Lecture 31

Fourier transforms and the Dirac delta function

In the previous section, great care was taken to restrict our attention to particular spaces of functions for which Fourier transforms are well-defined. That being said, it is often necessary to extend our definition of FTs to include "non-functions", including the Dirac "delta function". In this section, we also show, very briefly, the importance of the delta function in the analysis of functions that are defined on the entire real line **R**.

Recall that the delta function $\delta(x)$ is not a function in the usual sense. It has the following properties:

$$\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases} \tag{1}$$

with the additional feature that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{2}$$

Actually, the Dirac delta function is an example of a *distribution* – distributions are defined in terms of their integration properties. For any function $f(x)$ that is *continuous* at $x = 0$, the delta distribution is defined as

$$\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0). \tag{3}$$

If $f(x)$ is continuous at $x = x_0$, then

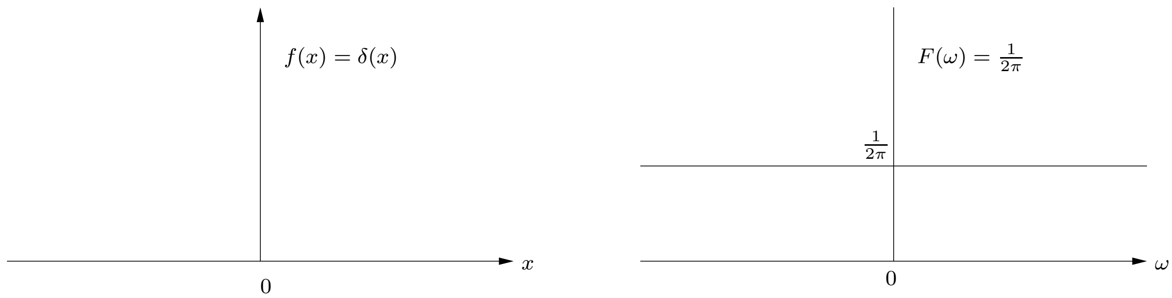
$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = f(x_0). \tag{4}$$

The Fourier transform of the Dirac distribution is easily calculated from the above property. For the distribution positioned at $x = 0$:

$$F(\omega) = \mathcal{F}(\delta(x)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)e^{i\omega x}\, dx = \frac{1}{2\pi}. \tag{5}$$

With reference to the sketches below, note that the delta function $\delta(x)$ is a perfect "spike", i.e., it is concentrated at $x = 0$, whereas its Fourier transform is a constant function for all $x \in \mathbf{R}$, i.e., it is "spread out" as much as possible.

This illustrates the complementarity between the "spatial domain", i.e., x-space (or "temporal domain," i.e., t-space, if x is replaced by t) and the "frequency domain, i.e., ω-space, Note also that



neither $\delta(x)$ nor its Fourier transform $F(\omega) = 1/(2\pi)$ belong to $L^2(\mathbf{R})$, the space of square integrable functions on $\mathbf{R}$.

Now suppose that the delta function is translated to $x = x_0$, i.e., we replace x with $x - x_0$:

$$F(\omega) = \mathcal{F}(\delta(x - x_0)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x - x_0)e^{i\omega x}\,dx = \frac{1}{2\pi}e^{i\omega x_0}. \tag{6}$$

Note that in all cases, $F(\omega)$ is *not* square-integrable. But, then again, the delta function $\delta(x - x_0)$ does not belong to $L^2(\mathbf{R})$ as well.

Let's now return to the formal definition of the Fourier transform of a function $f(x)$ for this course (assuming that the integral exists),

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx, \tag{7}$$

and the associated inverse Fourier transform,

$$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega. \tag{8}$$

As we discussed earlier, in Eq. (8), the terms $F(\omega)$ may be regarded as the expansion coefficients of $f(x)$ in terms of the basis functions

$$\phi_\omega(x) = e^{-i\omega x}. \tag{9}$$

In other words, we may view Eq. (8), when written as follows,

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\phi_\omega(x)\,d\omega, \tag{10}$$

as a continuous version of the following expansion of a function $g(x)$ that is defined over a finite interval $[a, b]$ and expressed in terms of a discrete set of functions $\phi_n(x)$, $n = 1, 2, \cdots$, that form a basis on $[a, b]$:

$$g(x) = \sum_{n=1}^{\infty} c_n\phi_n(x). \tag{11}$$

The discrete summation over the integer-valued index n in Eq. (11) has been replaced by a continuous integration over the real-valued index ω in Eq. (8). We'll have more to say about Eq. (8) later.

Let us now substitute our results for the Dirac delta function and its Fourier transform, i.e.,

$$f(x) = \delta(x), \quad F(\omega) = \frac{1}{2\pi}, \tag{12}$$

into Eq. (8):

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x}\, d\omega. \tag{13}$$

If we replace x with $x - x_0$, this equation becomes

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(x - x_0)}\, d\omega. \tag{14}$$

Eqs. (13) and (14) are known as the "integral representations" of the Dirac delta function. Note that the integrations are performed over the frequency variable ω.

Let us now consider the following case,

$$F(\omega) = \delta(\omega). \tag{15}$$

We wish to find the inverse Fourier transform of the Dirac delta function in ω-space. In other words, what is the function $f(x)$ such that $\mathcal{F}(f) = \delta(\omega)$? If we substitute $F(\omega) = \delta(\omega)$ into Eq. (8), then

$$f(x) = \int_{-\infty}^{\infty} \delta(\omega) e^{-i\omega x} d\,\omega. \tag{16}$$

But recall that an integration of the Dirac delta function $\delta(whatever)$ yields $f(whatever = 0)$. This means that

$$f(x) = e^{-i \cdot 0 \cdot x} = 1. \tag{17}$$

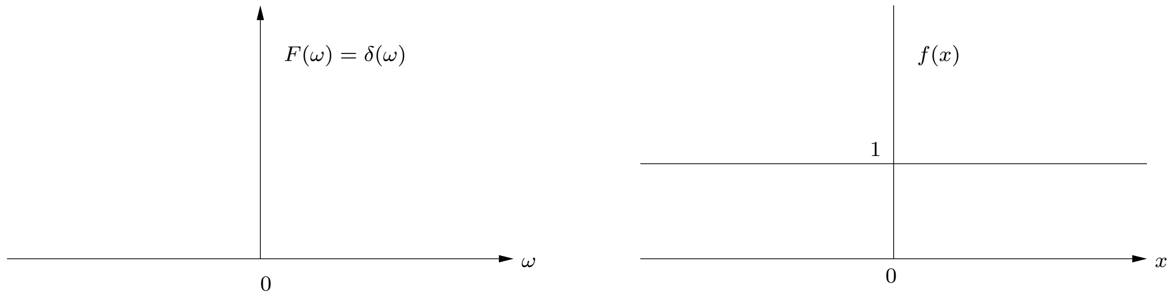
Therefore, the inverse Fourier transform of $\delta(\omega)$ is the function $f(x) = 1$. This time, the function $\delta(\omega)$ in frequency space is spiked, and its inverse Fourier transform $f(x) = 1$ is a constant function spread over the real line, as sketched in the figure below.

Let us now substitute this result into Eq. (7), i.e., $f(x) = 1$ and $F(\omega) = \delta(\omega)$. We then have

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x}\, dx. \tag{18}$$

Now let $\omega = \omega_1 - \omega_2$ so that the above equation becomes

$$\delta(\omega_1 - \omega_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega_1 - \omega_2)x} dx. \tag{19}$$



Let us rewrite this result slightly, i.e.,

$$\delta(\omega_1 - \omega_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega_1 x} e^{-i\omega_2 x} dx. \tag{20}$$

Eq. (20) may be viewed as an orthogonality relation for the functions $\phi_\omega(x) = e^{-i\omega x}$ defined earlier. Recalling the definition of the inner product in the Hilbert space $L^2(\mathbf{R})$, we may rewrite Eq. (20) as

$$\frac{1}{2\pi} \langle \phi_{\omega_1}, \phi_{\omega_2} \rangle = \delta(\omega_1 - \omega_2). \tag{21}$$

We may go one step further and define the functions

$$\psi_\omega(x) = \frac{1}{\sqrt{2\pi}} \phi_\omega(x) = \frac{1}{\sqrt{2\pi}} e^{-i\omega x}, \quad \omega \in \mathbf{R}, \tag{22}$$

so that Eq. (21) becomes

$$\langle \psi_{\omega_1}, \psi_{\omega_2} \rangle = \delta(\omega_1 - \omega_2). \tag{23}$$

Note that this may be viewed as a continuous version of the relations encountered for functions $\{\chi_n\}_{n=1}^{\infty}$ that form an *orthonormal* set on a finite interval $[a, b]$, i.e.,

$$\langle \chi_n, \chi_m \rangle = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases} \tag{24}$$

In going from Eq. (24) to Eq. (23), the discrete indices n and m become the continuous indices ω_1 and ω_2, and the constant 1 on the RHS becomes ∞ because of the Dirac delta function. This is the price to be paid in going from the discrete case to the infinite case. Moreover, note once again that the functions $\psi_\omega(x)$ are *not* elements of $L^2(\mathbf{R})$ even though they form a basis for the space! Physicists refer to these functions as (one-dimensional) *plane waves.*

Plane waves are very important in physics. It's not too hard to imagine that they would be important in classical optics or electromagnetism, where light or electromagnetic radiation is viewed in terms of waves. In quantum mechanics, plane waves are important in "scattering theory:" For example, a particle X travels toward another particle Y and interacts with it, thereby being "scattered" or deflected. The quantum mechanical wavefunction of the particle, before and after the interaction, may be expressed in terms of plane waves.

This agrees with comments made earlier in this lecture: If we choose to "expand" a function $f(x)$ defined over the real line $\mathbf{R}$ in terms of plane waves, integrating over its ω index, i.e.,

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} A(\omega)\psi_\omega(x)\, d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(\omega)e^{-i\omega x}\, d\omega, \end{aligned} \tag{25}$$

then the "coefficients" $A(\omega)$ of the expansion comprise, up to a constant, the Fourier transform of $f(x)$, cf. Eq. (8).

We conclude this section with a couple of intriguing results. Let us return to the integral representation of the Dirac delta function $\delta(x - x_0)$ in Eq. (14):

$$\delta(x - x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(x-x_0)}\, d\omega. \tag{26}$$

We'll first rewrite this equation as follows,

$$\begin{aligned} \delta(x - x_0) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{i\omega x_0}\frac{1}{\sqrt{2\pi}}e^{-i\omega x}\, d\omega \\ &= \int_{-\infty}^{\infty} \psi_\omega^*(x_0)\psi_\omega(x)\, d\omega. \end{aligned} \tag{27}$$

Note that this is an integration over ω involving functions evaluated at x and x_0 (which may be the same, or may not).

We claim that the above equation is the continuous analogue of the following result: Let $\{\chi_n\}_{n=1}^{\infty}$ be a set of orthonormal basis functions on an interval $[a, b]$, as given by Eq. (24). Then for any point $x_0 \in (a, b)$,

$$\delta(x - x_0) = \sum_{n=1}^{\infty} \chi_n^*(x)\chi_n(x_0), \tag{28}$$

This may be viewed as an eigenfunction expansion of the Dirac delta function $\delta(x - x_0)$. It is an important result that has applications in the solution of ODEs and PDEs, in context of *Green's functions*.

We leave the proof of this result as an exercise. Hint: Multiply each side of Eq. (28) by a continuous function $f(x)$ and consider the integral of each side over $\mathbf{R}$.

The two-dimensional Fourier transform

`Relevant section of text: 10.6.5`

The definition of the Fourier transform for a function of two variables, i.e., $f : \mathbf{R}^2 \to \mathbf{R}$, is a rather straightforward extension of the one-dimensional FT. It is notationally convenient to let (x_1, x_2) represent the two spatial variables, i.e., $f(x_1, x_2)$. Since there are two spatial variables, we must have two frequency variables, w_1 and w_2, which will also be written as a pair: $\omega = (\omega_1, \omega_2)$.

The Fourier transform of a function $f(x_1, x_2)$ is defined as

$$F(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2) e^{i\omega_1 x_1} e^{i\omega_2 x_2} dx_1\, dx_2. \tag{29}$$

Note that each variable introduces a factor of $1/(2\pi)$. The above equation can be written in a more compact way using vector notation, i.e.,

$$\vec{\omega} = (\omega_1, \omega_2), \quad \vec{r} = (x_1, x_2), \quad \text{so that} \quad \vec{\omega} \cdot \vec{r} = \omega_1 x_1 + \omega_2 x_2. \tag{30}$$

Then

$$F(\vec{\omega}) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} f(\vec{r}) e^{i\vec{\omega}\cdot\vec{r}} d\vec{r}, \tag{31}$$

where $d\vec{r} = dx_1 dx_2$ or $dx_2 dx_1$.

The 2D inverse Fourier transform will be defined as

$$f(\vec{x}) = \int \int_{\mathbf{R}^2} F(\vec{\omega}) e^{-i\vec{\omega}\cdot\vec{r}} d\vec{\omega}. \tag{32}$$

We now consider the heat equation on $\mathbf{R}^2$,

$$\frac{\partial u}{\partial t} = \nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \tag{33}$$

with initial condition

$$u(x_1, x_2, 0) = f(x_1, x_2). \tag{34}$$

The (unique) solution $u(x_1, x_2)$ to this problem may be produced using the FT in the same way as was done in 1D. Very briefly, we take 2D FTs of both sides of Eq. (33) – the rules for FTs of partial derivatives will apply again – to arrive at the following PDE for the FT $U(\omega_1, \omega_2)$ of u:

$$\frac{\partial U}{\partial t} = k\omega^2 U, \quad \text{where } \omega^2 = \omega_1^2 + \omega_2^2. \tag{35}$$

Once again, since only the partial time derivative appears in the equation, it may be solved as an ODE. The solution is easily found to be

$$U(\omega_1, \omega_2, t) = F(\omega_1, \omega_2)e^{-k\omega^2 t}, \tag{36}$$

where

$$F(\omega_1, \omega_2) = \mathcal{F}(f(x_1, x_2)). \tag{37}$$

We now take the inverse Fourier transform of each side to obtain $u(x_1, x_2)$. The IFT of the RHS is obtained from a two-dimensional Convolution Theorem (see text, p. 498). First of all, the IFT of the Gaussian on the RHS is assisted by the fact that it is separable, i.e.,

$$\begin{aligned} G(\omega_1, \omega_2) &= e^{-k\omega^2 t} \\ &= e^{-k\omega_1^2 t} e^{-k\omega_2^2 t} \\ &= G_1(\omega_1)G_2(\omega_2). \end{aligned} \tag{38}$$

The inverse FT of this function is then a product of the IFTs of G_1 and G_2, which we know from the one-dimensional case,

$$g_1(x_1) = \sqrt{\frac{\pi}{kt}}e^{-x_1^2/4kt}, \quad g_2(x_2) = \sqrt{\frac{\pi}{kt}}e^{-x_2^2/4kt}. \tag{39}$$

As a result, the solution will be a convolution of these functions and the initial data function $f(x_1, x_2)$. The final result is

$$\begin{aligned} u(x_1, x_2, t) &= \int\int_{\mathbf{R}^2} f(s_1, s_2)\frac{1}{4\pi kt}e^{-[(x_1-s_1)^2+(x_2-s_2)]/(4kt)}ds_1\, ds_2 \\ &= \int\int_{\mathbf{R}^2} f(s_1, s_2)h_t(x_1 - s_1, x_2 - s_2)ds_1\, ds_2. \end{aligned} \tag{40}$$

Here, $h_t(x_1, x_2)$ is the *two-dimensional heat kernel*, a two-dimensional, normalized Gaussian distribution. It is the product of the one-dimensional heat kernels in the x and y directions.

In the special case that the initial condition is concentrated at a point (x_0, y_0), i.e.,

$$f(x_1, x_2) = \delta(x_1 - a_1, x_2 - a_2), \tag{41}$$

then the solution $u(x,t)$ to the heat equation becomes

$$\begin{aligned} u(x_1,x_2,t) &= \int\int_{\mathbf{R}^2} \delta(s_1-a_1,s_2-a_2)\frac{1}{4\pi kt}e^{-[(x_1-s_1)^2+(x_2-s_2)]/(4kt)}ds_1\,ds_2 \\ &= \frac{1}{4\pi kt}e^{-[(x_1-a_1)^2+(x_2-a_2)^2]/(4kt)}, \qquad t>0. \end{aligned} \tag{42}$$

At time $t>0$, the temperature function $u(x_1,x_2,t)$ is a Gaussian function centered at (a_1,a_2) which spreads out with increasing time.

Lecture 32

Quasilinear PDEs and the method of "characteristics"

`Relevant section of text: 12.1-2`

A *quasilinear PDE* in the function $u(x,t)$ has the form

$$\frac{\partial u}{\partial t} + c(u,x,t)\frac{\partial u}{\partial x} = Q(u,x,t). \tag{43}$$

It is called quasilinear because the partial derivatives do not multiply each other. Note that the PDE can be nonlinear since the coefficient c could be a function of u. As well, the function Q could be nonlinear in u. The special form of the quasilinear PDE permits its reduction to a system of ODEs which can, at least in principle, be solved, as we show below.

Quasilinear PDEs arise in many one-dimensional applications, e.g., gas dynamics, and traffic flow on a highway. We shall examine some applications later in this section.

Motivating example: The wave equation

As a motivating example, we consider the second-order wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{44}$$

When considered on a finite interval, i.e., $x \in [0,L]$, we saw that the normal mode solutions can always be decomposed into forward and backward moving waves. The consequence is that all solutions may be written in the form

$$u(x,t) = f(x-ct) + g(x+ct). \tag{45}$$

On the real line $\mathbf{R}$, it can be shown, using Fourier transforms, that the solution to the wave equation satisfying the initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \tag{46}$$

is given by

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \tag{47}$$

We'll derive this result using the method of characteristics.

First, we shall rewrite the wave equation in (44) as follows,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{48}$$

We now "factor" the left-hand side, in the same way as would be done for a difference of squares,

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0. \tag{49}$$

The reader may check that the above is equivalent to the wave equation – the cross derivatives cancel out.

Note that we could have factored the wave equation in the other order, i.e.,

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0. \tag{50}$$

Now define the following functions

$$w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}, \qquad v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}. \tag{51}$$

Substitution of these functions into (49) and (50) yield the following PDEs,

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0. \tag{52}$$

This is a system of two first-order linear PDEs in w and v. These are special cases of quasilinear PDEs.

Let us now consider the first of the two equations in (52), a PDE in $w(x,t)$:

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0. \tag{53}$$

Now consider the rate of change of the function $w(x(t),t)$ as measured by a moving observer, "X," whose trajectory is given by $x = x(t)$. Note that this is a particular "sampling" of the values of $w(x,t)$ in the (x,t) plane.

From the Chain Rule, the rate of change observed by X is given by

$$\frac{d}{dt}w(x(t),t) = \frac{\partial w}{\partial t}(x(t),t) + \frac{\partial w}{\partial x}(x(t),t)\frac{dx}{dt}. \tag{54}$$

Let us examine each of these terms:

1. $\frac{d}{dt}w(x(t),t)$: The "total" or "substantial" derivative (also called the "material" derivative in continuum mechanics). This is the rate of change of w w.r.t. t as measured by the moving observer.

2. $\dfrac{\partial w}{\partial t}(x(t),t)$: The rate of change of w at the fixed position $x(t)$ at time t.

3. $\dfrac{\partial w}{\partial x}(x(t),t)\dfrac{dx}{dt}$: The rate of change of w due to the motion of the observer – in moving from a region with a lower/higher value of w to one with a higher/lower value of w.

We have arrived at the most important point of this section. Let us compare Eqs. (53) and (54): If the observer moves with velocity

$$\frac{dx}{dt} = c, \tag{55}$$

then the rate of change of w measured by the observer is zero, i.e.,

$$\frac{dw}{dt} = 0. \tag{56}$$

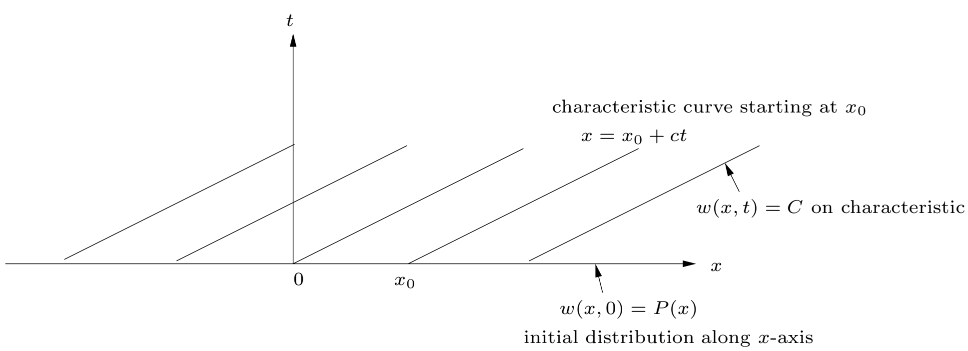
This implies that

$$w(x(t),t) = C, \tag{57}$$

a constant, along the trajectory $x(t)$. Integrating (55), we obtain the equation of the trajectory,

$$x(t) = x_0 + ct, \tag{58}$$

where x_0 is the intitial point of the trajectory. These trajectories are called "characteristic curves," or simply "characteristics". They form a family of curves.



Now suppose that $w(x,t)$ is prescribed at time $t = 0$ by the condition

$$w(x,0) = P(x), \qquad x \in \mathbf{R}. \tag{59}$$

The function $P(x)$ may be viewed as the initial condition of the quasilinear PDE for w. Since $w(x(t),t) = C$ on the trajectory $x = x_0 + ct$, it follows that

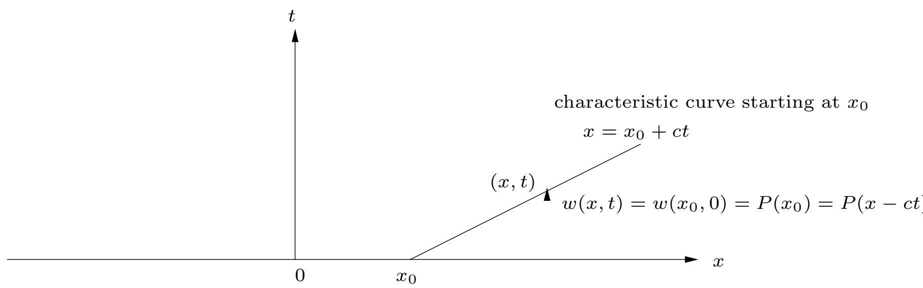
$$w(x,t) = w(x_0,0) = P(x_0). \tag{60}$$

But from the equation of the trajectory, (58), we may solve for x_0:

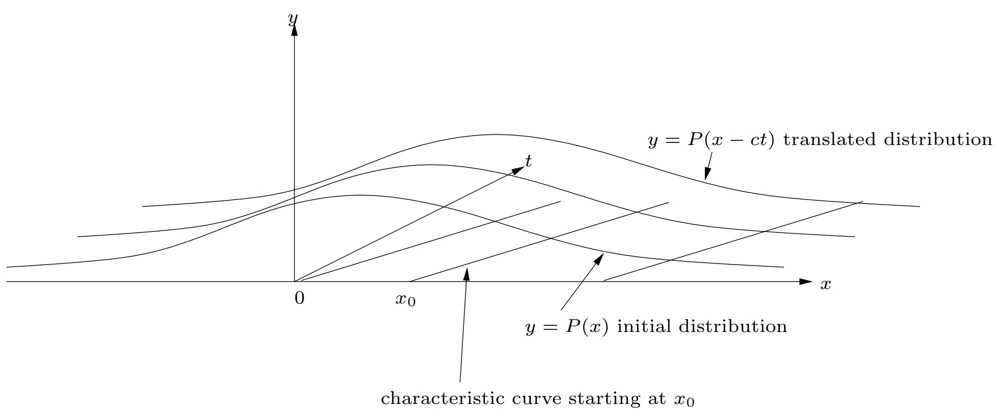
$$x_0 = x - ct, \tag{61}$$

so that the solution in (60) may be written as

$$w(x,t) = P(x - ct). \tag{62}$$



In order to visualize this result, we may interpret Eq. (62) in the following way: The graph of $w(x,t)$ vs. x at time t is simply the graph of $P(x)$ shifted to the right (assuming that $c > 0$) by ct: *The value of $w(x,t)$ is the value $P(x_0)$ back at $x_0 = x - ct$.*



We claim that $w(x,t)$ in (62) is the general solution to the PDE in $w(x,t)$. Just to check this, let us compute the derivatives:

$$\frac{\partial w}{\partial x} = \frac{\partial P}{\partial x} = P'(x-ct)\frac{\partial(x-ct)}{\partial x} = P'(x-ct), \tag{63}$$

$$\frac{\partial w}{\partial t} = \frac{\partial P}{\partial t} = P'(x-ct)\frac{\partial(x-ct)}{\partial t} = -cP'(x-ct). \tag{64}$$

It follows that

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = (-c+c)P'(x-ct) = 0. \tag{65}$$

The lines $x(t) = x_0 + ct$ represent one set of characterstic curves associated with the wave equation. Another set exists, corresponding to the PDE for $v(x,t)$:

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0. \tag{66}$$

Using similar methods to that above, it is straightforward to show that the equation for the characterstic curves is

$$\frac{dx}{dt} = -c, \qquad \rightarrow \qquad x(t) = x_0 - ct. \tag{67}$$

Therefore $x_0 = x + ct$, implying that the solution $v(x,t)$ is given by

$$v(x,t) = Q(x + ct), \tag{68}$$

where $Q(x)$ is the initial condition function. In this case, at time $t > 0$ the graph of $Q(x)$ is translated *leftward* by the amount ct to produce the graph of $v(x,t)$.

Lecture 33

The method of characteristics (cont'd)

`Relevant section of text: 12.3`

We'll look at a couple of examples to illustrate the effectiveness of the method of characteristics in solving quasilinear PDEs, and then return to complete our study of the wave equation.

Example 1: Consider the PDE

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0, \tag{69}$$

with initial condition

$$w(x,0) = P(x) = \cos x. \tag{70}$$

The observant reader may note that the above PDE is simply one of the two factors of the wave equation discussed in the previous lecture, with $c = -3$. But let's start from scratch.
Recall that if we assume that $w = w(x(t), t)$, i.e., $x(t)$ is the trajectory of an "observer", then the rate of change of w measured by the observer over the path $x(t)$ is given by

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt}. \tag{71}$$

For this problem,

$$\frac{dx}{dt} = -3, \tag{72}$$

which implies that the if the observer travels over the trajectory,

$$x(t) = x_0 - 3t, \tag{73}$$

then the observed rate of change of w is zero. This means that

$$w(x(t), t) = C = w(x_0, 0), \tag{74}$$

the value of w at the starting point $x(0) = x_0$. But the value of w at x_0 is given by the initial condition $P(x_0) = \cos(x_0)$. Thus

$$w(x(t), t) = \cos(x_0). \tag{75}$$

This is still not useful – we need to relate x_0 to $x(t)$. We simply solve for x_0 from the trajectory equation:

$$x_0 = x + 3t. \tag{76}$$

The final result is

$$w(x,t) = \cos(x+3t). \tag{77}$$

Let us quickly check if this result does, indeed, satisfy the original equation:

$$\frac{\partial w}{\partial t} = -3\sin(x+3t), \qquad \frac{\partial w}{\partial x} = -\sin(x+3t), \tag{78}$$

so that

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = -3\sin(x+3t) + 3\sin(x+3t) = 0. \tag{79}$$

Thus, the PDE is satisfied, and we have determined the solution.

Example 2: Given the PDE

$$\frac{\partial w}{\partial t} + 3t\frac{\partial w}{\partial x} = tw, \qquad w(x,0) = f(x). \tag{80}$$

Once again, we consider $w = w(x(t),t)$ as measured by an observer travelling on a trajectory $x(t)$. The measured rate of change of w is

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt}. \tag{81}$$

If we compare this result with the PDE we wish to solve, then if the observer travels on the trajectory – or *characteristic curve* – that satisfies

$$\frac{dx}{dt} = 3t, \tag{82}$$

then the rate of change of w measured by the observer is

$$\frac{dw}{dt} = tw. \tag{83}$$

Note that the right-hand side of the latter equation is not zero. As a result, the value of $w(x,t)$ on a characteristic curve will not be constant.

We can solve the DE for the characteristics by simple integration:

$$x(t) = x_0 + \frac{3}{2}t^2. \tag{84}$$

In this case, the characteristics are not straight lines, but parabolas in the xt-plane. Note that $x(0) = x_0$ is the initial point on the characteristic/trajectory.

We can also solve the ODE for w in terms of t on the characterstics – it is a separable first-order DE:

$$\frac{dw}{w} = t\,dt \quad \rightarrow \quad \ln w = \frac{t^2}{2} + C \quad \rightarrow \quad w = Ae^{t^2/2}, \tag{85}$$

where $A = e^C$ is an arbitrary constant.

It now remains to connect this expression of w to the initial condition function $f(x)$. From our solution

$$w(x,t) = Ae^{t^2/2}, \tag{86}$$

it follows that

$$w(x_0, 0) = Ae^0 = A. \tag{87}$$

But the original initial condition in the problem is

$$w(x_0, 0) = f(x_0). \tag{88}$$

Therefore, we conclude that

$$A = f(x_0). \tag{89}$$

Substituting this result into (86) gives

$$w(x,t) = f(x_0)e^{t^2/2}. \tag{90}$$

But once again, we must relate x_0 to x: We solve for x_0 in the equation for the characteristic curve:

$$x_0 = x - \frac{3}{2}t^2, \tag{91}$$

and substitute into (90) to arrive at the final result,

$$w(x,t) = f\left(x - \frac{3}{2}t^2\right)e^{t^2/2}. \tag{92}$$

Once again, it is a good idea to verify the result:

$$\frac{\partial w}{\partial x} = f'\left(x - \frac{3}{2}t^2\right)e^{t^2/2}, \tag{93}$$

$$\frac{\partial w}{\partial t} = f'\left(x - \frac{3}{2}t^2\right)(-3t)e^{t^2/2} + f\left(x - \frac{3}{2}t^2\right)te^{t^2/2}. \tag{94}$$

Substitution into the LHS of the PDE yields

$$\frac{\partial w}{\partial t} + 3t\frac{\partial w}{\partial t} = f\left(x - \frac{3}{2}t^2\right) te^{t^2/2} = tw. \tag{95}$$

Thus, the PDE is satisfied.

A couple of comments on the solution in Eq. (92):

1. Note that the solution $w(x,t)$ is obtained by translating the graph of the initial data $f(x)$ along the characteristic curve $x = x(t)$, as expected.

2. In addition, however, the value of f on this curve is multiplied by the factor $e^{t^2/2}$. This is a consequence of the fact that the RHS of the original PDE in this problem, i.e., Eq. (80) is **not** zero. As such, the *rate of change* of w, as measured by the observer travelling along the trajectory $x(t)$ is **not** zero. It is given by $tw(x,t)$. We solved for this rate of change, which accounts for the additional factor $e^{t^2/2}$.

   If the RHS of Eq. (80) were zero, then ODE representing the rate of change of w as measured by the observer would be

   $$\frac{dw}{dt} = 0, \tag{96}$$

   with solution

   $$w(x,t) = C = w(x_0,t) = f(x_0). \tag{97}$$

   From the equation of the characteristic, we would then obtain the solution

   $$w(x,t) = f\left(x - \frac{3}{2}t^2\right). \tag{98}$$

   In other words, the graph of $f(x)$ is shifted but not altered in magnitude.

Back to the wave equation

We now return to the wave equation in order to provide a complete solution in terms of characteristics. Recall that the equation was writen as follows:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0. \tag{99}$$

It was then factored as follows,

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)u=0. \tag{100}$$

We then defined the following functions

$$w=\frac{\partial u}{\partial t}-c\frac{\partial u}{\partial x}, \qquad v=\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}. \tag{101}$$

Substitution of these functions into (100) yielded the following PDEs,

$$\frac{\partial w}{\partial t}+c\frac{\partial w}{\partial x}=0, \qquad \frac{\partial v}{\partial t}-c\frac{\partial v}{\partial x}=0, \tag{102}$$

a system of two first-order linear PDEs in w and v. These are special cases of quasilinear PDEs.

These equations are easily treated by the method of characteristics. For the first of the two equations,

$$\frac{\partial w}{\partial t}+c\frac{\partial w}{\partial x}=0, \tag{103}$$

The DE for the characteristic curves is

$$\frac{dx}{dt}=c \quad \rightarrow \quad x=x_0+ct. \tag{104}$$

The general solution is $w(x_0,0)=P(x_0)$ or

$$w(x,t)=P(x-ct). \tag{105}$$

In a similar fashion, the general solution to the v equation is

$$v(x,t)=Q(x+ct). \tag{106}$$

We now use these general solutions to construct the general solution $u(x,t)$ of the wave equation. From (101), we may solve for $\partial u/\partial t$ and $\partial u/\partial x$:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{2}(w+v)=\frac{1}{2}P(x-ct)+\frac{1}{2}Q(x+ct),\\ \frac{\partial u}{\partial x} &= \frac{1}{2c}(v-w)=\frac{1}{2c}Q(x+ct)-\frac{1}{2c}P(x-ct).\end{aligned} \tag{107}$$

It follows from these results that $u(x,t)$ has the form

$$u(x,t)=F(x-ct)+G(x+ct). \tag{108}$$

Why? Because taking partial derivatives of F and G w.r.t t and x will give us back F and G, perhaps with a multiplicative factor.

A little calculation (Exercise) shows that

$$F'(s) = \frac{1}{2c}P(s), \qquad G'(s) = \frac{1}{2c}Q(s). \tag{109}$$

The exact relation is not important - what is important is that the general solution in (108) may be written as the sum of

1. a rightward-moving shape $F(x - ct)$, moving with speed c,

2. a leftward-moving shape $G(x + ct)$, moving with speed c.

This solution was first developed by d'Alembert in 1747.

We now seek to express the general solution (108) in terms of the initial conditions imposed on the solution to the wave equation,

$$u(x, 0) = f(x), \qquad \frac{\partial u}{\partial t}(x, 0) = g(x), \qquad -\infty < x < \infty. \tag{110}$$

From the first initial condition, we have the requirement

$$u(x, 0) = f(x) = F(x) + G(x). \tag{111}$$

To accomodate the second condition, we compute $\partial u/\partial t$:

$$\frac{\partial u}{\partial t} = -cF'(x - ct) + cG'(x + ct). \tag{112}$$

This implies that

$$\frac{\partial u}{\partial t}(x, 0) = g(x) = -cF'(x) + cG'(x), \tag{113}$$

or

$$\frac{g(x)}{c} = -F'(x) + G'(x). \tag{114}$$

We must now solve for F and G, so that we can produce the solution $u(x, t)$ in Eq. (108). The procedure that we shall adopt differs slightly from that used in the textbook.

Noting that Eq. (114) involves only functions of one variable, i.e., x, let us change x to s and integrate both sides from $s = 0$ to $s = x$. The result is

$$\frac{1}{c}\int_0^x g(s)\, ds = -F(x) + F(0) + G(x) - G(0). \tag{115}$$

We now compare this equation to Eq. (111) and note that $F(x)$ may be removed if we add the two equations (ignoring, of course, the $u(x,t)$ part):

$$f(x) + \frac{1}{c}\int_0^x g(s)\, ds = 2G(x) + F(0) - G(0). \tag{116}$$

We can now solve for $G(x)$ in terms of f and g:

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\, ds + \frac{1}{2}G(0) - \frac{1}{2}F(0). \tag{117}$$

We can remove $G(x)$ by subtracting Eq. (115) from Eq. (111:

$$f(x) - \frac{1}{c}\int_0^x g(s)\, ds = 2F(x) + G(0) - F(0). \tag{118}$$

Now we solve for $F(x)$:

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\, ds - \frac{1}{2}G(0) + \frac{1}{2}F(0). \tag{119}$$

We now have $F(x)$ and $G(x)$, which we can substitute into Eq. (108) to obtain the result,

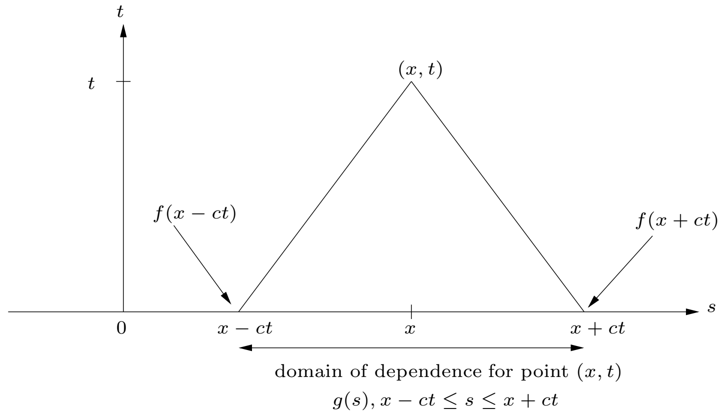
$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\, ds + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\, ds. \end{aligned} \tag{120}$$

or

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds. \tag{121}$$

This is an extremely interesting result. First of all, it tells us that the initial displacement distribution of the string, $f(x)$ is “split in half”, with one half moving rightward and the other half moving leftward. But there is another way to view these contributions. They tell us that the displacement $u(x,t)$ at position x at time t is determined, at least in part, by the initial displacements at positions $x - ct$ and $x + ct$. It is instructive to draw this dependence on a “space-time diagram”, as shown below.

The final term in Eq. (121) – the integral over $g(s)$ – shows that the initial velocity distribution over interval $[x - ct, x + ct]$ contributes to the displacement $u(x,t)$. For this reason, we refer to the interval $[x - ct, x + ct]$ as the *domain of dependence* of the point (x,t) in space-time.



$u(x,t)$ is determined from (i) $f(x-ct)$, (ii) $f(x+ct)$ and (iii) $g(s), x-ct \le s \le x+ct$

Appendix: Separation of variables?

During this lecture, a student asked if the solution to Example 2 could be obtained by the method of separation of variables. I invited him to explore this and to report the results in the next lecture. At the end of class, he told me that he had tried during the lecture to find a solution in this way but was unsuccessful. (Of course, I was very shocked and hurt that he would allow his attention to be taken away from my lecture!) Let's explore this further.

In fact, let's return to Example 1, the PDE,

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0, \tag{122}$$

with initial condition

$$w(x,0) = P(x) = \cos x. \tag{123}$$

Indeed, it would appear that this PDE is well suited for the separation-of-variables method, since it can be rewritten as

$$\frac{\partial w}{\partial t} = 3\frac{\partial w}{\partial x}. \tag{124}$$

If we assume a solution of the form,

$$w(x,t) = u(x)v(t), \tag{125}$$

and substitute into this PDE, we obtain

$$u(x)v'(t) = 3u'(x)v(t). \tag{126}$$

This, of course, can be written in separated form, i.e.,

$$\frac{v'(t)}{v(t)} = 3\frac{u'(x)}{u(x)} = \mu, \tag{127}$$

where μ denotes the separation constant. (Recall that x and t are independent variables, so the only way that the equality of these two independent sides can hold is if they are constant.) We may integrate to obtain

$$v(t) = v(0)e^{\mu t}, \qquad x(t) = x(0)e^{\frac{1}{3}\mu t}. \tag{128}$$

There seem to be a few problems with this approach. First of all, unlike the case of the heat and wave equations on finite intervals, it doesn't appear that we can determine particular values of the separation constant μ – there are no boundary conditions that allow us to do so. That being said, we would encounter the same type of problem with the heat and wave equations on the entire real line $\mathbf{R}$, so that is probably one reason that the method doesn't work.

Another problem is how to incorporate the initial condition into the separation-of-variables solution.

In summary, it looks like the method of separation of variables will not work for these problems as they are posed.