Counting the Real Numbers Between 0 and 1

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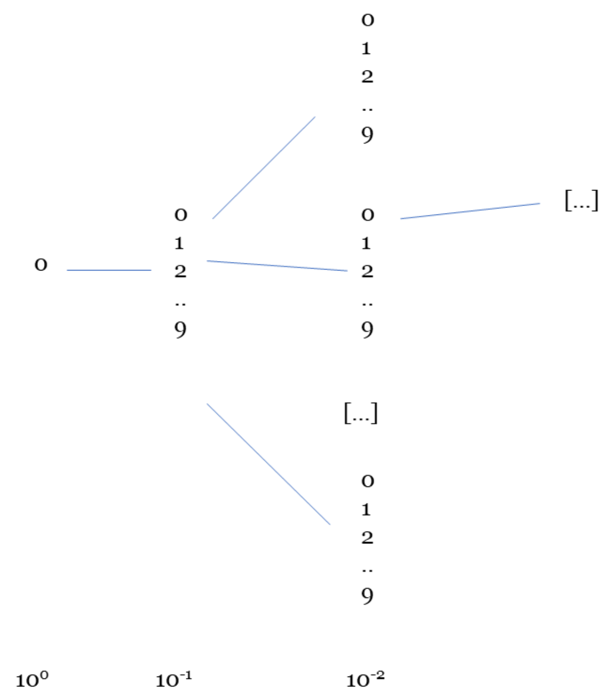
Georg Cantor defined countable infinite sets and uncountable ones. The set of integers on the number line are countable. The real numbers between 0 and 1 are defined as uncountable. The proof involves assuming you have a set of all the reals and then constructing a number that is not in that set. [1]

Another way to visualize this is to count on a number line. Integers are countable, you can start counting upwards from zero and no matter where you stop there are no gaps in sequence for the set that you have counted. For real numbers you cannot specify a starting number, e.g. some very small number 0.00…001, because someone can always inject another zero. However this approach is artificially constrained by thinking that the sequence must proceed from smaller to larger, and that at any given stopping point the set of all smaller numbers must have been fully listed.

Instead if we approach not a one-dimensional number line but rather a two-dimensional number graph the reals begin to look countable, at least by the 'no gaps thus far' definition. Consider this algorithm:

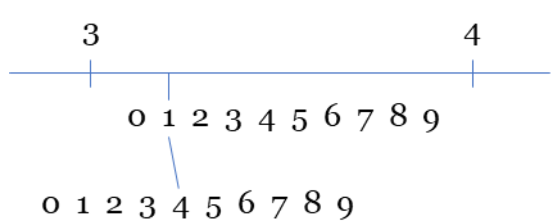
1. Construct a graph proceeding from left to right on a page
2. Begin with the root node zero on the left. Zero is not part of the set but is necessary to retain in each column as a leading zero for the next column.
3. Move to the right one column. Enumerate the digits 0 through 9.
4. Move to the right another column. For each digit in the previous column enumerate the digits 0..9.

Repeat step 4 until you decide to stop counting.



At any point you stop counting you have produced an exhaustive list of all permutations of all possible digits up to that point. Adding further precision *appends* items to the set, it does not *inject* into a prior gap. Thus there can be no construction of a number that has the same number of digits and does not exist in the set up until now. We can require the same number of digits because adding more digits means resuming counting. Once you stop counting the set that is behind you is complete.

The set of numbers in the graph increases exponentially. Numbers in this set can now be mapped in a one-to-one correspondence to other sets. This set can also contain irrational and transcendental numbers. To illustrate this concept, insert this graph (rotated to be vertically aligned) between two integers on a number line, for example:



Pi, e and the square root of 2 all can be represented on the graph at any precision. If the graph is continued ad infinitum then all permutations of digits between any two integers will be represented.

Computer scientists will recognize this graph as a B Tree. [2] Traversing B Trees is well known and many algorithms exist to that can traverse all nodes exhaustively, ensuring all values are read. Describing these algorithms is out of scope, however any of these could be used to output the list of numbers contained in the graph in ordinary decimal format, ensuring that none are missed.

Constructing real numbers in this manner creates countable sets. The approach requires that we change the concept of what it means to count up to a given point: previously this implied completeness *up to a given magnitude*, here it means *up to a given precision*.

References:

[1] https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

[2] https://en.wikipedia.org/wiki/B-tree