Introduction to Fourier Analysis

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A function $f(t)$ is a *periodic function of period* T if there exists a number $T > 0$

$$f(t+T) = f(t) \;\; \forall t.$$

If there is such a number, then the smallest one for which the equation holds is known as the *fundamental period* of the f.

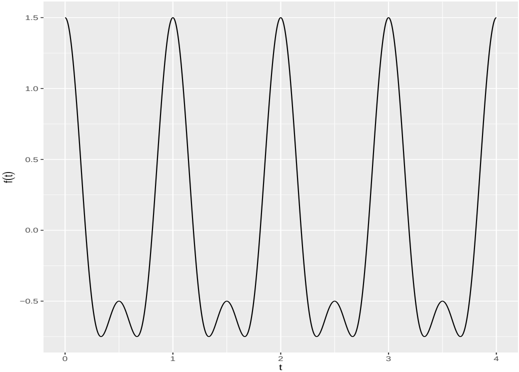
Every integer multiple of the fundamental period is also a period:

$$f(t+nT) = f(t), \;\; n = 0, \pm 1, \pm 2, \ldots$$

Consider the function

$$f(t) = \cos 2\pi t + \frac{1}{2} \cos 4\pi t$$

Visualising this:

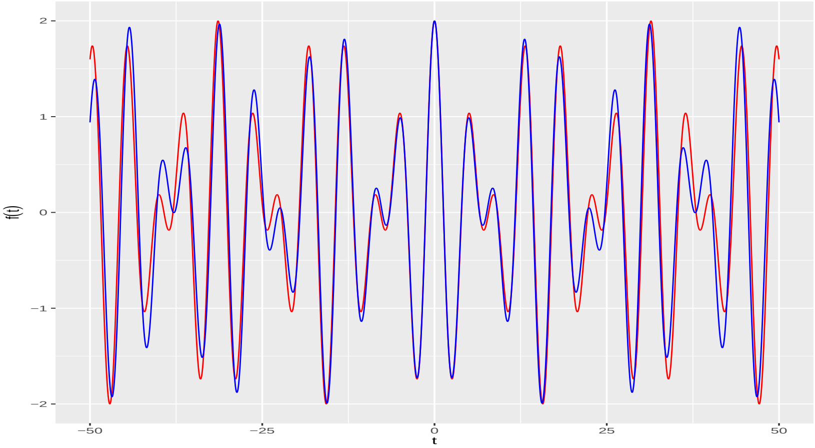


Is the sum of two periodic functions also periodic?

Mathematician? No.

Engineer? Yes.

$$f(t) = \cos t + \cos \sqrt{2}t$$



Building Blocks

Consider the harmonic oscillator: mass on a spring or current in an LC circuit (no resistance)

State is described by a single sinusoid:

$$f(t) = A\sin(2\pi\nu t + \phi),$$

where the parameters are *amplitude* A, *frequency* ν, and *phase* ϕ.

Building Blocks

Consider how temperature heats on a ring. Work via angle θ

$$T(\theta) = \sum_{n=1}^{N} A_n \sin(n\theta + \phi)$$

Time dependence is embedded in the co-efficients A_n.

More generally, use sin and cos:

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right).$$

Building Blocks

Re-write using complex exponentials:

$$\sum_{n=-N}^{N} c_n \exp(2\pi i n t)$$

Co-efficients c_n are complex, with $c_{-n} = \overline{c_n}$ and $|c_{-n}| = |c_n|$.

Fourier Series

We can represent any function of period 1 (and later more general size of periods) as a *Fourier series*:

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \exp(2\pi int)$$

where

$$\hat{f}(n) = \int_0^1 exp(-2\pi int)f(t)dt$$

and so

$$\hat{f}(0) = \int_0^1 f(t)dt, \text{ (average value of the function)}$$

Set of frequencies present in a signal is the *spectrum* of the signal

Note that the spectrum is the frequencies — not the values of $\hat{f}$ at the frequencies, such as $\hat{f}(\pm 2)$.

The sequence of squared magnitudes

$$|\hat{f}(0)|^2, |\hat{f}(\pm 1)|^2, |\hat{f}(\pm 2)|^2, \ldots$$

is the *power spectrum* or *energy spectrum*

If the period is T instead of 1, we modify the series as follows:

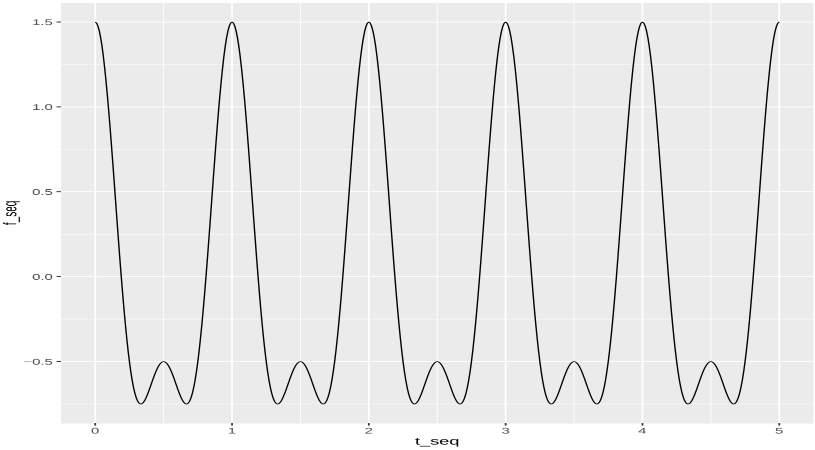
$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(2\pi i n t / T)$$

where the coefficients c_n are given by

$$c_n = \frac{1}{T} \int_0^T \exp(-2\pi i n t / T) f(t) dt.$$

Can also have limits from $-T/2$ to $T/2$

$$f(t) = \cos 2\pi t + \frac{1}{2}\cos 4\pi t$$



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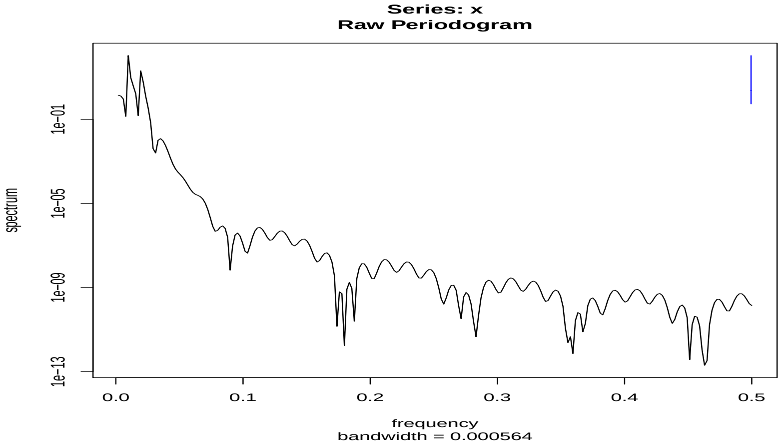

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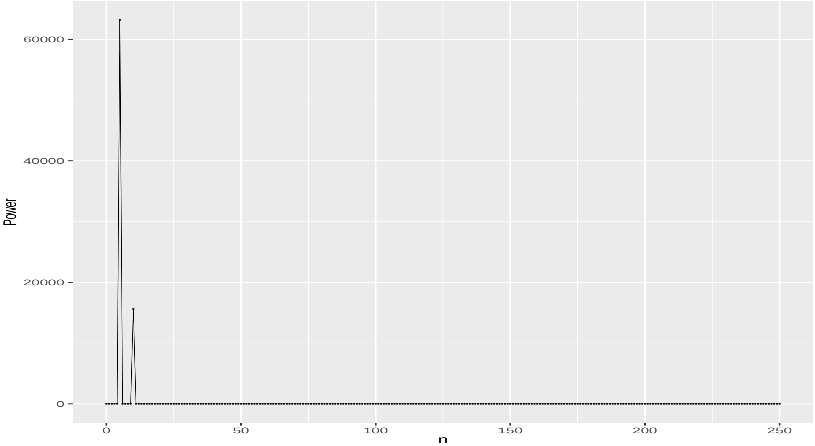
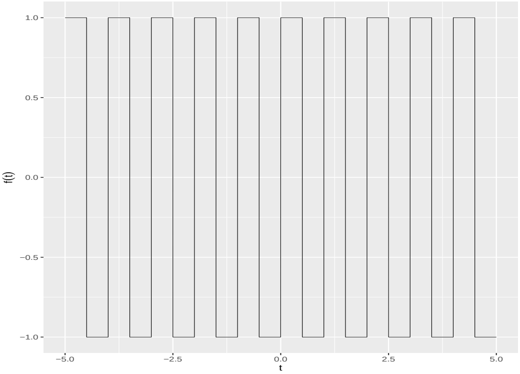


Does it work?

Consider the square wave of period 1. Can define it as:

$$f(t) = \begin{cases} +1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \end{cases}$$

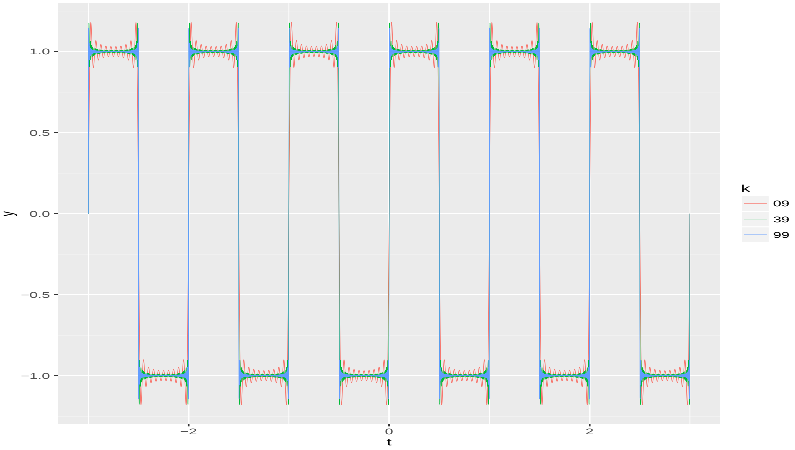


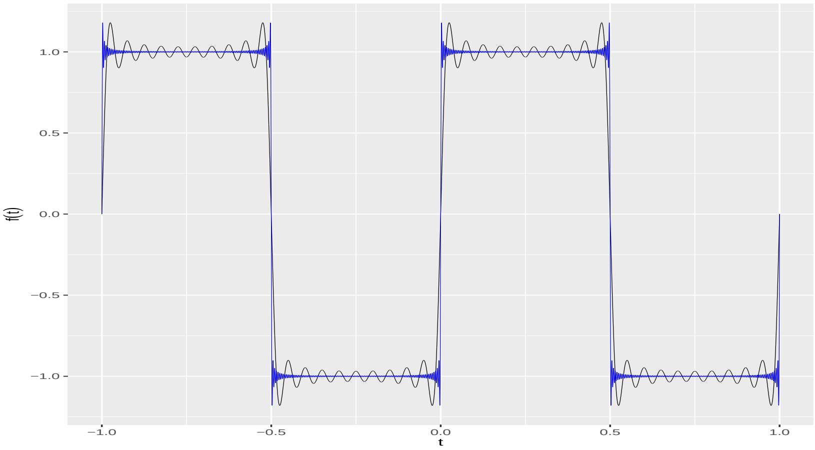
We can show that the Fourier series for this function is

$$\sum_{n \text{ odd}} \frac{2}{\pi i n} \exp(2\pi i n t).$$

This can be rewritten as

$$\frac{4}{\pi} \sum_{k=0}^{N} \frac{1}{2k+1} \sin 2\pi(2k+1)t.$$

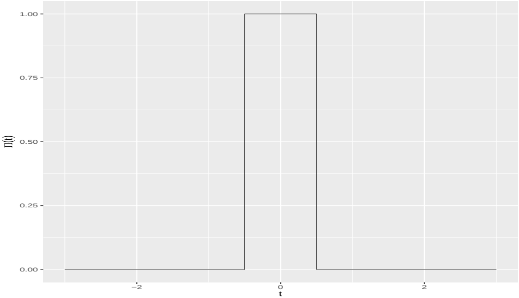

1.0
0.5
0.0
−0.5
−1.0
y
−2
0
2
t
k
09
39
99


f(t)
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
t

The Fourier Transform

Consider the rectangle function $\Pi(t)$:

$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| \geq \frac{1}{2} \end{cases}$$



The n-th Fourier coefficient is given by

$$c_n = \frac{1}{T}\int_0^T \exp(2\pi int/T)\, f(t)dt.$$

Some algebraic manipulation gives us

$$c_n = \frac{1}{\pi n}\sin\frac{\pi n}{T}.$$

Scale up for behaviour as $T \to \infty$:

$$\text{Transform of } \Pi(t) = T\frac{1}{\pi n}\sin\frac{\pi n}{T} = \frac{\sin(\pi n/T)}{\pi n/T}$$

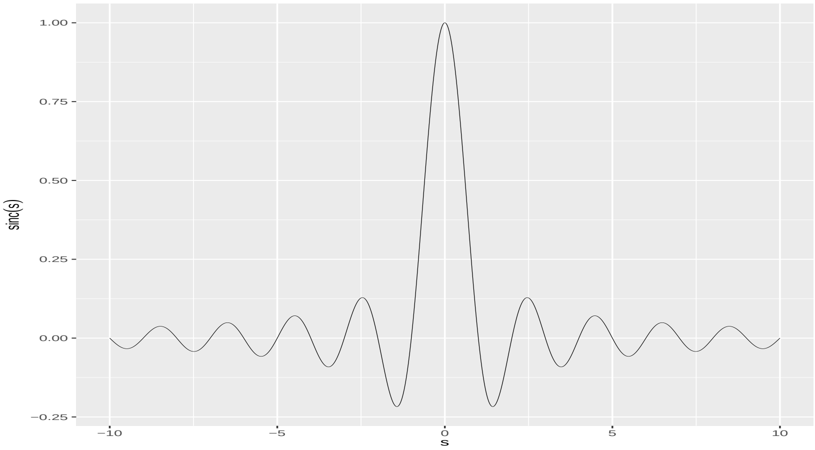
For T large, can make $s = n/T$ closer together.

For $T \to \infty$, can have s continuous and get the *Fourier Transform* of Π

$$\hat{\Pi}(s) = \int_{-\infty}^{\infty} \exp(2\pi i s t)\,\Pi(t)\,dt.$$

Thus

$$\hat{\Pi}(s) = \frac{\sin \pi s}{\pi s} = \text{sinc } s$$


1.00
0.75
0.50
0.25
0.00
−0.25
sinc(s)
−10
−5
0
5
10
s

Define the *Fourier Transform* of a function f, $\hat{f}$, to be

$$\hat{f}(s) = \int_{-\infty}^{\infty} \exp(-2\pi ist)\, f(t) dt.$$

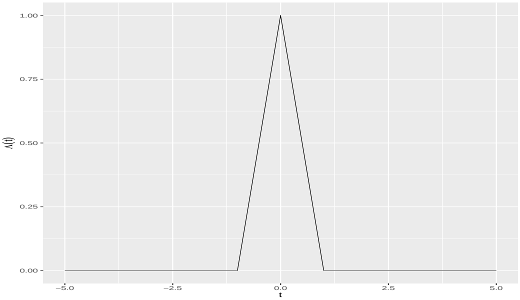
Similarly, the *inverse Fourier Tranform* or *Fourier inversion* of $\hat{f}$ is

$$f(t) = \int_{-\infty}^{\infty} \exp(2\pi ist)\, \hat{f}(s) ds.$$

Way to move from time domain to frequency domain, with $t \in \mathbb{R}$ and $s \in \mathbb{C}$

Fourier transform of the 'triangle' function, $\Lambda(t)$:

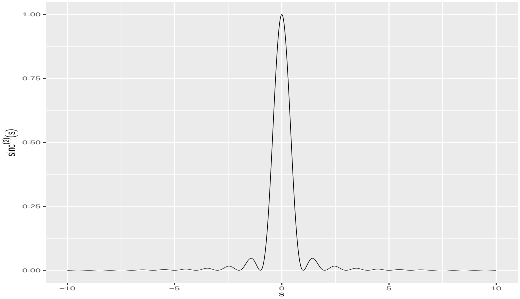
$$\Lambda(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$



The transform of $\Lambda(t)$ is closely related:

$$\mathcal{F}\Lambda(s) = \text{sinc}^2 s$$

This is due to convolution — important, but not covered here.



Summary

- Fourier series is a good way to look at periodic functions
- Fourier transforms moves signal from time (or spatial) domain back to frequency

Links

Based entirely on Stanford course

EE261 - The Fourier Transform and Its Applications

Stanford Engineering Everywhere

https://see.stanford.edu/Course/EE261

YouTube Playlist

https://www.youtube.com/playlist?list=PLB24BC7956EE040CD