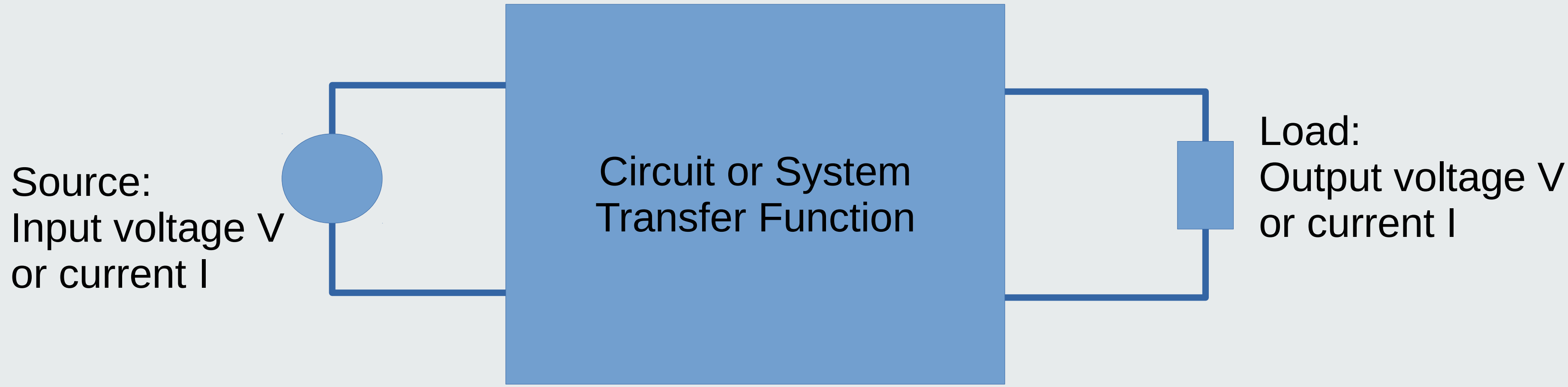
TÉCNICO LISBOA

Circuit Theory and Electronics Fundamentals

Lecture 8: Transfer Function

- What is a transfer function
- RC and RL transfer function types
- Amplitude as a function of frequency
- Phase as a function of frequency
- Using logarithmic scale
- Magnitude Bode plot
- Phase Bode plot
- Complex frequency and transfer function
- Octave Bode plots

What is a transfer function



- Transfer function: computes complex output from complex input
- We know how to compute the response for a ***sinusoidal*** input: magnitude and phase of output
- Signals can be decomposed into a series of sinusoidal signals of various frequencies (Fourier Analysis)
- A function that computes the sinusoidal output from a sinusoidal input can compute the system response because of ***linearity and time invariance***

Transfer function maths

$$x(u) \rightarrow y(u) \Rightarrow a\,x_a(u) + b\,x_b(u) \rightarrow a\,y_a(u) + b\,y_b(u)$$

Linearity

$$x(t) \rightarrow y(t) \Rightarrow x(t-\delta) \rightarrow y(t-\delta)$$

Time invariance

$$f(t) \Leftrightarrow F(s)$$

Time domain versus complex frequency domain representation

$$s = j\,\omega$$
$$f(t) \Leftrightarrow \widetilde{F}(\omega)$$

Sinusoidal time function maps to frequency dependent phasor

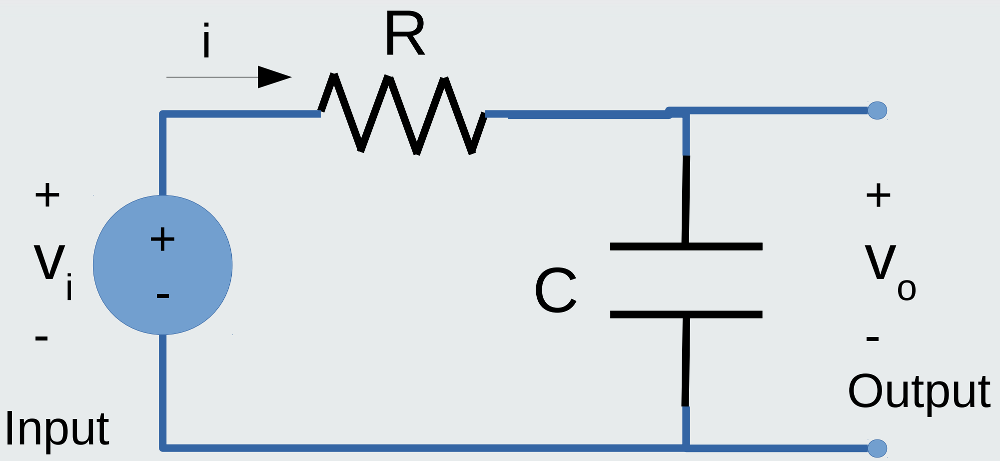
$$T(j\,\omega) : \widetilde{X}(j\,\omega) \rightarrow \widetilde{Y}(j\,\omega)$$
$$\widetilde{Y}(j\,\omega) = T(j\,\omega)\,\widetilde{X}(j\,\omega)$$
$$T(j\,\omega) = \frac{\widetilde{Y}(j\,\omega)}{\widetilde{X}(j\,\omega)}$$

The frequency response converts input phasor into output phasor

In LTI circuits phasors at a given frequency have linear relations

The frequency response is just the phasor quotient!

RC frequency response

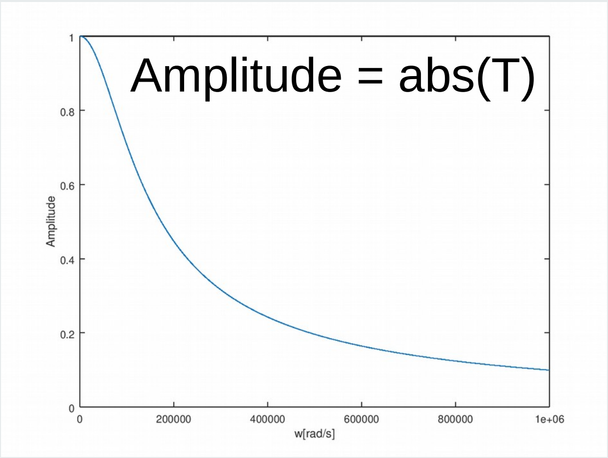


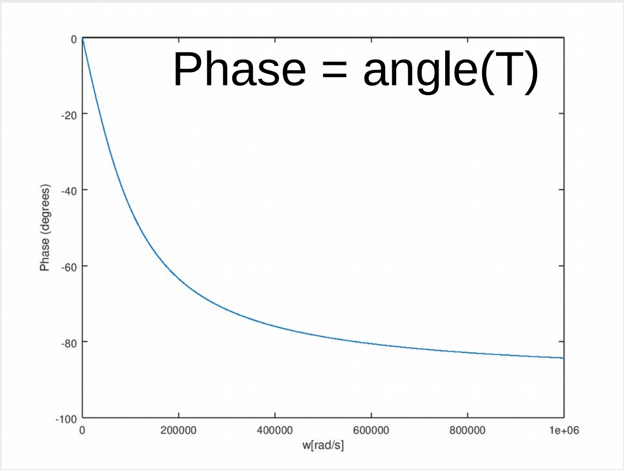
$$\tilde{V}_i = R\,i + \tilde{V}_o \;(KVL)$$

$$i = \frac{\tilde{V}_o}{Z_C} \;(Ohm's)$$

$$Z_C = \frac{1}{j\,\omega C}$$

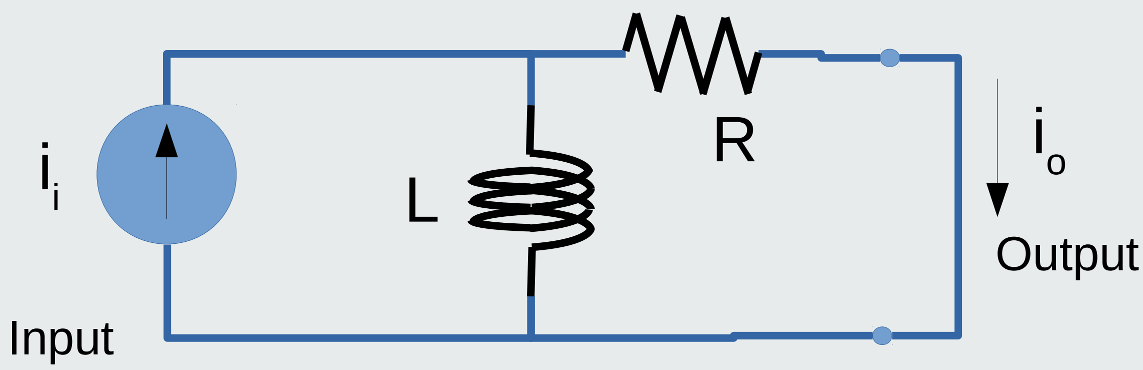
$$\tilde{V}_i = (1 + j\,\omega R C)\,\tilde{V}_o$$





$$T(j\,\omega) = \frac{\tilde{V}_o}{\tilde{V}_i} = \frac{1}{1 + j\,\omega R C}$$

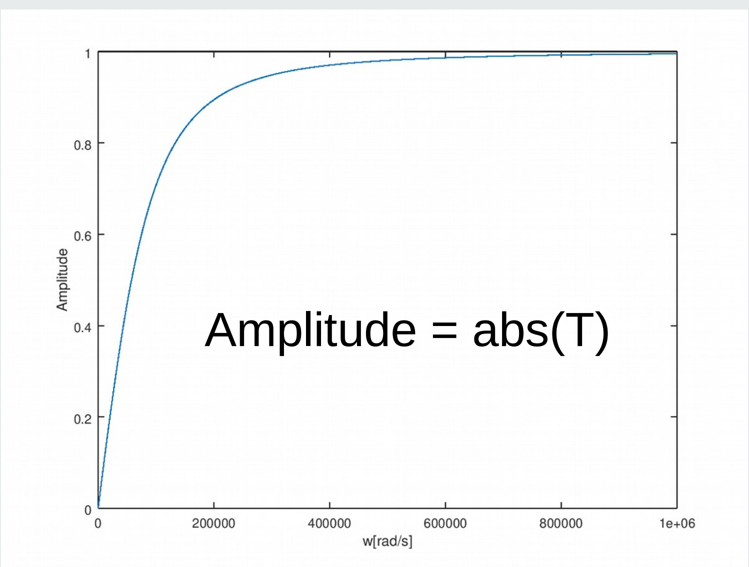
RL frequency response

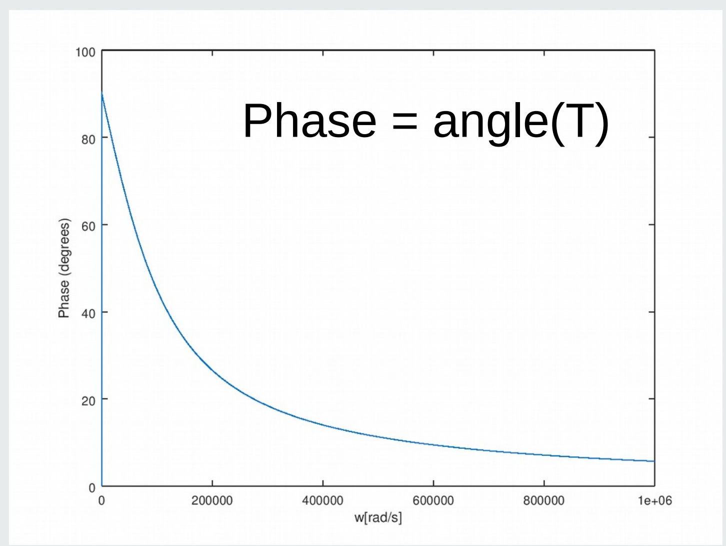


Current div.

$$\tilde{I}_o = \frac{\frac{1}{R}}{\frac{1}{R} + \frac{1}{j\omega L}} \tilde{I}_i$$

$$T(j\omega) = \frac{\tilde{I}_o}{\tilde{I}_i} = \frac{j\omega L}{R + j\omega L}$$





Using logarithmic scale

- Several plots use logarithmic scales for the X and Y axes
- Log scales are a convenient way to ***make the plot fit*** in the figure
- Log plots exploit common user behaviour:
  - User is particularly interested in a certain region of X and Y
  - User still wants to see the plot outside the region of interest but in less detail
- Log scale provides great visual detail around 1, which is represented by 0 in log scale: log(1) = 0
- Log scale provides less detail for very small or very large numbers but enables seeing a wide range

The decibel (dB)

$$X_{dB} = 20\log_{10}(X)$$
$$Y_{dB} = 20\log_{10}(Y)$$

The decibel (dB) is a convenient logarithmic scale historically used for sound levels

$$Y_{dB} = X_{dB} + 10 \Rightarrow 20\log_{10}(Y) = 20\log_{10}(X) + 10$$
$$\log_{10}\left(\frac{Y}{X}\right) = \frac{1}{2} \Rightarrow \frac{Y^2}{X^2} = 10$$

each 10dB increment corresponds to 10x more power

$$Y_{dB} = X_{dB} + 20 \Rightarrow 20\log_{10}(Y) = 20\log_{10}(X) + 20$$
$$\log_{10}\left(\frac{Y}{X}\right) = 1 \Rightarrow \frac{Y}{X} = 10$$

Each 20dB increment corresponds to 10x more amplitude

$$Y = T\,X \Rightarrow Y_{dB} = T_{dB} + X_{dB}$$

Multiplications are converted into additions

Expressing |T(jω)| in dBs

$$T(j\omega)=\frac{j\omega L}{R+j\omega L}$$

$$|T(j\omega)|=\frac{\omega L}{\sqrt{R^2+\omega^2 L^2}}$$

$$|T(j\omega)|_{dB}=20\log_{10}\left(\sqrt{\frac{\omega^2 L^2}{R^2+\omega^2 L^2}}\right)$$

$$|T(j\omega)|_{dB}=20\log_{10}(\omega)+20\log_{10}(L)-10\log_{10}(R^2+j\,\omega^2 L^2)$$

Magnitude Bode Plot

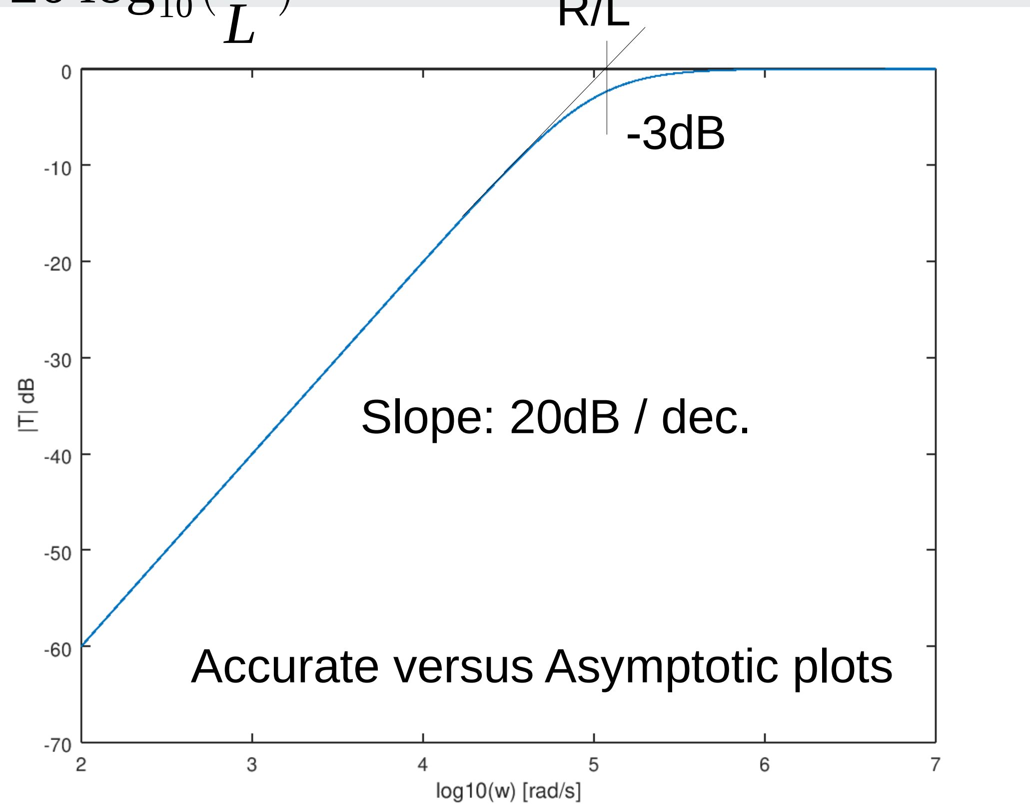
$$|T(j\omega)|_{dB} = 20\log_{10}(\omega) + 20\log_{10}(L) - 10\log_{10}(R^2 + \omega^2 L^2)$$

$$\omega \ll \frac{R}{L} \Rightarrow |T(j\omega)|_{dB} = 20\log_{10}(\omega) - 20\log_{10}\left(\frac{R}{L}\right)$$

$$\omega = \frac{R}{L} \Rightarrow |T(j\omega)|_{dB} = -3\,dB$$

$$\omega \gg \frac{R}{L} \Rightarrow |T(j\omega)|_{dB} = 0$$

Magnitude Bode plot: asymptotic plot of $|T(\log(\omega))|_{dB}$



Phase Bode Plot

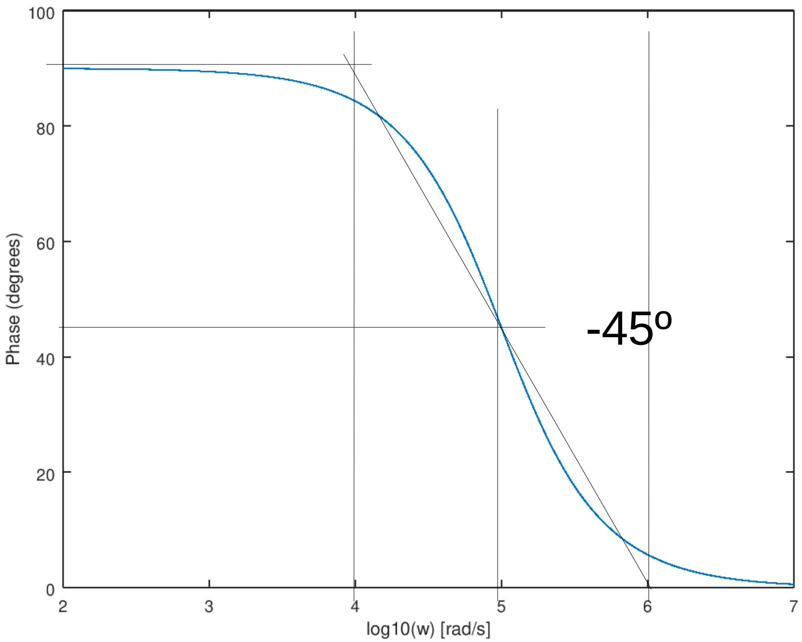
$$T(j\omega)=\frac{j\omega L}{R+j\omega L}$$

$$\sphericalangle T(j\omega)=\frac{\pi}{2}-\arctan\left(\omega\frac{L}{R}\right)$$

$$\omega\ll\frac{R}{L}\Rightarrow T(j\omega)=\frac{\pi}{2}$$

$$\omega=\frac{R}{L}\Rightarrow T(j\omega)=\frac{\pi}{4}$$

$$\omega\gg\frac{R}{L}\Rightarrow T(j\omega)=0$$



Accurate versus Asymptotic plots

Phase Bode plot: angle(T(log(ω)))

Complex frequency

$$T(j\omega)=\frac{j\omega L}{R+j\omega L}$$

Note j and ω always hang out together

$$s=\sigma+j\omega$$

jω is a particular value of the ***complex frequency s***

$$T(s)=\frac{sL+0}{sL+R}$$

Transfer functions are commonly expressed as a ratio of two polynomials in s

Using T(s) in Octave

- The time ***t*** is the <u>real variable</u> in a time <u>real function</u>
- The frequency ***ω*** is the <u>real variable</u> in the Fourier transform <u>complex function</u>
- The complex frequency ***s*** is the <u>complex variable</u> in the Laplace transform <u>complex function</u>
- The Laplace transform is more general than the Fourier transform and covers a wider range of time functions

- Octave can produce “Bode plots” and many other niceties if the user just inputs T(s)! – Let’s see how.

Conclusion

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