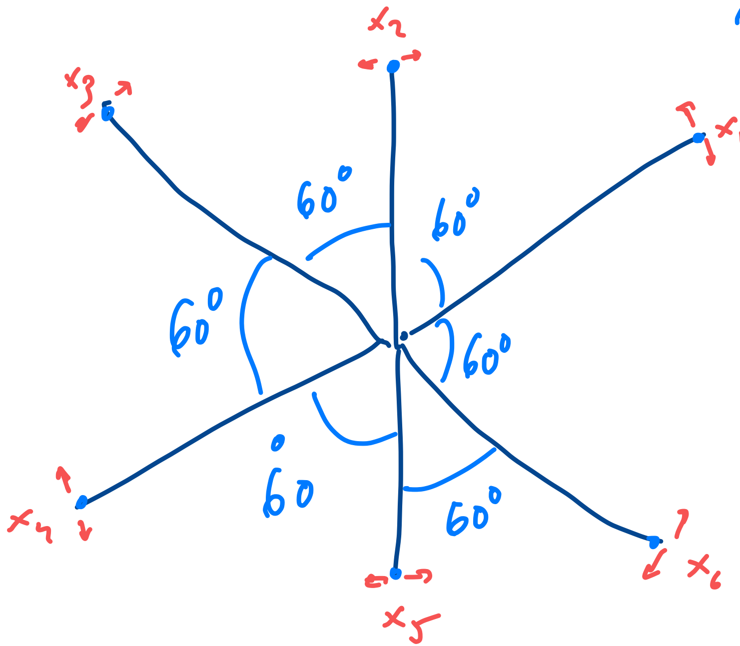
De volta ao exemplo da aula passada



$m \equiv m_1 = m_2 = m_3 = m_4 = m_5 = m_6$

acoplamentos entre osciladores só dependem da distância relativa

Εξισώσεις κιν. του μορίου

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$$\frac{d^2 X}{dt^2} = -\Pi^{-1} K X$$

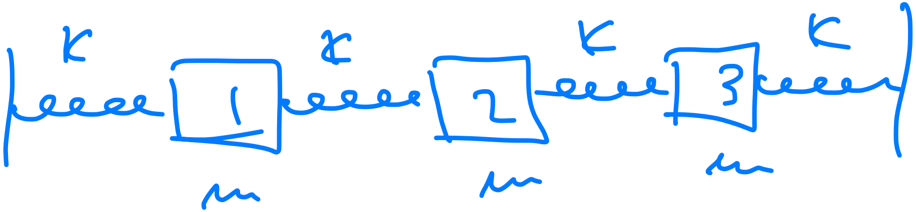
$$\Pi = m \mathbb{1} \rightarrow \Pi^{-1} = \frac{1}{m} \mathbb{1}$$

$$K^T = K$$

$$K = \begin{pmatrix} E & -B & -C & -D & -C & -B \\ -B & E & -B & -C & -D & -C \\ & & E & -B & -C & -D \\ & & & E & -B & -C \\ & & & & E & -B \\ & & & & & E \end{pmatrix}$$

problème masses simples

$$-M^{-1}KX$$



$$x_2$$

$$\vec{f}_{12} = +k x_2$$

$$K = \begin{pmatrix} 2k & -k & 0 \\ & 2k & -k \\ & & 2k \end{pmatrix}$$

utiliza simetria para simplificar o problema:

mud. cíclica de coord. $\equiv$ rotação de 60°

$$x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow x_4 \longrightarrow x_5 \longrightarrow x_6 \longrightarrow x_1$$

$$x' = S x$$

$$S = \begin{array}{c} \begin{array}{cccccc} & x_1 \to x_2 & & & & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

S é muito mais simples que $\Pi^{-1}K$

logo,

$$SA = \beta A \qquad S^6 = \mathbb{1}$$

rotazione per $60°$

$$S^6 A = A$$

$$\hookrightarrow S^5(SA) = S^5(\beta A) = \beta S^4(\underbrace{SA}_{A}) = \beta^6 A$$

$$A = \beta^6 A \implies \beta^6 = 1$$

$$\implies \beta = \sqrt[6]{1} \implies \beta = \beta_k = e^{i 2\pi k / 6}$$

$$k = 0, 1, 2, 3, 4, 5$$

para k tenho um modo normal!

$(k=0,1,2,3,4,5)$

$$S A^{k} = \beta_k A^{k}$$

autovalor ↓

$$\begin{pmatrix} A_2^k \\ A_3^k \\ A_4^k \\ A_5^k \\ A_6^k \\ A_1^k \end{pmatrix} = \beta_k \begin{pmatrix} A_1^k \\ A_2^k \\ A_3^k \\ A_4^k \\ A_5^k \\ A_6^k \end{pmatrix}$$

cada modo normal tem uma comp. arbitrária

↓ escolher

$$A_1^k = 1$$

$$A_2^k = \beta_k A_1^k = \beta_k$$

$$A_3^k = \beta_k A_2^k = \beta_k^2$$

$$\vdots$$

$$\boxed{A_j^k = (\beta_k)^{j-1}}$$

οι κόμβοι κινούνται πώς;

$$A^k = \begin{pmatrix} A_1^k \\ A_2^k \\ A_3^k \\ A_4^k \\ A_5^k \\ A_6^k \end{pmatrix} = \begin{pmatrix} 1 \\ e^{2ik\pi/6} \\ e^{4ik\pi/6} \\ e^{6ik\pi/6} \\ e^{8ik\pi/6} \\ e^{10ik\pi/6} \end{pmatrix}$$

as freq. próprias do sistema obtêm-se substituindo

$$M^{-1} K A^k = \omega_k^2 A^k$$

$$\Rightarrow \frac{1}{m}\Big[E - \underbrace{e^{2ik\pi/6}}_{e^{ik\pi/3}} B - e^{4ik\pi/6} C - e^{6ik\pi/6} D - e^{8ik\pi/6} C - \underbrace{e^{10ik\pi/6}}_{e^{5ik\pi/3} = e^{-ik\pi/3}} B\Big] = \omega_k^2$$

$\cos k\pi/3$

$$\Rightarrow \omega_k^2 = E/m - B/m \, 2\cos k\pi/3 - C/m \, 2\cos 2k\pi/3 - D/m \underbrace{(-1)^k}_{e^{ik\pi}}$$

$$\omega_k^2 = E/m - B/m \, 2\cos k\pi/3 - C/m \, 2\cos 2k\pi/3 - D/m \, (-1)^k$$

$k = 0, 1, 2, 3, 4, 5$

notons que

$\omega_5^2 = \omega_1^2$ mais $\beta_1 \neq \beta_5$

$\omega_2^2 = \omega_4^2$ $\quad\beta_2 \neq \beta_4$

lorsque

$$A^5 = (A^1)^* \qquad A^4 = (A^2)^*$$

$$\begin{pmatrix} 1 \\ e^{2i\,5\pi/6} \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ e^{2i\pi/6} \\ \\ \end{pmatrix}^* \Rightarrow \begin{pmatrix} 1 \\ e^{i5\pi/3} \\ \\ \end{pmatrix} = \begin{pmatrix} 1 \\ e^{-i\pi/3} \\ = \\ \end{pmatrix}$$

Sempre que os modos normais obtidos como vectores próprios de S sejam complexos, ocorrem sempre em pares c.c. tal que de cada par se podem construir 2 modos reais

$$A^k + (A^k)^* \qquad\qquad A^k - A^{k^*} \qquad\qquad k = 1, 2$$

e.g.

$$A^1 + (A^1)^* = A^1 + A^5 \qquad\qquad A^1 - A^5$$

e os outros A^0 e A^3

real ✓

$$A^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 \\ e^{i\pi} \\ e^{2i\pi} \\ e^{3i\pi} \\ e^{4i\pi} \\ e^{5i\pi} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Batimentos

(de volta às 2 pêndulos)

cond. iniciais $\longrightarrow$ bloco 1 com deslocamento d

$\longrightarrow$ " 2 na posição equil.

Sol. geral

$$X(t) = A^1 \left(b_1 \cos\omega_1 t + c_1 \sin\omega_1 t \right) + A^2 \left(b_2 \cos\omega_2 t + c_2 \sin\omega_2 t \right)$$

$$A^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

cond. iniciais

$$X(0) = \begin{pmatrix} d \\ 0 \end{pmatrix} = A^1 b_1 + A^2 b_2 = \begin{pmatrix} b_1 + b_2 \\ b_1 - b_2 \end{pmatrix} \Longrightarrow b_1 = b_2 = d/2$$

$$\dot{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \omega_1 A^1 e_1 + \omega_2 A^2 e_2$$

$$\omega_1, \omega_2 \neq 0$$

$$= \begin{pmatrix} \omega_1 e_1 + \omega_2 e_2 \\ \omega_1 e_1 - \omega_2 e_2 \end{pmatrix}$$

$$\Longrightarrow \boxed{e_1 = e_2 = 0}$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} d/2\,(\cos\omega_1 t + \cos\omega_2 t) \\ d/2\,(\cos\omega_1 t - \cos\omega_2 t) \end{pmatrix}$$

rels trigonométriques :

$$\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) = \frac{1}{2}\left(\cos a + \cos b\right)$$

important

$$\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) = \frac{1}{2}\left(\cos b - \cos a\right)$$

$$\Omega = \frac{\omega_1+\omega_2}{2} \qquad \delta\omega = \frac{\omega_2-\omega_1}{2}$$

$$\omega_1 = \sqrt{g/l}$$

$$\omega_2 = \sqrt{g/l + 2k/m}$$

$$x_1(t) = d \cos \Omega t \cos \delta\omega t$$

$$x_2(t) = d \sin \Omega t \sin \delta\omega t$$

oscilação c/ diferença de freq.

$$x_1(t) = d \cos \Omega t \, \overbrace{\cos \delta\omega t}$$

$$x_2(t) = d \, \underbrace{\sin \Omega t} \, \sin \delta\omega t$$

oscilação c/ freq. média

↓ muito mais lenta que a osc. com Ω

↓ modulação de amplitude

$$\boxed{\Omega \gg \delta\omega}$$

A modulação de amplitude (a osc. lenta) está desfasada (por 90°) para as massas 1 e 2