Instituto Superior Técnico

Sistemas de Processamento Digital de Sinais

2^{o} Semestre de 2012/2013

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FIR/IIR filter design

1) Design a low-pass ideal FIR filter with cutoff frequency $f_c = 1$ kHz, with $N = 8$ coefficients and linear phase using the frequency sampling method (non-recursive). Determine the most accurate Q format to implement the filter using fixed point arithmetic.

2) Design the same filter but using the recursive frequency sampling method.

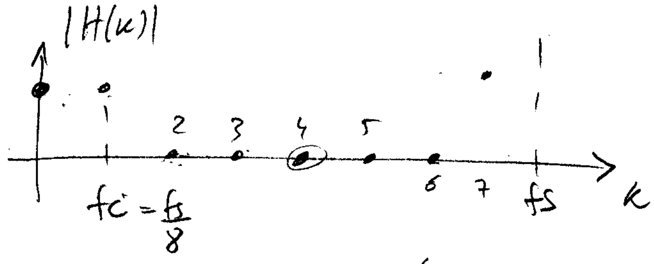
3) Design a low-pass IIR filter with unity DC gain and with two real poles at frequencies $f_{p_1} = 1$ kHz and $f_{p_2} = 2$ kHz using the impulse invariance method and the bilinear transformation method with pole frequency pre-distortion.

Note: Consider the sampling frequency to be $f_s = 8$ kHz.

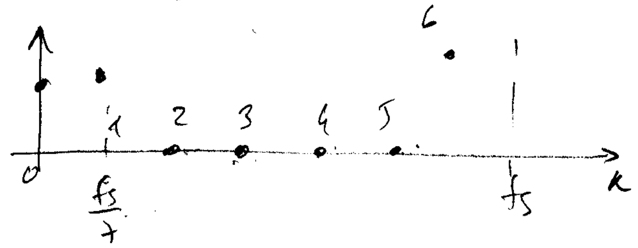
1) Filtro FIR passa-baixo ideal $f_c = 1kHz$, $N=8$

$f_k = \frac{k}{N} f_s \quad k=0,\dots,7$

N=8



N=7



Os coeficients obtêm-se de

$$h_n = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{j\frac{2\pi}{N}nk}$$

Fase linear $\longrightarrow \tau_g = \frac{N-1}{2}T_s \longrightarrow \phi(\omega_k) = -\frac{N-1}{2}T_s\,\omega_k =$

$$= -\frac{N-1}{2}T_s\,\frac{2\pi k}{N}f_s$$

$$= -\underbrace{\frac{N-1}{2}}_{\alpha}\,\frac{2\pi}{N}k$$

N par

Para $k = 0,\dots,\frac{N}{2}-1 \longrightarrow H(k) \longrightarrow \phi(\omega_k)$

$= \frac{N}{2}+1,\dots,N-1 \longrightarrow H^*(k) \longrightarrow -\phi(\omega_k)$

$k = \frac{N}{2} \longrightarrow H(k)$ é real $\Rightarrow \phi(\omega_{\frac{N}{2}}) = 0$

Então,

$$h_n = \frac{1}{N}\Big[|H(0)| + |H(\tfrac{N}{2})|\underbrace{e^{j\frac{2\pi}{N}n\frac{N}{2}}}_{e^{j\pi n}=(-1)^n} + \sum_{k=1}^{\frac{N}{2}-1}|H(k)|\Big(e^{-j\alpha\frac{2\pi k}{N}}e^{j\frac{2\pi}{N}nk} + e^{j\alpha\frac{2\pi}{N}k}\underbrace{e^{j\frac{2\pi}{N}n(N-k)}}_{e^{-j\frac{2\pi}{N}nk}}\Big)\Big]$$

$$= \frac{1}{N}\Big[|H(0)| + (-1)^n|H(\tfrac{N}{2})| + 2\sum_{k=1}^{\frac{N}{2}-1}|H(k)|\cos\Big[\frac{2\pi}{N}k(n-\alpha)\Big]\Big]$$

$$\underbrace{n = 0,\dots,N-1}_{\text{já podemos fazer isto porque consideramos a fase linear}}$$

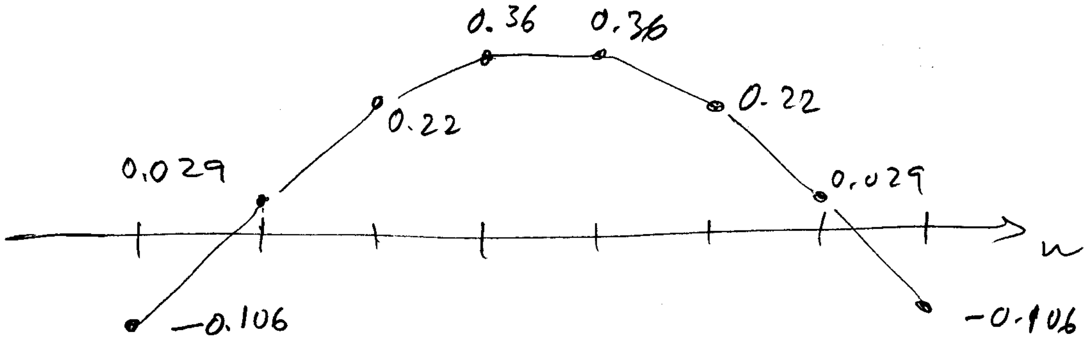
N impar

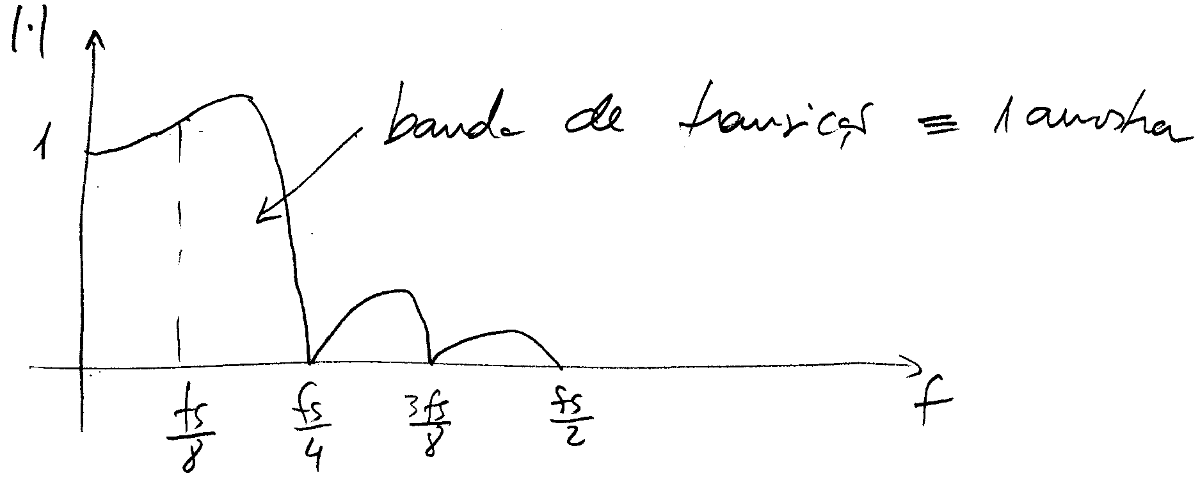
$$h_n = \frac{1}{N}\left[|H(0)| + 2 \sum_{k=1}^{\frac{N-1}{2}} |H(k)| \cos\left[\frac{2\pi}{N} k (n - \alpha)\right]\right] \qquad n = 0, \dots, N-1$$

No nosso caso $N = 8$

$$h_n = \frac{1}{8}\left[1 + 2 \sum_{k=1}^{3} |H(k)| \cos \frac{\pi}{4} k \Big(n - \underbrace{\tfrac{8-1}{2}}_{\alpha}\Big)\right]$$

$$= \frac{1}{8}\left[1 + 2 \cos \frac{\pi}{4}\left(n - \frac{7}{2}\right)\right]$$



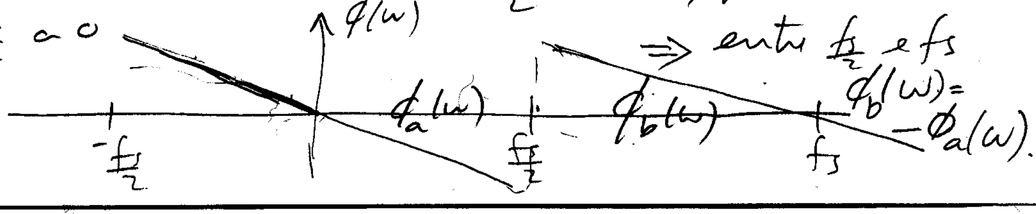


$$\sum_{i=0}^{7} |a_i| = 2 \times (0.106 + 0.029 + 0.22 + 0.36) = 1{,}43$$

$\Rightarrow$ utilizar formato Q14

Nota: Para a resposta h_n ser real, $|H(\omega)|$ tem de ter simetria par e $\phi(\omega)$ simetria ímpar. Portanto $\phi(-\omega) = -\phi(\omega)$ e no intervalo $\frac{f_s}{2}$ a f_s, $\phi(\omega) = \phi(-\omega')$ com $\omega' = -\frac{f_s}{2}$ a 0

$\Rightarrow$ entre $\frac{f_s}{2}$ e f_s $\phi_b(\omega) = -\phi_a(\omega)$.



2) $$H(z) = \sum_{n=0}^{N-1} h_n z^{-n} = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j\frac{2\pi}{N}nk} \right) z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}k} z^{-1} \right)^n$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \frac{1 - \left(e^{j\frac{2\pi}{N}k} z^{-1}\right)^N}{1 - e^{j\frac{2\pi}{N}k} z^{-1}} = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j\frac{2\pi}{N}k} z^{-1}}$$

$N=8$, par

$$= \frac{1-z^{-N}}{N} \left[\frac{|H(0)|}{1-z^{-1}} + \sum_{k=1}^{\frac{N}{2}-1} |H(k)| \left(\frac{e^{j\phi_k}}{1-e^{j\frac{2\pi}{N}k} z^{-1}} + \frac{e^{-j\phi_k}}{1-e^{-j\frac{2\pi}{N}k} z^{-1}} \right) \right]$$

Padmitindo $H(\frac{N}{2})=0$

$$= \frac{1-z^{-N}}{N} \left[\frac{|H(0)|}{1-z^{-1}} + \sum_{k=1}^{\frac{N}{2}-1} |H(k)| \frac{e^{j\phi_k}\left(1-e^{-j\frac{2\pi}{N}k} z^{-1}\right) + e^{-j\phi_k}\left(1-e^{j\frac{2\pi}{N}k} z^{-1}\right)}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}} \right]$$

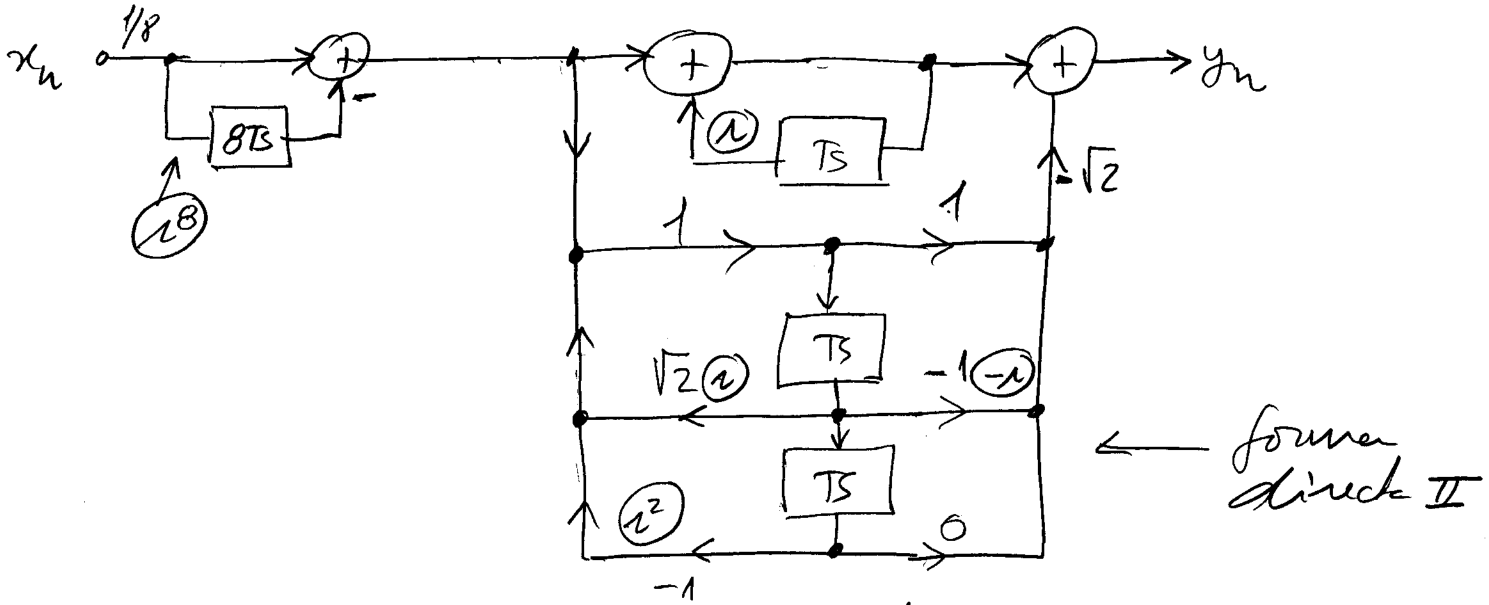
$$= \frac{1-z^{-N}}{N} \left[\frac{|H(0)|}{1-z^{-1}} + 2 \sum_{k=1}^{\frac{N}{2}-1} |H(k)| \frac{\cos\phi_k - \cos\left(\phi_k - \frac{2\pi}{N}k\right) z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}} \right]$$

$$= \frac{1-z^{-8}}{8} \left[\frac{1}{1-z^{-1}} + 2 \frac{\cos\phi_1 - \cos(\phi_1 - \frac{\pi}{4}) z^{-1}}{1 - 2\cos\frac{\pi}{4} z^{-1} + z^{-2}} \right]$$

$$\phi_k = -\frac{N-1}{2} \frac{2\pi}{N} k \quad \longrightarrow \quad \phi_1 = -\frac{7}{2}\frac{\pi}{4} = -\frac{7\pi}{8}$$

$$\cos\frac{7\pi}{8} = -\frac{1}{\sqrt{2}} \qquad \cos(\phi_1 - \frac{\pi}{4}) = \cos\left(-\frac{7\pi}{8} - \frac{2\pi}{8}\right) = \cos\frac{9\pi}{8} = -\frac{1}{\sqrt{2}}$$

$$= \frac{1-z^{-8}}{8} \left[\frac{1}{1-z^{-1}} - \sqrt{2} \frac{1-z^{-1}}{1-\sqrt{2}z^{-1}+z^{-2}} \right]$$



<u>Nota</u>: o filtro pode ser instável! Amostrando com $r z^{-1}$ com $r \lesssim 1$ ($r = 1-10^{-3}$ por exemplo)

$$H(z) = \frac{1 - r^8 z^{-1}}{8}\left[\frac{1}{1 - r z^{-1}} - \sqrt{2}\,\frac{1 - r z^{-1}}{1 - \sqrt{2} r z^{-1} + r^2 z^{-2}}\right]$$

$\boxed{\text{Conservação da resposta ao impulso}}$

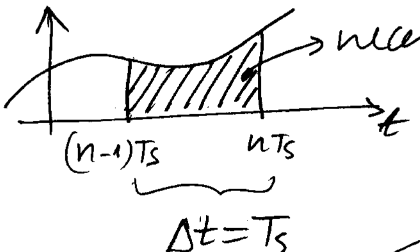
3) Pólos reais em $f_{p1} = 1\,kHz$ e $f_{p2} = 2\,kHz$

$$H(s) = \omega_{p1}\omega_{p2}\,\frac{1}{(s+\omega_{p1})(s+\omega_{p2})} = \frac{\omega_{p1}\omega_{p2}}{\omega_{p2}-\omega_{p1}}\left(\frac{1}{s+\omega_{p1}} - \frac{1}{s+\omega_{p2}}\right)$$

Aplicando a transformada de Laplace inversa

$$h(t) = \frac{\omega_{p1}\omega_{p2}}{\omega_{p2}-\omega_{p1}}\left(e^{-\omega_{p1}t} - e^{-\omega_{p2}t}\right)u(t) \quad \times T_s$$

muito importante para manter a energia

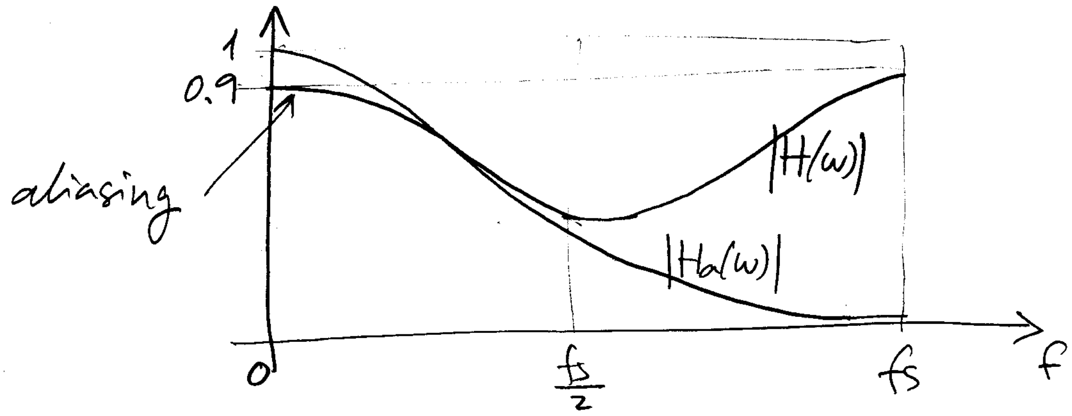


$h(t)_{t=nT_s}$

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} = K\left(\frac{1}{1-e^{-\omega_{p1}T_s}z^{-1}} - \frac{1}{1-e^{-\omega_{p2}T_s}z^{-1}}\right)$$

$$= K\,\frac{\left(-e^{-\omega_{p2}T_s} + e^{-\omega_{p1}T_s}\right)z^{-1}}{1-\left(e^{-\omega_{p1}T_s}+e^{-\omega_{p2}T_s}\right)z^{-1} + e^{-(\omega_{p1}+\omega_{p2})T_s}z^{-2}}, \quad K = T_s\,\frac{\omega_{p1}\omega_{p2}}{\omega_{p2}-\omega_{p1}}$$

$$= 0.3896 \frac{z^{-1}}{1 - 0.6638 z^{-1} + 0.09478 z^{-2}}$$



$\boxed{\text{Transformação bilinear}}$

$$s = \frac{2}{T_s} \frac{z-1}{z+1}$$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T_s}\frac{z-1}{z+1}} = \frac{\omega_{p1}\omega_{p2}}{\omega_{p2}-\omega_{p1}} \cdot \frac{1}{\left(\frac{2}{T_s}\frac{z-1}{z+1}+\omega_{p1}\right)\left(\frac{2}{T_s}\frac{z-1}{z+1}+\omega_{p2}\right)}$$

$$= \frac{\omega_{p1}\omega_{p2}}{\omega_{p2}-\omega_{p1}} \frac{\left(\frac{T_s}{2}\right)^2 (z+1)^2}{\left(z-1+\omega_{p1}\frac{T_s}{2}(z+1)\right)\left(z-1+\omega_{p2}\frac{T_s}{2}(z+1)\right)} =$$

$$= \frac{\omega_{p1}\omega_{p2}}{\omega_{p2}-\omega_{p1}} \left(\frac{T_s}{2}\right)^2 \frac{(z+1)^2}{(z-1)^2 + \omega_{p2}\frac{T_s}{2}(z^2-1) + \omega_{p1}\frac{T_s}{2}(z^2-1) + \omega_{p1}\omega_{p2}\left(\frac{T_s}{2}\right)^2 \times (z+1)^2}$$

$$= k \frac{1 + 2z^{-1} + z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} \rightarrow \text{funções de 2ª ordem}$$

Pré-distorção da escala de frequências

$$s = \frac{2}{T_s}\frac{z-1}{z+1} \rightarrow j\omega_a = \frac{2}{T_s} j \operatorname{tg} \frac{\omega_d T_s}{2} \rightarrow \boxed{\omega_a = \frac{2}{T_s} \operatorname{tg} \frac{\omega_d T_s}{2}}$$

$\operatorname{tg} x \geq x$

$$\omega_a \geq \frac{2}{T_s}\left(\frac{\omega_d T_s}{2}\right) = \omega_d$$

$$\left.\begin{aligned} \omega_{d1} = 2\pi \times 1\,\text{kHz} &\longrightarrow \omega_{a1} = 2\pi \times 1.0547\ \text{kHz} \\ \omega_{d2} = 2\pi \times 2\,\text{kHz} &\longrightarrow \omega_{a2} = 2\pi \times 2.5464\ \text{kHz} \end{aligned}\right\} \text{usar estas frequências no filtro analógico}$$

4) $$y_n = a_1 y_{n-1} + a_2 y_{n-2} + b_0 x_n + b_1 x_{n-1} + b_2 x_{n-2}$$

$$|y_n|_{max} = |a_1|\underbrace{|y_{n-1}|}_{1} + |a_2|\underbrace{|y_{n-2}|}_{1} + |b_0|\underbrace{|x_n|}_{1} + |b_1|\underbrace{|x_{n-1}|}_{1} + |b_2|\underbrace{|x_{n-2}|}_{1}$$

$$= |a_1| + |a_2| + |b_0| + |b_1| + |b_2|.$$

Usar este valor para estimar o formato aritmético. Mas é só uma estimativa.