# Homework 1 - Group 03

Planning, Learning and Intelligent Decision Making

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## Exercise 1.

1. The motion of the truck can be described by a Markov Chain $\mathcal{M} = (\mathcal{X}, \boldsymbol{P})$, where:

$$\mathcal{X} = \{RP,\ A,\ B,\ C,\ D,\ E,\ F\}$$

$$\boldsymbol{P} = \begin{array}{c} \\ RP \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{ccccccc} RP & A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

2. By the Chapman-Kolmogorov equations, we have, for $x \in \mathcal{X}$:

$$\begin{aligned} \mathbb{P}(\mathrm{x}_2 = x \mid \mathrm{x}_0 = RP) &= \sum_{y \in \mathcal{X}} \mathbb{P}(\mathrm{x}_1 = y \mid \mathrm{x}_0 = RP)\, \mathbb{P}(\mathrm{x}_2 = x \mid \mathrm{x}_1 = y) \\ &= \mathbb{P}(\mathrm{x}_2 = x \mid \mathrm{x}_1 = A) = \boldsymbol{P}(x \mid A) \end{aligned}$$

with the last equality coming from the fact that $A$ is the only possible state for time step 1. Thus, we have that the required probabilities are:

$$[\boldsymbol{P}][0,:] = \begin{array}{c} \begin{array}{ccccccc} RP & A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0.25 & 0 & 0.25 & 0.25 & 0.25 & 0 & 0 \end{bmatrix} \end{array}$$

3. Let $E = \{(x, y) \in \mathcal{X} \times \mathcal{X} :\ \boldsymbol{P}(y \mid x) > 0\}$ denote the set of states that correspond to sites with roads between them. Let $T_{xy}$ be a random variable that expresses the time spent travelling from site $x$ to site $y$ before returning to the recycling plant . We are thus asked to find:

$$\mathbb{E}\left[\sum_{(x,y)\in E} T_{xy}\right] = \mathbb{E}\left[\sum_{(x,y)\in E} t_{xy} N_{xy \mid \mathrm{RP}}\right] = \sum_{(x,y)\in E} t_{xy} \underbrace{\mathbb{E}\left[N_{xy \mid \mathrm{RP}}\right]}_{\eta_{xy \mid \mathrm{RP}}} \tag{1}$$

where $t_{xy}$ denotes the time needed to travel from site $x$ to site $y$ and $N_{xy \mid x^*}$ is a random variable that expresses the number of times the truck travels from site $x$ to site $y$ before returning to the recycling plant. Let also $T_{x^*}$ denote the first return time to $x^*$. We then have:

$$
\begin{aligned}
\eta_{xy \mid x^*} &= \mathbb{E}\left[\sum_{t=1}^{\infty} \mathbb{I}[\mathrm{x}_t = x,\ \mathrm{x}_{t+1} = y,\ T_{x^*} > t \mid \mathrm{x}_0 = x^*]\right] \\
&= \sum_{t=1}^{\infty} \mathbb{P}[\mathrm{x}_t = x,\ \mathrm{x}_{t+1} = y,\ T_{x^*} > t \mid \mathrm{x}_0 = x^*] \\
&= \sum_{t=1}^{\infty} \mathbb{P}[\mathrm{x}_{t+1} = y,\ T_{x^*} > t \mid \mathrm{x}_t = x,\ \mathrm{x}_0 = x^*]\, \mathbb{P}[\mathrm{x}_t = x \mid \mathrm{x}_0 = x^*] \\
&= \sum_{t=1}^{\infty} \mathbb{P}[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x,\ \mathrm{x}_0 = x^*]\, \mathbb{P}[T_{x^*} > t \mid \mathrm{x}_t = x,\ \mathrm{x}_0 = x^*]\, \mathbb{P}[\mathrm{x}_t = x \mid \mathrm{x}_0 = x^*] \qquad (2) \\
&= \sum_{t=1}^{\infty} \mathbb{P}[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x]\, \mathbb{P}[\mathrm{x}_t = x,\ T_{x^*} > t \mid \mathrm{x}_0 = x^*] \qquad (3) \\
&= \sum_{t=1}^{\infty} \boldsymbol{P}(y \mid x)\, \mathbb{P}[\mathrm{x}_t = x,\ T_{x^*} > t \mid \mathrm{x}_0 = x^*] \\
&= \boldsymbol{P}(y \mid x)\ \sum_{t=1}^{\infty} \mathbb{P}[\mathrm{x}_t = x,\ T_{x^*} > t \mid \mathrm{x}_0 = x^*] \\
&= \boldsymbol{P}(y \mid x)\ \mathbb{E}\left[\sum_{t=1}^{\infty} \mathbb{I}[\mathrm{x}_t = x,\ T_{x^*} > t \mid \mathrm{x}_0 = x^*]\right] \\
&= \boldsymbol{P}(y \mid x)\ \eta_{x \mid x^*}
\end{aligned}
$$

where $\eta_{x \mid x^*}$ denotes the expected number of visits to state $x$ between two visits to state $x^*$. Furthermore:

- In (2), we used the fact that the event $\{T_{x^*} > t\}$ depends only on $\{\mathrm{x}_i,\ 1 \leq i \leq t\}$, and thus it is independent of $\mathrm{x}_{t+1}$;
- In (3) we used the Markovian property.

In order to perform the desired computation, we must try to find out the stationary distribution $\boldsymbol{\mu^*}$ of this chain:

$$
\boldsymbol{\mu^*} = \boldsymbol{\mu^*}\boldsymbol{P} \Leftrightarrow \boldsymbol{\mu^*} = \begin{bmatrix} 0.0104 & 0.08333 & 0.0408 & 0.0729 & 0.0408 & 0.0747 & 0.0781 \end{bmatrix}
$$

Now, on one hand, this chain is **irreducible** (since all states are accessible from all states - this stems from the fact the roads are bidirectional). On another hand, this is chain is also **aperiodic**. This claim can be proven by successively computing powers of $\boldsymbol{P}$:

$$
\boldsymbol{P}^2 = \begin{bmatrix}
0.25 & 0.0 & 0.25 & 0.25 & 0.25 & 0.0 & 0.0 \\
0.0 & 0.625 & 0.0 & 0.0 & 0.0 & 0.125 & 0.25 \\
0.125 & 0.0 & 0.2917 & 0.125 & 0.2917 & 0.1667 & 0.0 \\
0.125 & 0.0 & 0.125 & 0.375 & 0.125 & 0.0 & 0.25 \\
0.125 & 0.0 & 0.2917 & 0.125 & 0.2917 & 0.1667 & 0.0 \\
0.0 & 0.25 & 0.1667 & 0.0 & 0.1667 & 0.4167 & 0.0 \\
0.0 & 0.3333 & 0.0 & 0.1667 & 0.0 & 0.0 & 0.5
\end{bmatrix}
$$

$$
\boldsymbol{P}^3 = \begin{bmatrix}
0.0 & 0.625 & 0.0 & 0.0 & 0.0 & 0.125 & 0.25 \\
0.1562 & 0.0 & 0.2396 & 0.2188 & 0.2396 & 0.0833 & 0.0625 \\
0.0 & 0.4792 & 0.0 & 0.0833 & 0.0 & 0.0625 & 0.375 \\
0.0 & 0.4375 & 0.0833 & 0.0 & 0.0833 & 0.2708 & 0.125 \\
0.0 & 0.4792 & 0.0 & 0.0833 & 0.0 & 0.0625 & 0.375 \\
0.0625 & 0.1667 & 0.0625 & 0.2708 & 0.0625 & 0.0 & 0.375 \\
0.0833 & 0.0833 & 0.25 & 0.0833 & 0.25 & 0.25 & 0.0
\end{bmatrix}
$$

$$
\boldsymbol{P}^4 = \begin{bmatrix}
0.1562 & 0.0 & 0.2396 & 0.2188 & 0.2396 & 0.0833 & 0.0625 \\
0.0 & 0.5052 & 0.0208 & 0.0417 & 0.0208 & 0.1302 & 0.2812 \\
0.1198 & 0.0417 & 0.2448 & 0.151 & 0.2448 & 0.1667 & 0.0312 \\
0.1094 & 0.0833 & 0.151 & 0.2448 & 0.151 & 0.0417 & 0.2188 \\
0.1198 & 0.0417 & 0.2448 & 0.151 & 0.2448 & 0.1667 & 0.0312 \\
0.0417 & 0.2604 & 0.1667 & 0.0417 & 0.1667 & 0.2604 & 0.0625 \\
0.0208 & 0.375 & 0.0208 & 0.1458 & 0.0208 & 0.0417 & 0.375
\end{bmatrix}
$$

$$
\boldsymbol{P}^5 = \begin{bmatrix}
0.0 & 0.5052 & 0.0208 & 0.0417 & 0.0208 & 0.1302 & 0.2812 \\
0.1263 & 0.0417 & 0.2201 & 0.1914 & 0.2201 & 0.1146 & 0.0859 \\
0.0104 & 0.4401 & 0.0208 & 0.0937 & 0.0208 & 0.0859 & 0.3281 \\
0.0208 & 0.3828 & 0.0938 & 0.0417 & 0.0938 & 0.1953 & 0.1719 \\
0.0104 & 0.4401 & 0.0208 & 0.0937 & 0.0208 & 0.0859 & 0.3281 \\
0.0651 & 0.2292 & 0.0859 & 0.1953 & 0.0859 & 0.0417 & 0.2969 \\
0.0937 & 0.1146 & 0.2187 & 0.1146 & 0.2187 & 0.1979 & 0.0417
\end{bmatrix}
$$

For example, we can see that $\boldsymbol{P}^2(A \mid A) = 0.625$ and $\boldsymbol{P}^5(A \mid A) = 0.0417$ and thus they are both different from 0, leading us to conclude that $\gcd\{\boldsymbol{P}^t(A \mid A)\,, t > 0\} = 1$ and, consequently, that the chain is aperiodic. Given the existence of a stationary distribution, this chain is also **positive**.
We are now equipped with the tools to perform our computation: given that the chain is **irreducible, aperiodic and positive**, the *Markov chain ergodic theorem* ensures us that the found stationary distribution is also **unique**. Since such stationary distribution $\boldsymbol{\mu^*}$ can be written as:

$$
\mu^*(x) = \frac{\eta_{x|x^*}}{\mathbb{E}[T_{x^*} \mid \mathrm{x}_0 = x^*]}
$$

Making the substitution $x = x^*$ in the expression above and taking into account that $\eta_{x^*|x^*} = 1$, we have that:

$$
\mathbb{E}[T_{x^*} \mid \mathrm{x}_0 = x^*] = \frac{1}{\mu^*(x^*)}
$$

and so we have that

$$
\eta_{x|x^*} = \mu^*(x)\,\mathbb{E}[T_{x^*} \mid \mathrm{x}_0 = x^*] = \frac{\mu^*(x)}{\mu^*(x^*)}
$$

Incorporating all this in equation (1), we have:

$$
\mathbb{E}\left[\sum_{(x,y)\in E} T_{xy}\right] = \sum_{(x,y)\in E} t_{xy}\, \boldsymbol{P}(y \mid x)\, \frac{\mu^*(x)}{\mu^*(x^*)}
$$

Hence, if the truck leaves the recycling plant at 10am on Monday, it is expected to return to the plant after:

$$
\begin{aligned}
&t_{RPA}\, \boldsymbol{P}(A \mid RP)\, \frac{\mu^*(RP)}{\mu^*(RP)} + t_{ARP}\, \boldsymbol{P}(RP \mid A)\, \frac{\mu^*(A)}{\mu^*(RP)} \\
&t_{AB}\, \boldsymbol{P}(B \mid A)\, \frac{\mu^*(A)}{\mu^*(RP)} + t_{BA}\, \boldsymbol{P}(A \mid B)\, \frac{\mu^*(B)}{\mu^*(RP)} \\
&t_{AC}\, \boldsymbol{P}(C \mid A)\, \frac{\mu^*(A)}{\mu^*(RP)} + t_{CA}\, \boldsymbol{P}(A \mid C)\, \frac{\mu^*(C)}{\mu^*(RP)} \\
&t_{AD}\, \boldsymbol{P}(D \mid A)\, \frac{\mu^*(A)}{\mu^*(RP)} + t_{DA}\, \boldsymbol{P}(A \mid D)\, \frac{\mu^*(D)}{\mu^*(RP)} \\
&t_{CE}\, \boldsymbol{P}(E \mid C)\, \frac{\mu^*(C)}{\mu^*(RP)} + t_{EC}\, \boldsymbol{P}(C \mid E)\, \frac{\mu^*(E)}{\mu^*(RP)} \\
&t_{BF}\, \boldsymbol{P}(F \mid B)\, \frac{\mu^*(B)}{\mu^*(RP)} + t_{FB}\, \boldsymbol{P}(B \mid F)\, \frac{\mu^*(F)}{\mu^*(RP)} \\
&t_{DF}\, \boldsymbol{P}(F \mid D)\, \frac{\mu^*(D)}{\mu^*(RP)} + t_{FD}\, \boldsymbol{P}(D \mid F)\, \frac{\mu^*(F)}{\mu^*(RP)} \\
&t_{EF}\, \boldsymbol{P}(F \mid E)\, \frac{\mu^*(E)}{\mu^*(RP)} + t_{FE}\, \boldsymbol{P}(E \mid F)\, \frac{\mu^*(F)}{\mu^*(RP)}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{0.0104}\Bigg( & 30\left(1 \times 0.0104 + \frac{1}{4} \times 0.08333\right) + 40\left(\frac{1}{4} \times 0.08333 + \frac{1}{2} \times 0.0408\right) \\
& 55\left(\frac{1}{4} \times 0.08333 + \frac{1}{2} \times 0.0729\right) + 70\left(\frac{1}{4} \times 0.08333 + \frac{1}{2} \times 0.0408\right) \\
& 55\left(\frac{1}{2} \times 0.0729 + \frac{1}{2} \times 0.0747\right) + 80\left(\frac{1}{2} \times 0.0408 + \frac{1}{3} \times 0.0781\right) \\
& 70\left(\frac{1}{2} \times 0.0408 + \frac{1}{3} \times 0.0781\right) + 20\left(\frac{1}{2} \times 0.0747 + \frac{1}{3} \times 0.0781\right)\Bigg) \\
& = 280 \text{ minutes} = \\
& = 4 \text{ hours } 40 \text{ minutes}
\end{aligned}
$$

Hence, it is expected for the truck to return to the plant at **2:40 pm on Monday**.