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Circuit Theory and Electronics Fundamentals

Lecture 4: Sinusoidal analysis

- General solution for RC and RL circuits
- Energy stored in C and L
- Self and mutual inductance
- The transformer
- Sinusoidal voltages and currents
- Complex representation
- Impedance and admittance

General solution for RC and RL circuits

- Circuits containing a single L or C and any number of voltage, current sources and resistors have the general solution below
- No need to solve differential equations!

$$x(t)=x(\infty)+[x(0)-x(\infty)]e^{-\frac{t}{\tau}}$$

- x(t) is any capacitor voltage or inductor current in the circuit
- τ is the time constant

$$\tau=RC$$

$$\tau=\frac{L}{R}$$

- R is the equivalent resistor ***seen*** by C or L when all sources are switched off.

Example RC circuit solving

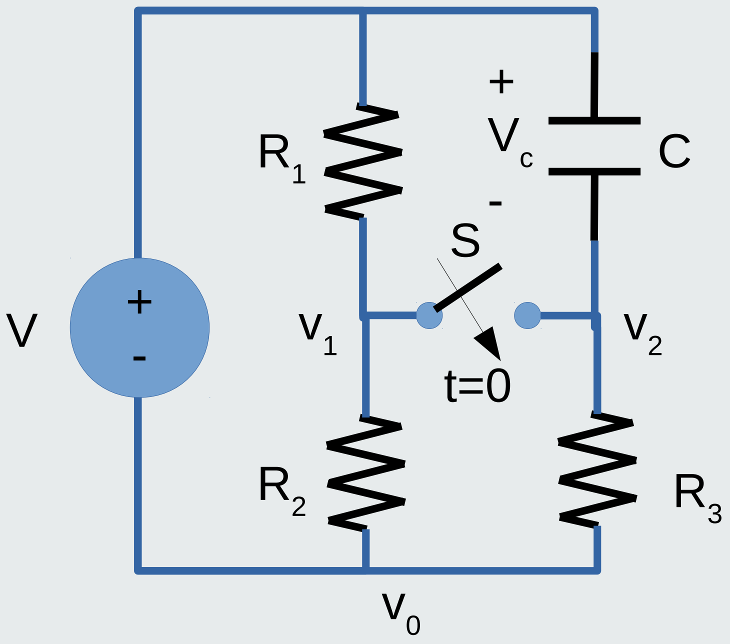
Problem: find $v_1(t)$ and $v_2(t)$

$t < 0$: switch S is open

$t=0$: switch S closes

$t>0$: switch remains closed, voltages and currents evolve

$t=\infty$: switch remains closed, voltages and currents stabilize in final values



RC Example: t < 0

$$v_2 = v_0 = 0$$

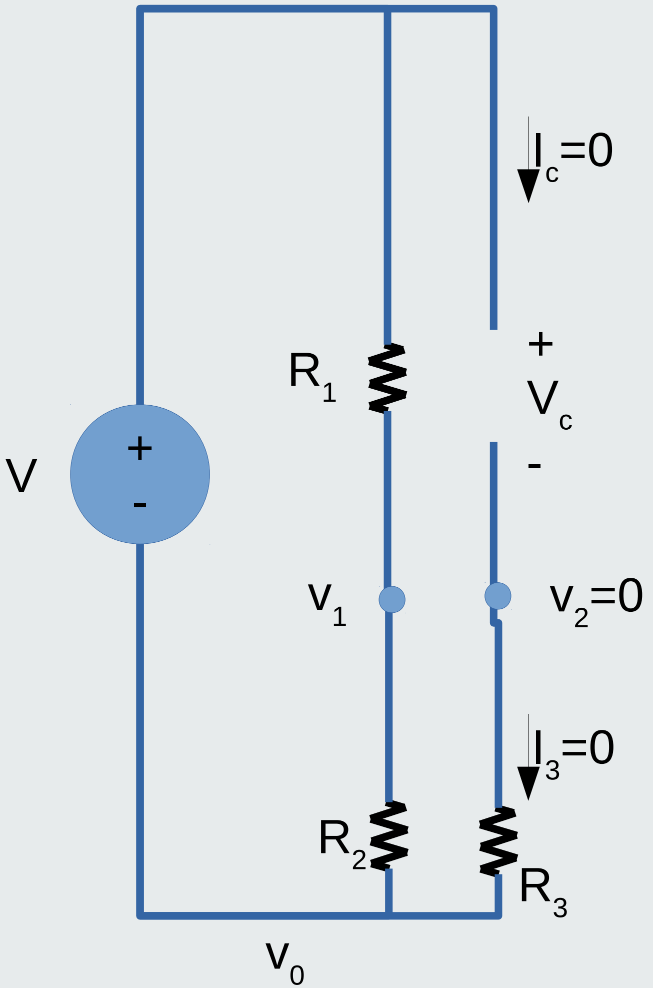
Switch is
open

$$v_1 = \frac{R_2}{R_1 + R_2} V$$

Voltage divider

$$v_c = V - v_2 = V$$

$$v_c(t<0) = V$$



RC Example: t=0

Switch closes, 1 and 2 become the same electrical node.

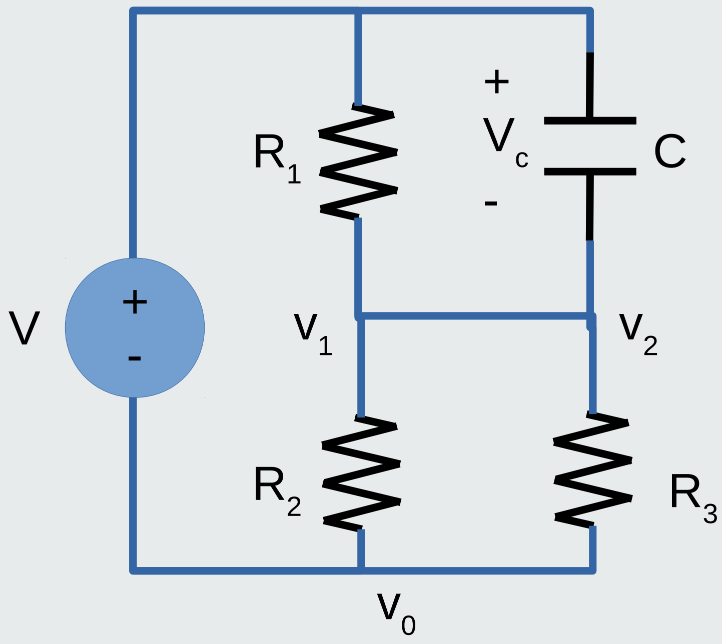
C will discharge through the resistors

This will take time: there is a time constant to be computed

Therefore:

$v_c(0) \; = v_c(t<0) \; = V$

$v_1(0) \; = \; v_2(0) = V - v_c(0) = V - V = 0$



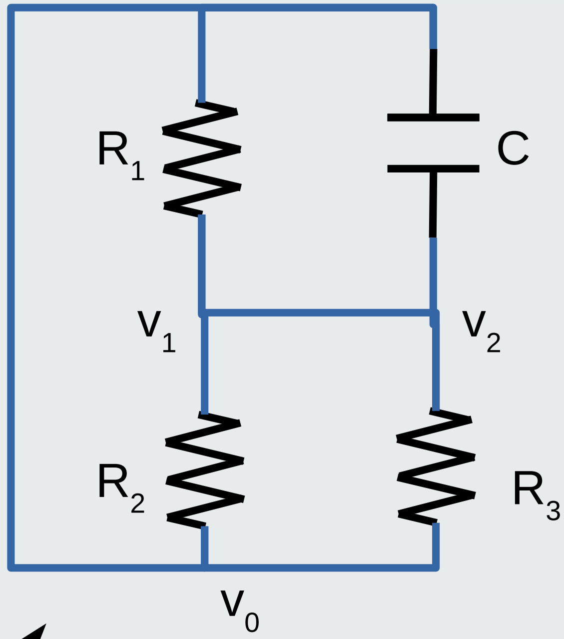
RC Example: time constant calculation

To compute the time constant, compute the equivalent resistance *as seen by C* when the independent sources are switched off

Switch-off Vs: replace it with a short-circuit

Compute the equivalent resistor seen at the terminals of C

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$



$$\tau = R\,C = \frac{C}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

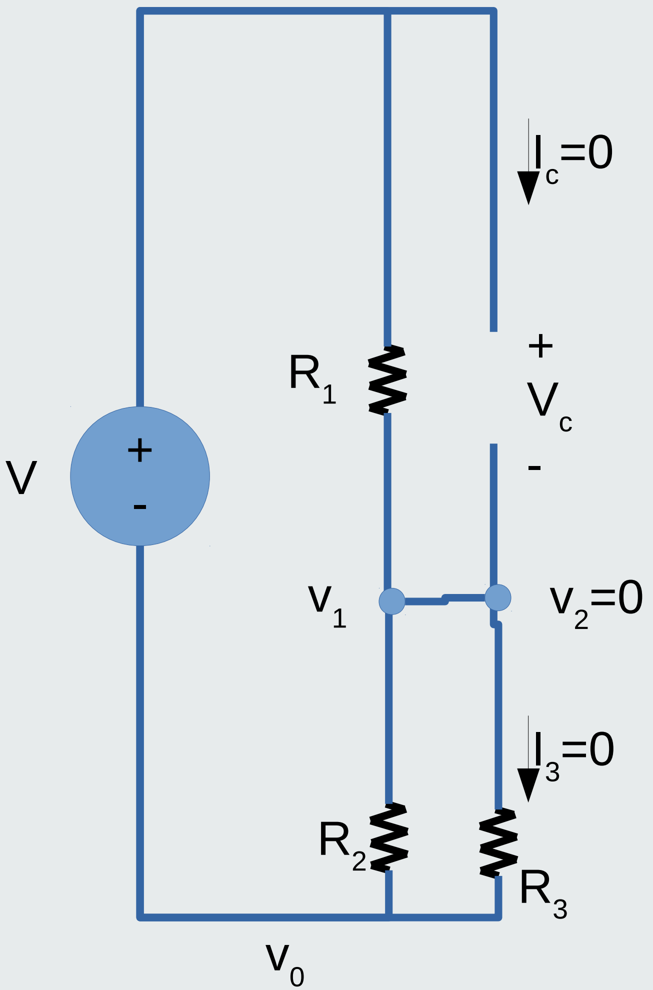
All resistors are in parallel with C

RC Example: t = ∞

Switch is closed, nodes 1 and 2 are connected

$$v_1(\infty)=v_2(\infty)=\frac{R_2\|R_3}{R_1+R_2\|R_3}V$$

Voltage divider



RC Example: t>0

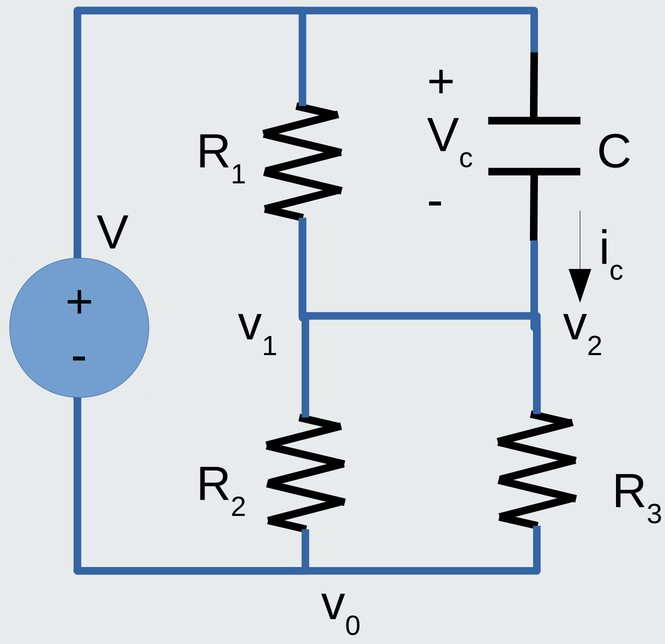
Switch is closed, C discharges through the resistors with time constant RC

Use general fornula:

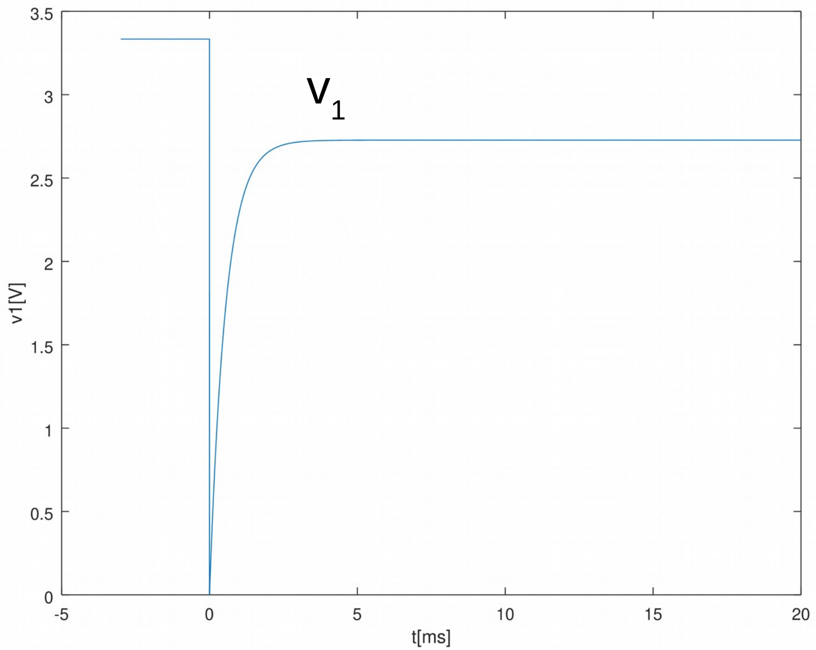
$$v_1(t) = v_2(t) = v_2(\infty) + [v_2(0) - v_2(\infty)]e^{-\frac{t}{\tau}}$$

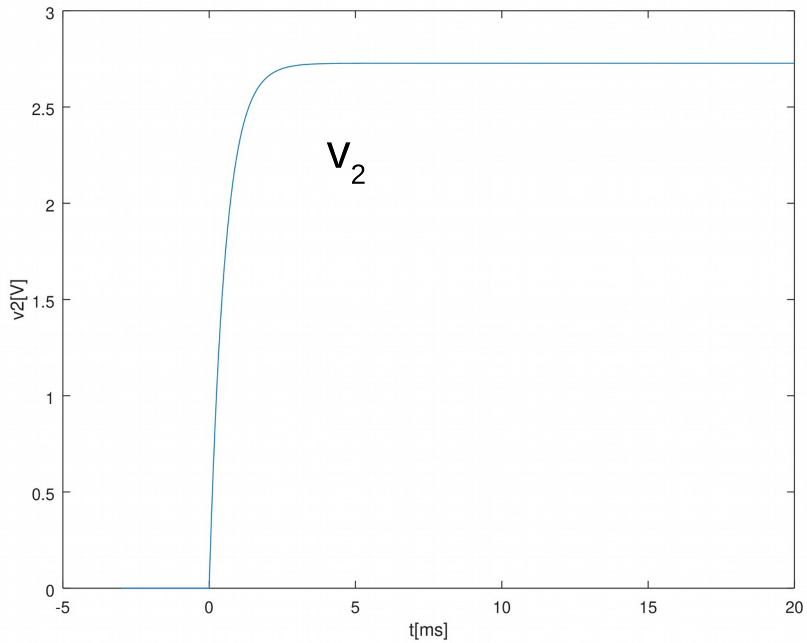
$$v_1(t) = v_2(t) = V\frac{R_2 \| R_3}{R_1 + R_2 \| R_3}\left(1 - e^{-\frac{t}{\tau}}\right)$$

Note values at t=0 and t=∞



RC Example: plot $v_1(t)$ and $v_2(t)$





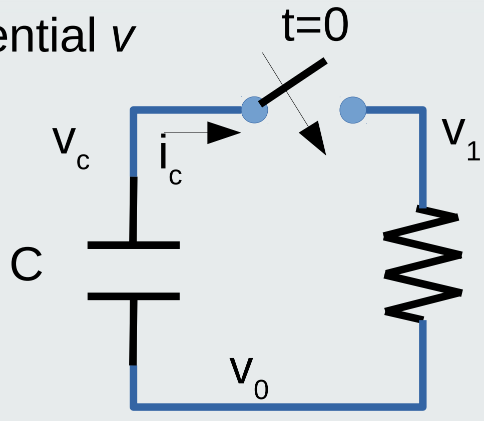
- V=5V, R_1=1kΩ, R_2=2k, R_3=3k, C=1μF => τ=1ms
- v_1 and v_2 are different for t < 0, and have the same value for t ≥ 0 (same node)
- C discharges (loses voltage): note that $v_C=V-v_2$ and v_2 increases
- v_1 has a discontinuity because C cannot charge or discharge instantly through R

Energy stored in C

$$de(t) = v(t)\,dq(t)$$ Energy delta for *dq* across potential *v*

$$p(t) = \frac{de(t)}{dt} = v(t)\frac{dq(t)}{dt}$$

$$p(t) = v(t)\,i(t)$$



Time constant
$\tau = RC$

t < 0: switch open

$$i_C = 0, v_c = V, v_1 = 0$$

$$v_1(t) = v_c(t) = V\,e^{-\frac{t}{\tau}}$$

Solution for t≥0

$$i_C(t) = \frac{V}{R}e^{-\frac{t}{\tau}}$$

t =0: switch closes

$$i_C(0) = V/R, v_c = v_1 = V$$

$$E_C = \int_{t=0}^{\infty} p(t)\,dt = \int_{t=0}^{\infty} \frac{V^2}{R}e^{-2\frac{t}{RC}}dt$$

t = ∞: switch closed

$$v_1 = v_c = 0\,V$$

$$E_C = -\left[\frac{V^2}{R}\frac{RC}{2}e^{-2\frac{t}{RC}}\right]_{t=0}^{t=\infty} = \frac{1}{2}C\,V^2$$

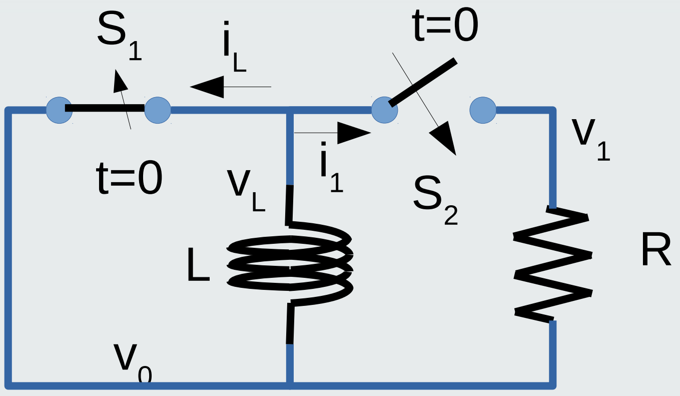
Energy previously stored in C and dissipated in R

Energy stored in L

$$p(t)=v(t)\,i(t)$$

Time constant

$\tau=L/R$



t < 0: S_1 closed, S_2 open

$$i_L=I,\ v_L=0,\ i_1=0,\ v_1=0$$

t =0: S_1 opens, S_2 closes

$$i_L=0,\ i_1=I,\ v_1=RI$$

t = ∞: S_1 open, S_2 closed

$$v_1=v_L=0,\ i_L=i_1=0$$

$$i_1(t)=I\,e^{-\frac{t}{\tau}}$$

$$v_L(t)=v_1(t)=RI\,e^{-\frac{t}{\tau}}$$

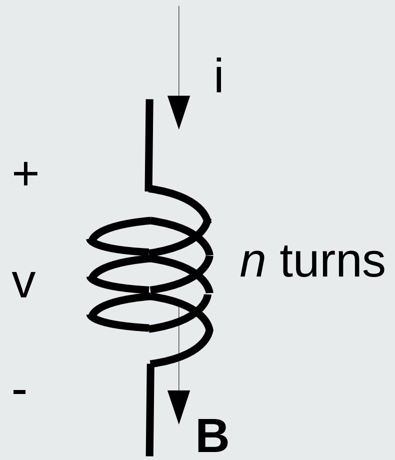
$$E_C=\int_{t=0}^{\infty}p(t)\,dt=\int_{t=0}^{\infty}R\,I^2 e^{-2\frac{t}{L/R}}\,dt$$

$$E_L=-\left[R\,I^2\frac{L}{2R}e^{-2\frac{t}{L/R}}\right]_{t=0}^{t=\infty}=\frac{1}{2}L\,I^2$$

Solution for t≥0

Energy previously stored in L and dissipated in R

Self-inductance



A current *i* creates magnetic field **B** and <u>*self*</u> induces a voltage *v* across the inductor

$$\Phi = \iint_A \vec{B} \cdot \vec{dA}$$

Flux

$$v = n\,d\frac{\Phi}{dt}$$

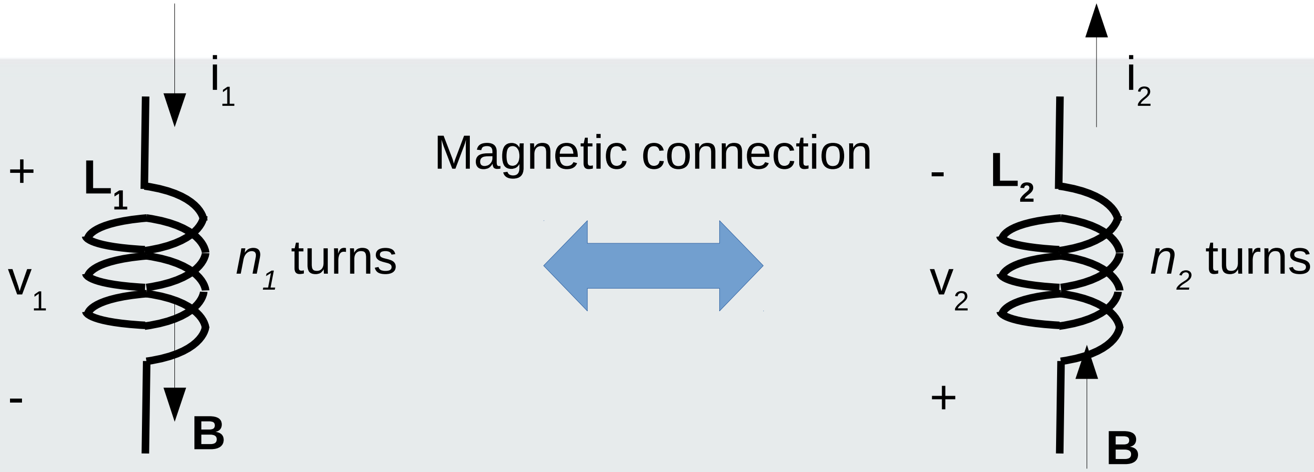
Self-induced voltage equals flux variation per time unit

$$n\Phi = Li$$

$$v = L\frac{di}{dt}$$

<u>***L***</u> is <u>self-inductance</u>, or simply <u>inductance</u>

Mutual inductance



$i_2=0 \Rightarrow$ Coil 2 open

$\Phi_2=\Phi_1=\Phi$ Perfect symmetry

$n_1\Phi_1=L_1 i_1$ Fluxes caused by 1 only

$n_2\Phi_2=L_M i_1$ **L_M is mutual inductance**

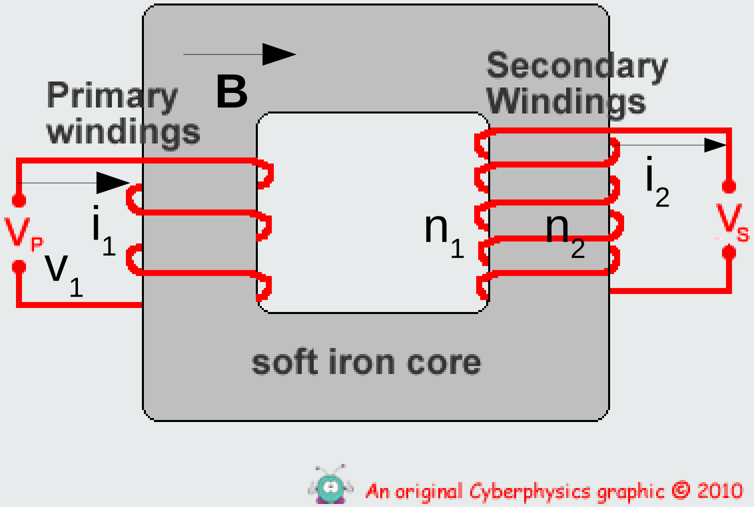
$\frac{n_1}{n_2}=\frac{L_1}{L_M}$ Previous 2 divided

$\frac{n_2}{n_1}=\frac{L_2}{L_M}$ Reciprocal for $i_1=0$

$$L_2=\left(\frac{n_1}{n_2}\right)^2 L_1$$

$$L_M=\sqrt{L_1 L_2}$$

The ideal transformer



The transformer is formed by 2 coils magnetically connected by a circular core!

The magnetic field **B** concentrates in the core for a ***perfect and symmetric connection***

$$\Phi_1 = \Phi_2$$ Perfect symmetry

$$\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix} = \begin{bmatrix} L_1 & L_M \\ L_M & L_2\end{bmatrix}\begin{bmatrix} i_1 \\ i_2\end{bmatrix}$$

Superposition Theorem

$$v_2 = \frac{n_2}{n_1} v_1$$ Transformer voltage law

$$i_2 = \frac{n_1}{n_2} i_1$$ Transformer current law

$$v_1 i_1 = v_2 i_2$$ Conservation of power

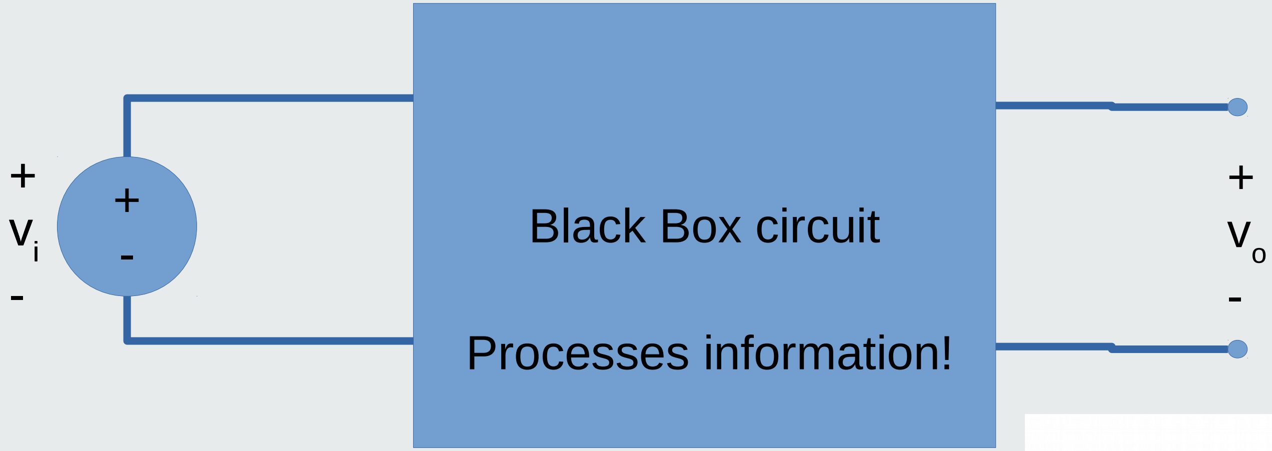
Sinusoidal analysis

- Circuits manipulate electrical signals
- Signals convey energy, information or both (vital functions)
- Mathematical methods can decompose signals into series or integrals of basic signals
- Fourier analysis: series of sinusoidal waves
- Laplace: integral of complex powers (generalizes Fourier analysis, Euler equation)

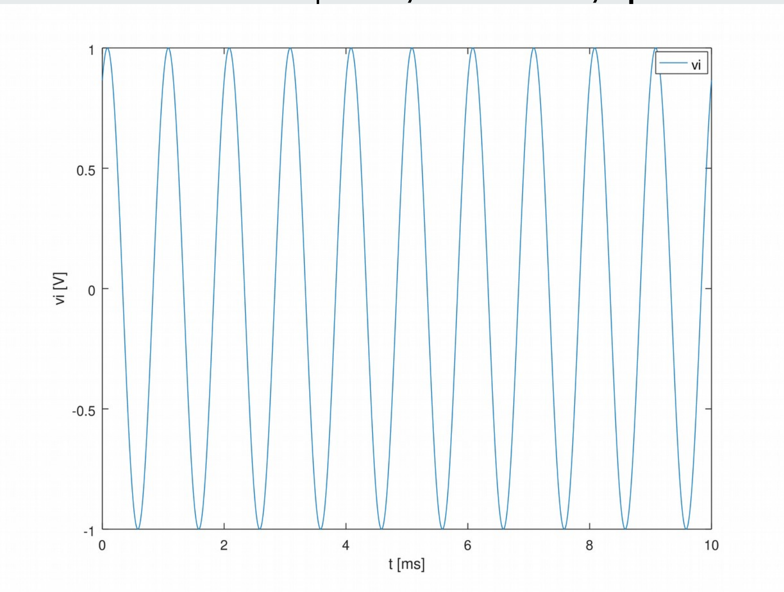
Do you know how the circuit responds to a sine wave?

If yes, then you know how the circuit responds to practically any signal!

Sinusoidal voltage source



V_i=1V, f=1KHz, φ=-π/6



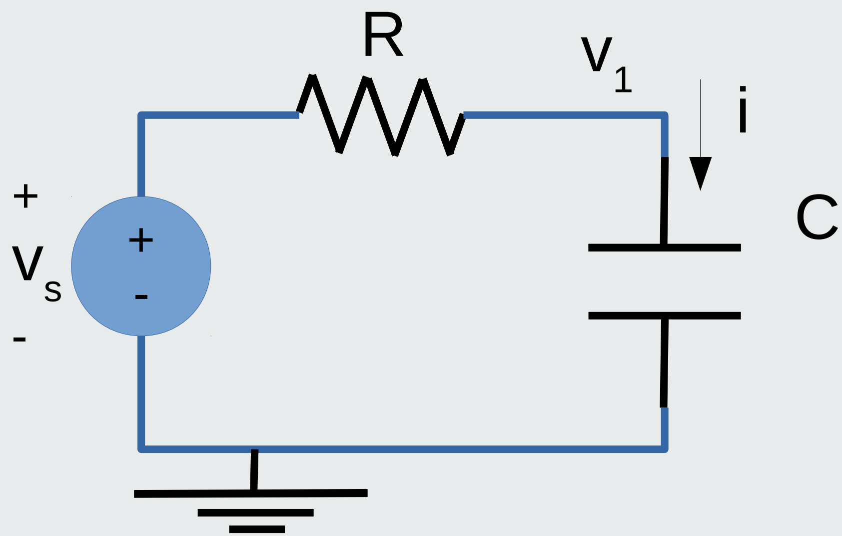
$$v_i(t) = V_i \cos(\omega t + \phi)$$

Vi: amplitude
ω: radian frequency
t: time
φ: phase or angle
f: frequency in hertz
T: period in second

$$\omega = 2\pi f = \frac{2\pi}{T}$$

RC series: forced solution with sinusoidal excitation

$$v_s(t)=V_s\cos(\omega t+\phi_s)$$



Sinusoidal excitation => sinusoidal forced solution at same frequency!

$$v_{1f}(t)=V_1\cos(\omega t+\phi_1)$$

We only need to determine V_1 and φ_1 … but how ?

$$v_s=RC\frac{dv_1}{dt}+v_1$$ KVL

$$V_s\cos(\omega t+\phi_s)=-\omega RC\,V_1\sin(\omega t+\phi_1)+V_1\cos(\omega t+\phi_1)$$

Solving forced solution with sinusoidal excitation

$$V_s \cos(\omega t + \phi_s) = -\omega RC\, V_1 \sin(\omega t + \phi_1) + V_1 \cos(\omega t + \phi_1)$$

Let’s choose 2 time instants to have 2 equations and thus solve for V1 and φ_1

$$\omega t = -\phi_1: \qquad V_s \cos(\phi_s - \phi_1) = V_1$$

$$\omega t = -\phi_1 + \frac{\pi}{2}: \qquad -V_s \sin(\phi_s - \phi_1) = -\omega RC\, V_1$$

$$\tan(\phi_s - \phi_1) = \omega RC$$

Obtained by dividing previous 2 equations

$$\phi_s - \phi_1 = atan(\omega RC)$$

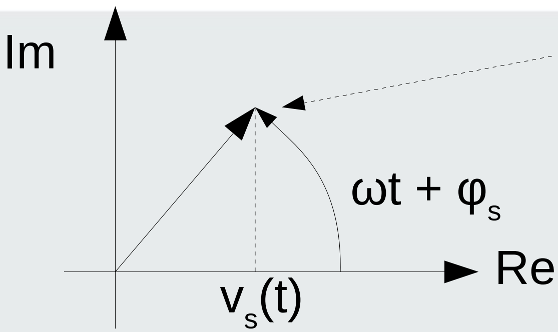
$$V_1 = V_s \cos(atan(\omega RC))$$

$$\frac{V_1}{V_s} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\cos^2(x) = \frac{1}{1 + \tan^2(x)}$$

For large circuits this is hard work: lots of trigonometry. There must be a better way!

Sinusoidal analysis with “complex” numbers



$$V_s e^{j(\omega t+\phi_s)}$$

Complex vector rotating with constant angular velocity ω, use Euler’s equation:

$$e^{jx}=\cos(x)+j\sin(x)$$

$$v_s=V_s\cos(\omega t+\phi_s)=\Re\left\{V_s e^{j(\omega t+\phi_s)}\right\}$$

$$v_1=V_1\cos(\omega t+\phi_1)=\Re\left\{V_1 e^{j(\omega t+\phi_1)}\right\}$$

- No need to carry the Real part around
- Work in complex domain, take real part in the end

$$v_s=RC\frac{dv_1}{dt}+v_1$$

KVL: replace real variables with complex variables

$$V_s e^{j(\omega t+\phi_s)}=j\omega RC\,V_1 e^{j(\omega t+\phi_1)}+V_1 e^{j(\omega t+\phi_1)}$$

Get rid of common factor $e^{j\omega t}$

$$V_s e^{j\phi_s}=(1+j\omega RC)V_1 e^{j\phi_1}$$

<u>Phasors</u>: complex vectors that represent sinusoidal signals; <u>magnitude/phase</u> info only

Complex amplitude aka Phasor

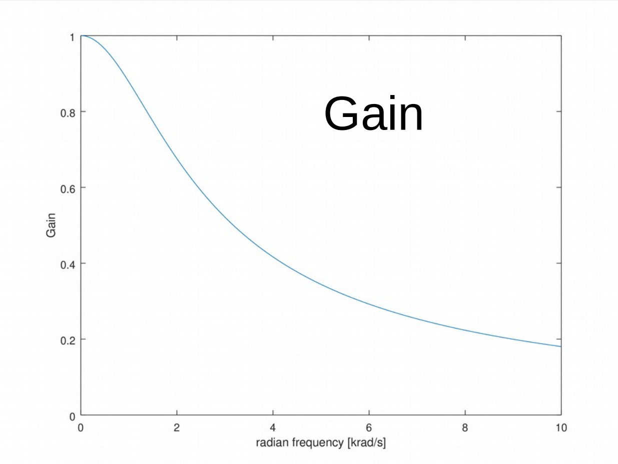
$$\tilde{V}_s = V_s e^{j\phi_s}, \quad \tilde{V}_1 = V_1 e^{j\phi_1} \Rightarrow$$

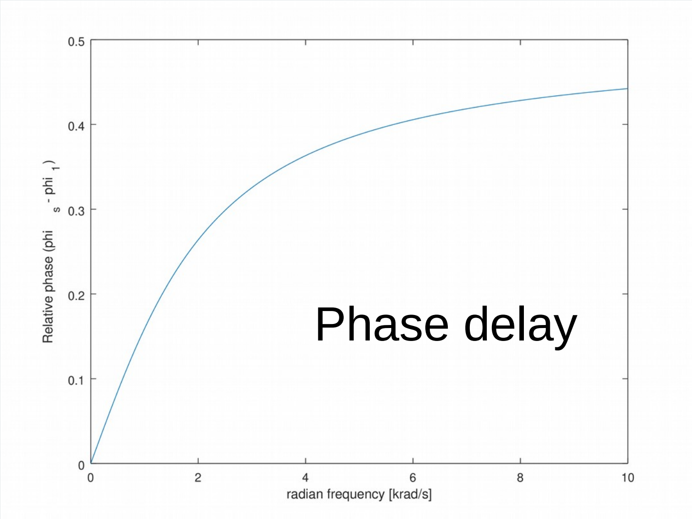
$$\tilde{V}_s = (1 + j\omega RC)\tilde{V}_1$$

$$\frac{\tilde{V}_1}{\tilde{V}_s} = \frac{1}{1 + j\omega RC} = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}} e^{j\,atan(\omega RC)}$$

$$\frac{V_1}{V_s} = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}$$ Gain

$$\phi_s - \phi_1 = atan(\omega RC)$$ Phase difference



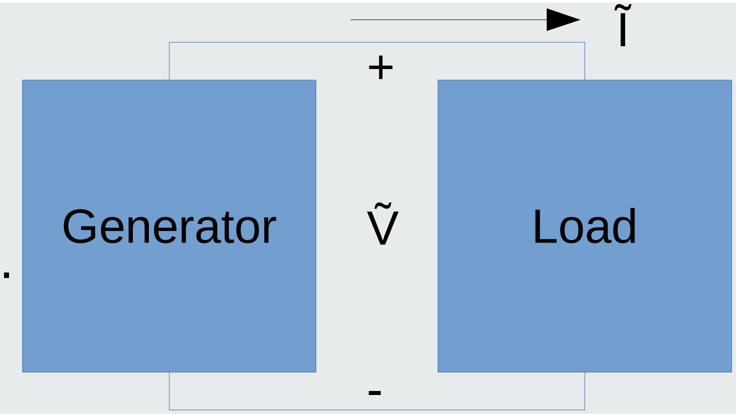


Impedance and admittance

Problem:

A ***Generator*** circuit applies sinusoidal voltage Ṽ to a ***Load*** (circuit that receives Ṽ).

What is the value of sinusoidal current Ĩ?



Answer: Ĩ = Ṽ / **Z** (Phasor Ohms Law!)

Z is called the ***impedance*** (resistance equivalent for phasors)
Z = R + j X (resistance is real part, reactance is imaginary part)

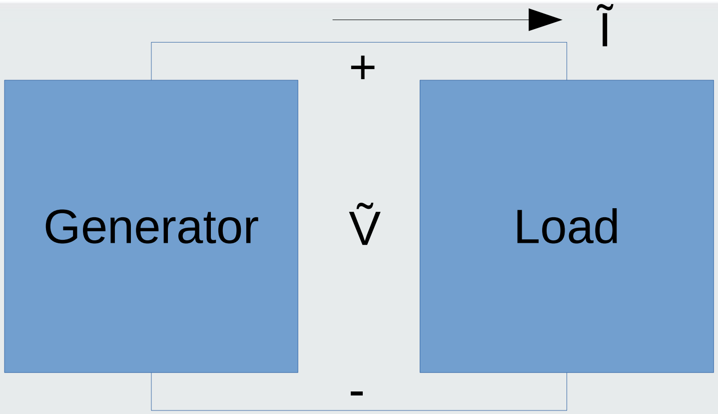
Y = 1/Z is called the ***admittance*** (conductance equivalent for phasors)
Y = G + jB (conductance is real part, susceptance is imaginary part)

Z and Y are complex quantities as they give the relationship between the amplitudes and phases of Ṽ and Ĩ

How to Compute Impedance

1) Compute the impedance of each component

2) Associate the components in series or parallel and compute the impedance between the two terminals of the load



3) component impedances:

$$i = v/R \Rightarrow \tilde{I}\, e^{j\omega t} = \frac{\tilde{V}\, e^{j\omega t}}{R} \Rightarrow Z_R = \frac{\tilde{V}}{\tilde{I}} = R$$

Resistor Impedance

$$i = C\frac{dv}{dt} \Rightarrow \tilde{I}\, e^{j\omega t} = C\frac{d}{dt}\left(\tilde{V}\, e^{j\omega t}\right) \Rightarrow Z_C = \frac{\tilde{V}}{\tilde{I}} = \frac{1}{j\omega C}$$

Capacitor Impedance

$$v = L\frac{di}{dt} \Rightarrow \tilde{V}\, e^{j\omega t} = L\frac{d}{dt}\left(\tilde{I}\, e^{j\omega t}\right) \Rightarrow Z_L = \frac{\tilde{V}}{\tilde{I}} = j\omega L$$

Inductor Impedance

Conclusion

- General solution for RC and RL circuits
- Power, energy stored in C and L
- Self-inductance and mutual inductance
- The ideal transformer
- Sinusoidal forced solution
- Sinusoidal forced solution using complex numbers
- Impedance and admittance