Instituto Superior Técnico

Digital Signal Processing Systems

FIR & IIR filter design

1. Using the Fourier series development method, design an ideal high-pass FIR filter with unity DC gain, cut-off at $f_{p_1} = 3$ kHz and order 9. What kind of coefficient symmetry can you expect? Compute the arithmetic format that should be used for the coefficients and for the filter output.

2. Design a low-pass IIR filter with unity DC gain and with two real poles at frequencies $f_{p_1} = 1$ kHz and $f_{p_2} = 2$ kHz using the impulse invariance method and the bilinear transformation method with pole frequency pre-distortion at the pole frequencies.

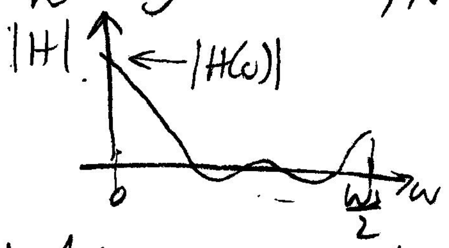
Note: Consider the sampling frequency to be $f_s = 8$ kHz.

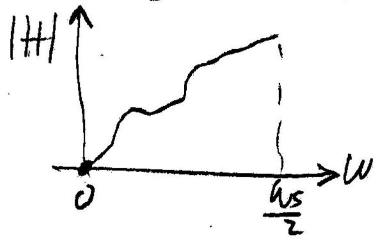
Ideal high-pass filter, $f_c=3\,kHz$, $f_s=8\,kHz$, $\tau_g=0.5\,ms$.

$\tau_g=\frac{N-1}{2}T_s \rightarrow N=2\tau_g f_s+1=2\times0.5\times10^{-3}\times8\times10^{3}+1=9$ coefficients. (odd)

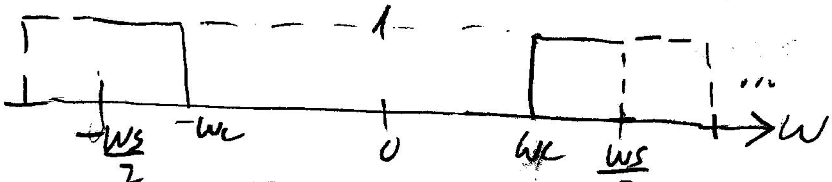
A high-pass filter requires $\left|H\left(\frac{\omega_s}{2}\right)\right|\neq 0$, so there are two possibilities

$h_n\rightarrow$ symmetric, N odd $\qquad$ $h_n\rightarrow$ anti-symmetric, N even





N odd $\Rightarrow$ symmetric coefficients.



$N=9=-4:4$

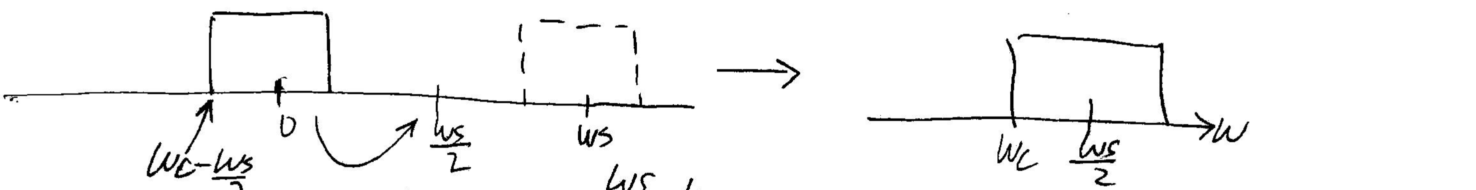
$$h_n=\frac{1}{\omega_s}\left[\int_{-\frac{\omega_s}{2}}^{-\omega_c}e^{jn\omega T_s}d\omega+\int_{\omega_c}^{\frac{\omega_s}{2}}e^{jn\omega T_s}d\omega\right]$$

$$=\frac{1}{\omega_s}\left[\frac{e^{jn\omega T_s}}{jnT_s}\Big|_{-\frac{\omega_s}{2}}^{-\omega_c}+\frac{e^{jn\omega T_s}}{jnT_s}\Big|_{\omega_c}^{\frac{\omega_s}{2}}\right]$$

$$=\frac{1}{2\pi}\left[\frac{e^{-jn\omega_c T_s}-e^{-jn\frac{\omega_s}{2}T_s}+e^{jn\frac{\omega_s}{2}T_s}-e^{jn\omega_c T_s}}{jn}\right]$$

$$=\frac{-\sin n\omega_c T_s+\sin n\pi}{\pi n}=\begin{cases}1-\frac{\omega_c T_s}{\pi}, & n=0\\ -\frac{\sin n\omega_c T_s}{\pi n}, & n\neq0\end{cases}$$

Another way: A high-pass filter is a low-pass filter shifted by $\frac{\omega_s}{2}$.



low pass filter: $h'_n=\frac{1}{\omega_c}\int_{-\frac{\omega_s}{2}+\omega_c}^{\frac{\omega_s}{2}-\omega_c}e^{jn\omega T_s}d\omega=\frac{\sin n\left(\frac{\omega_s}{2}-\omega_c\right)nT_s}{\pi n}$

$$=\frac{\sin n(\pi-\omega_c T_s)}{\pi n}=-\frac{\cos n\pi\,\sin n\omega_c T_s}{\pi n}=$$

$$=-(-1)^n\frac{\sin n\omega_c T_s}{\pi n},\quad n\neq0,\quad h'_0=\frac{\pi-\omega_c T_s}{\pi}=1-\frac{\omega_c T_s}{\pi}$$

low-pass

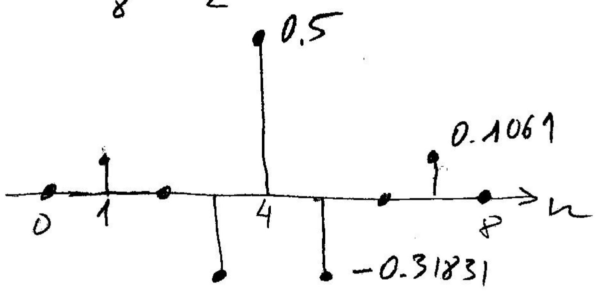
$\downarrow$

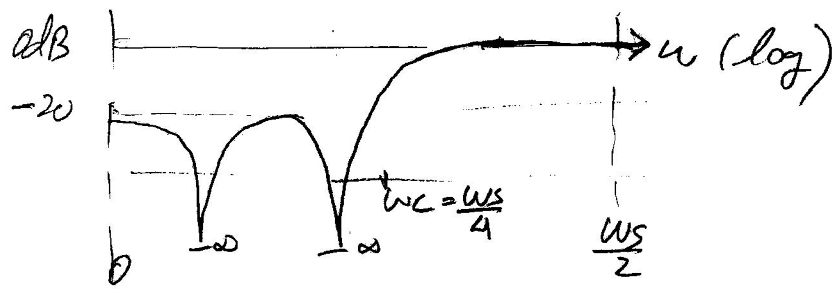
$$\underset{\substack{\uparrow\\ \text{high-pass}}}{h_n}=\underbrace{h'_n\times e^{j\frac{\omega_s}{2}nT_s}}_{\substack{\text{shifting}\equiv\text{modulation}\\ \text{by }\frac{\omega_s}{2}}}=h'_n e^{j\pi n}=h'_n(-1)^n=\begin{cases}-\frac{\sin n\omega_c T_s}{\pi n}, & n\neq0\\ 1-\frac{\omega_c T_s}{\pi}, & n=0\end{cases}$$

equal!

$h_0 = h_8 = 0$
$h_1 = h_7 = 0.1061$
$h_2 = h_6 = 0$
$h_3 = h_5 = -0.31831$
$h_4 = 0.5 = 1 - \frac{\omega_c T_s}{\pi} = \frac{1}{2}$

$\omega_c T_s = 2\pi \times \frac{2}{8} = \frac{\pi}{2}$





Arithmetic format:

$$y_n = \sum_{i=0}^{N-1} h_i\, x(n-i) \longrightarrow \text{Non recursive computation}$$

Input samples Q_m. The worst case is when all products add up with the same sign so

$$|y_n|_{max} = \sum_{i=0}^{N-1} |h_i| \underbrace{|x(n-i)|_{max}}_{x_{max}} = x_{max} \cdot \sum_{i=0}^{N-1} |h_i|$$

The format for y_n depends on $\sum_{i=0}^{N-1} |h_i| = 1.3488$ which is >1 but <2. So y_n has to be stored in Q_{m-1} (one more integer bit than the samples). The coefficients may be stored in Q_{n-1} where n is the number of bits of the words (this is because $|h_n| < 1$ for all n).

2 Lowpass filter, $f_c = 2\,kHz$, $|H(0)| = 1 \rightarrow H_{LP}(s) = \frac{\omega_c}{s + \omega_c}$

Impulse invariance:

1. $h(t) = \mathcal{L}^{-1}\{H(s)\} = \omega_c\, e^{-\omega_c t}\, u(t)$
2. $h_n = T_s\, h(nT_s) = \omega_c T_s\, e^{-\omega_c T_s n}\, u(n)$
3. $H(z) = \mathcal{Z}\{h_n\} = \omega_c T_s \sum_{n=0}^{\infty} e^{-\omega_c T_s n} z^{-n} = \frac{\overset{b_0}{\omega_c T_s}}{1 - \underset{a_1}{e^{-\omega_c T_s}} z^{-1}}$

$$= \frac{0.7854}{1 - 0.4559 z^{-1}}$$

$\omega=0,\ z=1,\ |H(1)| = \dfrac{0.7854}{1-0.4436\,z^{-1}} = 1.4436\,!\ (\gg 1)$

This is due to aliasing of the frequency response. Need to normalize so $b_0 = 0.7854/1.4436 = 0.5441$

$$H(z) = \frac{0.5441}{1-0.4559\,z^{-1}}$$

3. Bilinear transformation with $f_c = 1KHz$

pré-distortion : $\omega_a = \dfrac{2}{T_s}\,tg\,\dfrac{\omega_d T_s}{2} = 2f_s\,tg\left(\pi\dfrac{f_c}{f_s}\right) =$
$= 6627.4\ rad/s = 2\pi \times \underbrace{1054.8}_{>f_c}\ Krad/s$

$$H(z) = \left.\frac{\omega_a}{s+\omega_a}\right|_{s=\frac{2}{T_s}\frac{z-1}{z+1}} = \frac{\omega_a}{\frac{2}{T_s}\frac{z-1}{z+1}+\omega_a} = \frac{\omega_a\frac{T_s}{2}(z+1)}{z-1+\omega_a(z+1)\frac{T_s}{2}} =$$

$$= \frac{\frac{\omega_a T_s}{2}(z+1)}{z\left(1+\frac{\omega_a T_s}{2}\right)-\left(1-\frac{\omega_a T_s}{2}\right)} =$$

$$= \frac{\frac{\omega_a T_s}{2}}{1+\frac{\omega_a T_s}{2}}\cdot\frac{z+1}{z-\frac{1-\frac{\omega_a T_s}{2}}{1+\frac{\omega_a T_s}{2}}}$$

$$= \frac{\frac{\omega_a T_s}{2}}{1+\frac{\omega_a T_s}{2}}\cdot\frac{1+z^{-1}}{1+\frac{1-\frac{\omega_a T_s}{2}}{1+\frac{\omega_a T_s}{2}}z^{-1}}$$

$$= 0.2929\cdot\frac{1-z^{-1}}{1-0.4142\,z^{-1}}$$