Laboratório de Mecânica Oscilações e Ondas

Elementos sobre a série trignomética de Fourier

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1º Semestre, 2019/2020

Elementos sobre séries e transformadas de Fourier

Cópia das transparências

Considerações gerais

Considere-se $f(t)$ uma função definida no intervalo [-T/2; T/2]
Considere-se o polinómio $P(t)$

$$P(t) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k \cos(k\omega t + \varphi_k), \quad \text{com} \quad \omega = \frac{2\pi}{T}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$P(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos k\omega t + b_k \sin k\omega t)$$

$$\begin{cases} a_k = & A_k \cos \varphi_k \\ b_k = & -A_k \sin \varphi_k \end{cases}$$

Mínimo do erro quadrático médio

Para $P(t)$ representar $f(t)$ é necessário que e a_k e b_k minimizem o erro quadrático médio

$$\epsilon = \frac{1}{T}\int_{-T/2}^{T/2} \left[f(t) - P(t)\right]^2 \, dt$$

$$\begin{cases} \frac{\partial \epsilon}{\partial a_0} & = 0 \\ \frac{\partial \epsilon}{\partial a_k} & = 0 \\ \frac{\partial \epsilon}{\partial b_k} & = 0 \end{cases}$$

$$\begin{cases} -\frac{2}{T}\int_{-T/2}^{T/2} \left[f(t) - P(t)\right] \frac{\partial P(t)}{\partial a_0} \, dt & = 0 \\ -\frac{2}{T}\int_{-T/2}^{T/2} \left[f(t) - P(t)\right] \frac{\partial P(t)}{\partial a_k} \, dt & = 0 \\ -\frac{2}{T}\int_{-T/2}^{T/2} \left[f(t) - P(t)\right] \frac{\partial P(t)}{\partial b_k} \, dt & = 0 \end{cases}$$

Condição necessária de mínimo

$$\begin{cases} \dfrac{\partial P(t)}{\partial a_0} & = \frac{1}{2} \\ \dfrac{\partial P(t)}{\partial a_k} & = \cos k\omega t \\ \dfrac{\partial P(t)}{\partial b_k} & = \sin k\omega t \end{cases}$$

$$\begin{cases} \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt & = \frac{1}{T}\int_{-T/2}^{T/2} P(t)\,dt \\ \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega t)\,dt & = \frac{1}{T}\int_{-T/2}^{T/2} P(t)\cos(k\omega t)\,dt \\ \frac{1}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega t)\,dt & = \frac{1}{T}\int_{-T/2}^{T/2} P(t)\sin k\omega t\,dt \end{cases}$$

Valores médios de $P(t)$

$$\left\{\begin{array}{r} \frac{1}{T}\int_{-T/2}^{T/2} P(t)\,dt \\ \frac{1}{T}\int_{-T/2}^{T/2} P(t)\cos(k\omega t)\,dt \\ \frac{1}{T}\int_{-T/2}^{T/2} P(t)\sin(k\omega t)\,dt \end{array}\right. =$$

$$= \left\{\begin{array}{r} \frac{1}{T}\int_{-T/2}^{T/2}\left[\frac{a_0}{2}+\sum_{p=1}^{n}\left(a_p\cos(p\omega t)+b_p\sin(p\omega t)\right)\right]\,dt \\ \frac{1}{T}\int_{-T/2}^{T/2}\left[\frac{a_0}{2}+\sum_{p=1}^{n}\left(a_p\cos(p\omega t)+b_p\sin(p\omega t)\right)\right]\cos(k\omega t)\,dt \\ \frac{1}{T}\int_{-T/2}^{T/2}\left[\frac{a_0}{2}+\sum_{p=1}^{n}\left(a_p\cos(p\omega t)+b_p\sin(p\omega t)\right)\right]\sin(k\omega t)\,dt \end{array}\right.$$

Valores médios

$$\frac{1}{T}\int_{-T/2}^{T/2} \frac{a_0}{2}\,dt = \frac{a_0}{2}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} a_p \cos(p\omega t)\,dt = 0$$

$$\frac{1}{T}\int_{-T/2}^{T/2} a_p \sin(p\omega t)\,dt = 0$$

$$\frac{1}{T}\int_{-T/2}^{T/2} a_p \cos(p\omega t)\cos(k\omega t)\,dt$$

$$\cos(p\omega t)\cos(k\omega t) = \frac{1}{2}\left[\cos\left((p+k)\omega t\right) + \cos\left((p-k)\omega t\right)\right]$$

$$\begin{cases} \frac{1}{T}\int_{-T/2}^{T/2} a_p \frac{1}{2}\left[\cos\left((p+k)\omega t\right) + \cos\left((p-k)\omega t\right)\right]\,dt = 0 & p \neq k \\ \frac{1}{T}\int_{-T/2}^{T/2} a_p \frac{1}{2}\left[\cos(2p\omega t) + 1\right]\,dt = a_p/2 & p = k \end{cases}$$

Cálculo de a_0, a_k e b_k

$$\delta_{pk} = \begin{cases} 0, & p \neq k \\ 1, & p = k \end{cases}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} a_p \cos(p\omega t)\cos(k\omega t)\,dt = \delta_{pk}\frac{a_p}{2}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} b_p \sin(p\omega t)\sin(k\omega t)\,dt = \delta_{pk}\frac{b_p}{2}$$

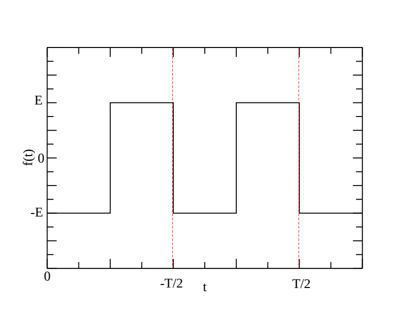
$$\frac{1}{T}\int_{-T/2}^{T/2} \sin(p\omega t)\cos(k\omega t)\,dt = 0$$

Cálculo de a_0, a_k e b_k

$$\left\{\begin{array}{r} \frac{1}{T}\int_{-T/2}^{T/2} P(t)\,dt = a_0/2 \\ \frac{1}{T}\int_{-T/2}^{T/2} P(t)\cos(k\omega t)\,dt = a_k/2 \\ \frac{1}{T}\int_{-T/2}^{T/2} P(t)\sin(k\omega t)\,dt = b_k/2 \end{array}\right. \quad \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{rl} \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt & = a_0/2 \\ \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega t)\,dt & = a_k/2 \\ \frac{1}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega t)\,dt & = b_k/2 \end{array}\right.$$

Onda quadrada



$$\begin{cases} a_0 = & \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt \\ a_k = & \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega t)\,dt \\ b_k = & \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega t)\,dt \end{cases}$$

Determinação de a_0, a_k e b_k

$$\begin{cases} a_0 = & \frac{2}{T}\left(\int_{-T/2}^{0}(-E)\,dt + \int_{0}^{T/2} E\,dt\right) \\ a_k = & \frac{2}{T}\left(\int_{-T/2}^{0}(-E)\cos(k\omega t)\,dt + \int_{0}^{T/2} E\cos k\omega t\,dt\right) \\ b_k = & \frac{2}{T}\left(\int_{-T/2}^{0}(-E)\sin(k\omega t)\,dt + \int_{0}^{T/2} E\sin k\omega t\,dt\right) \end{cases}$$

$$\begin{cases} a_0 = & \frac{2}{T}\left(\int_{0}^{T/2}(-E)\,dt + \int_{0}^{T/2} E\,dt\right) \\ a_k = & \frac{2}{T}\left(\int_{0}^{T/2}(-E)\cos(k\omega t)\,dt + \int_{0}^{T/2} E\cos k\omega t\,dt\right) \\ b_k = & \frac{2}{T}\left(\int_{0}^{T/2}(+E)\sin(k\omega t)\,dt + \int_{0}^{T/2} E\sin k\omega t\,dt\right) \end{cases}$$

Valores de a_0, a_k e b_k

$$\begin{cases} a_0 = & 0 \\ a_k = & 0 \\ b_k = & 2\frac{2}{T}\int_0^{T/2}(+E)\sin k\omega t\, dt \end{cases}$$

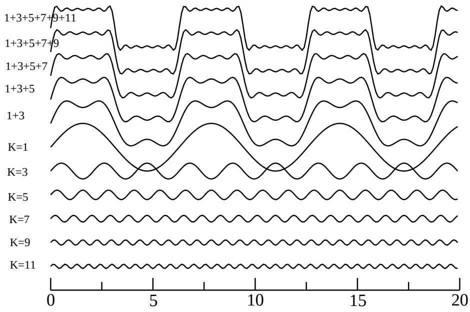
$$\begin{cases} a_0 = & 0 \\ a_k = & 0 \\ b_k = & 2\frac{2}{T}\frac{1}{k\omega}E\left[-\cos(k\omega t)\right]_0^{T/2} \end{cases}$$

$$b_k = \frac{2E}{k\pi}\left[1 - \cos(k\pi)\right]$$

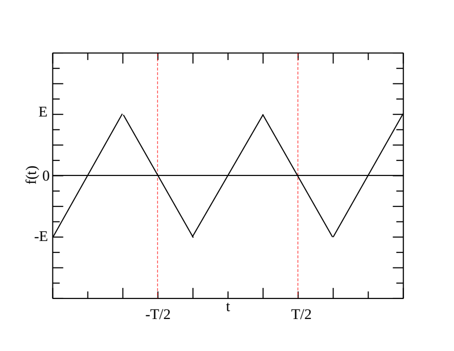
$$\begin{cases} b_k = 0, & k = 2p \\ b_k = \frac{4E}{k\pi}, & k = 2p+1 \end{cases}$$

$$P(t) = \frac{4E}{\pi}\sum_{k=1,3,5,..}^{n}\frac{1}{k}\sin(k\omega t)$$

Ilustração



Onda triangular



$$f(t) = \begin{cases} -\frac{4E}{T}t - 2E, & -\frac{T}{2} \le t \le -\frac{T}{4} \\ \frac{4E}{T}t, & -\frac{T}{4} \le t \le \frac{T}{4} \\ -\frac{4E}{T}t + 2E, & \frac{T}{4} \le t \le \frac{T}{2} \end{cases} \quad \Rightarrow f(t) = ct + d$$

Determinação de a_0, a_k e b_k

$$\left\{\begin{array}{rl} a_0 = & \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt \\ a_k = & \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega t)\,dt \\ b_k = & \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega t)\,dt \end{array}\right. \quad \Leftrightarrow$$

$$\text{Se } x = \omega t \text{ e } g(x) = \left\{\begin{array}{rl} 1, & \text{função par} \\ \cos(kx), & \text{função par} \\ \sin(kx), & \text{função impar} \end{array}\right.$$

$$\omega f(x) = cx + d \left\{\begin{array}{rlc} c = -\frac{2E}{\pi}, & d = -2E, & -\pi \leq x \leq -\frac{\pi}{2} \\ c = \frac{2E}{\pi}, & d = 0, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ c = -\frac{2E}{\pi}, & d = 2E, & \frac{\pi}{2} \leq x \leq \pi \end{array}\right.$$

$$a_0 = 0 \;\; a_k,b_k \sim \frac{1}{\pi}\int_{-\pi}^{\pi} (cx+d)g(x)\,dx$$

Valores de a_k e b_k

$$a_k, b_k \sim \frac{2E}{\pi} \left\{ \int_{-\pi}^{-\pi/2} \left(-\frac{x}{\pi} - 1 \right) g(x)\, dx + \int_{-\pi/2}^{0} \frac{x}{\pi} g(x)\, dx \right.$$

$$\left. \int_{0}^{\pi/2} \frac{x}{\pi} g(x)\, dx + \int_{\pi/2}^{\pi} \left(-\frac{x}{\pi} + 1 \right) g(x)\, dx \right\}$$

$$a_k, b_k \sim \frac{2E}{\pi} \left\{ \int_{\pi}^{\pi/2} \left(\frac{x}{\pi} - 1 \right) g(-x)\, (-dx) + \int_{\pi/2}^{0} \frac{x}{\pi} g(-x)\, dx \right.$$

$$\left. \int_{0}^{\pi/2} \frac{x}{\pi} g(x)\, dx + \int_{\pi/2}^{\pi} \left(-\frac{x}{\pi} + 1 \right) g(x)\, dx \right\}$$

Valores de a_k e b_k

$$a_k, b_k \sim \frac{2E}{\pi}\left\{\int_{\pi/2}^{\pi}\left(\frac{x}{\pi}-1\right)g(-x)\,dx+\int_0^{\pi/2}\frac{-x}{\pi}g(-x)\,dx\right.$$

$$\left.\int_0^{\pi/2}\frac{x}{\pi}g(x)\,dx+\int_{\pi/2}^{\pi}\left(-\frac{x}{\pi}+1\right)g(x)\,dx\right\}$$

$$a_k, b_k \sim \frac{2E}{\pi}\left\{\int_{\pi/2}^{\pi}\left(\frac{x}{\pi}-1\right)[g(-x)-g(x)]\,dx+\right.$$

$$\left.\int_0^{\pi/2}\frac{x}{\pi}[g(x)-g(-x)]\,dx\right\}$$

Série de Fourier para uma onda triangular

$$g(x) = \begin{cases} 1, & \text{função par} \\ \cos(kx), & \text{função par} \\ \sin(kx), & \text{função impar} \end{cases}$$

$$g(x) - g(-x) = \begin{cases} 0, & \text{função par} \\ 2\sin(kx), & \text{função impar} \end{cases}$$

$$a_k, b_k \sim \frac{2E}{\pi} \left\{ \int_{\pi/2}^{\pi} \left(\frac{x}{\pi} - 1 \right) [g(-x) - g(x)] \, dx + \int_0^{\pi/2} \frac{x}{\pi} [g(x) - g(-x)] \, dx \right\}$$

$a_0 = a_k = 0$

Série de Fourier para uma onda triangular

$$b_k = \frac{4E}{\pi}\left\{-\int_{\pi/4}^{\pi/2}\left(\frac{x}{\pi}-1\right)\sin(kx)\,dx + \int_0^{\pi/2}\frac{x}{\pi}\sin(kx)\,dx\right\}$$

Integrando por partes $G'(x) = g(x)$

$$\int_a^b f(x)g(x)\;dx = f(b)G(b) - f(a)G(a) - \int_a^b f'(x)G(x)\;dx$$

$g(x) = sin(kx) \Rightarrow G(x) = -cos(kx)/k$

$$b_k = \frac{8E}{k^2\pi^2}\sin(k\pi/2) = \frac{8E}{k^2\pi^2}(-1)^{(k-1)/2},\ \text{k impar}$$

$$P(t) = \frac{8E}{\pi^2}\sum_{k=1,3,5,..}^{n}\frac{(-1)^{(k-1)/2}}{k^2}\sin(k\omega t)$$

Transformada de Fourier

Transformada directa $f(t) \to F(\omega)$

$$F(\omega) = TF[f(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

Transformada inversa $F(\omega) \to f(t)$

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega$$

Teorema da Convolução

$$\begin{aligned} TF[f(t)g(t)] &= \tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty} f(t)g(t)e^{-j\omega t}\, dt \\ &= \tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_0)\exp^{j\omega_0 t}\, d\omega_0\, g(t)e^{-j\omega t}\, dt \\ &= \tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_0)g(t)e^{-j(\omega-\omega_0)t}\, dt\, d\omega_0 \end{aligned}$$

$$TF[f(t)g(t)] = \int_{-\infty}^{\infty} F(\omega_0)G(\omega-\omega_0)d\omega_0 = \int_{-\infty}^{\infty} F(\omega)G(\omega_0-\omega)d\omega$$

Função delta de Dirac

Exemplos elementares:
Função δ de Dirac $\delta(x - x_0)$

$$\delta(x - x_0) = \begin{cases} 0, & x \neq x_0 \\ +\infty, & x = x_0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1, \;\; \int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = f(x_0)$$

Função $\delta(\omega) = \lim_{a\to 0} \delta_a(\omega)$

$$TF\left[e^{-at^2}\right] = \frac{1}{2\sqrt{\pi a}} e^{-\omega^2/(4a)} = \delta_a(\omega)$$

$$\delta(\omega) = \lim_{a\to 0} \frac{1}{2\sqrt{\pi a}} e^{-\omega^2/(4a)}$$

$$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} \delta_a(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi a}} e^{-\omega^2/(4a)}\, d\omega = 1$$

$$\lim_{a\to 0} TF\left[e^{-at^2}\right] = \lim_{a\to 0} \delta_a(\omega)$$

$$TF[1] = \delta(\omega)$$

$f(t) = e^{j\omega_0 t}$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\, dt = \delta(\omega - \omega_0)$$

Transformada inversa de $F(\omega)$

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega = e^{j\omega_0 t}$$

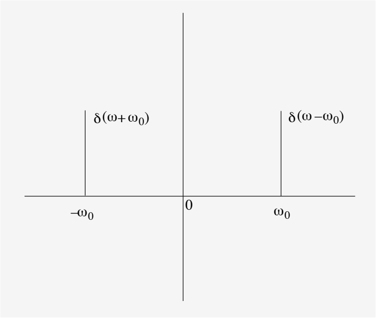
Exemplos elementares:

$$TF[\cos(\omega_o t)] = TF\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] = \frac{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)}{2}$$

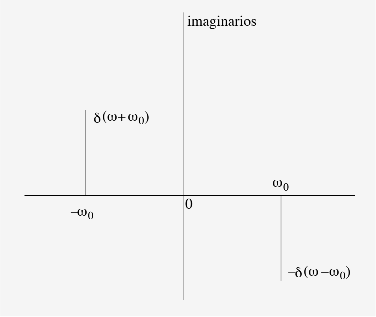
$$TF[\sin(\omega_o t)] = TF\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] = \frac{\delta(\omega - \omega_0) - \delta(\omega + \omega_0)}{2j}$$

$$TF[\sin(\omega_o t)] = \frac{-j\delta(\omega - \omega_0) + j\delta(\omega + \omega_0)}{2}$$

$$TF[\cos(\omega_o t)] = \frac{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)}{2}$$



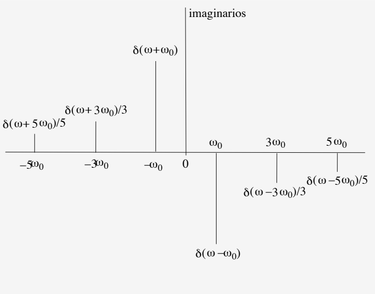
$$TF[\sin(\omega_o t)] = \frac{-j\delta(\omega - \omega_0) + j\delta(\omega + \omega_0)}{2}$$



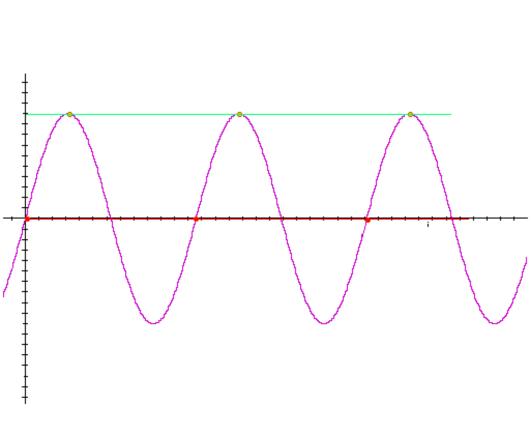
Onda quadrada

$$P(t) = \frac{4E}{\pi} \sum_{k=1,3,5,..}^{n} \frac{1}{k} \sin(k\omega_0 t)$$

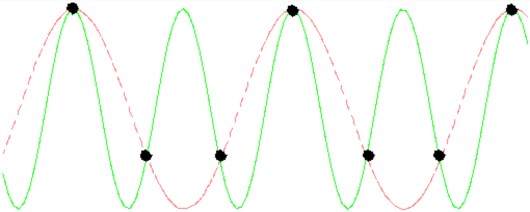
$$TF[P(t)] = \frac{2E}{\pi} \sum_{k=1,3,5,..}^{n} \frac{-j\delta(\omega - k\omega_0) + j\delta(\omega + k\omega_0)}{k}$$



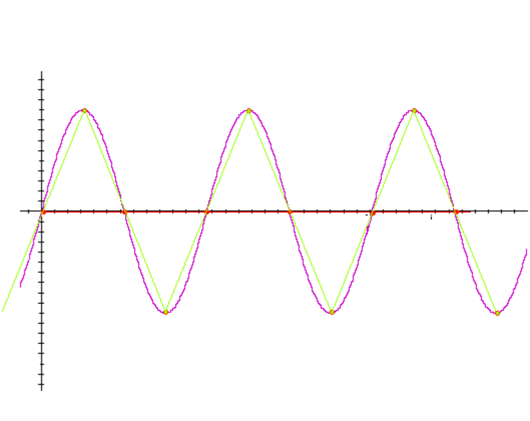
$$f_a = f$$



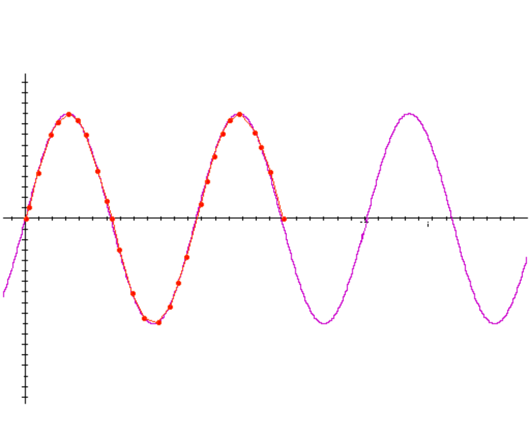
$f_a = 1.5f$



$f_a = 2f$ *(frequência crítica de Nyquist)*



$f_a > 2f \;\; (oversampling)$



Transformada discreta de Fourier

N pontos $h_k = h(t_k)$ obtidos a intervalos de tempo $t_k = k\Delta$, $(k = 0{,}1{,}2{,}3{,}...)$
N pontos $H_n = H(f_n)$ com

$$f_n = \frac{n}{N\Delta} \text{ com } n = -\frac{N}{2},...\frac{N}{2}$$

$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f_n t}\, dt \approx$$

$$\approx \sum_{k=0}^{N-1} h_k e^{-2j\pi f_n t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k e^{-2j\pi kn/N} = \Delta H_n$$

$$f_{N/2} = \frac{1}{2\Delta}$$

A frequência de amostragem $1/\Delta$ tem de ser pelo menos igual ao dobro da frequência máxima do espectro do sinal.
Temos de escolher bem o valor de Δ e de N

Transformada rápida de Fourier

O número de passos para calcular

$$H_n = \sum_{k=0}^{N-1} h_k e^{-2j\pi kn/N}$$

é da ordem de N^2
Com os algorítmos de FFT o número decresce para $N \log_2 N$

(Numerical Recipes in C, Cambrigde University Press)