Homework 2 - Group 076

Aprendizagem 2021/2022

1 Pen and Paper

1) Applying the linear basis function $\phi(\mathbf{x}) = (1, \|\mathbf{x}\|_2, \|\mathbf{x}\|_2^2, \|\mathbf{x}\|_2^3)$ (with $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$) to each instance $\mathbf{x}^{(i)}$ $(i = 1, ..., 8)$ in the training set, we get a new design matrix:

$$\mathbf{\Phi} = \begin{bmatrix} — & (\phi(\mathbf{x}^{(1)}))^T & — \\ — & (\phi(\mathbf{x}^{(2)}))^T & — \\ & \vdots & \\ — & (\phi(\mathbf{x}^{(8)}))^T & — \end{bmatrix} = \begin{bmatrix} 1.0 & 1.4142 & 2.0 & 2.8284 \\ 1.0 & 5.1962 & 27.0 & 140.2961 \\ 1.0 & 4.4721 & 20.0 & 89.4427 \\ 1.0 & 3.7417 & 14.0 & 52.3832 \\ 1.0 & 7.2801 & 53.0 & 385.8458 \\ 1.0 & 1.7321 & 3.0 & 5.1962 \\ 1.0 & 2.8284 & 8.0 & 22.6274 \\ 1.0 & 9.2195 & 85.0 & 783.6613 \end{bmatrix}$$

To learn the regression model, we must compute the weight vector $\mathbf{w}$ that minimizes the Sum of Squares error between the outputs $\mathbf{z} = \begin{bmatrix} 1 & 3 & 2 & 0 & 6 & 4 & 5 & 7 \end{bmatrix}^T$ and predictions $\hat{\mathbf{z}} = \mathbf{\Phi w}$ (i.e., $\mathbf{w} = (\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\mathbf{z}$):

$$\mathbf{\Phi}^T\mathbf{\Phi} = \begin{bmatrix} 8.0 & 35.8843 & 212.0 & 1482.2811 \\ 35.8843 & 212.0 & 1482.2811 & 11436.0 \\ 212.0 & 1482.2811 & 11436.0 & 93573.5164 \\ 1482.2811 & 11436.0 & 93573.5164 & 793976.0 \end{bmatrix}$$

$$(\mathbf{\Phi}^T\mathbf{\Phi})^{-1} = \begin{bmatrix} 8.1955 & -6.2313 & 1.3049 & -0.0793 \\ -6.2313 & 5.0781 & -1.1044 & 0.0686 \\ 1.3049 & -1.1044 & 0.2472 & -0.0157 \\ -0.0793 & 0.0686 & -0.0157 & 0.001 \end{bmatrix}$$

$$(\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T = \begin{bmatrix} 1.7686 & -0.0811 & -0.6694 & -1.0069 & 1.3794 & 0.9051 & -0.785 & -0.5107 \\ -1.0644 & -0.0319 & 0.5312 & 0.904 & -1.307 & -0.3922 & 0.8501 & 0.5101 \\ 0.1933 & 0.0436 & -0.0907 & -0.1868 & 0.3232 & 0.0524 & -0.1954 & -0.1395 \\ -0.0107 & -0.0043 & 0.0044 & 0.0109 & -0.0214 & -0.0022 & 0.0123 & 0.011 \end{bmatrix}$$

$$\mathbf{w} = (\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\mathbf{z} = \begin{bmatrix} 4.5835 & -1.6872 & 0.3377 & -0.0133 \end{bmatrix}^T$$

2) Similarly to the previous question, we compute the image of each instance $\mathbf{x}^{(i)}$ $(i \in \{1, 2\})$ from the testing set and place the image $\phi(\mathbf{x}^{(i)})$ in each row of the matrix $\mathbf{\Phi}$. Using the obtained weight vector $\mathbf{w}$, we have the following estimates vector:

$$\hat{\mathbf{z}} = \mathbf{\Phi w} = \begin{bmatrix} 1.0 & 2.0 & 4.0 & 8.0 \\ 1.0 & 2.4495 & 6.0 & 14.6969 \end{bmatrix} \begin{bmatrix} 4.5835 \\ -1.6872 \\ 0.3377 \\ -0.0133 \end{bmatrix} = \begin{bmatrix} 2.4536 \\ 2.2816 \end{bmatrix}$$

Computing the root mean square error, we have:

$$\text{RMSE}(\hat{\mathbf{z}}, \mathbf{z}) = \sqrt{\frac{1}{2}\sum_{i=1}^{2}(\hat{z}_i - z_i)^2} = 1.2567$$

3) To make an equal depth binarization of y_3, we compute the median of the values of y_3 of all the instances in the dataset:

$$\text{median}(\{x_3^{(i)}\}_{i=1}^{10}) = 2.5$$

and make the following transformation to y_3:

$$y_3 := \begin{cases} 0, & \text{if } y_3 < \text{median} \\ 1, & \text{otherwise} \end{cases}$$

Considering the given class targets, we have the following data table:

y_1	y_2	y_3	t
1	1	0	N
1	1	1	N
0	2	1	N
1	2	1	N
2	0	1	P
1	1	0	P
2	0	0	P
0	2	1	P

- Starting entropy:

$$H(t) = -\sum_{i=0}^{1} \frac{\#\{k|t^{(k)} = i\}}{8} \log_2\left(\frac{\#\{k|t^{(j)} = i\}}{8}\right) = -\left(\frac{1}{2}\log_2\left(\frac{1}{2}\right) + \frac{1}{2}\log_2\left(\frac{1}{2}\right)\right) = 1\ bit$$

- Variable conditional entropies and corresponding information gains ($\text{IG}(y_i) = H(t) - H(t|y_i)$):

$$\begin{aligned}
H(t|y_1) &= \sum_{i=0}^{2} \frac{\#\{k|y_1^{(k)} = i\}}{8} H(t|y_1 = i) \\
&= -\sum_{i=0}^{2} \frac{\#\{k|y_1^{(k)} = i\}}{8} \sum_{j=0}^{1} \frac{\#\{k|y_1^{(k)} = i, t^{(k)} = j\}}{\#\{k|t^{(k)} = j\}} \log_2\left(\frac{\#\{k|y_1^{(k)} = i, t^{(k)} = j\}}{\#\{k|t^{(k)} = j\}}\right) \\
&= -\frac{1}{4}\left(\frac{1}{2}\log_2\left(\frac{1}{2}\right) + \frac{1}{2}\log_2\left(\frac{1}{2}\right)\right) - \frac{1}{2}\left(\frac{3}{4}\log_2\left(\frac{3}{4}\right) + \frac{1}{4}\log_2\left(\frac{1}{4}\right)\right) - \frac{1}{2}\left(0\log_2(0) + 1\log_2(1)\right) \\
&= 0.6556\ bit \\
H(t|y_2) &= -\frac{1}{4}\left(0\log_2(0) + 1\log_2(1)\right) - \frac{3}{8}\left(\frac{2}{3}\log_2\left(\frac{2}{3}\right) + \frac{1}{3}\log_2\left(\frac{1}{3}\right)\right) - \frac{3}{8}\left(\frac{2}{3}\log_2\left(\frac{2}{3}\right) + \frac{1}{3}\log_2\left(\frac{1}{3}\right)\right) \\
&= 0.68872\ bit \\
H(t|y_3) &= -\frac{3}{8}\left(\frac{1}{3}\log_2\left(\frac{1}{3}\right) + \frac{2}{3}\log_2\left(\frac{2}{3}\right)\right) - \frac{5}{8}\left(\frac{3}{5}\log_2\left(\frac{3}{5}\right) + \frac{2}{5}\log_2\left(\frac{2}{5}\right)\right) = 0.95121\ bit
\end{aligned}$$

$$\text{IG}(y_1) = 1 - 0.6556 = 0.34436\ bit \quad \text{IG}(y_2) = 0.31128\ bit \quad \text{IG}(y_3) = 0.04879\ bit$$

Since y_1 yields the biggest information gain, it will take part in the first node in the decision tree and the following division in the dataset is obtained:

$y_1 = 0$			$y_1 = 1$			$y_1 = 2$		
y_2	y_3	t	y_2	y_3	t	y_2	y_3	t
2	1	N	1	0	N	0	1	P
2	1	P	1	1	N	0	0	P
			2	1	N			
			1	0	P			

Partition ($y_1 = 2$) has no uncertainty and we cannot reduce uncertainty in partition ($y_1 = 0$) (note that there is only one value of y_2 (resp. y_3), so $H(t|y_2, y_1 = 1) = H(t|y_1 = 1)$ (resp., $H(t|y_3, y_1 = 1)$), so there is no possible information gain). We thus proceed to divide partition ($y_1 = 1$) even further in an analogous way as before:

- Starting entropy:

$$\begin{aligned} H(t|y_1 = 1) &= -\sum_{i=0}^{1} \frac{\#\{k|t^{(k)} = i \wedge y_1^{(k)} = 1\}}{\#\{k|y_1^{(k)} = 1\}} \log_2 \left(\frac{\#\{k|t^{(k)} = i \wedge y_1^{k} = 1\}}{\#\{k|y_1^{(k)} = 1\}} \right) \\ &= -\left(\frac{3}{5} \log_2 \left(\frac{3}{4} \right) + \frac{1}{4} \log_2 \left(\frac{1}{4} \right) \right) = 0.688721\, bit \end{aligned}$$

- Variable conditional entropies and corresponding information gains (where $\mathrm{IG}(y_i|y_1 = 1) = H(t|y_1 = 1) - H(t|y_i, y_1 = 1)$):

$$H(t|y_2, y_1 = 1) = \sum_{i=0}^{2} \frac{\#\{k|y_2^{(k)} = i \wedge y_1^{(k)} = 1\}}{\#\{k|y_1^{(k)} = 1)\}} H(t|y_2 = i, y_1 = 1)$$

$$= -\sum_{i=0}^{2} \frac{\#\{k|y_2^{(k)} = i \wedge y_1^{(k)} = 1)\}}{\#\{k|y_1^{(k)} = 1\}} \sum_{j=0}^{1} \frac{\#\{k|t^{(k)} = j \wedge y_2^{(k)} = i \wedge y_1^{(k)} = 1)\}}{\#\{k|y_2^{(k)} = i \wedge y_1^{(k)} = 1\}} \log_2 \left(\frac{\#\{k|y_2^{(k)} = i \wedge y_1^{(k)} = 1)\}}{\#\{k|y_1^{(k)} = 1\}} \right)$$

$$= -\frac{3}{4} \left(\frac{2}{3} \log_2 \left(\frac{2}{3} \right) + \frac{1}{3} \log_2 \left(\frac{1}{3} \right) \right) - \frac{1}{4} \left(1 \log_2(1) + 0 \log_2(0) \right) = 0.688721\, bit$$

$$H(t|y_3, y_1 = 1) = -\frac{1}{2} \left(\frac{1}{2} \log_2 \left(\frac{1}{2} \right) + \frac{1}{2} \log_2 \left(\frac{1}{2} \right) \right) = 0.5\, bit$$

$$\mathrm{IG}(y_2|y_1 = 1) = 0.81127 - 0.688721 = 0.1225979\, bit \quad \mathrm{IG}(y_3|y_1 = 1) = 0.81127 - 0.5 = 0.31127\, bit$$

As y_3 provides the biggest information gain, it will originate a new partition in the partition $y_1 = 1$. The following divided dataset cannot be further partitioned (($y_1 = 1, y_3 = 1$) has no uncertainty and we can't obtain information gain from dividing ($y_1 = 1, y_3 = 0$) for the same argument as the applied above to ($y_1 = 0$)):

$y_1 = 0$			$y_1 = 1$				$y_1 = 2$		
y_2	y_3	t	$y_3 = 0$		$y_3 = 1$		y_2	y_3	t
2	1	N	y_2	t	y_2	t	0	1	P
2	1	P	1	N	1	N	0	0	P
			1	P	2	N			

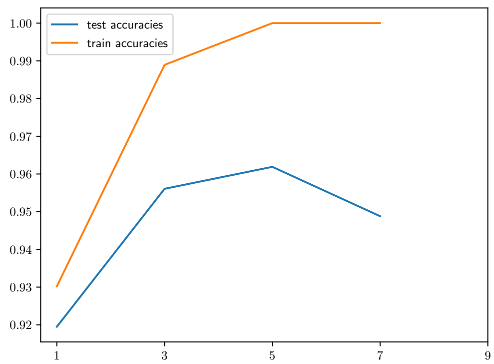
And thus we have the desired decision tree:

$$\begin{array}{ccccc} y_1 = 0 & & y_1 = 1 & & y_1 = 2 \\ \swarrow & & \downarrow & & \searrow \\ \text{N/P} & y_3 = 0 & & y_3 = 1 & \text{P} \\ & \downarrow & & \downarrow & \\ & \text{N/P} & & \text{N} & \end{array}$$

4) Considering the computed decision tree, for $\mathbf{x}^{(9)} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^T$ we have that $\hat{\mathbf{t}}(\mathbf{x}^{(9)}) = \text{P}$, and, for $\mathbf{x}^{(10)} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$, we have that $\hat{\mathbf{t}}(\mathbf{x}^{(10)}) = \text{N}$. Since $\mathbf{t}(\mathbf{x}^{(9)}) = \text{N}$ and $\mathbf{t}(\mathbf{x}^{(10)}) = \text{P}$ there are no true positives and no true negatives in the classification of these instances. Thus, accuracy on the testing data is 0 according to the formula Accuracy $= \frac{TP+TN}{P+N}$.

2 Programming and critical analysis

5) (i)



(ii)