Counting the Real Numbers Between 0 and 1

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Georg Cantor's work in set theory defined a method by which two infinite sets could be compared in size, as well as classifications for different sizes of infinity, such as countable and uncountable. The set of integers are defined as countable, whereas the real numbers between 0 and 1 are defined as uncountable. The proof for the latter involves assuming you have a list of all the reals and then constructing a number that is not in that set, using his diagonal argument. [1]

Another way to visualize the uncountable dilemma is to count on a number line. For integers you can start counting upwards from zero and no matter where you stop there are no gaps in sequence behind you, i.e. the set that you have just counted is complete. For real numbers you cannot specify a starting number, such as some very small number 0.00…001, because one can always inject another zero between the decimal and the 1.

However, this approach is artificially constrained by insisting that the sequence begin with the smallest number possible, a number that is already 'at infinity', which is problematic. Cantor's list of reals has the same issue. In his diagonal proof he assumes that all numbers in the list are also 'at infinity' with regards to their precision (moving to the right) but not with regards to their completeness (moving down the list). This inconsistency leads to the conclusion that the reals are uncountable. Summarizing:

Adding precision (digits to the right) without first requiring full enumeration of that precision (down the list) yields incorrect results about countability.

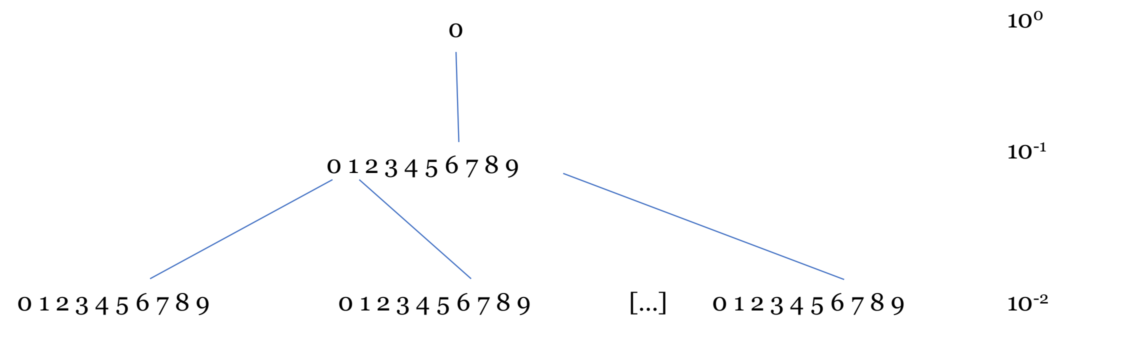
Method of Counting

Instead of using a one-dimensional number list we employ a two-dimensional graph. Consider the following algorithm:

1. Construct a graph beginning with zero as the root node
2. Move down one layer
3. For every node in the layer above, create one child node for each of the digits 0 through 9
4. Evaluate completeness for the layers thus enumerated

Repeat steps 2 – 4 until you decide to stop counting.

Figure 1 – Graph of the Digits of the Reals



At any point that you stop counting you have produced an exhaustive list of all permutations of all possible digits up to that point. Adding further precision *appends* items to the set, it does not *inject* into a prior gap.

Establishing a One-to-One Correspondence

If we traverse the graph above we can generate a list of real numbers as follows, taking one digit from each level of precision:

Real Numbers
0
~~0.0~~ 0.1 0.2 … 0.9
~~0.00~~ 0.01 0.02 … ~~0.10~~ 0.11 … 0.99
~~0.000~~ 0.001 0.002 … 0.999

Note 1: We can remove items with trailing zeros to the right, since they are duplicates with elements in the layer(s) above. (The graph contains placeholder zeros that become leading zeros for the layers below.)

Note 2: The quantity of numbers removed in a given layer is the same as the quantity of numbers in the layer above.

Using Cantor's diagonal algorithm on this list will produce duplicates. The only way to generate a missing number is to add more digits to the right, which is not allowed until you enumerate downward to the same level of precision.

Numbers in this set can now be mapped in a one-to-one correspondence to the integers.

Real Numbers	Graph Level	Precision	Count in Level = Current – Previous	Corresponding Integer
0	Level 0	10^0	$10^0 = 1$	0
0.1 0.2 … 0.9	Level 1	10^{-1}	$10^1 - 10^0 = 9$	1 2 … 9
0.01 0.02 … 0.09 0.11 … 0.99	Level 2	10^{-2}	$10^2 - 10^1 = 90$	10 11 … 18 19 … 99
0.001 0.002 … 0.999	Level 3	10^{-3}	$10^3 - 10^2 = 900$	100 101 … 999

For a given level N in the graph, the total count of numbers across all levels is 10^N, which is an exact mapping to the count of integers up to that level.

Conclusion

Constructing real numbers in this manner creates a countable set, one with as many elements as the integers. The approach requires that we change the concept of what it means to count up to a given point: count from lesser precision to greater precision, rather than attempting to enumerate numbers that are already at infinite precision.

References

[1] https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument