Instituto Superior Técnico

Sistemas de Processamento Digital de Sinais

Digital Signal Processing Systems

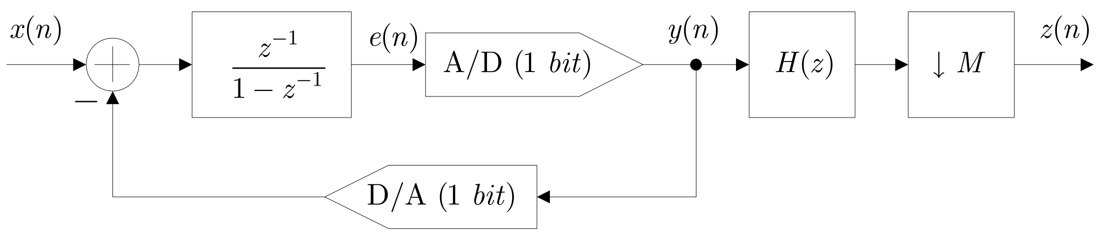
Sigma-Delta A/D converter

Consider a first order $\Sigma\Delta$ A/D converter for which the noise modulator is as depicted in figure 1a). The output 1-bit signal $y(n)$ is filtered by an ideal digital lowpass filter with transfer function $H(z)$ with magnitude represented in figure 1b) and then decimated by a factor M.

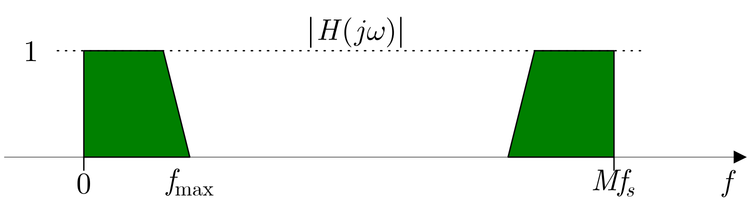
a) Determine the maximum signal-to-noise ratio (SNR) at the output $z(n)$ and plot its variation in dB as a function of the oversampling factor M.

b) Determine and plot the effective number of bits as a function of M.

c) Explain how many dB by octave of M are achieved by oversampling and by noise-shaping.



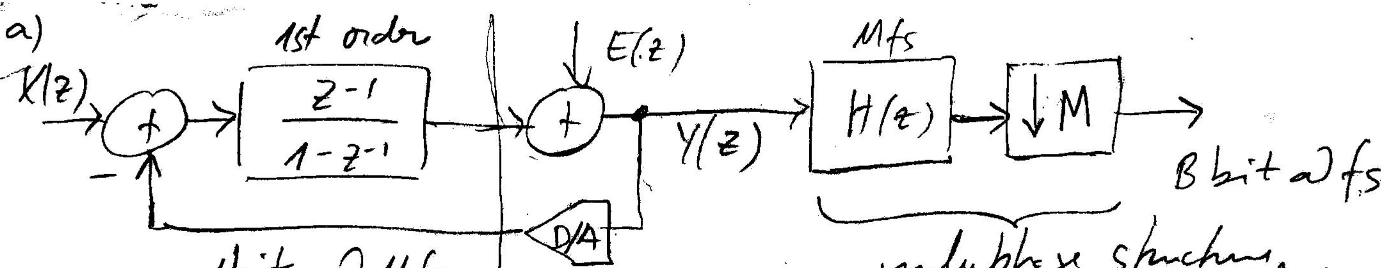
(a)



(b)

Figure 1: a) Sigma-Delta noise shaping model; b) Decimator magnitude filter response.

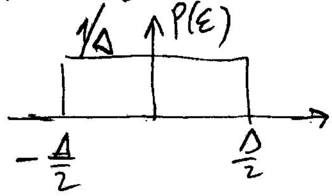
a)



polyphase structure, words with B bit

analog ⟷ digital

Quantization noise: $\Delta = \frac{2A}{2^B}$ = quantization step.



$$\bar{\varepsilon} = 0, \quad \sigma_\varepsilon^2 = \int_{-\Delta/2}^{\Delta/2} p(\varepsilon)(\varepsilon-\bar{\varepsilon})^2 d\varepsilon = \frac{\Delta^2}{12}$$

1 bit $\Rightarrow \Delta = A$ (quantize from $-A$ to A) $\Rightarrow \sigma_\varepsilon^2 = \frac{A^2}{12}$

Signal Power $\frac{A^2}{2}$ (sinewave) $\Rightarrow SNR_{in} = \frac{A^2/2}{A^2/12} = 6 = 7.8\,dB$

$(\approx 1.7 + 6B)\,dB$

$$Y(z) = \left(X(z) - Y(z)\right)\frac{z^{-1}}{1-z^{-1}} + E(z) \rightarrow Y(z)\underbrace{\left(1 + \frac{z^{-1}}{1-z^{-1}}\right)}_{\frac{1}{1-z^{-1}}} = \frac{z^{-1}}{1-z^{-1}}X(z) + E(z)$$

$$Y(z) = \underbrace{z^{-1}}_{\text{this is only a delay}} X(z) + \underbrace{(1-z^{-1})}_{\text{noise-shaping function } N(z) \text{ (high-pass)}} E(z)$$

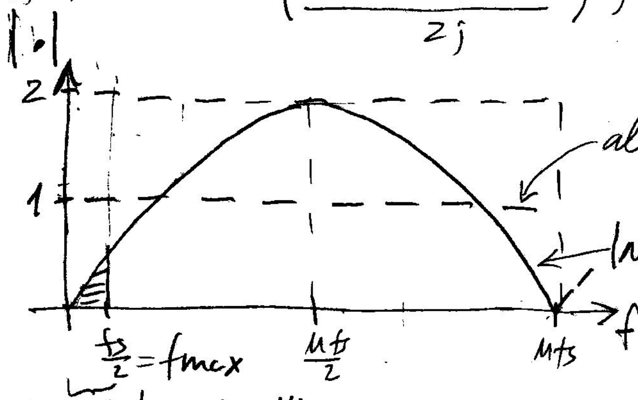
$N(z) = 1 - z^{-1}$, $N(j\omega) = 1 - e^{-j\omega\frac{T_S}{M}}$

careful! Noise shaper operates @ Mfs

$$N(j\omega) = e^{-j\frac{\omega T_S}{2M}}\left(\frac{e^{j\frac{\omega T_S}{2M}} - e^{-j\frac{\omega T_S}{2M}}}{2j}\right)2j = 2j\,e^{-j\frac{\omega T_S}{2M}}\sin\frac{\omega T_S}{2M} \rightarrow |N(j\omega)| = 2\left|\sin\frac{\omega T_S}{2M}\right|$$

linear phase

$\omega = 2\pi f$

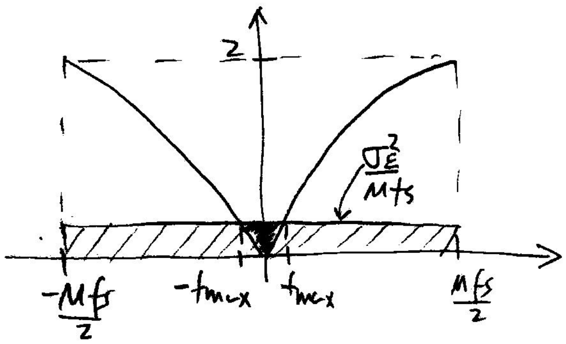


$|N(f)| = 2\left|\sin\frac{\pi f T_S}{M}\right|$

Quantization noise is white with power spectral density $S_\varepsilon(f) = \frac{\sigma_\varepsilon^2}{Mf_s}$, $|f| < \frac{Mf_s}{2}$

signal bandwidth

At the output: signal power remains $A^2/2$ (unchanged). Noise undergoes high pass filtering and power is reduced.



$$\sigma_{out}^2 = \int_{-f_{max}}^{f_{max}} S_\varepsilon(f)\,|N(f)|^2 df =$$

$$= 2\int_0^{f_{max}} \frac{\sigma_\varepsilon^2}{Mf_s}\left(2\left|\sin\frac{\pi f T_S}{M}\right|\right)^2 df =$$

$$= \frac{8\sigma_\varepsilon^2}{M f_s}\int_0^{f_{max}} \sin^2\left(\frac{\pi f T_s}{M}\right) df \approx \frac{8\sigma_\varepsilon^2}{M f_s}\int_0^{f_{max}}\left(\frac{\pi f T_s}{M}\right)^2 df = \frac{8\pi^2\sigma_\varepsilon^2}{M^3 f_s^3}\int_0^{f_{max}} f^2\, df$$

because $f_{max} \ll M f_s$ ($\sin x \approx x$)

$$= \frac{8\pi^2\sigma_\varepsilon^2}{M^3 f_s^3}\cdot\frac{f_{max}^3}{3} = \frac{8\pi^2\sigma_\varepsilon^2}{M^3 f_s^3}\cdot\frac{(f_s/2)^3}{3} = \frac{\pi^2}{3M^3}\sigma_\varepsilon^2 = \frac{\pi^2}{3M^3}\cdot\frac{A^2}{12} = \frac{\pi^2}{36M^3}A^2$$

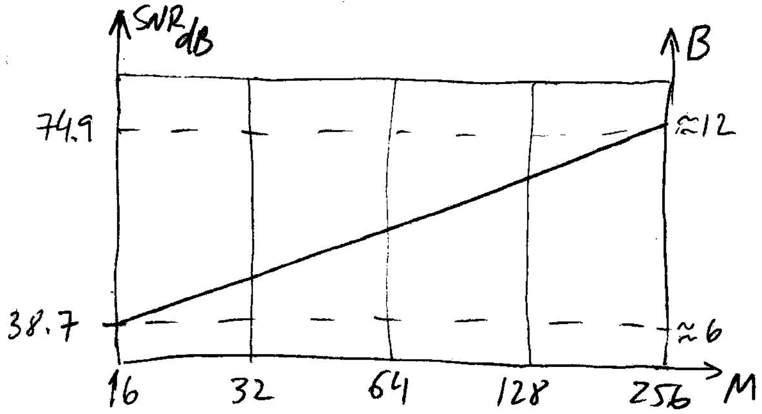
$$SNR_{out} = \frac{A^2/2}{\frac{\pi^2 A^2}{36M^3}} = \frac{18}{\pi^2}M^3, \quad SNR\,dB = 10\log_{10}\frac{18M^3}{\pi^2} = \underbrace{30\log_{10}M}_{=9\log_2 M} + 2.61$$

$= 9\log_2 M \equiv 9\,dB$ for each doubling of M.

b)

$$SNR = 1.76 + 6B = 9\log_2 M + 2.61 \Rightarrow$$

$$\Rightarrow B = \frac{9\log_2 M + 0.85}{6} \simeq \frac{3}{2}\log_2 M + 0.14 \approx \frac{3}{2}\log_2 M \equiv 1.5 \text{ bit for every doubling of } M$$



c) $9\log_2 M = (3+6)\log_2 M \equiv$

$(3+6)$ dB/octave

from noise shaping

from oversampling (each doubling of f_s reduces $S_\varepsilon(f)$ by half $\Rightarrow$ 3dB improvement on the SNR

For a $\underline{n}$th order shaping filter, a similar analysis shows that the SNR is proportional to M^{2n+1} so the SNR increase is

$$10\log_{10} M^{2n+1} = 10(2n+1)\log_{10} M, \text{ that is } (20n+10)\,dB/decade$$

or $3(2n+1) = (6n+3)$ dB/octave

oversampling

noise shaping