Estudo do Condensador

-Carga e descarga do condensador num circuito RC
-Variação com a frequência da permitividade elétrica de um dielétrico com perdas

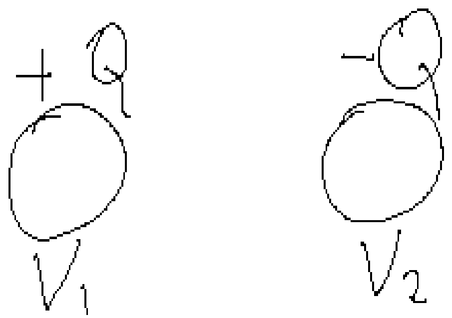
LET 2020

Sumário

- Conceitos básicos
- Tipos de condensadores
- Carga e descarga do condensador num circuito RC
- Campo elétrico na presença de matéria
- Análise da permitividade elétrica, equação de Clausius-Mossoti
- Modelo elementar para a polarização eletrónica
- Permitividade elétrica complexa e condensador com perdas

Estudo do Condensador, conceitos básicos

Dispositivo composto por 2 condutores em presença:



$$C = \frac{Q}{V_1 - V_2}$$

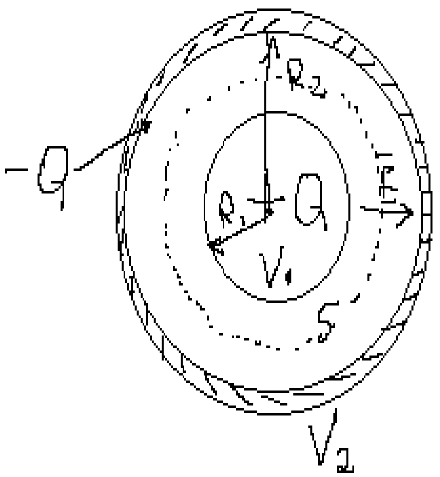
O condensador armazena carga e energia elétrica:

$$W_e = \frac{1}{2}\sum_{i=1}^{2} Q_i V_i = \frac{1}{2}(Q_1 V_1 + Q_2 V_2) = \frac{1}{2} Q (V_1 - V_2) = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C (V_1 - V_2)^2$$

Diferentes tipos de Condensadores

Condensador esférico

$$C = \frac{Q}{V_1 - V_2}$$



$$\oint_S \vec{E} \cdot \vec{N} ds = \frac{Q}{\varepsilon_0} \quad (\text{T Gauss})$$

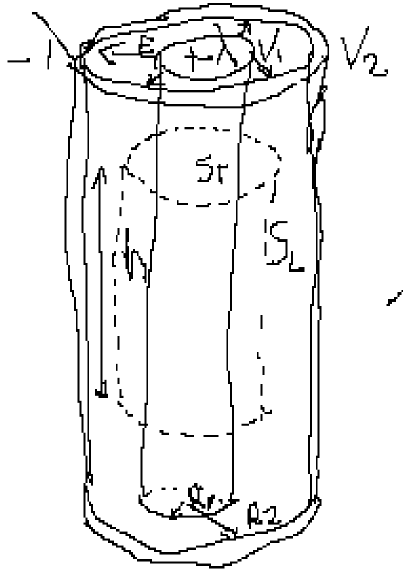
$$E \cdot S = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2}$$

$$V_1 - V_2 = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{r} = \int_{R_1}^{R_2} E \cdot dr = \frac{Q}{4\pi\varepsilon_0} \left[-\frac{1}{R_2} + \frac{1}{R_1} \right]$$

$$C = \frac{Q}{\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]} = \frac{4\pi\varepsilon_0}{\frac{1}{R_1} - \frac{1}{R_2}}$$

Condensador Cilíndrico



$$C/L = \frac{Q/L}{V_1 - V_2}$$

$$\int_{S_L + S_T} \bar{E}\,\bar{N}\,dS = \frac{\lambda h}{\varepsilon_0}$$

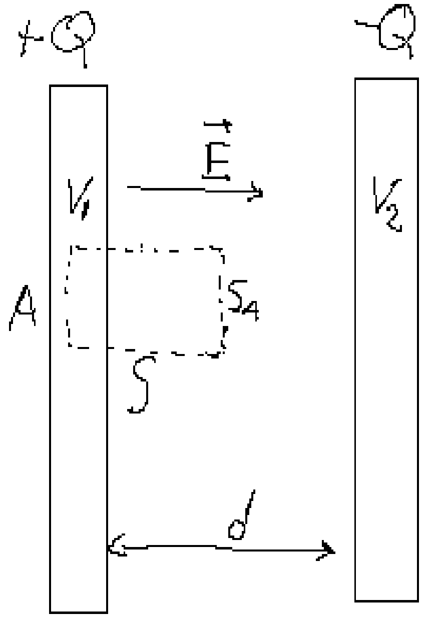
$$E S_L = \frac{\lambda h}{\varepsilon_0}$$

$$E = \frac{\lambda \cancel{h}}{2\pi r \cancel{h}\, \varepsilon_0}$$

$$V_1 - V_2 = \int_{R_1}^{R_2} \vec{E}\, d\vec{R} = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{R_2}{R_1}\right)$$

$$C/L = \frac{\lambda}{\frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{R_2}{R_1}\right)} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{R_2}{R_1}\right)}$$

Condensador plano



$$\sigma = \frac{Q}{A}$$

$$C = \frac{Q}{V_1 - V_2}$$

$$\int_S \bar{E}\,\bar{N}\,dS = \frac{\sigma S_A}{\varepsilon_0}$$

$$E S_A = \frac{\sigma S_A}{\varepsilon_0}$$

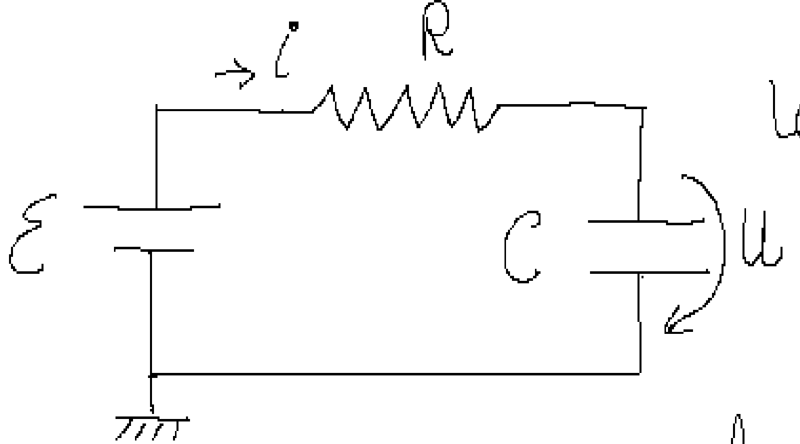
$$E = \frac{\sigma}{\varepsilon_0}$$

$$V_1 - V_2 = \int_1^2 \vec{E}\, d\vec{l} = E d$$

$$V_1 - V_2 = \frac{\sigma d}{\varepsilon_0}$$

$$C = \frac{\sigma A}{\frac{\sigma d}{\varepsilon_0}} = \frac{\varepsilon_0 A}{d}$$

Carga e descarga do condensador num circuito RC



$$C = \frac{Q}{U} \qquad i = \frac{dQ}{dt} = C\frac{dU}{dt}$$

Lei de Kirchoff:

$$\varepsilon = Ri + U \rightarrow \varepsilon = RC\frac{dU}{dt} + U$$

Eq. homogénea $0 = RC\frac{dU}{dt} + U \rightarrow U = U_0 e^{\lambda t} = U_0 e^{-\frac{t}{RC}}$

Eq. Característica $0 = RC\lambda + 1$

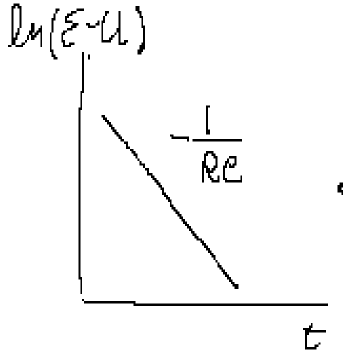
$\rightarrow \lambda = -\frac{1}{RC}$

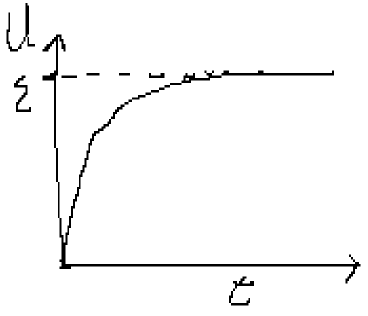
$$U = U_0 e^{-t/RC} + \varepsilon$$

$$U(0) = 0 \Rightarrow U_0 = -\varepsilon$$

$$U = \varepsilon\left(1 - e^{-t/RC}\right)$$

$$i = C\frac{dU}{dt} = \frac{\cancel{C}\varepsilon}{R\cancel{C}} e^{-t/RC}$$





Balanço energético

$$W_\varepsilon = \int_0^\infty \varepsilon i\, dt = \varepsilon \int_0^\infty i\, dt =$$

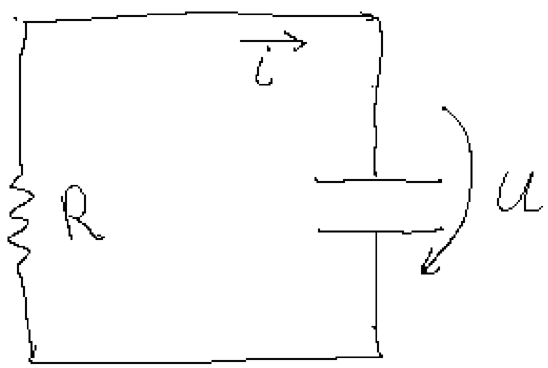
$$= \frac{\varepsilon^2}{R}\int_0^\infty e^{-t/RC} dt = \frac{\varepsilon^2}{R}\left.\frac{e^{-t/RC}}{-\frac{1}{RC}}\right|_0^\infty = \underline{\varepsilon^2 C}$$

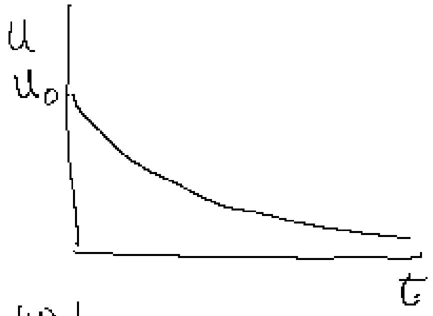
$$W_C = \frac{1}{2}CU^2(t=\infty) = \underline{\frac{1}{2}C\varepsilon^2}$$

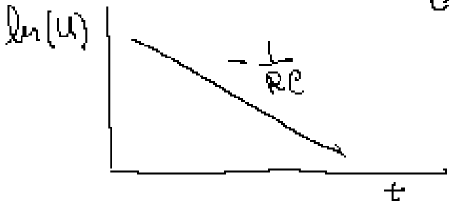
$$W_R = \int_0^\infty R i^2 dt = R\frac{\varepsilon^2}{R^2}\int_0^\infty e^{-\frac{2t}{RC}} dt = \underline{\frac{1}{2}C\varepsilon^2}$$

$$\boxed{W_\varepsilon = W_C + W_R}$$

Descarga do condensador num circuito RC







$$U = -Ri = -R\frac{dQ}{dt}$$

$$U = -RC\frac{dU}{dt}$$

$$RC\frac{dU}{dt} + U = 0 \rightarrow U = U_0 e^{-\frac{t}{RC}}$$

$$RCs + 1 = 0$$

$$s = -\frac{1}{RC}$$

$$\ln(U) = -\frac{t}{RC} + \ln(U_0)$$

$$i = C\frac{dU}{dt} = -\frac{U_0}{R} e^{-t/RC}$$

Balanço energético

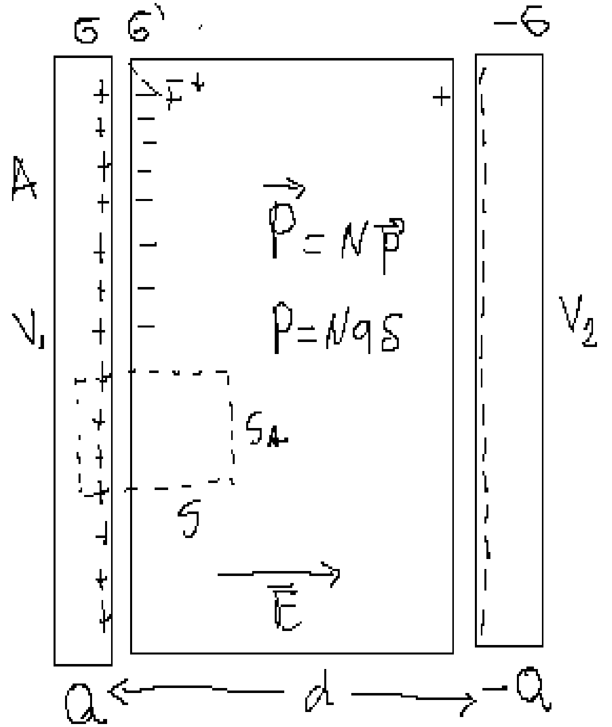
$$W_C = \frac{1}{2} C U_0^2$$

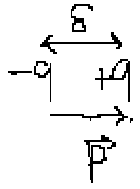
$$W_R = \int_0^\infty R i^2 dt = R\frac{U_0^2}{R^2}\int_0^\infty e^{-2t/RC} dt$$

$$W_R = \frac{1}{2} C U_0^2$$

$$\boxed{W_C = W_R}$$

Campo elétrico na presença de matéria, condensador plano com dielétrico





$p = q\delta$

$$\sigma' = \frac{Q'}{A} = \frac{-qn}{A} = \frac{-qN\delta A}{A} = -qN\delta$$

$$\sigma' = -P \qquad \sigma' = \vec{P}\cdot\vec{N}$$

$$E S_A = \frac{\sigma + \sigma'}{\varepsilon_0} S_A \rightarrow E = \frac{\sigma - P}{\varepsilon_0}$$

$$\varepsilon_0 E + P = \sigma \rightarrow \varepsilon_0 \vec{E} + \vec{P} = \vec{D}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \rightarrow \vec{D} = (\varepsilon_0 + \varepsilon_0 \chi_e)\vec{E} = \varepsilon \vec{E} = \varepsilon_0(1+\chi_e)\vec{E}$$

$\chi_e \rightarrow$ Susceptibilidade elétrica (adimensional)

$\varepsilon = \varepsilon_0(1+\chi_e) \rightarrow$ Permitividade elétrica ($F m^{-1}$)

$$E = \frac{\sigma - \varepsilon_0 \chi_e E}{\varepsilon_0} \rightarrow \varepsilon_0 E + \varepsilon_0 \chi_e E = \sigma \rightarrow E = \frac{\sigma}{\varepsilon_0(1+\chi_e)} = \frac{\sigma}{\varepsilon} \rightarrow C = \frac{Q}{V_1 - V_2} = \frac{\sigma A}{E d} = \frac{\varepsilon A}{d}$$

$$\varepsilon E = \sigma = D$$

Análise da permitividade elétrica

$$\varepsilon = \varepsilon_0(1+\chi_e) \quad \vec{P} = \varepsilon_0 \chi_e \vec{E}\,, \quad \vec{P} = N\vec{p}\,, \quad \vec{p} = \alpha \vec{E}_i$$

campo local

número de dipolos por unidade de volume

polarizabilidade molecular

Relação entre E_i e E

Sistemas pouco densos (gases)

$E_i = E$

Sistemas densos (líquidos e sólidos)

$$E_i = E + \frac{P}{3\varepsilon_0}$$

$$P = N\alpha E_i = \begin{cases} N\alpha E \Rightarrow \varepsilon_0 \chi_e = N\alpha \Rightarrow \varepsilon = \varepsilon_0 + N\alpha \quad (\text{gases}) \\[2ex] N\alpha\left(E + \dfrac{P}{3\varepsilon_0}\right) \Rightarrow P = \dfrac{N\alpha}{1-\dfrac{N\alpha}{3\varepsilon_0}} E \Rightarrow \varepsilon_0\chi_e = \dfrac{N\alpha}{1-\dfrac{N\alpha}{3\varepsilon_0}} \Rightarrow \varepsilon = \varepsilon_0 + \dfrac{N\alpha}{1-\dfrac{N\alpha}{3\varepsilon_0}} \end{cases}$$

Eq. de Clausius-Mossotti ou Lorentz-Lorenz

Fontes de polarização: $\begin{cases} \text{eletrônica} \\ \text{orientação de dipolos} \\ \text{iônica} \end{cases}$

Análise da Permitividade elétrica

Modelo elementar para a polarização eletrônica:

Eq. movimento p/ electrão no potencial do átomo

$$m\frac{d^2x}{dt^2}+\gamma'\frac{dx}{dt}+kx=qE_i \qquad E_i=\mathrm{Re}\left\{E_{i0}\,e^{i(\omega t+\phi_E)}\right\}=\mathrm{Re}\left\{\bar{E}_i\,e^{i\omega t}\right\} \qquad \bar{E}_i=E_{i0}\,e^{i\phi_E}$$

$$\frac{d^2x}{dt^2}+\frac{\gamma'}{m}\frac{dx}{dt}+\frac{k}{m}x=\frac{q}{m}E_i \qquad x=\mathrm{Re}\left\{x_0\,e^{i(\omega t+\phi_x)}\right\}=\mathrm{Re}\left\{\bar{x}\,e^{i\omega t}\right\} \qquad \bar{x}=x_0\,e^{i\phi_x}$$

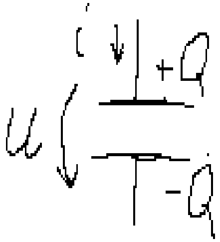
$$-\omega^2\bar{x}+i\omega\gamma\bar{x}+\omega_0^2\bar{x}=\frac{q}{m}\bar{E}_i \rightarrow \bar{x}=\frac{q/m\,\bar{E}_i}{\omega_0^2-\omega^2+i\omega\gamma}, \quad \bar{p}=q\bar{x}=\underbrace{\frac{q^2/m}{\omega_0^2-\omega^2+i\omega\gamma}}_{\alpha}\bar{E}_i$$

$$\gamma=\frac{\gamma'}{m}, \quad \omega_0^2=\frac{k}{m}$$

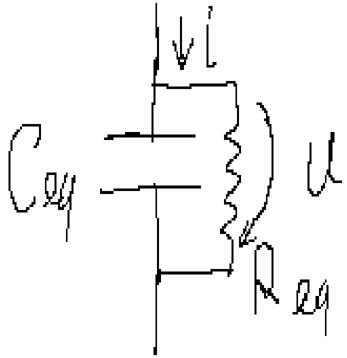
$$\alpha=\alpha(\omega)=\frac{q^2/m}{\omega_0^2-\omega^2+i\omega\gamma} \quad (\text{complexo}) \Rightarrow \varepsilon=\varepsilon_0+\frac{N\alpha(\omega)}{1-\frac{N\alpha(\omega)}{3\varepsilon_0}}=\varepsilon(\omega)\rightarrow \text{complexo}$$

Condensador plano com dielétrico de permitividade elétrica complexa

$$C = \frac{\varepsilon A}{d} = \frac{A}{d}(\varepsilon_r + j\varepsilon_i) = \frac{A}{d}\varepsilon_r + j\frac{A}{d}\varepsilon_i = C_r + jC_i \qquad j \equiv \sqrt{-1}$$



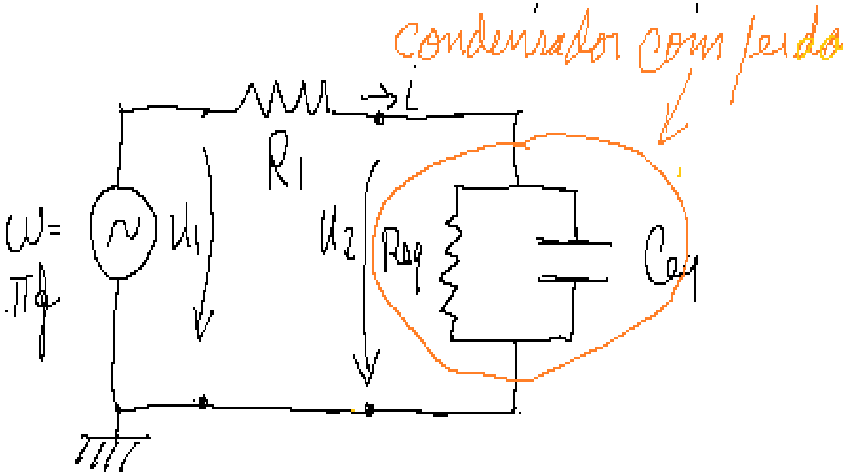
$$i = \frac{dQ}{dt} = C\frac{du}{dt} \rightarrow \bar{I} = j\omega\bar{Q} = j\omega C\bar{U} = j\omega\bar{U}(C_r + jC_i) = j\omega C_r\bar{U} - \omega C_i\bar{U}$$



$$i = \frac{u}{R_{eq}} + \frac{dQ}{dt} = \frac{u}{R_{eq}} + C_{eq}\frac{du}{dt} \rightarrow \bar{I} = \frac{\bar{U}}{R_{eq}} + j\omega C_{eq}\bar{U}$$

$$\rightarrow \begin{cases} C_{eq} = C_r \\ \dfrac{1}{R_{eq}} = -\omega C_i \end{cases} \rightarrow \begin{cases} \varepsilon_r \dfrac{A}{d} = C_{eq} \\ \varepsilon_i \dfrac{A}{d} = -\dfrac{1}{\omega R_{eq}} \end{cases}$$

Determinação experimental de $\varepsilon(\omega) = \varepsilon_r + j\varepsilon_i$ com condensador plano



$$\begin{cases} u_1 = R_1 i_1 + u_2 \\ i_1 = \dfrac{u_2}{R_{eq}} + C_{eq}\dfrac{du_2}{dt} \end{cases} \rightarrow \begin{cases} \dfrac{\bar{U}_1 - \bar{U}_2}{R_1} = \bar{I}_1 \\ \bar{I}_1 = \dfrac{\bar{U}_2}{R_{eq}} + j\omega C_{eq}\bar{U}_2 \end{cases} \quad \left\{ \dfrac{\bar{U}_1 - \bar{U}_2}{R_1} = \dfrac{\bar{U}_2}{R_{eq}} + j\omega C_{eq}\bar{U}_2 \right.$$

$$\frac{\bar{U}_1}{R_1} = \bar{U}_2\left(\frac{1}{R_1} + \frac{1}{R_{eq}} + j\omega C_{eq}\right) \rightarrow \frac{\bar{U}_1}{\bar{U}_2} = 1 + \frac{R_1}{R_{eq}} + j\omega R_1 C_{eq}$$

$$\frac{U_{1ef}}{U_{2ef}} = \left|\frac{\bar{U}_1}{\bar{U}_2}\right| = \sqrt{\left(1+\frac{R_1}{R_{eq}}\right)^2 + \omega^2 R_1^2 C_{eq}^2} \rightarrow C_{eq} = \frac{1}{R_1\omega}\sqrt{\left(\frac{U_{1ef}}{U_{2ef}}\right)^2 - \left(1+\frac{R_1}{R_{eq}}\right)^2} \qquad (2)$$

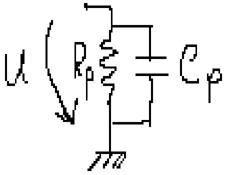
Potência dissipada no condensador com perdas: P:

$$P = \langle u_2 i \rangle = \left\langle \frac{u_2^2}{R_{eq}} \right\rangle = \frac{U_{2ef}^2}{R_{eq}} = \left\langle u_2 \frac{u_1 - u_2}{R_1} \right\rangle = \frac{\langle u_2 u_1 \rangle - \langle u_2^2 \rangle}{R_1} \rightarrow$$

$$\rightarrow R_{eq} = R_1 \frac{U_{2ef}^2}{\langle u_1 u_2 \rangle - U_{2ef}^2} \qquad (1)$$

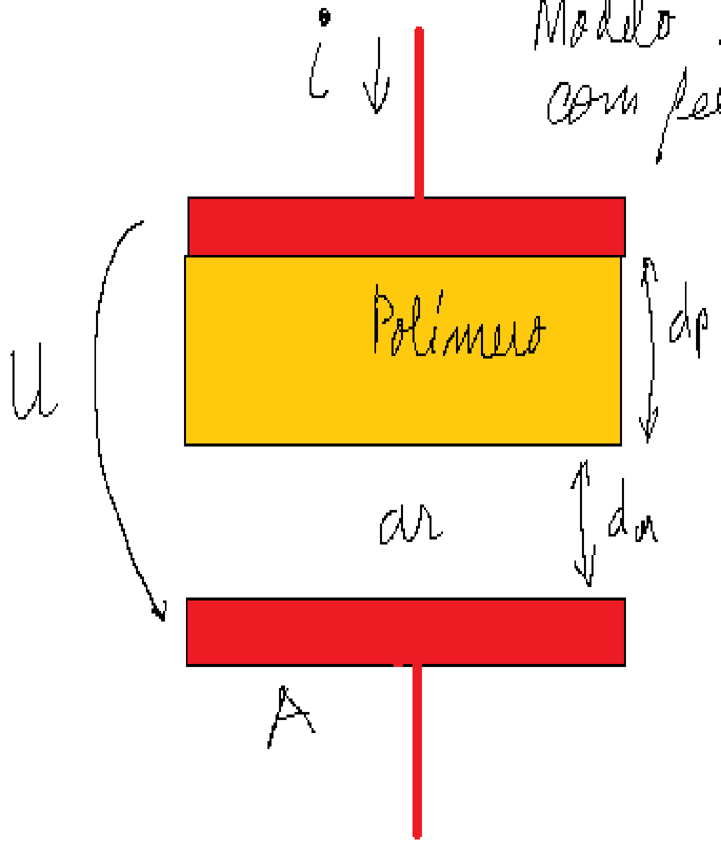
$$\langle x \rangle \equiv \frac{1}{T} \int_0^T x \, dt \,, \qquad X_{ef}^2 \equiv \frac{1}{T} \int_0^T x^2 \, dt$$

Nota: Circuito equivalente da ponta de prova do osciloscópio:

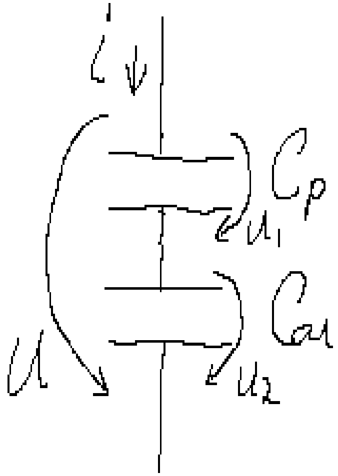


$R_p = 1\,M\Omega$, $C_p = 160\,pF$

Modelo mais completo p/ descrever o condensador com perdas em estudo no lab.



$\equiv$



$$i = C_T \frac{du}{dt} = C_T\left(\frac{du_1}{dt}+\frac{du_2}{dt}\right), \quad i = C_p \frac{du_1}{dt}$$

$$i = C_{ar}\frac{du_2}{dt}$$

$$i = C_T\left(\frac{i}{C_p}+\frac{i}{C_{ar}}\right)$$

$$\frac{1}{C_T}=\frac{1}{C_p}+\frac{1}{C_{ar}} \rightarrow \frac{1}{C_{Tr}+jC_{Ti}} - \frac{1}{C_{ar}} = \frac{1}{C_{pr}+jC_{pi}}$$

$$C_{pr}+jC_{pi} = \frac{C_{ar}(C_{Tr}+jC_{Ti})(C_{ar}-C_{Tr}+jC_{Ti})}{(C_{ar}-C_{Tr})^2+C_{Ti}^2}$$

$$C_{ar}=\varepsilon_0 \frac{A}{d_{ar}}$$

$$C_{Tr}=C_{eq}$$

$$C_{Ti}=-\frac{1}{\omega R_{eq}}$$

$$\begin{cases} C_{pr}=\varepsilon_{rp}\dfrac{A}{d_p} \\ C_{pi}=\varepsilon_{ip}\dfrac{A}{d_p} \end{cases}$$

FIM