of $\mathcal{P}_{\mathbb{Z}}(H)$ can be written as a formal sum $[P]-[Q]$ for non-empty integral polytopes $P,Q\subset H_1(H;\mathbb{R})$. Two such expressions $[P_1]-[Q_1]$ and $[P_2]-[Q_2]$ represents the same element if and only if $P_1+Q_2=P_2+Q_1$. Equivalently, two elements $x,y\in\mathcal{P}_{\mathbb{Z}}(H)$ are distinct if and only if there exists $\phi\colon H\to\mathbb{Z}$ such that their images under the induced map $\mathcal{P}_{\mathbb{Z}}(H)\to\mathcal{P}_{\mathbb{Z}}(\mathbb{Z})$ is different. This observation can be used to show that both $\mathcal{P}_{\mathbb{Z}}(H)$ and $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ are free abelian groups [FL17, Lemma 4.8].

(2) The face map F_ϕ extends naturally to a homomorphism on the polytope group:

$$F_\phi\colon \mathcal{P}_{\mathbb{Z}}(H)\to\mathcal{P}_{\mathbb{Z}}(H),\quad F_\phi([P]-[Q]) = [F_\phi(P)]-[F_\phi(Q)].$$

Since F_ϕ preserves the subgroup H, it descends to a homomorphism (denoted by the same symbol)

$$F_\phi\colon \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)\to\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H).$$

4.5.1. *Polytope homomorphism* $\mathbb{P}$. Let G be a finitely generated torsion-free group satisfying the Atiyah Conjecture. Let H be the free abelianization of G, then we have the short exact sequence

$$1\to K\to G\to H\to 1$$

and by Proposition 4.5 $\mathcal{D}_G$ is the field of fractions of the subring $\mathcal{D}_K * H$. Recall from Definition 4.6 that for any section $s\colon H\to G$, every $u\in\mathcal{D}_K*H$ has a unique expression

$$u=\sum_{h\in H} x_h\cdot s(h),\quad x_h\in\mathcal{D}_K,$$

and its support $\mathrm{supp}(u) := \{h\in H \mid x_h\neq 0\}$ is independent of the choice of s.

Definition 4.23 (Polytope map)**.** For any $u \in (\mathcal{D}_K * H)^\times$, define $\mathbb{P}(u) \in \mathcal{P}_\mathbb{Z}(H)$ to be the element represented by the convex hull of $\operatorname{supp}(u)$ in $\mathbb{R} \otimes_\mathbb{Z} H$. The *polytope homomorphism* is defined as

$$\mathbb{P}\colon \mathcal{D}_G^\times \to \mathcal{P}_\mathbb{Z}(H), \quad \mathbb{P}(uv^{-1}) := \mathbb{P}(u) - \mathbb{P}(v)$$

for all $u, v \in (\mathcal{D}_K * H)^\times$.

It is shown in [FL19, Lemma 6.4] that $\mathbb{P}$ is a well-defined homomorphism and

$$\mathbb{P}(z_1 z_2) = \mathbb{P}(z_1) + \mathbb{P}(z_2)$$

for all $z_1, z_2 \in \mathcal{D}_G^\times$.

The following commutative diagram is immediate from the definition.

$$\begin{tikzcd} \mathcal{D}_G^\times \arrow[r, "L_\phi"] \arrow[d, "\mathbb{P}"] & \mathcal{D}_G^\times \arrow[d, "\mathbb{P}"] \\ \mathcal{P}_\mathbb{Z}(H) \arrow[r, "F_\phi"] & \mathcal{P}_\mathbb{Z}(H). \end{tikzcd}$$

Recall that $K_1(\mathcal{D}_G)$ is the abelianization of $\mathcal{D}_G^\times$ and $\mathrm{Wh}(\mathcal{D}_G) = K_1(\mathcal{D}_G)/[\pm G]$, then the polytope homomorphism naturally induces (by abuse of notation)

$$\mathbb{P}\colon \mathrm{Wh}(\mathcal{D}_\mathrm{G}) \to \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H).$$

We then obtain the commutative diagram for the induced homomorphisms

$$\begin{tikzcd} \mathrm{Wh}(\mathcal{D}_G) \arrow[r, "L_\phi"] \arrow[d, "\mathbb{P}"] & \mathrm{Wh}(\mathcal{D}_G) \arrow[d, "\mathbb{P}"] \\ \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H) \arrow[r, "F_\phi"] & \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H). \end{tikzcd}$$

We now state a key result relating the universal L^2-torsion of an admissible 3-manifold to its dual Thurston norm ball.

Theorem 4.24 ([FL17, Theorem 4.37])**.** *Let M be an admissible 3-manifold which is not homeomorphic to $S^1 \times D^2$. Then*

$$[B_x^*(M)] = 2 \cdot \mathbb{P}(\tau_u^{(2)}(M)) \in \mathcal{P}_\mathbb{Z}^{\mathrm{Wh}}(H_1(M)_f),$$

where $B_x^*(M) \subset H_1(M;\mathbb{R})$ is the dual Thurston norm ball, and $\tau_u^{(2)}(M) \in \mathrm{Wh}(\mathcal{D}_{\pi_1(M)})$ is the universal L^2-torsion of M.

4.6. **Half of Theorem 1.2.** We now prove the "if" part of Theorem 1.2. A stronger version of the "only if" part will be stated and proved in Theorem 5.16 after establishing Theorem 1.3.

Theorem 4.25 (The "if" part of Theorem 1.2)**.** *Suppose M is an admissible 3-manifold such that M is not a closed graph manifold without an NPC metric. If $\phi \in H^1(M;\mathbb{R})$ is a nonzero class such that $L_\phi \tau_u^{(2)}(M) = 1 \in \mathrm{Wh}(\mathcal{D}_{\pi_1(M)})$, then ϕ is fibered.*

Proof. If M is homeomorphic to a solid torus then any nonzero class is fibered. Assume therefore that M is not a solid torus. The assumption on M implies that $\pi_1(M)$ is virtually special by a combination of work (c.f. [AFW15, Section 4.7] and references therein). In particular, by Agol's criterion for virtual fibering [Ago08] there exists a regular finite covering $\overline{M} \to M$ such that the pull back class $\overline{\phi}$ lies in the closure of a fibered cone. Let G be the fundamental group of M and $\pi_1(\overline{M}) =: L < G$. By the restriction property of Theorem 3.12(4),

$$\tau_u^{(2)}(\overline{M}) = \operatorname{res}_L^G \tau_u^{(2)}(M) \in \mathrm{Wh}(\mathcal{D}_L).$$

Applying $L_{\bar\phi}$ to both sides and using Theorem 4.18, we obtain

$$\begin{aligned} L_{\bar\phi}(\tau_u^{(2)}(\overline{M})) &= L_{\bar\phi}(\operatorname{res}_L^G \tau_u^{(2)}(M)) \\ &= \operatorname{res}_L^G(L_\phi \tau_u^{(2)}(M)) = 1 \in \mathrm{Wh}(\mathcal{D}_L). \end{aligned}$$

Now consider the polytope map $\mathbb{P} : \mathrm{Wh}(\mathcal{D}_L) \to \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(L;\mathbb{Z})_f)$,

$$\begin{aligned} 0 &= \mathbb{P}(L_{\bar\phi}(\tau_u^{(2)}(\overline{M}))) \\ &= F_{\bar\phi}\mathbb{P}(\tau_u^{(2)}(\overline{M})) \quad \text{by commutativity} \\ &= 2 \cdot F_{\bar\phi}[B_x^*(\overline{M})] \quad \text{by Theorem 4.24.} \end{aligned}$$

Since $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(\overline{M};\mathbb{R}))$ is torsion-free by [FL17, Lemma 4.8], it follows that

$$F_{\bar\phi}[B_x^*(\overline{M})] = 0,$$

therefore $F_{\bar\phi}B_x^*(\overline{M})$ is a vertex. By Lemma 4.21 $\bar\phi$ lies in a top dimensional Thurston cone of $H^1(\overline{M};\mathbb{R})$. Since $\bar\phi$ was chose to lie in the closure of a fibered cone, and the boundary of a fibered cone consists of Thurston cones of strictly lower dimensions, $\bar\phi$ must lie in a fibered cone itself. Therefore $\bar\phi$ is a fibered class for $\overline{M}$ and consequently ϕ is a fibered class for M. □

5. Universal L^2-torsion for taut sutured manifolds

In this section, we first briefly recall the terminologies of the sutured manifold theory, then discuss the universal L^2-torsion of a taut sutured manifold and prove the decomposition formula Theorem 1.3, which is used to complete the proof of Theorem 1.2. After that, we use a doubling trick to prove Theorem 1.4.

5.1. **Background on sutured manifold theory.** Throughout this section N will be an arbitrary compact, oriented 3-manifold.

Definition 5.1 (Taut surfaces)**.** Given a compact orientable surface Σ with path-components $\Sigma_1, \dots, \Sigma_k$ we define its *complexity* as

$$\chi_-(\Sigma) := \sum_{i=1}^{k} \max\{0, -\chi(\Sigma_i)\}.$$

A properly embedded oriented surface Σ in N is *taut* if Σ is incompressible, and has minimal complexity among all properly embedded oriented surfaces representing the homology class $[\Sigma, \partial\Sigma] \in H_2(N, \nu(\partial\Sigma);\mathbb{Z})$, where $\nu(\partial\Sigma)$ is a regular neighborhood of $\partial\Sigma$ in ∂N.

Definition 5.2 (Sutured manifold)**.** A *sutured manifold* (N, R_-, R_+, γ) is a compact oriented 3-manifold N with a decomposition of its boundary into two subsurfaces R_+ and R_- along their common boundary γ which we call the *suture*. The orientation on $R_\pm$ is defined in the way that the normal vector of R_+ points out of N and the normal vector of R_- points inward of N. The boundary orientations of $R_\pm$ on γ coincide and induce an orientation of the suture γ. We often abbreviate (N, R_-, R_+, γ) as (N, γ) where no confusions arises.

A sutured manifold (N, R_-, R_+, γ) is called *taut* if N is irreducible and $R_\pm$ are both taut surfaces (viewed as properly embedded surfaces after pushing slightly into N).

Before introducing sutured manifold decompositions, we establish some notations. Let S be a (possibly disconnected) properly embedded oriented surface in N. Choose a product neighborhood $S \times (-1, 1)$ of S in N and denote by $N \backslash\backslash S := N \setminus S \times (-1, 1)$ the complement. Let S_+ (resp. S_-) be the components of $S \times \{-1\} \cup S \times \{1\}$ in $N \backslash\backslash S$ whose normal vector points out of (resp. into) N'. This notation ensures that $(N\backslash\backslash S, S_+, S_-, \emptyset)$ forms a sutured manifold when N is closed.

Definition 5.3 (Decomposition surface)**.** Let (N, R_-, R_+, γ) be a sutured manifold. A properly embedded surface S in N is called S a *decomposition surface* if ∂S is transverse to γ, no component of ∂S bounds a disk in $R_\pm$ and no component of S is a disk D with $\partial D \subset R_\pm$.

Definition 5.4 (Sutured manifold decomposition)**.** Given an oriented decomposition surface S for (N, R_-, R_+, γ), the *sutured manifold decomposition*

$$(N, R_-, R_+, \gamma) \overset{S}{\rightsquigarrow} (N', R'_-, R'_+, \gamma')$$

is defined by:

$$\begin{aligned} N' &= N\backslash\backslash S, \\ R'_+ &= (R_+ \cup S_+) \cap N', \\ R'_- &= (R_- \cup S_-) \cap N', \\ \gamma' &= \partial R'_+ = \partial R'_-. \end{aligned}$$

See Figure 3–4 for illustrations. A sutured manifold decomposition $(N, \gamma) \overset{S}{\rightsquigarrow} (N', \gamma')$ is called *taut* if (N', γ') is taut. Gabai [Gab87, Lemma 0.4] proved that if $(N, \gamma) \overset{S}{\rightsquigarrow} (N', \gamma')$ is a taut sutured decomposition then (N, γ) is also taut.

We make the following observations:

(1) Following [AD19], we consider sutures as simple closed curves and exclude torus sutures, unlike many other sources where the suture are disjoint union of annuli and tori. In fact, if a sutured manifold (N, γ) contains toral sutures, these can be absorbed into $R_\pm$ without affecting the universal L^2-torsion:

Lemma 5.5. *If (N, γ) is a taut sutured manifold and $T \subset \gamma$ is a torus component. Then $\tau_u^{(2)}(N, R_+) = \tau_u^{(2)}(N, R_+ \cup T)$.*

Proof. This follows from the Sum Formula 3.12 and the fact that $\tau_u^{(2)}(T) = 1$. ☐

Accordingly, the definition of sutured manifold decomposition is slightly modified. In the classical definition (c.f. [Gab83, Gab87]), a decomposition surface may include boundary components that are cores of a sutured annulus. Such boundary components can be isotoped to entirely contained in $R_+ \cup R_-$, yielding isomorphic sutured manifolds after decomposition.

(2) Let $(N, R_+, R_-, \gamma) \overset{S}{\rightsquigarrow} (N', R'_+, R'_-, \gamma')$ be a taut sutured decomposition, then S is incompressible in N. To see this, note that R'_+ is formed by gluing the surfaces S_+ and $R_+ \cap N'$ along $S_+ \cap R_+$, which consists of arcs and boundary circles of S_+. By assumption, no boundary circles of S_+ bound a disk in R_+ or in S_+, so the closed curves of $S_+ \cap R_+$ are homotopically nontrivial in both S_+ and $R_+ \cap N'$. By Van Kampen Theorem the surface S_+ is π_1-injective in R'_+, hence π_1-injective in N' since R'_+ is incompressible in N'. This shows that S admits no compressing disk in N.

(3) Given a taut sutured decomposition $(N, \gamma) \overset{S}{\rightsquigarrow} (N', \gamma')$. Since S is incompressible in N, the inclusion of each component of N' into N induces a monomorphism on fundamental groups.

5.2. **Universal L^2-torsion for taut sutured 3-manifolds.**

Theorem 5.6 ([Her23]). *Let (N,γ) be a sutured 3-manifold with infinite fundamental group. Suppose that N is irreducible and $R_\pm$ are both incompressible. Then (N,γ) is taut if and only if the pair (N,R_+) is L^2-acyclic.*

Proof. Assume first that (N,γ) is taut. Since N has infinite fundamental group and is irreducible, no components of $R_\pm$ are disks or spheres (otherwise N would be a 3-ball, contradicting $\pi_1(N)$ infinite). Therefore the complexity of $R_\pm$ equals $-\chi(R_\pm)$, and tautness implies $\chi(R_+)=\chi(R_-)$. We apply [Her23, Theorem 1.1] to conclude that (N,R_+) is L^2-acyclic.

Conversely, if (N,R_+) is L^2-acyclic, then the Euler characteristic $\chi(N,R_+)=\chi(N)-\chi(R_+)$ is zero. Since $\chi(N)=\frac{1}{2}\chi(\partial N)=\frac{1}{2}(\chi(R_+)+\chi(R_-))$, it follows that $\chi(R_+)=\chi(R_-)$. Another application of [Her23, Theorem 1.1] shows that (N,γ) is taut. □

The reader might wonder why we prefer (N,R_+) than (N,R_-). In fact, these two pairs are dual to each other by the following Proposition 5.7. We prefer the pair (N,R_+) since it better suits our convention of orientations.

Proposition 5.7. *Let (N,R_+,R_-,γ) be a sutured manifold. Suppose the pair (N,R_+) is L^2-acyclic. Then (N,R_-) is also L^2-acyclic. Moreover*

$$\tau_u^{(2)}(N,R_+)=(\tau_u^{(2)}(N,R_-))^*.$$

Proof. The proof adapts the argument in [Mil62, Page 139] and does not make essential use of L^2-theory. Choose a smooth triangulation of N such that R_+ and R_- are subcomplexes. Let $\widehat{N}$ be the universal cover and let $\widehat{R}_\pm$ be the preimages of $R_\pm$. Consider the dual cellular complex $\widehat{N}'$ of $\widehat{N}$. Each cell of $\widehat{N}$ is canonically dual to a cell of $\widehat{N}'\setminus\partial\widehat{N}'$, as illustrated in Figure 2 below.

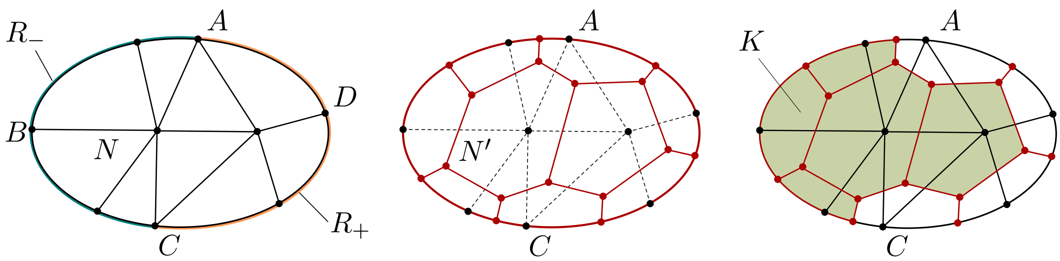

Figure 2. *An illustration in one lower dimension. Left: A simplicial complex N whose boundary is a union of a subcomplex R_- (the arc $\widehat{ABC}$) and a subcomplex R_+ (the arc $\widehat{ADC}$). Middle: the dual cellular complex N' (red). Right: the subcomplex $K\subset N'$ consisting of cells disjoint from R_+. The complex N' deformation retracts to K along a product neighborhood of R_+.*

More explicitly, define the intersection pairing

$$p\colon C_*(\widehat{N})\times C_{3-*}(\widehat{N}',\partial\widehat{N}')\to\mathbb{Z}G,\quad p(\sigma,\sigma')=\sum_{g\in G}\langle\sigma,g\sigma'\rangle\cdot g$$

where $\langle\sigma,g\sigma'\rangle$ is the intersection number of σ and $g\sigma'$. One can verify the identities:

$$p(g\sigma,\sigma')=g\cdot p(\sigma,\sigma'),$$
$$p(\sigma,g\sigma')=p(\sigma,\sigma')\cdot g^{-1},$$
$$p(\partial\sigma,\sigma')=\pm p(\sigma,\partial\sigma').$$

The pairing is non-degenerate in the sense that $\sigma\mapsto p(\sigma,*)$ gives the isomorphism of $\mathbb{Z}G$-chain complexes $C_*(\widehat{N})\cong C^{3-*}(\widehat{N}',\partial N')$. A cell of $\widehat{R}_+$ is canonically dual to a cell of $\widehat{N}'\setminus\partial\widehat{N}'$ whose closure intersects $\widehat{R}_+$ non-trivially, and vise versa. Let K be the union of the cells of N' whose closure is disjoint from R_+, and let $\widehat{K}$ be its preimage in the universal cover. Then we have the induced non-degenerate pairing:

$$C_*(\widehat{N},\widehat{R}_+)\times C_{3-*}(\widehat{K},\partial\widehat{N}')\to\mathbb{Z}G,$$

and hence an isomorphism of $\mathbb{Z}G$-chain complexes $C_*(\widehat{N},\widehat{R}_+)\cong C^{3-*}(\widehat{K},\partial\widehat{N}')$. By Proposition 3.7 the finite based free $\mathbb{Z}G$-chain complex $C_*(\widehat{K},\partial\widehat{N}')$ is L^2-acyclic and

$$\tau_u^{(2)}(C_*(\widehat{N},\widehat{R}_+))=(\tau_u^{(2)}(C_*(\widehat{K},\partial\widehat{N}')))^*.$$

Therefore $\tau_u^{(2)}(N,R_+)=(\tau_u^{(2)}(K,\partial N'))^*$. The pair $(K,\partial N')$ is a deformation retract of the pair (N',R'_-) (after barycentric-subdividing N if necessary), so

$$\tau_u^{(2)}(N,R_+)=(\tau_u^{(2)}(N',R'_-))^*.$$

This shows that the pair (N,R_-) is L^2-acyclic with $\tau_u^{(2)}(N,R_+)=(\tau_u^{(2)}(N,R_-))^*$, as the universal L^2-torsion of (N,R_-) is independent of its cellular structure. □

5.3. **Turaev's algorithm.** Let $(N,\gamma)\overset{S}{\rightsquigarrow}(N',\gamma')$ be a taut sutured decomposition. To simplify notation, we abbreviate the pair $(N',R'_+)=(N\backslash\backslash S,(S_+\cup R_+)\cap(N\backslash\backslash S))$ as $(N\backslash\backslash S,S_+\cup R_+)$, with the understanding that a CW-pair (X,Y) always means $(X,X\cap Y)$.

Turaev's algorithm [Tur02] is a procedure that modifies S into S', without changing the push-forward of the universal L^2-torsion $i_*\tau_u^{(2)}(N',R'_+)$.

Lemma 5.8. *Let (N,γ) be a taut sutured manifold. Suppose there is a decomposition surface S such that:*

1. *The sutured manifold decomposition $(N,R_+,R_-,\gamma)\overset{S}{\rightsquigarrow}(N',R'_+,R'_-,\gamma')$ is taut.*
2. *S is separating and N' is a disjoint union of two (possibly disconnected) manifolds N_0 and N_1.*
3. *$S_-\subset N_0$ and $S_+\subset N_1$.*

Then $\tau_u^{(2)}(N,R_+)=i_*\tau_u^{(2)}(N',R'_+)$ where $i\colon N'\hookrightarrow N$ is the inclusion.

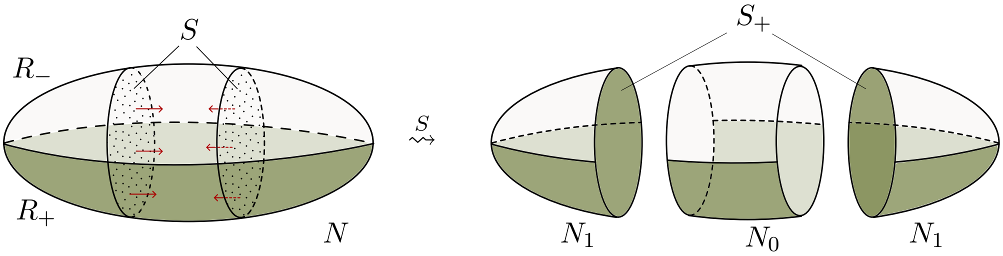

Figure 3. *Decomposing N along a separating decomposition surface S (dotted region in left figure, with normal direction indicated by red arrows). The R_+-regions of the sutured manifolds are shown in green.*

Proof. Consider the short exact sequence of chain complexes

$$0\to C_*(\widehat{N_0},\widehat{R_+})\to C_*(\widehat{N},\widehat{R_+})\to C_*(\widehat{N},\widehat{N_0\cup R_+})\to 0$$

and a natural isomorphism $C_*(\widehat{N},\widehat{N_0\cup R_+})=C_*(\widehat{N_1},\widehat{S_+\cup R_+})$. Note that $N_0\cap R_+$ is the R_+-region of N_0 and $(S_+\cup R_+)\cap N_1$ is the R_+-region of N_1. Since both N_0 and

N_1 are taut sutured manifolds by assumption, Theorem 5.6 implies that all three chain complexes are L^2-acyclic, and by the Sum Formula (Proposition 3.5),

$$\tau_u^{(2)}(N, R_+) = (i_0)_* \tau_u^{(2)}(N_0, R_+) \cdot (i_1)_* \tau_u^{(2)}(N_1, S_+ \cup R_+)$$

where $i_0 : N_0 \hookrightarrow N$ and $i_1 : N_1 \hookrightarrow N$ are the inclusions. The right-hand side equals $i_* \tau_u^{(2)}(N', R'_+)$, completing the proof. $\square$

Definition 5.9 (Weighted surface)**.** A *weighted surface* $\widehat{S}$ in a compact oriented 3-manifold N is a finite collection of pairs (S_i, w_i), $i = 1, \dots, n$, where each S_i is a connected properly embedded oriented surface, each w_i is a positive integer, and the surfaces S_i are pairwise disjoint. The *realization* of $\widehat{S}$ is the properly embedded oriented surface

$$\bar{S} := \bigcup_{i=1}^{n} w_i \cdot S_i$$

where $w_i \cdot S_i$ denotes the union of w_i parallel copies of S_i. The *reduction* of $\widehat{S}$ is the surface

$$S := \bigcup_{i=1}^{n} S_i.$$

A weighted surface is called a *weighted decomposition surface* if its realization is a decomposition surface.

If $\widehat{S}$ is a weighted decomposition surface, then $N\backslash\backslash\bar{S}$ is a disjoint union of $N\backslash\backslash S$ and some copies of $S_i \times I$. Consequently, the sutured decomposition along $\bar{S}$ is taut if and only if the sutured decomposition along S is taut.

The following Proposition 5.10 is a generalization of [BAFH22, Proposition 3.3].

Proposition 5.10. *Let (N, R_+, R_-, γ) be a taut sutured manifold and let $(N, \gamma) \overset{\Sigma}{\rightsquigarrow} (N', \gamma')$ be a taut sutured decomposition. Then there exists a weighted decomposition surface $\widehat{S}$ in N (with realization $\bar{S}$ and reduction S) such that*

1. *$N\backslash\backslash S$ is connected,*
2. *$[\bar{S}] = [\Sigma] \in H_2(N, \partial N; \mathbb{Z})$,*
3. *the sutured decomposition of N along S is taut,*
4. *$i_* \tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+) = j_* \tau_u^{(2)}(N\backslash\backslash\Sigma, \Sigma_+ \cup R_+)$ where i, j are the natural inclusions of $N\backslash\backslash S$ and $N\backslash\backslash\Sigma$ into N.*

Proof. For any weighted surface $\widehat{S}$ in N, define $c(\widehat{S}) := \#\pi_0(N\backslash\backslash S)$. Begin by taking $\widehat{S}$ to be the surface Σ with weight 1 assigned to each component, then $\bar{S} = S = \Sigma$, and $\widehat{S}$ satisfies (2)–(4). It suffices to show that given any weighted surface $\widehat{S}$ with $c(\widehat{S}) > 1$ satisfying (2)–(4), then there exists a weighted surface $\widehat{T}$ satisfying (2)–(4) with $c(\widehat{T}) < c(\widehat{S})$.

Given such $\widehat{S}$. Since $c(\widehat{S}) > 1$ there is a component $C \subset S$ such that $C_\pm$ lie in different components of $N\backslash\backslash S$. Choose C to be such a component with minimal weight w. Let M_0 (resp. M_1) be the component of $N\backslash\backslash S$ containing C_- (resp. C_+). Let $C_1 = C, C_2, \dots, C_k$ be the components of S whose normal direction point out of M_1 and let $D_1, \dots, D_l$ be the components of S whose normal direction point into M_1 (it is possible that $C_i = D_j$ for some i, j, meaning both sides of C_i both belong to M_1). Then

$$[C_1] + \cdots + [C_k] = [D_1] + \cdots + [D_l] \in H_2(N, \partial N; \mathbb{Z}).$$

We change the weights of $\widehat{S}$ by increasing the weights of $D_1, \dots, D_l$ by w and decreasing the weights of $C_1, \dots, C_k$ by w. If a component's weight becomes zero, we discard this component. Denote the new weighted surface by $\widehat{T}$. Clearly, $[\bar{S}] = [\bar{T}] \in H_2(N, \partial N; \mathbb{Z})$.

Since T is a subcollection of S and the decomposition along S is taut, the decomposition along T is also taut. Thus, (2) and (3) hold for $\widehat{T}$. Moreover, $c(\widehat{T}) < c(\widehat{S})$ because the component C is discarded and M_0 and M_1 lie in the same component of $N\backslash\backslash T$.

It remains to check (4). Let $S_0 = S \setminus T$. Then $N\backslash\backslash S$ is obtained from the sutured manifold decomposition of $N\backslash\backslash T$ along S_0. By construction, S_0 separates $N\backslash\backslash T$; in particular, it separates M_0 from M_1 and $(S_0)_+ \subset M_1$. Apply Lemma 5.8 to $N_0 := (N\backslash\backslash T) - M_1$ and $N_1 := M_1$, we have

$$i'_*\tau_u^{(2)}(N\backslash\backslash T, T_+ \cup R_+) = \tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+)$$

where $i' : N\backslash\backslash T \hookrightarrow N\backslash\backslash S$ is the inclusion. It follows that

$$(i \circ i')_*\tau_u^{(2)}(N\backslash\backslash T, T_+ \cup R_+) = i_*\tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+) = j_*\tau_u^{(2)}(N\backslash\backslash \Sigma, \Sigma_+ \cup R_+)$$

where $i \circ i', i, j$ are the inclusions of $N\backslash\backslash T$, $N\backslash\backslash S$ and $N\backslash\backslash\Sigma$ into N, respectively. This verifies (4) for $\widehat{T}$ and completes the proof. □

5.4. **Decomposition formula for universal L^2-torsion.** In this section we prove Theorem 1.3.

Definition 5.11 (Leading term of matrices)**.** Let G be a finitely generated torsion-free group satisfying Atiyah Conjecture. Let $\phi \in H^1(G;\mathbb{R})$ be an 1-cohomology class. There are natural homomorphisms

$$\delta_\phi\colon (\mathbb{Z}G)^\times \to \mathbb{R}, \quad L_\phi\colon (\mathbb{Z}G)^\times \to (\mathbb{Z}G)^\times$$

defined by considering the terms with minimal ϕ-value (see Section 4.3). For a nonzero matrix A over $\mathbb{Z}G$, let $\delta_\phi(A)$ be the minimum of $\delta_\phi(A_{ij})$ over all nonzero entries A_{ij}. Then A admits a unique decomposition

$$A = L_\phi(A) + (A - L_\phi(A))$$

where every group element g appearing in $L_\phi(A)$ satisfies $\phi(g) = \delta_\phi(A)$, and any group element h in $(A - L_\phi(A))$ satisfies $\phi(h) > \delta_\phi(A)$.

We define $L_\phi(A) = 0$ if A is a zero matrix. Otherwise $L_\phi(A)$ is always nonzero.

Remark 5.12. By Theorem 4.13, if $L_\phi(A)$ is invertible over $\mathcal{D}_G$, then A is also invertible over $\mathcal{D}_G$ and $L_\phi(\det A) = \det(L_\phi(A)) \in K_1(\mathcal{D}_G)$.

Definition 5.13 (Leading term of chain complexes)**.** For any finite based free $\mathbb{Z}G$-chain complex

$$C_* = (0 \longrightarrow C_n \xrightarrow{A_n} \cdots \xrightarrow{A_2} C_1 \xrightarrow{A_1} C_0 \longrightarrow 0),$$

define

$$L_\phi(C_*) = (0 \to C_n \xrightarrow{L_\phi(A_n)} \cdots \xrightarrow{L_\phi(A_2)} C_1 \xrightarrow{L_\phi(A_1)} C_0 \longrightarrow 0).$$

One can verify that if $AB = 0$, then $L_\phi(A)L_\phi(B) = 0$. In particular $L_\phi(C_*)$ is a well-defined finite based free $\mathbb{Z}G$-chain complex.

Lemma 5.14. *Suppose G is a finitely generated torsion-free group satisfying the Atiyah Conjecture. Let $\phi \in H^1(G;\mathbb{R})$ be an 1-cohomology class and let*

$$C_* = (0 \longrightarrow C_n \xrightarrow{A_n} \cdots \xrightarrow{A_2} C_1 \xrightarrow{A_1} C_0 \longrightarrow 0)$$

be a finite based free $\mathbb{Z}G$-chain complex. If $L_\phi(C_*)$ is L^2-acyclic, then C_* is also L^2-acyclic and

$$\tau_u^{(2)}(L_\phi(C_*)) = L_\phi(\tau_u^{(2)}(C_*)) \in \widetilde{K_1}(\mathcal{D}_G).$$

Proof. Since $L_\phi(C_*)$ is L^2-acyclic, there exists a non-degenerate matrix chain $\mathcal{A}$ of $L_\phi(C_*)$ by Theorem 3.22. Let $B_1,\dots,B_n$ be the associated submatrices of $L_\phi(A_1),\dots,L_\phi(A_n)$, then each B_i is a weak isomorphism, and

$$\tau_u^{(2)}(L_\phi(C_*))=\prod_{i=1}^n \det{}_r(B_i)^{(-1)^n}\in \widetilde{K}_1(\mathcal{D}_G)$$

Let $C_1,\dots,C_n$ be the submatrices of $A_1,\dots,A_n$ associated to the same matrix chain $\mathcal{A}$. Since $B_i\neq 0$, we have $L_\phi(C_i)=B_i$. By Theorem 4.13 each C_i is a weak isomorphism and

$$L_\phi \det{}_r(C_i)=\det{}_r(B_i)\in \widetilde{K}_1(\mathcal{D}_G).$$

Therefore $\mathcal{A}$ is a non-degenerate matrix chain of C_*. Applying Theorem 3.22 again,

$$\tau_u^{(2)}(C_*)=\prod_{i=1}^n \det{}_r(C_i)^{(-1)^n}\in \widetilde{K}_1(\mathcal{D}_G)$$

and consequently $\tau_u^{(2)}(L_\phi(C_*))=L_\phi(\tau_u^{(2)}(C_*))$. ☐

Theorem 5.15 (Theorem 1.3)**.** *Let $(N,R_+,R_-,\gamma)\overset{\Sigma}{\rightsquigarrow}(N',R'_+,R'_-,\gamma')$ be a taut sutured decomposition and let $\phi\in H^1(N;\mathbb{Z})$ be the Poincaré dual of the surface Σ, then*

$$j_*\tau_u^{(2)}(N',R'_+)=L_\phi\tau_u^{(2)}(N,R_+)$$

where $j\colon N'\hookrightarrow N$ is the natural inclusion.

Proof. By Proposition 5.10, there is a weighted decomposition surface $\widehat{S}$ in N such that $N\backslash\backslash S$ is connected, $\phi=PD([\bar S,\partial\bar S])\in H^1(N;\mathbb{Z})$, and

$$j_*\tau_u^{(2)}(N',R'_+)=i_*\tau_u^{(2)}(N\backslash\backslash S,S_+\cup R_+)$$

where $i\colon N\backslash\backslash S\to N$ is the inclusion. We are left to show that

$$L_\phi(\tau_u^{(2)}(N,R_+))=i_*\tau_u^{(2)}(N\backslash\backslash S,S_+\cup R_+).$$

Chose a CW-structure for N such that $S\times I$ and $R_\pm$ are subcomplexes. Fix a base point $p\in N\setminus(S\times I)$. For each cell σ in the CW-structure of N, choose a path γ_σ (shown in red in Figure 4) connecting p and σ such that

- γ_σ is disjoint with S_- if $\sigma\subset N\setminus S_-$,
- γ_σ is disjoint with S_+ if $\sigma\subset S_-$.

Lift the base point p to $\hat p$ in the universal cover $\widehat{N}$ and lift each cell σ to $\hat\sigma$ using the path γ_σ. The cells $\hat\sigma$ form a basis for the finite based free $\mathbb{Z}[\pi_1(N)]$-chain complex $C_*(\widehat{N})$. Now consider the finite based chain complex $C_*(\widehat{N},\widehat{R_+})$, for each $k=0,1,2,3$ the chain group admits the following direct sum decomposition

$$C_k(\widehat{N},\widehat{R_+})=C_k(\widehat{N\backslash\backslash S},\widehat{S_+\cup R_+})\oplus C_k(\widehat{S\times I},\widehat{S_-\cup R_+}).$$

Accordingly, the boundary homomorphism $\partial_k\colon C_k(\widehat{N},\widehat{R_+})\to C_{*-1}(\widehat{N},\widehat{R_+})$ admits the following decomposition

$$\partial_k=\begin{pmatrix}\partial_k^1 & \partial_k^2\\ \partial_k^3 & \partial_k^4\end{pmatrix},$$
$$\partial_k^1\colon C_k(\widehat{N\backslash\backslash S},\widehat{S_+\cup R_+})\to C_{k-1}(\widehat{N\backslash\backslash S},\widehat{S_+\cup R_+}),$$
$$\partial_k^2\colon C_k(\widehat{N\backslash\backslash S},\widehat{S_+\cup R_+})\to C_{k-1}(\widehat{S\times I},\widehat{S_-\cup R_+}),$$
$$\partial_k^3\colon C_k(\widehat{S\times I},\widehat{S_-\cup R_+})\to C_{k-1}(\widehat{N\backslash\backslash S},\widehat{S_+\cup R_+}),$$
$$\partial_k^4\colon C_k(\widehat{S\times I},\widehat{S_-\cup R_+})\to C_{k-1}(\widehat{S\times I},\widehat{S_-\cup R_+}).$$

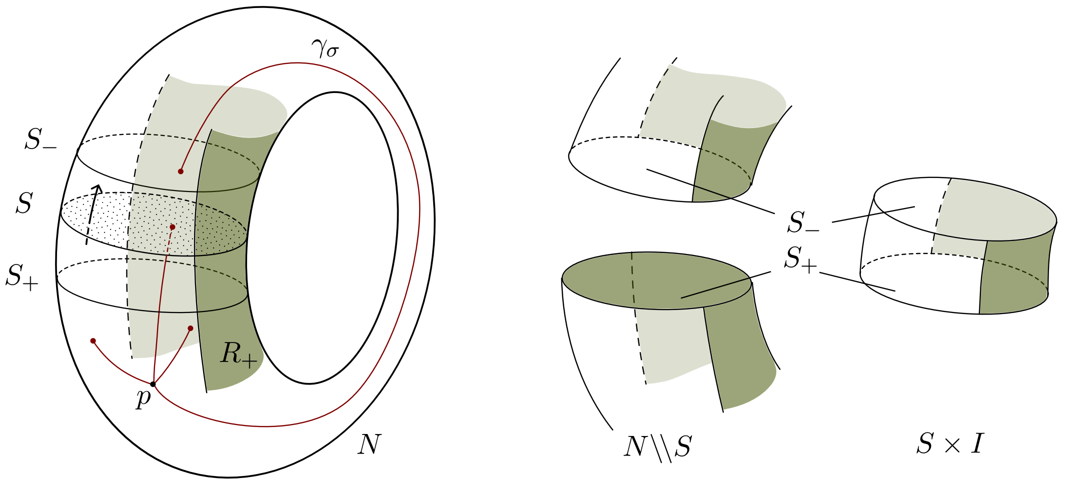

Figure 4. *Consider N as the union of $N\backslash\backslash S$ and $S \times I$.*

In particular, if $\hat{\sigma}$ is a basis element of $C_k(\widehat{S \times I}, \widehat{S_- \cup R_+})$ then there are basis elements τ_i (allowing repetitions) of $C_{k-1}(\widehat{N\backslash\backslash S}, \widehat{S_+ \cup R_+})$ and $g_i \in \pi_1(N)$ such that

$$\partial_k^3(\hat{\sigma}) = \sum_i g_i \hat{\tau}_i.$$

The cells $\hat{\tau}_i$ must lie in $\widehat{S}_-$. By our choice of basis, each g_i satisfies $\phi(g_i) > 0$. A similar argument shows that any group element h appearing in $\partial_k^1, \partial_k^2$ or ∂_k^4 satisfies $\phi(h) = 0$. It follows that

$$L_\phi(\partial_*) = \begin{pmatrix} \partial_*^1 & \partial_*^2 \\ 0 & \partial_*^4 \end{pmatrix}$$

and we obtain a short exact sequence of chain complexes

$$0 \to C_*(\widehat{N\backslash\backslash S}, \widehat{S_+ \cup R_+}) \to L_\phi(C_*(\widehat{N}, \widehat{R_+})) \to C_*(\widehat{S \times I}, \widehat{S_- \cup R_+}) \to 0.$$

By the product formula 3.5 and the induction property Theorem 3.12, we have

$$\tau_u^{(2)}(L_\phi(C_*(\widehat{N}, \widehat{R_+}))) = i_*\tau_u^{(2)}(N\backslash\backslash S, S_- \cup R_+) \cdot i'_*\tau_u^{(2)}(S \times I, S_- \cup R_+),$$

where $i : N\backslash\backslash S \hookrightarrow N$ and $i' : S \times I \hookrightarrow N$ are the inclusions. The left-hand side equals $L_\phi(\tau_u^{(2)}(N, R_+))$ by Lemma 5.14. On the right-hand side, since $(S_- \cup R_+) \cap (S \times I)$ deformation retracts onto S_-, we have $\tau_u^{(2)}(S \times I, S_- \cup R_+) = \tau_u^{(2)}(S \times I, S_-) = 1$. Therefore, $L_\phi(\tau_u^{(2)}(N, R_+)) = i_*\tau_u^{(2)}(N\backslash\backslash S, S_+ \cup R_+)$, completing the proof. □

The (stronger) "only if" part of Theorem 1.2 follows easily.

Theorem 5.16 (The "only if" part of Theorem 1.2)**.** *Suppose M is an admissible 3-manifold and $\phi \in H^1(M;\mathbb{R})$ is a fibered class. Then $L_\phi\tau_u^{(2)}(M) = 1 \in \mathrm{Wh}(\mathcal{D}_G)$.*

Proof. By Proposition 4.9(7), there exists a rational fibered class $\psi \in H^1(M;\mathbb{Q})$ such that $L_\phi(\tau_u^{(2)}(M)) = L_\psi(\tau_u^{(2)}(M))$. Choose a positive integer n such that $n\psi$ is an integral fibered class, and let S be a fiber surface dual to $n\psi$, then $M\backslash\backslash S = S \times [0,1]$ is a product. By Theorem 1.3 we have

$$L_{n\psi}\tau_u^{(2)}(M) = j_*\tau_u^{(2)}(M\backslash\backslash S, S_+) = 1,$$

where $j\colon M\backslash\backslash S \hookrightarrow M$ is the inclusion. It follows that $L_\phi\tau_u^{(2)}(M) = L_{n\psi}\tau_u^{(2)}(M) = 1$. □

Combined with Theorem 4.25, this completes the proof of Theorem 1.2.

5.5. **Proof of Theorem 1.4.** We use a refined doubling trick to show that universal L^2-torsion detects product sutured manifolds.

Definition 5.17 (Double of taut sutured manifolds)**.** Let (N, R_+, R_-, γ) be a taut sutured manifold and let $f\colon R_+ \to R_+$ be an orientation-preserving homeomorphism. Let $(N, \overline{R}_+, \overline{R}_-, \bar{\gamma})$ be the sutured manifold with the same underlying oriented manifold as N, but with R_+ and R_- interchanged (see Figure 5 below). That is,

$$\bar{\gamma} = -\gamma, \quad \overline{R}_+ = -R_-, \quad \overline{R}_- = -R_+.$$

The *double of N with monodromy f* is defined to be the admissible 3-manifold

$$DN_f = (N, \gamma) \cup (N, \bar{\gamma}) / \sim$$

formed by gluing together the two sutured manifolds as follows: identify R_- and $\overline{R}_+$ via the identity map and identify R_+ and $\overline{R}_-$ via f.

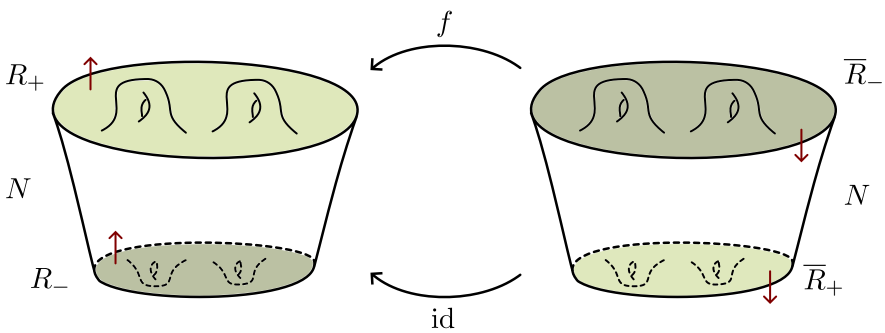

Figure 5. *Double of a sutured manifold N with monodromy f*

Lemma 5.18. *If (N, R_+, R_-, γ) is a taut sutured manifold with R_+ and R_- both non-empty. If not every component of $R_\pm$ is a torus, then there exists $f\colon R_+ \to R_+$ such that DN_f is not a closed graph manifold.*

Proof. If the suture γ is non-empty, then DN_f has non-empty torus boundary for any f. If $\pi_1(N)$ is finite then N is a product 3-ball and has non-empty sutured annuli.

Now assume N has infinite fundamental group and $\gamma = \emptyset$. Then each component of $R_\pm$ is a closed surfaces with positive genus. By hypothesis, $R_\pm$ contains a component of genus at least 2. Since $\chi(R_+) = \chi(R_-)$, the surface R_+ must contain such a component and we call it Σ. Define

$$f = (\phi, \mathrm{id}), \quad \phi\colon \Sigma \to \Sigma, \quad \mathrm{id}\colon R_+ \setminus \Sigma \to R_+ \setminus \Sigma$$

where ϕ is to be specified later. Let M be the manifold obtained by gluing together (N, γ) and $(N, \bar{\gamma})$ via the identity on $R_- \to \overline{R}_+$ and on $R_+ \setminus \Sigma \to \overline{R}_- \setminus (-\Sigma)$. Then ∂M consists of two copies of Σ (denoted Σ_1, Σ_2) that remain unglued. The double DN_f is obtained by identifying Σ_1 and Σ_2 via ϕ.

It remains to prove the following Claim:

Claim. *Let M be a compact, orientable, irreducible 3-manifold with $\partial M = \Sigma_1 \sqcup \Sigma_2$, where Σ_1, Σ_2 are connected incompressible surfaces of genus $g \geqslant 2$. Then there exists a homeomorphism $\phi\colon \Sigma_1 \to \Sigma_2$ such that the closed manifold M_ϕ, obtained by gluing Σ_1 to Σ_2 via ϕ, is not a graph manifold.*

Proof of Claim. If $M = \Sigma_1 \times I$, then a pseudo-Anosov homeomorphism ϕ suffices by Thurston's Hyperbolization Theorem. Now suppose M is not a product.

By the Characteristic Submanifold Theorem [JS79, Chapter V], there exists a submanifold $X \subset M$ that is a disjoint union of Seifert fibered spaces and I-bundles over surfaces, such that any incompressible torus and annulus of M can be homotoped into X. The intersections $X \cap \Sigma_1$ and $X \cap \Sigma_2$ are incompressible *proper* subsurfaces of Σ_1 and Σ_2 respectively; otherwise, if (say) $\Sigma_1 \subset X$, then Σ_1 is contained in an I-bundle component P of X. Since $\partial P \subset \partial M$, we must have $P = M$, contradicting the assumption that M is not a product.

Choose an arbitrary component of $\partial(X \cap \Sigma_i)$ and denote it by C_i. Then C_i is an essential simple closed curve on Σ_i, $i = 1, 2$. Choose a homeomorphism $\phi\colon \Sigma_1 \to \Sigma_2$ such that the curves $\phi(C_1)$ and C_2 fill Σ_2. This can be achieved by taking $\phi = \psi^n \circ \phi_0$, where $\phi_0\colon \Sigma_1 \to \Sigma_2$ is any homeomorphism, $\psi\colon \Sigma_2 \to \Sigma_2$ is pseudo-Anosov homeomorphism and n sufficiently large. Then the distance between $[\phi(C_1)]$ and $[C_2]$ in the curve complex of Σ_2 becomes arbitrarily large [MM99, Proposition 4.6], implying that $\phi(C_1)$ and C_2 fills Σ_2.

We now show that M_ϕ is not a graph manifold. Suppose $T \subset M_\phi$ is an essential torus. After isotopy we assume that T is transverse to $\Sigma = \Sigma_1 = \Sigma_2$ in M_ϕ, and any component of $T \cap \Sigma$ (if nonempty) is an essential simple closed curves on Σ. If $T \cap \Sigma \neq \emptyset$ and let C be such a intersection curve, then $T \setminus C$ is an essential annulus in M, so the image of C on $\Sigma_2 \subset \partial M$ can be homotoped into both $X \cap \Sigma_2$ and $\phi(X \cap \Sigma_1)$. Hence the geometric intersection numbers $i(C, \phi(C_1))$ and $i(C, C_2)$ are both zero, contradicting the fact that $\phi(C_1)$ and C_2 fill Σ_2.

Thus, any essential torus T in M_f can be isotoped to be disjoint with Σ. Let Y be the JSJ-piece of M_ϕ containing Σ. If Y were Seifert fibered, then Σ would be a horizontal surface in Y since it is essential with genus $\geqslant 2$. The circle fiber over any essential simple closed curves of Σ would yield essential tori in Y intersecting Σ, a contradiction. Therefore Y is hyperbolic and M_ϕ is not a closed graph manifold. □

Theorem 5.19 (Theorem 1.4)**.** *Let (N, R_+, R_-, γ) be a taut sutured manifold with R_+ and R_- both non-empty. Then (N, γ) is a product sutured manifold if and only if $\tau_u^{(2)}(N, R_+) = 1 \in \mathrm{Wh}(\mathcal{D}_{\pi_1(N)})$.*

Proof. If (N, γ) is a product sutured manifold then $\tau_u^{(2)}(N, R_+) = 1$ by Proposition 3.23.

Now we suppose $\tau_u^{(2)}(N, R_+) = 1$ and we show that N is a product.

Case 1: Each component of $R_\pm$ is a torus. Then N is an admissible 3-manifold and $\tau_u^{(2)}(N) = \tau_u^{(2)}(N, R_+) = 1$. By Theorem 1.2, every nonzero cohomology class is fibered, and its Thurston norm vanishes by Theorem 4.24. Hence N is homeomorphic to one of the following: the solid torus $D^2 \times S^1$, the thickened torus $T^2 \times I$, or the twisted I-bundle over Klein bottle $K \widetilde{\times} I$. Since R_+ and R_- are both non-empty, it follows that

$$(N, R_+, R_-, \gamma) = (T^2 \times I, T^2 \times \{1\}, T^2 \times \{0\}, \emptyset)$$

which is a product sutured manifold.

Case 2: Not every component of $R_\pm$ is a torus. Choose a homeomorphism $f\colon R_+ \to R_+$ as in Lemma 5.18 such that the double DN_f is not a closed graph manifold. Then $R_+ \cup R_-$ is a Thurston norm minimizing surface dual to a cohomology class ϕ. By Theorem 1.3,

$$L_\phi \tau_u^{(2)}(DN_f) = (j_1)_* \tau_u^{(2)}(N, R_+) \cdot (j_2)_* \tau_u^{(2)}(N, \overline{R}_+)$$

where j_1, j_2 are the natural inclusions. By Proposition 5.7,

$$\tau_u^{(2)}(N, R_+) = \tau_u^{(2)}(N, R_-) = \tau_u^{(2)}(N, \overline{R}_+) = 1,$$

so $L_\phi \tau_u^{(2)}(DN_f) = 1$. By Theorem 1.2, ϕ is a fibered class and hence $R_+ \cup R_-$ is a fiber surface. Therefore, (N, γ) is a product sutured manifold. □

6. Applications of the universal L^2-torsions

6.1. **Universal L^2-torsion for groups homomorphisms.** Let G_1, G_2 be torsion-free groups satisfying the Atiyah Conjecture. Assume that their Whitehead groups $\mathrm{Wh}(G_i)$ vanish and they have finite classifying spaces X_i, $i = 1, 2$. Then any homomorphism $\varphi\colon G_1 \to G_2$ determines a homotopy class of mapping $\Phi\colon X_1 \to X_2$. We call Φ a *realization of* φ.

Definition 6.1 (Universal L^2-torsion for group homomorphisms)**.** Let $\varphi\colon G_1 \to G_2$ be a homomorphism and let $\Phi\colon X_1 \to X_2$ be its realization. The *universal L^2-torsion of* φ is defined to be $\tau_u^{(2)}(\varphi) := \tau_u^{(2)}(\Phi) \in \mathrm{Wh}(\mathcal{D}_{G_2}) \sqcup \{0\}$. We say φ is an *L^2-homology equivalence* if $\tau_u^{(2)}(\varphi) \neq 0$.

If X_1', X_2' are different choice of classifying spaces then we have the following commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "\Phi"] \arrow[d, "\psi_1"'] & X_2 \arrow[d, "\psi_2"] \\ X_1' \arrow[r, "\Phi'"] & X_2' \end{tikzcd}$$

where $\psi_i\colon X_i \to X_i'$ are homotopy equivalences (hence simple-homotopy equivalences since the Whitehead groups vanish) and Φ' is the realization of φ with respect to the classifying spaces X_1', X_2'. By Lemma 3.17 we know that

$$\tau_u^{(2)}(\Phi') = (\psi_2)_* \tau_u^{(2)}(\Phi),$$

hence the definition of $\tau_u^{(2)}(\varphi)$ does not depend on the choice of classifying spaces X_i.

6.1.1. *Homomorphism between free groups.* Let $F_1 = \langle x_1, \ldots, x_n \rangle$ and $F_2 = \langle y_1, \ldots, y_m \rangle$ be finitely generated free groups. They satisfy the Atiyah Conjecture and have trivial Whitehead groups [Sta65]. Let us explicitly calculate $\tau_u^{(2)}(\varphi)$ for a homomorphism $\varphi\colon F_1 \to F_2$ under the given basis.

Let $X_1 = \vee_{i=1}^n S^1$ and $X_2 = \vee_{i=1}^m S^1$ be the wedge of circles. The space X_1 is given the usual CW-structure with one 0-cell p and n 1-cells $e_1, \ldots, e_n$. Identify the fundamental group $\pi_1(X_1, p)$ with F_1 in such a way that $x_i = [e_i]$. Similarly, X_2 is given the CW-structure with one 0-cell q and m 1-cells $f_1, \ldots, f_m$ and $y_i = [f_i]$. Form the wedge space $X_1 \vee X_2$ by identifying $p \in X$ and $q \in X_2$, attach a 2-cell σ_i whose boundary is the concatenation of $\Phi(e_i)$ and e_i^{-1} for each $i = 1, \ldots, n$. The resulting CW-complex M_Φ is simple-homotopy equivalent to the mapping cylinder of Φ, the realization of φ. Let $\widehat{M_\Phi}$ be the universal cover of M_Φ and let $\widehat{X_1}$ be the preimage of $X_1 \subset M_\Phi$. Fix a lifting $\hat{p} \in \widehat{M_\Phi}$ of p and lift the other cells with respect to the base point $\hat{p}$. Then we have the following $\mathbb{Z}F_2$-chain complex

$$(\dagger) \qquad C_*(\widehat{M_\Phi}, \widehat{X_1}) = (0 \to \mathbb{Z}F_2\langle \hat{\sigma}_1, \ldots, \hat{\sigma}_n \rangle \xrightarrow{J_\varphi} \mathbb{Z}F_2\langle \hat{f}_1, \ldots, \hat{f}_m \rangle \to 0 \to 0).$$

The square matrix J_φ is called the *Fox Jacobian* of φ with respect to the basis $\langle x_1, \ldots, x_n \rangle$ and $\langle y_1, \ldots, y_m \rangle$. Recall that the *Fox derivative* $\frac{\partial}{\partial y_i}\colon \mathbb{Z}F_2 \to \mathbb{Z}F_2$, $i = 1, \ldots, m$ are $\mathbb{Z}$-linear maps characterized by the following two properties:

- $\frac{\partial}{\partial y_i} 1 = 0$ and $\frac{\partial}{\partial y_i} y_j = \delta_{ij}$.
- $\frac{\partial}{\partial y_i}(uv) = \frac{\partial}{\partial y_i} u + u \cdot \frac{\partial}{\partial x_i} v$ for all $u, v \in F_2$.

The entries of J_φ are then given by the Fox derivatives

$$J_{ij} = \frac{\partial \varphi(x_i)}{\partial y_j} \in \mathbb{Z}F_2, \quad 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m.$$

Proposition 6.2. *Let $\varphi\colon F_1 \to F_2$ be a homomorphism between finitely generated free groups. Then:*

(1) $\tau_u^{(2)}(\varphi) = \det_w(J_\varphi)$ *where J_φ is the Fox Jacobian of φ with respect to any choice of basis of F_1 and F_2. In particular φ is an L^2-homology equivalence if and only if J_φ is a weak isomorphism.*

(2) $\tau_u^{(2)}(\varphi) = 1$ *if φ is an isomorphism;* $\tau_u^{(2)}(\varphi) = 0$ *if $m \neq n$ or φ is not injective.*

(3) *Let $\psi\colon F_2 \to F_3$ be a homomorphism to a finitely generated free group F_3. Suppose φ, ψ are L^2-homology equivalences then* $\tau_u^{(2)}(\psi \circ \varphi) = \psi_* \tau_u^{(2)}(\varphi) \cdot \tau_u^{(2)}(\psi)$.

Proof. Firstly, (1) is immediate from Equation (†) and Definition 6.1.

For (2), if φ is an isomorphism, then $\Phi\colon X_1 \to X_2$ is a homotopy equivalence. Since the Whitehead group of a free group is trivial, Φ is a simple homotopy equivalence and $\tau_u^{(2)}(\varphi) = \tau_u^{(2)}(\Phi) = 1$ by Proposition 3.15(3). If $m \neq n$, then J_φ is not a square matrix and clearly $\tau_u^{(2)}(\varphi) = 0$. The proof of (2) is finished once we establish the following Lemma 6.3.

Lemma 6.3. *If $n = m$ and φ is not injective, then the Fox Jacobian J_φ is not a weak isomorphism.*

Proof of Lemma 6.3. Suppose the contrary that φ is not injective, then there is a reduced word $w \in F_1$ such that $\varphi(w) = 1$. Let $w = x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$ be such a word with shortest length $k \geqslant 1$, where $i_1, \dots, i_k \in \{1, \dots, n\}$ and $\epsilon_1, \dots, \epsilon_k \in \{\pm 1\}$. We may assume that $x_{i_1}^{\epsilon_1} = x_1$ and $x_{i_k}^{\epsilon_k} \neq x_1^{-1}$. Denote by $w_s := x_{i_1}^{\epsilon_1} \cdots x_{i_s}^{\epsilon_s}$, $s = 1, \dots, k$ to be the prefix of w of length s and set $w_0 = 1$. For any $j = 1, \dots, n$, apply $\frac{\partial}{\partial y_j}$ to both sides of the identity $\varphi(x_{i_1})^{\epsilon_1} \cdots \varphi(x_{i_k})^{\epsilon_k} = 1$, we have

$$\sum_{s=1}^{k} u_s \cdot \frac{\partial \varphi(x_{i_s})}{\partial y_j} = 0, \quad \text{where} \quad u_s = \begin{cases} \varphi(w_{s-1}), & \epsilon_s = 1, \\ -\varphi(w_s), & \epsilon_s = -1. \end{cases}$$

Note that u_s is independent of j. Rearranging the identities, we have

$$\sum_{i=1}^{n} U_i \cdot \frac{\partial \varphi(x_i)}{\partial y_j} = 0, \quad j = 1, \dots, n$$

where U_i is the sum of all u_s such that $i_s = i$. Therefore

$$(U_1, U_2, \dots, U_n) \cdot J_\varphi = 0.$$

We prove that $U_1 \neq 0$ and this implies that J_φ is not a weak isomorphism. Let $1 \leqslant s_1 < s_2 < \cdots < s_r \leqslant k$ be the indices such that $i_{s_1} = \cdots = i_{s_r} = 1$. By assumption $s_1 = 1$. Then

$$\begin{aligned} U_1 &= 1 + u_{s_2} + \cdots + u_{s_r} \\ &= 1 \pm \varphi(w_{s_2'}) \pm \cdots \pm \varphi(w_{s_r'}) \end{aligned}$$

where each $w_{s_j'}$ equals either w_{s_j} or $w_{s_j - 1}$, depending on the sign of ϵ_{s_j}. Since w is reduced and does not end in x_1^{-1}, we have $0 < s_2' < \cdots < s_r' < k$. We claim that the elements $1, \varphi(w_{s_2'}), \dots, \varphi(w_{s_r'})$ are pairwise distinct. Indeed, if $\varphi(w_{s_p'}) = \varphi(w_{s_q'})$ for some $p < q$ then the reduced word $\varphi(w_{s_p'})^{-1}\varphi(w_{s_q'})$ would be a non-trivial element of $\ker \phi$ of length shorter than k, contradicting the minimality of k. Therefore U_1 is a nonzero element of $\mathbb{Z}F_2$, completing the proof. ☐

It remains to prove Proposition 6.2(3). Note that φ and ψ are injective by (2) and the result follows from Proposition 3.15(4). ☐

It is an interesting question to characterize when a homomorphism $\varphi\colon F_1 \to F_2$ is an L^2-homology equivalence. A finitely generated subgroup H of a free group F is called *compressed* if for any subgroup L of F containing H we have $\operatorname{rank} H \leqslant \operatorname{rank} L$. The following characterization given by [JZ24] is notable since it does not involve any L^2-theories in its statement.

Theorem 6.4. *Let $\varphi\colon F_1 \to F_2$ be a homomorphism between finitely generated free groups of the same rank. Then φ is an L^2-homology equivalence if and only if φ is injective with compressed image $\operatorname{im}\varphi \subset F_2$.*

Proof. If φ is not injective then $\tau_u^{(2)}(\varphi) = 0$ by Proposition 6.2(2). Now we assume that φ is injective and aim to show that $\tau_u^{(2)}(\varphi) \neq 0$ if and only if $\operatorname{im}\varphi$ is compressed in F_2.

Let $\Phi\colon X_1 \to X_2$ be the topological realization of φ and let M_Φ be the mapping cylinder. Then $\tau_u^{(2)}(\varphi) \neq 0$ if and only if the pair (M_Φ, X_1) is L^2-acyclic. By the homology long exact sequence this is equivalent to

$$H_1(X_1; \mathcal{D}_{F_2}) \to H_1(M_\Phi; \mathcal{D}_{F_2})$$

being injective (note that $\operatorname{rank} H_1(X_1; \mathcal{D}_{F_2}) = \operatorname{rank} H_1(M_\Phi; \mathcal{D}_{F_2}) = \operatorname{rank} F_2 - 1$). By [JZ24, Corollary 1.2] this is equivalent to $\pi_1(X_1) \subset \pi_1(M_\varphi)$ being compressed, i.e. $\operatorname{im}\varphi \subset F_2$ is compressed. □

Conjecture 6.5. *A homomorphism φ between finitely generated free groups is an isomorphism if and only if $\tau_u^{(2)}(\varphi) = 1$.*

6.2. **3-dimensional handlebodies.** In the remaining part of this paper, a "handlebody" refers to a compact connected orientable 3-manifold obtained from attaching 1-handles to a 3-ball. The boundary of a handlebody is a connected closed surface. The homeomorphism type of a handlebody is determined by the genus of its boundary. A *genus-g handlebody H_g* refers to a handlebody whose boundary is a genus $g \geqslant 0$ surface. Note that H_g deformation retracts to the one-point union of g circles.

Fix $g \geqslant 1$. Consider the sutured manifold (H_g, R_+, R_-, γ) such that R_+ and R_- are both connected and $\chi(R_+) = \chi(R_-)$. It is clear that $\chi(R_+) = \chi(R_-) = 1 - g$, and the fundamental group of $R_\pm$ are both isomorphic to a free group of rank g.

Proposition 6.6. *Suppose (H_g, R_+, R_-, γ) is a sutured manifold such that R_+, R_- are both connected and $\chi(R_+) = \chi(R_-)$ (we do not assume that $R_\pm$ are incompressible). Then:*

1. *$\tau_u^{(2)}(H_g, R_+) = \tau_u^{(2)}(\varphi)$ where $\varphi\colon \pi_1(R_+) \to \pi_1(H_g)$ is the inclusion-induced homomorphism.*
2. *(H_g, R_+, R_-, γ) is a taut sutured manifold if and only if $\tau_u^{(2)}(H_g, R_+) \neq 0$.*
3. *(H_g, R_+, R_-, γ) is a product sutured manifold if and only if $\tau_u^{(2)}(H_g, R_+) = 1$.*

Proof. For (1), Let $\Phi\colon R_+ \to H_g$ be the inclusion map. Note that R_+ and H_g are classifying spaces for $\pi_1(R_+)$ and $\pi_1(H_g)$, respectively. Therefore

$$\tau_u^{(2)}(H_g, R_+) = \tau_u^{(2)}(\Phi) = \tau_u^{(2)}(\varphi).$$

For (2), the forward direction follows from Theorem 5.6. Now suppose $\tau_u^{(2)}(H_g, R_+)$ is nonzero. By Proposition 5.7 we know that $\tau_u^{(2)}(H_g, R_-)$ is also nonzero. Then it follows from (1) and Proposition 6.2(2) that $R_\pm \subset H_g$ are both incompressible surfaces. Applying Theorem 5.6 again implies that (H_g, γ) is taut.

Finally, (3) is a direct corollary of (2) and Theorem 1.4. □

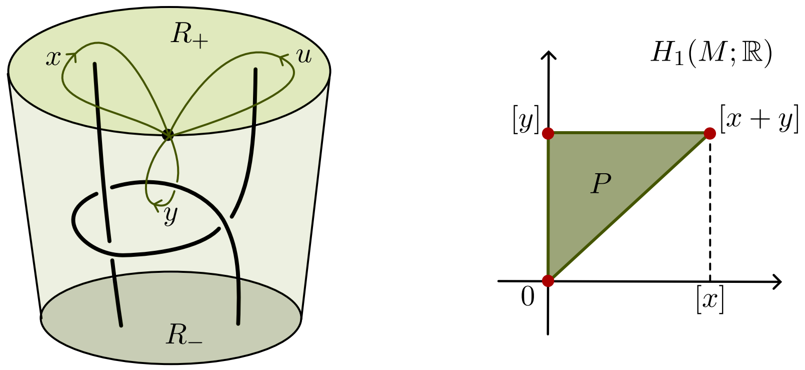

Figure 6. *A sutured manifold M and a representative P of its L^2-polytope in* $H_1(M;\mathbb{R})$

Example 6.7. Consider a sutured manifold M as depicted in the left of Figure 6. It is a 3-ball with two arcs removed, and three sutures separate ∂M into two pairs of pants R_+ and R_-. The manifold M is homeomorphic to a genus-2 handlebody whose fundamental group is generated by the loops x and y. The fundamental group of R_+ is generated by the loops x and u where $u = yxyx^{-1}y^{-1}$ in $\pi_1(M)$. Let $\varphi\colon \pi_1(R_+) \to \pi_1(M)$ be the homomorphism induced by inclusion, then under the basis $\pi_1(R_+) = \langle x, u\rangle$, $\pi_1(M) = \langle x, y\rangle$, we have

$$J_\varphi = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y - yxyx^{-1} & 1 + yx - u \end{pmatrix}.$$

Therefore by Proposition 6.6,

$$\tau_u^{(2)}(M, R_+) = \det_w(J_\varphi) = [1 + yx - u].$$

The polytope map $\mathbb{P}\colon \operatorname{Wh}(\mathcal{D}_{\pi_1(M)}) \to \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(M;\mathbb{Z}))$ sends $\tau_u^{(2)}(M, R_+)$ to a polytope $[P]$, where P is the convex hull of $\{0, [u] = [y], [x+y]\}$. Proposition 6.6 implies that M is a taut sutured manifold but not a product sutured manifold.

Example 6.8 (n-chain link)**.** For each $n \geqslant 3$, the n-chain link L_n is an alternating link obtained by linking n unknots in a cyclic way. A diagram for the case $n = 4$ is shown in Figure 7(a). Consider the natural Seifert surface Σ as illustrated in Figure 7(b), where the positive-side of Σ is in light green and the negative-side of Σ is in dark green. The surface Σ is obtained from two disks by attaching n twisted-bands, and it deformation retracts to a wedge of $(n-1)$ circles.

The complement $S^3\backslash\backslash\Sigma$ is a handle body of genus $(n-1)$, whose boundary is a union of two copies of Σ, namely Σ_+ and Σ_-. Choose a free basis for the fundamental group $\pi_1(S^3\backslash\backslash\Sigma) = \langle x_1, \ldots, x_{n-1}\rangle$ where each x_i is represented by a red loop in Figure 7(b). A free basis for $\pi_1(\Sigma_+)$ is represented by the blue loops $u_1, \ldots, u_{n-1}$ in Figure 7(c). By pushing u_i slightly into the positive direction, we obtain its image u_i^+ under the inclusion $\Sigma_+ \hookrightarrow S^3\backslash\backslash\Sigma$, which is represented by $x_i x_{i+1}^{-1}$ (we assume that $x_n := x_1^{-1}\cdots x_{n-1}^{-1}$, see Figure 7(b)). Let $\varphi\colon \pi_1(\Sigma_+) \to \pi_1(S^3\backslash\backslash\Sigma)$ be the homomorphism induced by inclusion, then

$$\operatorname{im}\varphi = \langle x_1x_2^{-1}, x_2x_3^{-1}, \ldots, x_{n-2}x_{n-1}^{-1}, x_{n-1}^2 x_{n-2}\cdots x_2x_1\rangle \subset \langle x_1, \ldots, x_{n-1}\rangle$$

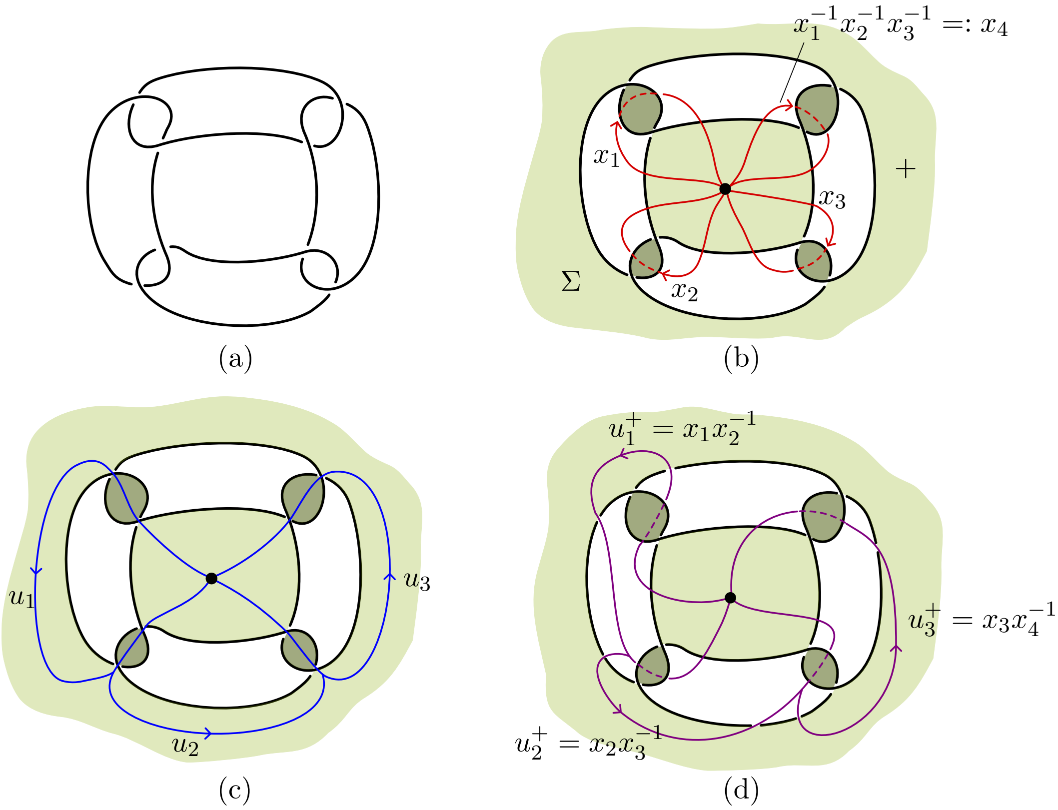

Figure 7. *The n-chain link example, where $n = 4$*

and the Fox Jacobian of φ is

$$J_\varphi = \begin{pmatrix} 1 & -x_1x_2^{-1} & & & \\ & 1 & -x_2x_3^{-1} & & \\ & & \ddots & & \\ & & & 1 & -x_{n-2}x_{n-1}^{-1} \\ x_{n-1}^2x_{n-2}\cdots x_2 & x_{n-1}^2x_{n-2}\cdots x_3 & \cdots & x_{n-1}^2 & 1+x_{n-1} \end{pmatrix}.$$

Lemma 6.9. *For $n \geqslant 3$, let*

$$B_n = \begin{pmatrix} 1 & -s_1 & & & \\ & 1 & -s_2 & & \\ & & \ddots & & \\ & & & 1 & -s_{n-2} \\ f_1 & f_2 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix}$$

be a matrix over a skew field. Then its Dieudonné determinant is given by

$$\det(B_n) = [f_1s_1s_2\cdots s_{n-2} + f_2s_2s_3\cdots s_{n-2} + \cdots f_{n-2}s_{n-2} + f_{n-1}].$$

Proof. For the base case $n = 3$, we have

$$\det(B_3) = \det\begin{pmatrix} 1 & -s_1 \\ f_1 & f_2 \end{pmatrix} = [f_1s_1 + f_2].$$

For general n, we left-multiply the first row by $(-f_1)$ and add it to the last row to eliminate the bottom-left entry. Thus

$$\det(B_n) = \det \begin{pmatrix} 1 & -s_2 & & \\ & \ddots & & \\ & & 1 & -s_{n-2} \\ f_1 s_1 + f_2 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix}.$$

The result then follows by induction on n. □

Now apply Lemma 6.9 to J_φ, note that $s_i s_{i+1} \cdots s_{n-2} = x_i x_{n-1}^{-1}$ for $i \leqslant n-2$. It follows that

$$f_i s_i s_{i+1} \cdots s_{n-2} = x_{n-1}^2 x_{n-2} \cdots x_i x_{n-1}^{-1}, \quad 1 \leqslant i < n-1$$

and $f_{n-1} = 1 + x_{n-1}$. Define $y_i := x_{n-1} x_{n-2} \cdots x_i$, $i = 1, \ldots, n-1$. Then $\{y_1, \ldots, y_{n-1}\}$ forms another free basis of $\pi_1(S^3 \backslash\backslash \Sigma)$. By Proposition 6.6,

$$\begin{aligned} \tau_u^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+) &= \det_w(J_\varphi) \\ &= [x_{n-1} \cdot (y_1 + y_2 + \cdots + y_{n-1} + 1) \cdot x_{n-1}^{-1}] \\ &= [y_1 + y_2 + \cdots + y_{n-1} + 1]. \end{aligned}$$

The polytope map $\mathbb{P}$ sends the universal L^2-torsion $\tau_u^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+)$ to an $(n-1)$-simplex in $H_1(S^3 \backslash\backslash \Sigma; \mathbb{R})$ spanned by vertices $\{0, [y_1], \ldots, [y_{n-1}]\}$. By Proposition 6.6 $S^3 \backslash\backslash \Sigma$ is a taut sutured manifold and Σ is a norm-minimizing Seifert surface for L. Moreover, $\tau_u^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+) \neq 1$ and L is not fibered as an oriented link.

Remark 6.10. Suppose G is group satisfying the Determinant Conjecture (see [Lüc02, Section 13] for a details), then the Fuglede–Kadison determinant defines a homomorphism $\det_{\mathcal{N}G} \colon \mathrm{Wh}(\mathcal{D}_G) \to \mathbb{R}_+$. For any L^2-acyclic finite CW-complex X with $\pi_1(X) = G$, applying $\det_{\mathcal{N}G}$ to the universal L^2-torsion $\tau_u^{(2)}(X)$ yields the L^2-torsion $\tau^{(2)}(X)$. Let G be the fundamental group $\pi_1(S^3 \backslash\backslash \Sigma)$. According to [BA22],

$$\det_{\mathcal{N}G}([1 + y_1 + \cdots + y_{n-1}]) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}}.$$

Therefore we obtain the L^2-torsion $\tau^{(2)}(S^3 \backslash\backslash \Sigma, \Sigma_+) = (n-1)^{\frac{n-1}{2}} / n^{\frac{n-2}{2}}$.

For an admissible 3-manifold N and a cohomology class $\phi \in H^1(N; \mathbb{Z})$, the L^2-*Alexander torsion* is a function $\tau^{(2)}(N, \phi) \colon \mathbb{R}_+ \to [0, +\infty)$ [DFL16]. This function has a well-defined "degree" which equals to the Thurston norm of ϕ [FL19, Liu17], and a "leading coefficient" $C(N, \phi) \geqslant 1$ [Liu17]. It is proved in [Dua25] that $C(N, \phi)$ equals the L^2-torsion of the pair $(N \backslash\backslash \Sigma, \Sigma_+)$ where Σ is a norm-minimizing surface dual to ϕ. Let X_n be the n-chain link complement and let $\phi \in H^1(X_n; \mathbb{Z})$ be the Poincaré dual of Σ. Then

$$C(X_n, \phi) = \tau^{(2)}(X_n, \Sigma_+) = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}} \sim \sqrt{n/e}, \quad \text{as } n \to +\infty.$$

Hence, the n-chain link complements X_n form an infinite family of hyperbolic manifolds for which $C(X_n, \phi) > 1$ for some nonzero class $\phi \in H^1(X_n; \mathbb{Z}) \setminus \{0\}$, answering a question raised in [BAFH22, Conjecture 1.7].

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