Exemplo de Formulário de Matemática Computacional

Representação de números em sistemas de ponto flutuante

$$x = \sigma(0.a_1 a_2 ...)_\beta \beta^t,\ a_1 \neq 0; \quad \tilde{x} = \mathrm{fl}(x) \in \mathbb{F}(\beta, n, t_{min}, t_{max})$$

$$|e_{\tilde{x}}| \leq \beta^{t-n}, \quad |\delta_{\tilde{x}}| \leq \beta^{1-n} := \epsilon_M, \quad \text{em arredondamento por corte}$$

$$|e_{\tilde{x}}| \leq \frac{1}{2}\beta^{t-n}, \quad |\delta_{\tilde{x}}| \leq \frac{1}{2}\beta^{1-n} := \epsilon_M, \quad \text{em arredondamento simétrico}$$

Propagação de erros em funções e algoritmos $\quad (x, \tilde{x} \in \mathbb{R}^n, \quad x \approx \tilde{x})$

$$\delta_{f(\tilde{x})} \approx \sum_{k=1}^{n} p_{f,k}(x)\delta_{\tilde{x}_k}, \quad p_{f,k}(x) := \frac{x_k \frac{\partial f}{\partial x_k}(x)}{f(x)}$$

$$\delta_{f_{\mathbb{F}}(\tilde{x})} \approx \sum_{k=1}^{n} p_{f,k}(x)\delta_{\tilde{x}_k} + \sum_{k=1}^{m} q_k \delta_{\mathrm{arr}_k}$$

Majoração de erros na aproximação de zeros $\quad (z, \tilde{z} \in \mathbb{R}, \quad z \approx \tilde{z})$

$$|z - \tilde{z}| \leq \frac{|f(\tilde{z})|}{\min_I |f'|},\ I = \mathrm{int}(z; \tilde{z})$$

Métodos iterativos para equações não-lineares

Método da bisseção: $\quad f(a_n)f(b_n) < 0, \quad x_{n+1} = \dfrac{a_n + b_n}{2}$

$$|z - x_n| \leq \frac{b-a}{2^n}, \quad |z - x_{n+1}| \leq |x_{n+1} - x_n|$$

Método de Newton: $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$z - x_{n+1} = -\frac{f''(\xi_n)}{2f'(x_n)}(z - x_n)^2,\ \xi_n \in \mathrm{int}(z; x_n)$$

$$|z - x_n| \leq \frac{1}{\mathbb{K}}\left(\mathbb{K}|z - x_0|\right)^{2^n},\ \mathbb{K} := \frac{\max|f''|}{2\min|f'|}$$

Método da secante: $\quad x_{n+1} = x_n - \dfrac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$

$$z - x_{n+1} = -\frac{f''(\mu_n)}{2f'(\nu_n)}(z - x_n)(z - x_{n-1}),\ \mu_n \in \mathrm{int}(x_n; x_{n-1}),\ \nu_n \in \mathrm{int}(z; x_n; x_{n-1})$$

$$|z - x_n| \leq \frac{1}{\mathbb{K}}\left(\mathbb{K}\max\{|z - x_{-1}|, |z - x_0|\}\right)^{\phi_n},\ \phi_{-1} = \phi_0 = 1, \phi_n = \phi_{n-1} + \phi_{n-2}$$

Método do ponto fixo: $x_{n+1} = g(x_n)$

$$|z - x_n| \leq L^n |z - x_0|, \quad |z - x_n| \leq \frac{L^n}{1-L}|x_1 - x_0|, \quad |z - x_{n+1}| \leq \frac{L}{1-L}|x_{n+1} - x_n|$$

Normas e Condicionamento

$$\|\mathbf{A}\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| \qquad \mathrm{cond}(\mathbf{A}) = \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\|$$

$$\|\mathbf{A}\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| \qquad \|\delta_{\tilde{\mathbf{x}}}\| \leq \frac{\mathrm{cond}(\mathbf{A})}{1 - \mathrm{cond}(\mathbf{A})\|\delta_{\tilde{\mathbf{A}}}\|}\left(\|\delta_{\tilde{\mathbf{A}}}\| + \|\delta_{\tilde{\mathbf{b}}}\|\right)$$

$$\|\mathbf{A}\|_2 = (\rho(\mathbf{A}^T\mathbf{A}))^{1/2}$$

Métodos iterativos para sistemas lineares

$$\mathbf{Ax} = \mathbf{b} \Leftrightarrow \mathbf{x} = \mathbf{Cx} + \mathbf{d} \qquad \mathbf{x}^{(n+1)} = \mathbf{C}\mathbf{x}^{(n)} + \mathbf{d}$$

$$\|\mathbf{z} - \mathbf{x}^{(n)}\| \leq \|\mathbf{C}\|^n \|\mathbf{z} - \mathbf{x}^{(0)}\|, \qquad \|\mathbf{z} - \mathbf{x}^{(n)}\| \leq \frac{\|\mathbf{C}\|^n}{1 - \|\mathbf{C}\|}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$$

$$\|\mathbf{z} - \mathbf{x}^{(n+1)}\| \leq \frac{\|\mathbf{C}\|}{1 - \|\mathbf{C}\|}\|\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}\|$$

Método de Jacobi: $\mathbf{C} = -\mathbf{D_A}^{-1}(\mathbf{L_A} + \mathbf{U_A})$;

Método de Gauss-Seidel: $\mathbf{C} = -(\mathbf{L_A} + \mathbf{D_A})^{-1}\mathbf{U_A}$;

Método SOR: $\mathbf{C} = -(\mathbf{L_A} + \omega^{-1}\mathbf{D_A})^{-1}(\mathbf{U_A} + (1-\omega^{-1})\mathbf{D_A})$

$$\mathbf{x}^{(n+1)} = (1-\omega)\mathbf{x}^{(n)} + \omega\mathbf{D_A}^{-1}(\mathbf{b} - \mathbf{L_A}\mathbf{x}^{(n+1)} - \mathbf{U_A}\mathbf{x}^{(n)})$$

Método de Newton para sistemas não-lineares

$$\mathbf{J}(\mathbf{x}^{(n)})\Delta\mathbf{x}^{(n)} = -\mathbf{f}(\mathbf{x}^{(n)}) \qquad \mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta\mathbf{x}^{(n)}$$

Interpolação polinomial

Fórmula de Lagrange: $\ell_i(x) = \prod_{j=0, j\neq i}^{n} \left(\frac{x - x_j}{x_i - x_j}\right) \qquad p_n(x) = \sum_{i=0}^{n} y_i\, \ell_i(x)$

Fórmula de Newton com dif. divididas:

$$\begin{cases} y[x_i] = y_i, \quad i = 0, ..., n \\ y[x_i, ..., x_{i+k}] = \dfrac{y[x_{i+1}, ..., x_{i+k}] - y[x_i, ..., x_{i+k-1}]}{x_{i+k} - x_i}, \quad i = 0 : n-k, \quad k = 1 : n \end{cases}$$

$$p_n(x) = y[x_0] + \sum_{i=1}^{n} y[x_0, ..., x_i] \prod_{j=0}^{i-1}(x - x_j)$$

Erro de interpolação: $e_n(x) = f[x_0, \cdots, x_n, x] \prod_{j=0}^{n}(x - x_j) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n}(x - x_j)$

Mínimos quadrados

$$\begin{bmatrix} (\phi_1, \phi_1) & \dots & (\phi_1, \phi_m) \\ \dots & \dots & \dots \\ (\phi_m, \phi_1) & \dots & (\phi_m, \phi_m) \end{bmatrix} \begin{bmatrix} c_1 \\ \dots \\ c_m \end{bmatrix} = \begin{bmatrix} (\phi_1, y) \\ \dots \\ (\phi_m, y) \end{bmatrix}$$

$$(\phi_i, \phi_j) = \sum_{k=1}^{n} \phi_i(x_k)\phi_j(x_k), \quad (\phi_i, y) = \sum_{k=1}^{n} \phi_i(x_k) y_k$$

Integração numérica

Regra dos trapézios:

$$T_N(f) = \frac{h}{2}\left[f(x_0) + f(x_N) + 2\sum_{i=1}^{N-1} f(x_i)\right] \quad E_N^T(f) = -\frac{(b-a)h^2}{12} f''(\xi)$$

Regra de Simpson:

$$S_N(f) = \frac{h}{3}\left[f(x_0) + f(x_N) + 4\sum_{i=1}^{N/2} f(x_{2i-1}) + 2\sum_{i=1}^{N/2-1} f(x_{2i})\right] \quad E_N^S(f) = -\frac{(b-a)h^4}{180} f^{(4)}(\xi)$$

Métodos numéricos para equações diferenciais

Método de Euler explícito: $y_{i+1} = y_i + hf(t_i, y_i)$

$$|y(t_i) - y_i| \leq \frac{hM}{2L}\left[e^{L(t_i - t_0)} - 1\right], \qquad |y''(t)| \leq M,\ t \in [t_0, t_i]$$

Método de Taylor de ordem 2:

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2}\left[\frac{\partial f}{\partial t}(t_i, y_i) + f(t_i, y_i)\frac{\partial f}{\partial y}(t_i, y_i)\right]$$

Métodos de Runge-Kutta de ordem 2:

$$y_{i+1} = y_i + h\left[(1 - \frac{1}{2\alpha})K_1 + \frac{1}{2\alpha}K_2\right]$$

$$K_1 = f(t_i, y_i) \qquad K_2 = f(t_i + \alpha h, y_i + \alpha h K_1)$$