TÉCNICO LISBOA

Circuit Theory and Electronics Fundamentals

Lecture 9: Second Order Circuits

- LC loop: natural solution,
- LC loop: energy swing
- RLC series: natural solution for 2 real roots, 1 real root, 2 complex roots
- RLC circuits: other configurations
- RLC circuits: forced solution
- Other second order circuits

LC loop

$$i = C\frac{dv}{dt}$$ Capacitor law

$$v = -L\frac{di}{dt}$$ Inductor law

$$-i + C\frac{dv}{dt} = 0$$ KCL

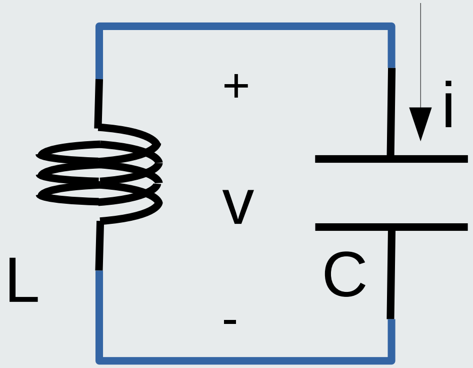
$$i + LC\frac{d^2 i}{dt^2} = 0$$ 2nd order Linear Ordinary Differential Equation (LODE)

$$1 + LC\,s^2 = 0$$ Characteristic equation

$$s_{1,2} = \pm j\frac{1}{\sqrt{LC}} = \pm j\,\omega_n$$ Natural frequencies are purely imaginary

$$\omega_n = \frac{1}{\sqrt{LC}}$$ Resonant frequency

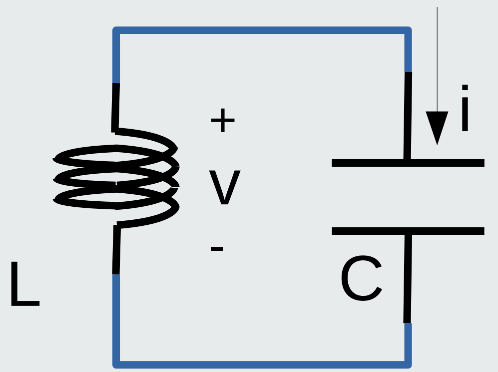
$$\begin{cases} i(t) = A_1 e^{j\omega_n t} + A_2 e^{-j\omega_n t} \\ v(t) = -L A_1 s_1 e^{j\omega_n t} + L A_2 s_2 e^{-j\omega_n t} \end{cases}$$ General solutions for current and voltage



LC loop: integration constants from initial conditions

$$i(t) = A_1 e^{j\omega_n t} + A_2 e^{-j\omega_n t}$$

$$v(t) = -L A_1 j\omega_n e^{j\omega_n t} + L A_2 j\omega_n e^{-j\omega_n t}$$



$$i(0) = A_1 + A_2 \qquad (1)$$

$$v(0) = -j\sqrt{\frac{L}{C}} A_1 + j\sqrt{\frac{L}{C}} A_2$$

$$j\, v(0)\sqrt{\frac{C}{L}} = A_1 - A_2 \qquad (2)$$

$$A_2 = A_1^* = \frac{A}{2} e^{j\alpha}$$

$$A = \sqrt{i(0)^2 + v(0)^2}$$

$$\alpha = \arctan\left(\frac{v(0)}{i(0)}\right)$$

$$\begin{cases} A_1 = \frac{1}{2}(i(0) + jv(0)) = \frac{A}{2} e^{j\alpha} \\ A_2 = \frac{1}{2}(i(0) - jv(0)) = \frac{A}{2} e^{-j\alpha} \end{cases}$$

Sum (1) and (2)

Subtract (1) and (2)

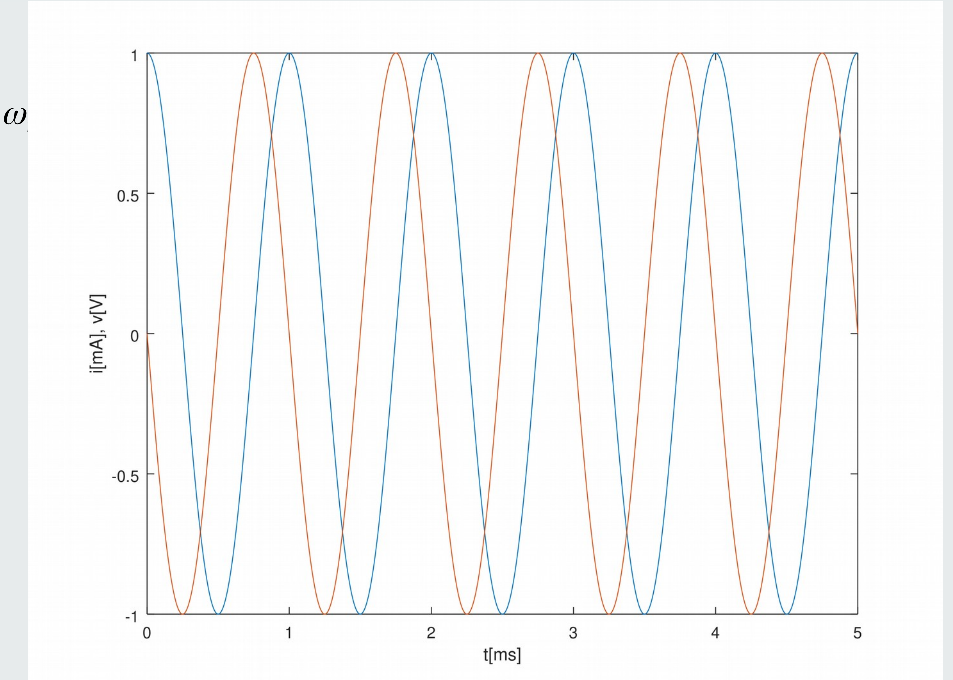
Natural solution plots

$$\begin{cases} i(t) = A_1 e^{j\omega_n t} + A_2 e^{-j\omega_n t} \\ v(t) = -L A_1 j \omega_n e^{j\omega_n t} + L A_2 j \omega_n e^{-j\omega} \end{cases}$$

$$\begin{cases} A_1 = \dfrac{A}{2} e^{j\alpha} \\ A_2 = \dfrac{A}{2} e^{-j\alpha} \end{cases}$$

$$\begin{cases} i(t) = A \cos(\omega_n t + \alpha) \\ v(t) = A \sqrt{\dfrac{L}{C}} \sin(\omega_n t + \alpha) \end{cases}$$

$$\begin{cases} i(0) = 1 \\ v(0) = 0 \end{cases}$$



- Current delayed by 90º vs voltage
- Maximum current (voltage) corresponds to minimum voltage (current)

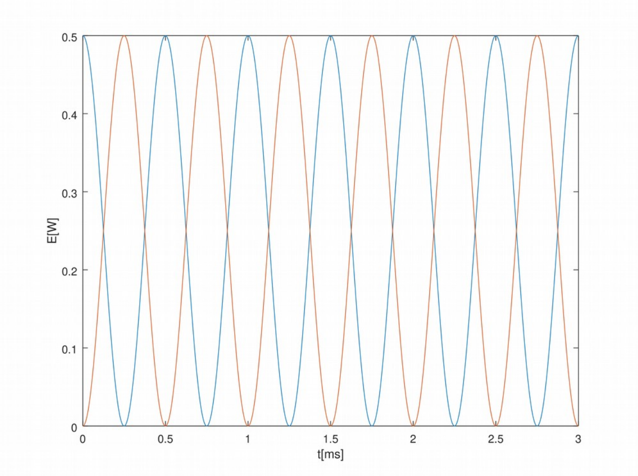
LC loop energy swing

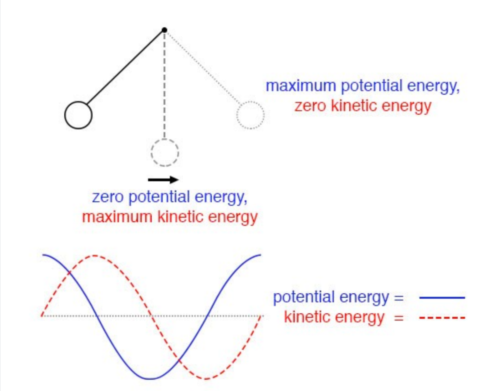
$$\begin{cases} E_L(t) = \frac{1}{2} L i^2(t) \\ E_C(t) = \frac{1}{2} C v^2(t) \end{cases}$$

$$\begin{cases} E_L(t) = \frac{1}{2} L A^2 \cos^2(\omega t) \\ E_C(t) = \frac{1}{2} C A^2 \frac{L}{C} \sin^2(\omega t) \end{cases}$$

$$\begin{cases} max\, E_L = \frac{1}{2} L A^2 \\ max\, E_C = \frac{1}{2} L A^2 \end{cases}$$

$$E_L + E_C = \frac{1}{2} L A^2$$

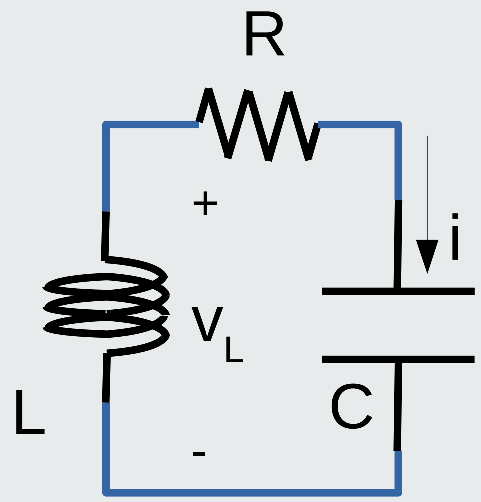




<u>Mechanical analogy</u>
Kinetic energy – current
Potential energy – voltage

- Energy swings between C and L
- Energy sum is constant
- When potential energy is max in C (L), it is null in L (C)

RLC series circuit



$$-v_L + R\,i + v_C = 0$$ KVL

$$L\frac{di}{dt} + R\,i + v_C = 0$$ Inductor law

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\,i = 0$$ Apply derivative to both sides

$$s^2 + \frac{R}{L}\,s + \frac{1}{LC} = 0$$ Characteristic equation

$$s_{1,2} = -\frac{1}{2}\frac{R}{L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$ Natural (eventually) complex frequencies

3 cases: 2 real roots; 1 real root; 2 complex conjugate roots!

RLC series circuit: 2 real frequencies

$$s_{1,2} = -\frac{1}{2}\frac{R}{L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

Natural frequencies

$$\frac{R^2}{L^2} > \frac{4}{LC}$$

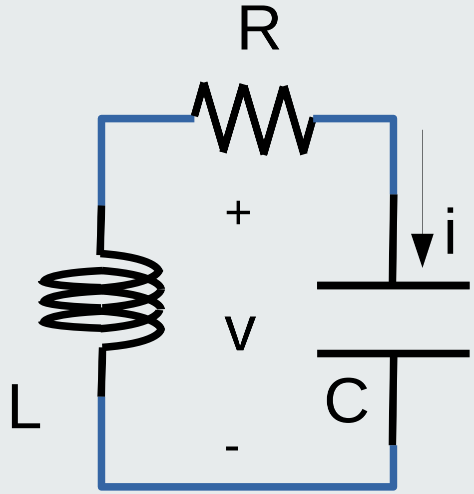
Case of real negative roots

$$i(t) = Ae^{s_1 t} + Be^{s_2 t}$$

General solution.

$$\begin{cases} i_L(0) = A + B \\ v_C(0) = -Ri_L(0) - L\dfrac{di_L}{dt}(0) \end{cases}$$

A and B can be determined by initial conditions



$$\begin{cases} A = \dfrac{(R + Ls_2)\, i_L(0) + v_C(0)}{L(s_1 - s_2)} \\ B = \dfrac{(R + Ls_1)\, i_L(0) + v_C(0)}{L(s_2 - s_1)} \end{cases}$$

<u>Over damped solution</u>

RLC series circuit: 1 real frequency

$$s_{1,2} = -\frac{1}{2}\frac{R}{L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

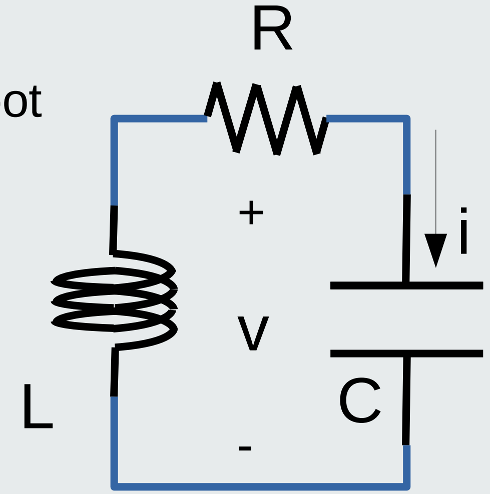
Natural frequencies

$$\frac{R^2}{L^2} = \frac{4}{LC}$$

Case of single real negative root

$$i(t) = (A + Bt)e^{s_1 t}$$

General solution.



$$\begin{cases} i_L(0) = A \\ v_C(0) = -Ri_L(0) - L\dfrac{di_L}{dt}(0) \end{cases}$$

A and B can be determined by initial conditions

$$\begin{cases} A = i_L(0) \\ B = \dfrac{(R + Ls_1)\, i_L(0) + v_C(0)}{-L} \end{cases}$$

<u>Critically damped solution</u>

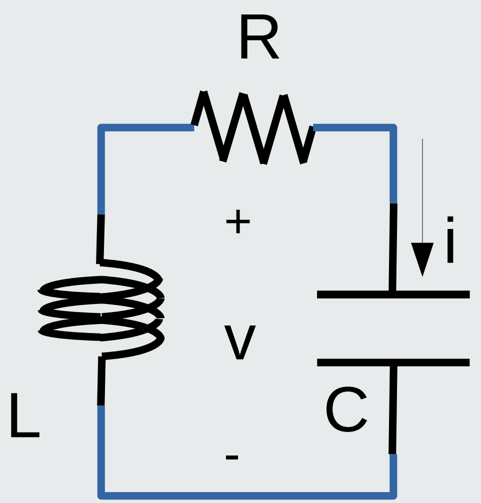
RLC series circuit: 2 complex conjugate frequencies

$$s_{1,2}=-\frac{1}{2}\frac{R}{L}\pm\frac{1}{2}\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}$$

Natural complex frequencies

$$\frac{R^2}{L^2}<\frac{4}{LC}$$

Case of complex conjugate roots



$$\alpha=\frac{R}{2}L \quad \omega_n=\frac{1}{2}\sqrt{\frac{4}{LC}-\frac{R^2}{L^2}}$$

α is ***damping factor***

ω_n is ***oscillating frequency***

$$i(t)=Ae^{-\alpha t}\cos(\omega_n t+B)$$

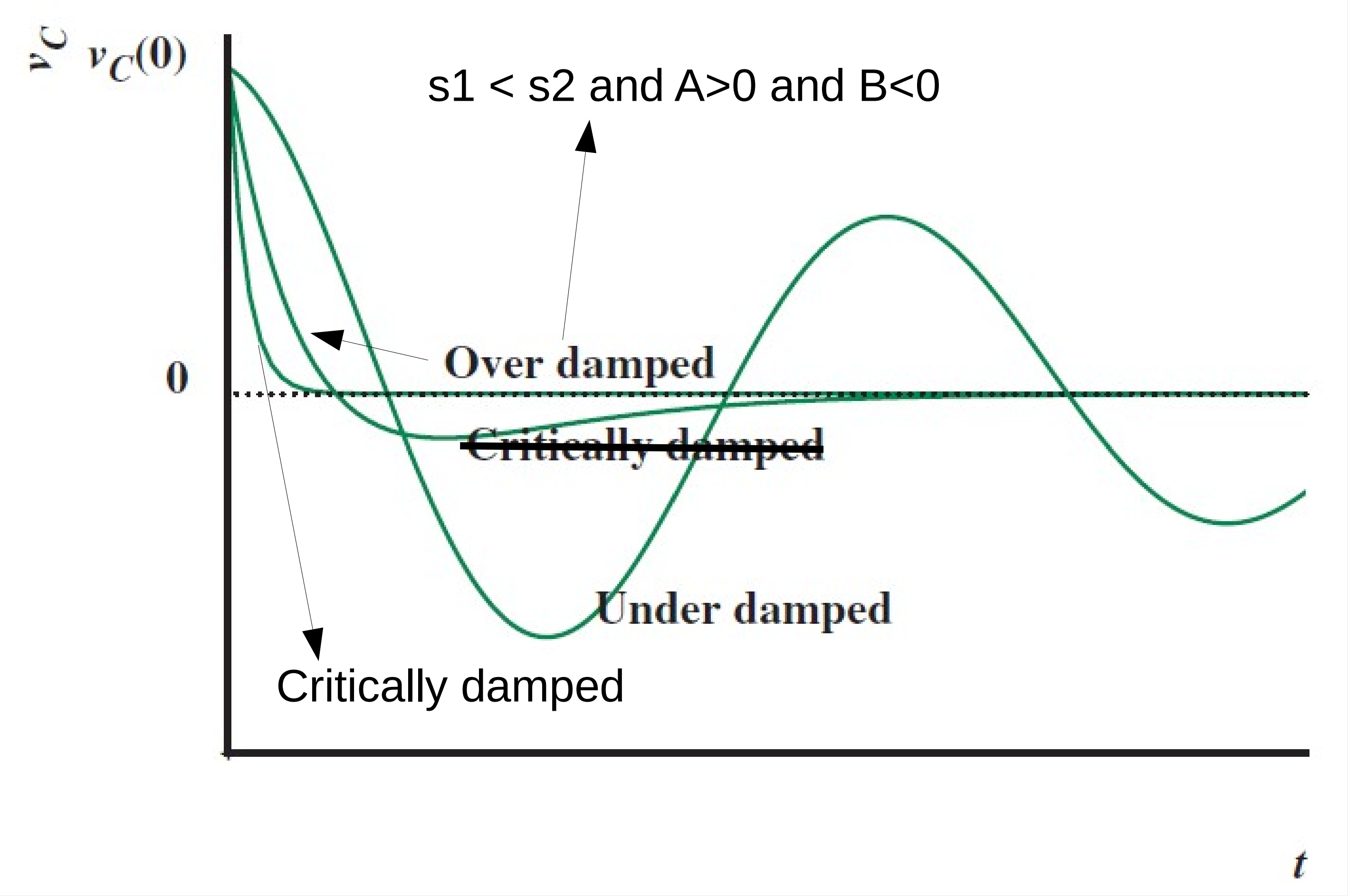
General solution.

$$\begin{cases} i_L(0)=A\cos(B) \\ v_C(0)=-Ri_L(0)-L\dfrac{di_L}{dt}(0) \end{cases}$$

Under damped solution

A and B can be determined by initial conditions

RLC series circuit plots



RLC circuits: other configurations

$$i = C\frac{dv}{dt}$$ Capacitor law

$$v = Ri$$ Ohm's law

$$v = L\frac{di}{dt}$$ Inductor law

$$i_L + \frac{v}{R} + i_C = 0$$ KCL

$$i_L + \frac{L}{R}\frac{di_L}{dt} + C\frac{dv}{dt} = 0$$

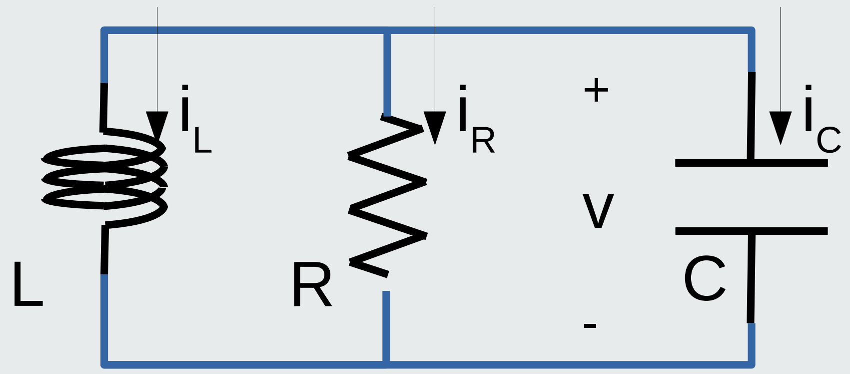
$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = 0$$

2nd order Linear Ordinary Differential Equation (LODE)

$$1 + \frac{L}{R}s + LC\,s^2 = 0$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Characteristic equation



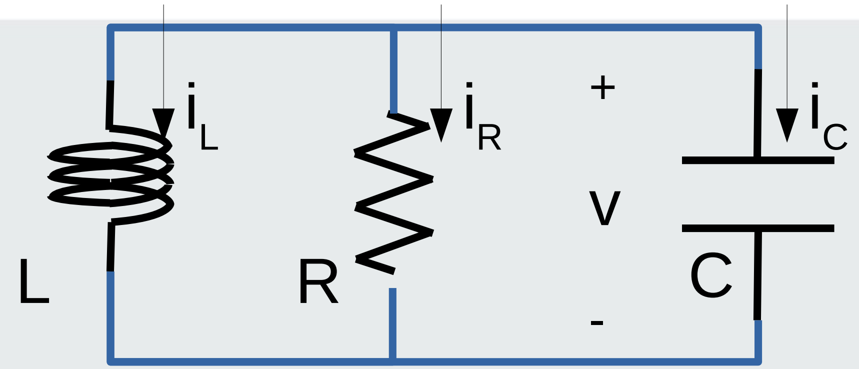
RLC parallel circuit

RLC parallel circuit

Characteristic equation

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

$$s_{1,2}=-\frac{1}{2RC}\pm\frac{1}{2}\sqrt{\frac{1}{R^2C^2}-\frac{4}{LC}}$$



RLC parallel circuit

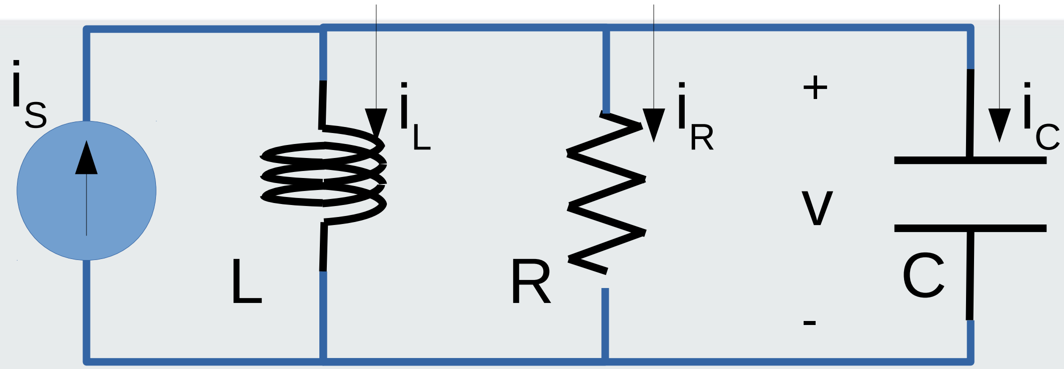
$$i_L(t)=Ae^{s_1t}+Be^{s_2t}$$ Solution for 2 real roots

$$i_L(t)=(A+Bt)e^{s_1t}$$ Solution for 1 real root

$$i_L(t)=Ae^{-\alpha t}\cos(\omega_n t+B)$$ Solution for 2 complex conjugate roots

Find A and B using initial conditions $i_L(0)$ and $v_C(0)$

RLC circuits: forced solution



RLC parallel circuit

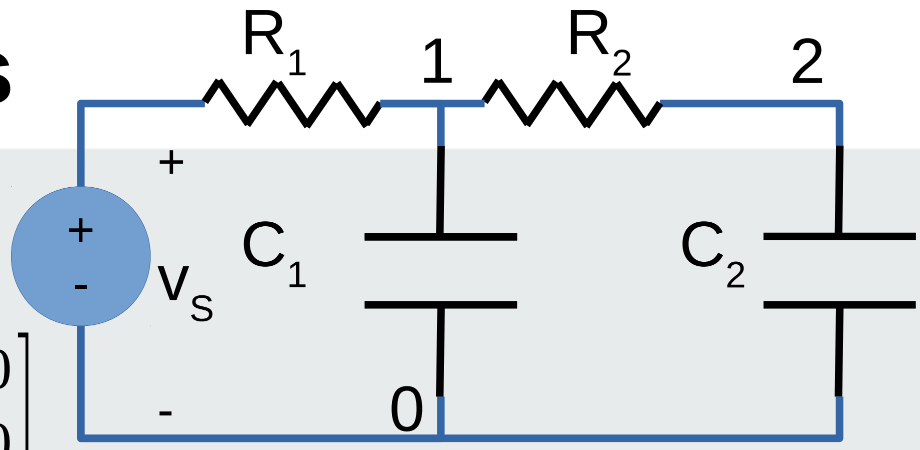
$$\tilde{I}_L + \frac{\tilde{V}}{R} + \tilde{I}_C = \tilde{I}_S$$ KCL

$$\frac{\tilde{V}}{j\omega L} + \frac{\tilde{V}}{R} + \frac{\tilde{V}}{\frac{1}{j\omega C}} = \tilde{I}_S$$ Ohm's

$$\tilde{V} = \frac{1}{\frac{1}{j\omega L} + \frac{1}{R} + j\omega C}\tilde{I}_S$$ Phasor form

$$v(t) = \Re\left\{\tilde{V} e^{j\omega t}\right\}$$
$$v(t) = |\tilde{V}|\cos\left(\omega t + \sphericalangle \tilde{V}\right)$$ Time solution

Other second order circuits



Second order RC circuit

KCL:

$$\begin{bmatrix} G_1+G2+C_1\frac{d}{dt} & -G2 \\ -G2 & G2+C_2\frac{d}{dt} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\frac{d}{dt}X=AX$$

First order matrix LODE

Other second order circuits

$$\frac{d}{dt}X = A\,X$$ First order matrix LODE

$$det\left(A - sI\right) = 0$$ Characteristic equation (I is identity matrix)

$$\begin{vmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-s & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2}-s \end{vmatrix} = 0$$

$$\begin{cases} s_1 = -f_1\left(R_{1,}R_{2,}C_{1,}C_2\right) \\ s_2 = -f_2\left(R_{1,}R_{2,}C_{1,}C_2\right) \end{cases}$$

2 real negative roots aka Eigen Values

$$\left[\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s\right]\left[\frac{1}{R_2C_2}+s\right]-\frac{1}{R_2^2C_1C_2} = 0$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A\,E_1 e^{s_1 t} + B\,E_2 e^{s_2 t}$$

Integration constants

Eigen Vectors

Compute Eigen Vectors

Eigen vector for eigen value s_1

$$(A - s_1 I) E_1 = 0$$

$$\begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) - s_1 & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} - s_1 \end{bmatrix} \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = 0$$

$$q_1 = K_1 p_1$$

$$E_1 = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1 \\ K_1 \end{bmatrix}$$

Eigen vector for eigen value s_2

$$(A - s_2 I) E_2 = 0$$

$$\begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) - s_2 & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} - s_2 \end{bmatrix} \begin{bmatrix} p_1 \\ q_2 \end{bmatrix} = 0$$

$$q_2 = K_2 p$$

$$E_2 = \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ K_2 \end{bmatrix}$$

Eigen vector is any vector whose coordinates are related as shown

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = A E_1 e^{s_1 t} + B E_2 e^{s_2 t}$$

General solution

$$\begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix} = A E_1 + B E_2$$

Use initial conditions to compute A and B and get particular solution using this system of 2 equations

Conclusion

- LC loop: natural solution
- LC loop: energy swing
- RLC series: natural solution for 2 real roots, 1 real root, 2 complex conjugate roots
- RLC circuits: other configurations
- RLC circuits: forced solution
- Other second order circuits