Circuits Theory and Electronic Fundamentals

Integrated Master in Engineering Physics, IST, University of Lisbon

Lab 2: RC Circuit Analysis

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Contents

1 Introduction

The objective of this laboratory assignment is to analyse the RC circuit shown in Figure 1. As shown below, the nodes have been numbered, current directions have been assigned to all branches and potential $0V$ has been assigned to one of the nodes. By running the Python script `t2_datagen.py`, the values shown in Table 1 have been obtained.

In Section 2, a theoretical analysis of the circuit and the results obtained with the Octave math tool are presented. In Section 3, the results obtained using the Ngspice simulation tool are shown. The conclusions of this study are outlined in Section 4, in which the theoretical results obtained in Section 2 are compared to those presented in Section 3.

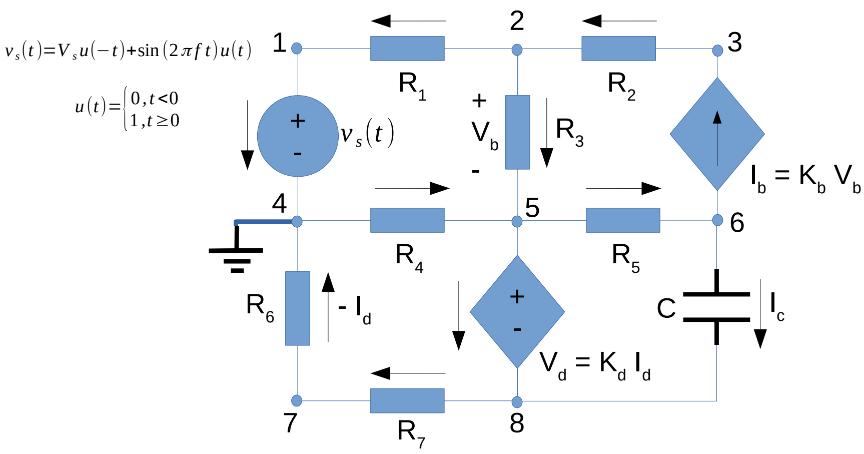


Figure 1: Circuit to be analysed in this laboratory assignment.

Designation	**Value [V, kΩ, mS or μF]**
R_1	1.04053890347
R_2	2.00185929606
R_3	3.06593231919
R_4	4.15163583349
R_5	3.03409481751
R_6	2.05654586148
R_7	1.00587575204
V_s	5.16821048288
C	1.0127707267
K_b	7.29055867767
K_d	8.22649929708

Table 1: Values obtained by running the file `t2_datagen.py`. Resistances R_i and constant K_d are in kΩ, voltage V_s is in volts, capacitance C is in microfarads and constant K_b is in milisiemens.

2 Theoretical Analysis

2.1 Exercise 1

In this exercise, the circuit shown in Figure 1 is analysed theoretically for $t < 0$, by using the node method. The Kirchhoff Current Law (KCL) states that the sum of the currents converging or diverging in a node is null. Using KCL and Ohm's Law (which can also be written as $I = VG$) in nodes not connected to voltage sources and additional equations for nodes related by voltage sources, it is possible to obtain a linear system from which the node voltages V_1 to V_8 are obtained. Using these values and Ohm's Law, the currents in all branches can be determined.

It is worth mentioning that the current passing through the capacitor is given by $i_c = C\frac{dv_c}{dt}$. However, it is assumed that the voltages in the capacitor's terminals have already achieved static values a long time ago, thus $i_c = 0$.

The following linear system is obtained:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 0 & -\frac{1}{R_3} & 0 & 0 & 0 \\ 0 & -K_b-\frac{1}{R_2} & \frac{1}{R_2} & 0 & K_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_3} & 0 & 0 & \frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5} & -\frac{1}{R_5} & -\frac{1}{R_7} & \frac{1}{R_7} \\ 0 & K_b & 0 & 0 & -\frac{1}{R_5}-K_b & \frac{1}{R_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_6}+\frac{1}{R_7} & -\frac{1}{R_7} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{K_d}{R_6} & -1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{pmatrix} = \begin{pmatrix} V_s \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

The matrix shown above isn't symmetrical because of the presence of dependent sources in this circuit. The values obtained by solving the linear system above, using Octave, are shown in Table 2.

Designation	Value [A or V]
I_1	-2.3414328285E-04
I_2	-2.4510896948E-04
I_3	-1.0965686632E-05
I_4	-1.1942751099E-03
I_5	-2.4510896948E-04
I_6	-9.6013182701E-04
I_7	-9.6013182701E-04
I_b	-2.4510896948E-04
I_c	4.3368086899E-19
I_{V_s}	-2.3414328285E-04
I_{V_d}	-9.6013182701E-04
V_1	5.1682104829E+00
V_2	4.9245752881E+00
V_3	4.4339016190E+00
V_5	4.9581953411E+00
V_6	5.7018791952E+00
V_7	-1.9745551353E+00
V_8	-2.9403284589E+00

Table 2: Values of node voltages (in volts) and currents (in amperes). Current I_i corresponds to the current passing through resistance R_i. Node 4 is connected to GND, thus it isn't necessary to show its voltage value ($V_4 = 0$) above.

2.2 Exercise 2

In this exercise, the goal is to obtain R_{eq}, the value of the equivalent resistance as seen from the capacitor's terminals. According to Thévenin's Theorem, it is possible to replace the linear sub-circuit connected to the capacitor's terminals by a voltage source connected with the resistor. In order to calculate R_{eq} by Thévenin's Theorem, we must have $V_s = 0$ (all independent sources of the sub-circuit connected to the capacitor's terminals must be switched off). By replacing the capacitor with a voltage source $V_x = V_6 - V_8$, where V_6 and V_8 are the voltages determined in Section 2.1, the current I_x supplied by V_x can be calculated by applying the node method to this new circuit - using Ohm's Law, the equivalent resistance is computed as $R_x = \frac{V_x}{I_x}$. The linear system and the node voltages and branch currents in this case are the following:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 0 & -\frac{1}{R_3} & 0 & 0 & 0 \\ 0 & -K_b-\frac{1}{R_2} & \frac{1}{R_2} & 0 & K_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & K_b-\frac{1}{R_3} & 0 & 0 & \frac{1}{R_3}+\frac{1}{R_4}-K_b & 0 & -\frac{1}{R_7} & \frac{1}{R_7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_6}+\frac{1}{R_7} & \frac{1}{R_7} \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{K_d}{R_6} & -1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V_x \\ 0 \\ 0 \end{pmatrix} \quad (2)$$

Designation	Value [A or V]
I_1	-0.0000000000E+00
I_2	0.0000000000E+00
I_3	-0.0000000000E+00
I_4	0.0000000000E+00
I_5	-2.8483643966E-03
I_6	0.0000000000E+00
I_7	-0.0000000000E+00
I_b	0.0000000000E+00
I_{V_x}	-2.8483643966E-03
I_{V_s}	-0.0000000000E+00
I_{V_d}	2.8483643966E-03
V_1	0.0000000000E+00
V_2	-0.0000000000E+00
V_3	0.0000000000E+00
V_5	0.0000000000E+00
V_6	8.6422076540E+00
V_7	0.0000000000E+00
V_8	-0.0000000000E+00

Table 3: Values of node voltages (in volts) and branch currents (in amperes). Current I_i corresponds to the current passing through resistance R_i.

Now, the current going through the equivalent resistor can be calculated as $I_x = I_b - I_5 = -I_{V_x}$ (KCL in node 6). It must be stated that the current going through R_{eq} (i.e., from + to - in R_{eq}) is flowing in the oposite direction of the current "passing through" voltage source V_x, i.e., the current going from + to - in V_x, thus the equation shown before.

Finally, one can compute the time constant as $\tau = R_x C$. The final results are shown in Table 4.

Designation	Value [A or V or Ω or s]
V_x	8.6422076540E+00
I_x	2.8483643966E-03
R_{eq}	3.0340948175E+03
τ	3.0728424132E-03

Table 4: Values determined for V_x [V], I_x [A], R_{eq} [Ω] and τ [s].

This procedure must be done in order to determine R_{eq}, which is used to obtain the time constant. This static analysis also allows us to obtain the initial condition $v_6(0)$, which will also be used to compute the natural and final solutions, because we may consider that, for $t = 0$, $v_s = 0$ and $V_6 - V_8 = V_x$. It is worth noting that, according to the expression for $v_s(t)$ given in Figure 1, $v_s(0) = V_s$, so the change in value for v_s would occur instantaneously afterwards, in an instant $t = \delta$; however, it is valid to consider it to be happening at $t = 0$, because it doesn't change the overall solution.

2.3 Exercise 3

Using the results from the previous exercise, we can infer that the circuit shown in Figure 1 can be simplified into a circuit with the independent voltage source, a resistance R_{eq} and the capacitor. We also have that $v_{6n}(+\infty) = 0$, because, as seen in Table 3, $v_8 = 0$, and it will continue to be that way, and the capacitor will eventually completely discharge. Thus, the general form for the wanted natural solution, as learnt in class, is given by

$$v_{6n}(t) = v_{6n}(+\infty) + [v_{6n}(0) - v_{6n}(+\infty)]e^{-\frac{t}{RC}} = v_{6n}(0)e^{-\frac{t}{RC}} \tag{3}$$

Where t stands for time and R for the (equivalent) resistance. The graph shown below is obtained by plotting Eq. 3 in Octave.

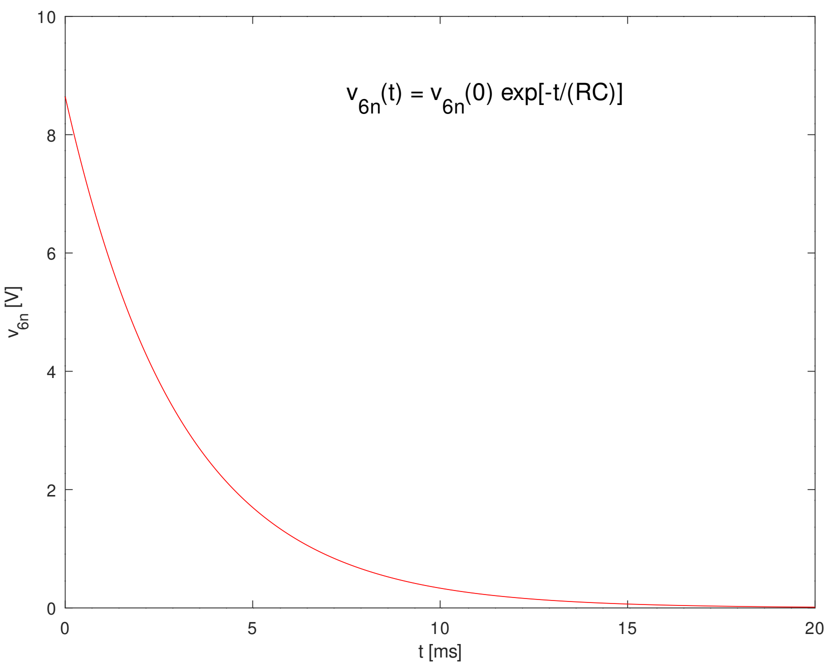


Figure 2: Natural solution $v_{6n}(t)$ in the time interval [0,20] ms.

2.4 Exercise 4

In this exercise, the forced solution for the voltage in node 6, $v_{6f}(t)$, will be determined for the time interval [0,20] ms. Because we are dealing with a forced solution with sinusoidal excitation, given by $v_s(t) = sin(2\pi ft)$, with $f = 1$kHz, it becomes much more efficient to use phasors. As suggested, a phasor voltage source $V_s = 1$ will be used. The correspondant phase is $\phi_{V_s} = 0$; therefore, in this forced solution analysis, the voltages will be considered as given by sin functions, as opposed to cos functions, as was learnt in class. This won't change the overall results, but will be important to take into account when analysing the phases plotted in Sections 2.6 and 3.5. In addition to this, C is replaced by its impedance, Z_C. This impedance is given by $Z_C = \frac{1}{j\omega C}$, with j being the imaginary unit and $\omega = 2\pi f$.

By running nodal analysis as in the previous sections, the following linear system is obtained:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 0 & -\frac{1}{R_3} & 0 & 0 & 0 \\ 0 & -K_b-\frac{1}{R_2} & \frac{1}{R_2} & 0 & K_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_3} & 0 & 0 & \frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5} & -\frac{1}{R_5}-jwC & -\frac{1}{R_7} & \frac{1}{R_7}+jwC \\ 0 & K_b & 0 & 0 & -K_b-\frac{1}{R_5} & \frac{1}{R_5}+jwC & 0 & -jwC \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_6}+\frac{1}{R_7} & -\frac{1}{R_7} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{K_d}{R_6} & -1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{pmatrix} = \begin{pmatrix} Vs \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

The complex amplitudes (phasors) in the nodes, obtained by solving the linear system above, are shown below.

Designation	**Value [V]**
V_1	1.0000000000E+00 + (0.0000000000E+00) i
V_2	9.5285888692E-01 + (5.3839669869E-17) i
V_3	8.5791815826E-01 + (9.6386837769E-16) i
V_4	0.0000000000E+00 + (0.0000000000E+00) i
V_5	9.5936405020E-01 + (-8.5138154658E-18) i
V_6	-5.6445199131E-01 + (-8.6377545528E-02) i
V_7	-3.8205780160E-01 + (3.3905477482E-18) i
V_8	-5.6892583392E-01 + (5.0488962587E-18) i

Table 5: Complex values of the phasors in the nodes (in volts).

Having the value of the phasor V_6, the respective sine function is easily obtained with Octave. It has the same frequency as the voltage source's and its phase and amplitude are given by the phasor shown in Table 5. The sinusoidal forced solution is plotted in Figure 3.

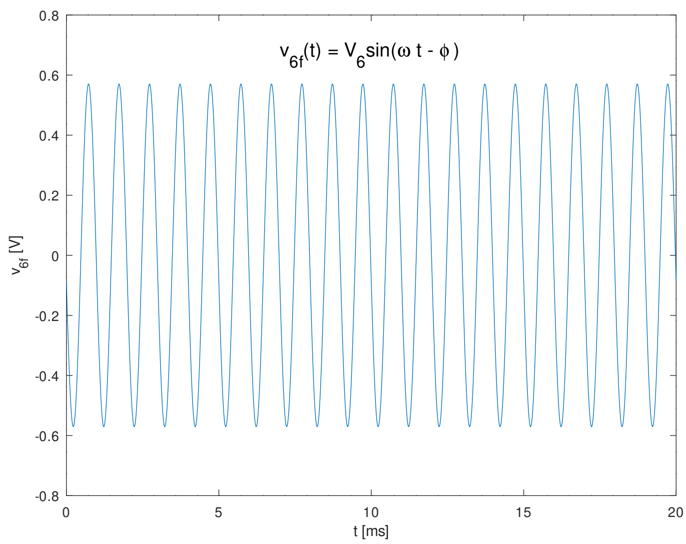


Figure 3: Forced solution $v_{6f}(t)$ in the time interval [0,20] ms. To be noted that V_6 written above refers to the absolute value of the phasor, not the phasor itself. The phase ϕ is the symmetric of the phasor's argument.

2.5 Exercise 5

In Figure 4, the final solution for $v_6(t)$ is plotted alongside $v_s(t)$, which is given by the branch function presented in Figure 1. The value of $v_6(t)$ is given by $v_{6n}(t)+v_{6f}(t)$ for $t > 0$ (obtained by superimposing the natural and forced solutions, previously determined), by the value obtained in Section 2.1 for $t < 0$ and by the value obtained in Section 2.2 for $t = 0$.

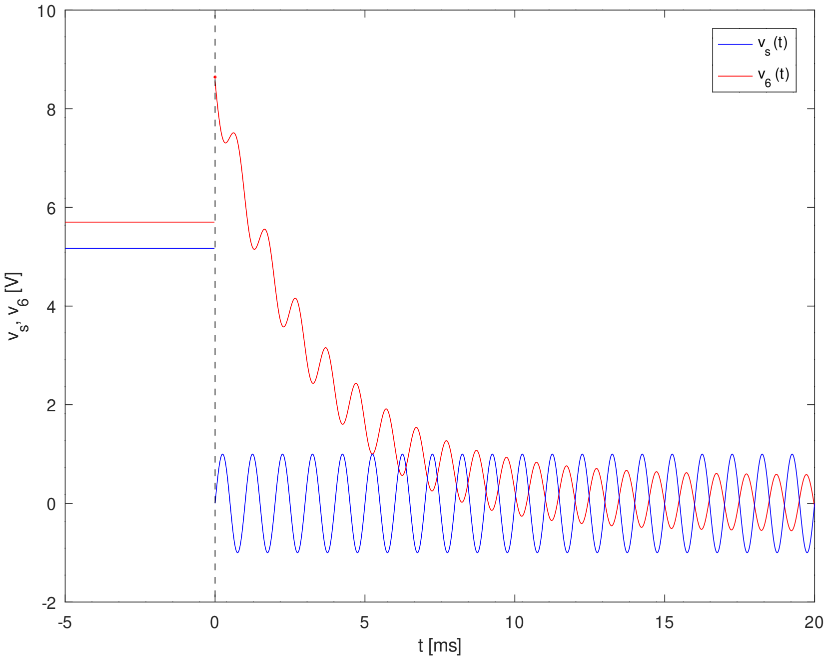
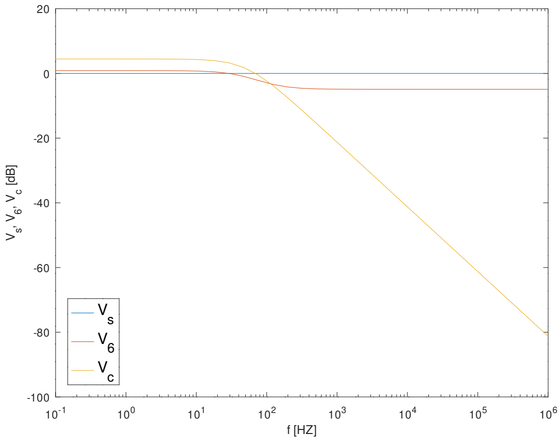


Figure 4: Final solution $v_6(t)$ and $v_s(t)$ in time interval [-5, 20] ms.

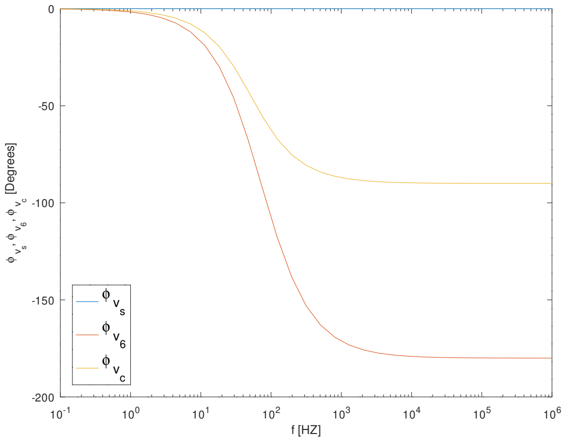
We can notice that neither v_6 nor v_s are continuous functions. What must be continuous is v_c, the voltage drop in the capacitor's terminals; the voltages in the nodes may have discontinuities in order to make sure this stays true. A capacitor cannot be discharged instantaneously: it does it through its equivalent resistor according to the time constant τ.

2.6 Exercise 6

Finally, the magnitudes and the phases of the frequency responses $v_c(f) = v_6(f) - v_8(f)$, $v_6(f)$ and $v_s(f)$ will be plotted for the frequency range 0.1 Hz to 1 MHz. The magnitude and phase of $v_c(f)$ are easily inferred by its respective branch function, given in Figure 1. In its turn, using Octave and the linear system given in Section 2.4, it was possible to get expressions for v_6 and v_8 in terms of frequency, from which the wanted plots were obtained.



(a) Magnitudes of $v_s(f)$, $v_6(f)$ and $v_c(f)$.



(b) Phases of $v_s(f)$, $v_6(f)$ and $v_c(f)$.

The results obtained in Figures (5a) and (5b) seem to behave as predicted. The magnitude and phase of v_s are constant, as they are given by the sine function shown in Figure 1. Again, it is worth mentioning that we are considering the phases of sin functions, not cos functions - this does not change the overall solutions, but it does shift the phases by -$\frac{\pi}{2}$. The impedance of the capacitor, given by $Z_c = \frac{1}{j\omega C}$, with $\omega = 2\pi f$, decreases as the frequency increases. Therefore, the magnitude of the voltage in the capacitor will tend to zero - because the graph is plotted in dB, it tends to $-\infty$. In fact, this magnitude is given by $V_c = \frac{V_s}{\sqrt{1+\omega^2 R^2 C^2}}$, thus its decrease. We can also see that the magnitude of V_6 tends to change to a particular value as the frequency increases; we know that this must be happening in a way that makes it so that the magnitude of $V_6 - V_8 (= V_c)$ tends to zero.

As for the phase of v_c, we can see that it tends to $-\frac{\pi}{2}$ rad $= -90^o$ as the frequency increases. This was to be expected, because the phase delay of the voltage in the capacitor is given by $\phi_{v_c} = \phi_{v_s} - atan(\omega RC) = -atan(2\pi f RC)$. On the other hand, the phase of v_6 seems to tend to $-\pi$ rad, i.e., the voltage in node 6 tends to be in phase opposition with v_s for high frequencies. Something like this had been observed in Figure 4, in which the final solution for $v_6(t)$ and $v_s(t)$ had been plotted for $f = 1$ kHz.

3 Simulation Analysis

3.1 Exercise 1

Table 6 shows the simulated operating point results for the circuit presented in Figure 1. Again, currents designated below as I_i refer to the currents passing through the respective resistances, R_i.

Theoretical		Simulation	
Designation	**Value [A or V]**	**Designation**	**Value [A or V]**
I_1	-2.3414328285E-04	I_1	-2.341432828e-04
I_2	-2.4510896948E-04	I_2	-2.451089695e-04
I_3	-1.0965686632E-05	I_3	-1.096568663e-05
I_4	-1.1942751099E-03	I_4	-1.194275110e-03
I_5	-2.4510896948E-04	I_5	-2.451089695e-04
I_6	-9.6013182701E-04	I_6	-9.601318270e-04
I_7	-9.6013182701E-04	I_7	-9.601318270e-04
I_b	-2.4510896948E-04	I_b	-2.451089695e-04
I_c	4.3368086899E-19	I_c	0.0000000000e+00
I_{V_s}	-2.3414328285E-04	I_{V_s}	-2.341432828e-04
I_{V_d}	-9.6013182701E-04	I_{V_d}	-9.601318270e-04
V_1	5.1682104829E+00	V_1	5.1682104829e+00
V_2	4.9245752881E+00	V_2	4.9245752881e+00
V_3	4.4339016190E+00	V_3	4.4339016190e+00
V_5	4.9581953411E+00	V_5	4.9581953411e+00
V_6	5.7018791952E+00	V_6	5.7018791952e+00
V_7	-1.9745551353E+00	V_7	-1.974555135e+00
V_8	-2.9403284589E+00	V_8	-2.940328459e+00

Table 6: Exercise 1 - comparison between theoretical analysis and operating point analysis's results. Currents I_i are in amperes; voltages V_i are in volts.

Comparing the results, we can notice almost no differences. This was to be expected, since the circuit has no time dependency - meaning it's equal at any point in time. There is only a small difference between the two values of I_c, although it is negligible.

3.2 Exercise 2

Now we can approach the second point of our simulation where we see how the system behaves when $v_s(0) = 0$ and the capacitor is replaced by a voltage souce V_X=V(6)-V(8), where the voltages are taken from the simulation column of Table 6. We use this method since we want to start by determining the natural solution of the system.

Theoretical	
Designation	Value [A or V]
I_1	-0.0000000000E+00
I_2	0.0000000000E+00
I_3	-0.0000000000E+00
I_4	0.0000000000E+00
I_5	-2.8483643966E-03
I_6	0.0000000000E+00
I_7	-0.0000000000E+00
I_b	0.0000000000E+00
I_{V_x}	-2.8483643966E-03
I_{V_s}	-0.0000000000E+00
I_{V_d}	2.8483643966E-03
V_1	0.0000000000E+00
V_2	-0.0000000000E+00
V_3	0.0000000000E+00
V_5	0.0000000000E+00
V_6	8.6422076540E+00
V_7	0.0000000000E+00
V_8	-0.0000000000E+00

Simulation	
Designation	Value [A or V]
I_1	0.000000e+00
I_2	0.000000e+00
I_3	0.000000e+00
I_4	0.000000e+00
I_5	-2.84836e-03
I_6	0.000000e+00
I_7	0.000000e+00
I_b	0.000000e+00
I_x	-2.84836e-03
I_{V_s}	0.000000e+00
I_{V_d}	2.848364e-03
V_1	0.000000e+00
V_2	0.000000e+00
V_3	0.000000e+00
V_5	0.000000e+00
V_6	8.642208e+00
V_7	0.000000e+00
V_8	0.000000e+00

Table 7: Exercise 2 comparison. Operating point analysis table. Currents I_i are in amperes; voltages V_i are in volts.

From Table 7 we want the values of potential in nodes v6 and v8 which are their values when t=0. This means that we now have everything we need (initial conditions) to determine the natural solution of the system for $t > 0$, which is what we will do next.

3.3 Exercise 3

Now we do a transient analisis of the values. Figure 6 shows the simulated transient analysis results obtained for the voltage in node 6, when V_s is still at 0V. This represents the discharge of the condenser threw the system, meaning the natural solution.

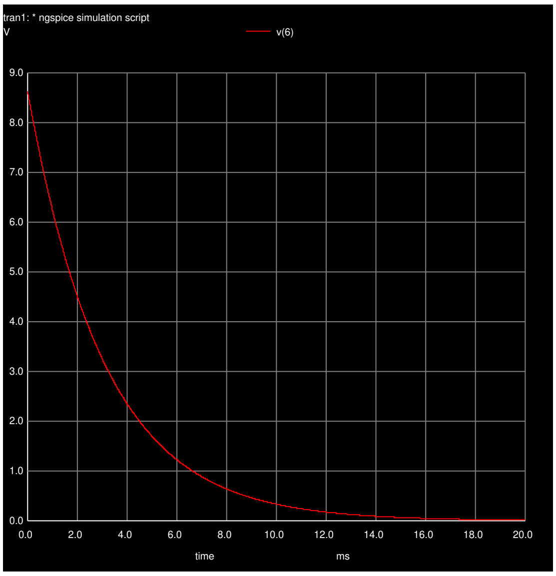


Figure 6: Natural response - value of $v_6(t)$ in the time interval [0,20] ms.

As we can see the solution is an exponential as theory tells us and very similar to the solution in the Theoretical section. We can also clearly see the inicial voltage in the beggining of the graph, which is the one calculated in Exercise 2 confirming that simulation is indeed correct.

3.4 Exercise 4

Having known the inicial conditions and how the system behaves naturally we now add the equation $v_s(t) = sin(2\pi f t)$ (true for $t > 0$) for v_s. We now analise the circuit exactly as represented in Figure 1, having the natural solution added to the forced solution.

In Figure 7 we can see two nodes represented. Node 6 where we observe the natural and forced solutions together and how that afects it's potential over time (the response) and the stimulus in node 1 since V_s is the one responsible for the forced solution. For this particular analisis f=1kHz and all the other values used where taken from the previous sections.

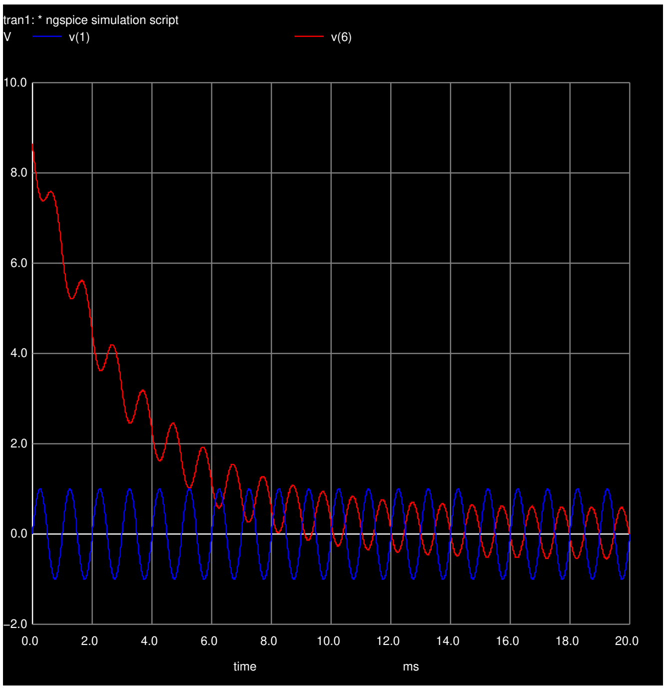
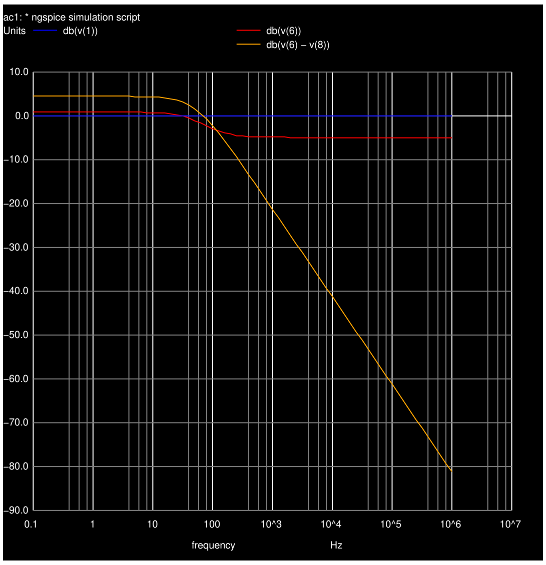


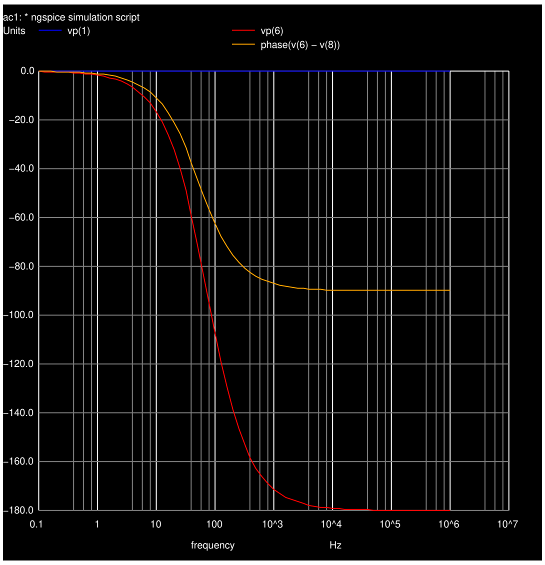
Figure 7: Forced sinusoidal response and stimulus on node 6 in time interval [0,20] ms.

As expected the graph obtained is the sum of an exponential and ans cos() function which correlates with theoretical results and the graph obtained in Figure 4 as they are identical.

3.5 Exercise 5



(a) Frequency analysis - magnitudes of $v_6(f)$, $v_s(f)$ and $v_c(f) = v_6(f) - v_8(f)$ in interval f=[0.1,10e+6] Hz and in dB.



(b) Frequency analysis - phases of $v_6(f)$, $v_s(f)$ and $v_c(f) = v_6(f) - v_8(f)$ in interval f=[0.1,10e+6] Hz and in degrees.

4 Conclusion

In this laboratory assignment, the objective of analysing the given circuit has been achieved. Static analysis has been performed both theoretically, using the Octave maths tool, and by circuit simulation, using the Ngspice tool. It is possible to verify that the simulation results matched the theoretical results precisely. The reason for this perfect match is the fact that this is a straightforward circuit containing only linear components, so the theoretical and simulation models shall not differ. Only algebraic equations are involved.

For more complex components, like transistors, it should be expected that the theoretical and simulation models could differ, but this is not the case in this lab assignment.