Instituto Superior Técnico

Sistemas de Processamento Digital de Sinais

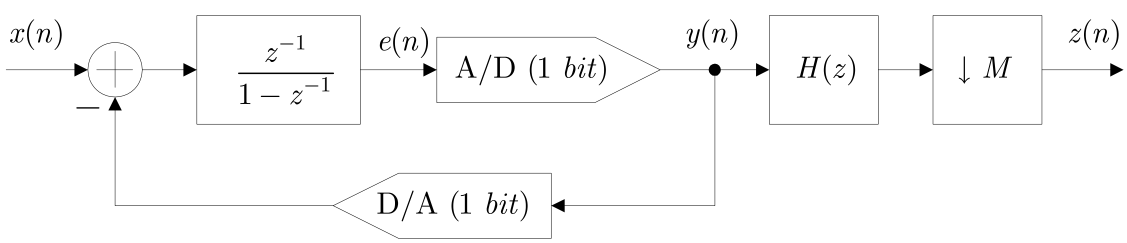
Sigma-Delta A/D converter

Consider a first order $\Sigma\Delta$ A/D converter for which the noise modulator is as depicted in figure 1a). The output 1-bit signal $y(n)$ is filtered by an ideal digital lowpass filter with transfer function $H(z)$ with magnitude represented in figure 1b) and then decimated by a factor M.

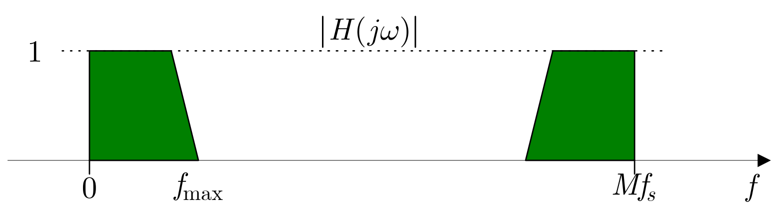
a) Determine the maximum signal-to-noise ratio (SNR) at the output $z(n)$ and plot its variation in dB as a function of the oversampling factor M.

b) Determine and plot the effective number of bits as a function of M.

c) Explain how many dB by octave of M are achieved by oversampling and by noise-shaping.

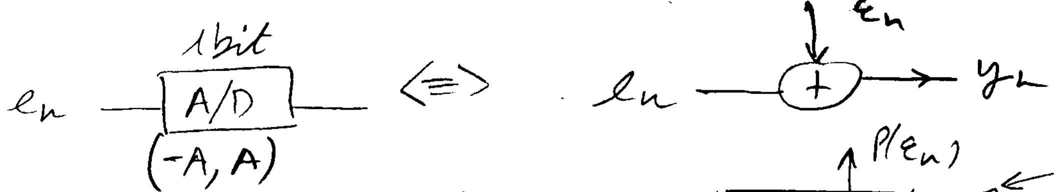


(a)

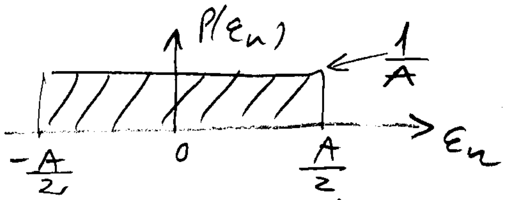


(b)

Figure 1: a) Sigma-Delta noise shaping model; b) Decimator magnitude filter response.



↑ tensão de saída

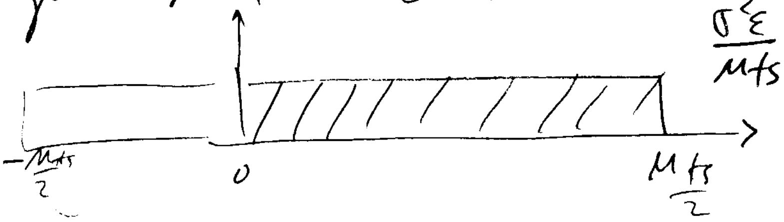


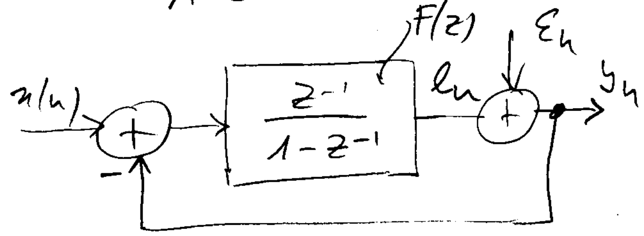
$f_s = 2 f_{max} \rightarrow$ Nyquist

$M f_s \rightarrow$ freq. a que o conversor funciona.

Densidade espectral de ruído de quantificação $S_\varepsilon(f)$

$$\sigma^2_{\varepsilon_n} = \int_{-A}^{A} \varepsilon^2 p(\varepsilon)\, d\varepsilon = \frac{1}{A}\int_{-A/2}^{A/2} \varepsilon^2 d\varepsilon = \frac{1}{A}\left.\frac{\varepsilon^3}{3}\right|_{-\frac{A}{2}}^{\frac{A}{2}} = \frac{1}{A}\frac{1}{3}\left(\frac{A}{2}\right)^3 \cdot 2 = A^2/12$$





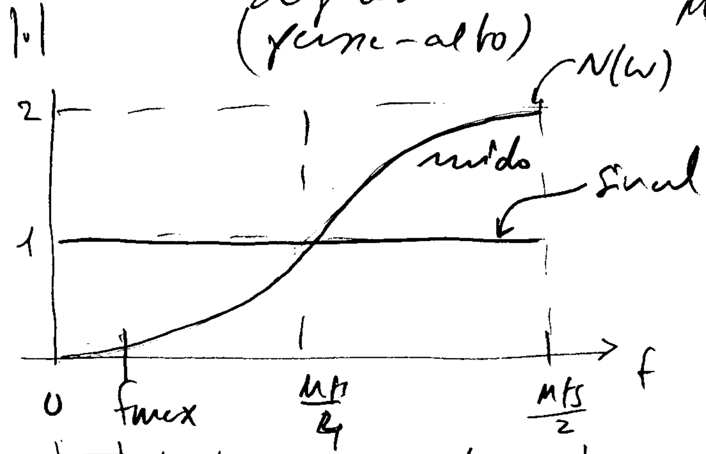
$$Y(z) = \mathcal{E}(z) + [X(z) - Y(z)]F(z)$$

$$Y(z)[1 + F(z)] = \mathcal{E}(z) + F(z)X(z)$$

$$Y(z) = \frac{1}{1+F(z)}\mathcal{E}(z) + \frac{F(z)}{1+F(z)}X(z)$$

$$1 + \frac{z^{-1}}{1-z^{-1}} = \frac{1}{1-z^{-1}}$$

$$= \underbrace{(1 - z^{-1})\mathcal{E}(z)}_{\text{diferenciador (passa-alto)}} + \underbrace{z^{-1}}_{\text{atraso de 1 amostra ao ritmo } Mf_s} X(z)$$



só esta banda interessa!

$$|z^{-1}|_{z = e^{j\omega \frac{T_s}{M}}} = 1$$

$$|1 - z^{-1}|_{z = e^{j\omega \frac{T_s}{M}}} = \left|1 - e^{-j\omega\frac{T_s}{M}}\right| = \left|2\sin\frac{\omega T_s}{2M}\right| = N(\omega)$$

$$2\pi f \frac{T_s}{2M} = \frac{\pi f T_s}{M}$$

$$= |\quad|^2 \times$$

Potência de ruído na banda

$$\sigma^2_{\varepsilon_{banda}} = \int_{-f_{max}}^{f_{max}} S_\varepsilon(f)\,df = 2\frac{\sigma^2_\varepsilon}{Mf_s}\int_0^{f_{max}}\left(2\sin\frac{2\pi f T_s}{2M}\right)^2 df$$

$\sin x \approx x$

$$\approx \frac{8\sigma^2_\varepsilon}{Mf_s}\int_0^{f_{max}}\left(\frac{\pi f T_s}{M}\right)^2 df = \frac{8\pi^2}{Mf_s}\frac{T_s^2}{M^2}\left.\frac{f^3}{3}\right|_0^{f_{max}} = \frac{8\pi^2}{Mf_s}\frac{T_s^2}{M^2}\frac{f_{max}^3}{3}\sigma^2_\varepsilon$$

$$f_s = 2 f_{max} \rightarrow f_{max} = \frac{f_s}{2}$$

$$\sigma^2_{\varepsilon_{banda}} = 8\pi^2 \frac{T_s^2}{M^2} \sigma^2_\varepsilon \left(\frac{f_s}{2}\right)^3 \times \frac{1}{3 M f_s} = \frac{\pi^2}{3M^3} \times \sigma^2_\varepsilon \rightarrow \frac{\pi^2}{3M^3}\sigma^2_\varepsilon$$

$$= \frac{\pi^2}{3M^3}\,\frac{A^2}{12} = \frac{\pi^2}{36M^3}A^2$$

Pot. de sinal $= A^2/2$

$$SNR = \frac{A^2/2}{\frac{\pi^2}{36M^3}A^2} = \frac{18}{\pi^2}M^3$$

$$SNR_{dB} = 10\log_{10}\frac{18M^3}{\pi^2} = 30\log_{10} M + \underbrace{10\log_{10}\left(\frac{18}{\pi^2}\right)}_{2.61\,dB}$$

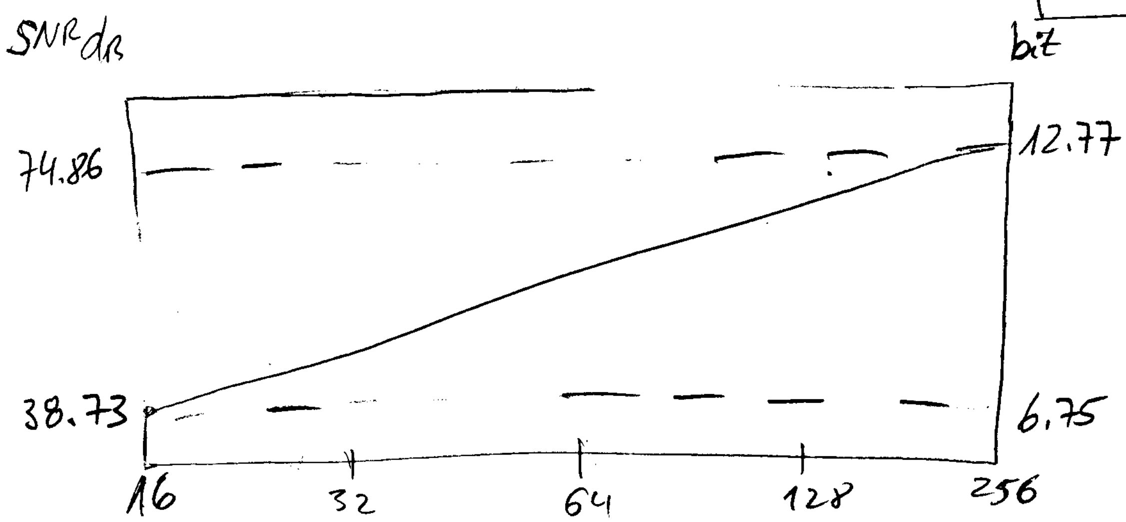
$$30\log_{10} M = 30\log_{10} 2 \,\log_2 M$$
$$\cong 9\log_2 M$$

9 dB/oitava

3dB (oversampling)
6dB (filtro)

$$SNR = 6n_b - 1{,}76\ dB$$
$$= 9\log_2 M + 2.61$$

$$\boxed{n_b = \frac{9\log_2 M + 4.37}{6} = \frac{3}{2}\log_2 M + 0.73}$$



$M \equiv$ factor de oversampling